Chapter 1

On the variance of the Madogram with extreme value copula

1.1 Introduction

1.1.1 Context

Management on environmental ressources often requires the analysis of multivariate (or univariate) extreme values. Suppose we would like to examine the water level of the Seine in Paris. In the classical theory, one is often interested in the behaviour of the mean or average. This average will then be described through the expected value $\mathbb{E}[X]$ of the distribution. The central limit theorem yiels, under some assumptions on the moments, the asymptotic behavior of the sample mean $\bar{X}$. This result can be used to provide a confidence interval for $\mathbb{E}[X]$ for a level $\alpha \in [0,1]$. But in case of water level, it can be just as important to estimate tail probabilities. Furthermore, what if the second moment $\mathbb{E}[X^2]$ or even the mean is not finite? Then the central limit theorem does not apply and the classical theory, carryied by the normal distribution, is no longer relevant [Beirlant et al., 2004].

Some extreme events, such as heavy precipitation or wind speed has spatial characteristics and geostatisticians are striving to better understand the physical processes in hand. In geostatistics, we often consider $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, S a set of locations and $(E, \mathcal{E})$ a measurable state space. We define on this probability space a stochastic process $X = \{X_s, s \in S\}$ with values on $(E, \mathcal{E})$. It is classical to define the following second-order statistic (see [Carlo, 2008] chapter 1.3 for definition and basic properties) :

$$\gamma(h) = \frac{1}{2}\mathbb{E}[|X(s+h) - X(s)|^2]$$

where $\{X(s), s \in S\}$ represents a spatial and stationnary process with a well defined covariance function. This function is called the semi-variogram. With respect to extremes, this definition is not well adapted because a second order statistic is difficult to interpret inside the framework of extreme value theory. To ensure that we always work with finite moment quantities, the following type of first-order of variogram is introduced :

$$\nu(h) = \frac{1}{2}\mathbb{E}[|F(X(s+h)) - F(X(s))|]$$

Where $F(u) = \mathbb{P}(X(s) \leq u)$. Let us define the pairwise extremal dependence function (section 4.3 of [Coles et al., 1999]) such as :

$$V_h(x,y) = \int_0^1 max(\frac{w}{x}, \frac{1-w}{y})2dH(w)$$

where x, y are two reals. It has been shown ([Cooley et al., 2006]) that $\nu(h)$ fully characterizes the extremal coefficient $V_h(1,1)$ since we have the following relationship :

$$V_h(1,1) = \frac{1+2\nu(h)}{1-2\nu(h)}$$

Then, the estimation and the study of the madogram gives us an estimator and a analysis of the extremal coefficient's estimator. This way of thinking was reproduced by [Marcon et al., 2017] using a multivariate madogram in order to estimate the Pickands dependence function. This method extend [Capéraà et al., 1997] which propose a non parametric estimator to estimate the Pickands dependence function for bivariate extreme value copulas. Let's go back to the estimation of our extremal coefficient, his main drawback is that it only focuses on the values $V_h(x,x)$ but does not provide any information about $V_h(x,y)$ for $x \neq y$. To overpass this drawback [Naveau et al., 2009] introduce the λ-Madogram defined as :

$$\nu(h,\lambda) = \frac{1}{2}\mathbb{E}[\left|F^{\lambda}(X(s+h)) - F^{1-\lambda}(X(s))\right|]$$

for every $\lambda \in (0,1)$. It is shown in the same paper that the λ-madogram fully characterizes the dependence function $V_h(x,y)$ with the following relationship,

$$V_h(\lambda, 1-\lambda) = \frac{c(\lambda)+\nu(h,\lambda)}{1-c(\lambda)-\nu(h,\lambda)}$$

Furthermore, this statistic kept our attention because it can be seen as a dissimilarity measure among bivariate maximas to be used in a clustering algorithm [Bernard et al., 2013]. This first chapter aims to study the variance of the λ-madogram with the fewest possible assumptions. In our knowledge, only [Guillou et al., 2014] has computed the variance of the sole madogram with independent copula and found 1/90.

1.1.2 Notations

Let (X,Y) be a bivariate random vector with joint distribution function $H(x,y)$ and marginal distribution function $F(x)$ and $G(y)$. A function $C : [0,1]^2 \rightarrow [0,1]$ is called a *bivariate copula* if it is the restriction to $[0,1]^2$ of a bivariate distribution function whose marginals are given by the uniform distribution on the interval $[0,1]$. Since the work of Sklar, it is well known that every distribution function H can be decomposed as $H(x,y) = C(F(x), G(y))$, for all $(x,y) \in \mathbb{R}^2$.

Let $(X_t)_{t=1,\dots,T}$ be an i.i.d. sample of a bivariate random vectors whose underlying copula is denoted by C and whose margins by F, G. We will write the generalized inverse function of F (respectively G) as $F^{\leftarrow}(u) = inf\{v \in \mathbb{R} | F(v) \geq u\}$ (respectively $G^{\leftarrow}(u) = inf\{v \in \mathbb{R} | G(v) \geq u\}$) where $0 < u, v < 1$. Given $\mathcal{X} \subset \mathbb{R}^2$, let $l^{\infty}(\mathcal{X})$ denote the spaces of bounded real-valued function on $\mathcal{X}$. We define by $D(\mathcal{X})$ the Skorokhod space of functions x with values on $\mathcal{X}$ which are *càdlàg*. For $f : \mathcal{X} \rightarrow \mathbb{R}$, let $||f||_{\infty} = sup_{x\in\mathcal{X}}|f(x)|$. The arrows "$\xrightarrow{a.s.}$", "$\xrightarrow{d}$" and "$\rightsquigarrow$" denote almost sure convergence, convergence in distribution of random vectors and weak convergence of functions in $l^{\infty}(\mathcal{X})$.

This chapter is organized as follows, in section 1.2, we introduce our estimator and we discuss its properties. In section 1.3, we investigate the finite-sample performance the estimator by means of Monte Carlo simulations. All proofs are deferred to the appendices.

1.2 Theory

1.2.1 Weak convergence of the Madogram

We consider the bivariate extreme value copula which can be written in the following form [Gudendorf and Segers, 2009].

$$C(u,v) = (uv)^{A(log(v)/log(uv))} \tag{1.1}$$

for all $u, v \in [0,1]$ and where $A(\cdot)$ is the Pickands dependence function, *i.e.*, $A : [0,1] \longrightarrow [1/2, 1]$ is convex and satisfies $max(t, 1-t) \leq A(t) \leq 1$, $\forall t \in [0,1]$. Following [Fermanian et al., 2004], to guarantee the weak convergence of our empirical copula process, we make the following assumptions.

Condition 1. *(i) The bivariate distribution function H has continuous margins F, G and copula C.*

(ii) The derivative of the Pickands dependence function $A'(t)$ exists and is continuous on $(0,1)$.

(iii) The limits $\lim_{u\rightarrow 0^+} \frac{\partial C(u,v)}{\partial u}$ for every $v \in [0,1]$ and $\lim_{v \rightarrow 0^+} \frac{\partial C(u,v)}{\partial v}$ for every $u \in [0,1]$ exists.

The first Condition guarantee the uniqueness of the representation $H(x,y) = C(F(x), G(y))$ on the range of (F,G). Under Condition 1 (ii), the first-order partial derivatives of C with respect to u and is continous on the set $\{(u,v) \in [0,1]^2 : 0 < u < 1\}$. Indeed, we have

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{C(u,v)}{u}\left(A(log(v)/log(uv)) - A'(log(v)/log(uv))\frac{log(v)}{log(uv)}\right), & \text{if}\, u,v>0 \\ 0, & \text{if}\, v = 0, \quad 0<u<1 \end{cases}$$

$$\frac{\partial C(u,v)}{\partial v} = \begin{cases} \frac{C(u,v)}{v}\left(A(log(v)/log(uv)) + A'(log(v)/log(uv))\frac{log(u)}{log(uv)}\right), & \text{if}\, u,v>0 \\ 0, & \text{if}\, u = 0, \quad 0<v<1 \end{cases}$$

The properties of A imply $0 \leq A(t) - tA'(t) \leq 1$ and $0 \leq A(t) + (1-t)A'(t) \leq 1$ where $t = log(v)/log(uv)$ (see [Segers, 2012]). Therefore if $v \downarrow 0$, then $\partial C(u,v)/\partial u \rightarrow 0$ as required. The Madogram is an estimator commonly used with extrema due to his relation with the pairwise extremal dependence coefficient (see [Cooley et al., 2006], [Guillou et al., 2014]). In this study, we aim to analyze the variance structure of the λ-Madogram defined in [Naveau et al., 2009] such as :

Definition 1. *Let $(X_t)_{t=1,\dots,T}$ be an i.i.d. sample of bivariate random vectors with unknown margins F and G. A λ-FMadogram is the quantity defined by :*

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|] \tag{1.2}$$

Having a sample $(X_1, Y_1), \dots, (X_T, Y_T)$ of T bivariate vector with unknown margins F and G, we construct the empirical distribution function :

$$\hat{H}_T(x, y) = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{\{X_t \leq x, Y_t \leq y\}}$$

and let $\hat{F}_T(x)$ and $\hat{G}_T(y)$ be its associated marginal distributions, that is,

$$\hat{F}_T(x) = \hat{H}_T(x, +\infty) \quad \text{and} \quad \hat{G}_T(y) = \hat{H}_T(+\infty, y) \quad -\infty < x, y < +\infty$$

Based on these identical and independent copies $(X_1, Y_t), \dots, (X_T, Y_T)$, it is natural to define the following estimator of the λ-Madogram:

$$\hat{\nu}_T(\lambda) = \frac{1}{2T}\sum_{t=1}^{T} |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| \tag{1.3}$$

We define the empirical copula function $\hat{C}_T(u, v)$ by

$$\hat{C}_T(u, v) = \hat{H}_T(\hat{F}_T^{\leftarrow}(u), \hat{G}_T^{\leftarrow}(v)), \quad 0 \leq u, v \leq 1,$$

and the (ordinary) empirical copula process

$$\mathbb{C}_T(u, v) = \sqrt{n}(\hat{C}_T - C)(u, v), \quad 0 \leq u, v \leq 1,$$

The weak convergence of $\mathbb{C}_T$ has already been proved by [Fermanian et al., 2004] using previous results on the Hadamard differentiability of the map $\phi : D([0, 1]^2) \rightarrow l^{\infty}([0, 1]^2)$ which transforms the cdf H into its copula function C (see lemma 3.9.28 from [van der Vaart and Wellner, 1996]). We recall the theorem for convenience.

Theorem 1 (Theorem 3 of [Fermanian et al., 2004]). *Suppose that H has continuous marginal distribution functions and that the copula function $C(x, y)$ has continuous partial derivatives. THen the empirical copula process $\{\hat{Z}_T(u, v), 0 \leq u, v \leq 1\}$ converges weakly to a Gaussian process $\{N_C(u, v), 0 \leq u, v \leq 1\}$ in $l^{\infty}([0, 1]^2)$.*

The limiting Gaussian process can be written as

$$N_C(u, v) = B_C(u, v) - \dot{C}_1(u, v)B_C(u, 1) - \dot{C}_2(u, v)B_C(1, v) \tag{1.4}$$

where B_C is a brownian bridge in $[0, 1]^2$ with covariance function

$$\mathbb{E}[B_C(u, v)B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$$

Under the assumptions defined in condition 1, the following proposition from [Naveau et al., 2009] hold.

Proposition 1 (Proposition 3 of [Naveau et al., 2009]). *Suppose that conditions 1 holds and let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then:*

$$T^{-1/2}\sum_{t=1}^{T}(J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u, v)dJ(u, v)$ where $N_C(u, v)$ is defined by equation (1.4) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Breton, 2020]). The special case, $J(x, y) = \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (1.3) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]^2} N_C(u, v)dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y)dJ(x, y) = \frac{1}{2}\int_{[0,1]} f(0, y^{1/(1-\lambda)})dy + \frac{1}{2}\int_{[0,1]} f(x^{1/\lambda}, 0)dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)})dx \tag{1.5}$$

for all bounded-measurable function $f : [0, 1]^2 \mapsto \mathbb{R}$.

Some details explaining equation (1.5) are given in lemma A.2 in appendix. Using the properties of a Copula fonction, we are able to write the term $Var(\int_{[0,1]^2} N_C(u, v)dJ(u, v)$ in a more practical way. This result is resumed with the followint proposition.

Proposition 2. *Let $N_C(u, v)$ the process defined in Equation (1.4) and $J(x, y) = |x^{\lambda} - y^{1-\lambda}|$, then :*

$$Var(\int_{[0,1]^2} N_C(u, v)dJ(u, v)) = Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) \tag{1.6}$$

Using extreme value copula, we want to compute the following integral:

$$Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) = Var(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du - int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du$$
$$- \int_{[0,1]} B_C(1, v^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v})$$

Notice that, on sections, the extreme value copula is a polynom, *i.e*:

$$C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) = u^{\frac{A(\lambda)}{\lambda(1-\lambda)}}$$

Furthermore, we have the same pattern for partial derivatives:

$$\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} = \frac{C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{u^{\frac{1}{\lambda}}}(A(\lambda) - A'(\lambda)\lambda)$$
$$\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} = \frac{C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{v^{\frac{1}{1-\lambda}}}(A(\lambda) + A'(\lambda)(1-\lambda))$$

Let $\mathcal{A}$ be the space of Pickands dependence functions. We will denote by $\kappa(\lambda, A)$ and $\zeta(\lambda, A)$ two functional such as :

$$\kappa\colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto A(\lambda) - A'(\lambda)\lambda$$

$$\zeta\colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto A(\lambda) + A'(\lambda)(1-\lambda)$$

Furthermore, the integral $\int_{[0,1]}\int_{[0,1]} C(u,v)dudv$ does not admit, in general, a closed form. In order to tackle this problem, we use the following inequality from extreme value copula :

$$uv \leq C(u,v) \leq min(u,v), \quad 0 \leq u,v \leq 1 \tag{1.7}$$

We note, for notational convenience the following functionals

$$f\colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)}\right)^2$$

Using properties of the extreme value copula and the bounds of equation (1.7) permit us to bound the variance of the scaled λ-Madogram $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda)$.

Theorem 2. *For $\lambda \in (0,1)$, let $A_1(\lambda) = A(\lambda)/\lambda$, $A_2(\lambda) = A(\lambda)/(1-\lambda)$. Then $Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du)$ is given by :*

$$f(\lambda,A)\left(\frac{A(\lambda)}{A(\lambda)+2\lambda(1-\lambda)} + \frac{\kappa(\lambda,A)^2(1-\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-1\lambda)} + \frac{\zeta(\lambda,A)^2\lambda}{2A(\lambda)-\lambda+2\lambda(1-\lambda)}\right)$$
$$-2\kappa(\lambda,A)f(\lambda,A)\left(\frac{(1-\lambda)^2 - A(\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)}\right) - 2\kappa(\lambda,A)\lambda(1-\lambda)\int_{[0,\lambda]}\left[A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)-s\lambda+1\right]^{-2}ds$$
$$-2\zeta(\lambda,A)f(\lambda,A)\left(\frac{\lambda^2 - A(\lambda)}{2A(\lambda)-\lambda+2\lambda(1-\lambda)}\right) - 2\zeta(\lambda,A)\lambda(1-\lambda)\int_{[\lambda,1]}\left[A(s)+s(A_1(\lambda)-1-\lambda)-(1-s)(1-\lambda)+1\right]^{-2}ds$$
$$+2f(\lambda,A)\kappa(\lambda,A)\zeta(\lambda,A) + 2\kappa(\lambda,A)\zeta(\lambda,A)\lambda(1-\lambda)\int_{[0,1]}\left[A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)+s(A_1(\lambda)-\lambda-1)+1\right]^{-2}ds$$

From theorem 2, we are able to infer the closed form of the λ-Madogram's variance in the case of an independent Copula, *i.e.* when $C(u,v) = uv$. Indeed, we just have to take $A(t) = 1$ for every $t \in [0,1]$. This result is summarised on the following statement:

Proposition 3. *Under the framework of theorem 2 and if we take $C(u,v) = uv$, the independent copula, then the asymptotic variance of $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda))$ has the following form*

$$Var(\int_{[0,1]^2} N_C(u,v)dJ(u,v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2\left(\frac{1}{1+2\lambda(1-\lambda)} - \frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} - \frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right)$$

with $\lambda \in [0,1]$

1.3 Simulation

This section present some simulation to support our findings. All codes are available on this github [1].

1.3.1 Simulation of Copula and compute numerically the asymptotic variance.

In our simulation, we sample our data from a Gumbel Copula where his Pickands dependance function are defined by the following :

$$A(t) = (t^\theta + (1-t)^\theta)^{1/\theta} \tag{1.8}$$

with $t \in [0,1]$ and $\theta \geq 1$. If $\theta = 1$, we retrieve the independent copula and if θ goes to infinity, we obtain the maximal dependency case with $C(u,v) = max(u,v)$. Furthermore, this copula satisfies Condition 1 (ii) and it's derivative is given by:

$$A'(t) = (t^{\theta-1} - (1-t)^{\theta-1})(t^\theta + (1-t)^\theta)^{(1/\theta)-1} \tag{1.9}$$

The Gumbel Copula has the particularity to be the unique object to be an Archimedean and an extreme value copula. To remind, an Archimedean copula can be written as :

$$C(u,v) = \phi^{\leftarrow}(\phi(u) + \phi(v))$$

With $\phi : [0,1] \rightarrow [0, \infty[$. The function ϕ should be scrictly decreasing and convex and satisfy $\phi(1) = 0$. Let $\phi(t) = (-log(t))^\theta$ the generator function of Gumbel Copula. As ϕ is continuously differentiable on $(0,1]$ and $phi'(0_+) = -\infty$, then the first order partial derivatives of C are given by :

$$\frac{\partial C(u,v)}{\partial u} = \frac{\phi'(u)}{\phi'(C(u,v))}, \quad v \in [0,1], 0 < u < 1, \quad \frac{\partial C(u,v)}{\partial v} = \frac{\phi'(v)}{\phi'(C(u,v))}, \quad u \in [0,1], 0 < v < 1$$

As $\lim_{x \rightarrow \infty} \phi^{\leftarrow}(x+y)/\phi^{\leftarrow}(y) = 1$ for all $y \in \mathbb{R}_+$, we have $\lim_{u \downarrow 0^+} \partial C(u,v)/\partial u = 1$ for every $v \in (0,1]$. Hence, Condition 1 (iii) is verified. Remark that, whereas $\partial C(u,v)/\partial u = 0$ as $v = 0$, it follows that $\partial C(u,v)/\partial u$ cannot be extended continuously to the point $(0,0)$. The reasoning for the other partial derivatives follows in a completely similar fashion, so the details are omitted

We recall the main algorithm to sample from an archimedean distribution below :

Algorithm 1 Sample from a Gumbel Copula

1: **procedure** GUMBEL COPULA
2: Sample v_1, v_2 independently from a random variable uniformly distributed on [0,1]
3: Set $K_C(w) = w - \frac{\phi(w)}{\phi'(t)}$
4: solve $K_C(w) = v_2$ for $0 < w < 1$
5: Set $u_1 = \phi^{-1}(v_1^{\frac{1}{\theta}}\phi(w))$ and $u_2 = \phi^{-1}((1-v_1)^{\frac{1}{\theta}}\phi(w))$
6: return u_1, u_2

The algorithm was implemented in Python from scratch (see repository in the git) or we can use the function BiCopSim from the package VineCopula in R (using the value 4 for the Gumbel's family).

Furthermore, using proposition 2, we are able to compute numerically the theoretical variance of the λ-Madogram. We just have to compute numerically the variance of the three process $\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du$, $\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\dot{C}_j(u,v)du$, for $j \in \{1,2\}$ and their covariances.

1.3.2 Monte Carlo Simulation

In this section, we give details how we implemented our Monte Carlo algorithm to study the variance structure of our estimator. We denote by M the number of reproduction of our experiment, T denotes the number of observation we have in a sample and n is the number of the subdivision of the $[0,1]$ segment. The general procedure is given by the following algorithm :

This algorithm is implemented either in R (see extreme_value_copula.R in github) either in python (same file with .py at the end).

1.3.3 Results

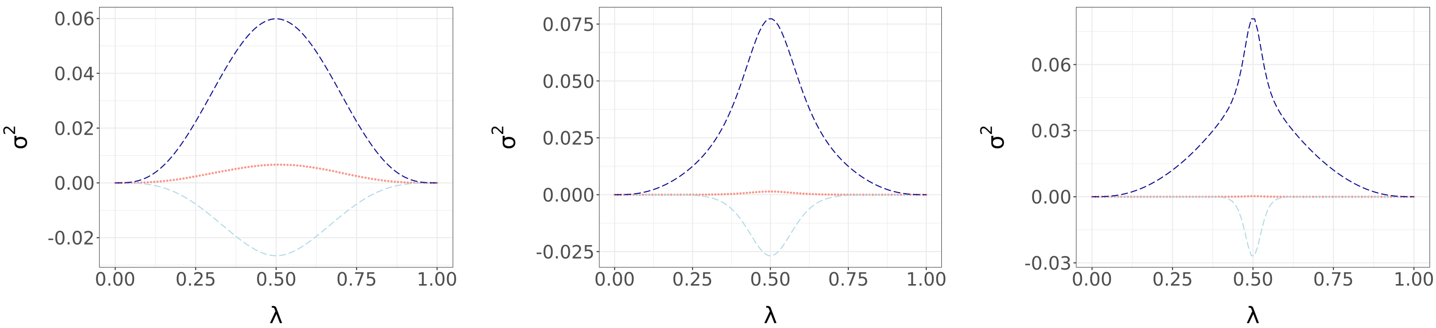
We present here three simulation with $M = 1000$, $T = 64$ and $n = 100$. The upper and lower bounds are respectively the dashed lightblue and the darkblue curves. While in salmon are the variances computed from the thousand different estimator $\sqrt{T}\hat{\nu}(\lambda)$ for each lambda in a subdivision of the segment $[0,1]$. We can notice that the bounds computed are non informative because the upper one is way higher than the theoretical and the lower still negative for every $\lambda \in (0,1)$.

[1] https://github.com/Aleboul/var_FMado

Algorithm 2 λ-Madogram estimator

procedure FMADO(M, T, n)
    System Initialization
    Let $0 = t_1 < \cdots < t_n = 1$ be a subdivision of $[0,1]$
    for $\lambda = t_1 : t_n$ **do**
        for $i = 0 : M$ **do**
            observe a realization $(U_1, V_1), \dots, (U_T, V_T)$ from C
            for every $i \in \{1, \dots, T\}$, set $(X_i, Y_i) = (F^{\leftarrow}(U_i), G^{\leftarrow}(V_i))$
            Compute the empirical cumulative distribution function $\hat{F}_T$, $\hat{G}_T$ from $(X_1, \dots, X_T)$ and $(Y_1, \dots, Y_T)$
            Compute the λ-Madogram $\hat{\nu}_T(\lambda)$
            store the value $\hat{\nu}_T(\lambda)$ the result in a output vector
    Return output vector

Figure 1.1: Bounds for Gumbel copula $\theta \in \{1.5, 5, 15\}$



Appendix A

Proofs of Chapter 1

A.1 Study of the Pickands dependence function

Lemma A.1. *Using properties of the Pickands dependence function, we have that*

$$0 \leq \kappa(\lambda, A) \leq 1, \quad 0 \leq \zeta(\lambda, A) \leq 1, \quad 0 < u, v < 1$$

Furthermore, if A admits a second derivate, $\kappa(\cdot, A)$ (resp $\zeta(\cdot, A)$) is a decreasing function (resp an increasing function).

Proof First, using that the graph of a (differentiable) convex function lies above all of its tangents and using that $A(t) \geq t$ gives, for $0 < t < 1$:

$$A'(t) \leq \frac{A(1) - A(t)}{t - 1} = \frac{1 - A(t)}{t - 1} \leq 1$$

Same reasoning using $A(t) \geq 1 - t$ leads to:

$$A'(t) \geq \frac{A(t) - A(0)}{t - 0} = \frac{A(t) - 1}{t} \geq -1$$

Let's fall back to κ and ζ. If we suppose that A admits a second derivative, the derivative of κ (resp ζ) with respect to λ gives:

$$\kappa'(\lambda, A) = -\lambda A''(\lambda) < 0, \quad \zeta'(\lambda, A) = (1 - \lambda)A''(\lambda) > 0 \quad \forall \lambda \in [0, 1]$$

Using $\kappa(0) = 1$, $\kappa(1) = 1 - A'(1) \geq 0$ gives $0 \leq \kappa(\lambda, A) \leq 1$. As $\zeta(0) = 1 + A'(0) \geq 0$ and $\zeta(1) = 1$, we have $0 \leq \zeta(\lambda, A) \leq 1$. That is the statement.

Now, we can obtain the same result while removing the hypothesis of A admits a second derivative. As A is a convex function, for $x, y \in [0, 1]$, we may have the following inequality:

$$A(x) \geq A(y) + A'(y).(x - y)$$

Take $x = 0$ and $y = \lambda$ gives

$$1 \geq A(\lambda) - \lambda A'(\lambda) = \kappa(\lambda)$$

Now, using that $-\lambda A'(\lambda) \geq -\lambda$, clearly

$$A(\lambda) - \lambda A'(\lambda) \geq A(\lambda) - \lambda \geq 0$$

As $A(\lambda) \geq max(\lambda, 1 - \lambda)$. We thus obtain our statement.

A.2 A first lemma for equation (1.5)

Lemma A.2. *For all bounded-measurable function $f : [0, 1]^2 \mapsto \mathbb{R}$, if $J(s, t) = |s^\lambda - t^{1-\lambda}|$, then the following integral satisfies:*

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_{[0,1]} f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_{[0,1]} f(x^{1/\lambda}, 0) dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

Proof Let A a element of $\mathcal{B}([0, 1]^2)$. We can pick an element of the form $A = [0, s] \times [0, t]$, where $s, t \in [0, 1]$ and $\lambda \in [0, 1]$. Let us introduce the following indicator function :

$$f_{s,t}(x, y) = \mathbb{1}_{\{(x,y) \in [0,1]^2, 0 \leq x \leq s, 0 \leq y \leq t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x, y) dJ(x, y) = J(s, t) - J(0, 0) = |s^\lambda - t^{1-\lambda}|$$

in other hand, using the equality $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - min(x,y)$, one has to show

$$\begin{aligned}
\frac{1}{2}|s^{\lambda} - t^{1-\lambda}| &= \frac{s^{\lambda}}{2} + \frac{y^{1-\lambda}}{2} - min(s^{\lambda}, t^{1-\lambda}) \\
&= \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} f_{s,t}(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx
\end{aligned}$$

Notice that the class :

$$\mathcal{E} = \{A \in \mathcal{B}([0,1]^2) : \int_{[0,1]^2} \mathbb{1}_A(x,y)dJ(x,y) = \int_{[0,1]} \mathbb{1}_A(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} \mathbb{1}_A(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} \mathbb{1}_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\}$$

contain the class $\mathcal{P}$ of all closed pavements of $[0,1]^2$. It is otherwise a monotone class (or λ-system). Hence as the class $\mathcal{P}$ of closed pavement is a π-system, the class monotone theorem ensure that $\mathcal{E}$ contains the sigma-field generated by $\mathcal{P}$, that is $\mathcal{B}([0,1]^2)$.

This result holds for simple function $f(x,y) = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i}$ where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathcal{B}([0,1]^2)$ for all $i \in \{1,\dots,n\}$. We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function $f : [0,1]^2 \mapsto \mathbb{R}$ considering $f = f_+ - f_-$ with $f_+ = max(f,0)$ and $f_- = min(-f,0)$. We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral.

A.3 Proof of prosition 2

In order to prove our proposition, we introduce two lemmas.

Lemma A.3. *Let $(B_C(u,v))_{u,v \in [0,1]^2}$ a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$. Let $a, b \in [0,1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}[\int_{[0,1]} B_C(u,a)du \int_{[0,1]} B_C(b,u)du] = 0$$

Proof Without loss of generality, suppose that $a = 0$ and $b \in [0,1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned}
\mathbb{E}[\int_{[0,1]} B_C(u,0)du \int_{[0,1]} B_C(b,u)du] &= \mathbb{E}[\int_{[0,1]}\int_{[0,1]} B_C(u,0)B_C(b,v)dudv] \\
&= \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u,0)B_C(b,v)]dudv
\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u,0)B_C(b,v)] = C(u \wedge v, 0) - C(u,0)C(b,v)$$

We recall that, by definition, a copula satisfy $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$. Then, the equation below is equal to 0. Our conclusion directly follows.

Lemma A.4. *Let $N_C(u,v)$ the process defined in equation (1.4) and $a, b \in [0,1]$ fixed. If $a = 0$ or $b = 0$, then :*

$$\mathbb{E}[\int_{[0,1]} N_C(a,u)du \int_{[0,1]} N_C(u,b)du] = 0$$

Proof Without loss of generality, let a = 0. Using the definition of $N_C(u,v)$, we have

$$N_C(0,u) = B_C(0,u) - \frac{\partial C(0,u)}{\partial u} B_C(0,1) - \frac{\partial C(0,u)}{\partial v} B_C(1,u)$$

Which is well defined if we consider, for a fixed $v \in [0,1]$

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{\partial C(u,v)}{\partial u}, & \text{if}\, u > 0 \\ \lim\limits_{u \to 0^+} \frac{\partial C(u,v)}{\partial v}, & \text{if}\, u = 0,\ v \in (0,1] \end{cases} \tag{A.1}$$

The continuous extension of $\frac{\partial C(u,v)}{\partial u}(\cdot, v)$ on $[0,1]$ while we have used 1 (iii) for the existence of the right limit. We do the same for $\frac{\partial C(u,v)}{\partial v}(u, \cdot)$. We have :

$$\begin{aligned}
\mathbb{E}\left[\int_{[0,1]} N_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] &= \mathbb{E}\left[\int_{[0,1]} B_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] \\
&- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial u} B_C(0,1)du \int_{[0,1]} N_C(u,b)du\right] \\
&- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right]
\end{aligned}$$

Using preceding lemma, we got that the two first terms are equal to zero. Only the last term should be discuss. Remember that $\frac{\partial C(0,u)}{\partial v} = 0$ for all $u \in]0,1]$, as we integrate with respect to the lebesgue measure, the set $\{0\}$ is of measure 0 because it is a countable set, then :

$$\mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = \mathbb{E}\left[\int_{]0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = 0$$

These two results gives us the proposition.

A.4 Proof of theorem 2

We are able to compute the variance for each process and they are given by the following expressions :

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) = f(\lambda, A)\left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)}\right)$$

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = f(\lambda, A)\left(\frac{\kappa^2(\lambda, A)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)}\right)$$

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial v}du\right) = f(\lambda, A)\left(\frac{\zeta^2(\lambda, A)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right)$$

We now compute the covariance :

$$cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(v^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

$$= \int_{[0,1]}\int_{[0,v]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv + \int_{[0,1]}\int_{[v,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

for the first one, we have :

$$\int_{[0,1]}\int_{[0,v]} (C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)\left(\frac{1-\lambda}{2A(\lambda) + (2\lambda - 1)(1-\lambda)}\right)$$

For the second part, using Fubini, we have :

$$\int_{[0,1]}\int_{[0,u]} (C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu$$

for the right hand side of the "minus" sign, we may compute :

$$\int_{[0,1]}\int_{[0,u]} C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)$$

The last one still difficult to handle,

$$\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \tag{A.2}$$

Following the proof of proposition 3.3 from [Genest and Segers, 2009], the substitution $v^{\frac{1}{\lambda}} = x$ and $u^{\frac{1}{1-\lambda}} = y$ yield

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu \\
&= \lambda(1-\lambda)\int_{[0,1]}\int_{[0,y^{\frac{1-\lambda}{\lambda}}]} C(x,y)\frac{\partial C(x, x^{\frac{\lambda}{1-\lambda}})}{\partial u} x^{\lambda-1}y^{-\lambda}dxdy \\
&= \lambda(1-\lambda)\kappa(\lambda, A)\int_{[0,1]}\int_{[0,y^{\frac{1-\lambda}{\lambda}}]} C(x,y)x^{\frac{A(\lambda)}{1-\lambda}-(1-\lambda)-1}y^{-\lambda}dxdy
\end{aligned}$$

Next, use the substitution $x = w^{1-s}$ and $y = w^s$. Note that $w = xy \in [0,1]$, $s = log(y)/log(xy) \in [0,1]$, $C(x,y) = w^{A(s)}$ and the Jacobian of the transformation is $-log(w)$. As the constraint $x < y^{-1+1/\lambda}$ reduces to $s < \lambda$, the integral becomes:

$$\begin{aligned}
&-\lambda(1-\lambda)\kappa(\lambda, A)\int_{[0,\lambda]}\int_{[0,1]} w^{A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)-s\lambda} log(w)dwds \\
&= \lambda(1-\lambda)\kappa(\lambda, A)\int_{[0,\lambda]} \left[A(s) + (1-s)(A_2(\lambda) - 1 - (1-\lambda)) - s\lambda + 1\right]^{-2} ds
\end{aligned}$$

Let's continue with computing the following integral :

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,1]}\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)B_C(1, v^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dudv\right] \\
&= \int_{[0,1]}\int_{[0,1]} \left(C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) - u^{\frac{1}{\lambda}}v^{\frac{1}{1-\lambda}}\right)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dudv
\end{aligned}$$

The second term can be easily handled and its value is given by :

$$\int_{[0,1]}\int_{[0,1]} u^{\frac{1}{\lambda}}v^{\frac{1}{1-\lambda}}\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dudv = f(\lambda, A)\kappa(\lambda, A)\zeta(\lambda, A)$$

For the first, use the substitutions $u^{\frac{1}{\lambda}} = x$ and $v^{\frac{1}{1-\lambda}} = y$. This yields :

$$\lambda(1-\lambda)\int_{[0,1]}\int_{[0,1]} C(x,y)\frac{\partial C(x, x^{\frac{\lambda}{1-\lambda}})}{\partial u}\frac{\partial C(y^{\frac{1-\lambda}{\lambda}}, y)}{\partial v} x^{\lambda-1}y^{-\lambda}dxdy$$

Then, make the substitutions $x = w^{1-s}$, $y = w^s$ that were used for the preceding integral gives :

$$\begin{aligned}
&-\lambda(1-\lambda)\kappa(\lambda, A)\zeta(\lambda, A)\int_{[0,1]}\int_{[0,1]} w^{A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)+s(A_1(\lambda)-\lambda-1)} log(w)dwds \\
&= \lambda(1-\lambda)\kappa(\lambda, A)\zeta(\lambda, A)\int_{[0,1]} \left[A(s) + (1-s)(A_2(\lambda) - (1-\lambda) - 1) + s(A_1(\lambda) - \lambda - 1) + 1\right]^{-2} ds
\end{aligned}$$

The last covariance requires the same tools as used before, it is left to the reader. It then suffices to use the bilinearity of the covariance and to assemble the various terms to conclude.

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