

Let (X, Y) be a bivariate random vector with joint distribution function $H(x, y)$ and continuous marginal distribution function $F(x)$ and $G(y)$. Its associated copula C is defined by $H(x, y) = C\{F(x), G(y)\}$. Since F and G are continuous, the copula C is unique and we can write $C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v))$ for $0 \leq u, v \leq 1$ and where $F^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | F(v) \geq u\}$ and $G^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | G(v) \geq u\}$ are the generalized inverse functions of F and G respectively.

We suppose that we observe sequentially a quadruple $(I_t, J_t, I_t X_t, J_t Y_t)$ for $t \in \{1, \dots, T\}$. At each $t \in \{1, \dots, T\}$, one of both entries may be missing. The observations consist of a sample of independent, identically distributed quadruple (I, J, X, Y) of law \mathbb{P} :

$$(I_t, J_t, I_t X_t, J_t Y_t) \quad t \in \{1, \dots, T\}$$

The indicator variables I_t (respectively J_t) is equal to 1 or 0 according to wheter X_t or Y_t is observed or not. The probability of observing a realisation partially or completely is denoted by $p_X = \mathbb{P}(I_t = 1) > 0$, $p_Y = \mathbb{P}(J_t = 1) > 0$ and $p_{XY} = \mathbb{P}(I_t = 1, J_t = 1) > 0$.

Let us define :

$$C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v)) = \varphi(H)(u, v)$$

The function \hat{H}_T corresponds to the empirical distribution function of the sample $(X_1, Y_1), \dots, (X_T, Y_T)$

$$\hat{H}_T(u, v) = \frac{\sum_{t=1}^T 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t}{\sum_{t=1}^T I_t J_t}$$

We define also the corresponding empirical distribution functions in the case of missing data :

$$\hat{F}_T(u) = \frac{\sum_{t=1}^T 1_{\{X_t \leq u\}} I_t}{\sum_{t=1}^T I_t}$$

$$\hat{G}_T(v) = \frac{\sum_{t=1}^T 1_{\{Y_t \leq v\}} J_t}{\sum_{t=1}^T J_t}$$

We have all tools in hand to define the *hybrid copula estimator* ([Seg14]) :

$$\hat{C}_{T,H}(u, v) = \hat{H}_T(\hat{F}_T(u), \hat{G}_T(v)) \quad (1)$$

Given a rate $r_T > 0$ and $r_T \rightarrow \infty$ as $T \rightarrow \infty$, the normalized estimation error of the hybrid copula estimator is :

$$Z_T(u, v) = \sqrt{T} \{ \hat{C}_{T,H}(u, v) - C(u, v) \} \quad (2)$$

Condition 1. We suppose for all $t \in \{1, \dots, T\}$, the pairs (I_t, J_t) and (X_t, Y_t) are independent, the data are missing completely at random. Furthermore, we suppose that there exist at least one $t \in \{1, \dots, T\}$ such that $I_t J_t \neq 0$.

Proposition 1. Under hypothesis 1, $\hat{H}_T, \hat{F}_T, \hat{G}_T$ are consistant estimators of H, F, G .

Démonstration. We check the consistency for \hat{H}_T . By independence, we have

$$\mathbb{E}[T^{-1} \sum_{t=1}^T I_t J_t] = T^{-1} \sum_{t=1}^T \mathbb{E}[I_t J_t] = p_{XY}$$

So, by applying the law of large numbers, we have :

$$T^{-1} \sum_{t=1}^T I_t J_t \longrightarrow p_{XY} \quad a.s. \quad as \quad T \rightarrow \infty$$

Then, we now use the first hypothesis to get :

$$T^{-1} \sum_{t=1}^T \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t] = T^{-1} \sum_{t=1}^T \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}}] \mathbb{E}[I_t J_t] = H(x, y) p_{XY}$$

By applying again the law of large numbers, we derive :

$$\sum_{t=1}^T 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t \longrightarrow H(x, y) p_{XY} \quad a.s. \quad as \quad T \rightarrow \infty$$

We can now apply the continuous mapping theorem to the function $f : (x, y) \mapsto \frac{x}{y}$ which are continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \setminus 0$ to conclude that :

$$\hat{H}_T(x, y) \longrightarrow H(x, y) \quad a.s. \quad as \quad T \rightarrow \infty$$

□

Condition 2. 1. The bivariate distribution function H has continuous margins F, G and copula C .
2. The first order partial derivatives $\dot{C}_1(u, v) = \frac{\partial C}{\partial u}(u, v)$ and $\dot{C}_2(u, v) = \frac{\partial C}{\partial v}(u, v)$ exists and is continuous on the set $\{(u, v) \in [0, 1]^2, 0 < u, v < 1\}$

Condition 3. There exists $\gamma_t > 0$ and $r_t > 0$ such that $r_t \longrightarrow \infty$ as $t \rightarrow \infty$ such that in the space $l^\infty(\mathbb{R}^2) \otimes (l^\infty(\mathbb{R}), l^\infty(\mathbb{R}))$ equipped with the topology of uniform convergence, we have the joint weak convergence

$$(r_t(\hat{H}_T - H); r_t(\hat{F}_t - F), r_t(\hat{G}_t - G)) \rightsquigarrow (\alpha \circ (F, G), \beta_1 \circ F, \beta_2 \circ G)$$

The stochastic processes α and β_j take values in $l^\infty([0, 1]^2)$ and $l^\infty([0, 1])$ respectively, and are such that $\alpha \circ F$ and $\beta_j \circ F_j$ have continuous trajectories on $[-\infty, \infty]^2$ and $[-\infty, \infty]$ almost surely.

Theorem 1 (Theorem 2.3 and example 3.5 in [Seg14]). If conditions 2.1 and 2.2 holds, then uniformly in $u \in [0, 1]^2$,

$$r_T \{\hat{C}_T(u, v) - C(u, v)\} = r_T \{\hat{H}_T((F, G)^\leftarrow(u, v) - C(u, v)\} \quad (3)$$

$$- \dot{C}_1(u, v) r_T \{\hat{F}_T(F^\leftarrow(u)) - u\} 1_{(0,1)}(u) \quad (4)$$

$$- \dot{C}_2(u, v) r_T \{\hat{G}_T(G^\leftarrow(v)) - v\} 1_{(0,1)}(v) + o_{\mathbb{P}}(1) \quad (5)$$

as $T \rightarrow \infty$. Hence in $l^\infty([0, 1]^2)$ equipped with the supremum norm, as $T \rightarrow \infty$,

$$(r_T \{\hat{C}_T(u, v) - C(u, v)\})_{u, v \in [0, 1]^2} \rightsquigarrow (\alpha(u, v) - \dot{C}_1(u, v) \beta_1(u) - \dot{C}_2(u, v) \beta_2(v))_{u, v \in [0, 1]^2}$$

We denote by $S_C(u, v)$ the process defined by $\forall (u, v) \in [0, 1]^2$:

$$S_C(u, v) = \alpha(u, v) - \dot{C}_1(u, v) \beta_1(u) - \dot{C}_2(u, v) \beta_2(v)$$

We denote by \mathbb{G} a \mathbb{P} -Brownian bridge, the process are defined by :

$$\beta_1(u) = p_X^{-1} \mathbb{G}(1_{X \leq F^\leftarrow(u), I=1} - u 1_{I=1})$$

$$\beta_2(v) = p_Y^{-1} \mathbb{G}(1_{Y \leq G^\leftarrow(v), J=1} - v 1_{J=1})$$

$$\alpha(u, v) = (p_{XY})^{-1} \mathbb{G}(1_{X \leq F^\leftarrow(u)} 1_{Y \leq G^\leftarrow(v), I=1, J=1} - C(u, v) 1_{I=1, J=1})$$

Lemma 1. The covariance function of the process $\beta_1(u)$, $\beta_2(v)$ and $\alpha(u, v)$ are : for $(u, u_1, u_2, v, v_1, v_2) \in [0, 1]^6$,

$$\text{cov}[\beta_1(u_1), \beta_1(u_2)] = p_X^{-1} \{u_1 \wedge u_2 - u_1 u_2\}$$

$$\text{cov}[\beta_2(v_1), \beta_2(v_2)] = p_Y^{-1} \{v_1 \wedge v_2 - v_1 v_2\}$$

$$\text{cov}[\beta_1(u), \beta_2(v)] = \frac{p_{XY}}{p_X p_Y} \{C(u, v) - uv\}$$

and

$$\text{cov}[\alpha(u_1, v_1), \alpha(u_2, v_2)] = p_{XY}^{-1} \{C(u_1 \wedge u_2, v_1 \wedge v_2) - C(u_1, v_1) C(u_2, v_2)\}$$

$$\text{cov}[\alpha(u_1, v), \beta_1(u_2)] = p_X^{-1} \{C(u_1 \wedge u_2, v) - C(u_1, v) u_2\}$$

$$\text{cov}[\alpha(u, v_1), \beta_2(v_2)] = p_Y^{-1} \{C(u, v_1 \wedge v_2) - C(u, v_1) v_2\}$$

Démonstration. We do the proof only for the first process. We know that the covariance of a \mathbb{P} -Gaussian process is given by $\mathbb{G}(f)\mathbb{G}(g) = \mathbb{P}(fg) - \mathbb{P}(f)\mathbb{P}(g)$ where f, g are measurable functions. Now, using that, we have :

$$\begin{aligned} \text{cov}[\beta_1(u_1), \beta_1(u_2)] &= p_X^{-2} \mathbb{E} [\mathbb{G}(1_{X \leq F^{\leftarrow}(u_1), I=1} - u_1 1_{I=1}) \mathbb{G}(1_{X \leq F^{\leftarrow}(u_2), I=1} - u_2 1_{I=1})] \\ &= p_X^{-2} (\mathbb{P} [(1_{X \leq F^{\leftarrow}(u_1), I=1} - u_1 1_{I=1})(1_{X \leq F^{\leftarrow}(u_2), I=1} - u_2 1_{I=1})]) \\ &= p_X^{-2} (\mathbb{P}(I=1) \mathbb{P}(X \leq F^{\leftarrow}(u_1), X \leq F^{\leftarrow}(u_2)) - u_1 u_2 \mathbb{P}(I=1)) \\ &= p_X^{-1} (u_1 \wedge u_2 - u_1 u_2) \end{aligned}$$

□

Remark 1. If we consider the empiric copula, theorem 1 gives us the weak convergence of this process to a brownian bridge $N_C(u, v)$ defined by, $\forall (u, v) \in [0, 1]$

$$N_C(u, v) = B_C(u, v) - \dot{C}_1(u, v)B_C(u, 1) - \dot{C}_2(u, v)B_C(1, v) \quad (6)$$

where B_C is a brownian bridge in $[0, 1]^2$ with covariance function

$$\mathbb{E}[B_C(u, v)B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$$

for each $0 \leq u, v, u', v' \leq 1$. This result is well known since 2004 due to [FRW04].

Definition 1. Let $(X_1, Y_1), \dots, (X_T, Y_T)$ a T bivariate random vectors with unknown margins F and G . A λ -FMadogram is the quantity defined by :

$$\nu(\lambda) = \frac{1}{2} \mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \quad (7)$$

We estimative the λ -FMadogram with the following quantity :

$$\hat{\nu}_T(\lambda) = \frac{1}{2 \sum_{t=1}^T I_t J_t} \sum_{t=1}^T |\hat{F}_T^\lambda(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| I_t J_t \quad (8)$$

Proposition 2 (Proposition 3 of [NGCD09]). Suppose that conditions 2 holds and that $\sum_{t=1}^T I_t J_t = T$ (no missing data). Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then

$$T^{-1/2} \sum_{t=1}^T (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u, v) dJ(u, v)$ where $N_C(u, v)$ is defined by equation (6) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Bre20]). The special case, $J(x, y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$ provide the weak of convergence of the λ -Madogram estimator defined by (8) :

$$T^{1/2} \{ \hat{\nu}_T(\lambda) - \frac{1}{2} \mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \}$$

converge in distribution to $\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \quad (9)$$

for all bounded-measurable function $f : [0, 1]^2 \mapsto \mathbb{R}$.

We add some elements in order to prove the identity (9).

Lemma 2. For all bounded-measurable function $f : [0, 1]^2 \mapsto \mathbb{R}$, if $J(s, t) = |s^\lambda - t^{1-\lambda}|$, then the following integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

Démonstration. Let A a element of $\mathcal{B}([0, 1]^2)$. We can pick an element of the form $A = [0, s] \times [0, t]$, where $s, t \in [0, 1]$ and $\lambda \in [0, 1]$. Let us introduce the following indicator function :

$$f_{s,t}(x, y) = 1_{\{(x,y) \in [0,1]^2, 0 \leq x \leq s, 0 \leq y \leq t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x, y) dJ(x, y) = J(s, t) - J(0, 0) = |s^\lambda - t^{1-\lambda}|$$

in other hand, using the equality $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - \min(x, y)$, one has to show

$$\begin{aligned} \frac{1}{2}|s^\lambda - t^{1-\lambda}| &= \frac{s^\lambda}{2} + \frac{y^{1-\lambda}}{2} - \min(s^\lambda, t^{1-\lambda}) \\ &= \int_0^1 f_{s,t}(x^{\frac{1}{\lambda}}, 0) dx + \int_0^1 f_{s,t}(0, y^{\frac{1}{1-\lambda}}) dy - \int_0^1 f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}}) dx \end{aligned}$$

Notice that the class :

$$\mathcal{E} = \{A \in \mathcal{B}([0, 1]^2) : \int_{[0,1]^2} 1_A(x, y) dJ(x, y) = \int_0^1 1_A(x^{\frac{1}{\lambda}}, 0) dx + \int_0^1 1_A(0, y^{\frac{1}{1-\lambda}}) dy - \int_0^1 1_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}}) dx\}$$

contain the class \mathcal{P} of all closed pavements of $[0, 1]^2$. It is otherwise a monotone class (or λ -system). Hence as the class \mathcal{P} of closed pavement is a π -system, the class monotone theorem ensure that \mathcal{E} contains the sigma-field generated by \mathcal{P} , that is $\mathcal{B}([0, 1]^2)$.

This result holds for simple $f(x, y) = \sum_{i=1}^n \lambda_i 1_{A_i}$ where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathcal{B}([0, 1]^2)$ for all $i \in \{1, \dots, n\}$. We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function $f : [0, 1]^2 \mapsto \mathbb{R}$ considering $f = f_+ - f_-$ with $f_+ = \max(f, 0)$ and $f_- = \min(-f, 0)$. We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral. \square

Furthermore, as the limiting process is the linear transformation of a tight gaussian process, we know from [vdVW96] that it is Gaussian. Before going further, we want to detail the structure of the variance of the limiting process. Doing that, we introduce the following lemma :

Lemma 3. Let $(B_C(u, v))_{u,v \in [0,1]^2}$ a brownian bridge with covariance function defined by :

$$\mathbb{E}[B_C(u, v) B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v) C(u', v')$$

for each $0 \leq u, v, u', v' \leq 1$. Let $a, b \in [0, 1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}[\int_0^1 B_C(u, a) du \int_0^1 B_C(b, u) du] = 0$$

Démonstration. Without loss of generality, suppose that $a = 0$ and $b \in [0, 1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned} \mathbb{E}[\int_0^1 B_C(u, 0) du \int_0^1 B_C(b, u) du] &= \mathbb{E}[\int_0^1 \int_0^1 B_C(u, 0) B_C(b, v) dudv] \\ &= \int_0^1 \int_0^1 \mathbb{E}[B_C(u, 0) B_C(b, v)] dudv \end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u, 0) B_C(b, v)] = C(u \wedge v, 0) - C(u, 0) C(b, v)$$

We recall that, by definition, a copula satisfy $C(u, 0) = C(0, u) = 0$ for every $u \in [0, 1]$. Then, the equation below is equal to 0. Our conclusion directly follows. \square

Using this lemma, we can infer the following proposition :

Proposition 3. Let $N_C(u, v)$ the process defined in equation (6) and $a, b \in [0, 1]$ fixed. If $a = 0$ or $b = 0$, then :

$$\mathbb{E}[\int_0^1 N_C(u, a)du \int_0^1 N_C(b, u)du] = 0$$

With this proposition, we can infer a better form of the variance of our limiting process :

Theorem 2. Let $N_C(u, v)$ the process defined in equation (6) and $J(x, y) = |x^\lambda - y^{1-\lambda}|$, then :

$$\text{Var}(\int_{[0,1]^2} N_C(u, v)dJ(u, v)) = \text{Var}(\int_0^1 N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) \quad (10)$$

Démonstration. Recall that, with J defined as in the statement that :

$$\int_{[0,1]^2} N_C(u, v)dJ(u, v) = \frac{1}{2} \int_0^1 N_C(0, v^{1/(1-\lambda)})dv + \frac{1}{2} \int_0^1 N_C(u^{1/\lambda}, 0)du - \int_0^1 N_C(u^{1/\lambda}, u^{1/(1-\lambda)})du$$

Taking the variance and using the proposition 3 gives that only the variance of the third term matters. \square

We are now able to detail precisely the variance of the limiting process with a given Copula. This is the purpose of the following proposition :

Proposition 4. Under the framework of theorem 2 and if we take $C(u, v) = uv$, the independent copula, then the variance of the lambda FMadogram has the following form

$$\text{Var}(\int_{[0,1]^2} N_C(u, v)dJ(u, v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} - \frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} - \frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right)$$

with $\lambda \in [0, 1]$

Démonstration. With direct computing and using the same techniques used in lemma 1, we obtain that :

$$\begin{aligned} \text{Var} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} \right) \\ \text{Var} \left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} \right) \\ \text{Var} \left[\int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right) \\ \text{cov} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} \right) \\ \text{cov} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right) \\ \text{cov} \left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du \right] &= 0 \end{aligned}$$

Using the identity $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{cov}(X, Y)$ gives the desired result. \square

Combining theorem 1 and proposition 2 gives the following result :

Theorem 3. Suppose that the assumption of theorem 1 holds. Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then

$$T^{1/2} \left(\frac{\sum_{t=1}^T J(\hat{F}_T(X_t), \hat{G}_T(Y_t))I_t J_t}{\sum_{t=1}^T I_t J_t} - \mathbb{E}[J(F(X), G(Y))] \right)$$

converges in distribution to $\int_{[0,1]^2} S_C(u, v)dJ(u, v)$ where $S_C(u, v)$ is defined in theorem 1. The special case, $J(x, y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$ provide the weak of convergence of the λ -Madogram estimator defined by (8) :

$$T^{1/2} \left(\hat{\nu}_T(\lambda) - \frac{1}{2} \mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \right)$$

converge in distribution to $\int_{[0,1]^2} S_C(u, v) dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

for all bounded functions f .

Démonstration. Let us introduce the empirical measure define by :

$$\mathbb{P}_n = \sum_{t=1}^T \delta_{\mathcal{X}}((\hat{F}_T(X_t), \hat{G}_T(Y_t)), I_t, J_t, \sum_{t=1}^n I_t J_t)$$

Where $\mathcal{X} = \mathbb{R}^2 \times \{0, 1\} \times \{0, 1\} \times \{0, \dots, T\}$, we consider the following function :

$$f: \mathcal{X} \rightarrow \mathbb{R} \\ ((x, y), I, J, K) \mapsto \frac{1_{\{[0, u] \times [0, v]\}}(x, y) I J}{K}$$

Then, we have :

$$\bar{C}_{T,H}(u, v) := \mathbb{P}_n f = \int_{\mathcal{X}} f(x) d\mathbb{P}_n(x) = \frac{1}{\sum_{t=1}^T I_t J_t} \sum_{t=1}^T 1_{\{\hat{F}_T(X_t) \leq u, \hat{G}_T(Y_t) \leq v\}} I_t J_t$$

Where $\bar{C}_{T,H}(u, v)$ corresponds to the càdlàg version of the ordinary hybrid copula process $C_{T,H}(u, v)$. The problem is that we can't write the following quantity :

$$\sqrt{n} \left(\frac{1}{\sum_{t=1}^T I_t J_t} \sum_{i=1}^n J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))] \right)$$

as an integral of the form :

$$\sqrt{n} \left(\int_{[0,1]^2} J(u, v) d(\bar{C}_{T,H} - C)(u, v) \right)$$

□

Proposition 5. Under the framework of theorem 3 and if take $C(u, v) = uv$, the independent opula, then the variance of the λ -FMadogram has the following form

$$\text{Var}(\int_{[0,1]^2} N_C(u, v) dJ(u, v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1}{p_{XY}(1+2\lambda(1-\lambda))} - \frac{1-\lambda}{p_X(1+\lambda+2\lambda(1-\lambda))} - \frac{\lambda}{p_Y(2-\lambda+2\lambda(1-\lambda))} \right)$$

We now consider the bivariate extreme value copula which can be written in the following form (See Segers extreme value copulas)

$$C(u, v) = (uv)^{A(\log(v)/\log(uv))} \quad (11)$$

for all $u, v \in [0, 1]$ and where $A(\cdot)$ is the pickhands dependence function. With this copula, we want to compute the following integral :

$$\begin{aligned} \text{Var}(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du) &= \text{Var}(\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du - \int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \\ &\quad - \int_0^1 B_C(1, v^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dv) \end{aligned}$$

We have for the following

$$\text{Var} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du \right] = \left(\frac{\lambda(1-\lambda)}{A(1-\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{A(1-\lambda)}{A(1-\lambda) + 2\lambda(1-\lambda)} \right)$$

$$\begin{aligned} \text{Var} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \right] &= \left(\frac{\lambda(1-\lambda)}{A(1-\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{\kappa^2(1-\lambda)}{2A(1-\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} \right) \\ \text{Var} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \right] &= \left(\frac{\lambda(1-\lambda)}{A(1-\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{\zeta^2 \lambda}{2A(1-\lambda) - \lambda + 2\lambda(1-\lambda)} \right) \end{aligned}$$

Where $\kappa := A(1-\lambda) + A'(1-\lambda)(1-\lambda)$ and $\zeta = A(1-\lambda) - A'(1-\lambda)(1-\lambda)$. We now compute the covariance :

$$\begin{aligned} \text{cov} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du, \int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \right] &= \int_0^1 \int_0^1 \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) B_C(v^{\frac{1}{\lambda}}, 1)] \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dudv \\ &= \int_0^1 \int_0^u \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) B_C(u^{\frac{1}{\lambda}}, 1)] \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dudv + \int_0^1 \int_v^1 \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) B_C(u^{\frac{1}{\lambda}}, 1)] \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dudv \end{aligned}$$

for the first one, we have :

$$\int_0^1 \int_0^u (C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) v^{\frac{1}{\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dv du = \frac{\kappa}{2} \left(\frac{\lambda(1-\lambda)}{A(1-\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2A(1-\lambda) + \lambda(1-\lambda)} \right)$$

For the second part, using Fubini, we have :

$$\int_0^1 \int_0^u (C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) v^{\frac{1}{\lambda}}) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} dv du$$

for the right hand side of the "minus" sign, we may compute :

$$\int_0^1 \int_0^u C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) v^{\frac{1}{\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dv du = \frac{\kappa}{2} \left(\frac{\lambda(1-\lambda)}{A(1-\lambda) + \lambda(1-\lambda)} \right)^2$$

The last one still difficult to handle

$$\int_0^1 \int_0^u C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dv du \quad (12)$$

Lemma 4. *We have the following inequalities :*

$$f_\kappa(\lambda) \leq \int_0^1 \int_0^u C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dv du \leq \quad (13)$$

with $f_\kappa(\lambda) := \frac{(\lambda(1-\lambda))^2 \kappa(\lambda)}{(A(1-\lambda)-1+4\lambda)(A(1-\lambda)-(1-\lambda)^2)}$

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