Condition 1. *1. The bivariate distribution function H has continuous margins F, G and copula C.*

2. The first order partial derivatives $\dot{C}_1(u,v) = \frac{\partial C}{\partial u}(u,v)$ and $\dot{C}_2(u,v) = \frac{\partial C}{\partial v}(u,v)$ exists and is continuous on the set $\{(u,v) \in [0,1]^2, 0 < u, v < 1\}$

Definition 1. *Let $(X_1, Y_1), \dots, (X_T, Y_T)$ a T bivariate random vectors with unknown margins F and G. A λ-FMadogram is the quantity defined by :*

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|] \tag{1}$$

We estimative the λ-FMadogram with the following quantity :

$$\hat{\nu}_T(\lambda) = \frac{1}{2T}\sum_{t=1}^{T} |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| \tag{2}$$

Proposition 1 (Proposition 3 of [NGCD09])**.** *Suppose that conditions 2 holds and that $\sum_{t=1}^{T} I_t J_t = T$ (no missing data). Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then*

$$T^{-1/2}\sum_{t=1}^{T}(J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u,v) dJ(u,v)$ where $N_C(u,v)$ is defined by equation (15) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Bre20]). The special case, $J(x,y) = \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (2) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]^2} N_C(u,v) dJ(u,v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2}\int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2}\int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \tag{3}$$

for all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$.

We add some elements in order to prove the identity (3).

Lemma 1. *For all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$, if $J(s,t) = |s^{\lambda} - t^{1-\lambda}|$, then the following integral satisfies :*

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2}\int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2}\int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

Démonstration. Let A a element of $\mathcal{B}([0,1]^2)$. We can pick an element of the form $A = [0,s] \times [0,t]$, where $s, t \in [0,1]$ and $\lambda \in [0,1]$. Let us introduce the following indicator function :

$$f_{s,t}(x,y) = 1_{\{(x,y) \in [0,1]^2, 0 \leq x \leq s, 0 \leq y \leq t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x,y) dJ(x,y) = J(s,t) - J(0,0) = |s^{\lambda} - t^{1-\lambda}|$$

in other hand, using the equality $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - min(x,y)$, one has to show

$$\begin{aligned}\frac{1}{2}|s^{\lambda} - t^{1-\lambda}| &= \frac{s^{\lambda}}{2} + \frac{y^{1-\lambda}}{2} - min(s^{\lambda}, t^{1-\lambda}) \\ &= \int_0^1 f_{s,t}(x^{\frac{1}{\lambda}}, 0) dx + \int_0^1 f_{s,t}(0, y^{\frac{1}{1-\lambda}}) dy - \int_0^1 f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}}) dx\end{aligned}$$

Notice that the class :

$$\mathcal{E} = \{A \in \mathcal{B}([0,1]^2) : \int_{[0,1]^2} 1_A(x,y) dJ(x,y) = \int_0^1 1_A(x^{\frac{1}{\lambda}}, 0) dx + \int_0^1 1_A(0, y^{\frac{1}{1-\lambda}}) dy - \int_0^1 1_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}}) dx\}$$

contain the class $\mathcal{P}$ of all closed pavements of $[0,1]^2$. It is otherwise a monotone class (or λ-system). Hence as the class $\mathcal{P}$ of closed pavement is a π-system, the class monotone theorem ensure that $\mathcal{E}$ contains the sigma-field generated by $\mathcal{P}$, that is $\mathcal{B}([0,1]^2)$.

This result holds for simple function $f(x,y) = \sum_{i=1}^{n} \lambda_i 1_{A_i}$ where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathcal{B}([0,1]^2)$ for all $i \in \{1, \dots, n\}$. We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function $f : [0,1]^2 \mapsto \mathbb{R}$ considering $f = f_+ - f_-$ with $f_+ = max(f, 0)$ and $f_- = min(-f, 0)$. We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral. □

Furthermore, as the limiting process is the linear transformation of a tight gaussian process, we know from [vdVW96] that it is Gaussian. Before going further, we want to detail the structure of the variance of the limiting process. Doing that, we introduce the following lemma :

Lemma 2. *Let $(B_C(u,v))_{u,v\in[0,1]^2}$ a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u\wedge u', v\wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$. Let $a, b \in [0,1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}[\int_0^1 B_C(u,a)du \int_0^1 B_C(b,u)du] = 0$$

Démonstration. Without loss of generality, suppose that $a = 0$ and $b \in [0,1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned}\mathbb{E}[\int_0^1 B_C(u,0)du \int_0^1 B_C(b,u)du] &= \mathbb{E}[\int_0^1\int_0^1 B_C(u,0)B_C(b,v)dudv]\\ &= \int_0^1\int_0^1 \mathbb{E}[B_C(u,0)B_C(b,v)]dudv\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u,0)B_C(b,v)] = C(u\wedge v, 0) - C(u,0)C(b,v)$$

We recall that, by definition, a copula satisfy $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$. Then, the equation below is equal to 0. Our conclusion directly follows. □

Using this lemma, we can infer the following proposition :

Proposition 2. *Let $N_C(u,v)$ the process defined in equation (15) and $a, b \in [0,1]$ fixed. If $a = 0$ or $b = 0$, then :*

$$\mathbb{E}[\int_0^1 N_C(u,a)du \int_0^1 N_C(b,u)du] = 0$$

With this proposition, we can infer a better form of the variance of our limiting process :

Theorem 1. *Let $N_C(u,v)$ the process defined in equation (15) and $J(x,y) = |x^\lambda - y^{1-\lambda}|$, then :*

$$Var(\int_{[0,1]^2} N_C(u,v)dJ(u,v)) = Var(\int_0^1 N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) \tag{4}$$

Démonstration. Recall that, with J defined as in the statement that :

$$\int_{[0,1]^2} N_C(u,v)dJ(u,v) = \frac{1}{2}\int_0^1 N_C(0, v^{1/(1-\lambda)})dv + \frac{1}{2}\int_0^1 N_C(u^{1/\lambda}, 0)du - \int_0^1 N_C(u^{1/\lambda}, u^{1/(1-\lambda)})du$$

Taking the variance and using the proposition 2 gives that only the variance of the third term matters. □

We are now able to detail precisely the variance of the limiting process with a given Copula. This is the purpose of the following proposition :

Proposition 3. *Under the framework of theorem 2 and if we take $C(u,v) = uv$, the independent copula, then the variance of the lambda FMadogram has the following form*

$$Var(\int_{[0,1]^2} N_C(u,v)dJ(u,v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} - \frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} - \frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right)$$

with $\lambda \in [0,1]$

Démonstration. With direct computing and using the same techniques used is lemma 1, we obtain that :

$$Var\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right] = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2\left(\frac{1}{1+2\lambda(1-\lambda)}\right)$$

$$Var\left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du\right] = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2\left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)}\right)$$

$$Var\left[\int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du\right] = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2\left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right)$$

$$cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du\right] = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)}\right)$$

$$cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}), du \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du\right] = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right)$$

$$cov\left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du\right] = 0$$

Using the identity $Var(X-Y) = Var(X) + Var(Y) - 2cov(X,Y)$ gives the desired result. □

We now consider the bivariate extreme value copula which can be written in the following form (See Segers extreme value copulas)

$$C(u,v) = (uv)^{A(log(v)/log(uv))} \tag{5}$$

for all $u, v \in [0,1]$ and where $A(\cdot)$ is the Pickhands dependence function. The partial derivatives of this copula are given by :

$$\frac{\partial C(u,v)}{\partial u} = \frac{C(u,v)}{u}\left(A(log(v)/log(uv)) - A'(log(v)/log(uv))\frac{log(v)}{log(uv)}\right)$$

$$\frac{\partial C(u,v)}{\partial v} = \frac{C(u,v)}{v}\left(A(log(v)/log(uv)) + A'(log(v)/log(uv))\frac{log(v)}{log(uv)}\right)$$

With this copula, we want to compute the following integral :

$$Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) = Var(\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du - \int_0^1 B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du$$
$$- \int_0^1 B_C(1, v^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v})$$

Notice that, on sections, the extreme value copula is a polynom, i.e :

$$C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) = u^{\frac{A(\lambda)}{\lambda(1-\lambda)}}$$

Furthermore, we have the same pattern for partial derivatives :

$$\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} = \frac{C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{u^{\frac{1}{\lambda}}}(A(\lambda) - A'(\lambda)\lambda)$$

$$\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} = \frac{C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{v^{\frac{1}{1-\lambda}}}(A(\lambda) + A'(\lambda)\lambda)$$

Let $\mathcal{A}$ be the space of Pickhands dependence functions. We will denote by $\kappa(\lambda, A)$ and $\zeta(\lambda, A)$ two functional such as :

$$\begin{aligned}\kappa\colon & [0,1]\times\mathcal{A} \to \mathbb{R} \\ & (\lambda, A) \mapsto A(\lambda) - A'(\lambda)\lambda\end{aligned}$$

$$\begin{aligned}\zeta\colon & [0,1]\times\mathcal{A} \to \mathbb{R} \\ & (\lambda, A) \mapsto A(\lambda) + A'(\lambda)\lambda\end{aligned}$$

(Ces deux définitions ont besoin d'un peu de travail pour rigoureusement définir l'espace d'arrivé).
We may compute the variance for each process and they are given by the following expressions :

$$Var\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right] = \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2 \left(\frac{1}{A(\lambda)+2\lambda(1-\lambda)}\right)$$

$$Var\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right] = \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2 \left(\frac{\kappa^2(\lambda, A)(1-\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)}\right)$$

$$Var\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right] = \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2 \left(\frac{\zeta^2(\lambda, A)\lambda}{2A(\lambda)-\lambda+2\lambda(\lambda)}\right)$$

We now compute the covariance :

$$cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})du, \int_0^1 B_C(u^{\frac{1}{\lambda}},1)\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}du\right] = \int_0^1\int_0^1 \mathbb{E}[B_C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})B_C(v^{\frac{1}{\lambda}},1)]\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$
$$= \int_0^1\int_0^v \mathbb{E}[B_C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}},1)]\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dudv + \int_0^1\int_v^1 \mathbb{E}[B_C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}},1)]\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

for the first one, we have :

$$\int_0^1\int_0^v (C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dudv = \frac{\kappa(\lambda,A)}{2}\left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2\left(\frac{1-\lambda}{2A(\lambda)+(2\lambda-1)(1-\lambda)}\right)$$

For the second part, using Fubini, we have :

$$\int_0^1\int_0^u (C(v^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dvdu$$

for the right hand side of the "minus" sign, we may compute :

$$\int_0^1\int_0^u C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}}\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dvdu = \frac{\kappa(\lambda,A)}{2}\left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2$$

The last one still difficult to handle

$$\int_0^1\int_0^u C(v^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \tag{6}$$

Lemma 3. *We have the following inequalities :*

$$\frac{\kappa(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+\lambda+2\lambda(1-\lambda))} \leq \int_0^1\int_0^u C(v^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \leq f_\kappa(\lambda,A)$$

$$\frac{\zeta(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+1-\lambda+2\lambda(1-\lambda))} \leq \int_0^1\int_0^u C(u^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial v}dvdu \leq f_\zeta(\lambda,A)$$

$$\left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2\zeta(\lambda,A)\kappa(\lambda,A) \leq \int_0^1\int_0^1 C(u^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial v}dvdu$$
$$\leq \frac{(\lambda(1-\lambda))^2\zeta(\lambda,A)\kappa(\lambda,A)}{A(\lambda)(A(\lambda)+\lambda(1-\lambda))}$$

with

$$f_\kappa(\lambda,A) = \begin{cases} \frac{\kappa(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+2\lambda(1-\lambda))} & if \lambda \leq 1/2 \\ f_1(\lambda,A,\kappa,\lambda)+f_2(\lambda,A,\kappa,\lambda) & if\lambda > 1/2 \end{cases}$$

$$f_\zeta(\lambda,A) = \begin{cases} f_1(\lambda,A,\zeta,1-\lambda)+f_2(\lambda,A,\zeta,1-\lambda) & if\lambda \leq 1/2 \\ \frac{\zeta(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+2\lambda(1-\lambda))} & if\lambda > 1/2 \end{cases}$$

where we denote by $f_1(\lambda,A,\kappa,x) = \frac{x(1-x)^3\kappa(\lambda,A)}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+(1-x))}$ *and :*

$$f_2(\lambda,A,\kappa,x) = \frac{x(1-x)\kappa(\lambda,A)}{A(\lambda)-(1-x)+\lambda(1-\lambda)}\left[\frac{x(1-x)}{A(\lambda)+x-(1-x)+2\lambda(1-\lambda)} - \frac{(1-x)^2}{A(\lambda)+1-x}\right]$$

Démonstration. This proof is really technical and can be omitted in first lecture. Because the tools used for on integral is the same for the three, we detail the algebra for the first one. Recall that $A(t)$ is a function which satisfies the following inequalities $t \vee 1-t \leq A(t) \leq 1$ for $t \in [0,1]$. Hence, as u,v are elements of $[0,1]$, we can bound the extreme value copula by $uv \leq C(u,v) \leq u \wedge v$. We thus obtain the fondamental inequality that we will use for bouding our untractable integrals :

$$v^{\frac{1}{\lambda}}u^{\frac{1}{1-\lambda}} \leq C(v^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}}) \leq v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}}$$

We want to compute now the following integral which is now tractable :

$$\int_0^1\int_0^u v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}}\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dvdu$$

We have to consider two cases :

— if $\lambda \leq 1/2$, then $v \leq u \leq u^{\frac{\lambda}{1-\lambda}}$ and $v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} = v^{\frac{1}{\lambda}}$

— if $\lambda > 1/2$, we should consider two more cases

— if $0 \leq v \leq u^{\frac{\lambda}{1-\lambda}}$, then $v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} = v^{\frac{1}{\lambda}}$

— if $u^{\frac{\lambda}{1-\lambda}} \leq v \leq u$, then $v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} = u^{\frac{1}{1-\lambda}}$

For the first case, we compute :

$$\int_0^1 \int_0^u v^{\frac{1}{\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = \frac{(\lambda(1-\lambda))^2 \kappa(\lambda, A)}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+2\lambda(1-\lambda))}$$

For the second case, we may compute, in one hand :

$$\int_0^1 \int_0^{u^{\frac{\lambda}{1-\lambda}}} v^{\frac{1}{\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = \frac{\lambda(1-\lambda)^3 \kappa(\lambda, A)}{(A(\lambda)+\lambda(1-\lambda)(A(\lambda)+2\lambda(1-\lambda))}$$

On the other hand, we have :

$$\int_0^1 \int_{u^{\frac{\lambda}{1-\lambda}}}^u v^{\frac{1}{1-\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = f_2(\lambda, \kappa, \lambda)$$

We then obtain the upper bound of our integral. For the lower bound, we just have to compute the following integral :

$$\int_0^1 \int_0^u v^{\frac{1}{\lambda}} u^{\frac{1}{1-\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = \frac{(\lambda(1-\lambda))^2 \zeta(\lambda, A) \kappa(\lambda, A)}{A(\lambda)(A(\lambda)+\lambda(1-\lambda))}$$

□

A direct consequence from this lemma and the remark is given by the proposition which is following :

Proposition 4. *If we consider an extreme value copula, then under condition 2, we can control the covariance between the random variables* $\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})$, $\int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du$ *and* $\int_0^1 B_C(1, u^{\frac{1}{1-\lambda}}) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial v} du$ *and this control is given by, in one hand :*

$$\begin{aligned} &\frac{\kappa(\lambda, A)}{2} \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)} \right)^2 \left(\frac{1-\lambda}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)} + \frac{A(\lambda)-\lambda}{A(\lambda)+\lambda+2\lambda(1-\lambda)} \right) \leq \\ &cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) du, \int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \right] \\ &\leq f_\kappa(\lambda, A) + \kappa(\lambda, A) \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)} \right)^2 \left(\frac{(1-\lambda)^2 - A(\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)} \right) \end{aligned}$$

on the other :

$$\begin{aligned} &\frac{\zeta(\lambda, A)}{2} \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2A(\lambda)-\lambda+2\lambda(1-\lambda)} + \frac{A(\lambda)-(1-\lambda)}{A(\lambda)+1-\lambda+2\lambda(1-\lambda)} \right) \leq \\ &cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du, \int_0^1 B_C(1, u^{\frac{1}{1-\lambda}}) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial v} du \right] \\ &\leq f_\zeta(\lambda, A) + \zeta(\lambda, A) \left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)} \right)^2 \left(\frac{\lambda^2 - A(\lambda)}{2A(\lambda)-\lambda+2\lambda(1-\lambda)} \right) \end{aligned}$$

and finally

$$\begin{aligned} 0 &\leq cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du, \int_0^1 B_C(1, v^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dv \right] \\ &\leq \left(\frac{\lambda(1-\lambda)}{A(1-\lambda)+\lambda(1-\lambda)} \right)^2 \frac{\zeta(\lambda, A)\kappa(\lambda, A)\lambda(1-\lambda)}{A(\lambda)} \end{aligned}$$

Démonstration. Again, we show the main elements of proof for the first covariance. Recall that, by definition of our Brownian bridge, we have :

$$\begin{aligned} cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) du, \int_0^1 B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \right] &= \int_0^1 \int_0^v (C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) v^{\frac{1}{\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dudv \\ &+ \int_0^1 \int_0^u (C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) v^{\frac{1}{\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu \end{aligned} \quad (7)$$

The first term on the right is tractable and it's value is given by :

$$\frac{\kappa(\lambda, A)}{2} f(\lambda, A) \left(\frac{1-\lambda}{2A(\lambda) + (2\lambda - 1)(1-\lambda)} \right) \tag{8}$$

The second term of (7) cannot be computed directly, but we know a lower bound whose value is :

$$\frac{\kappa(\lambda, A)(\lambda(1-\lambda))^2}{(A(\lambda) + \lambda(1-\lambda))(A(\lambda) + \lambda + 2\lambda(1-\lambda))} - \frac{\kappa(\lambda, A)}{2} f(\lambda, A) = \frac{\kappa(\lambda, A)}{2} f(\lambda, A) \left(\frac{A(\lambda) - \lambda}{A(\lambda) + \lambda + 2\lambda(1-\lambda)} \right)$$

The sum of the two previous term gives us the lower bound. For the upper bound, we know from lemma 3 that the second term of (7) is bounded above by :

$$f_\kappa(\lambda, A) - \frac{\kappa(\lambda, A)}{2} f(\lambda, A)$$

The sum of this quantity with equality (8) gives :

$$f_\kappa(\lambda, A) + \kappa(\lambda, A) f(\lambda, A) \left(\frac{(1-\lambda)^2 - A(\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} \right)$$

That is the statement. We finish the proof. $\square$

We have all tools now to write the main theorem of this part :

Theorem 2. *Consider an extreme value copula $C(u,v)$. Under condition 1, $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda))$ is weakly convergent to a centered Gaussian random variable with variance $Var\left[\int_0^1 N_c(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du\right]$. Furthermore, we can control this term and the controls are given by :*

$$\mathcal{L}(\lambda) \leq Var\left[\int_0^1 N_c(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du\right] \leq \mathcal{U}(\lambda) \tag{9}$$

where we respectively define $\mathcal{U}$ and $\mathcal{A}$ by :

$$\begin{aligned}
\mathcal{L}(\lambda, A) = & \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)} + \frac{\kappa^2(\lambda)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} + \frac{\zeta^2(\lambda)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} \right) \\
& - 2\kappa(\lambda, A) \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{(1-\lambda)^2 - A(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} \right) - 2f_\kappa(\lambda, A) \\
& - 2\zeta(\lambda, A) \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{\lambda^2 - A(\lambda)}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} \right) - 2f_\zeta(\lambda, A)
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}(\lambda, A) = & \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)} + \frac{\kappa^2(\lambda)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} + \frac{\zeta^2(\lambda)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} \right) \\
& - \kappa(\lambda, A) \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{1-\lambda}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} + \frac{A(\lambda) - \lambda}{A(\lambda) + \lambda + 2\lambda(1-\lambda)} \right) \\
& - \zeta(\lambda, A) \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} + \frac{A(\lambda) - (1-\lambda)}{A(\lambda) + 1 - \lambda + 2\lambda(1-\lambda)} \right) \\
& + 2 \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2 \frac{\zeta(\lambda, A)\kappa(\lambda, A)\lambda(1-\lambda)}{A(\lambda)}
\end{aligned}$$

Let (X, Y) be a bivariate random vector with joint distribution function $H(x, y)$ and continuous marginal distribution function $F(x)$ and $G(y)$. Its associated copula C is defined by $H(x, y) = C\{F(x), G(y)\}$. Since F and G are continuous, the copula C is unique and we can write $C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v))$ for $0 \leq u, v \leq 1$ and where $F^{\leftarrow}(u) = inf\{v \in \mathbb{R} | F(v) \geq u\}$ and $G^{\leftarrow}(u) = inf\{v \in \mathbb{R} | G(v) \geq u\}$ are the generalized inverse functions of F and G respectively.

We suppose that we observe sequentially a quadruple $(I_t, J_t, I_t X_t, J_t Y_t)$ for $t \in \{1, \dots, T\}$. At each $t \in \{1, \dots, T\}$, one of both entries may be missing. The observations consist of a sample of independent, identically distributed quadruple (I, J, X, Y) of law $\mathbb{P}$:

$$(I_t, J_t, I_t X_t, J_t Y_t) \quad t \in \{1, \dots, T\}$$

The indicator variables I_t (respectively J_t) is equal to 1 or 0 according to wheter X_t or Y_t is observed or not. The probability of observing a realisation partially or completely is denoted by $p_X = \mathbb{P}(I_t = 1) > 0$, $p_Y = \mathbb{P}(J_t = 1) > 0$ and $p_{XY} = \mathbb{P}(I_t = 1, J_t = 1) > 0$.

Let us define :

$$C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v)) = \varphi(H)(u, v)$$

The function $\hat{H}_T$ corresponds to the empirical distribution function of the sample $(X_1, Y_1), \dots, (X_T, Y_T)$

$$\hat{H}_T(u, v) = \frac{\sum_{t=1}^{T} 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t}{\sum_{t=1}^{T} I_t J_t}$$

We define also the corresponding empirical distribution functions in the case of missing data :

$$\hat{F}_T(u) = \frac{\sum_{t=1}^{T} 1_{\{X_t \leq u\}} I_t}{\sum_{t=1}^{T} I_t}$$

$$\hat{G}_T(v) = \frac{\sum_{t=1}^{T} 1_{\{Y_t \leq v\}} J_t}{\sum_{t=1}^{T} J_t}$$

We have all tools in hand to define the *hybrid copula estimator* ([Seg14]) :

$$\hat{C}_{T,H}(u, v) = \hat{H}_T(\hat{F}_T(u), \hat{G}_T(v)) \tag{10}$$

Given a rate $r_T > 0$ and $r_T \to \infty$ as $T \to \infty$, the normalized estimation error of the hybrid copula estimator is :

$$Z_T(u, v) = \sqrt{T}\{\hat{C}_{T,H}(u, v) - C(u, v)\} \tag{11}$$

Condition 2. *We suppose for all $t \in \{1, \dots, T\}$, the pairs (I_t, J_t) and (X_t, Y_t) are independent, the data are missing completely at random. Furthermore, we suppose that there exist at least one $t \in \{1, \dots, T\}$ such that $I_t J_t \neq 0$.*

Proposition 5. *Under hypothesis 1, $\hat{H}_T, \hat{F}_T, \hat{G}_T$ are consistant estimators of H, F, G.*

Démonstration. We check the consistency for $\hat{H}_T$. By independence, we have

$$\mathbb{E}[T^{-1} \sum_{t=1}^{T} I_t J_t] = T^{-1} \sum_{t=1}^{T} \mathbb{E}[I_t J_t] = p_{XY}$$

So, by applying the law of large numbers, we have :

$$T^{-1} \sum_{t=1}^{T} I_t J_t \longrightarrow p_{XY} \quad a.s. \quad as \quad T \to \infty$$

Then, we now use the first hypothesis to get :

$$T^{-1} \sum_{t=1}^{T} \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t] = T^{-1} \sum_{t=1}^{T} \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}}] \mathbb{E}[I_t J_t] = H(x, y) p_{XY}$$

By applying again the law of large numbers, we derive :

$$\sum_{t=1}^{T} 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t \longrightarrow H(x, y) p_{XY} \quad a.s. \quad as \quad T \to \infty$$

We can now apply the continuous mapping theorem to the function $f : (x, y) \mapsto \frac{x}{y}$ which are continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \setminus 0$ to conclude that :

$$\hat{H}_T(x, y) \longrightarrow H(x, y) \quad a.s. \quad as \quad T \to \infty$$

$\square$

Condition 3. *There exists $\gamma_t > 0$ and $r_t > 0$ such that $r_t \longrightarrow \infty$ as $t \to \infty$ such that in the space $l^\infty(\mathbb{R}^2) \otimes (l^\infty(\mathbb{R}), l^\infty(\mathbb{R}))$ equipped with the topology of uniform convergence, we have the joint weak convergence*

$$(r_t(\hat{H}_T - H); r_t(\hat{F}_t - F), r_t(\hat{G}_t - G)) \rightsquigarrow (\alpha \circ (F, G), \beta_1 \circ F, \beta_2 \circ G)$$

The stochastic processes α and β_j take values in $l^\infty([0,1]^2)$ and $l^\infty([0,1])$ respectively, and are such that $\alpha \circ F$ and $\beta_j \circ F_j$ have continuous trajectories on $[-\infty, \infty]^2$ and $[-\infty, \infty]$ almost surely.

Theorem 3 (Theorem 2.3 and example 3.5 in [Seg14])**.** *If conditions 2.1 and 2.2 holds, then uniformly in $u \in [0,1]^2$,*

$$\begin{aligned} r_T\{\hat{C}_T(u,v) - C(u,v)\} &= r_T\{\hat{H}_T((F,G)^{\leftarrow}(u,v) - C(u,v)\} && (12)\\ &- \dot{C}_1(u,v) r_T\{\hat{F}_T(F^{\leftarrow}(u)) - u\} 1_{(0,1)}(u) && (13)\\ &- \dot{C}_2(u,v) r_T\{\hat{G}_T(G^{\leftarrow}(v)) - v\} 1_{(0,1)}(v) + o_{\mathbb{P}}(1) && (14) \end{aligned}$$

as $T \to \infty$. Hence in $l^\infty([0,1]^2)$ equipped with the supremum norm, as $T \to \infty$,

$$(r_T\{\hat{C}_T(u,v) - C(u,v)\})_{u,v \in [0,1]^2} \rightsquigarrow (\alpha(u,v) - \dot{C}_1(u,v)\beta_1(u) - \dot{C}_2(u,v)\beta_2(v))_{u,v \in [0,1]^2}$$

We denote by $S_C(u,v)$ the process defined by $\forall (u,v) \in [0,1]^2$:

$$S_C(u,v) = \alpha(u,v) - \dot{C}_1(u,v)\beta_1(u) - \dot{C}_2(u,v)\beta_2(v)$$

We denote by $\mathbb{G}$ a $\mathbb{P}$-Brownian bridge, the process are defined by :

$$\begin{aligned} \beta_1(u) &= p_X^{-1} \mathbb{G}(1_{X \le F^{\leftarrow}(u), I=1} - u 1_{I=1})\\ \beta_2(v) &= p_Y^{-1} \mathbb{G}(1_{Y \le G^{\leftarrow}(v), J=1} - v 1_{J=1})\\ \alpha(u,v) &= (p_{XY})^{-1} \mathbb{G}(1_{X \le F^{\leftarrow}(u)} 1_{Y \le G^{\leftarrow}(v), I=1, J=1} - C(u,v) 1_{I=1, J=1}) \end{aligned}$$

Lemma 4. *The covariance function of the process $\beta_1(u)$, $\beta_2(v)$ and $\alpha(u,v)$ are : for $(u, u_1, u_2, v, v_1, v_2) \in [0,1]^6$,*

$$\begin{aligned} cov[\beta_1(u_1), \beta_1(u_2)] &= p_X^{-1}\{u_1 \wedge u_2 - u_1 u_2\}\\ cov[\beta_2(v_1), \beta_2(v_2)] &= p_Y^{-1}\{v_1 \wedge v_2 - v_1 v_2\}\\ cov[\beta_1(u), \beta_2(v)] &= \frac{p_{XY}}{p_X p_Y}\{C(u,v) - uv\} \end{aligned}$$

and

$$\begin{aligned} cov[\alpha(u_1,v_1), \alpha(u_2,v_2)] &= p_{XY}^{-1}\{C(u_1 \wedge u_2, v_1 \wedge v_2) - C(u_1,v_1)C(u_2,v_2)\}\\ cov[\alpha(u_1,v), \beta_1(u_2)] &= p_X^{-1}\{C(u_1 \wedge u_2, v) - C(u_1,v)u_2\}\\ cov[\alpha(u,v_1), \beta_2(v_2)] &= p_Y^{-1}\{C(u, v_1 \wedge v_2) - C(u,v_1)v_2\} \end{aligned}$$

Démonstration. For the weak convergence of the processes $\hat{F}_T(x)$, $\hat{G}_T(y)$ and $\hat{H}_T(x,y)$, all is explained in [Seg14] (page 15). We just want to add some line of algebra. These lines are made for the first process as the method is similar for the others. For $(x,y) \in \mathbb{R}^2$,

$$\hat{F}_T(x) = \frac{p_X(x) + T^{-1/2}\mathbb{G}_T g_{1,x}}{p_X + T^{-1/2}\mathbb{G}_T f_1}$$

We may obtain :

$$\begin{aligned} p_X(\hat{F}_T(x) - F(x)) &= T^{-1/2}(\mathbb{G}_T(g_{1,x}) - \mathbb{G}_T(f_1)\hat{F}_T(x))\\ &= T^{-1/2}(\mathbb{G}_T(g_{1,x} - F(x)f_1)) + T^{-1/2}\mathbb{G}(f_1)(F(x) - \hat{F}_T(x)) \end{aligned}$$

Multiplying by $\sqrt{T}$ and dividing by p_X gives :

$$\sqrt{T}(\hat{F}_T(x) - F(x)) = p_X^{-1}(\mathbb{G}_T(g_{1,x} - F(x)f_1)) + p_X^{-1}\mathbb{G}(f_1)(F(x) - \hat{F}_T(x))$$

Take a closer look at the second term in the right hand side. By the central limit theorem, we have that $\mathbb{G}_n(f_1) \rightsquigarrow \mathcal{N}(0, \mathbb{P}(f_1 - \mathbb{P}f_1))$, applying the law of the large number gives us that $(F(x) - \hat{F}_T(x)) = o_{\mathbb{P}}(1)$. With the help of Slutksy theorem, we must claim that :

$$\sqrt{T}(\hat{F}_T(x) - F(x)) = p_X^{-1}(\mathbb{G}_T(g_{1,x} - F(x)f_1)) + o_{\mathbb{P}}(1)$$

As a consequence, we obtain the following limit process of the theorem :

$$\beta_1(u) = p_X^{-1}\mathbb{G}(1_{X \le F^{\leftarrow}(u), I=1} - u 1_{I=1})$$

We know that the covariance of a $\mathbb{P}$-Gaussian process is given by $\mathbb{G}(f)\mathbb{G}(g) = \mathbb{P}(fg) - \mathbb{P}(f)\mathbb{P}(g)$ where f, g are measurable functions. Now, using that, we have :

$$\begin{aligned}
cov[\beta_1(u_1), \beta_1(u_2)] &= p_X^{-2}\mathbb{E}\left[\mathbb{G}(1_{X \le F^{\leftarrow}(u_1), I=1} - u_1 1_{I=1})\mathbb{G}(1_{X \le F^{\leftarrow}(u_2), I=1} - u_2 1_{I=1})\right] \\
&= p_X^{-2}(\mathbb{P}\left[(1_{X \le F^{\leftarrow}(u_1), I=1} - u_1 1_{I=1})(1_{X \le F^{\leftarrow}(u_2), I=1} - u_2 1_{I=1})\right]) \\
&= p_X^{-2}(\mathbb{P}(I=1)\mathbb{P}(X \le F^{\leftarrow}(u_1), X \le F^{\leftarrow}(u_2)) - u_1 u_2 \mathbb{P}(I=1)) \\
&= p_X^{-1}(u_1 \wedge u_2 - u_1 u_2)
\end{aligned}$$

□

Remark 1. *If we consider the empiric copula, theorem 3 gives us the weak convergence of this process to a brownian bridge $N_C(u,v)$ defined by, $\forall (u,v) \in [0,1]$*

$$N_C(u,v) = B_C(u,v) - \dot{C}_1(u,v)B_C(u,1) - \dot{C}_2(u,v)B_C(1,v) \tag{15}$$

where B_C is a brownian bridge in $[0,1]^2$ with covariance function

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$$

for each $0 \le, u, v, u', v' \le 1$. This result is well known since 2004 due to [FRW04].

Definition 2. *In case of missing data, we consider the following estimator of the λ-FMadogram :*

$$\hat{\nu}_T(\lambda) = \frac{1}{2\sum_{t=1}^T I_t J_t} \sum_{t=1}^T |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| I_t J_t \tag{16}$$

Proposition 6. *Under the framework given by condition 2 2, we have the following convergence*

$$\hat{\nu}_T(\lambda) \to \nu(\lambda) \quad a.s. \quad as \quad T \to \infty$$

Démonstration. Using proposition 5 and the continuous mapping theorem, we have that :

$$\frac{1}{2T}\sum_{t=1}^T |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| I_t J_t \xrightarrow[T \to +\infty]{a.s.} \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|] p_{XY}$$

The law of large number gives us :

$$\sum_{t=1}^T I_t J_t \xrightarrow[T \to +\infty]{a.s.} p_{XY} > 0$$

Applying again the continuous mapping theorem gives us the statement. □

Combining theorem 3 and proposition 1 gives the following result :

Theorem 4. *Suppose that the assumption of theorem 3 holds. Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then*

$$T^{1/2}\left(\frac{\sum_{t=1}^T J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) I_t J_t}{\sum_{t=1}^T I_t J_t} - \mathbb{E}[J(F(X), G(Y))]\right)$$

converges in distribution to $\int_{[0,1]^2} S_C(u,v) dJ(u,v)$ where $S_C(u,v)$ is defined in theorem 3. The special case, $J(x,y) = \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (2) :

$$T^{1/2}\left(\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|]\right)$$

converge in distribution to $\int_{[0,1]^2} S_C(u,v) dJ(u,v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2}\int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2}\int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

for all bounded functions f.

Démonstration. Let us introduce the empirical measure define by :

$$\mathbb{P}_n = \sum_{t=1}^{T} \delta_{((\hat{F}_T(X_t),\hat{G}_T(Y_t)),I_i,J_i,\sum_{t=1}^{n} I_t J_t)}(\mathcal{X})$$

Where $\mathcal{X} = [0,1]^2 \times \{0,1\} \times \{0,1\} \times \{0,\dots,T\}$, we consider the following function :

$$\begin{aligned} f \colon \mathcal{X} &\to \mathbb{R} \\ ((x,y),I,J,K) &\mapsto \frac{1_{\{[0,u]\times[0,v]\}}(x,y)IJ}{K} \end{aligned}$$

Then, we have :

$$\bar{C}_{T,H}(u,v) := \mathbb{P}_n f = \int_{\mathcal{X}} f(x) d\mathbb{P}_n(x) = \frac{1}{\sum_{t=1}^{T} I_t J_t} \sum_{t=1}^{T} 1_{\{\hat{F}_T(X_t)\leq u, \hat{G}_T(Y_t)\leq v\}} I_t J_t$$

Where $\bar{C}_{T,H}(u,v)$ corresponds to the càdlàg version of the ordinary hybrid copula process $C_{T,H}(u,v)$. The problem is that we can't write the following quantity :

$$\sqrt{T}(\frac{1}{\sum_{t=1}^{T} I_t J_t} \sum_{t=1}^{n} J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) I_t J_t - \mathbb{E}[J(F(X), G(Y))])$$

as an integral of the form :

$$\sqrt{T}(\int_{[0,1]^2} J(u,v) d(\bar{C}_{T,H} - C)(u,v))$$

Indeed, without missing data ($p_Y = p_Y = p_{XY} = 1$), we have $\bar{C}_T(u,v) = \frac{1}{T}\sum_{t=1}^{T} 1_{\{\hat{F}_T(X_t)\leq u, \hat{G}_T(Y_t)\leq v\}}$ and we can write :

$$\int_{[0,1]^2} J(u,v) d\bar{C}_T(u,v) = \frac{1}{T}\sum_{t=1}^{T} \int_{[0,1]^2} J(u,v) d\delta_{\{(\hat{F}_T(X_t),\hat{G}_T(Y_t))\}} = \frac{1}{T}\sum_{i=t}^{T} J(\hat{F}_T(X_t), \hat{G}_T(Y_t))$$

In the case of missing data, this is more complicated (see remark 4 of [DF20]) □

Proposition 7. *Under the framework of theorem 4 and if take $C(u,v) = uv$, the independent opula, then the variance of the λ-FMadogram has the following form*

$$Var(\int_{[0,1]^2} N_C(u,v) dJ(u,v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1}{p_{XY}(1+2\lambda(1-\lambda))} - \frac{1-\lambda}{p_X(1+\lambda+2\lambda(1-\lambda))} - \frac{\lambda}{p_Y(2-\lambda+2\lambda(1-\lambda))}\right)$$

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