

Let  $(X, Y)$  be a bivariate random vector with joint distribution function  $H(x, y)$  and continuous marginal distribution function  $F(x)$  and  $G(y)$ . Its associated copula  $C$  is defined by  $H(x, y) = C\{F(x), G(y)\}$ . Since  $F$  and  $G$  are continuous, the copula  $C$  is unique and we can write  $C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v))$  for  $0 \leq u, v \leq 1$  and where  $F^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | F(v) \geq u\}$  and  $G^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | G(v) \geq u\}$  are the generalized inverse functions of  $F$  and  $G$  respectively. At each  $t \in \{1, \dots, T\}$ , we suppose that one of both entries may be missing. The observations consist of a sample of independent, identically distributed quadruples

$$(I_t, J_t, I_t X_t, J_t Y_t) \quad t \in 1, \dots, T$$

The indicator variables  $I_t$  (respectively  $J_t$ ) is equal to 1 or 0 according to whether  $X_t$  or  $Y_t$  is observed or not. We suppose that the indicator functions  $I_t$  and  $J_t$  are independent. The probability of observing a realisation partially or completely is denoted by  $p_X = \mathbb{P}(I_t = 1) > 0$  and  $p_Y = \mathbb{P}(J_t = 1) > 0$ .

We define :

$$C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v)) = \varphi(H)(u, v)$$

and,

$$Z_T(u, v) = \sqrt{T}\{\hat{C}_T(u, v) - C(u, v)\} \quad (1)$$

where  $\hat{H}_T$  corresponds to the empirical distribution function of the sample  $(X_1, Y_1), \dots, (X_T, Y_T)$

$$\hat{H}_T(u, v) = \frac{\sum_{t=1}^T 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t}{\sum_{t=1}^T I_t J_t}$$

We define also the corresponding empirical distribution functions in the case of missing data :

$$\begin{aligned} \hat{F}_T(u) &= \frac{\sum_{t=1}^T 1_{\{X_t \leq u\}} I_t}{\sum_{t=1}^T I_t} \\ \hat{G}_T(v) &= \frac{\sum_{t=1}^T 1_{\{Y_t \leq v\}} J_t}{\sum_{t=1}^T J_t} \end{aligned}$$

**Condition 1.** We suppose for all  $t \in \{1, \dots, T\}$  :

$$\mathbb{E}[X_t Y_t I_t J_t] = \mathbb{E}[X_t Y_t] \mathbb{E}[I_t J_t]$$

Furthermore, we suppose that there exist at least one  $t \in \{1, \dots, T\}$  such that  $I_t J_t \neq 0$ .

**Proposition 1.** Under hypothesis 1,  $\hat{H}_T, \hat{F}_T, \hat{G}_T$  are consistant estimators of  $H, F, G$ .

*Démonstration.* We check the consistency for  $\hat{H}_T$ . By independence, we have

$$\mathbb{E}[T^{-1} \sum_{t=1}^T I_t J_t] = T^{-1} \sum_{t=1}^T \mathbb{E}[I_t] \mathbb{E}[J_t] = p_X p_Y$$

So, by applying the law of large numbers, we have :

$$T^{-1} \sum_{t=1}^T I_t J_t \longrightarrow p_X p_Y \quad a.s. \quad as \quad T \rightarrow \infty$$

Then, we now use the first hypothesis to get :

$$T^{-1} \sum_{t=1}^T \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t] = T^{-1} \sum_{t=1}^T \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}}] \mathbb{E}[I_t J_t] = H(x, y) p_X p_Y$$

By applying again the law of large numbers, we derive :

$$\sum_{t=1}^T 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t \longrightarrow H(x, y) p_X p_Y \quad a.s. \quad as \quad T \rightarrow \infty$$

We can now apply the continuous mapping theorem to the function  $f : (x, y) \mapsto \frac{x}{y}$  which are continuous on  $\mathbb{R}_+ \times \mathbb{R}_+ \setminus 0$  to conclude that :

$$\hat{H}_T(x, y) \longrightarrow H(x, y) \quad a.s. \quad as \quad T \rightarrow \infty$$

□

**Condition 2.** 1. The bivariate distribution function  $H$  has continuous margins  $F, G$  and copula  $C$ .

2. The first order partial derivatives  $\dot{C}_1(u, v) = \frac{\partial C}{\partial u}(u, v)$  and  $\dot{C}_2(u, v) = \frac{\partial C}{\partial v}(u, v)$  exists and is continuous on the set  $\{(u, v) \in [0, 1]^2, 0 < u, v < 1\}$

**Condition 3.** There exists  $\gamma_t > 0$  and  $r_t > 0$  such that  $r_t \rightarrow \infty$  as  $t \rightarrow \infty$  such that in the space  $l^\infty(\mathbb{R}^2) \otimes (l^\infty(\mathbb{R}), l^\infty(\mathbb{R}))$  equipped with the topology of uniform convergence, we have the joint weak convergence

$$(r_t(\hat{H}_T - H); r_t(\hat{F}_t - F), r_t(\hat{G}_t - G)) \rightsquigarrow (\alpha \circ (F, G), \beta_1 \circ F, \beta_2 \circ G)$$

The stochastic processes  $\alpha$  and  $\beta_j$  take values in  $l^\infty([0, 1]^2)$  and  $l^\infty([0, 1])$  respectively, and are such that  $\alpha \circ F$  and  $\beta_j \circ F_j$  have continuous trajectories on  $[-\infty, \infty]^2$  and  $[-\infty, \infty]$  almost surely.

**Theorem 1** (Theorem 2.3 and example 3.5 in [Seg14]). If conditions 2.1 and 2.2 holds, then uniformly in  $u \in [0, 1]^2$ ,

$$r_T\{\hat{C}_T(u, v) - C(u, v)\} = r_T\{\hat{H}_T((F, G)^\leftarrow(u, v) - C(u, v)\} \quad (2)$$

$$- \dot{C}_1(u, v)r_T\{\hat{F}_T(F^\leftarrow(u)) - u\}1_{(0,1)}(u) \quad (3)$$

$$- \dot{C}_2(u, v)r_T\{\hat{G}_T(G^\leftarrow(v)) - v\}1_{(0,1)}(v) + o_P(1) \quad (4)$$

as  $T \rightarrow \infty$ . Hence in  $l^\infty([0, 1]^2)$  equipped with the supremum norm, as  $T \rightarrow \infty$ ,

$$(r_T\{\hat{C}_T(u, v) - C(u, v)\})_{u,v \in [0,1]^2} \rightsquigarrow (\alpha(u, v) - \dot{C}_1(u, v)\beta_1(u) - \dot{C}_2(u, v)\beta_2(v))_{u,v \in [0,1]^2}$$

We denote by  $S_C(u, v)$  the process defined on the right-hand side in the weak convergence from above.

**Remark 1.** If we consider the empiric copula, theorem 1 gives us the weak convergence of this process to a brownian bridge  $N_C(u, v)$  defined by,  $\forall(u, v) \in [0, 1]$

$$N_C(u, v) = B_C(u, v) - \dot{C}_1(u, v)B_C(u, 1) - \dot{C}_2(u, v)B_C(1, v) \quad (5)$$

where  $B_C$  is a brownian bridge in  $[0, 1]^2$  with covariance function

$$\mathbb{E}[B_C(u, v)B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$$

for each  $0 \leq u, v, u', v' \leq 1$ . This result is well known since 2004 due to [FRW04].

In our specific case with missing data, (ref) shows that  $r_T\{\hat{C}_T(u, v) - C(u, v)\}$  is weakly convergent toward  $\alpha(u, v) - \dot{C}_1(u, v)\beta_1(u) - \dot{C}_2(u, v)\beta_2(v)$  where  $\beta_1(u) = p_X^{-1}\mathbb{G}(1_{X \leq F^\leftarrow(u)} - u1_{I=1})$ ,  $\beta_2(v) = p_Y^{-1}\mathbb{G}(1_{Y \leq G^\leftarrow(v)} - v1_{J=1})$  and  $\alpha(u, v) = (p_X p_Y)^{-1}\mathbb{G}(1_{X \leq F^\leftarrow(u)} 1_{Y \leq G^\leftarrow(v)} - C(u, v)1_{I=1}1_{J=1})$ . Furthermore, by these expressions, we can detail the structure of the covariance matrix between the three processes.

**Definition 1.** Let  $(X_1, Y_1), \dots, (X_T, Y_T)$  a  $T$  bivariate random vectors with unknown margins  $F$  and  $G$ . A  $\lambda$ -FMadogram is the quantity defined by :

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \quad (6)$$

We estimative the  $\lambda$ -FMadogram with the following quantity :

$$\hat{\nu}_T(\lambda) = \frac{1}{2 \sum_{t=1}^T I_t J_t} \sum_{t=1}^T |\hat{F}_T^\lambda(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| I_t J_t \quad (7)$$

**Proposition 2** (Proposition 3 of [NGCD09]). Suppose that conditions 2 holds and that  $\sum_{t=1}^T I_t J_t = T$  (no missing data). Let  $J$  be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then

$$T^{-1/2} \sum_{t=1}^T (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to  $\int_{[0,1]} N_C(u, v) dJ(u, v)$  where  $N_C(u, v)$  is defined by equation (5) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Bre20]). The special case,  $J(x, y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$  provide the weak of convergence of the  $\lambda$ -Madogram estimator defined by (7) :

$$T^{1/2}\{\hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|]\}$$

converge in distribution to  $\int_{[0,1]^2} N_C(u, v) dJ(u, v)$  where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx \quad (8)$$

for all bounded-measurable function  $f : [0, 1]^2 \mapsto \mathbb{R}$ .

We add some elements in order to prove the identity (8).

**Lemma 1.** *For all bounded-measurable function  $f : [0, 1]^2 \mapsto \mathbb{R}$ , if  $J(s, t) = |s^\lambda - t^{1-\lambda}|$ , then the following integral satisfies :*

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

*Démonstration.* Let  $A$  a element of  $\mathcal{B}([0, 1]^2)$ . We can pick an element of the form  $A = [0, s] \times [0, t]$ , where  $s, t \in [0, 1]$  and  $\lambda \in [0, 1]$ . Let us introduce the following indicator function :

$$f_{s,t}(x, y) = 1_{\{(x,y) \in [0,1]^2, 0 \leq x \leq s, 0 \leq y \leq t\}}$$

Then, for this function, we have in one hand :

$$\int_{[0,1]^2} f_{s,t}(x, y) dJ(x, y) = J(s, t) - J(0, 0) = |s^\lambda - t^{1-\lambda}|$$

in other hand, using the equality  $\frac{|x-y|}{2} = \frac{x}{2} + \frac{y}{2} - \min(x, y)$ , one has to show

$$\begin{aligned} \frac{1}{2}|s^\lambda - t^{1-\lambda}| &= \frac{s^\lambda}{2} + \frac{y^{1-\lambda}}{2} - \min(s^\lambda, t^{1-\lambda}) \\ &= \int_0^1 f_{s,t}(x^{\frac{1}{\lambda}}, 0) dx + \int_0^1 f_{s,t}(0, y^{\frac{1}{1-\lambda}}) dy - \int_0^1 f_{s,t}(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}}) dx \end{aligned}$$

Notice that the class :

$$\mathcal{E} = \{A \in \mathcal{B}([0, 1]^2) : \int_{[0,1]^2} 1_A(x, y) dJ(x, y) = \int_0^1 1_A(x^{\frac{1}{\lambda}}, 0) dx + \int_0^1 1_A(0, y^{\frac{1}{1-\lambda}}) dy - \int_0^1 1_A(x^{\frac{1}{\lambda}}, x^{\frac{1}{1-\lambda}}) dx\}$$

contain the class  $\mathcal{P}$  of all closed pavements of  $[0, 1]^2$ . It is otherwise a monotone class (or  $\lambda$ -system). Hence as the class  $\mathcal{P}$  of closed pavement is a  $\pi$ -system, the class monotone theorem ensure that  $\mathcal{E}$  contains the sigma-field generated by  $\mathcal{P}$ , that is  $\mathcal{B}([0, 1]^2)$ .

This result holds for simple  $f(x, y) = \sum_{i=1}^n \lambda_i 1_{A_i}$  where  $\lambda_i \in \mathbb{R}$  and  $A_i \in \mathcal{B}([0, 1]^2)$  for all  $i \in \{1, \dots, n\}$ . We then can prove the identity for positive measurable function by approximation with an increasing sequence of simple function using Beppo-Levy theorem. This identity is then extended to measurable function  $f : [0, 1]^2 \mapsto \mathbb{R}$  considering  $f = f_+ - f_-$  with  $f_+ = \max(f, 0)$  and  $f_- = \min(-f, 0)$ . We take the function bounded-measurable in order that the left hand size of the equality is well defined as a Lebesgue-Stieljes integral.  $\square$

Furthermore, as the limiting process is the linear transformation of a tight gaussian process, we know from [vdVW96] that it is Gaussian. Before going further, we want to detail the structure of the variance of the limiting process. Doing that, we introduce the following lemma :

**Lemma 2.** *Let  $(B_C(u, v))_{u, v \in [0, 1]^2}$  a brownian bridge with covariance function defined by :*

$$\mathbb{E}[B_C(u, v) B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v) C(u', v')$$

for each  $0 \leq u, v, u', v' \leq 1$ . Let  $a, b \in [0, 1]$  fixed, if  $a = 0$  or  $b = 0$ , then wet get the following equality :

$$\mathbb{E}[\int_0^1 B_C(u, a) du \int_0^1 B_C(b, u) du] = 0$$

*Démonstration.* Without loss of generality, suppose that  $a = 0$  and  $b \in [0, 1]$ . Using the linearity of the integral, we obtain :

$$\begin{aligned}\mathbb{E}\left[\int_0^1 B_C(u, 0)du \int_0^1 B_C(b, u)du\right] &= \mathbb{E}\left[\int_0^1 \int_0^1 B_C(u, 0)B_C(b, v)dudv\right] \\ &= \int_0^1 \int_0^1 \mathbb{E}[B_C(u, 0)B_C(b, v)]dudv\end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u, 0)B_C(b, v)] = C(u \wedge v, 0) - C(u, 0)C(b, v)$$

We recall that, by definition, a copula satisfy  $C(u, 0) = C(0, u) = 0$  for every  $u \in [0, 1]$ . Then, the equation below is equal to 0. Our conclusion directly follows.  $\square$

Using this lemma, we can infer the following proposition :

**Proposition 3.** *Let  $N_C(u, v)$  the process defined in equation (5) and  $a, b \in [0, 1]$  fixed. If  $a = 0$  or  $b = 0$ , then :*

$$\mathbb{E}\left[\int_0^1 N_C(u, a)du \int_0^1 N_C(b, u)du\right] = 0$$

With this proposition, we can infer a better form of the variance of our limiting process :

**Theorem 2.** *Let  $N_C(u, v)$  the process defined in equation (5) and  $J(x, y) = |x^\lambda - y^{1-\lambda}|$ , then :*

$$Var\left(\int_{[0,1]^2} N_C(u, v)dJ(u, v)\right) = Var\left(\int_0^1 N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) \quad (9)$$

*Démonstration.* Recall that, with  $J$  defined as in the statement that :

$$\int_{[0,1]^2} N_C(u, v)dJ(u, v) = \frac{1}{2} \int_0^1 N_C(0, v^{1/(1-\lambda)})dv + \frac{1}{2} \int_0^1 N_C(u^{1/\lambda}, 0)du - \int_0^1 N_C(u^{1/\lambda}, u^{1/(1-\lambda)})du$$

Taking the variance and using the proposition 3 gives that only the variance of the third term matters.  $\square$

We are now able to detail precisely the variance of the limiting process with a given Copula. This is the purpose of the following proposition :

**Proposition 4.** *Under the framework of theorem 2 and if we take  $C(u, v) = uv$ , the independent copula, then the variance of the lambda FMadogram has the following form*

$$Var\left(\int_{[0,1]^2} N_C(u, v)dJ(u, v)\right) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} - \frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} - \frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right)$$

with  $\lambda \in [0, 1]$

*Démonstration.* With direct computing and using the same techniques used in lemma 1, we obtain that :

$$\begin{aligned}Var\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)}\right) \\ Var\left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du\right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)}\right) \\ Var\left[\int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du\right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right) \\ cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du\right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)}\right) \\ cov\left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du\right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)}\right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)}\right) \\ cov\left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1)du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}})du\right] &= 0\end{aligned}$$

Using the identity  $Var(X - Y) = Var(X) + Var(Y) - 2cov(X, Y)$  gives the desired result.  $\square$

Combining theorem 1 and proposition 2 gives the following result :

**Proposition 5.** *Suppose that the assumption of theorem 1 holds. Let  $J$  be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then*

$$T^{1/2} \left( \frac{\sum_{t=1}^T J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) I_t J_t}{\sum_{t=1}^T I_t J_t} - \mathbb{E}[J(F(X), G(Y))] \right)$$

*converges in distribution to  $\int_{[0,1]^2} S_C(u, v) dJ(u, v)$  where  $S_C(u, v)$  is defined in theorem 1. The special case,  $J(x, y) = \frac{1}{2} |x^\lambda - y^{1-\lambda}|$  provide the weak of convergence of the  $\lambda$ -Madogram estimator defined by (7) :*

$$T^{1/2} \left( \hat{\nu}_T(\lambda) - \frac{1}{2} \mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \right)$$

*converge in distribution to  $\int_{[0,1]^2} S_C(u, v) dJ(u, v)$  where the latter integral satisfies :*

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

*for all bounded functions  $f$ .*

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