Chapter 1

On the variance of the Madogram with extreme value copula

1 Introduction

1.1 Context

Management on environmental ressources often requires the analysis of multivariate of a extreme values. In the classical theory, one is often interested in the behaviour of the mean or average of a random variable X defined on a probabilised space $(\Omega, \mathcal{A}, \mathbb{P})$. This average will then be described through the expected value $\mathbb{E}[X]$ of the distribution. The central limit theorem yields, under some assumptions on the moments, the asymptotic behavior of the sample mean $\bar{X}$. This result can be used to provide a confidence interval for $\mathbb{E}[X]$ for a level $\alpha \in [0,1]$. But in case of extreme events, it can be just as important to estimate tail probabilities. Furthermore, what if the second moment $\mathbb{E}[X^2]$ or even the mean is not finite? Then the central limit theorem does not apply and the classical theory, carryied by the normal distribution, is no longer relevant [Beirlant et al., 2004].
Some extreme events, such as heavy precipitation or wind speed has spatial characteristics and geostatisticians are striving to better understand the physical processes in hand. In geostatistics, we often consider $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, S a set of locations and $(E, \mathcal{E})$ a measurable state space. We define on this probability space a stochastic process $X = \{X_s, s \in S\}$ with values on $(E, \mathcal{E})$. It is classical to define the following second-order statistic as the variogram (see [Gaetan and Guyon, 2008] chapter 1.3 for definition and basic properties) :

$$2\gamma(h) = \mathbb{E}[|X(s+h) - X(s)|^2]$$

where $\{X(s), s \in S\}$ represents a spatial and stationnary process with a well defined covariance function. The function $\gamma(\cdot)$ is called the semi-variogram of X. With respect to extremes, this definition is not well adapted because a second order statistic is difficult to interpret inside the framework of extreme value theory or may not even be defined. To ensure that we always work with finite moment quantities, the following type of first-order variogram is introduced :

$$\nu(h) = \frac{1}{2}\mathbb{E}[|F(X(s+h)) - F(X(s))|] \tag{1.1}$$

Where $F(u) = \mathbb{P}(X(s) \leq u)$ is named as the FMadogram. Let us define the pairwise extremal dependence function (section 4.3 of [Coles et al., 1999]) such as :

$$V_h(x,y) = \int_0^1 max\left(\frac{w}{x}, \frac{1-w}{y}\right) 2dH(w)$$

where x, y are two reals. It has been shown ([Cooley et al., 2006]) that $\nu(h)$ fully characterizes the extremal coefficient $V_h(1,1)$ since, using $2^{-1}|a-b| = max(a,b) - 2^{-1}(a+b)$, we have the following relationship,

$$V_h(1,1) = \frac{1+2\nu(h)}{1-2\nu(h)} \tag{1.2}$$

Then, the estimation and the study of the madogram gives us an estimator of the extremal coefficient's estimator. We point out that this identity whose equation (1.2) is raised also permits a multivariate extension in higher dimension of the FMadogram. This approach was tackled by [Marcon et al., 2017] using a multivariate madogram in order to estimate the Pickands dependence function. This method extend [Capéraà et al., 1997] which proposes a non parametric estimator to estimate the Pickands dependence function for bivariate extreme value copulas (these two objects will be defined through the paper). Let's go back to the estimation of our extremal coefficient, his main drawback is that it only focuses on the values $V_h(x,x)$ but does not provide any information about $V_h(x,y)$, for $x \neq y$. To overpass this drawback [Naveau et al., 2009] introduce the λ-FMadogram defined as,

$$\nu(h,\lambda) = \frac{1}{2}\mathbb{E}[\left|F^{\lambda}(X(s+h)) - F^{1-\lambda}(X(s))\right|] \tag{1.3}$$

for every $\lambda \in (0,1)$. It is shown in the same paper that the λ-madogram fully characterizes the dependence function $V_h(x,y)$ with the following relationship,

$$V_h(\lambda, 1-\lambda) = \frac{c(\lambda) + \nu(h,\lambda)}{1 - c(\lambda) - \nu(h,\lambda)}$$

Furthermore, this statistic kept our attention because it can be seen as a dissimilarity measure among bivariate maxima to be used in a clustering algorithm [Bernard et al., 2013]. This first chapter aims to study the variance of the λ-madogram with the fewest possible assumptions. In our knowledge, only [Guillou et al., 2014] has computed the variance of the sole madogram in equation (1.1) by assuming independency and found 1/90.

2 Notations and preliminaries

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and (X,Y) be a bivariate random vector with joint distribution function H and marginal distribution function F and G. A function $C : [0,1]^2 \rightarrow [0,1]$ is called a bivariate copula if it is the restriction to $[0,1]^2$ of a bivariate distribution function whose marginals are given by the uniform distribution on the interval $[0,1]$. Since the work of Sklar [Sklar, 1959], it is well known that every distribution function H can be decomposed as $H(x,y) = C(F(x), G(y))$, for all $(x,y) \in \mathbb{R}^2$.
Let $(X_t)_{t=1,\dots,T}$ be an i.i.d. sample of a bivariate random vectors whose underlying copula is denoted by C and whose margins by F, G. For $x, y \in \mathbb{R}$, let $x \wedge y = min(x,y)$ and $x \vee y = max(x,y)$. We will write the generalized inverse function of F (respectively G) as $F^{\leftarrow}(u) = inf\{v \in \mathbb{R} | F(v) \geq u\}$ (respectively $G^{\leftarrow}(u) = inf\{v \in \mathbb{R} | G(v) \geq u\}$) where

$0 < u, v < 1$. Given $\mathcal{X} \subset \mathbb{R}^2$, let $l^\infty(\mathcal{X})$ denote the spaces of bounded real-valued function on $\mathcal{X}$. We define by $D(\mathcal{X})$ the Skorokhod space of functions x with values on $\mathcal{X}$ which are *càdlàg*. For $f : \mathcal{X} \rightarrow \mathbb{R}$, let $||f||_\infty = sup_{x \in \mathcal{X}}|f(x)|$. The arrows "$\xrightarrow{a.s.}$", "$\xrightarrow{d}$" and "$\rightsquigarrow$" denote almost sure convergence, convergence in distribution of random vectors and weak convergence of functions in $l^\infty(\mathcal{X})$.
This chapter is organized as follows, in section 3, we introduce our estimator and we discuss its properties. Explicit formula for the asymptotic variance is also given. In section 4, we investigate the finite-sample performance the estimator by means of Monte Carlo simulations. All proofs are deferred to the final section.

3 Weak convergence of the λ-FMadogram

We consider the bivariate extreme value copula which can be written in the following form [Gudendorf and Segers, 2009].

$$C(u, v) = (uv)^{A(log(v)/log(uv))} \tag{1.4}$$

for all $u, v \in [0, 1]$ and where $A(\cdot)$ is the Pickands dependence function, *i.e.*, $A : [0, 1] \longrightarrow [1/2, 1]$ is convex and satisfies $t \vee (1-t) \leq A(t) \leq 1, \forall t \in [0, 1]$. Following [Fermanian et al., 2004], to guarantee the weak convergence of our empirical copula process, we introduce the following assumptions.

Assumption A. *(i) The bivariate distribution function H has continuous margins F, G and copula C.*

(ii) The derivative of the Pickands dependence function $A'(t)$ exists and is continuous on $(0, 1)$.

(iii) The limit $\lim_{t \rightarrow 0^+} A'(t)$ exist.

The Assumption A (i) guarantee the uniqueness of the representation $H(x, y) = C(F(x), G(y))$ on the range of (F, G). Under Assumption A (ii), the first-order partial derivatives of C with respect to u and v exists and are continous on the set $\{(u, v) \in [0, 1]^2 : 0 < u < 1\}$. Indeed, we have

$$\frac{\partial C(u, v)}{\partial u} = \begin{cases} \frac{C(u,v)}{u} \left(A(\frac{log(v)}{log(uv)}) - A'(\frac{log(v)}{log(uv)}) \frac{log(v)}{log(uv)} \right), & \text{if}\, u, v > 0 \\ 0, & \text{if}\, v = 0, \quad 0 < u < 1 \end{cases}$$

$$\frac{\partial C(u, v)}{\partial v} = \begin{cases} \frac{C(u,v)}{v} \left(A(\frac{log(v)}{log(uv)}) + A'(\frac{log(v)}{log(uv)}) \frac{log(u)}{log(uv)} \right), & \text{if}\, u, v > 0 \\ 0, & \text{if}\, u = 0, \quad 0 < v < 1 \end{cases}$$

The properties of A imply $0 \leq A(t) - tA'(t) \leq 1$ and $0 \leq A(t) + (1-t)A'(t) \leq 1$ where $t = log(v)/log(uv)$ (see section 5.1). Therefore if $v \searrow 0$, then $\partial C(u, v)/\partial u \rightarrow 0$ as required. The Madogram is an estimator commonly used with extrema due to his relation with the pairwise extremal dependence coefficient (see [Cooley et al., 2006], [Guillou et al., 2014]). In this study, we aim to analyze the asymptotic variance structure of the λ-FMadogram defined in [Naveau et al., 2009] such as :

Definition 1. *Let (X, Y) be a bivariate random vectors with margins F and G. A λ-FMadogram is the quantity defined by :*

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^{\lambda}(X) - G^{1-\lambda}(Y)|] \tag{1.5}$$

Having a sample $(X_1, Y_1), \ldots, (X_T, Y_T)$ of T bivariate vector with unknown margins F and G, we construct the empirical distribution function :

$$\hat{H}_T(x, y) = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{\{X_t \leq x, Y_t \leq y\}},$$

and let $\hat{F}_T(x)$ and $\hat{G}_T(y)$ be its associated marginal distributions, that is,

$$\hat{F}_T(x) = \hat{H}_T(x, +\infty) \quad \text{and} \quad \hat{G}_T(y) = \hat{H}_T(+\infty, y) \quad -\infty < x, y < +\infty$$

Based on these identical and independent copies $(X_1, Y_1), \ldots, (X_T, Y_T)$, it is natural to define the following estimator of the λ-Madogram:

$$\hat{\nu}_T(\lambda) = \frac{1}{2T}\sum_{t=1}^{T}\left|\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)\right| \tag{1.6}$$

Concerning the boundaries of the estimators, the λ-FMadogram defined by (1.5) satisfies $\nu(0) = \nu(1) = 0.25$. Hence we can force our estimator to satisfy $\hat{\nu}_T(0) = \hat{\nu}_T(1) = 0.25$. This leads to the following definition :

$$\begin{aligned}\hat{\nu}_T^*(\lambda) = &\frac{1}{2T}\sum_{t=1}^{T}\left|\hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t)\right| - \frac{\lambda}{2T}\sum_{t=1}^{T}\{1 - \hat{F}_T^{\lambda}(X_t)\} \\ &- \frac{1-\lambda}{2T}\sum_{t=1}^{T}\{1 - \hat{G}_T^{1-\lambda}(Y_t)\} + \frac{1}{2}\frac{1-\lambda+\lambda^2}{(2-\lambda)(1+\lambda)}\end{aligned} \tag{1.7}$$

In the following proposition, we establish some properties of the λ-FMadogram.

Proposition 1. *Let $(X_1, Y_1), \ldots, (X_T, Y_T)$ a sample of $\mathbb{R}^2$-valued independent random vectors. We have, for each $\lambda \in [0, 1]$,*

(i) $0 \leq \nu(\lambda) \leq 1$,

(ii) $\hat{\nu}_T(\lambda) \xrightarrow{a.s.} \nu(\lambda)$,

(iii) with probability at least $1 - \delta$:

$$\left|\underset{\lambda \in (0,1)}{sup}\ \hat{\nu}_T(\lambda) - \mathbb{E}[\underset{\lambda \in (0,1)}{sup}\ \hat{\nu}_T(\lambda)]\right| \leq \frac{2}{\sqrt{2T}}\sqrt{log(2/\delta)}.$$

(iv) $\nu(0) = \nu(1) = 0.25$*, and if* $\lambda \in (0, 1)$*,*

$$\frac{1}{2}\left(\frac{\lambda}{1+\lambda} + \frac{1-\lambda}{1+1-\lambda}\right) - \frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)} \tag{1.8}$$

We define the empirical copula function $\hat{C}_T(u,v)$ by,

$$\hat{C}_T(u,v) = \hat{H}_T(\hat{F}_T^{\leftarrow}(u), \hat{G}_T^{\leftarrow}(v)), \quad 0 \leq u, v \leq 1,$$

and the (ordinary) empirical copula process,

$$\mathbb{C}_T(u,v) = \sqrt{n}(\hat{C}_T - C)(u,v), \quad 0 \leq u, v \leq 1,$$

The weak convergence of $\mathbb{C}_T$ has already been proved by [Fermanian et al., 2004] using previous results on the Hadamard differentiability of the map $\phi : D([0,1]^2) \rightarrow l^{\infty}([0,1]^2)$ which transforms the cumulative distribution function H into its copula function C (see lemma 3.9.28 from [van der Vaart and Wellner, 1996]). Some auxiliary results concerning the convergence of $\mathbb{C}_T$ to a gaussian process and the convergence of the λ-FMadogram to a centered Gaussian law are recalled in Appendix. The limiting Gaussian process of $\mathbb{C}_T$ can be written as

$$N_C(u,v) = B_C(u,v) - \frac{\partial C}{\partial u}(u,v) B_C(u,1) - \frac{\partial C}{\partial u}(u,v) B_C(1,v) \tag{1.9}$$

where B_C is a brownian bridge in $[0,1]^2$ with covariance function

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v')$$

Furthermore, the asymptotic variance of the λ-FMadogram can be written as:

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2} \int_{[0,1]} f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_{[0,1]} f(x^{1/\lambda}, 0) dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \tag{1.10}$$

Some details explaining equation (1.10) are given in lemma 3 at the end. Using the properties of a Copula fonction, we are able to write the term $Var(\int_{[0,1]^2} N_C(u,v))dJ(u,v)$ in a more practical way. This result is resumed with the following proposition.

Proposition 2. *Let $N_C(u,v)$ the process defined in Equation (1.9) and $J(x,y) = |x^{\lambda} - y^{1-\lambda}|$, then :*

$$Var\left(\int_{[0,1]^2} N_C(u,v) dJ(u,v)\right) = Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du\right) \tag{1.11}$$

Using extreme value copula, we want to compute the following integral:

$$\begin{aligned} Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du\right) = Var(&\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du - \int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1) \frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} du \\ &- \int_{[0,1]} B_C(1, v^{\frac{1}{1-\lambda}}) \frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}) \end{aligned}$$

Notice that, on sections, the extreme value copula is of the form, $i.e$:

$$C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) = u^{\frac{A(\lambda)}{\lambda(1-\lambda)}}$$

Furthermore, we have the same pattern for partial derivatives:

$$\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u} = \frac{C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{u^{\frac{1}{\lambda}}} (A(\lambda) - A'(\lambda)\lambda)$$

$$\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v} = \frac{C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{v^{\frac{1}{1-\lambda}}} (A(\lambda) + A'(\lambda)(1-\lambda))$$

Let $\mathcal{A}$ be the space of Pickands dependence functions. We will denote by $\kappa(\lambda, A)$ and $\zeta(\lambda, A)$ two functional such as :

$$\kappa \colon [0,1] \times \mathcal{A} \to [0,1], \quad (\lambda, A) \mapsto A(\lambda) - A'(\lambda)\lambda$$

$$\zeta \colon [0,1] \times \mathcal{A} \to [0,1], \quad (\lambda, A) \mapsto A(\lambda) + A'(\lambda)(1-\lambda)$$

Furthermore, the integral $\int_{[0,1]^2} C(u,v)duv$ does not admit, in general, a closed form. But we are able to express it with respect to a simple integral of the Pickands dependence function. We note, for notational convenience the following functionals

$$f \colon [0,1] \times \mathcal{A} \to [0,1], \quad (\lambda, A) \mapsto \left(\frac{\lambda(1-\lambda)}{A(\lambda) + \lambda(1-\lambda)} \right)^2$$

For a fixed $\lambda \in (0,1)$, using properties of the extreme value copula permit us to give an explicit formulas of the asymptoc variance of the scaled λ-Madogram $\sqrt{T}\,(\hat{\nu}_T(\lambda) - \nu(\lambda))$.

Theorem 1. *For $\lambda \in (0,1)$, let $A_1(\lambda) = A(\lambda)/\lambda$, $A_2(\lambda) = A(\lambda)/(1-\lambda)$. Then $Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})d\right.$ is given by :*

$$\begin{aligned}
& f(\lambda, A)\left(\frac{A(\lambda)}{A(\lambda) + 2\lambda(1-\lambda)} + \frac{\kappa(\lambda, A)^2(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-1\lambda)} + \frac{\zeta(\lambda, A)^2\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} \right) \\
& - 2\kappa(\lambda, A)f(\lambda, A)\left(\frac{(1-\lambda)^2 - A(\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)} \right) - 2\kappa(\lambda, A)\lambda(1-\lambda)\int_{[0,\lambda]} \left[A(s) + (1-s)(A_2(\lambda) - (1-\lambda) - 1) - s\lambda + 1\right]^{-2} ds \\
& - 2\zeta(\lambda, A)f(\lambda, A)\left(\frac{\lambda^2 - A(\lambda)}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)} \right) - 2\zeta(\lambda, A)\lambda(1-\lambda)\int_{[\lambda,1]} \left[A(s) + s(A_1(\lambda) - 1 - \lambda) - (1-s)(1-\lambda) + 1\right]^{-2} ds \\
& - 2f(\lambda, A)\kappa(\lambda, A)\zeta(\lambda, A) + 2\kappa(\lambda, A)\zeta(\lambda, A)\lambda(1-\lambda)\int_{[0,1]} \left[A(s) + (1-s)(A_2(\lambda) - (1-\lambda) - 1) + s(A_1(\lambda) - \lambda - 1) + 1\right]^{-2} ds
\end{aligned}$$

From theorem 1, we are able to infer the closed form of the λ-Madogram's variance in the case of an independent Copula, *i.e.* when $C(u,v) = uv$. Indeed, we just have to take $A(t) = 1$ for every $t \in [0,1]$. This result is summarised on the following statement:

Proposition 3. *Under the framework of theorem 1 and if we take $C(u,v) = uv$, the independent copula, then the asymptotic variance of $\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda))$ has the following form, for $\lambda \in (0,1)$*

$$Var(\int_{[0,1]^2} N_C(u,v)dJ(u,v)) = \left(\frac{\lambda(1-\lambda)}{1 + \lambda(1-\lambda)} \right)^2 \left(\frac{1}{1 + 2\lambda(1-\lambda)} - \frac{1-\lambda}{1 + \lambda + 2\lambda(1-\lambda)} - \frac{\lambda}{2 - \lambda + 2\lambda(1-\lambda)} \right)$$

Now, one may consider the weak convergence of the stochastic process $(\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda)))_{\lambda \in [0,1]}$ in $l^\infty([0,1])$. To establish such a result, we use empirical processes arguments as formulated in [van der Vaart and Wellner, 1996]. This allows us to show the following theorem.

Theorem 2. *Let $\lambda \in [0,1]$. Under the framework of Theorem 1, then in $l^\infty([0,1])$, as $T \to \infty$,*

$$\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda)) \rightsquigarrow \left(-\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du \right)_{\lambda \in [0,1]}$$

4 Simulation

4.1 Comparison with several models for the bivariate case

We present several models that would be used in the simulation section in order to assess our findings remains in finite-sample settings.

1. The asymmetric logistic model defined by the following dependence function :

$$A(t) = (1-\psi_1)t + (1-\psi_2)(1-t) + [(\psi_1 t)^\theta + (\psi_2(1-t))^\theta]^{\frac{1}{\theta}}$$

   with parameters $\theta \in [1, \infty[$, ψ_1, $\psi_2 \in [0,1]$. The special case $\psi_1 = \psi_2 = 1$ gives us the symmetric model of Gumbel. In the symmetric model, as we retrieve the independent case when $\theta = 1$, the dependence between the two variables is stronger as θ goes to infinity.

2. The asymmetric negative logistic model, namely,

$$A(t) = 1 - [\{\psi_1(1-t)\}^{-\theta} + (\psi_2 t)^{-\theta}]^{-\frac{1}{\theta}}$$

   with parameters $\theta \in (0, \infty)$, $\psi_1, \psi_2 \in (0,1]$. The special case $\psi_1 = \psi_2 = 1$ returns the symmetric negative logistic of Galambos.

3. The asymmetric mixel model :

$$A(t) = 1 - (\theta + \kappa)t + \theta t^2 + \kappa t^3$$

   with parameters θ and κ satisfying $\theta \geq 0$, $\theta + 3\kappa \geq 0$, $\theta + \kappa \leq 1$, $\theta + 2\kappa \leq 1$. The special case $\kappa = 0$ and $\theta \in [0,1]$ yields the symmetric mixed model.

4. The model of Hüsler and Reiss [Hüsler and Reiss, 1989],

$$A(t) = (1-t)\Phi\left(\theta + \frac{1}{2\theta}log(\frac{1-t}{t})\right) + t\Phi\left(\theta + \frac{1}{2\theta}log(\frac{t}{1-t}\right)$$

   where $\theta \in (0, \infty)$ and Φ is the standard normal distribution function. As θ goes to 0^+, the dependence between the two variables is stronger. When θ goes to infinity, we are near indepence.

5. The t-EV model [Demarta and McNeil, 2005], in which

$$\begin{aligned} A(w) &= w t_{\chi+1}(z_w) + (1-w)t_{\chi+1}(z_{1-w}) \\ z_w &= (1+\chi)^{1/2}[w/(1-w)^{\frac{1}{\chi}} - \theta](1-\theta^2)^{-1/2} \end{aligned}$$

   with parameters $\chi > 0$, and $\theta \in (-1,1)$, where $t_{\chi+1}$ is the distribution function of a Student-t random variable with $\chi + 1$ degrees of freedom.

A vast Monte Carlo study was used to confirm that the conclusion of Section 3 remain in finite-sample settings. Specifically, for each $\lambda \in (0,1)$, 500 random samples of size $n = 256$ were generated from the Gumbel copula with $\theta \in \{1.0, 1.5, 2.0, 2.5, 3.0\}$. For each sample, the λ-FMadogram estimators were computed where the margins are unknown. For each estimator, the empirical version of the variance,

$$Var\left(\sqrt{T}\left(\hat{\nu}(\lambda) - \nu(\lambda)\right)\right)$$

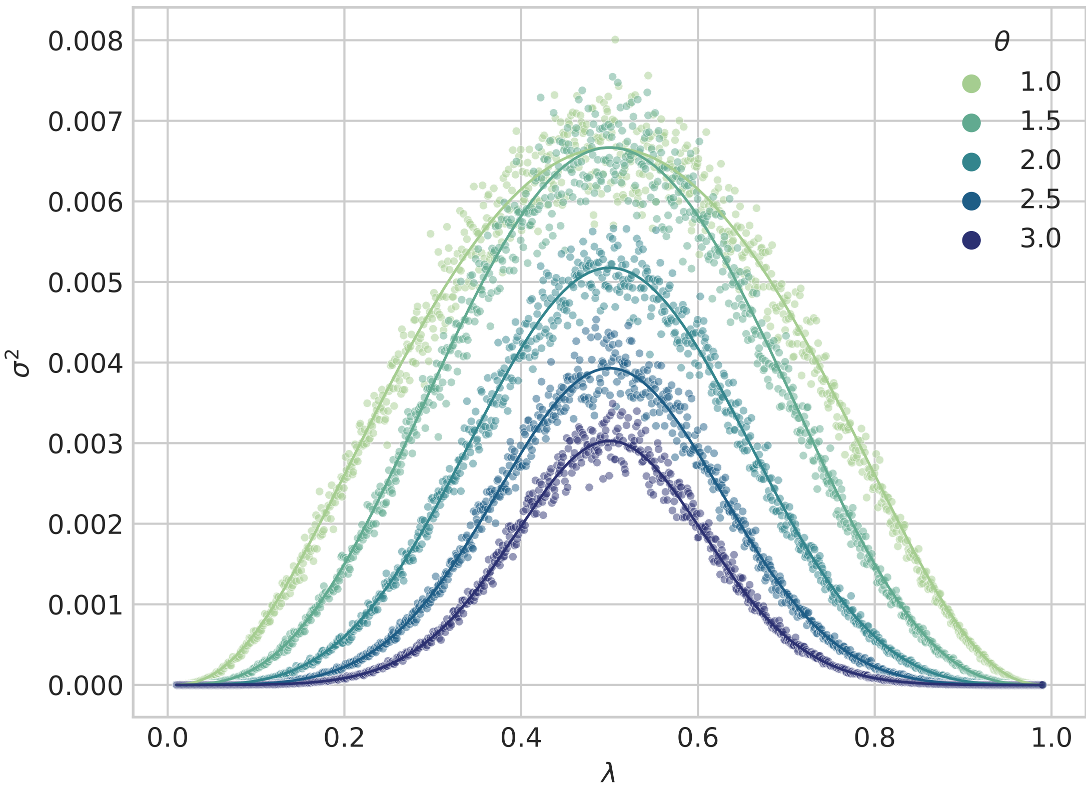


Figure 1: Variance ($\times 256$) of the estimators $\hat{\nu}(\lambda)$ based on 500 samples of size $T = 256$ from the Gumbel copula with $\theta = \{1.0, 1.5, 2.0, 2.5, 3.0\}$ chosen in such a way that $\lambda \in \{i/1000 : i = 0.01, \dots, 0.99\}$.

was computed by taking the variance over the 500 samples. For each estimator, we represent its theoretical asymptotic variance using the integral exhibits in Theorem 1. Similar results were obtained for many other extreme-value dependence models (see figure 2), we can note the following :

1. When A is symmetric, one would expect the asymptotic variance of the estimator to reach its maximum at $\lambda = 1/2$. Such is not always the case, however, as illustrated by the t-EV model.

2. In the asymmetric negative logistic model, the asymptotic of the λ-FMadogram is close to zero for all $\lambda \in [0, 0.3]$. This is due to the fact that $A(\lambda) \approx 1 - t$ for this model.

Remarks 1. and 2. are also observed in [Genest and Segers, 2009]. We propose in Figure 3 the theoritical asymptotic variance depending of θ and λ for six model of extreme-value copula. The parameters are chosen accordingly to [Genest and Segers, 2009].

4.2 Non-monotonicity of the variance with respect to the dependence parameter

Looking at our simulations, one may make the third remark :

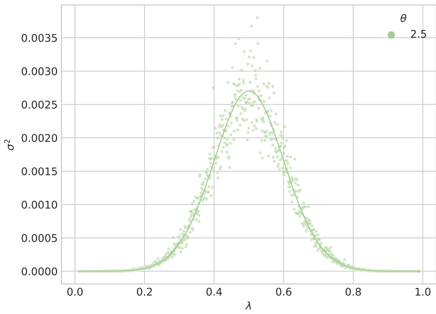
3. Interestingly, as our variables (X, Y) are more positively dependent (in Figure 1, as θ increase), then asymptotic variance is, for every $\lambda \in (0, 1)$ lower or equal than the asymptotic variance in the independent case. As shown in the proof section 5.7 where we exhibits a counter example, it is not always the case.

The Figure 4 shows the same model with different values of θ and different scale for λ. The moving average is computed out of 10 empirical variances for each θ. Each theoretical asymptotic variance depending of λ is fitted by it's empirical counterpart represented by the moving average. As the dependency parameter θ increases, we can find some λ for which the asymptotic variance is greater than the asymptotic variance in case of independance. That figure support our counter-example that draws the same conclusion. Also, Figure 5 depict the asymptotic variance for a fixed $\lambda = 0.5$ for the symetric mixed model with a varying $\theta \in [0, 1]$. When $\theta = 0$, we are turning back to the independent copula $C(u, v) = uv$ and it's asymptotic variance given by $1/150$ for this value of λ. When the vector (X, Y) are becoming positively dependent, *i.e.* when θ increase, the asymptotic variance for this given λ increase also, but after a certain threshold which depends of the dependence structure, the variance starts to decrease.

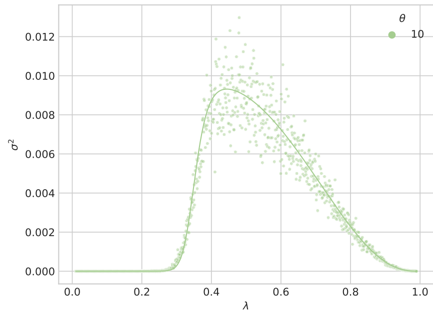
4.3 Estimation on Max-Stable processes

To determine the quality of the λ-FMadogram for estimating the pairwise depence of maxima in space, [Naveau et al., 2009] compute on a particular class of simulated max-stable random fields. They focus on the Smith's max-stable process [Smith, 2005]. We recall the bivariate distribution for the max-stable process model proposed by Smith is equal to :

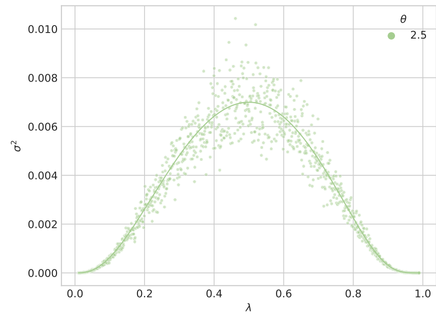
$$\mathbb{P}\left(X(s) \leq u, X(s+h) \leq v\right) = exp\left[-\frac{1}{u}\Phi(\frac{a}{2} + \frac{1}{a}log(\frac{v}{u})) - \frac{1}{v}\Phi(\frac{a}{2} + \frac{1}{a}log(\frac{u}{v}))\right] \quad (1.12)$$



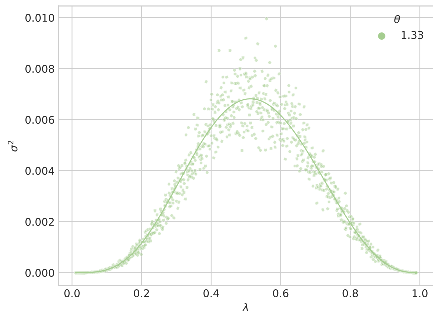
(a) **Asym. Neg. Logistic ($\theta = 10$, $\psi_1 = 1.0$, $\psi_2 = 1.0$)**



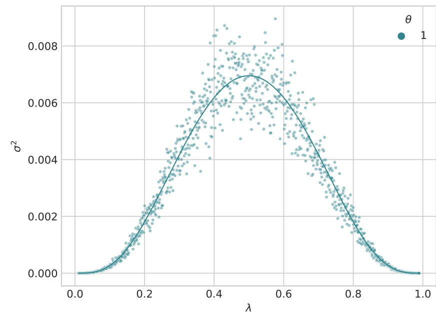
(b) **Asym. Neg. Logistic ($\theta = 10$, $\psi_1 = 0.5$, $\psi_2 = 1.0$)**



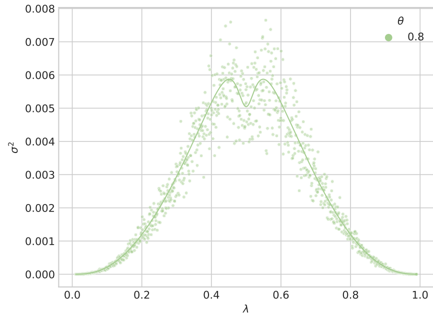
(c) **Asymmetric Logistic ($\theta = 5/2$, $\psi_1 = 0.1$, $\psi_2 = 1.0$)**



(d) **Asymmetric Mixed ($\theta = 1.00$, $\kappa = -0.33$)**

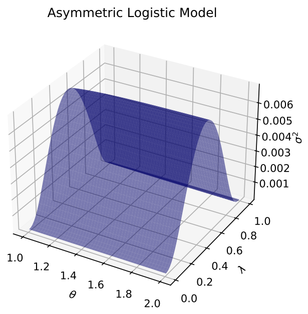


(e) **Hüsler-Reiss ($\theta = 1$)**

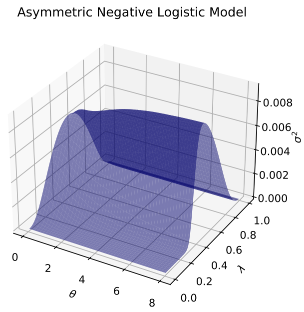


(f) **t-EV ($\theta = 0.8$, $\chi = 0.2$)**

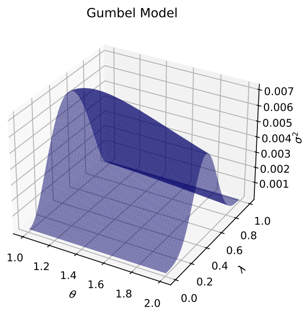
Figure 2: Graph, as a function of λ, of the asymptotic variances of the estimators of the λ-FMadogram for six extreme-value copula models. The empirical of the variance ($\times 256$) based on 500 samples of size $T = 256$.



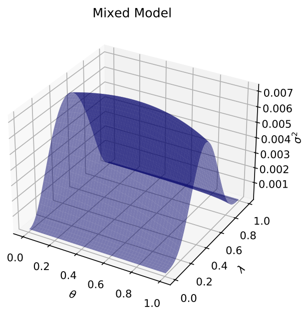
(a) **Asym. Logistic** ($\psi_1 = 0.1$, $\psi_2 = 1.0$)



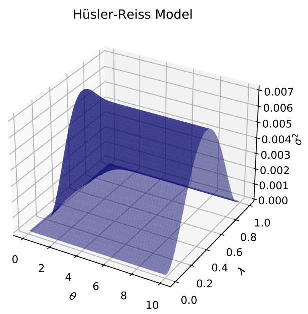
(b) **Asym. Neg. Logistic** ($\psi_1 = 0.5$, $\psi_2 = 1.0$)



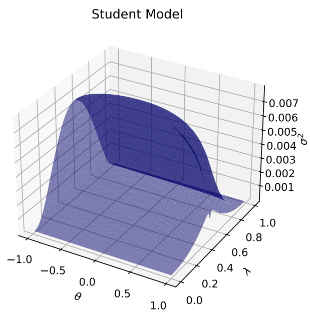
(c) **Gumbel** ($\psi_1 = 0.1$, $\psi_2 = 1.0$)



(d) **Symmetric Mixed** ($\kappa = 0.0$)



(e) **Hüsler-Reiss**



(f) **t-EV** ($\chi = 0.2$)

Figure 3: Graph, as a function of λ and the θ, of the asymptotic variance of the λ-FMadogram for six extreme-value copula models.

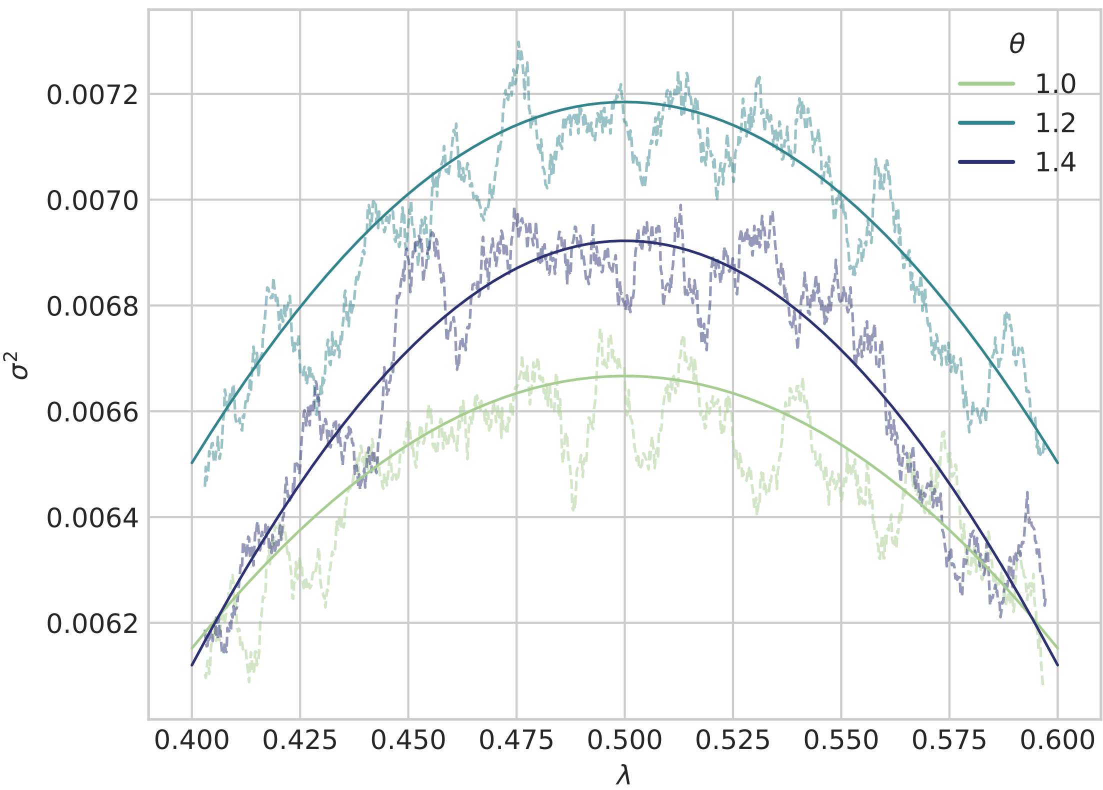


Figure 4: Variance ($\times 256$) of the estimators $\hat{\nu}(\lambda)$ based on 500 samples of size $T = 256$ from the Gumbel copula with $\theta = \{1.0, 1.2, 1.4\}$ chosen in such a way that $\lambda \in \{i/1000 : i = 0.4, \dots, 0.6\}$. The dotted lines are moving averages made out of the 1000 empirical estimators of the variance.

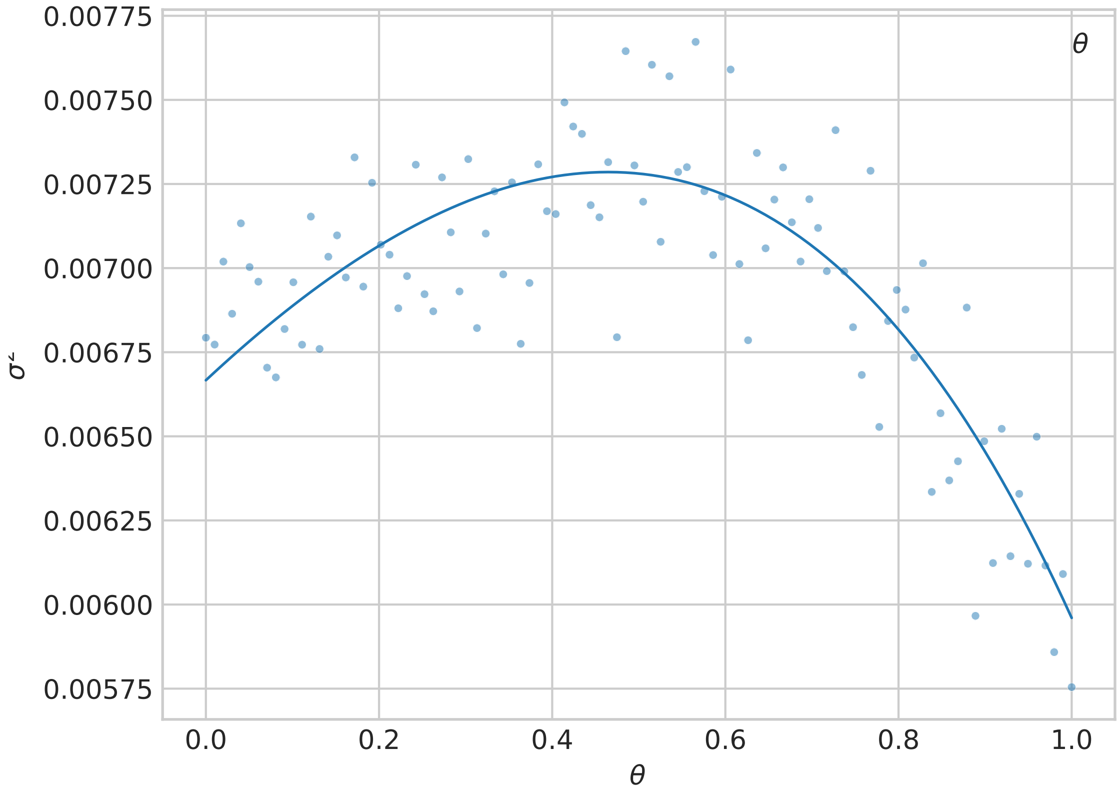


Figure 5: Variance ($\times 512$) of the estimators $\hat{\nu}(\lambda)$ based on 2000 sample of size $T = 512$ from the symetric mixed model with $\lambda = 0.5$ chosen in such a way that $\theta \in \{i/100, i = 0.0, \ldots 1.0\}$. The solid line is the asymptotic variance computed numerically using Theorem 1.

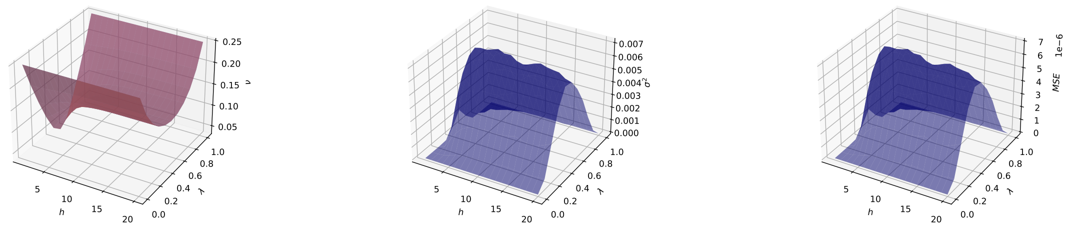
Where Φ denotes the standard normal distribution function, with $a^2 = (h^\top \Sigma^{-1} h)$ and Σ is a covariance matrix. In case of isotropic field, we set $\Sigma = \sigma^2 I_2$. We recall that, for this kind of process, the pairwise extremal dependence function $V_h(\cdot,\cdot)$ is given by :

$$V_h(u,v) = \frac{1}{u}\Phi(\frac{a}{2} + \frac{1}{a} log(\frac{v}{u})) + \frac{1}{v}\Phi(\frac{a}{2} + \frac{1}{a} log(\frac{u}{v})) \tag{1.13}$$

Furthermore, for a max-stable process, the theoretical value of the λ-FMadogram is given by,

$$\nu(h,\lambda) = \frac{V_h(\lambda, 1-\lambda)}{1 + V_h(\lambda, 1-\lambda)} - c(\lambda) \tag{1.14}$$

With $c(\lambda) = 3/\{2(1+\lambda)(1+1-\lambda)\}$ and for any $\lambda \in (0,1)$. This statement was shown in Proposition 1 of [Naveau et al., 2009]. The λ-FMadogram was estimated independently for each simulated field by using (1.7) with $T = 1024$. The z-axis corresponds to the error of the estimator and the xy-space, $[0,20] \times [0,1]$, represents the distance h and parameter λ. In smith's model, the pairwise dependence function between two locations s and $s+h$ decreases as the distance h between these two points increases.



(a) **Estimated value** (b) **variance** (c) **Mean Squared Error**

Figure 6: Simulation results obtained by generating 300 independently and identically distributed Smith random fields. The dependence struce is characterized by (1.12) with $\Sigma = 25 I_2$. Panel 6a show the estimated and the true λ-FMadogram. Panel 6b show the estimated variance ($\times 1024$) of the λ-FMadogram. Panel 6c depicts the mean squared error between the true and estimated λ-FMadogram for all h and λ.

The surface in Figure 6a provides the mean values of the estimated λ-FMadogram in blue, the true quantity is given by the surface in red. Figure 6c indicates the mean squared error between the estimated λ-FMadogram. As expected, the error is close to zero at the two boundary planes $\lambda = 0$ and $\lambda = 1$ by construction of the estimator. The largest mean squared errors are obtained where $\lambda = 0.5$, expecially for very small distances, *i.e.* near $h = 0$. This behaviour is now well known from our discussion.

5 Technical section

5.1 Study of the Pickands dependence function

Lemma 1. *Using properties of the Pickands dependence function, we have that*

$$0 \leq \kappa(\lambda, A) \leq 1, \quad 0 \leq \zeta(\lambda, A) \leq 1, \quad 0 < u, v < 1$$

Furthermore, if A admits a second derivative, $\kappa(\cdot, A)$ (resp. $\zeta(\cdot, A)$) is a decreasing function (resp. an increasing function).

Proof First, using that the graph of a (differentiable) convex function lies above all of its tangents and using that $A(t) \geq t$ gives, for $0 < t < 1$:

$$A'(t) \leq \frac{A(1) - A(t)}{t - 1} = \frac{1 - A(t)}{t - 1} \leq 1$$

Same reasoning using $A(t) \geq 1 - t$ leads to:

$$A'(t) \geq \frac{A(t) - A(0)}{t - 0} = \frac{A(t) - 1}{t} \geq -1$$

Let's fall back to κ and ζ. If we suppose that A admits a second derivative, the derivative of κ (resp ζ) with respect to λ gives:

$$\kappa'(\lambda, A) = -\lambda A''(\lambda) < 0, \quad \zeta'(\lambda, A) = (1 - \lambda)A''(\lambda) > 0 \quad \forall \lambda \in [0, 1]$$

Using $\kappa(0) = 1$, $\kappa(1) = 1 - A'(1) \geq 0$ gives $0 \leq \kappa(\lambda, A) \leq 1$. As $\zeta(0) = 1 + A'(0) \geq 0$ and $\zeta(1) = 1$, we have $0 \leq \zeta(\lambda, A) \leq 1$. That is the statement.

Now, we can obtain the same result while removing the hypothesis of A admits a second derivative. As A is a convex function, for $x, y \in [0, 1]$, we may have the following inequality:

$$A(x) \geq A(y) + A'(y).(x - y)$$

Take $x = 0$ and $y = \lambda$ gives

$$1 \geq A(\lambda) - \lambda A'(\lambda) = \kappa(\lambda)$$

Now, using that $-\lambda A'(\lambda) \geq -\lambda$, clearly

$$A(\lambda) - \lambda A'(\lambda) \geq A(\lambda) - \lambda \geq 0$$

As $A(\lambda) \geq max(\lambda, 1 - \lambda)$. We thus obtain our statement.

5.2 Proof of Proposition 1

The first statement results from the definition of the λ-FMadogram. The estimator $\hat{\nu(\lambda)}$ is strongly consistent since it holds

$$\begin{aligned}
&\left| \frac{1}{2T} \sum_{t=1}^{T} \left| \hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t) \right| - \frac{1}{2} \mathbb{E} \left| F^{\lambda}(X) - G^{1-\lambda}(Y) \right| \right| \\
&\leq \left| \frac{1}{2T} \sum_{t=1}^{T} \left| \hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t) \right| - \frac{1}{2T} \sum_{t=1}^{T} \left| F^{\lambda}(X_t) - G^{1-\lambda}(Y_t) \right| \right| \\
&+ \left| \frac{1}{2T} \sum_{t=1}^{T} \left| F^{\lambda}(X_t) - G^{1-\lambda}(Y_t) \right| - \frac{1}{2} \mathbb{E} \left| F^{\lambda}(X) - G^{1-\lambda}(Y) \right| \right|
\end{aligned}$$

The second term converges almost surely to zero by the strong Law of Large Numbers. For the first term, we have

$$\begin{aligned}
&\left| \frac{1}{2T} \sum_{t=1}^{T} \left| \hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t) \right| - \frac{1}{2T} \sum_{t=1}^{T} \left| F^{\lambda}(X_t) - G^{1-\lambda}(Y_t) \right| \right| \\
&\leq \frac{1}{2T} \sum_{t=1}^{T} \left| \left| \hat{F}_T^{\lambda}(X_t) - \hat{G}_T^{1-\lambda}(Y_t) \right| - \left| F^{\lambda}(X_t) - G^{1-\lambda}(Y_t) \right| \right| \\
&\leq \frac{1}{2T} \sum_{t=1}^{T} \left| \hat{F}_T^{\lambda}(X_t) - F^{\lambda}(X_t) - \left(\hat{G}_T^{1-\lambda}(Y_t) - G^{1-\lambda}(Y_t) \right) \right| \\
&\leq \frac{1}{2T} \sum_{t=1}^{T} \left| \hat{F}_T^{\lambda}(X_t) - F^{\lambda}(X_t) \right| + \left| \hat{G}_T^{1-\lambda}(Y_t) - G^{1-\lambda}(Y_t) \right|
\end{aligned}$$

which converges almost surely to zero according to the strong Law of Large Numbers. For the third statement, we begin by a lemma,

Lemma 2. *Let $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_T)$ with $\mathbb{X}_i = (X_i, Y_i)$ for every $i \in \{1, \ldots, T\}$ where (X_i, Y_i) be independent random vector with values in $\mathbb{R}^2$, we note,*

$$f(\mathbb{X}) = \underset{\lambda \in (0,1)}{sup} \frac{1}{T} \sum_{i=1}^{T} \nu_i(\lambda)$$

with $\nu_i(\lambda) = 1/2 \left| F^{\lambda}(X_i) - G^{1-\lambda}(Y_i) \right|$. Let $\mathbb{X}_i'$ be $\mathbb{X}$ except for the i-th coordinate : (X_i, Y_i) is changed for (X_i', Y_i'). Then

$$\underset{\mathbb{X} \in E^T, (\mathbb{X}_i', \mathbb{Y}_i') \in E_i}{sup} |f(\mathbb{X}) - f(\mathbb{X}_i')| \leq \frac{2}{T}, \quad \forall i \in \{1, \ldots, T\}$$

Proof We have, by definition of f:

$$|f(\mathbb{X}) - f(\mathbb{X}_i')| = \left| \underset{\lambda \in (0,1)}{sup} \frac{1}{T} \sum_{i=1}^{T} \nu_i(\lambda) - \underset{\lambda \in (0,1)}{sup} \frac{1}{T} (\sum_{i=1, i \neq i'}^{T} \nu_i(\lambda) + \nu_i'(\lambda)) \right|$$

Using the inequality $| \underset{\lambda \in (0,1)}{sup} f(\lambda) - \underset{\lambda \in (0,1)}{sup} g(\lambda)| \leq \underset{\lambda \in (0,1)}{sup} |f(\lambda) - g(\lambda)|$ gives :

$$\begin{aligned}
|f(\mathbb{X}) - f(\mathbb{X}_i')| &\leq \underset{\lambda \in (0,1)}{sup} \frac{1}{T} |\nu_i(\lambda) - \nu_i'(\lambda)| \\
&\leq \underset{\lambda \in (0,1)}{sup} \frac{1}{T} (|\nu_i(\lambda)| + |\nu_i'(\lambda)|) \leq \frac{2}{T}
\end{aligned}$$

We Thus obtain (iii) using Proposition A.2. To show (iv), we make of use of a Lemma we can be found in [Marcon et al., 2017] and sketch the proof for self-consistency in Appendix. For $\lambda \in [0, 1]$, we define the following function,

$$\nu_{\lambda} \colon [0, 1]^2 \to [0, 1], \quad (u, v) \mapsto u^{\lambda} \wedge v^{1-\lambda} - \frac{1}{2}(u^{\lambda} + v^{1-\lambda})$$

Using Lemma A.1 and the equality $|u^\lambda - v^{1-\lambda}| = u^\lambda \wedge v^{1-\lambda} - 1/2(u^\lambda + v^{1-\lambda})$ gives,

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\left(\int_{[0,1]} C(x^{\frac{1}{\lambda}}, 1)dx + \int_{[0,1]} C(1, x^{\frac{1}{1-\lambda}})dx\right) - \int_{[0,1]} C(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx \\ &= \frac{1}{2}\left(\int_{[0,1]} x^{\frac{1}{\lambda}}dx + \int_{[0,1]} x^{\frac{1}{1-\lambda}}dx\right) - \int_{[0,1]} x^{\frac{A(\lambda)}{\lambda(1-\lambda)}}dx \\ &= \frac{1}{2}\left(\frac{\lambda}{1+\lambda} + \frac{1-\lambda}{1+1-\lambda}\right) - \frac{\lambda(1-\lambda)}{A(\lambda)+\lambda(1-\lambda)}\end{aligned}$$

That is our statement.

5.3 A Lemma for equation (1.10)

Lemma 3. *For all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$, if $J(s,t) = |s^\lambda - t^{1-\lambda}|$, then the following integral satisfies:*

$$\int_{[0,1]^2} f(x,y)dJ(x,y) = \frac{1}{2}\int_{[0,1]} f(0, y^{1/(1-\lambda)})dy + \frac{1}{2}\int_{[0,1]} f(x^{1/\lambda}, 0)dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)})dx$$

Proof Let A a element of $\mathcal{B}([0,1]^2)$. We can pick an element of the form $A = [0,s] \times [0,t]$, where $s,t \in [0,1]$ and $\lambda \in [0,1]$. Let us introduce the following indicator function :

$$f_{s,t}(x,y) = \mathbb{1}_{\{(x,y)\in[0,1]^2, 0\le x\le s, 0\le y\le t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x,y)dJ(x,y) = J(s,t) - J(0,0) = |s^\lambda - t^{1-\lambda}|$$

in other hand, using the equality $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - min(x,y)$, one has to show

$$\begin{aligned}\frac{1}{2}|s^\lambda - t^{1-\lambda}| &= \frac{s^\lambda}{2} + \frac{y^{1-\lambda}}{2} - min(s^\lambda, t^{1-\lambda}) \\ &= \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, 0)dx + \int_{[0,1]} f_{s,t}(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\end{aligned}$$

Notice that the class :

$$\begin{aligned}\mathcal{E} = \{A \in \mathcal{B}([0,1]^2) : \int_{[0,1]^2} \mathbb{1}_A(x,y)dJ(x,y) &= \int_{[0,1]} \mathbb{1}_A(x^{\frac{1}{\lambda}}, 0)dx \\ &+ \int_{[0,1]} \mathbb{1}_A(0, y^{\frac{1}{1-\lambda}})dy - \int_{[0,1]} \mathbb{1}_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\}\end{aligned}$$

contain the class $\mathcal{P}$ of all closed pavements of $[0,1]^2$. It is otherwise a monotone class (or λ-system). Hence as the class $\mathcal{P}$ of closed pavement is a π-system, the class monotone theorem ensure that $\mathcal{E}$ contains the sigma-field generated by $\mathcal{P}$, that is $\mathcal{B}([0,1]^2)$.

This result holds for simple function $f(x,y) = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$ where $\lambda_i \in \mathbb{R}$ and $A_i \in \mathcal{B}([0,1]^2)$ for all $i \in \{1,\ldots,n\}$. We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function $f : [0,1]^2 \mapsto \mathbb{R}$ considering $f = f_+ - f_-$ with $f_+ = max(f,0)$ and $f_- = min(-f,0)$. We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral.

5.4 Proof of Proposition 2

In order to prove our proposition, we introduce two lemmas.

Lemma 4. *Let $(B_C(u,v))_{u,v\in[0,1]^2}$ a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u,v)B_C(u',v')] = C(u\wedge u', v\wedge v') - C(u,v)C(u',v')$$

for each $0 \leq, u, v, u', v' \leq 1$. Let $a, b \in [0,1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}[\int_{[0,1]} B_C(u,a)du \int_{[0,1]} B_C(b,u)du] = 0$$

Proof Without loss of generality, suppose that $a = 0$ and $b \in [0,1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned}\mathbb{E}[\int_{[0,1]} B_C(u,0)du \int_{[0,1]} B_C(b,u)du] &= \mathbb{E}[\int_{[0,1]}\int_{[0,1]} B_C(u,0)B_C(b,v)dudv] \\ &= \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u,0)B_C(b,v)]dudv\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u,0)B_C(b,v)] = C(u\wedge v, 0) - C(u,0)C(b,v)$$

We recall that, by definition, a copula satisfy $C(u,0) = C(0,u) = 0$ for every $u \in [0,1]$. Then, the equation below is equal to 0. Our conclusion directly follows.

Lemma 5. *Let $N_C(u,v)$ the process defined in equation (1.9) and $a, b \in [0,1]$ fixed. If $a = 0$ or $b = 0$, then :*

$$\mathbb{E}[\int_{[0,1]} N_C(a,u)du \int_{[0,1]} N_C(u,b)du] = 0$$

Proof Without loss of generality, let a = 0. Using the definition of $N_C(u,v)$, we have

$$N_C(0,u) = B_C(0,u) - \frac{\partial C(0,u)}{\partial u}B_C(0,1) - \frac{\partial C(0,u)}{\partial v}B_C(1,u)$$

Which is well defined if we consider, for a fixed $v \in [0,1]$

$$\frac{\partial C(u,v)}{\partial u} = \begin{cases} \frac{\partial C(u,v)}{\partial u}, & \text{if } u > 0 \\ \lim\limits_{u\to 0^+} \frac{\partial C(u,v)}{\partial v}, & \text{if } u = 0,\ v \in (0,1] \end{cases} \tag{1.15}$$

The continuous extension of $\frac{\partial C(u,v)}{\partial u}(\cdot, v)$ on $[0,1]$ while we have used A (iii) for the existence of the right limit. We do the same for $\frac{\partial C(u,v)}{\partial v}(u,\cdot)$. We have :

$$\begin{aligned}\mathbb{E}\left[\int_{[0,1]} N_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] &= \mathbb{E}\left[\int_{[0,1]} B_C(0,u)du \int_{[0,1]} N_C(u,b)du\right] \\ &- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial u}B_C(0,1)du \int_{[0,1]} N_C(u,b)du\right] \\ &- \mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial v}B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right]\end{aligned}$$

Using preceding lemma, we got that the two first terms are equal to zero. Only the last term should be discuss. Remember that $\frac{\partial C(0,u)}{\partial v} = 0$ for all $u \in]0,1]$, as we integrate with respect to the lebesgue measure, the set $\{0\}$ is of measure 0 because it is a countable set, then :

$$\mathbb{E}\left[\int_{[0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = \mathbb{E}\left[\int_{]0,1]} \frac{\partial C(0,u)}{\partial v} B_C(1,u)du \int_{[0,1]} N_C(u,b)du\right] = 0$$

These two results gives us the proposition.

5.5 Proof of Theorem 1

We are able to compute the variance for each process and they are given by the following expressions :

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) = f(\lambda, A)\left(\frac{1}{A(\lambda) + 2\lambda(1-\lambda)}\right)$$

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) = f(\lambda, A)\left(\frac{\kappa^2(\lambda, A)(1-\lambda)}{2A(\lambda) - (1-\lambda) + 2\lambda(1-\lambda)}\right)$$

$$Var\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial v}du\right) = f(\lambda, A)\left(\frac{\zeta^2(\lambda, A)\lambda}{2A(\lambda) - \lambda + 2\lambda(1-\lambda)}\right)$$

We now compute the covariance :

$$\begin{aligned}
&cov\left(\int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_{[0,1]} B_C(u^{\frac{1}{\lambda}}, 1)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}du\right) \\
&= \int_{[0,1]}\int_{[0,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(v^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv \\
&= \int_{[0,1]}\int_{[0,v]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv \\
&+ \int_{[0,1]}\int_{[v,1]} \mathbb{E}[B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})B_C(u^{\frac{1}{\lambda}}, 1)]\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv
\end{aligned}$$

for the first one, we have :

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,v]} (C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dudv \\
&\qquad = \frac{\kappa(\lambda, A)}{2} f(\lambda, A)\left(\frac{1-\lambda}{2A(\lambda) + (2\lambda-1)(1-\lambda)}\right)
\end{aligned}$$

For the second part, using Fubini, we have :

$$\int_{[0,1]}\int_{[0,u]} (C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) - C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu$$

for the right hand side of the "minus" sign, we may compute :

$$\int_{[0,1]}\int_{[0,u]} C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})v^{\frac{1}{\lambda}}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu = \frac{\kappa(\lambda, A)}{2}f(\lambda, A)$$

The last one still difficult to handle,

$$\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \tag{1.16}$$

Following the proof of proposition 3.3 from [Genest and Segers, 2009], the substitution $v^{\frac{1}{\lambda}} = x$ and $u^{\frac{1}{1-\lambda}} = y$ yield

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,u]} C(v^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial u}dvdu \\
&= \lambda(1-\lambda)\int_{[0,1]}\int_{[0,y^{\frac{1-\lambda}{\lambda}}]} C(x,y)\frac{\partial C(x, x^{\frac{\lambda}{1-\lambda}})}{\partial u}x^{\lambda-1}y^{-\lambda}dxdy \\
&= \lambda(1-\lambda)\kappa(\lambda, A)\int_{[0,1]}\int_{[0,y^{\frac{1-\lambda}{\lambda}}]} C(x,y)x^{\frac{A(\lambda)}{1-\lambda}-(1-\lambda)-1}y^{-\lambda}dxdy
\end{aligned}$$

Next, use the substitution $x = w^{1-s}$ and $y = w^s$. Note that $w = xy \in [0,1]$, $s = log(y)/log(xy) \in [0,1]$, $C(x,y) = w^{A(s)}$ and the Jacobian of the transformation is $-log(w)$. As the constraint $x < y^{-1+1/\lambda}$ reduces to $s < \lambda$, the integral becomes:

$$\begin{aligned}
&-\lambda(1-\lambda)\kappa(\lambda, A)\int_{[0,\lambda]}\int_{[0,1]} w^{A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)-s\lambda}log(w)dwds \\
&= \lambda(1-\lambda)\kappa(\lambda, A)\int_{[0,\lambda]} [A(s)+(1-s)(A_2(\lambda)-1-(1-\lambda))-s\lambda+1]^{-2}\,ds
\end{aligned}$$

Let's continue with computing the following integral :

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,1]}\int_{[0,1]} B_C(u^{\frac{1}{\lambda}},1)B_C(1,v^{\frac{1}{1-\lambda}})\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}dudv\right] \\
&= \int_{[0,1]}\int_{[0,1]} \left(C(u^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}}) - u^{\frac{1}{\lambda}}v^{\frac{1}{1-\lambda}}\right)\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}dudv
\end{aligned}$$

The second term can be easily handled and its value is given by :

$$\int_{[0,1]}\int_{[0,1]} u^{\frac{1}{\lambda}}v^{\frac{1}{1-\lambda}}\frac{\partial C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})}{\partial u}\frac{\partial C(v^{\frac{1}{\lambda}}, v^{\frac{1}{1-\lambda}})}{\partial v}dudv = f(\lambda, A)\kappa(\lambda, A)\zeta(\lambda, A)$$

For the first, use the substitutions $u^{\frac{1}{\lambda}} = x$ and $v^{\frac{1}{1-\lambda}} = y$. This yields :

$$\lambda(1-\lambda)\int_{[0,1]}\int_{[0,1]} C(x,y)\frac{\partial C(x, x^{\frac{\lambda}{1-\lambda}})}{\partial u}\frac{\partial C(y^{\frac{1-\lambda}{\lambda}}, y)}{\partial v}x^{\lambda-1}y^{-\lambda}dxdy$$

Then, make the substitutions $x = w^{1-s}$, $y = w^s$ that were used for the preceding integral gives :

$$\begin{aligned}
&-\lambda(1-\lambda)\kappa(\lambda, A)\zeta(\lambda, A)\int_{[0,1]}\int_{[0,1]} w^{A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)+s(A_1(\lambda)-\lambda-1)}log(w)dwds \\
&= \lambda(1-\lambda)\kappa(\lambda, A)\zeta(\lambda, A)\int_{[0,1]} [A(s)+(1-s)(A_2(\lambda)-(1-\lambda)-1)+s(A_1(\lambda)-\lambda-1)+1]^{-2}\,ds
\end{aligned}$$

The last covariance requires the same tools as used before, it is left to the reader. It then suffices to use the bilinearity of the covariance and to assemble the various terms to conclude.

5.6 Proof of Theorem 2

This results comes from the proof of the Theorem 2.4 of [Marcon et al., 2017]. Following their approch, we express the empirical λ-FMadogram $\hat{\nu}_T(\lambda)$ in terms of the empirical copula and exploiting known results. Let us note :

$$\hat{\nu}_T(\lambda) = \frac{1}{T}\sum_{t=1}^{T} \nu_\lambda(\hat{F}_T(X_t), \hat{G}_T(Y_t)) = \int_{[0,1]^2} \nu_\lambda(u,v) d\hat{C}_T(u,v)$$

$$\nu(\lambda) = \int_{[0,1]^2} \nu_\lambda(u,v) dC(u,v)$$

Using lemma A.1, we obtain :

$$\sqrt{T}(\hat{\nu}_T(\lambda) - \nu(\lambda)) = \frac{1}{2}\left(\int_{[0,1]} \mathbb{C}_T(x^{\frac{1}{\lambda}}, 1) + \int_{[0,1]} \mathbb{C}(1, x^{\frac{1}{1-\lambda}})dx\right) - \int_{[0,1]} \mathbb{C}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx$$

Consider the function $\phi\colon l^\infty([0,1]^2) \to l^\infty([0,1]]), f \mapsto \phi(f)$, defined by

$$(\phi(f))(\lambda) = \frac{1}{2}\left(\int_{[0,1]} f(x^{\frac{1}{\lambda}}, 1)dx + \int_{[0,1]} f(1, x^{\frac{1}{1-\lambda}})dx\right) - \int_{[0,1]} f(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx$$

This function is linear and bounded thus continuous. The continous mapping theorem (Theorem 1.3.6 of [van der Vaart and Wellner, 1996]) implies, as $T \to \infty$

$$\sqrt{T}(\hat{\nu}_T - \nu) = \phi(\mathbb{C}_T) \rightsquigarrow \phi(N_C)$$

in $l^\infty([0,1])$. Note that $N_C(u,1) = N_C(1,v) = 0$ almost surely for every $(u,v) \in [0,1]^2$. Indeed for the second one,

$$N_C(1,v) = B_C(1,v) - \frac{\partial C}{\partial v}(1,v) B_C(1,v)$$

and $\partial C/\partial v(1,v)$ is well defined due to Assumption A (iii) and is equal 1 almost surely. Then, we have

$$\sqrt{T}(\hat{\nu}_T - \nu) \rightsquigarrow \left(-\int_{[0,1]} N_C(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx\right)$$

That is our statement.

5.7 A counter example against of variance's monotony with respect to positive dependence

First, notive that, under dependency condition, the variance of the Madogram evaluated in $\lambda = 0.5$ is equal to $1/150$.

Lemma 6. *Let us consider $C(u,v) = 1 - \theta t + \theta t^2$ where $\theta \in [0,1]$. If we take $\lambda = 0.5$, there exist $\theta \in (0,1)$ such that*

$$Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) > \frac{1}{150} \tag{1.17}$$

Proof For this dependence function, we have immediately :

$$\kappa(\lambda, A) = 1 - \theta\lambda^2, \quad \zeta(\lambda, A) = 1 - \theta(1-\lambda)^2$$

For $\lambda = 0.5$, we notice that $\kappa(0.5, A) = \zeta(0.5, A)$. By a simple change of variable, we notice that :

$$\int_0^{0.5} [A(s)+(1-s)(2A(0.5)-0.5-1)-0.5s+1]^{-2}ds = \int_{0.5}^{1} [A(s)+s(2A(0.5)-0.5-1)-0.5(1-s)+1]^{-2}ds$$

By simple substitution, we have for the chosen copula that

$$\int_{0.5}^{1} [A(s)+s(2A(0.5)-0.5-1)-0.5(1-s)+1]^{-2}ds = \int_{0.5}^{1} [\frac{3}{2} - s(\theta+1-2A(0.5)) + s^2\theta]ds$$

Let us take $\theta = 2A(0.5) - 1$, which implies by direct computation that $\theta = 2/3 > 0$. Let us make of use of this lemma :

Lemma 7. *Let a, b be two reals. Note $I_n = \int_{\mathbb{R}} (ax^2+b)^n dx$, then :*

$$I_n = \frac{2n-3}{2b(n-1)} I_{n-1} + \frac{x}{2b(n-1)(ax^2+b)}$$

Proof An integration by parts gives and some algebra gives:

$$\begin{aligned} I_{n-1} &= \frac{x}{(ax^2+b)^{n-1}} + 2(n-1)\int_{\mathbb{R}} \frac{ax^2}{(ax^2+b)^n}dx \\ &= \frac{x}{(ax^2+b)^{n-1}} + 2(n-1)I_{n-1} - 2b(n-1)I_n \end{aligned}$$

Solving the equation for I_n gives the result.

We want to compute the following quantity :

$$\int_{0.5}^{1} [\frac{3}{2} + s^2\frac{2}{3}]^{-2}ds$$

The lemma gives :

$$\begin{aligned} \int_{0.5}^{1} [\frac{3}{2} + s^2\frac{2}{3}]^{-2}ds &= 36\int_{0.5}^{1} [4s^2+9]^{-2}ds \\ &= 2\left(\frac{7}{20} + \int_{0.5}^{1} (4s^2+9)^{-1}ds\right) \\ &= 2\left(\frac{7}{20} + \frac{1}{6}\int_{1/3}^{2/3} \frac{1}{u^2+1}du\right) \end{aligned}$$

Where we have made the substitution $u = 2s/3$ in the third line. Then :

$$\int_{0.5}^{1} [\frac{3}{2} + s^2\frac{2}{3}]^{-2}ds = 2\left[\frac{7}{20} + \frac{1}{6}(atan(2/3) - atan(1/3))\right] \approx 0.142596$$

For the last integral, we have, by substitution for $\lambda = 0.5$ and $\theta = 2/3$:

$$\int_0^1 [A(s)+(1-s)(2A(0.5)-0.5-1)+s(2A(0.5)-0.5-1)+1]^{-2}ds = \int_0^1 [\frac{13}{6} - \frac{2}{3}s + \frac{2}{3}s^2]^{-2}ds$$

Then, we are able to compute :

$$\int_0^1 [\frac{13}{6} - \frac{2}{3}s + \frac{2}{3}s^2]^{-2} ds = 36 \int_0^1 (13 - 4s + 4s^2) ds \overset{u=(2s-1)}{=} 36 \int_0^1 ((2s-1)^2 + 12)^2 ds$$
$$= 18 \int_{-1}^1 (u^2+12)^{-2} du \overset{\text{Lemma}}{=} \frac{3}{4} \left(\frac{2}{13} + \int_{-1}^1 \frac{1}{u^2+12} du \right)$$
$$\overset{v=u/(2\sqrt{3})}{=} \frac{6}{52} + \frac{3}{8\sqrt{3}} \int_{\frac{-1}{2\sqrt{3}}}^{\frac{1}{2\sqrt{3}}} \frac{1}{v^2+1} dv$$

$$\int_0^1 [\frac{13}{6} - \frac{2}{3}s + \frac{2}{3}s^2]^{-2} ds = \frac{\sqrt{3}}{8} \left(atan(\frac{1}{2\sqrt{3}}) - atan(-\frac{1}{2\sqrt{3}}) \right) + \frac{6}{52} \approx 0.23707$$

Summing all the components of the variance gives $Var\left(\int_{[0,1]} N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du\right) \approx 0.00713 > 1/150$, which gives our counterexample.

Bibliography

[Beirlant et al., 2004] Beirlant, J., Goegebeur, Y., Segers, J., and Teugels, J. (2004). *Statistics of Extremes: Theory and Applications*. Wiley. Pagination: 522.

[Bernard et al., 2013] Bernard, E., Naveau, P., Vrac, M., and Mestre, O. (2013). Clustering of Maxima: Spatial Dependencies among Heavy Rainfall in France. *Journal of Climate*, 26(20):7929–7937.

[Breton, 2020] Breton, J.-C. (2020). Processus stochastiques.

[Capéraà et al., 1997] Capéraà, P., Fougères, A.-L., and Genest, C. (1997). A nonparametric estimation procedure for bivariate extreme value copulas. *Biometrika*, 84(3):567–577.

[Coles et al., 1999] Coles, S., Heffernan, J., and Tawn, J. (1999). Dependence measures for extreme value analyses. *Extremes*, 2:339 – 365.

[Cooley et al., 2006] Cooley, D., Naveau, P., and Poncet, P. (2006). *Variograms for spatial max-stable random fields*, pages 373–390. Springer New York, New York, NY.

[Demarta and McNeil, 2005] Demarta, S. and McNeil, A. J. (2005). The t copula and related copulas. *International Statistical Review*, 73(1):111–129.

[Fermanian et al., 2004] Fermanian, J.-D., Radulovic, D., and Wegkamp, M. (2004). Weak convergence of empirical copula processes. *Bernoulli*, 10(5):847 – 860.

[Gaetan and Guyon, 2008] Gaetan, C. and Guyon, X. (2008). *Modélisation et statistique spatiales*. Mathématiques & applications. Springer, Berlin Heidelberg New York.

[Genest and Segers, 2009] Genest, C. and Segers, J. (2009). Rank-based inference for bivariate extreme-value copulas. *The Annals of Statistics*, 37(5B):2990 – 3022.

[Gudendorf and Segers, 2009] Gudendorf, G. and Segers, J. (2009). Extreme-value copulas.

[Guillou et al., 2014] Guillou, A., Naveau, P., and Schorgen, A. (2014). Madogram and asymptotic independence among maxima. *Revstat Statistical Journal*, 12:119–134.

[Hüsler and Reiss, 1989] Hüsler, J. and Reiss, R.-D. (1989). Maxima of normal random vectors: Between independence and complete dependence. *Statistics & Probability Letters*, 7(4):283–286.

[Marcon et al., 2017] Marcon, G., Padoan, S., Naveau, P., Muliere, P., and Segers, J. (2017). Multivariate nonparametric estimation of the pickands dependence function using bernstein polynomials. *Journal of Statistical Planning and Inference*, 183:1–17.

[Naveau et al., 2009] Naveau, P., Guillou, A., Cooley, D., and Diebolt, J. (2009). Modeling pairwise dependence of maxima in space. *Biometrika*, 96(1):1–17.

[Sklar, 1959] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8:229–231.

[Smith, 2005] Smith, R. L. (2005). Max-stable processes and spatial extremes.

[van der Vaart and Wellner, 1996] van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Process: With Applications to Statistics*.

Appendix A

Auxiliary results

Theorem A.1 (Theorem 3 of [Fermanian et al., 2004])**.** *Suppose that H has continuous marginal distribution functions and that the copula function $C(x,y)$ has continuous partial derivatives. Then the empirical copula process $\{\mathbb{C}_T(u,v), 0 \leq u,v \leq 1\}$ converges weakly to a Gaussian process $\{N_C(u,v), 0 \leq u,v \leq 1\}$ in $l^\infty([0,1]^2)$.*

Under the assumptions defined in Assumption A, the following proposition from [Naveau et al., 2009] hold.

Proposition A.1 (Proposition 3 of [Naveau et al., 2009])**.** *Suppose that Assumptions A holds and let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then:*

$$T^{-1/2} \sum_{t=1}^{T} (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u,v) dJ(u,v)$ where $N_C(u,v)$ is defined by equation (1.9) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Breton, 2020]). The special case, $J(x,y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$ provide the weak of convergence of the λ-Madogram estimator defined by (1.6) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to $\int_{[0,1]^2} N_C(u,v) dJ(u,v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x,y) dJ(x,y) = \frac{1}{2} \int_{[0,1]} f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_{[0,1]} f(x^{1/\lambda}, 0) dx - \int_{[0,1]} f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \tag{A.1}$$

for all bounded-measurable function $f : [0,1]^2 \mapsto \mathbb{R}$.

Let $E_1, \dots, E_T$ is a finite sequence of separable spaces. Let $E^T = E_1 \times \cdots \times E_T$. A function f from E^T into $\mathbb{R}$ is said to be of bounded difference if for some nonnegative constants $c_1, \dots, c_T$,

$$\underset{x_1,\dots,x_T \in E^T, x_i' \in E_i}{sup} |f(x_1,\dots,x_T) - f(x_1,\dots,x_{i-1}, x_i', x_{i+1},\dots,x_T)| \leq c_i, \quad \forall i \in 1,\dots,T$$

Proposition A.2. *(McDiarmid's inequality) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathbb{X} = (\mathbb{X}_1, \dots, \mathbb{X}_T)$ be a random vector with independent components and values in E^T. Let f be any bounded difference function from E^T to $\mathbb{R}$. Set $Z = f(X)$, then for any $t > 0$*

$$\mathbb{P}(Z - \mathbb{E}Z \geq tc\sqrt{T}) \leq e^{-2t^2} \tag{A.2}$$

Lemma A.1. *(Lemma A.1 of [Marcon et al., 2017]) For $\lambda \in [0,1]$, let H be any distribution function in $[0,1]^2$, we have :*

$$\int_{[0,1]^2} \nu_\lambda(u,v) dH(u,v) = \frac{1}{2}\left(\int_{[0,1]} H(x^{\frac{1}{\lambda}}, 1)dx + \int_{[0,1]} H(1, x^{\frac{1}{1-\lambda}})dx\right) - \int_{[0,1]} H(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}})dx \tag{A.3}$$

Proof We have,

$$u^\lambda \wedge v^{1-\lambda} = 1 - \int_{[0,1]} \mathbb{1}_{\{u^\lambda \leq x, v^{1-\lambda} \leq x\}} dx = 1 - \int_{[0,1]} \mathbb{1}_{\{u \leq x^{\frac{1}{\lambda}}, v \leq x^{\frac{1}{1-\lambda}}\}} dx,$$

using the same techniques, we may have,

$$\frac{1}{2}(u^\lambda + v^{1-\lambda}) = 1 - \frac{1}{2}\left(\int_{[0,1]} \mathbb{1}_{\{u \leq x^{\frac{1}{\lambda}}\}} + \mathbb{1}_{\{v \leq x^{\frac{1}{1-\lambda}}\}}\right),$$

We obtain by substracting the two terms above and integration with respect to H,

$$\int_{[0,1]^2} v_\lambda(u,v) dH(u,v) = \frac{1}{2}\int_{[0,1]^2}\int_{[0,1]} \left(\mathbb{1}_{\{u \leq x^{\frac{1}{\lambda}}\}} + \mathbb{1}_{\{v \leq x^{\frac{1}{1-\lambda}}\}}\right) dH(u,v)dx - \int_{[0,1]^2}\int_{[0,1]} \mathbb{1}_{\{u \leq x^{\frac{1}{\lambda}}, v \leq x^{\frac{1}{1-\lambda}}\}} dH(u,v)dx$$

Applying Fubini lead us to the conclusion.