Introduction

Definitions and Notation

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathbf{X} = (X_1, \dots, X_d)$ be a d-dimensional random vector with values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. This random vector has a joint distribution function G and the margins of G are denoted by $F_j(x) = \mathbb{P}\{X_j \leq x_j\}$ for all $x_j \in \mathbb{R}$ and $j \in \{1, \dots, d\}$. A function $C : [0,1]^d \to [0,1]$ is called a d-dimensional copula if it is the restriction to $[0,1]^d$ of a distribution function whose margins are given by the uniform distribution on the interval $[0,1]$. Since the work of [Sklar, 1959], it is well known that every distribution function G can be decomposed as $G(\mathbf{x}) = C(F_1(x_d), \dots, F_d(x_d))$, for all $\mathbf{x} \in \mathbb{R}^d$. Under the framework of extreme, the notion of copulas leads to the so-called extreme value copulas.

Definition 1 ([Gudendorf and Segers, 2010])**.** *A d-dimensional copula C is an extreme-value copula if and only if it admits a representation of the form*

$$C(\boldsymbol{u}) = exp\left(-\ell(-log(u_1), \dots, -log(u_d)\right), \quad \boldsymbol{u} \in (0,1]^d, \tag{1}$$

with $\ell : [0,\infty)^d \to [0,\infty)$ the stable tail dependence function.

We denote by Δ^{d-1} the $(d-1)$-simplex unit simplex. The tail dependence function ℓ is convex, homogeneous of order one, that is $\ell(cx_1, \dots, cx_d) = c\ell(x_1, \dots, x_d)$ for $c > 0$ and satisfies $\max(x_1, \dots, x_d) \leq \ell(x_1, \dots, x_d) \leq x_1 + \dots + x_d$, $\forall (x_1, \dots, x_d) \in [0,\infty)^d$. By homogeneity, it is characterized by the *Pickands dependence function* $A : \Delta^{d-1} \to [1/d, 1]$, which is the restriction of ℓ to the unit simplex :

$$\ell(x_1, \dots, x_d) = (x_1 + \dots + x_d)A(w_1, \dots, w_d), \quad w_j = \frac{x_j}{x_1 + \dots + x_d}, \tag{2}$$

for $(x_1, \dots, x_d) \in [0,\infty)^d \setminus \{0\}$. Notice that, for every $\mathbf{w} \in \Delta^{d-1}$

$$C(u^{w_1}, \dots, u^{w_d}) = u^{A(\mathbf{w})}. \tag{3}$$

Assume that its copula C, is an extreme-value copula with stable tail dependence function ℓ and Pickands dependence function A. The $\mathbf{w}$-madogram is defined as follows.

Definition 2 ([Marcon et al., 2017])**.** *Let $\boldsymbol{X}$ be a random vector with marginal distribution functions $F_1, \dots F_d$. The multivariate $\mathbf{w}$-madogram ($\mathbf{w} \in \Delta^{d-1}$), denoted by $\nu(\mathbf{w})$, is defined as*

$$\nu(\mathbf{w}) = \mathbb{E}\left[\bigvee_{j=1}^{d} \{F_j(X_j)\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d} \{F_j(X_j)\}^{1/w_j}\right]$$

if $w_j = 0$ and $0 < u < 1$, then $u^{1/w_j} = 0$ by convention.

Starting from independent and identically distributed *i.i.d.* copies $\mathbf{X}_1, \dots, \mathbf{X}_n$ of $\mathbf{X}$, suppose we observe a $2d$-tuple such as

$$(\mathbf{I}_i \mathbf{X}_i, \mathbf{X}_i), \quad i \in \{1, \dots, n\}, \tag{4}$$

where $\mathbf{I}_i \mathbf{X}_i = (X_{i,1} I_{i,1}, \dots, X_{i,d} I_{i,d})$ and $I_{i,j} = 0$ if $X_{i,j}$ is missing, otherwise $I_{i,j} = 1$, *i.e.* at each $i \in \{1, \dots, n\}$, several entries may be missing. The probability of observing a realization partially or completely is denoted by $p_j = \mathbb{P}(I_{i,j} = 1) > 0$, $\forall j \in \{1, \dots, d\}$ and by $\mathbf{p} = \mathbb{P}(I_{i,1} = 1, \dots, I_{i,d} = 1) > 0$. Let us now define the empirical cumulative distribution of X_i (resp. G) in case of missing data,

$$\hat{F}_{n,j}(x_j) = \frac{\sum_{i=1}^n \mathbb{1}_{\{X_i \leq x_i\}} I_{i,j}}{\sum_{i=1}^n I_{i,j}}, \quad \forall x_j \in \mathbb{R}. \tag{5}$$
$$\hat{G}_n(\mathbf{x}) = \frac{\sum_{i=1}^n \mathbb{1}_{\{X_{i,1} \leq x_1, \dots, X_{i,d} \leq x_d\}} \Pi_{i=1}^d I_{i,j}}{\sum_{i=1}^n \Pi_{j=1}^d I_{i,j}}, \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

Here, we weight the estimator by the number of observed data which is a natural estimator if divided by n of probabilities of missing. We have all tools in hand to recall the definition of the *hybrid copula estimator* introduced by [Segers, 2015] under missing data framework,

$$\hat{C}_n^{\mathcal{H}}(u, v) = \hat{G}_n(\hat{F}_{n,1}^{\leftarrow}(u_1), \dots, \hat{F}_{n,d}^{\leftarrow}(u_d)), \quad \forall \mathbf{u} \in [0,1]^d.$$

Here, we write the generalized inverse function of F as $F^{\leftarrow}(u) = \inf\{v \in \mathbb{R} | F(v) \geq u\}$ where $0 < u, v < 1$. The normalized estimation error of the hybrid copula estimator is

$$\mathbb{C}_n^{\mathcal{H}}(\mathbf{u}) = \sqrt{n} \left(\hat{C}_n^{\mathcal{H}}(\mathbf{u}) - C(\mathbf{u}) \right), \quad \mathbf{u} \in [0,1]^d.$$

In order to shorten formulas, the notation

$$\mathbf{u}_j(t) := (u_1, \dots, u_{i-1}, t, u_{i+1}, \dots, u_d),$$
$$\mathbf{u}_{jk}(s,t) := (u_1, \dots, u_{i-1}, s, u_{i+1}, \dots, u_{j-1}, t, u_{j+1}, \dots, u_d),$$

will be adopted for $s, t \in [0,1]$, $(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d) \in [0,1]^d$ and $j < k$.

Also, the following notations are used. Given $\mathcal{X} \subset \mathbb{R}^2$, let $\ell^\infty(\mathcal{X})$ denote the spaces of bounded real-valued function on $\mathcal{X}$. For $f : \mathcal{X} \to \mathbb{R}$, let $||f||_\infty = sup_{x \in \mathcal{X}} |f(x)|$. Here, we use the abbreviation $Q(f) = \int f dQ$ for a given measurable function f and signed measure Q. The arrows $\overset{a.s.}{\to}$, $\overset{d}{\to}$ denote almost sure convergence and convergence in distribution of random vectors. Weak convergence of a sequence of maps will be understood in the sense of J.Hoffman-Jørgensen (see Part 1 in the monograph by [van der Vaart and Wellner, 1996]).

Given that $t \in \mathbb{N}^*, X, X_t$ are maps from $(\Omega, \mathcal{A}, \mathbb{P})$ into a metric space $\mathcal{X}$ and that X is Borel measurable, $(X_t)_{t\geq 1}$ is said to converge weakly to X if $\mathbb{E}^* f(X_t) \to \mathbb{E} f(X)$ for every bounded continuous real-valued function f defined on $\mathcal{X}$, where $\mathbb{E}^*$ denotes outer expectation in the event that X_t may not be Borel measurable. In what follows, weak convergence is denoted by $X_t \rightsquigarrow X$.

1 Non parametric estimation of the Madogram with missing data

Under the notation of the introduction, we assume that the copula C is of extreme value type as in Definition 1. On the condition that the first-order partial derivatives of the copula function C exists and are continuous on a subset of the unit hypercube, [Segers, 2012] obtained weak convergence of the normalized estimation error of the empirical copula process. To obtain this condition, we make the following assumption as suggested in [Segers, 2012] in Example 5.3.

Assumption A.

(i) The bivariate distribution function G has continuous margins $F_1, \dots, F_d$

(ii) For every $j \in \{1, \dots, d\}$, the first-order partial derivative $\dot{\ell}_j$ of ℓ with respect to x_j exists ans is continuous on set $\{x \in [0,\infty)^d : x_j > 0\}$.

The Assumption A *(i)* guarantees that the representation $G(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d))$ is unique on the range of $(F_1, \dots, F_d)$. Under the Assumption A *(ii)*, the first-order partial derivatives of C with respect to u_j exists and are continuous on the set $\{\mathbf{u} \in [0,1]^d : 0 < u_j < 1\}$. We now define our estimator of Equation (2) in the general context (allowing missing data).

Definition 3. *Let $(\boldsymbol{I}_i \boldsymbol{X}_i, \boldsymbol{I}_i)_{i=1}^n$ be a sample given by Equation (4), we define the hybrid estimator of the* $\mathbf{w}$*-FMadogram by*

$$\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) = \frac{1}{\sum_{i=1}^n \Pi_{j=1}^d I_{i,j}} \sum_{i=1}^n \left[\bigvee_{j=1}^d \left\{ \hat{F}_{n,j}(X_{i,j}) \right\}^{1/w_j} - \frac{1}{d} \sum_{j=1}^d \left\{ \hat{F}_{n,j}(X_{i,j}) \right\}^{1/w_j} \right] \Pi_{j=1}^d I_{i,j}, \tag{6}$$

where $\hat{F}_{n,j}(x_j)$ are defined on Equation (5).

The idea raised here is to estimate the margins by the complete series for each variables but estimate $\nu(\mathbf{w})$ only based on the time period where all series were recorded simultaneously. One may verify that in the complete data framework, *i.e.* when $p_j = 1, \forall j \in \{1, \dots, d\}$

and $\mathbf{p} = 1$ we retrieve the $\mathbf{w}$-madogram such as defined in [Marcon et al., 2017], namely

$$\hat{\nu}_n(\mathbf{w}) = \frac{1}{n} \sum_{m=1}^{n} \left[\bigvee_{i=1}^{d} \left\{ \hat{F}_{n,i}(X_{m,i}) \right\}^{1/w_j} - \frac{1}{d} \sum_{i=1}^{d} \left\{ \hat{F}_{n,i}(X_{m,i}) \right\}^{1/w_j} \right],$$

with $\hat{F}_{n,j}(x_j)$ the empirical cumulative distribution function of X_j.

Remark 1. *Our estimator defined in (6) does not verify $\hat{\nu}_T^{\mathcal{H}}(\boldsymbol{e}_j) = (d-1)/2d$ while $\nu(\boldsymbol{e}_j)$ does. In addition, the variance at $\boldsymbol{e}_j$ does not equal 0. Indeed, suppose that we evaluate this statistic at $\boldsymbol{w} = \boldsymbol{e}_j$, we thus obtain the following quantity :*

$$\hat{\nu}_n^{\mathcal{H}}(\boldsymbol{e}_j) = \frac{1}{\sum_{i=1}^{n} \Pi_{j=1}^{d} I_{i,j}} \sum_{i=1}^{n} \left[\hat{F}_{n,j}(X_{i,j}) - \frac{1}{d} \hat{F}_{n,j}(X_{i,j}) \right] \Pi_{j=1}^{d} I_{i,j}.$$

In this situation, the sample $(X_{i,-j})_{i=1}^{n}$ is taken into account through the indicators sequence $(I_{i,-j})_{i=1}^{n}$ and induce a supplementary variance when estimating.

We can force our estimator as in [Naveau et al., 2009] to satisfy these endpoint conditions. This leads to the following corrected estimator.

Definition 4. *Under the notation of Definition 3, we define the hybrid corrected estimator of the $\boldsymbol{w}$-FMadogram by*

$$\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) = \hat{\nu}_n(\mathbf{w}) + \sum_{j=1}^{d} \left[\frac{w_j(d-1)}{d} \frac{w_j}{1+w_j} - \frac{w_j(d-1)}{d \sum_{m=1}^{n} \Pi_{j=1}^{d} I_{i,j}} \sum_{i=1}^{n} \left\{ \hat{F}_{n,j}(X_{i,j}) \right\}^{1/w_j} \Pi_{j=1}^{d} I_{i,j} \right]. \tag{7}$$

Let us now introduce a condition on the missing mechanism :

Assumption B. *We suppose for all $i \in \{1, \dots, n\}$, the vector $(\boldsymbol{I}_i)$ and $(\boldsymbol{X}_i)$ are independent, the data are missing completely at random (**MCAR**). Furthermore, we suppose that there exists at least one $i \in \{1, \dots, n\}$ such that $\Pi_{j=1}^{d} I_{i,j} \neq 0$.*

Under this Assumption, we state the strong consistency of our hybrid estimator of the $\mathbf{w}$-FMadogram.

Proposition 1 (**Strong consistency**). *Let $(\boldsymbol{I}_i \boldsymbol{X}_i, \boldsymbol{X}_i)_{i=1}^{n}$ a i.i.d sample given by Equation (4). We have, under Assumption B for a fixed $\mathbf{w} \in \Delta^{d-1}$, as $n \to \infty$*

$$\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) \xrightarrow{a.s.} \nu(\mathbf{w}), \quad \hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) \xrightarrow{a.s.} \nu(\mathbf{w}).$$

Details on the proof are given in Section 2. We present with Theorem 1 our main result concerning the weak convergence of the following processes

$$\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\lambda)-\nu(\lambda)\right), \quad \sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}}(\lambda)-\nu(\lambda)\right). \tag{8}$$

Without missing data, the weak convergence of the normalized estimation error of the empirical copula process has been proved by [Fermanian et al., 2004] under a more restrictive condition than Assumption A. With the hybrid copula estimator, we need a technical assumption in order to guarantee the weak convergence of the process $\mathbb{C}_T^{\mathcal{H}}$ (see [Segers, 2015]). Before spelled it, we note for convenience the marginal distribution and quantile functions into vector valued functions $\mathbf{F}$ and $\mathbf{F}^{\leftarrow}$:

$$\mathbf{F}(\mathbf{x})=(F_1(x_1),\ldots,F_d(x_d)), \quad \mathbf{x}\in\mathbb{R}^d, \quad \mathbf{F}^{\leftarrow}(\mathbf{u})=(F_1^{\leftarrow}(u_1),\ldots,F^{\leftarrow}(u_d)), \quad \mathbf{u}\in[0,1]^d.$$

Assumption C. *In the space $\ell^\infty(\mathbb{R}^d)\otimes(\ell^\infty(\mathbb{R}),\ldots,\ell^\infty(\mathbb{R}))$ equipped with the topology of uniform convergence, we have the joint weak convergence*

$$\begin{aligned}\Big(\sqrt{n}(\hat{G}_n-G);\sqrt{n}(\hat{F}_{n,1}-F)_1,\ldots,\sqrt{n}(\hat{F}_{n,d}-F_d)\Big)&\\ \rightsquigarrow\left(\alpha\circ\boldsymbol{F},\beta_1\circ F_1,\ldots,\beta_d\circ F_d\right).&\end{aligned}$$

The stochastic processes α and $\beta_j, j\in\{1,\ldots,d\}$ take values in $\ell^\infty([0,1]^d)$ and $\ell^\infty([0,1])$ respectively, and are such that $\alpha\circ F$ and $\beta_j\circ F_j$ have continuous trajectories on $[-\infty,\infty]^d$ and $[-\infty,\infty]$ almost surely.

Under Assumptions A and C, the stochastic process $\mathbb{C}_T^{\mathcal{H}}$ converges weakly to the tight Gaussian process S_C defined by,

$$S_C(\mathbf{u})=\alpha(\mathbf{u})-\sum_{i=1}^d\dot{C}_j(\mathbf{u})\beta_j(u_j), \quad \forall\mathbf{u}\in[0,1]^d.$$

Considering the same statistical framework and missing mechanism as [Segers, 2015] (in Example 3.5) but in higher dimension, we show that the processes α, β_j takes the following closed form

$$\begin{aligned}\beta_j(u_j)&=p_j^{-1}\mathbb{G}\left(\mathbb{1}_{X_j\le F_j^{\leftarrow}(u_j),I_j=1}-u_j\mathbb{1}_{I_j=1}\right),\\ \alpha(\mathbf{u})&=p^{-1}\mathbb{G}\left(\mathbb{1}_{\mathbf{X}\le\mathbf{F}^{\leftarrow}(\mathbf{u})}\mathbb{1}_{\mathbf{I}=\mathbf{1}}-C(\mathbf{u})\mathbb{1}_{\mathbf{I}=\mathbf{1}}\right),\end{aligned}$$

Where $\mathbb{G}$ is a tight Gaussian process. Furthermore, we are able to compute their covariance functions given in the following lemma.

Lemma 1. *The covariance function of the process $\beta_i(u_i)$, $\alpha(\boldsymbol{u})$ are, for $(\boldsymbol{u},\boldsymbol{v},v_k)\in[0,1]^{2d+1}$,* *and for* $j < k$

$$
\begin{aligned}
cov\left(\beta_j(u_j), \beta_j(v_j)\right) &= p_j^{-1}\left(u_j \wedge v_j - u_j v_j\right), \\
cov\left(\beta_j(u_j), \beta_k(v_k)\right) &= \frac{p_{jk}}{p_j p_k}\left(C(\boldsymbol{1}_{j,k}(u_j, v_k)) - u_j v_k\right),
\end{aligned}
$$

and

$$
\begin{aligned}
cov\left(\alpha(\boldsymbol{u}), \alpha(\boldsymbol{v})\right) &= \boldsymbol{p}^{-1}\left(C(\boldsymbol{u} \wedge \boldsymbol{v}) - C(\boldsymbol{u})C(\boldsymbol{v})\right), \\
cov\left(\alpha(\boldsymbol{u}), \beta_j(v_j)\right) &= p_j^{-1}\left(C(\boldsymbol{u}_j(u_j \wedge v_j)) - C(\boldsymbol{u})v_j\right).
\end{aligned}
$$

Where we denote by $\mathbf{u} \wedge \mathbf{v}$ the vector of componentwise minima and $p_{jk} = \mathbb{P}(I_j = 1, I_k = 1)$. Proof of Lemma 1 is deferred to Section 2.

We have all tools in hand to consider the weak convergence of the stochastic processes in Equation (8).

Theorem 1 (**Functional central limit theorem with missing data**). *Under Assumptions A, B, C we have the weak convergence in* $\ell^{\infty}(\Delta^{d-1})$ *for the hybrid estimator defined in (6) and (7), as* $n \to \infty$,

$$
\begin{aligned}
\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right) \rightsquigarrow &\left(\frac{1}{d}\sum_{j=1}^{d}\int_{[0,1]} \alpha(\boldsymbol{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx \right. \\
&\left. - \int_{[0,1]} S_C(x^{w_1}, \dots, x^{w_d})dx\right)_{\lambda \in [0,1]},
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right) \rightsquigarrow &\left(\frac{1}{d}\sum_{j=1}^{d}(1 + w_j(d-1))\int_{[0,1]} \alpha(\boldsymbol{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx \right. \\
&\left. - \int_{[0,1]} S_C(x^{w_1}, \dots, x^{w_d})dx\right)_{\lambda \in [0,1]}.
\end{aligned}
$$

Proof is postponed in Section 2.

Ici[1], nous nous posons dans le cas de données complètes. Le cas général peut être déduit ensuite, mais il faut d'abord voir si le raisonnement est correct. Pour un $\mathbf{w} \in \Delta^{d-1}$ fixé, la loi de $\sqrt{n}(\nu_n(\mathbf{w}) - \nu(\mathbf{w}))$ suit une Gaussienne centrée (car transformation linéaire continue

[1] J'écris en français tout paragraphes qui vont être modifiés

d'un processus Gaussien tendu) et sa variance est donnée par :

$$Var(\int_{[0,1]} N_C(u^{w_1}, \dots, u^{w_d}) du)$$

Proposition 2 (**Boulin, 2021**). *Je pense avoir une forme close de la variance et celle-ci est décomposée comme suit :*

$$Var(\int_{[0,1]} N_C(u^{w_1}, \dots, u^{w_d}) du) = \sigma_1^2(\boldsymbol{w}) + \sum_{i=1}^d \gamma_i^2(\boldsymbol{w}) - 2\sum_{i=1}^d \sigma_{1i}(\boldsymbol{w}) + 2\sum_{i<j} \gamma_{ij}(\boldsymbol{w}).$$

Technical details are available on Section 2.

2 Proof

Lemma 2. *We have,* $\forall i \in \{1, \dots, n\}$

$$\left| \bigvee_{j=1}^d \left\{ \hat{F}_{n,j}(X_{i,j}) \right\}^{1/w_j} - \bigvee_{j=1}^d \{F_j(X_j)\}^{1/w_j} \right| \leq \sup_{i \in \{1,\dots,d\}} \left| \hat{F}_{n,j}(X_{i,j})^{1/w_j} - F_j(X_j)^{1/w_j} \right|.$$

Proof The lemma becomes trivial once we write, $\forall i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$

$$\begin{aligned} \left\{ \hat{F}_{n,j}(X_{i,j}) \right\}^{1/w_j} &= F_j(X_j)^{1/w_j} + \hat{F}_{n,j}(X_{i,j})^{1/w_j} - F_j(X_j)^{1/w_j}, \\ &\leq F_j(X_j)^{1/w_j} + \sup_{j \in \{1,\dots,d\}} \left| \hat{F}_{n,j}(X_{i,j})^{1/w_j} - F_j(X_j)^{1/w_j} \right|, \\ &\leq \bigvee_{j=1}^d \{F_j(X_j)\}^{1/w_j} + \sup_{j \in \{1,\dots,d\}} \left| \hat{F}_{n,j}(X_{i,j})^{1/w_j} - F_j(X_j)^{1/w_j} \right|. \end{aligned}$$

Taking the max over $j \in \{1, \dots, d\}$ gives

$$\bigvee_{i=1}^d \left\{ \hat{F}_{n,j}(X_{i,j}) \right\}^{1/w_j} - \bigvee_{j=1}^d \{F_j(X_j)\}^{1/w_j} \leq \sup_{j \in \{1,\dots,d\}} \left| F_{n,j}(X_{i,j})^{1/w_j} - F_j(X_j)^{1/w_j} \right|.$$

Moreover, by symmetry of $\hat{F}_{n,j}$ and F_j, the second ones follows similarly. □

Proof of Proposition 1 We write, for notational convenience $n_i = \Pi_{j=1}^d I_{i,j}$ and $N = \sum_{i=1}^n n_i$. We prove it for $\hat{\nu}_n^{\mathcal{H}}(\mathbf{w})$ as the strong consistency for $\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w})$ use the same arguments. The estimator $\hat{\nu}_n(\mathbf{w})$ is strongly consistent since it holds

$$\begin{aligned}|\hat{\nu}_n(\mathbf{w}) - \nu(\mathbf{w})| &= \left|\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu_n(\mathbf{w}) + \nu_n(\mathbf{w}) - \nu(\mathbf{w})\right|, \\ &\leq \left|\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu_n(\mathbf{w})\right| + |\nu_n(\mathbf{w}) - \nu(\mathbf{w})|,\end{aligned}$$

where

$$\nu_n(\mathbf{w}) = \frac{1}{N}\sum_{i=1}^{n}\left(\bigvee_{j=1}^{d}\{F_j(X_{i,j})\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d}\{F_j(X_{i,j})\}^{1/w_j}\right)n_i$$

By direct application of Assumption B and the law of large number, we have that

$$|\nu_n(\mathbf{w}) - \nu(\mathbf{w})| \overset{a.s.}{\to} 0$$

For the second term, we write :

$$\begin{aligned}|\hat{\nu}_n(\mathbf{w}) - \nu(\mathbf{w})| \leq & \frac{1}{N}\sum_{i=1}^{n}\left|\bigvee_{j=1}^{d}\hat{F}_{n,j}(X_{i,j})^{1/w_j} - \bigvee_{j=1}^{d}F_j(X_{i,j})^{1/w_j}\right|n_i \\ &+ \frac{1}{Nd}\sum_{i=1}^{n}\sum_{j=1}^{d}\left|\hat{F}_{n,j}(X_{i,j})^{1/w_j} - F_j(X_{i,j})^{1/w_j}\right|n_j \\ \leq & 2\sup_{j\in\{1,\ldots,d\}}\sup_{i\in\{1,\ldots,n\}}\left|\hat{F}_{n,j}(X_{i,j})^{1/w_j} - F_j(X_{i,j})^{1/w_j}\right|,\end{aligned}$$

Where we used Lemma 2 to obtain the second inequality. The right term converges almost surely to zero by Glivencko-Cantelli. □

Proof of Lemma 1 Following [Segers, 2015] Example 3.5, we consider the function from $\{0,1\}^d \times \mathbb{R}^d$ into $\mathbb{R}$: for $\mathbf{x} \in \mathbb{R}^d$,

$$\begin{aligned}f_j(\mathbf{I}, \mathbf{X}) = \mathbb{1}_{\{I_j=1\}}, &\quad g_{j,x_j}(\mathbf{I}, \mathbf{X})\mathbb{1}_{\{X_j\leq x_j, I_j=1\}}, \\ f_{d+1} = \Pi_{j=1}^{d} f_j, &\quad g_{d+1,\mathbf{x}} = \Pi_{j=1}^{d} g_{j,x_j}.\end{aligned}$$

Let P denote the common distribution of the tuple $(\mathbf{I}, \mathbf{X})$. The collection of functions

$$\mathcal{F} = \{f_1, \ldots, f_d, f_{d+1}\} \cup \bigcup_{i=1}^{d}\{g_{i,x_i}, x_i \in \mathbb{R}\} \cup \{g_{d+1,\mathbf{x}}, \mathbf{x} \in \mathbb{R}^d\}$$

is a finite union of VC-classes and thus P-Donsker (see Chapter 2.6 of [van der Vaart and Wellner, 1996]).

The empirical process $\mathbb{G}_n$ defined by

$$\mathbb{G}_n(f) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} f(\mathbf{I}_i, \mathbf{X}_i) - \mathbb{E}[f(\mathbf{I}_i, \mathbf{X}_i)]\right), \quad f \in \mathcal{F},$$

converges in $\ell^\infty(\mathcal{F})$ to a P-browian bride $\mathbb{G}$. For $\mathbf{x} \in \mathbb{R}^d$,

$$\begin{aligned}\hat{F}_{n,j}(x_j) &= \frac{p_j F_j(x_j) + n^{-1/2}\mathbb{G}_n g_{j,x_j}}{p_j + n^{-1/2}\mathbb{G}_n f_j}, \\ \hat{G}_n(\mathbf{x}) &= \frac{pG(\mathbf{x}) + n^{-1/2}\mathbb{G}_n g_{d+1,\mathbf{x}}}{p + n^{-1/2}\mathbb{G}_n f_{d+1}}\end{aligned}$$

We obtain for the second one

$$\begin{aligned}p\left(\hat{G}_n(\mathbf{x}) - G(x)\right) &= n^{-1/2}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}}) - \hat{G}_n(\mathbf{x})\mathbb{G}_n(f_{d+1})\right), \\ &= n^{-1/2}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}} - G(x)f_{d+1})\right) - n^{-1/2}\mathbb{G}_n(f_{d+1})(\hat{G}_n(\mathbf{x}) - G(\mathbf{x}))\end{aligned}$$

We thus have

$$\sqrt{n}\left(\hat{G}_n(\mathbf{x}) - G(x)\right) = p^{-1}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}} - G(x)f_{d+1})\right) - p^{-1}\mathbb{G}_n(f_{d+1})(\hat{G}_n(\mathbf{x}) - G(\mathbf{x}))$$

Applying the central limit theorem gives that $\mathbb{G}_n(f_{d+1}) \xrightarrow{d} \mathcal{N}(0, \mathbb{P}(f_{d+1} - \mathbb{P}f_{d+1})^2)$, the law of large numbers gives also $\hat{G}_n(\mathbf{x}) - G(\mathbf{x}) = \circ_{\mathbb{P}}(1)$. Using Slutsky's lemma gives us

$$\sqrt{n}\left(\hat{G}_n(\mathbf{x}) - G(x)\right) = p^{-1}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}} - G(x)f_{d+1})\right) + \circ_{\mathbb{P}}(1).$$

Similar reasoning might be applied to the margins, as a consequence, Condition B is fulfilled with for $\mathbf{u} \in [0,1]^d$,

$$\begin{aligned}\beta_j(u_j) &= p_j^{-1}\mathbb{G}\left(g_{j,F_j^{\leftarrow}(u_j)} - u_j f_j\right), \\ \alpha(\mathbf{u}) &= \mathbf{p}^{-1}\mathbb{G}\left(g_{d+1,\mathbf{F}^{\leftarrow}(\mathbf{u})} - C(\mathbf{u})f_{d+1}\right).\end{aligned}$$

Let us compute one covariance function, the method still the same for the others, without loss of generality, suppose that $j < k$, we have for $u_j, v_k \in [0,1]$

$$
\begin{aligned}
cov(\beta_j(u_j), \beta_k(v_k)) &= \mathbb{E}\left[p_j^{-1}\mathbb{G}\left(g_{j,F_j^{\leftarrow}(u_j)} - u_j f_j\right) p_k^{-1}\mathbb{G}\left(g_{k,F_k^{\leftarrow}(v_k)} - v_k f_k\right)\right], \\
&= \frac{1}{p_j p_k}\mathbb{E}\left[\mathbb{G}\left(g_{j,F_j^{\leftarrow}(u_i)} - u_j f_j\right)\mathbb{G}\left(g_{k,F_k^{\leftarrow}(v_j)} - v_k f_k\right)\right], \\
&= \frac{1}{p_j p_k}\mathbb{P}\left\{X_j \leq F_j^{\leftarrow}(u_j), X_k \leq F_k^{\leftarrow}(v_k), I_j = 1, I_k = 1\right\} - \frac{p_{jk}}{p_j p_k} u_j v_k, \\
&= \frac{1}{p_j p_k}\mathbb{P}\left\{X_j \leq F_j^{\leftarrow}(u_j), X_k \leq F_k^{\leftarrow}(v_k)\right\}\mathbb{P}\left\{I_j = 1, I_k = 1\right\} - \frac{p_{jk}}{p_j p_k} u_j v_k, \\
&= \frac{p_{jk}}{p_j p_k}\left(C(\mathbf{1}_{jk}(u_j, v_k)) - u_j v_k\right).
\end{aligned}
$$

Hence the result. $\square$

Proof of Theorem 1 We do the proof for $\nu_n^{\mathcal{H}*}$ as the proof for $\nu_n^{\mathcal{H}}$ is similar. Using that $\mathbb{E}[F_i(X_i)^\alpha] = (1+\alpha)^{-1}$ for $\alpha \neq 1$, we can write $\nu(\mathbf{w})$ as :

$$
\begin{aligned}
\nu(\mathbf{w}) =& \mathbb{E}\left[\bigvee_{j=1}^{d}\{F_j(X_j)\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d}\{F_j(X_j)\}^{1/w_j}\right] + \\
& \sum_{j=1}^{d}\left(\frac{w_j(d-1)}{d}\frac{w_j}{1+w_j} - \frac{w_j(d-1)}{d}\mathbb{E}\left[F_j(X_j)^{1/w_j}\right]\right), \\
=& \mathbb{E}\left[\bigvee_{j=1}^{d}\{F_j(X_j)\}^{1/w_j}\right] - \frac{1}{d}\sum_{j=1}^{d}(1+w_j(d-1))\mathbb{E}\left[F_j(X_j)^{1/w_j}\right] + c(\mathbf{w}),
\end{aligned}
$$

with $c(\mathbf{w}) = d^{-1}\sum_{j=1}^{d} w_j/(1+w_j)$. Let us note by $g_{\mathbf{w}}$ the function defined as

$$
g_{\mathbf{w}} : [0,1]^d \to [0,1], \quad \mathbf{u} \mapsto \bigvee_{i=1}^{d} u_j^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d}(1+w_j(d-1))u_j^{1/w_j}.
$$

We are to write our estimator of the $\mathbf{w}$-madogram and the $\mathbf{w}$-madogram in missing data framework as an integral with respect to the hybrid copula estimator and the copula function. We thus have:

$$
\begin{aligned}
&\nu_n^{\mathcal{H}*}(\mathbf{w}) = \frac{1}{N}\sum_{m=1}^{n} g_{\mathbf{w}}\left(\hat{\mathbf{F}}_n(\mathbf{X}_m)\right)\Pi_{j=1}^{d} I_{i,j} + c(\mathbf{w}) = \int_{[0,1]^d} g_{\mathbf{w}}(\mathbf{u})\, d\hat{C}_n^{\mathcal{H}}(\mathbf{u}) + c(\mathbf{w}), \\
&\nu(\mathbf{w}) = \int_{[0,1]^d} g_{\mathbf{w}}(\mathbf{u})\, dC(\mathbf{u}) + c(\mathbf{w}).
\end{aligned}
$$

Where $\hat{\mathbf{F}}_n(\mathbf{X}_m) = (\hat{F}_{n,1}(X_{m,1}), \dots, \hat{F}_{n,d}(X_{m,d}))$. We obtain, proceeding as in Theorem 2.4

of [Marcon et al., 2017] :

$$\sqrt{n}\left(\nu_n^{\mathcal{H}*}(\mathbf{w})-\nu(\mathbf{w})\right)=\frac{1}{d}\sum_{j=1}^{d}(1+w_j(d-1))\int_{[0,1]}\mathbb{C}_n^{\mathcal{H}}(\mathbf{1}_j(x^{w_j}))dx - \int_{[0,1]}\mathbb{C}_n^{\mathcal{H}}\left(x^{w_1},\dots,x^{w_d}\right)dx.$$

Consider the function $\phi:\ell^\infty([0,1]^d)\to\ell^\infty(\Delta^{d-1}), f\mapsto\phi(f)$, defined by

$$(\phi)(f)(\mathbf{w})=\frac{1}{d}\sum_{j=1}^{d}(1+w_j(d-1))\int_{[0,1]}f(\mathbf{1}_j(x^{w_j}))dx-\int_{[0,1]}f(x^{w_1},\dots,x^{w_d})dx.$$

this function is linear and bounded thus continuous. The continous mapping theorem (Theorem 1.3.6 of [van der Vaart and Wellner, 1996]) implies, as $n\to\infty$

$$\sqrt{n}(\hat{\nu}_n^{\mathcal{H}*}-\nu)=\phi(\mathbb{C}_n^{\mathcal{H}})\rightsquigarrow\phi(S_C),$$

in $\ell^\infty(\Delta^{d-1})$. We note that $S_C(\mathbf{1}_j(x^{w_j}))=\alpha(\mathbf{1}_j(x^{w_j}))-\beta_j(u_j)$ and we obtain our statement. □

Lemma 3. *If $\ell(x_1,\dots,x_d)$ is homogeneous of degree 1, then for any $i\in\{1,\dots,d\}$ the partial derivative $\dot{\ell}_j(x_1,\dots,x_d)$ is homogeneous of degree 0.*

Proof of Proposition 2 We have $\forall j\in\{1,\dots,d\}$

$$\dot{C}_j(\mathbf{u})=\frac{C(\mathbf{u})}{u_j}\dot{\ell}_j(-log(u_1),\dots,-log(u_d)).$$

Furthermore, using Lemma 3, we have

$$\begin{aligned}\dot{C}_j(u^{w_1},\dots,u^{w_d})&=\frac{u^{A(\mathbf{w})}}{u^{w_j}}\dot{\ell}_j(-w_1log(u),\dots,-w_dlog(u))=\frac{u^{A(\mathbf{w})}}{u^{w_j}}\dot{\ell}_j(-w_1,\dots,-w_d)\\&=\frac{u^{A(\mathbf{w})}}{u^{w_j}}\mu_k(\mathbf{w}).\end{aligned}$$

Now, let us compute

$$\sigma_1^2(\mathbf{w})=\mathbb{E}\left[\int_{[0,1]}B_C(u^{w_1},\dots,u^{w_d})du\int_{[0,1]}B_C(v^{w_1},\dots,v^{w_d})dv\right].$$

Using linearity of the integral and the definition of the covariance function of B_C, we obtain

$$\sigma_1^2(\mathbf{w}) = 2\int_{[0,1]}\int_{[0,v]} u^{A(\mathbf{w})}(1-v^{A(\mathbf{w})})duv = \frac{1}{(1+A(\mathbf{w}))^2}\frac{A(\mathbf{w})}{2+A(\mathbf{w})}.$$

The quantity γ_j^2 is defined by the following

$$\gamma_j^2 = \mathbb{E}\left[\int_{[0,1]} B_C(\mathbf{1}_j(u^{w_j}))\dot{C}_j(u^{w_1},\dots,u^{w_d})du \int_{[0,1]} B_C(\mathbf{1}_j(v^{w_j}))\dot{C}_j(v^{w_1},\dots,v^{w_d})dv\right].$$

It is clear that

$$\begin{aligned}\gamma_j^2 &= 2\int_{[0,1]}\int_{[0,v]} u^{w_j}(1-v^{w_j})\mu_j(\mathbf{w})\mu_j(\mathbf{w})u^{A(\mathbf{w})-w_j}v^{A(\mathbf{w})-w_j}duv,\\ &= \left(\frac{\mu_j(\mathbf{w})}{1+A(\mathbf{w})}\right)^2 \frac{w_j}{2A(\mathbf{w})+1+1-w_j}.\end{aligned}$$

We now deal with cross product terms, the first we define is

$$\begin{aligned}\sigma_{1j} &= \mathbb{E}\left[\int_{[0,1]} B_C(u^{w_1},\dots,u^{w_d})du \int_{[0,1]} B_C(\mathbf{1}_j(v^{w_j}))\dot{C}_j(v^{w_1},\dots,v^{w_d})dv\right],\\ &= \int_{[0,1]^2}\left(C(u^{w_1},\dots,(u\wedge v)^{w_j},\dots,u^{w_d}) - u^{A(\mathbf{w})}v^{w_j}\right)\dot{C}_j(v^{w_1},\dots,v^{w_d})duv.\end{aligned}$$

Under the cube $[0,1]\times[0,v]$, we have

$$\begin{aligned}\sigma_{1j} &= \int_{[0,1]\times[0,v]}\left(C(u^{w_1},\dots,u^{w_j},\dots,u^{w_d}) - u^{A(\mathbf{w})}v^{w_j}\right)\dot{C}_j(v^{w_1},\dots,v^{w_d})duv,\\ &= \int_{[0,1]\times[0,v]} u^{A(\mathbf{w})}(1-v^{w_j})v^{A(\mathbf{w})-w_j}\mu_j(\mathbf{w})duv = \frac{\mu_j(\mathbf{w})}{2(1+A(\mathbf{w}))}\frac{w_j}{2A(\mathbf{w})+1+1-w_j}.\end{aligned}$$

Under the cube $[0,1]\times[0,u]$, we have for the right term

$$\int_{[0,1]\times[0,u]} u^{A(\mathbf{w})}v^{w_j}v^{A(\mathbf{w})-w_j}\mu_j(\mathbf{w})dvu = \frac{\mu_j(\mathbf{w})}{2(1+A(\mathbf{w}))^2}.$$

For the left term, by definition, we have

$$\int_{[0,1]\times[0,u]} C(u^{w_1},\dots,v^{w_j},\dots,u^{w_d})\dot{C}_j(v^{w_1},\dots,v^{w_d})dvu.$$

Let us consider the substitution $x = v^{w_j}$ and $y = u^{1-w_j}$, we obtain

$$\frac{1}{w_j(1-w_j)} \int_{[0,1]} \int_{[0,y^{w_j/(1-w_j)}]} C\left(y^{w_1/(1-w_j)}, \dots, x, \dots, y^{w_d/(1-w_j)}\right) \\ \times \dot{C}_j\left(x^{w_1/w_j}, \dots, x^{w_d/w_j}\right) x^{(1-w_j)/w_j} y^{w_j/(1-w_j)} dxy.$$

Let us compute the quantity

$$\dot{C}_j(x^{w_1/w_j}, \dots, x^{w_d/w_j}) = \frac{C(x^{w_1/w_j}, \dots, x^{w_d/w_j})}{x} \mu_j(\mathbf{w}).$$

Using Equation (1), we have

$$\begin{aligned} C(x^{w_1/w_j}, \dots, x^{w_d/w_j}) &= exp\left(-\ell\left(-\frac{log(x)}{w_j}w_1, \dots, \frac{log(x)}{w_j}w_d\right)\right) \\ &= exp\left(-\frac{log(x)}{w_j}\ell\left(-w_1, \dots, -w_d\right)\right) = x^{A(\mathbf{w})/w_j} =: x^{A_j(\mathbf{w})}. \end{aligned}$$

Where we use the homogeneity of order one of ℓ and that $-\ell(-w_1, \dots, -w_d) = A(\mathbf{w})$ because of Equation (2) and that $\mathbf{w} \in \Delta^{d-1}$. Now, consider the substitution $x = w^{1-s}$ and $y = w^s$, the jacobian of this transformation is given by $-log(w)$, we have

$$-\frac{\mu_j(\mathbf{w})}{w_j(1-w_j)} \int_{[0,1]} \int_{[0,1-w_j]} C\left(w^{sw_1/(1-w_j)}, \dots, w^{1-s}, \dots, w^{sw_d/(1-w_j)}\right) \\ \times w^{(1-s)\left[A_j(\mathbf{w}) + \frac{1-w_j}{w_j} - 1\right] + s\frac{w_j}{1-w_j}} log(w) dsw.$$

We now compute the quantity

$$C\left(w^{sw_1/(1-w_j)}, \dots, w^{1-s}, \dots, w^{sw_d/(1-w_j)}\right).$$

Using the same methods as above, we have

$$\begin{aligned} &C\left(w^{sw_1/(1-w_j)}, \dots, w^{1-s}, \dots, w^{sw_d/(1-w_j)}\right) \\ &= exp\left(-\ell\left(-\frac{sw_1}{1-w_j}log(w), \dots, -(1-s)log(w), \dots, -\frac{sw_d}{1-w_j}log(w)\right)\right), \\ &= exp\left(-log(w)\ell\left(-\frac{sw_1}{1-w_j}, \dots, -(1-s), \dots, -\frac{sw_d}{1-w_j}\right)\right). \end{aligned}$$

Now, using that $\mathbf{w} \in \Delta^{d-1}$, remark that $s\sum_{i \neq j} w_i/(1-w_j) = s$, we have, using Equation (2)

$$-\ell\left(-\frac{sw_1}{1-w_j}, \dots, -(1-s), \dots, -\frac{sw_d}{1-w_j}\right) = A\left(\frac{sw_1}{1-w_j}, \dots, \frac{sw_d}{1-w_j}\right).$$

Where we set $1-s$ in the j-th components of the Pickands dependence function A. So we have

$$\sigma_{1i} = -\frac{\mu_j(\mathbf{w})}{w_j(1-w_j)} \int_{[0,1-w_j]} \int_{[0,1]} w^{A\left(\frac{sw_1}{1-w_j},\dots,\frac{sw_d}{1-w_j}\right)+(1-s)\left(A_j(\mathbf{w})+\frac{1-w_j}{w_j}-1\right)+s\frac{w_j}{1-w_j}} log(w) dws,$$

$$= \frac{\mu_j(\mathbf{w})}{w_j(1-w_j)} \int_{[0,1-w_j]} \left[A\left(\frac{sw_1}{1-w_j},\dots,\frac{sw_d}{1-w_j}\right) + (1-s)\left(A_j(\mathbf{w}) + \frac{1-w_j}{w_j} - 1\right) + s\frac{w_j}{1-w_j} + 1\right]^{-2} ds.$$

No further simplifications can be obtained. For $j<k$, let us define the quantity γ_{jk} such as

$$\gamma_{jk} = \mathbb{E}\left[\int_{[0,1]} B_C(\mathbf{1}_j(u^{w_j}))\dot{C}_j(u^{w_1},\dots,u^{w_d}) du B_C(\mathbf{1}_k(v^{w_k}))\dot{C}_k(v^{w_1},\dots,v^{w_d}) dv \right].$$

Again, we have

$$\gamma_{ij} = \int_{[0,1]^2} \left(C(\mathbf{1}_{jk}(u^{w_j}, v^{w_j})) - u^{w_j}v^{w_j}\right) \dot{C}_j(u^{w_1},\dots,u^{w_d})\dot{C}_k(v^{w_1},\dots,v^{w_d}) duv$$

We set $x = u^{w_j}$ and $y = v^{w_k}$, the left side become

$$\gamma_{ij} = \frac{1}{w_j(1-w_k)} \int_{[0,1]^2} C(\mathbf{1}_{jk}(x,y)) \times \dot{C}_j(x^{w_1/w_j},\dots,x^{w_d/w_j})\dot{C}_k(y^{w_1/w_k},\dots,y^{w_d/w_k}) x^{(1-w_j)/w_j} y^{(1-w_k)/w_k} dxy,$$

$$= \frac{\mu_j(\mathbf{w})\mu_k(\mathbf{w})}{w_j w_k} \int_{[0,1]^2} C(\mathbf{1}_{jk}(x,y)) x^{A_j(\mathbf{w})+(1-w_j)/w_j - 1} y^{A_k(\mathbf{w})+(1-w_k)/w_k-1} dxy.$$

Now, we set $x = w^{1-s}$ and $y = w^s$ and we obtain

$$\gamma_{jk} = \frac{\mu_j(\mathbf{w})\mu_k(\mathbf{w})}{w_j w_k} \int_{[0,1]} \left[A(0,\dots,s,\dots,0) + (1-s)\left(A_j(\mathbf{w}) + \frac{1-w_j}{w_j} - 1\right) + s\left(A_k(\mathbf{w}) + \frac{1-w_k}{w_k} + 1\right)\right]^{-2} ds.$$

Where we set $1-s$ at the ith component of the Pickands. The right side of the expression is given by

$$\int_{[0,1]^2} u^{w_j} v^{w_k} \dot{C}_j(u^{w_1},\dots,u^{w_d})\dot{C}_k(v^{w_1},\dots,v^{w_d}) duv = \frac{\mu_j(\mathbf{w})\mu_k(\mathbf{w})}{(1+A(\mathbf{w}))^2}.$$

Hence the result. □

References

[Fermanian et al., 2004] Fermanian, J.-D., Radulović, D., and Wegkamp, M. (2004). Weak convergence of empirical copula processes. *Bernoulli*, 10(5):847–860.

[Gudendorf and Segers, 2010] Gudendorf, G. and Segers, J. (2010). Extreme-value copulas. In *Copula theory and its applications*, volume 198 of *Lect. Notes Stat. Proc.*, pages 127–145. Springer, Heidelberg.

[Marcon et al., 2017] Marcon, G., Padoan, S., Naveau, P., Muliere, P., and Segers, J. (2017). Multivariate nonparametric estimation of the pickands dependence function using bernstein polynomials. *Journal of Statistical Planning and Inference*, 183:1–17.

[Naveau et al., 2009] Naveau, P., Guillou, A., Cooley, D., and Diebolt, J. (2009). Modelling pairwise dependence of maxima in space. *Biometrika*, 96(1):1–17.

[Segers, 2012] Segers, J. (2012). Asymptotics of empirical copula processes under non-restrictive smoothness assumptions. *Bernoulli*, 18(3):764 – 782.

[Segers, 2015] Segers, J. (2015). Hybrid copula estimators. *J. Statist. Plann. Inference*, 160:23–34.

[Sklar, 1959] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8:229–231.

[van der Vaart and Wellner, 1996] van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Process: With Applications to Statistics*. Springer.