

Let (X, Y) be a bivariate random vector with joint distribution function $H(x, y)$ and continuous marginal distribution function $F(x)$ and $G(y)$. Its associated copula C is defined by $H(x, y) = C\{F(x), G(y)\}$. Since F and G are continuous, the copula C is unique and we can write $C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v))$ for $0 \leq u, v \leq 1$ and where $F^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | F(v) \geq u\}$ and $G^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | G(v) \geq u\}$ are the generalized inverse functions of F and G respectively. At each $t \in \{1, \dots, T\}$, we suppose that one of both entries may be missing. The observations consist of a sample of independent, identically distributed quadruples

$$(I_t, J_t, I_t X_t, J_t Y_t) \quad t \in 1, \dots, T$$

The indicator variables I_t (respectively J_t) is equal to 1 or 0 according to whether X_t or Y_t is observed or not. We suppose that the indicator functions I_t and J_t are independent. The probability of observing a realisation partially or completely is denoted by $p_X = \mathbb{P}(I_t = 1) > 0$ and $p_Y = \mathbb{P}(J_t = 1) > 0$.

We define :

$$C(u, v) = H(F^{\leftarrow}(u), G^{\leftarrow}(v)) = \varphi(H)(u, v)$$

and,

$$Z_T(u, v) = \sqrt{T}\{\hat{C}_T(u, v) - C(u, v)\} \quad (1)$$

where \hat{H}_T corresponds to the empirical distribution function of the sample $(X_1, Y_1), \dots, (X_T, Y_T)$

$$\hat{H}_T(u, v) = \frac{\sum_{t=1}^T 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t}{\sum_{t=1}^T I_t J_t}$$

We define also the corresponding empirical distribution functions in the case of missing data :

$$\begin{aligned} \hat{F}_T(u) &= \frac{\sum_{t=1}^T 1_{\{X_t \leq u\}} I_t}{\sum_{t=1}^T I_t} \\ \hat{G}_T(v) &= \frac{\sum_{t=1}^T 1_{\{Y_t \leq v\}} J_t}{\sum_{t=1}^T J_t} \end{aligned}$$

Condition 1. We suppose for all $t \in \{1, \dots, T\}$:

$$\mathbb{E}[X_t Y_t I_t J_t] = \mathbb{E}[X_t Y_t] \mathbb{E}[I_t J_t]$$

Furthermore, we suppose that there exist at least one $t \in \{1, \dots, T\}$ such that $I_t J_t \neq 0$.

Proposition 1. Under hypothesis 1, $\hat{H}_T, \hat{F}_T, \hat{G}_T$ are consistant estimators of H, F, G .

Démonstration. We check the consistency for \hat{H}_T . By independence, we have

$$\mathbb{E}[T^{-1} \sum_{t=1}^T I_t J_t] = T^{-1} \sum_{t=1}^T \mathbb{E}[I_t] \mathbb{E}[J_t] = p_X p_Y$$

So, by applying the law of large numbers, we have :

$$T^{-1} \sum_{t=1}^T I_t J_t \longrightarrow p_X p_Y \quad a.s. \quad as \quad T \rightarrow \infty$$

Then, we now use the first hypothesis to get :

$$T^{-1} \sum_{t=1}^T \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t] = T^{-1} \sum_{t=1}^T \mathbb{E}[1_{\{X_t \leq u, Y_t \leq v\}}] \mathbb{E}[I_t J_t] = H(x, y) p_X p_Y$$

By applying again the law of large numbers, we derive :

$$\sum_{t=1}^T 1_{\{X_t \leq u, Y_t \leq v\}} I_t J_t \longrightarrow H(x, y) p_X p_Y \quad a.s. \quad as \quad T \rightarrow \infty$$

We can now apply the continuous mapping theorem to the function $f : (x, y) \mapsto \frac{x}{y}$ which are continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \setminus 0$ to conclude that :

$$\hat{H}_T(x, y) \longrightarrow H(x, y) \quad a.s. \quad as \quad T \rightarrow \infty$$

□

Condition 2. 1. The bivariate distribution function H has continuous margins F, G and copula C .

2. The first order partial derivatives $\dot{C}_1(u, v) = \frac{\partial C}{\partial u}(u, v)$ and $\dot{C}_2(u, v) = \frac{\partial C}{\partial v}(u, v)$ exists and is continuous on the set $\{(u, v) \in [0, 1]^2, 0 < u, v < 1\}$

Condition 3. There exists $\gamma_t > 0$ and $r_t > 0$ such that $r_t \longrightarrow \infty$ as $t \rightarrow \infty$ such that in the space $l^\infty(\mathbb{R}^2) \otimes (l^\infty(\mathbb{R}), l^\infty(\mathbb{R}))$ equipped with the topology of uniform convergence, we have the joint weak convergence

$$(r_t(\hat{H}_T - H); r_t(\hat{F}_t - F), r_t(\hat{G}_t - G)) \rightsquigarrow (\alpha \circ (F, G), \beta_1 \circ F, \beta_2 \circ G)$$

The stochastic processes α and β_j take values in $l^\infty([0, 1]^2)$ and $l^\infty([0, 1])$ respectively, and are such that $\alpha \circ F$ and $\beta_j \circ F_j$ have continuous trajectories on $[-\infty, \infty]^2$ and $[-\infty, \infty]$ almost surely.

Theorem 1 (Theorem 2.3 and example 3.5 in [Seg14]). If conditions 2.1 and 2.2 holds, then uniformly in $u \in [0, 1]^2$,

$$r_T \{\hat{C}_T(u, v) - C(u, v)\} = r_T \{\hat{H}_T((F, G)^\leftarrow(u, v) - C(u, v)\} \quad (2)$$

$$- \dot{C}_1(u, v) r_T \{\hat{F}_T(F^\leftarrow(u)) - u\} 1_{(0,1)}(u) \quad (3)$$

$$- \dot{C}_2(u, v) r_T \{\hat{G}_T(G^\leftarrow(v)) - v\} 1_{(0,1)}(v) + o_{\mathbb{P}}(1) \quad (4)$$

as $T \rightarrow \infty$. Hence in $l^\infty([0, 1]^2)$ equipped with the supremum norm, as $T \rightarrow \infty$,

$$(r_T \{\hat{C}_T(u, v) - C(u, v)\})_{u,v \in [0,1]^2} \rightsquigarrow (\alpha(u, v) - \dot{C}_1(u, v) \beta_1(u) - \dot{C}_2(u, v) \beta_2(v))_{u,v \in [0,1]^2}$$

We denote by $N_C(u, v)$ the process defined on the right-hand side in the weak convergence from above.

Remark 1. If we consider the empiric copula, theorem 1 gives us the weak convergence of this process to a brownian bridge $N_C(u, v)$ defined by, $\forall(u, v) \in [0, 1]$

$$N_C(u, v) = B_C(u, v) - \dot{C}_1(u, v)B_C(u, 1) - \dot{C}_2(u, v)B_C(1, v) \quad (5)$$

where B_C is a brownian bridge in $[0, 1]^2$ with covariance function

$$\mathbb{E}[B_C(u, v)B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$$

for each $0 \leq u, v, u', v' \leq 1$. This result is well known since 2004 due to [FRW04].

In our specific case with missing data, (ref) shows that $r_T\{\hat{C}_T(u, v) - C(u, v)\}$ is weakly convergent toward $\alpha(u, v) - \dot{C}_1(u, v)\beta_1(u) - \dot{C}_2(u, v)\beta_2(v)$ where $\beta_1(u) = p_X^{-1}\mathbb{G}(1_{X \leq F^{\leftarrow}(u)} - u1_{I=1})$, $\beta_2(v) = p_Y^{-1}\mathbb{G}(1_{Y \leq G^{\leftarrow}(v)} - v1_{J=1})$ and $\alpha(u, v) = (p_X p_Y)^{-1}\mathbb{G}(1_{X \leq F^{\leftarrow}(u)} 1_{Y \leq G^{\leftarrow}(v)} - C(u, v)1_{I=1}1_{J=1})$. Furthermore, by these expressions, we can detail the structure of the covariance matrix between the three processes.

Definition 1. Let $(X_1, Y_1), \dots, (X_T, Y_T)$ a T bivariate random vectors with unknown margins F and G . A λ -FMadogram is the quantity defined by :

$$\nu(\lambda) = \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \quad (6)$$

We estimative the λ -FMadogram with the following quantity :

$$\hat{\nu}_T(\lambda) = \frac{1}{2 \sum_{t=1}^T I_t J_t} \sum_{t=1}^T |\hat{F}_T^\lambda(X_t) - \hat{G}_T^{1-\lambda}(Y_t)| I_t J_t \quad (7)$$

Proposition 2 (Proposition 3 of [NGCD09]). Suppose that conditions 2 holds and that $\sum_{t=1}^T I_t J_t = T$ (no missing data). Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then

$$T^{-1/2} \sum_{t=1}^T (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]} N_C(u, v) dJ(u, v)$ where $N_C(u, v)$ is defined by equation (5) and the integral is well defined as a Lebesgue-Stieltjes integral (see section 5.2 of [Bre20]). The special case, $J(x, y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$ provide the weak of convergence of the λ -Madogram estimator defined by (7) :

$$T^{1/2} \{ \hat{\nu}_T(\lambda) - \frac{1}{2}\mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \}$$

converge in distribution to $\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

for all bounded functions f .

Furthermore, as the limiting process is the linear transformation of a tight gaussian process, we know from [vdVW96] that it is Gaussian. Before going further, we want to detail the structure of the variance of the limiting process. Doing that, we introduce the following lemma :

Lemma 1. Let $(B_C(u, v))_{u, v \in [0, 1]^2}$ a brownian bridge with covariance function defined by :

$$\mathbb{E}[B_C(u, v)B_C(u', v')] = C(u \wedge u', v \wedge v') - C(u, v)C(u', v')$$

for each $0 \leq u, v, u', v' \leq 1$. Let $a, b \in [0, 1]$ fixed, if $a = 0$ or $b = 0$, then wet get the following equality :

$$\mathbb{E}\left[\int_0^1 B_C(u, a)du \int_0^1 B_C(b, u)du\right] = 0$$

Démonstration. Without loss of generality, suppose that $a = 0$ and $b \in [0, 1]$. Using the linearity of the integral, we obtain :

$$\begin{aligned} \mathbb{E}\left[\int_0^1 B_C(u, 0)du \int_0^1 B_C(b, u)du\right] &= \mathbb{E}\left[\int_0^1 \int_0^1 B_C(u, 0)B_C(b, v)dudv\right] \\ &= \int_0^1 \int_0^1 \mathbb{E}[B_C(u, 0)B_C(b, v)]dudv \end{aligned}$$

We then use the definition of the covariance function of our Brownian bridge, we have

$$\mathbb{E}[B_C(u, 0)B_C(b, v)] = C(u \wedge v, 0) - C(u, 0)C(b, v)$$

We recall that, by definition, a copula satisfy $C(u, 0) = C(0, u) = 0$ for every $u \in [0, 1]$. Then, the equation below is equal to 0. Our conclusion directly follows. \square

Using this lemma, we can infer the following proposition :

Proposition 3. Let $N_C(u, v)$ the process defined in equation (5) and $a, b \in [0, 1]$ fixed. If $a = 0$ or $b = 0$, then :

$$\mathbb{E}\left[\int_0^1 N_C(u, a)du \int_0^1 N_C(b, u)du\right] = 0$$

With this proposition, we can infer a better form of the variance of our limiting process :

Theorem 2. Let $N_C(u, v)$ the process defined in equation (5) and $J(x, y) = |x^\lambda - y^{1-\lambda}|$, then :

$$Var\left(\int_{[0, 1]^2} N_C(u, v)dJ(u, v)\right) = Var\left(\int_0^1 N_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}})du\right) \quad (8)$$

Démonstration. Recall that, with J defined as in the statement that :

$$\int_{[0, 1]^2} N_C(u, v)dJ(u, v) = \frac{1}{2} \int_0^1 N_C(0, v^{1/(1-\lambda)})dv + \frac{1}{2} \int_0^1 N_C(u^{1/\lambda}, 0)du - \int_0^1 N_C(u^{1/\lambda}, u^{1/(1-\lambda)})du$$

Taking the variance and using the proposition 3 gives that only the variance of the third term matters. \square

We are now able to detail precisely the variance of the limiting process with a given Copula. This is the purpose of the following proposition :

Proposition 4. Under the framework of theorem 2 and if we take $C(u, v) = uv$, the independent copula, then the variance of the lambda FMadogram has the following form

$$\text{Var}(\int_{[0,1]^2} N_C(u, v) dJ(u, v)) = \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} - \frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} - \frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right)$$

with $\lambda \in [0, 1]$

Démonstration. With direct computing and using the same techniques used in lemma 1, we obtain that :

$$\begin{aligned} \text{Var} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1}{1+2\lambda(1-\lambda)} \right) \\ \text{Var} \left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1) du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} \right) \\ \text{Var} \left[\int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}}) du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right) \\ \text{cov} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du, \int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1) du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{1-\lambda}{1+\lambda+2\lambda(1-\lambda)} \right) \\ \text{cov} \left[\int_0^1 B_C(u^{\frac{1}{\lambda}}, u^{\frac{1}{1-\lambda}}) du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}}) du \right] &= \left(\frac{\lambda(1-\lambda)}{1+\lambda(1-\lambda)} \right)^2 \left(\frac{\lambda}{2-\lambda+2\lambda(1-\lambda)} \right) \\ \text{cov} \left[\int_0^1 u^{\frac{1}{1-\lambda}} B_C(u^{\frac{1}{\lambda}}, 1) du, \int_0^1 u^{\frac{1}{\lambda}} B_C(1, u^{\frac{1}{1-\lambda}}) du \right] &= 0 \end{aligned}$$

Using the identity $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{cov}(X, Y)$ gives the desired result. \square

Combining theorem 1 and proposition 2 gives the following result :

Proposition 5. Suppose that the assumption of theorem 1 holds. Let J be a function of bounded variation, continuous from above and with discontinuities of the first kind. Then

$$\left(\sum_{t=1}^T I_t J_t \right)^{-1/2} \sum_{t=1}^T (J(\hat{F}_T(X_t), \hat{G}_T(Y_t)) I_t J_t - \mathbb{E}[J(F(X), G(Y))])$$

converges in distribution to $\int_{[0,1]^2} N_C(u, v) dJ(u, v)$ where $N_C(u, v)$ is defined in theorem 1. The special case, $J(x, y) = \frac{1}{2}|x^\lambda - y^{1-\lambda}|$ provide the weak of convergence of the λ -Madogram estimator defined by (7) :

$$\left(\sum_{t=1}^T I_t J_t \right)^{1/2} \left\{ \hat{\nu}_T(\lambda) - \frac{1}{2} \mathbb{E}[|F^\lambda(X) - G^{1-\lambda}(Y)|] \right\}$$

converge in distribution to $\int_{[0,1]} N_C(u, v) dJ(u, v)$ where the latter integral satisfies :

$$\int_{[0,1]^2} f(x, y) dJ(x, y) = \frac{1}{2} \int_0^1 f(0, y^{1/(1-\lambda)}) dy + \frac{1}{2} \int_0^1 f(x^{1/\lambda}, 0) dx - \int_0^1 f(x^{1/\lambda}, x^{1/(1-\lambda)}) dx$$

for all bounded functions f .

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