Let (X, Y) be a bivariate random vector with joint distribution function $H(x, y)$ and continuous marginal distribution function $F(x)$ and $G(y)$. Its associated copula C is defined by $H(x, y) = C\{F(x), G(y)\}$. Since F and G are continuous, the copula C is unique and we can write $C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v))$ for $0 \leq u, v \leq 1$ and where $F^{\leftarrow}(u) = inf\{v \in \mathbb{R} | F(v) \geq u$ and $G \leftarrow (u) = inf\{v \in \mathbb{R} | G(v) \geq u\}$ are the generalized inverse functions of F and G respectively. At each $t \in \{1, \ldots, T\}$, we suppose that one of both entries may be missing. The observations consist of a sample of independent, identically distributed quadruples

$$(I_t, J_t, I_t X_t, J_t Y_t) \quad t \in 1, \ldots, T$$

The indicator variables I_t (respectively J_t) is equal to 1 or 0 according to wheter X_t or Y_t is observed or not. We suppose that the indicator functions I_t and J_t are independent. The probability of observing a realisation partially or completely is denoted by $p_X = \mathbb{P}(I_t = 1) > 0$ and $p_Y = \mathbb{P}(J_t = 1) > 0$.

We define :

$$C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v)) = \varphi(H)(u, v)$$

and,

$$Z_T(u, v) = \sqrt{T}\{\hat{C}_T(u, v) - C(u, v)\} \tag{1}$$

where $\hat{H}_T$ corresponds to the empirical distribution function of the sample $(X_1, Y_1), \ldots, (X_T, Y_T)$

$$\hat{H}_T(u, v) = \frac{\sum_{t=1}^{T} 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t}{\sum_{t=1}^{T} I_t J_t}$$

We define also the corresponding empirical distribution functions in the case of missing data :

$$\hat{F}_T(u) = \frac{\sum_{t=1}^{T} 1_{\{X_t \leq u\}} I_t}{\sum_{t=1}^{T} I_t}$$

$$\hat{G}_T(v) = \frac{\sum_{t=1}^{T} 1_{\{Y_t \leq v\}} J_t}{\sum_{t=1}^{T} J_t}$$

Condition 1. *We suppose for all $t \in \{1, \ldots, T\}$:*

$$\mathbb{E}[X_t Y_t I_t J_t] = \mathbb{E}[X_t Y_t] \mathbb{E}[I_t J_t]$$

Furthermore, we suppose that there exist at least one $t \in \{1, \ldots, T\}$ such that $I_t J_t \neq 0$.

Proposition 1. *Under hypothesis 1, $\hat{H}_T, \hat{F}_T, \hat{G}_T$ are consistant estimators of H, F, G.*

Démonstration. We check the consistency for $\hat{H}_T$. By independence, we have

$$\mathbb{E}[T^{-1} \sum_{t=1}^{T} I_t J_t] = T^{-1} \sum_{t=1}^{T} \mathbb{E}[I_t] \mathbb{E}[J_t] = p_X p_Y$$

So, by applying the law of large numbers, we have :

$$T^{-1}\sum_{t=1}^{T} I_t J_t \longrightarrow p_X p_Y \quad a.s. \quad as \quad T\to\infty$$

Then, we now use the first hypothesis to get :

$$T^{-1}\sum_{t=1}^{T} \mathbb{E}[1_{\{X_t\le u, Y_t\le v\}} I_t J_t] = T^{-1}\sum_{t=1}^{T} \mathbb{E}[1_{\{X_t\le u, Y_t\le v\}}]\mathbb{E}[I_t J_t] = H(x,y)p_X p_Y$$

By applying again the law of large numbers, we derive :

$$\sum_{t=1}^{T} 1_{\{X_t\le u, Y_t\le v\}} I_t J_t \longrightarrow H(x,y)p_X p_Y \quad a.s. \quad as \quad T\to\infty$$

We can now apply the continuous mapping theorem to the function $f:(x,y)\mapsto \frac{x}{y}$ which are continuous on $\mathbb{R}_+\times\mathbb{R}_+\setminus 0$ to conclude that :

$$\hat{H}_T(x,y)\longrightarrow H(x,y) \quad a.s. \quad as \quad T\to\infty$$

$\square$

Condition 2. *1. The bivariate distribution function H has continuous margins F, G and copula C.*

2. The first order partial derivatives $\dot{C}_1(u,v) = \frac{\partial C}{\partial u}(u,v)$ and $\dot{C}_2(u,v) = \frac{\partial C}{\partial v}(u,v)$ exists and is continuous on the set $\{(u,v)\in[0,1]^2, 0<u,v<1\}$

Condition 3. *There exists $\gamma_t > 0$ and $r_t > 0$ such that $r_t \longrightarrow \infty$ as $t\to\infty$ such that in the space $l^\infty(\mathbb{R}^2)\otimes(l^\infty(\mathbb{R}), l^\infty(\mathbb{R}))$ equipped with the topology of uniform convergence, we have the joint weak convergence*

$$(r_t(\hat{H}_T - H); r_t(\hat{F}_t - F), r_t(\hat{G}_t - G)) \rightsquigarrow (\alpha\circ(F,G), \beta_1\circ F, \beta_2\circ G)$$

The stochastic processes α and β_j take values in $l^\infty([0,1]^2)$ and $l^\infty([0,1])$ respectively, and are such that $\alpha\circ F$ and $\beta_j\circ F_j$ have continuous trajectories on $[-\infty,\infty]^2$ and $[-\infty,\infty]$ almost surely.

Theorem 1 (Theorem 2.3 and example 3.5 in [Seg14])**.** *If conditions 2.1 and 2.2 holds, then uniformly in $u\in[0,1]^2$,*

$$\begin{aligned} r_T\{\hat{C}_T(u,v) - C(u,v)\} &= r_T\{\hat{H}_T((F,G)^{\leftarrow}(u,v) - C(u,v)\} && (2)\\ &\quad - \dot{C}_1(u,v) r_T\{\hat{F}_T(F^{\leftarrow}(u)) - u\}1_{(0,1)}(u) && (3)\\ &\quad - \dot{C}_2(u,v) r_T\{\hat{G}_T(G^{\leftarrow}(v)) - v\}1_{(0,1)}(v) + \circ_{\mathbb{P}}(1) && (4)\end{aligned}$$

as $T\to\infty$. Hence in $l^\infty([0,1]^2)$ equipped with the supremum norm, as $T\to\infty$,

$$(r_T\{\hat{C}_T(u,v) - C(u,v)\})_{u,v\in[0,1]^2} \rightsquigarrow (\alpha(u,v) - \dot{C}_1(u,v)\beta_1(u) - \dot{C}_2(u,v)\beta_2(v))_{u,v\in[0,1]^2}$$

We denote by $N_C(u,v)$ the process defined on the right-hand side in the weak convergence from above.

Remark 1. *If we consider the empiric copula, theorem 1 gives us the weak convergence of this process to a brownian bridge $N_C(u,v)$ defined by, $\forall(u,v) \in [0,1]$*

$$N_C(u,v) = B_C(u,v) - \dot{C}_1(u,v)B_C(u,1) - \dot{C}_2(u,v)B_C(1,v) \tag{5}$$

where B_C is a brownian bridge in $[0,1]^2$ with covariance function

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$. This result is well known since 2004 due to [FRW04].

In our specific case with missing data, (ref) shows that $r_T\{\hat{C}_T(u,v) - C(u,v)\}$ is weakly convergent toward $\alpha(u,v) - \dot{C}_1(u,v)\beta_1(u) - \dot{C}_2(u,v)\beta_2(v)$ where $\beta_1(u) = p_X^{-1}\mathbb{G}(1_{X \leq F^{\leftarrow}(u)} - u1_{I=1})$, $\beta_2(v) = p_Y^{-1}\mathbb{G}(1_{Y \leq G^{\leftarrow}(v)} - v1_{J=1})$ and $\alpha(u,v) = (p_X p_Y)^{-1}\mathbb{G}(1_{X \leq F^{\leftarrow}(u)}1_{Y \leq G^{\leftarrow}(v)} - C(u,v)1_{I=1}1_{J=1})$. Furthermore, by these expressions, we can detail the structure of the covariance matrix between the three processes.

Definition 1. *Let $(X_1,Y_1), \dots, (X_T,Y_T)$ a T bivariate random vectors with unknown margins F and G. A λ-FMadogram is the quantity defined by :*

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|] \tag{6}$$

We estimative the λ-FMadogram with the following quantity :

$$\hat{\nu}_T(\lambda) = \frac{1}{2\sum_{t=1}^T I_t J_t}\sum_{t=1}^T |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)|I_t J_t \tag{7}$$

Proposition 2 (Proposition 3 of [NGCD09])**.** *Suppose that conditions 2 holds and that $\sum_{t=1}^T I_t J_t = T$ (no missing data). Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then*

$$T^{-1/2}\sum_{t=1}^T (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u,v)dJ(u,v)$ where $N_C(u,v)$ is defined by equation (5) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Bre20]). The special case, $J(x,y) = \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (7) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]^2} N_C(u,v)dJ(u,v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x,y)dJ(x,y) = \frac{1}{2}\int_0^1 f(0, y^{1/(1-\lambda)})dy + \frac{1}{2}\int_0^1 f(x^{1/\lambda}, 0)dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)})dx$$

for all bounded functions f.

Furthermore, as the limiting process is the linear transformation of a tight gaussian process, we know from [vdVW96] that it is Gaussian. Before going further, we want to detail the structure of the variance of the limiting process. Doing that, we introduce the following lemma :

Lemma 1. *Let* $(B_C(u,v))_{u,v\in[0,1]^2}$ *a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u\wedge u', v\wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$*. Let* $a, b \in [0,1]$ *fixed, if* $a = 0$ *or* $b = 0$*, then wet get the following equality :*

$$\mathbb{E}[\int_0^1 B_C(u,a)du \int_0^1 B_C(b,u)du] = 0$$

Démonstration. Without loss of generality, suppose that $a = 0$ and $b \in [0,1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned}\mathbb{E}[\int_0^1 B_C(u,0)du \int_0^1 B_C(b,u)du] &= \mathbb{E}[\int_0^1 \int_0^1 B_C(u,0)B_C(b,v)dudv] \\ &= \int_0^1 \int_0^1 \mathbb{E}[B_C(u,0)B_C(b,v)]dudv\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u,0)B_C(b,v)] = C(u\wedge v, 0) - C(u,0)C(b,v)$$

We recall that, by definition, a copula satisfy $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$. Then, the equation below is equal to 0. Our conclusion directly follows. □

Using this lemma, we can infer the following proposition :

Proposition 3. *Let* $N_C(u,v)$ *the process defined in equation (5) and* $a, b \in [0,1]$ *fixed. If* $a = 0$ *or* $b = 0$*, then :*

$$\mathbb{E}[\int_0^1 N_C(u,a)du \int_0^1 N_C(b,u)du] = 0$$

With this proposition, we can infer a better form of the variance of our limiting process :

Theorem 2. *Let* $N_C(u,v)$ *the process defined in equation (5) and* $J(x,y) = |x^\lambda - y^{1-\lambda}$*, then :*

$$Var(\int_{[0,1]^2} N_C(u,v)dJ(u,v)) = Var(\int_0^1 N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) \tag{8}$$

Démonstration. Recall that, with J defined as in the statement that :

$$\int_{[0,1]^2} N_C(u,v)dJ(u,v) = \frac{1}{2}\int_0^1 N_C(0, v^{1/(1-\lambda)})dv + \frac{1}{2}\int_0^1 N_C(u^{1/\lambda}, 0)du - \int_0^1 N_C(u^{1/\lambda}, u^{1/(1-\lambda)})du$$

Taking the variance and using the proposition 3 gives that only the variance of the third term matters. □

Combining theorem 1 and proposition 2 gives the following result :

Proposition 4. *Suppose that the assumption of theorem 1 holds. Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then*

$$(\sum_{t=1}^{T} I_t J_t)^{-1/2} \sum_{t=1}^{T} (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) I_t J_t - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]^2} N_C(u,v) dJ(u,v)$ where $N_C(u,v)$ is defined in theorem 1. The special case, $J(x,y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (7) :

$$(\sum_{t=1}^{T} I_t J_t)^{1/2} \{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]} N_C(u,v) dJ(u,v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2}\int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2}\int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

for all bounded functions f.

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