Chapter 1

On the variance of the Madogram with extreme value copula

1.1 Introduction

Let (X, Y) be a bivariate random vector with joint distribution function $H(x, y)$ and continuous marginal distribution function $F(x)$ and $G(y)$. Its associated copula C is defined by $H(x, y) = C\{F(x), G(y)\}$. Since F and G are continuous, the copula C is unique and we can write $C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v))$ for $0 \leq u, v \leq 1$ and where $F^{\leftarrow}(u) = inf\{v \in \mathbb{R} | F(v) \geq u\}$ and $G^{\leftarrow}(u) = inf\{v \in \mathbb{R} | G(v) \geq u\}$ are the generalized inverse functions of F and G respectively.

1.2 Theory

1.2.1 Weak convergence of the Madogram

We consider the bivariate extreme value copula which can be written in the following form [Gudendorf and Segers, 2009].

$$C(u, v) = (uv)^{A(log(v)/log(uv))} \tag{1.1}$$

for all $u, v \in [0, 1]$ and where $A(\cdot)$ is the Pickhands dependence function, i.e. $A : [0, 1] \longrightarrow [1/2, 1]$ is convex and satisfies $max(t, 1-t) \leq A(t) \leq 1$ for every $t \in [0, 1]$. Following [Fermanian et al., 2004], to guarantee the weak convergence of our empirical copula process, we make the following assumptions.

Condition 1. *(i) The bivariate distribution function H has continuous margins F, G and copula C.*

(ii) The derivative of the Pickhands dependence function $A'(t)$ exists and is continuous on $(0, 1)$.

(iii) The limits $\underset{u \rightarrow 0^+}{lim} \frac{\partial C(u,v)}{\partial u}$ for every $v \in [0, 1]$ and $\underset{v \rightarrow 0^+}{lim} \frac{\partial C(u,v)}{\partial v}$ for every $u \in [0, 1]$ exists.

Under Condition 1 (ii), the first-order partial derivatives of C with respect to u and is continous on the set $\{(u, v) \in [0, 1]^2 : 0 < u < 1\}$. Indeed, we have

$$\frac{\partial C(u, v)}{\partial u} = \begin{cases} \frac{C(u,v)}{u} \left(A(log(v)/log(uv)) - A'(log(v)/log(uv)) \frac{log(v)}{log(uv)} \right), & \text{if } u, v > 0 \\ 0, & \text{if } v = 0, \quad 0 < u < 1 \end{cases}$$

$$\frac{\partial C(u, v)}{\partial v} = \begin{cases} \frac{C(u,v)}{v} \left(A(log(v)/log(uv)) + A'(log(v)/log(uv)) \frac{log(u)}{log(uv)} \right), & \text{if } u, v > 0 \\ 0, & \text{if } u = 0, \quad 0 < v < 1 \end{cases}$$

The properties of A imply $0 \leq A(t) - tA'(t) \leq 1$ and $0 \leq A(t) + (1-t)A'(t) \leq 1$ where $t = logv/log(uv)$ (see [Segers, 2012]). Therefore if $v \downarrow 0$, then $\partial C(u, v)/\partial u \rightarrow 0$ as required. The Madogram is an estimator commonly used with extrema due to his relation with the pairwise extremal dependence coefficient (see [Cooley et al., 2006], [Guillou et al., 2014]). In this study, we aim to analyze the variance structure of the λ-Madogram defined in [Naveau et al., 2009] such as :

Definition 1. *Let $(X_1, Y_1), \ldots, (X_T, Y_T)$ a T bivariate random vectors with unknown margins F and G. A λ-FMadogram is the quantity defined by :*

$$\nu(\lambda) = \frac{1}{2} \mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|] \tag{1.2}$$

Having a sample $(X_1, Y_1), \dots, (X_T, Y_T)$ of T bivariate vector with unknown margins F and G, it is natural to define the following estimator of the λ-Madogram :

$$\hat{\nu}_T(\lambda) = \frac{1}{2T} \sum_{t=1}^{T} |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| \tag{1.3}$$

Based on these independent copies $(X_1, Y_t), \dots, (X_T, Y_T)$, we construct the empirical distribution function :

$$\hat{H}_T(x, y) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{X_t \leq x, Y_t \leq y\}}$$

and let $\hat{F}_T(x)$ and $\hat{G}_T(y)$ be its associated marginal distributions, that is,

$$\hat{F}_T(x) = \hat{H}_T(x, +\infty) \quad \text{and} \quad \hat{G}_T(y) = \hat{H}_T(+\infty, y) \quad -\infty < x, y < +\infty$$

We define the empirical copula function $\hat{C}_T(u, v)$ by

$$\hat{C}_T(u, v) = \hat{H}_T(\hat{F}_T^{\leftarrow}(u), \hat{G}_T^{\leftarrow}(v)), \quad 0 \leq u, v \leq 1$$

and the (ordinary) empirical copula process

$$\hat{Z}_T(u, v) = \sqrt{n}(\hat{C}_T - C)(u, v), \quad 0 \leq u, v \leq 1$$

The weak convergence of $\hat{Z}_T$ has already been proved by [Fermanian et al., 2004] using previous results on the Hadamard differentiability of the map $\phi : D([0,1]^2) \to l^{\infty}([0,1]^2)$ which transforms the cdf H into its copula function C (see lemma 3.9.28 from [van der Vaart and Wellner, 1996]). We recall the theorem for convenience.

Theorem 1 (Theorem 3 of [Fermanian et al., 2004])**.** *Suppose that H has continuous marginal distribution functions and that the copula function $C(x, y)$ has continuous partial derivatives. THen the empirical copula process $\{\hat{Z}_T(u, v), 0 \leq u, v \leq 1\}$ converges weakly to a Gaussian process $\{N_C(u, v), 0 \leq u, v \leq 1\}$ in $l^{\infty}([0,1]^2)$.*

The limiting Gaussian process can be written as

$$N_C(u, v) = B_C(u, v) - \dot{C}_1(u, v) B_C(u, 1) - \dot{C}_2(u, v) B_C(1, v) \tag{1.4}$$

where B_C is a brownian bridge in $[0,1]^2$ with covariance function

$$\mathbb{E}[B_C(u, v) B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v) C(u', v')$$

Under the assumptions defined in condition 1, the following proposition from [Naveau et al., 2009] hold.

Proposition 1 (Proposition 3 of [Naveau et al., 2009])**.** *Suppose that conditions 1 holds and let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then:*

$$T^{-1/2} \sum_{t=1}^{T} (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u, v) dJ(u, v)$ where $N_C(u, v)$ is defined by equation (1.4) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Breton, 2020]). The special case, $J(x, y) = \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (1.3) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_{[0,1]} f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_{[0,1]} f(x^{1/\lambda}, 0) dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \tag{1.5}$$

for all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$.

Some details explaining equation (1.5) are given in lemma A.2 in appendix. Using the properties of a Copula fonction, we are able to write the term $Var(\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ in a more practical way. This result is resumed with the followint proposition.

Proposition 2. *Let $N_C(u, v)$ the process defined in Equation (1.4) and $J(x, y) = |x^{\lambda} - y^{1-\lambda}|$, then :*

$$Var(\int_{[0,1]^2} N_C(u, v) dJ(u, v)) = Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du) \tag{1.6}$$

Using extreme value copula, we want to compute the following integral:

$$Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) = Var(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du - int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du$$
$$- \int_{[0,1]} B_C(1, v^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v})$$

Notice that, on sections, the extreme value copula is a polynom, i.e :

$$C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) = u^{\frac{A(\lambda)}{\lambda(1-\lambda)}}$$

Furthermore, we have the same pattern for partial derivatives:

$$\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} = \frac{C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{u^{\frac{1}{\lambda}}}(A(\lambda) - A'(\lambda)\lambda)$$
$$\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} = \frac{C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{v^{\frac{1}{1-\lambda}}}(A(\lambda) + A'(\lambda)(1-\lambda))$$

Let $\mathcal{A}$ be the space of Pickhands dependence functions. We will denote by $\kappa(\lambda, A)$ and $\zeta(\lambda, A)$ two functional such as :

$$\kappa \colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto A(\lambda) - A'(\lambda)\lambda$$

$$\zeta \colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto A(\lambda) + A'(\lambda)(1-\lambda)$$

Furthermore, the integral $\int_{[0,1]}\int_{[0,1]} C(u,v)dudv$ does not admit, in general, a closed form. In order to tackle this problem, we use the following inequality from extreme value copula :

$$uv \leq C(u,v) \leq min(u,v), \quad 0 \leq u, v \leq 1 \tag{1.7}$$

We note, for notational convenience the following functionals

$$f \colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto f(\lambda, A)$$

Using properties of the extreme value copula and the bounds of equation (1.7) permit us to bound the variance of the scaled λ-Madogram $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda)$.

Theorem 2. *Consider an extreme value copula as in equation (1.1). Under Condition 1, $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda))$ is weakly convergent to a centered Gaussian random variable with variance $Var\left(\int_{[0,1]} N_c(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right)$ such that*

$$\mathcal{L}(\lambda) \leq Var\left[\int_{[0,1]} N_c(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right] \leq \mathcal{U}(\lambda) \tag{1.8}$$

where we respectively define $\mathcal{U}$ and $\mathcal{A}$ by :

$$\begin{aligned}
\mathcal{L}(\lambda, A) = {} & f(\lambda, A)\left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)} + \frac{\kappa^2(\lambda)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} + \frac{\zeta^2(\lambda)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right) \\
& - 2\kappa(\lambda, A)f(\lambda, A)\left(\frac{(1-\lambda)^2 - A(\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)}\right) - 2f_\kappa(\lambda, A) \\
& - 2\zeta(\lambda, A)f(\lambda, A)\left(\frac{\lambda^2 - A(\lambda)}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right) - 2f_\zeta(\lambda, A)
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}(\lambda, A) = {} & f(\lambda, A)\left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)} + \frac{\kappa^2(\lambda)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} + \frac{\zeta^2(\lambda)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right) \\
& - \kappa(\lambda, A)f(\lambda, A)\left(\frac{1-\lambda}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} + \frac{A(\lambda) - \lambda}{A(\lambda) + \lambda + 2\lambda(1-\lambda)}\right) \\
& - \zeta(\lambda, A)f(\lambda, A)\left(\frac{\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} + \frac{A(\lambda) - (1-\lambda)}{A(\lambda) + 1 - \lambda + 2\lambda(1-\lambda)}\right) \\
& + 2f(\lambda, A)\frac{\zeta(\lambda, A)\kappa(\lambda, A)\lambda(1-\lambda)}{A(\lambda)}
\end{aligned}$$

From theorem 2, we are able to infer the closed form of the λ-Madogram's variance in the case of an independent Copula, i.e. when $C(u,v) = uv$. Indeed, we just have to take $A(t) = 1$ for every $t \in [0,1]$ and look at the upper bound while erasing the contribution of the covariance between $int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du$ and $int_{[0,1]} B_C(1, v^{\frac{1}{1-\lambda}}) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial v} du$ because it is worth zero with independency. This result is summarised on the following statement:

Proposition 3. *Under the framework of theorem 2 and if we take $C(u,v) = uv$, the independent copula, then the asymptotic variance of $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda))$ has the following form*

$$Var(\int_{[0,1]^2} N_C(u,v) dJ(u,v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} - \frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} - \frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right)$$

with $\lambda \in [0,1]$

1.3 Simulation

This section present some simulation to support our findings. All codes are available on this github[1].

1.3.1 Simulation of a Gumbel Copula

In our simulation, we sample our data from a Gumbel Copula where his Pickhands dependance function are defined by the following :

$$A(t) = (t^\theta + (1-t)^\theta)^{1/\theta} \tag{1.9}$$

with $t \in [0,1]$ and $\theta \geq 1$. If $\theta = 1$, we retrieve the independent copula and if θ goes to infinity, we obtain the maximal dependency case with $C(u,v) = max(u,v)$. Furthermore, this copula satisfies Condition 1 (ii) and it's derivative is given by:

$$A'(t) = (t^{\theta-1} - (1-t)^{\theta-1})(t^\theta + (1-t)^\theta)^{(1/\theta)-1} \tag{1.10}$$

The Gumbel Copula has the particularity to be the unique object to be an Archimedean and an extreme value copula. To remind, an Archimedean copula can be written as :

$$C(u,v) = \phi^{\leftarrow}(\phi(u) + \phi(v))$$

With $\phi : [0,1] \rightarrow [0, \infty[$. The function ϕ should be scrictly decreasing and convex and satisfy $\phi(1) = 0$. Let $\phi(t) = (-log(t))^\theta$ the generator function of Gumbel Copula. As ϕ is continuously differentiable on $(0,1]$ and $phi'(0_+) = -\infty$, then the first order partial derivatives of C are given by :

$$\frac{\partial C(u,v)}{\partial u} = \frac{\phi'(u)}{\phi'(C(u,v))}, \quad v \in [0,1], 0 < u < 1, \qquad \frac{\partial C(u,v)}{\partial v} = \frac{\phi'(v)}{\phi'(C(u,v))}, \quad u \in [0,1], 0 < v < 1$$

As $\lim_{x \to \infty} \phi^{\leftarrow}(x+y)/\phi^{\leftarrow}(y) = 1$ for all $y \in \mathbb{R}_+$, we have $\lim_{u \downarrow 0^+} \partial C(u,v)/\partial u = 1$ for every $v \in (0,1]$. Hence, Condition 1 (iii) is verified. Remark that, whereas $\partial C(u,v)/\partial u = 0$ as $v = 0$, it follows that $\partial C(u,v)/\partial u$ cannot be extended continuously to the point $(0,0)$. The reasoning for the other partial derivatives follows in a completely similar fashion, so the details are omitted

We recall the main algorithm to sample from this distribution below :

Algorithm 1 Sample from a Gumbel Copula

```
procedure GUMBEL COPULA
    Sample v1, v2 independently from a random variable uniformly distributed on [0,1]
    Set K_C(w) = w - φ(w)/φ'(t)
    solve K_C(w) = v2 for 0 < w < 1
    Set u1 = φ^{-1}(v1^{1/θ} φ(w)) and u2 = φ^{-1}((1 - v1)^{1/θ} φ(w))
    return u1, u2
```

The algorithm was implemented in Python from scratch (see *gumbel.py* in the github) or we can use the function BiCopSim from the package VineCopula in R (using the value 4 for the family's parameter).

1.3.2 Monte Carlo Simulation

In this section, we give details how we implemented our Monte Carlo algorithm to study the variance structure of our estimator. We denote by M the number of reproduction of our experiment, T denotes the number of observation we have in a sample and n is the number of the subdivision of the $[0,1]$ segment. The general procedure is given by the following algorithm :

This algorithm is implemented either in R (see extreme_value_copula.R in github) either in python (same file with .py at the end).

[1] https://github.com/Aleboul/LJAD

Algorithm 2 λ-Madogram estimator

procedure FMADO(M, T, n)
    System Initialization
    Let $0 = t_1 < \cdots < t_n = 1$ be a subdivision of $[0, 1]$
    for $\lambda = t_1 : t_n$ **do**
        for $i = 0 : M$ **do**
            observe a realization $(U_1, V_1), \ldots, (U_T, V_T)$ from C
            for every $i \in \{1, \ldots, T\}$, set $(X_i, Y_i) = (F^{\leftarrow}(U_i), G^{\leftarrow}(V_i))$
            Compute the empirical cumulative distribution function $\hat{F}_T$, $\hat{G}_T$ from $(X_1, \ldots, X_T)$ and $(Y_1, \ldots, Y_T)$
            Compute the λ-Madogram $\hat{\nu}_T(\lambda)$
            store the value $\hat{\nu}_T(\lambda)$ the result in a output vector
    Return output vector

1.3.3 Results

We present here three simulation with $M = 1000$, $T = 64$ and $n = 100$. The upper and lower bounds are respectively the dashed lightblue and the darkblue curves. While in salmon are the variances computed from the thousand different estimator $\sqrt{T}\hat{\nu}(\lambda)$ for each lambda in a subdivision of the segment $[0, 1]$.

In figure 1.4, the case $\theta = 1$ is depicted (independance case). We add the variance of the scaled estimator as a straight black line.

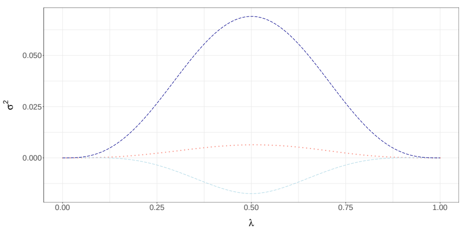

Figure 1.1: $\theta = 1.5$

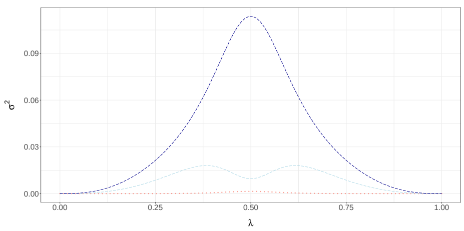

Figure 1.2: $\theta = 5$

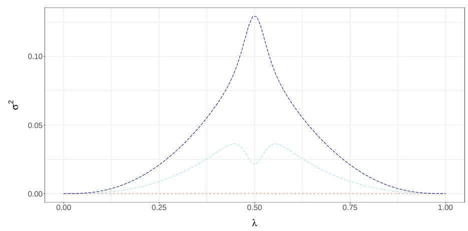

Figure 1.3: $\theta = 15$

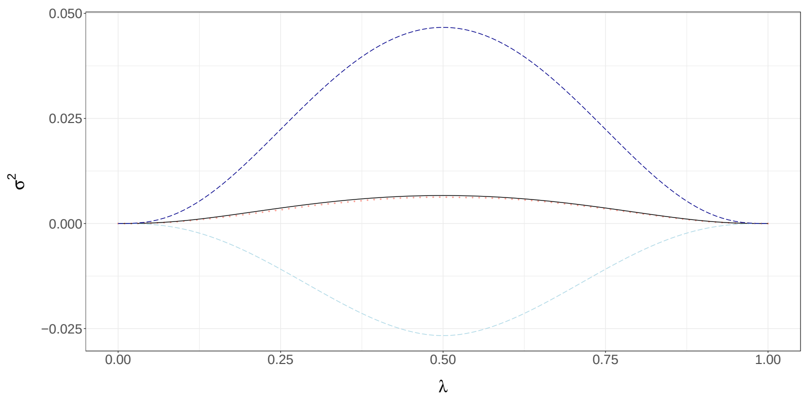

Figure 1.4: $\theta = 1$

Appendix A

Proofs of Chapter 1

A.1 Study of the Pickhands dependence function

Lemma A.1. *Using properties of the Pickhands dependence function, we have that*

$$0 \leq \kappa(\lambda, A) \leq 1, \quad 0 \leq \zeta(\lambda, A) \leq 1, \quad 0 < u, v < 1$$

Furthermore, if A admits a second derivate, $\kappa(\cdot, A)$ (resp $\zeta(\cdot, A)$) is a decreasing function (resp an increasing function).

Proof First, using that the graph of a (differentiable) convex function lies above all of its tangents and using that $A(t) \geq t$ gives, for $0 < t < 1$:

$$A'(t) \leq \frac{A(1) - A(t)}{t-1} = \frac{1 - A(t)}{t-1} \leq 1$$

Same reasoning using $A(t) \geq 1 - t$ leads to:

$$A'(t) \geq \frac{A(t) - A(0)}{t-0} = \frac{A(t) - 1}{t} \geq -1$$

Let's fall back to κ and ζ. If we suppose that A admits a second derivative, the derivative of κ (resp ζ) with respect to λ gives:

$$\kappa'(\lambda, A) = -\lambda A''(\lambda) < 0, \quad \zeta'(\lambda, A) = (1-\lambda)A''(\lambda) > 0 \quad \forall \lambda \in [0,1]$$

Using $\kappa(0) = 1$, $\kappa(1) = 1 - A'(1) \geq 0$ gives $0 \leq \kappa(\lambda, A) \leq 1$. As $\zeta(0) = 1 + A'(0) \geq 0$ and $\zeta(1) = 1$, we have $0 \leq \zeta(\lambda, A) \leq 1$. That is the statement.

Now, we can obtain the same result while removing the hypothesis of A admits a second derivative. As A is a convex function, for $x, y \in [0,1]$, we may have the following inequality:

$$A(x) \geq A(y) + A'(y).(x-y)$$

Take $x = 0$ and $y = \lambda$ gives

$$1 \geq A(\lambda) - \lambda A'(\lambda) = \kappa(\lambda)$$

Now, using that $-\lambda A'(\lambda) \geq -\lambda$, clearly

$$A(\lambda) - \lambda A'(\lambda) \geq A(\lambda) - \lambda \geq 0$$

As $A(\lambda) \geq max(\lambda, 1-\lambda)$. We thus obtain our statement.

A.2 A first lemma for equation (1.5)

Lemma A.2. *For all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$, if $J(s,t) = |s^\lambda - t^{1-\lambda}|$, then the following integral satisfies:*

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2}\int_{[0,1]} f(0, y^{1/(1-\lambda)})dy + \frac{1}{2}\int_{[0,1]} f(x^{1/\lambda}, 0)dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)})dx$$

Proof Let A a element of $\mathcal{B}([0,1]^2)$. We can pick an element of the form $A = [0,s] \times [0,t]$, where $s, t \in [0,1]$ and $\lambda \in [0,1]$. Let us introduce the following indicator function :

$$f_{s,t}(x,y) = \mathbb{1}_{\{(x,y) \in [0,1]^2, 0 \leq x \leq s, 0 \leq y \leq t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x,y) dJ(x,y) = J(s,t) - J(0,0) = |s^\lambda - t^{1-\lambda}|$$

in other hand, using the equality $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - min(x,y)$, one has to show

$$\begin{aligned}\frac{1}{2}|s^{\lambda} - t^{1-\lambda}| &= \frac{s^{\lambda}}{2} + \frac{y^{1-\lambda}}{2} - min(s^{\lambda}, t^{1-\lambda}) \\ &= int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} f_{s,t}(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\end{aligned}$$

Notice that the class :

$$\mathcal{E} = \{A \in \mathcal{B}([0,1]^2) : \int_{[0,1]^2} 1_A(x,y)dJ(x,y) = int_{[0,1]} 1_A(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} 1_A(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} 1_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\}$$

contain the class $\mathcal{P}$ of all closed pavements of $[0,1]^2$. It is otherwise a monotone class (or λ-system). Hence as the class $\mathcal{P}$ of closed pavement is a π-system, the class monotone theorem ensure that $\mathcal{E}$ contains the sigma-field generated by $\mathcal{P}$, that is $\mathcal{B}([0,1]^2)$.

This result holds for simple function $f(x,y) = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i}$ where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathcal{B}([0,1]^2)$ for all $i \in \{1, \ldots, n\}$. We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function $f : [0,1]^2 \mapsto \mathbb{R}$ considering $f = f_+ - f_-$ with $f_+ = max(f,0)$ and $f_- = min(-f,0)$. We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral.

A.3 Proof of prosition 2

In order to prove our proposition, we introduce two lemmas.

Lemma A.3. *Let $(B_C(u,v))_{u,v \in [0,1]^2}$ a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$. Let $a, b \in [0,1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}[\int_{[0,1]} B_C(u,a)du \int_{[0,1]} B_C(b,u)du] = 0$$

Proof Without loss of generality, suppose that $a = 0$ and $b \in [0,1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned}\mathbb{E}[\int_{[0,1]} B_C(u,0)du \int_{[0,1]} B_C(b,u)du] &= \mathbb{E}[\int_{[0,1]} \int_{[0,1]} B_C(u,0)B_C(b,v)dudv] \\ &= \int_{[0,1]} \int_{[0,1]} \mathbb{E}[B_C(u,0)B_C(b,v)]dudv\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u,0)B_C(b,v)] = C(u \wedge v, 0) - C(u,0)C(b,v)$$

We recall that, by definition, a copula satisfy $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$. Then, the equation below is equal to 0. Our conclusion directly follows.

Lemma A.4. *Let $N_C(u,v)$ the process defined in equation (1.4) and $a, b \in [0,1]$ fixed. If $a = 0$ or $b = 0$, then :*

$$\mathbb{E}[\int_{[0,1]} N_C(a,u)du \int_{[0,1]} N_C(u,b)du] = 0$$

Proof Without loss of generality, let a = 0. Using the definition of $N_C(u,v)$, we have

$$N_C(0,u) = B_C(0,u) - \frac{\partial C(0,u)}{\partial u} B_C(0,1) - \frac{\partial C(0,u)}{\partial u} B_C(1,u)$$

Which is well defined if we consider, for a fixed $v \in [0,1]$

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{\partial C(u,v)}{\partial u}, & \text{if } u > 0 \\ \lim\limits_{u \to 0^+} \frac{\partial C(u,v)}{\partial u}, & \text{if } u = 0 \end{cases} \tag{A.1}$$

The continuous extension of $\frac{\partial C(u,v)}{\partial u}(\cdot, v)$ on $[0,1]$ while we have used 1 (iii) for the existence of the right limit. We do the same for $\frac{\partial C(u,v)}{\partial v}(u,\cdot)$. We have :

$$\begin{aligned}\mathbb{E}\left[\int_{[0,1]} N_C(a,u)du \int_{[0,1]} N_C(u,b)du\right] &= \mathbb{E}\left[\int_{[0,1]} B_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] \\ &- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial u} B_C(0,1)du \int_{[0,1]} N_C(u,b)du\right] \\ &- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial u} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right]\end{aligned}$$

Using preceding lemma, we got that the two first terms are equal to zero. Only the last term should be discuss. Remember that $\frac{\partial C(0,u)}{\partial u} = 0$ for all $u \in]0,1[$, as we integrate with respect to the lebesgue measure, the set $\{\{0\},\{1\}\}$ is of measure 0 because it is a countable set, then :

$$\mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial u} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = \mathbb{E}\left[\int_{]0,1[} \frac{\partial C(0,u)}{\partial u} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = 0$$

These two results gives us the proposition.

A.4 Proof of theorem 2

We are able to compute the variance for each process and they are given by the following expressions :

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) = f(\lambda, A)\left(\frac{1}{A(\lambda)+2\lambda(1-\lambda)}\right)$$

$$Var\left(int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = f(\lambda, A)\left(\frac{\kappa^2(\lambda, A)(1-\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)}\right)$$

$$Var\left(int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = f(\lambda, A)\left(\frac{\zeta^2(\lambda, A)\lambda}{2A(\lambda)-\lambda+2\lambda(1-\lambda)}\right)$$

We now compute the covariance :

$$cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(v^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

$$= \int_{[0,1]}\int_{[0,v]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv + \int_{[0,1]}\int_{[v,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

for the first one, we have :

$$\int_{[0,1]}\int_{[0,v]} (C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)\left(\frac{1-\lambda}{2A(\lambda)+(2\lambda-1)(1-\lambda)}\right)$$

For the second part, using Fubini, we have :

$$\int_{[0,1]}\int_{[0,u]} (C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu$$

for the right hand side of the "minus" sign, we may compute :

$$\int_{[0,1]}\int_{[0,u]} C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)$$

The last one still difficult to handle

$$\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \tag{A.2}$$

Lemma A.5. *We have the following inequalities :*

$$\frac{\kappa(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+\lambda+2\lambda(1-\lambda))} \leq \int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu \leq f_\kappa(\lambda, A)$$

$$\frac{\zeta(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+1-\lambda+2\lambda(1-\lambda))} \leq \int_{[0,1]}\int_{[0,u]} C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dvdu \leq f_\zeta(\lambda, A)$$

$$\begin{aligned}\left(\frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\right)^2 \zeta(\lambda,A)\kappa(\lambda,A) &\leq \int_{[0,1]}\int_{[0,1]} C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} dvdu \\ &\leq \frac{(\lambda(1-\lambda))^2\zeta(\lambda,A)\kappa(\lambda,A)}{A(\lambda)(A(\lambda)+\lambda(1-\lambda))}\end{aligned}$$

with

$$f_\kappa(\lambda, A) = \begin{cases} \frac{\kappa(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+2\lambda(1-\lambda))} & if \lambda \leq 1/2 \\ f_1(\lambda, A, \kappa, \lambda) + f_2(\lambda, A, \kappa, \lambda) & if \lambda > 1/2 \end{cases}$$

$$f_\zeta(\lambda, A) = \begin{cases} f_1(\lambda, A, \zeta, 1-\lambda) + f_2(\lambda, A, \zeta, 1-\lambda) & if \lambda \leq 1/2 \\ \frac{\zeta(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+2\lambda(1-\lambda))} & if \lambda > 1/2 \end{cases}$$

where we denote by $f_1(\lambda, A, \kappa, x) = \frac{x(1-x)^3\kappa(\lambda,A)}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+(1-x))}$ *and :*

$$f_2(\lambda, A, \kappa, x) = \frac{x(1-x)\kappa(\lambda,A)}{A(\lambda)-(1-x)+\lambda(1-\lambda)} \left[\frac{x(1-x)}{A(\lambda)+x-(1-x)+2\lambda(1-\lambda)} - \frac{(1-x)^2}{A(\lambda)+1-x}\right]$$

Proof As the tools used for one integral are the same for the three, we detail the algebra for the first one. Recall that $A(t)$ is a function which satisfies the following inequalities $t \vee 1-t \leq A(t) \leq 1$ for $t \in [0,1]$. Hence, as u, v are elements of $[0,1]$, we can bound the extreme value copula by $uv \leq C(u,v) \leq u \wedge v$. We thus obtain the fondamental inequality that we will use for bouding our untractable integrals:

$$v^{\frac{1}{\lambda}} u^{\frac{1}{1-\lambda}} \leq C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) \leq v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}}$$

We want to compute now the following integral which is now tractable :

$$\int_{[0,1]}\int_{[0,u]} v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu$$

We have to consider two cases :

- if $\lambda \leq 1/2$, then $v \leq u \leq u^{\frac{\lambda}{1-\lambda}}$ and $v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} = v^{\frac{1}{\lambda}}$
- if $\lambda > 1/2$, we should consider two more cases
  - if $0 \leq v \leq u^{\frac{\lambda}{1-\lambda}}$, then $v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} = v^{\frac{1}{\lambda}}$
  - if $u^{\frac{\lambda}{1-\lambda}} \leq v \leq u$, then $v^{\frac{1}{\lambda}} \wedge u^{\frac{1}{1-\lambda}} = u^{\frac{1}{1-\lambda}}$

For the first case, we compute :

$$\int_{[0,1]}\int_{[0,u]} v^{\frac{1}{\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = \frac{(\lambda(1-\lambda))^2\kappa(\lambda,A)}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+2\lambda(1-\lambda))}$$

For the second case, we may compute, in one hand :

$$\int_{[0,1]}\int_{[0,u^{\frac{\lambda}{1-\lambda}}]} v^{\frac{1}{\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = \frac{\lambda(1-\lambda)^3\kappa(\lambda,A)}{(A(\lambda)+\lambda(1-\lambda)(A(\lambda)+2\lambda(1-\lambda))}$$

On the other hand, we have :

$$\int_{[0,1]}\int_{[u^{\frac{\lambda}{1-\lambda}},u]} v^{\frac{1}{1-\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = f_2(\lambda, \kappa, \lambda)$$

We then obtain the upper bound of our integral. For the lower bound, we just have to compute the following integral :

$$\int_{[0,1]}\int_{[0,u]} v^{\frac{1}{\lambda}} u^{\frac{1}{1-\lambda}} \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu = \frac{(\lambda(1-\lambda))^2\zeta(\lambda,A)\kappa(\lambda,A)}{A(\lambda)(A(\lambda)+\lambda(1-\lambda))}$$

A direct consequence from this lemma is given by the proposition which is following :

Lemma A.6. *If we consider an extreme value copula, then under condition* **??**, *we can control the covariance between the random variables* $\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})$, $int_{[0,1]} B_C(u^{\frac{1}{\lambda}},1)\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}du$ *and* $int_{[0,1]} B_C(1,u^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial v}du$ *such that*

$$\begin{aligned}&\frac{\kappa(\lambda,A)}{2}f(\lambda,A)\left(\frac{1-\lambda}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)}+\frac{A(\lambda)-\lambda}{A(\lambda)+\lambda+2\lambda(1-\lambda)}\right)\leq\\&cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})du, int_{[0,1]}B_C(u^{\frac{1}{\lambda}},1)\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}du\right)\\&\leq f_\kappa(\lambda,A)+\kappa(\lambda,A)f(\lambda,A)\left(\frac{(1-\lambda)^2-A(\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)}\right)\end{aligned}$$

on the other :

$$\begin{aligned}&\frac{\zeta(\lambda,A)}{2}f(\lambda,A)\left(\frac{\lambda}{2A(\lambda)-\lambda+2\lambda(1-\lambda)}+\frac{A(\lambda)-(1-\lambda)}{A(\lambda)+1-\lambda+2\lambda(1-\lambda)}\right)\leq\\&cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})du, int_{[0,1]}B_C(1,u^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial v}du\right)\\&\leq f_\zeta(\lambda,A)+\zeta(\lambda,A)f(\lambda,A)\left(\frac{\lambda^2-A(\lambda)}{2A(\lambda)-\lambda+2\lambda(1-\lambda)}\right)\end{aligned}$$

and finally

$$\begin{aligned}0&\leq cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},1)\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}du, int_{[0,1]}B_C(1,v^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial v}dv\right)\\&\leq\left(\frac{\lambda(1-\lambda)}{A(1-\lambda)+\lambda(1-\lambda)}\right)^2\frac{\zeta(\lambda,A)\kappa(\lambda,A)\lambda(1-\lambda)}{A(\lambda)}\end{aligned}$$

Proof Again, we show the main elements of proof for the first covariance. Recall that, by definition of our Brownian bridge, we have :

$$\begin{aligned}cov\left[\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})du, int_{[0,1]}B_C(u^{\frac{1}{\lambda}},1)\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}du\right]&=\int_{[0,1]}\int_{[0,v]}(C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})-C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dudv\\&+\int_{[0,1]}\int_{[0,u]}(C(v^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})-C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}},v^{\frac{1}{1-\lambda}})}{\partial u}dvdu\end{aligned}\tag{A.3}$$

The first term on the right is tractable and it's value is given by :

$$\frac{\kappa(\lambda,A)}{2}f(\lambda,A)\left(\frac{1-\lambda}{2A(\lambda)+(2\lambda-1)(1-\lambda)}\right)\tag{A.4}$$

The second term of (A.3) cannot be computed directly, but we know a lower bound whose value is :

$$\frac{\kappa(\lambda,A)(\lambda(1-\lambda))^2}{(A(\lambda)+\lambda(1-\lambda))(A(\lambda)+\lambda+2\lambda(1-\lambda))}-\frac{\kappa(\lambda,A)}{2}f(\lambda,A)=\frac{\kappa(\lambda,A)}{2}f(\lambda,A)\left(\frac{A(\lambda)-\lambda}{A(\lambda)+\lambda+2\lambda(1-\lambda)}\right)$$

The sum of the two previous term gives us the lower bound. For the upper bound, we know from lemma A.5 that the second term of (A.3) is bounded above by :

$$f_\kappa(\lambda,A)-\frac{\kappa(\lambda,A)}{2}f(\lambda,A)$$

The sum of this quantity with equality (A.4) gives :

$$f_\kappa(\lambda,A)+\kappa(\lambda,A)f(\lambda,A)\left(\frac{(1-\lambda)^2-A(\lambda)}{2A(\lambda)-(1-\lambda)+2\lambda(1-\lambda)}\right)$$

That is the statement. We finish the proof.

Proof of theorem 2. If X,Y,Z are three random variables, using the bilinearity of the covariance gives :

$$Var(X-Y-Z)=V(X)+V(Y)+V(Z)-2cov(X,Y)-2cov(X,Z)+2cov(Y,Z)$$

setting respectively X,Y,Z as $\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})du$, $int_{[0,1]}B_C(u^{\frac{1}{\lambda}},1)\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial u}du$, $int_{[0,1]}B_C(1,v^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}},u^{\frac{1}{1-\lambda}})}{\partial v}du$ and using their respective bounds gives us the statement.

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