Chapter 1

On the variance of the Madogram with extreme value copula

1 Introduction

1.1 Context

Management on environmental ressources often requires the analysis of multivariate (or univariate) extreme values. Suppose we would like to examine the water level of the Seine in Paris. In the classical theory, one is often interested in the behaviour of the mean or average. This average will then be described through the expected value $\mathbb{E}[X]$ of the distribution. The central limit theorem yiels, under some assumptions on the moments, the asymptotic behavior of the sample mean $\bar{X}$. This result can be used to provide a confidence interval for $\mathbb{E}[X]$ for a level $\alpha \in [0,1]$. But in case of water level, it can be just as important to estimate tail probabilities. Furthermore, what if the second moment $\mathbb{E}[X^2]$ or even the mean is not finite? Then the central limit theorem does not apply and the classical theory, carryied by the normal distribution, is no longer relevant [Beirlant et al., 2004].

Some extreme events, such as heavy precipitation or wind speed has spatial characteristics and geostatisticians are striving to better understand the physical processes in hand. In geostatistics, we often consider $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, S a set of locations and $(E, \mathcal{E})$ a measurable state space. We define on this probability space a stochastic process $X = \{X_s, s \in S\}$ with values on $(E, \mathcal{E})$. It is classical to define the following second-order statistic (see [Carlo, 2008] chapter 1.3 for definition and basic properties) :

$$\gamma(h) = \frac{1}{2}\mathbb{E}[|X(s+h) - X(s)|^2]$$

where $\{X(s), s \in S\}$ represents a spatial and stationnary process with a well defined covariance function. This function is called the semi-variogram. With respect to extremes, this definition is not well adapted because a second order statistic is difficult to interpret inside the framework of extreme value theory. To ensure that we always work with finite moment quantities, the following type of first-order of variogram is introduced :

$$\nu(h) = \frac{1}{2}\mathbb{E}[|F(X(s+h)) - F(X(s))|]$$

Where $F(u) = \mathbb{P}(X(s) \leq u)$. Let us define the pairwise extremal dependence function (section 4.3 of [Coles et al., 1999]) such as :

$$V_h(x,y) = \int_0^1 max(\frac{w}{x}, \frac{1-w}{y})2dH(w)$$

where x, y are two reals. It has been shown ([Cooley et al., 2006]) that $\nu(h)$ fully characterizes the extremal coefficient $V_h(1,1)$ since we have the following relationship :

$$V_h(1,1) = \frac{1+2\nu(h)}{1-2\nu(h)}$$

Then, the estimation and the study of the madogram gives us an estimator and a analysis of the extremal coefficient's estimator. This way of thinking was reproduced by [Marcon et al., 2017] using a multivariate madogram in order to estimate the Pickands dependence function. This method extend [Capéraà et al., 1997] which propose a non parametric estimator to estimate the Pickands dependence function for bivariate extreme value copulas. Let's go back to the estimation of our extremal coefficient, his main drawback is that it only focuses on the values $V_h(x,x)$ but does not provide any information about $V_h(x,y)$ for $x \neq y$. To overpass this drawback [Naveau et al., 2009] introduce the λ-Madogram defined as :

$$\nu(h,\lambda) = \frac{1}{2}\mathbb{E}[\left|F^{\lambda}(X(s+h)) - F^{1-\lambda}(X(s))\right|]$$

for every $\lambda \in (0,1)$. It is shown in the same paper that the λ-madogram fully characterizes the dependence function $V_h(x,y)$ with the following relationship,

$$V_h(\lambda, 1-\lambda) = \frac{c(\lambda)+\nu(h,\lambda)}{1-c(\lambda)-\nu(h,\lambda)}$$

Furthermore, this statistic kept our attention because it can be seen as a dissimilarity measure among bivariate maximas to be used in a clustering algorithm [Bernard et al., 2013]. This first chapter aims to study the variance of the λ-madogram with the fewest possible assumptions. In our knowledge, only [Guillou et al., 2014] has computed the variance of the sole madogram with independent copula and found 1/90.

1.2 Notations

Let (X,Y) be a bivariate random vector with joint distribution function $H(x,y)$ and marginal distribution function $F(x)$ and $G(y)$. A function $C : [0,1]^2 \to [0,1]$ is called a *bivariate copula* if it is the restriction to $[0,1]^2$ of a bivariate distribution function whose marginals are given by the uniform distribution on the interval $[0,1]$. Since the work of Sklar, it is well known that every distribution function H can be decomposed as $H(x,y) = C(F(x), G(y))$, for all $(x,y) \in \mathbb{R}^2$.

Let $(X_t)_{t=1,\dots,T}$ be an i.i.d. sample of a bivariate random vectors whose underlying copula is denoted by C and whose margins by F, G. For $x, y \in \mathbb{R}$, let $x \wedge y = min(x,y)$ and $x \vee y = max(x,y)$. We will write the generalized inverse function of F (respectively G) as $F^{\leftarrow}(u) = inf\{v \in \mathbb{R} | F(v) \geq u\}$ (respectively $G^{\leftarrow}(u) = inf\{v \in \mathbb{R} | G(v) \geq u\}$) where $0 < u, v < 1$. Given $\mathcal{X} \subset \mathbb{R}^2$, let $l^\infty(\mathcal{X})$ denote the spaces of bounded real-valued function on $\mathcal{X}$. We define by $D(\mathcal{X})$ the Skorokhod space of functions x with values on $\mathcal{X}$ which are *càdlàg*. For $f : \mathcal{X} \to \mathbb{R}$, let $||f||_\infty = sup_{x \in \mathcal{X}} |f(x)|$. The arrows "$\overset{a.s.}{\to}$", "$\overset{d}{\to}$" and "$\rightsquigarrow$" denote almost sure convergence, convergence in distribution of random vectors and weak convergence of functions in $l^\infty(\mathcal{X})$.

This chapter is organized as follows, in section 2, we introduce our estimator and we discuss its properties. Explicit formula for the asymptotic variance is also given. In section 3, we investigate the finite-sample performance the estimator by means of Monte Carlo simulations. All proofs are deferred to the appendices.

2 Theory

We consider the bivariate extreme value copula which can be written in the following form [Gudendorf and Segers, 2009].

$$C(u,v) = (uv)^{A(log(v)/log(uv))} \tag{1.1}$$

for all $u, v \in [0,1]$ and where $A(\cdot)$ is the Pickands dependence function, *i.e.*, $A : [0,1] \longrightarrow [1/2, 1]$ is convex and satisfies $max(t, 1-t) \leq A(t) \leq 1$, $\forall t \in [0,1]$. Following [Fermanian et al., 2004], to guarantee the weak convergence of our empirical copula process, we make the following assumptions.

Condition 1. *(i) The bivariate distribution function H has continuous margins F, G and copula C.*

(ii) The derivative of the Pickands dependence function $A'(t)$ exists and is continuous on $(0,1)$.

(iii) The limits $\lim\limits_{u \to 0^+} \frac{\partial C(u,v)}{\partial u}$ for every $v \in [0,1]$ and $\lim\limits_{v \to 0^+} \frac{\partial C(u,v)}{\partial v}$ for every $u \in [0,1]$ exists.

The first Condition guarantee the uniqueness of the representation $H(x,y) = C(F(x), G(y))$ on the range of (F, G). Under Condition 1 (ii), the first-order partial derivatives of C with respect to u and is continous on the set $\{(u,v) \in [0,1]^2 : 0 < u < 1\}$. Indeed, we have

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{C(u,v)}{u}\left(A(log(v)/log(uv)) - A'(log(v)/log(uv))\frac{log(v)}{log(uv)}\right), & \text{if } u,v > 0 \\ 0, & \text{if } v = 0, \quad 0 < u < 1 \end{cases}$$

$$\frac{\partial C(u,v)}{\partial v} = \begin{cases} \frac{C(u,v)}{v}\left(A(log(v)/log(uv)) + A'(log(v)/log(uv))\frac{log(u)}{log(uv)}\right), & \text{if } u,v > 0 \\ 0, & \text{if } u = 0, \quad 0 < v < 1 \end{cases}$$

The properties of A imply $0 \leq A(t) - tA'(t) \leq 1$ and $0 \leq A(t) + (1-t)A'(t) \leq 1$ where $t = log(v)/log(uv)$ (see [Segers, 2012]). Therefore if $v \downarrow 0$, then $\partial C(u,v)/\partial u \to 0$ as required. The Madogram is an estimator commonly used with extrema due to his relation with the pairwise extremal dependence coefficient (see [Cooley et al., 2006], [Guillou et al., 2014]). In this study, we aim to analyze the variance structure of the λ-Madogram defined in [Naveau et al., 2009] such as :

Definition 1. *Let $(X_t)_{t=1,\dots,T}$ be an i.i.d. sample of bivariate random vectors with unknown margins F and G. A λ-FMadogram is the quantity defined by :*

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \tag{1.2}$$

Having a sample $(X_1, Y_1), \dots, (X_T, Y_T)$ of T bivariate vector with unknown margins F and G, we construct the empirical distribution function :

$$\hat{H}_T(x, y) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{X_t \leq x, Y_t \leq y\}}$$

and let $\hat{F}_T(x)$ and $\hat{G}_T(y)$ be its associated marginal distributions, that is,

$$\hat{F}_T(x) = \hat{H}_T(x, +\infty) \quad \text{and} \quad \hat{G}_T(y) = \hat{H}_T(+\infty, y) \quad -\infty < x, y < +\infty$$

Based on these identical and independent copies $(X_1, Y_t), \dots, (X_T, Y_T)$, it is natural to define the following estimator of the λ-Madogram:

$$\hat{\nu}_T(\lambda) = \frac{1}{2T} \sum_{t=1}^{T} |\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| \tag{1.3}$$

We define the empirical copula function $\hat{C}_T(u, v)$ by

$$\hat{C}_T(u, v) = \hat{H}_T(\hat{F}_T^{\leftarrow}(u), \hat{G}_T^{\leftarrow}(v)), \quad 0 \leq u, v \leq 1,$$

and the (ordinary) empirical copula process

$$\mathbb{C}_T(u, v) = \sqrt{n}(\hat{C}_T - C)(u, v), \quad 0 \leq u, v \leq 1,$$

The weak convergence of $\mathbb{C}_T$ has already been proved by [Fermanian et al., 2004] using previous results on the Hadamard differentiability of the map $\phi : D([0,1]^2) \rightarrow l^{\infty}([0,1]^2)$ which transforms the cdf H into its copula function C (see lemma 3.9.28 from [van der Vaart and Wellner, 1996]). We recall the theorem for convenience.

Theorem 1 (Theorem 3 of [Fermanian et al., 2004])**.** *Suppose that H has continuous marginal distribution functions and that the copula function $C(x, y)$ has continuous partial derivatives. THen the empirical copula process $\{\hat{Z}_T(u, v), 0 \leq u, v \leq 1\}$ converges weakly to a Gaussian process $\{N_C(u, v), 0 \leq u, v \leq 1\}$ in $l^{\infty}([0,1]^2)$.*

The limiting Gaussian process can be written as

$$N_C(u, v) = B_C(u, v) - \dot{C}_1(u, v) B_C(u, 1) - \dot{C}_2(u, v) B_C(1, v) \tag{1.4}$$

where B_C is a brownian bridge in $[0,1]^2$ with covariance function

$$\mathbb{E}[B_C(u, v) B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v) C(u', v')$$

Under the assumptions defined in condition 1, the following proposition from [Naveau et al., 2009] hold.

Proposition 1 (Proposition 3 of [Naveau et al., 2009])**.** *Suppose that conditions 1 holds and let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then:*

$$T^{-1/2} \sum_{t=1}^{T} (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u, v) dJ(u, v)$ where $N_C(u, v)$ is defined by equation (1.4) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Breton, 2020]). The special case, $J(x, y) = \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (1.3) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_{[0,1]} f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_{[0,1]} f(x^{1/\lambda}, 0) dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \tag{1.5}$$

for all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$.

Some details explaining equation (1.5) are given in lemma A.2 in appendix. Using the properties of a Copula fonction, we are able to write the term $Var(\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ in a more practical way. This result is resumed with the followint proposition.

Proposition 2. *Let $N_C(u, v)$ the process defined in Equation (1.4) and $J(x, y) = |x^{\lambda} - y^{1-\lambda}|$, then :*

$$Var(\int_{[0,1]^2} N_C(u, v) dJ(u, v)) = Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du) \tag{1.6}$$

Using extreme value copula, we want to compute the following integral:

$$Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du) = Var(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du - int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du$$
$$- \int_{[0,1]} B_C(1, v^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v})$$

Notice that, on sections, the extreme value copula is a polynom, *i.e*:

$$C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) = u^{\frac{A(\lambda)}{\lambda(1-\lambda)}}$$

Furthermore, we have the same pattern for partial derivatives:

$$\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} = \frac{C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{u^{\frac{1}{\lambda}}}(A(\lambda) - A'(\lambda)\lambda)$$
$$\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} = \frac{C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{v^{\frac{1}{1-\lambda}}}(A(\lambda) + A'(\lambda)(1-\lambda))$$

Let $\mathcal{A}$ be the space of Pickands dependence functions. We will denote by $\kappa(\lambda, A)$ and $\zeta(\lambda, A)$ two functional such as :

$$\kappa \colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto A(\lambda) - A'(\lambda)\lambda$$

$$\zeta \colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto A(\lambda) + A'(\lambda)(1-\lambda)$$

Furthermore, the integral $\int_{[0,1]}\int_{[0,1]} C(u,v)dudv$ does not admit, in general, a closed form. But we are able to express it with respect to a simple integral of the Pickands dependence function. We note, for notational convenience the following functionals

$$f \colon [0,1] \times \mathcal{A} \to [0,1]$$
$$(\lambda, A) \mapsto \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)}\right)^2$$

For a fixed $\lambda \in (0,1)$, using properties of the extreme value copula permit us to give an explicit formulas of the asymptoc variance of the scaled λ-Madogram $\sqrt{T}\,(\hat{\nu}_T(\lambda) - \nu(\lambda))$.

Theorem 2. *For $\lambda \in (0,1)$, let $A_1(\lambda) = A(\lambda)/\lambda$, $A_2(\lambda) = A(\lambda)/(1-\lambda)$. Then $Var(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du)$ is given by :*

$$f(\lambda, A)\left(\frac{A(\lambda)}{A(\lambda) + 2\lambda(1-\lambda)} + \frac{\kappa(\lambda, A)^2(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-1\lambda)} + \frac{\zeta(\lambda, A)^2\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right)$$
$$- 2\kappa(\lambda, A)f(\lambda, A)\left(\frac{(1-\lambda)^2 - A(\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)}\right) - 2\kappa(\lambda, A)\lambda(1-\lambda)\int_{[0,\lambda]}\left[A(s) + (1-s)(A_2(\lambda) - (1-\lambda) - 1) - s\lambda + 1\right]^{-2} ds$$
$$- 2\zeta(\lambda, A)f(\lambda, A)\left(\frac{\lambda^2 - A(\lambda)}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right) - 2\zeta(\lambda, A)\lambda(1-\lambda)\int_{[\lambda,1]}\left[A(s) + s(A_1(\lambda) - 1 - \lambda) - (1-s)(1-\lambda) + 1\right]^{-2} ds$$
$$- 2f(\lambda, A)\kappa(\lambda, A)\zeta(\lambda, A) + 2\kappa(\lambda, A)\zeta(\lambda, A)\lambda(1-\lambda)\int_{[0,1]}\left[A(s) + (1-s)(A_2(\lambda) - (1-\lambda) - 1) + s(A_1(\lambda) - \lambda - 1) + 1\right]^{-2} ds$$

From theorem 2, we are able to infer the closed form of the λ-Madogram's variance in the case of an independent Copula, *i.e.* when $C(u,v) = uv$. Indeed, we just have to take $A(t) = 1$ for every $t \in [0,1]$. This result is summarised on the following statement:

Proposition 3. *Under the framework of theorem 2 and if we take $C(u,v) = uv$, the independent copula, then the asymptotic variance of $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda))$ has the following form*

$$Var(\int_{[0,1]^2} N_C(u,v)dJ(u,v)) = \left(\frac{\lambda(1-\lambda)}{1 + \lambda(1-\lambda)}\right)^2\left(\frac{1}{1 + 2\lambda(1-\lambda)} - \frac{1-\lambda}{1 + \lambda + 2\lambda(1-\lambda)} - \frac{\lambda}{2 - \lambda + 2\lambda(1-\lambda)}\right)$$

with $\lambda \in [0,1]$

3 Simulation

A vast Monte Carlo study was used to confirm that the conlusiosn of Section 2 remain in finite-sample settings. Specifically, for each $\lambda \in (0,1)$, 500 random samples of size $n = 256$ were generated from the Gumbel copula with $\theta \in \{1.0, 1.5, 2.0, 2.5, 3.0\}$. For each sample, the λ-FMadogram estimators were computed where the margins are unknown. For each estimator, the empirical version of the variance,

$$var\left(\sqrt{T}\left(\hat{\nu}(\lambda) - \nu(\lambda)\right)\right)$$

was computed by taking the variance over the 500 samples. For each estimator, we represent it's theoretical asymptotic variance using the integral exhibits in Theorem 2.

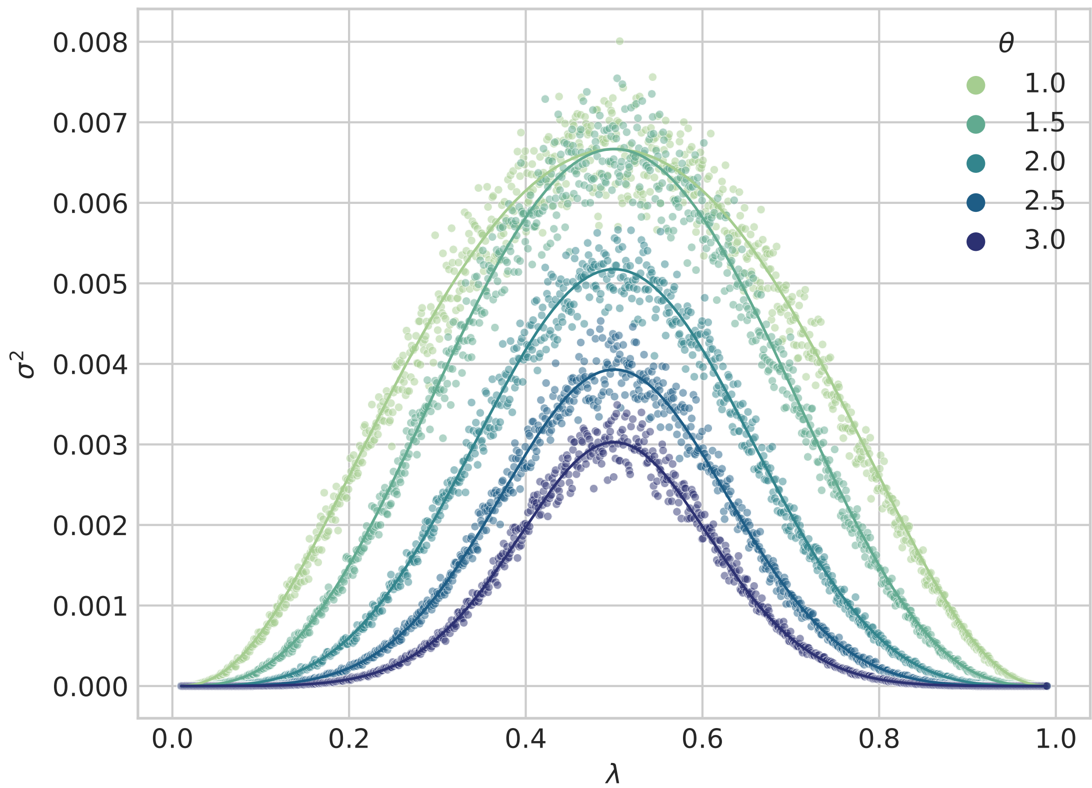


Figure 1: Variance ($\times 256$) of the estimators $\hat{\nu}(\lambda)$ based on 500 samples of size $T = 256$ from the Gumbel copula with $\theta = \{1.0, 1.5, 2.0, 2.5, 3.0\}$ chosen in such a way that $\lambda \in \{i/1000 : i = 0.01, \dots, 0.99\}$.

In figure 1, the normalized variance (*i.e.* multiplied by n) is plotted as a function of λ. Furthermore, we juxtapose on each curve the empirical estimation of the variance ($\times 256$) of the estimators $\hat{\nu}(\lambda)$ based on 500 samples of size $T = 256$ from the Gumbel Copula with a varying θ. Similar results were obtained for manu other extreme-value dependence models (see figure 2). Further, note the following (same remark as [Genest and Segers, 2009]):

1. When A is symmetric, one would expect the asymptotic variance of an estimator to reach its maximum at $\lambda = 1/2$. Such is not always the case, however, as illustrated by the t-EV model.

2. In the asymmetric negative logistic model, the asymptotic of the λ-FMadogram is close to zero for all $\lambda \in [0, 0.3]$. This is due to the fact that $A(\lambda) \approx 1 - t$ for this model.

3. We can thought that, as our variables (X, Y) are more positively dependent (in Figure 1, as θ increase), then asymptotic variance is, for $\lambda \in (0,1)$ lower or equal than the asymptotic variance in the independent case. As shown in appendix section 5 where we exhibits a counter example, it is not always the case. The Figure 2

We recall the following definitions of the extreme-value copula models:

1. The asymmetric logistic model defined by the following dependence function :

$$(1-\psi_1)t + (1-\psi_2)(1-t) + [(\phi_1 t)^\theta + (psi_2(1-t))^\theta]^{\frac{1}{\theta}}$$

with parameters $\theta \in [1, \infty[$, ψ_1, $\psi_2 \in [0,1]$. The special case $\psi_1 = \psi_2 = 1$ gives us the symmetric model of Gumbel.

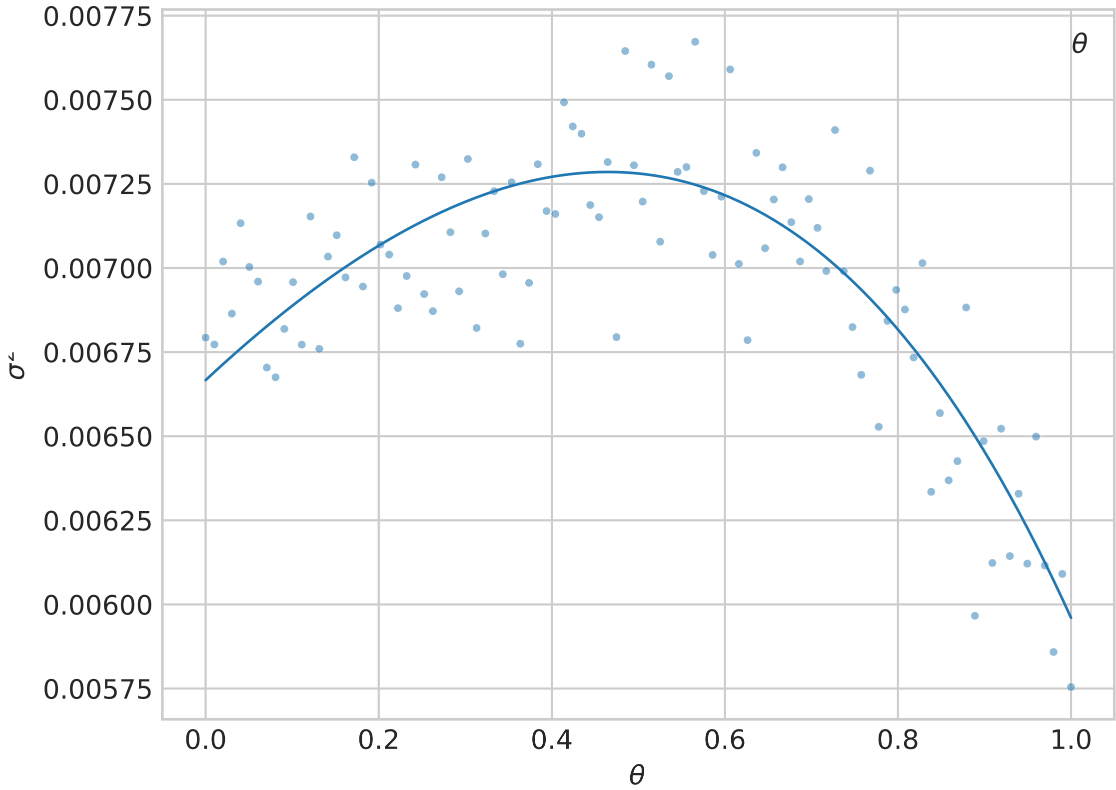


Figure 2: Variance ($\times 512$) of the estimators $\hat{\nu}(\lambda)$ based on 2000 sample of size $T = 512$ from the assymetric mixed model with $\lambda = 0.5$ chosen in such a way that $\theta\{i/100, i = 0.0, \ldots 1.0\}$. The solid line is the asymptotic variance computed numerically using Theorem 2.

2. The asymmetric negative logistic model, namely,

$$A(t) = 1 - [\{\psi_1(1-t)\}^{-\theta} + (\psi_2)^{-\theta}]^{-\frac{1}{\theta}}$$

with parameters $\theta \in (0, \infty)$, $\psi_1, \psi_2 \in (0, 1]$. The special case $\psi_1 = \psi_2 = 1$ returns the symmetric negative logistic of Galambos.

3. The asymmetric mixel model :

$$A(t) = 1 - (\theta + \kappa)t + \theta t^2 + \kappa t^3$$

with parameters θ and κ satisfying $\theta \geq 0$, $\theta + 3\kappa \geq 0$, $\theta + \kappa \leq 1$, $\theta + 2\kappa \leq 1$. The special case $\kappa = 0$ and $\theta \in [0, 1]$ yields the symmetric mixed model.

4. The model of Hüsler and Reiss [Hüsler and Reiss, 1989],

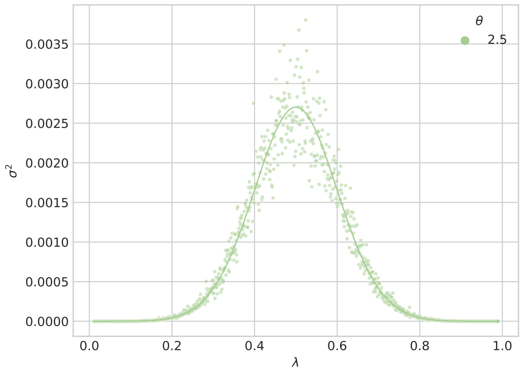
$$A(t) = (1-t)\Phi\left(\theta + \frac{1}{2\theta}log(\frac{1-t}{t})\right) + t\Phi\left(\theta + \frac{1}{2\theta}log(\frac{t}{1-t}\right)$$

where $\theta \in (0, \infty)$ and Φ is the standard normal distribution function.

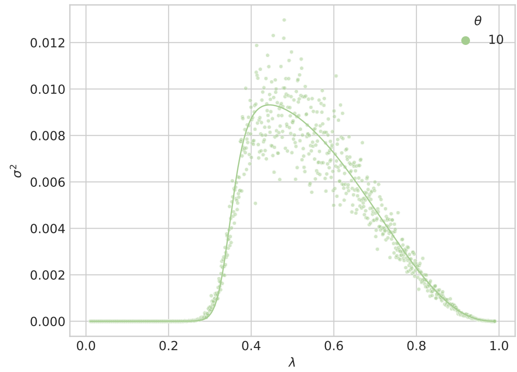
5. The t-EV model [Demarta and McNeil, 2005], in which

$$A(w) = wt_{\chi+1}(z_w) + (1-w)t_{\chi+1}(z_{1-w})$$
$$z_w = (1+\chi)^{1/2}[w/(1-w)^{\frac{1}{\chi}} - \theta](1-\theta^2)^{-1/2}$$

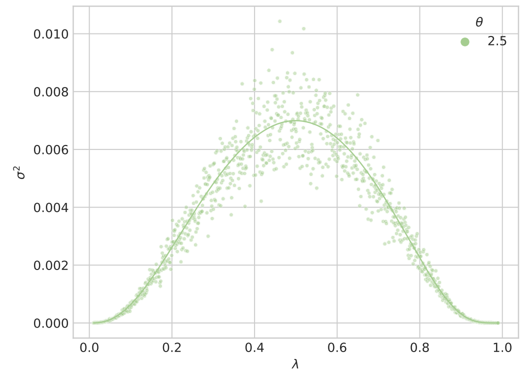
with parameters $\chi > 0$, and $\theta \in (-1, 1)$, where $t_{\chi+1}$ is the distribution function of a Student-t random variable with $\chi + 1$ degrees of freedom.



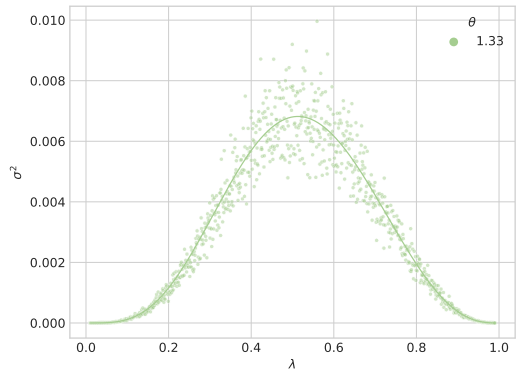
(a) **Asym. Neg. Logistic ($\theta = 10$, $\psi_1 = 1.0$, $\psi_2 = 1.0$)**



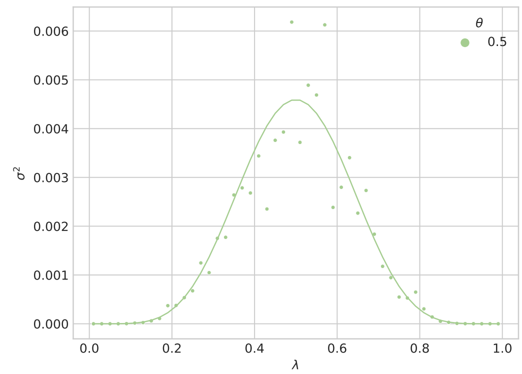
(b) **Asym. Neg. Logistic ($\theta = 10$, $\psi_1 = 0.5$, $\psi_2 = 1.0$)**



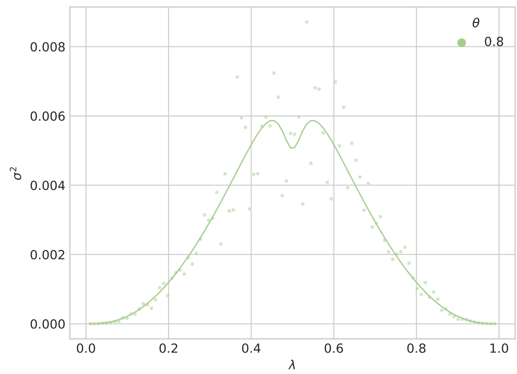
(c) **Asymmetric Logistic ($\theta = 5/2$, $\psi_1 = 0.1$, $\psi_2 = 1.0$)**



(d) **Asymmetric Mixed ($\theta = 1.00$, $\kappa = -0.33$)**



(e) **Hüsler-Reiss ($\theta = 1$)**



(f) **t-EV ($\theta = 0.8$, $\chi = 0.2$)**

Figure 3: Graph, as a function of λ, of the asymptotic variances of the estimators of the λ-FMadogram for six extreme-value copula models. The empirical of the variance ($\times 256$) based on 500 samples of size $T = 256$.

Appendix A

Proofs of Chapter 1

1 Study of the Pickands dependence function

Lemma A.1. *Using properties of the Pickands dependence function, we have that*

$$0 \leq \kappa(\lambda, A) \leq 1, \quad 0 \leq \zeta(\lambda, A) \leq 1, \quad 0 < u, v < 1$$

Furthermore, if A admits a second derivate, $\kappa(\cdot, A)$ (resp $\zeta(\cdot, A)$) is a decreasing function (resp an increasing function).

Proof First, using that the graph of a (differentiable) convex function lies above all of its tangents and using that $A(t) \geq t$ gives, for $0 < t < 1$:

$$A'(t) \leq \frac{A(1) - A(t)}{t - 1} = \frac{1 - A(t)}{t - 1} \leq 1$$

Same reasoning using $A(t) \geq 1 - t$ leads to:

$$A'(t) \geq \frac{A(t) - A(0)}{t - 0} = \frac{A(t) - 1}{t} \geq -1$$

Let's fall back to κ and ζ. If we suppose that A admits a second derivative, the derivative of κ (resp ζ) with respect to λ gives:

$$\kappa'(\lambda, A) = -\lambda A''(\lambda) < 0, \quad \zeta'(\lambda, A) = (1 - \lambda)A''(\lambda) > 0 \quad \forall \lambda \in [0, 1]$$

Using $\kappa(0) = 1$, $\kappa(1) = 1 - A'(1) \geq 0$ gives $0 \leq \kappa(\lambda, A) \leq 1$. As $\zeta(0) = 1 + A'(0) \geq 0$ and $\zeta(1) = 1$, we have $0 \leq \zeta(\lambda, A) \leq 1$. That is the statement.

Now, we can obtain the same result while removing the hypothesis of A admits a second derivative. As A is a convex function, for $x, y \in [0, 1]$, we may have the following inequality:

$$A(x) \geq A(y) + A'(y).(x - y)$$

Take $x = 0$ and $y = \lambda$ gives

$$1 \geq A(\lambda) - \lambda A'(\lambda) = \kappa(\lambda)$$

Now, using that $-\lambda A'(\lambda) \geq -\lambda$, clearly

$$A(\lambda) - \lambda A'(\lambda) \geq A(\lambda) - \lambda \geq 0$$

As $A(\lambda) \geq max(\lambda, 1 - \lambda)$. We thus obtain our statement.

2 A first lemma for equation (1.5)

Lemma A.2. *For all bounded-measurable function $f : [0, 1]^2 \mapsto \mathbb{R}$, if $J(s, t) = |s^\lambda - t^{1-\lambda}|$, then the following integral satisfies:*

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_{[0,1]} f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_{[0,1]} f(x^{1/\lambda}, 0) dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

Proof Let A a element of $\mathcal{B}([0, 1]^2)$. We can pick an element of the form $A = [0, s] \times [0, t]$, where $s, t \in [0, 1]$ and $\lambda \in [0, 1]$. Let us introduce the following indicator function :

$$f_{s,t}(x, y) = \mathbb{1}_{\{(x,y) \in [0,1]^2, 0 \leq x \leq s, 0 \leq y \leq t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x, y) dJ(x, y) = J(s, t) - J(0, 0) = |s^\lambda - t^{1-\lambda}|$$

in other hand, using the equality $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - min(x,y)$, one has to show

$$\begin{aligned}\frac{1}{2}|s^{\lambda} - t^{1-\lambda}| &= \frac{s^{\lambda}}{2} + \frac{y^{1-\lambda}}{2} - min(s^{\lambda}, t^{1-\lambda}) \\ &= \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} f_{s,t}(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\end{aligned}$$

Notice that the class :

$$\mathcal{E} = \{A \in \mathcal{B}([0,1]^2) : \int_{[0,1]^2} \mathbb{1}_A(x,y)dJ(x,y) = \int_{[0,1]} \mathbb{1}_A(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} \mathbb{1}_A(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} \mathbb{1}_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\}$$

contain the class $\mathcal{P}$ of all closed pavements of $[0,1]^2$. It is otherwise a monotone class (or λ-system). Hence as the class $\mathcal{P}$ of closed pavement is a π-system, the class monotone theorem ensure that $\mathcal{E}$ contains the sigma-field generated by $\mathcal{P}$, that is $\mathcal{B}([0,1]^2)$.

This result holds for simple function $f(x,y) = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i}$ where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathcal{B}([0,1]^2)$ for all $i \in \{1,\dots,n\}$. We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function $f : [0,1]^2 \mapsto \mathbb{R}$ considering $f = f_+ - f_-$ with $f_+ = max(f,0)$ and $f_- = min(-f,0)$. We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral.

3 Proof of prosition 2

In order to prove our proposition, we introduce two lemmas.

Lemma A.3. *Let $(B_C(u,v))_{u,v\in[0,1]^2}$ a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$. Let $a, b \in [0,1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}[\int_{[0,1]} B_C(u,a)du \int_{[0,1]} B_C(b,u)du] = 0$$

Proof Without loss of generality, suppose that $a = 0$ and $b \in [0,1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned}\mathbb{E}[\int_{[0,1]} B_C(u,0)du \int_{[0,1]} B_C(b,u)du] &= \mathbb{E}[\int_{[0,1]}\int_{[0,1]} B_C(u,0)B_C(b,v)dudv] \\ &= \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u,0)B_C(b,v)]dudv\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u,0)B_C(b,v)] = C(u \wedge v, 0) - C(u,0)C(b,v)$$

We recall that, by definition, a copula satisfy $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$. Then, the equation below is equal to 0. Our conclusion directly follows.

Lemma A.4. *Let $N_C(u,v)$ the process defined in equation (1.4) and $a, b \in [0,1]$ fixed. If $a = 0$ or $b = 0$, then :*

$$\mathbb{E}[\int_{[0,1]} N_C(a,u)du \int_{[0,1]} N_C(u,b)du] = 0$$

Proof Without loss of generality, let a = 0. Using the definition of $N_C(u,v)$, we have

$$N_C(0,u) = B_C(0,u) - \frac{\partial C(0,u)}{\partial u}B_C(0,1) - \frac{\partial C(0,u)}{\partial v}B_C(1,u)$$

Which is well defined if we consider, for a fixed $v \in [0,1]$

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{\partial C(u,v)}{\partial u}, & \text{if } u > 0 \\ \lim\limits_{u\to 0^+} \frac{\partial C(u,v)}{\partial v}, & \text{if } u = 0,\ v \in (0,1] \end{cases} \tag{A.1}$$

The continuous extension of $\frac{\partial C(u,v)}{\partial u}(\cdot, v)$ on $[0,1]$ while we have used 1 (iii) for the existence of the right limit. We do the same for $\frac{\partial C(u,v)}{\partial v}(u, \cdot)$. We have :

$$\begin{aligned}\mathbb{E}\left[\int_{[0,1]} N_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] &= \mathbb{E}\left[\int_{[0,1]} B_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] \\ &- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial u} B_C(0,1)du \int_{[0,1]} N_C(u,b)du\right] \\ &- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right]\end{aligned}$$

Using preceding lemma, we got that the two first terms are equal to zero. Only the last term should be discuss. Remember that $\frac{\partial C(0,u)}{\partial v} = 0$ for all $u \in]0,1]$, as we integrate with respect to the lebesgue measure, the set $\{0\}$ is of measure 0 because it is a countable set, then :

$$\mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = \mathbb{E}\left[\int_{]0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = 0$$

These two results gives us the proposition.

4 Proof of theorem 2

We are able to compute the variance for each process and they are given by the following expressions :

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) = f(\lambda, A)\left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)}\right)$$

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = f(\lambda, A)\left(\frac{\kappa^2(\lambda, A)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)}\right)$$

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial v}du\right) = f(\lambda, A)\left(\frac{\zeta^2(\lambda, A)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right)$$

We now compute the covariance :

$$cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(v^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

$$= \int_{[0,1]}\int_{[0,v]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv + \int_{[0,1]}\int_{[v,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv$$

for the first one, we have :

$$\int_{[0,1]}\int_{[0,v]} (C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)\left(\frac{1-\lambda}{2A(\lambda) + (2\lambda - 1)(1-\lambda)}\right)$$

For the second part, using Fubini, we have :

$$\int_{[0,1]}\int_{[0,u]} (C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu$$

for the right hand side of the "minus" sign, we may compute :

$$\int_{[0,1]}\int_{[0,u]} C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)$$

The last one still difficult to handle,

$$\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \tag{A.2}$$

Following the proof of proposition 3.3 from [Genest and Segers, 2009], the substitution $v^{\frac{1}{\lambda}} = x$ and $u^{\frac{1}{1-\lambda}} = y$ yield

$$
\begin{aligned}
&\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u} dvdu \\
&= \lambda(1-\lambda)\int_{[0,1]}\int_{[0,y^{\frac{1-\lambda}{\lambda}}]} C(x,y)\frac{\partial C(x, x^{\frac{\lambda}{1-\lambda}})}{\partial u} x^{\lambda-1}y^{-\lambda}dxdy \\
&= \lambda(1-\lambda)\kappa(\lambda,A)\int_{[0,1]}\int_{[0,y^{\frac{1-\lambda}{\lambda}}]} C(x,y)x^{\frac{A(\lambda)}{1-\lambda}-(1-\lambda)-1}y^{-\lambda}dxdy
\end{aligned}
$$

Next, use the substitution $x = w^{1-s}$ and $y = w^s$. Note that $w = xy \in [0,1]$, $s = log(y)/log(xy) \in [0,1]$, $C(x,y) = w^{A(s)}$ and the Jacobian of the transformation is $-log(w)$. As the constraint $x < y^{-1+1/\lambda}$ reduces to $s < \lambda$, the integral becomes:

$$
\begin{aligned}
&-\lambda(1-\lambda)\kappa(\lambda,A)\int_{[0,\lambda]}\int_{[0,1]} w^{A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)-s\lambda}log(w)dwds \\
&= \lambda(1-\lambda)\kappa(\lambda,A)\int_{[0,\lambda]} [A(s)+(1-s)(A_2(\lambda)-1-(1-\lambda))-s\lambda+1]^{-2}\, ds
\end{aligned}
$$

Let's continue with computing the following integral :

$$
\begin{aligned}
&\mathbb{E}\left[\int_{[0,1]}\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},1)B_C(1,v^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}dudv\right] \\
&= \int_{[0,1]}\int_{[0,1]} \left(C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) - u^{\frac{1}{\lambda}}v^{\frac{1}{1-\lambda}}\right)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}dudv
\end{aligned}
$$

The second term can be easily handled and its value is given by :

$$
\int_{[0,1]}\int_{[0,1]} u^{\frac{1}{\lambda}}v^{\frac{1}{1-\lambda}}\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}dudv = f(\lambda,A)\kappa(\lambda,A)\zeta(\lambda,A)
$$

For the first, use the substitutions $u^{\frac{1}{\lambda}} = x$ and $v^{\frac{1}{1-\lambda}} = y$. This yields :

$$
\lambda(1-\lambda)\int_{[0,1]}\int_{[0,1]} C(x,y)\frac{\partial C(x, x^{\frac{\lambda}{1-\lambda}})}{\partial u}\frac{\partial C(y^{\frac{1-\lambda}{\lambda}}, y)}{\partial v}x^{\lambda-1}y^{-\lambda}dxdy
$$

Then, make the substitutions $x = w^{1-s}$, $y = w^s$ that were used for the preceding integral gives :

$$
\begin{aligned}
&-\lambda(1-\lambda)\kappa(\lambda,A)\zeta(\lambda,A)\int_{[0,1]}\int_{[0,1]} w^{A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)+s(A_1(\lambda)-\lambda-1)}log(w)dwds \\
&= \lambda(1-\lambda)\kappa(\lambda,A)\zeta(\lambda,A)\int_{[0,1]} [A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)+s(A_1(\lambda)-\lambda-1)+1]^{-2}\, ds
\end{aligned}
$$

The last covariance requires the same tools as used before, it is left to the reader. It then suffices to use the bilinearity of the covariance and to assemble the various terms to conclude.

5 A counter example against of variance's monotony with respect to positive dependence

First, notive that, under dependency condition, the variance of the Madogram evaluated in $\lambda = 0.5$ is equal to 1/150.

Lemma A.5. *Let us consider $C(u,v) = 1 - \theta t + \theta t^2$ where $\theta \in [0,1]$. If we take $\lambda = 0.5$, there exist $\theta \in (0,1)$ such that*

$$
Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) > \frac{1}{150} \tag{A.3}
$$

Proof For this dependence function, we have immediately :

$$
\kappa(\lambda,A) = 1-\theta\lambda^2, \quad \zeta(\lambda,A) = 1-\theta(1-\lambda)^2
$$

For $\lambda = 0.5$, we notice that $\kappa(0.5,A) = \zeta(0.5,A)$. By a simple change of variable, we notice that :

$$
\int_0^{0.5} [A(s)+(1-s)(2A(0.5)-0.5-1)-0.5s+1]^{-2}ds = \int_{0.5}^1 [A(s)+s(2A(0.5)-0.5-1)-0.5(1-s)+1]^{-2}ds
$$

By simple substitution, we have for the chosen copula that

$$\int_{0.5}^{1} [A(s) + s(2A(0.5) - 0.5 - 1) - 0.5(1 - s) + 1]^{-2} ds = \int_{0.5}^{1} [\frac{3}{2} - s(\theta + 1 - 2A(0.5)) + s^2\theta] ds$$

Let us take $\theta = 2A(0.5) - 1$, which implies by direct computation that $\theta = 2/3 > 0$. We omit the details, but we can compute :

$$\int_{0.5}^{1} [\frac{3}{2} + s^2\frac{2}{3}]^{-2} ds = 2\left[\frac{7}{20} + \frac{1}{6}(atan(2/3) - atan(1/3))\right] \approx 0.142596$$

For the last integral, we have, by substitution for $\lambda = 0.5$ and $\theta = 2/3$:

$$\int_{0}^{1} [A(s) + (1 - s)(2A(0.5) - 0.5 - 1) + s(2A(0.5) - 0.5 - 1) + 1]^{-2} ds = \int_{0}^{1} [\frac{13}{6} - \frac{2}{3}s + \frac{2}{3}s^2]^{-2} ds$$

Omitting again the technical details, we are able to compute :

$$\int_{0}^{1} [\frac{13}{6} - \frac{2}{3}s + \frac{2}{3}s^2]^{-2} ds = \frac{\sqrt{3}}{8}\left(atan(\frac{1}{2\sqrt{3}}) - atan(-\frac{1}{2\sqrt{3}})\right) \approx 0.23707$$

Summing all the components of the variance gives $Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du\right) \approx 0.00713 > 1/150$, which gives our counterexample.

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