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Abstract

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1. Introduction

Management of environmental ressources often requires the analysis of multivariate extreme values. In climate studies, extreme events represent a major challenge due to their consequences. The problem of missing data is present in many fields in particular in environmental research (see [27]), usually due to instruments, communication and processing errors. In a time series setting, the observation periods of a multivariate series could be different and overlap only partially. The problem of estimating when unequal amounts of data are available to each variable is of interest in many applications for financial economics where data cannot be generated as neatly overlapping samples (see [19]). Also, the machine learning community has to design prediction procedure which handle missing values, *e.g.* [15]. In this paper, we consider inference methods for assessing extremal dependencies involving variables with missing values. We are particularly interested in the dependence structure of multivariate extreme value distribution. Formally, this concept is defined as follows.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathbf{X} = (X_1, \dots, X_d)$ be a d-dimensional random vector with values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, with $d \geq 2$. This random vector has a joint distribution function F and its margins are denoted by $F_j(x) = \mathbb{P}\{X_j \leq x\}$ for all $x \in \mathbb{R}$ and $j \in \{1, \dots, d\}$. A function $C : [0,1]^d \to [0,1]$ is called a d-dimensional copula if it is the restriction to $[0,1]^d$ of a distribution function whose margins are given by the uniform distribution on the interval $[0,1]$. Since the work of [23], it is well known that every distribution function F can be decomposed as $F(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d))$, for all $\mathbf{x} \in \mathbb{R}^d$ and the copula C is unique if the marginals are continuous. Under the framework of extreme, the notion of copulas leads to the so-called extreme value copulas (see [7])

$$C(\mathbf{u}) = exp\left(-\ell(-ln(u_1), \dots, -ln(u_d))\right), \quad \mathbf{u} \in (0,1]^d, \tag{1}$$

with $\ell : [0,\infty)^d \to [0,\infty)$ the stable tail dependence function which is convex, homogeneous of order one, namely $\ell(cx_1, \dots, cx_d) = c\ell(x_1, \dots, x_d)$ for $c > 0$ and satisfies $\max(x_1, \dots, x_d) \leq \ell(x_1, \dots, x_d) \leq x_1 + \cdots + x_d$, $\forall (x_1, \dots, x_d) \in [0,\infty)^d$. Denote by $\Delta^{d-1} = \{(w_1, \dots, w_d) \in [0,1]^d : w_1 + \cdots + w_d = 1\}$ the unit simplex. By homogeneity, ℓ is characterized by the *Pickands dependence function* $A : \Delta^{d-1} \to [1/d, 1]$, which is the restriction of ℓ to the unit simplex Δ^{d-1} :

$$\ell(x_1, \dots, x_d) = (x_1 + \cdots + x_d) A(w_1, \dots, w_d), \quad w_j = \frac{x_j}{x_1 + \cdots + x_d}, \tag{2}$$

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for $j \in \{2, \dots, d\}$ and $w_1 = 1 - w_2 - \cdots - w_d$ with $(x_1, \dots, x_d) \in [0, \infty)^d \setminus \{\mathbf{0}\}$. Notice that, for every $\mathbf{w} \in \Delta^{d-1}$ and $u \in]0, 1[$

$$C(u^{w_1}, \dots, u^{w_d}) = u^{A(\mathbf{w})}. \tag{3}$$

Based on the madogram concept from geostatistics, the λ-madogram is introduced in [17] to capture bivariate extremal dependencies. This quantity leads to its extension in higher dimension the $\mathbf{w}$-madogram defined in [16]

$$\nu(\mathbf{w}) = \mathbb{E}\left[\bigvee_{j=1}^{d} \left\{F_j(X_j)\right\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d} \left\{F_j(X_j)\right\}^{1/w_j}\right], \tag{4}$$

if $w_j = 0$ and $0 < u < 1$, then $u^{1/w_j} = 0$ by convention. The $\mathbf{w}$-madogram can be interpreted as the L_1-distance between the maximum and the average of the uniform margins $F_1(X_1), \dots, F_d(X_d)$ elevated to the inverse of the corresponding weights $w_1, \dots, w_d$. This quantity describes the dependence structure between extremes by its relation with the Pickands dependence function as stated by the Proposition 2.2 of *loc. cit.*, namely

$$A(\mathbf{w}) = \frac{\nu(\mathbf{w}) + c(\mathbf{w})}{1 - \nu(\mathbf{w}) - c(\mathbf{w})}, \tag{5}$$

with $c(\mathbf{w}) = d^{-1} \sum_{j=1}^{d} w_j/(1 + w_j)$. Through this relation, it contributes to the vast literature of the estimation of the Pickands dependence function for bivariate extreme value copula (see [20], [3], [1] or [13]) but also multivariate extreme value copula, *e.g.* [8]. Also, a test for assessing asymptotic independence in dimension $d \geq 2$ has been designed based on the $\mathbf{w}$-madogram (see [10]). Several methods for handling missing values in the framework of extremes have been proposed for univariate time series; see *e.g.* [5, 12]. However, handling missing values in the context of multivariate extreme values with $d \geq 2$ still in their infancy. The main contribution of this paper is to give an estimator of the $\mathbf{w}$-madogram involving variables with missing values.

In order to shorten formulas, notations

$$\mathbf{u}_j(t) := (u_1, \dots, u_{j-1}, t, u_{j+1}, \dots, u_d),$$
$$\mathbf{u}_{jk}(s, t) := (u_1, \dots, u_{j-1}, s, u_{j+1}, \dots, u_{k-1}, t, u_{k+1}, \dots, u_d),$$

will be adopted for $s, t \in [0, 1]$, $(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_d) \in [0, 1]^{d-1}$ and $j, k \in \{1, \dots, d\}$ with $j < k$. The notation $\mathbf{1}$ (resp. $\mathbf{0}$) corresponds to the d-dimensional vector composed out of 1 (resp. 0). Similarly, we define $\mathbf{1}_j(s)$, $\mathbf{0}_j(s)$, $\mathbf{1}_{jk}(s, t)$ and $\mathbf{0}_{jk}(s, t)$ with the same idea of previous notations of this paragraph.

The following notations are also used. Given $\mathcal{X} \subset \mathbb{R}^d$, let $\ell^\infty(\mathcal{X})$ denote the space of bounded real-valued functions on $\mathcal{X}$. For $f : \mathcal{X} \to \mathbb{R}$, let $\|f\|_\infty = sup_{x \in \mathcal{X}} |f(x)|$. Here, we use the abbreviation $Q(f) = \int f dQ$ for a given measurable function f and signed measure Q. The arrows $\overset{a.s.}{\to}$, $\overset{d}{\to}$ denote almost sure convergence and convergence in distribution of random vectors. Weak convergence of a sequence of maps will be understood in the sense of J.Hoffman-Jørgensen (see Part 1 in the monograph by [26]). Given that $n \in \mathbb{N}^*, X, X_n$ are maps from $(\Omega, \mathcal{A}, \mathbb{P})$ into a metric space $\mathcal{X}$ and that X is Borel measurable, $(X_n)_{n \geq 1}$ is said to converge weakly to X if $\mathbb{E}^* f(X_n) \to \mathbb{E} f(X)$ for every bounded continuous real-valued function f defined on $\mathcal{X}$, where $\mathbb{E}^*$ denotes outer expectation in the event that X_n may not be Borel measurable. In what follows, weak convergence is denoted by $X_n \rightsquigarrow X$.

In this paper, we propose in Section 2 estimators of the $\mathbf{w}$-madogram suitable to the missing data framework. We state the weak convergence of the depicted estimators. Explicit formula for the asymptotic variance are also given. In section 3, a simulation study will provide evidence that the conclusions remain valid in the finite-sample framework. All the proofs are postponed to Section 4.

2. Non parametric estimation of the Madogram with missing data

We assume that the copula C is an extreme value copula as in Equation (1). We consider independent and identically distributed *i.i.d.* copies $\mathbf{X}_1, \dots, \mathbf{X}_n$ of $\mathbf{X}$. In presence of missing data, we do not observe a complete vector $\mathbf{X}_i$ for $i \in \{1, \dots, n\}$. We introduce $\mathbf{I}_i \in \{0, 1\}^d$ which satisfies, $\forall j \in \{1, \dots, d\}$, $I_{i,j} = 0$ if $X_{i,j}$ is not observed. To formalize incomplete observations, we introduce the incomplete vector $\tilde{\mathbf{X}}_i$ with values in the product space $\bigotimes_{j=1}^d(\mathbb{R} \cup \{\mathrm{NA}\})$ such as

$$\tilde{X}_{i,j} = X_{i,j}I_{i,j} + \mathrm{NA}(1 - I_{i,j}), \quad i \in \{1, \ldots, n\},\ j \in \{1, \ldots, d\}.$$

We thus suppose that we observe a $2d$-tuple such as

$$(\mathbf{I}_i, \tilde{\mathbf{X}}_i), \quad i \in \{1, \ldots, n\}, \tag{6}$$

i.e. at each $i \in \{1, \ldots, n\}$, several entries may be missing. We also suppose that for all $i \in \{1, \ldots, n\}$, $\mathbf{I}_i$ are *i.i.d* copies from $\mathbf{I} = (I_1, \ldots, I_d)$ where I_j is distributed according to a Bernoulli random variable $\mathcal{B}(p_j)$ with $p_j = \mathbb{P}(I_j = 1)$ for $j \in \{1, \ldots, d\}$. We denote by p the probability of observing completely a realization from $\mathbf{X}$, that is $p = \mathbb{P}(I_1 = 1, \ldots, I_d = 1)$. Let us now define the empirical cumulative distribution in case of missing data, we write for notational convenience $\{\tilde{\mathbf{X}}_i \leq \mathbf{x}\} := \{\tilde{X}_{i,1} \leq x_1, \ldots, \tilde{X}_{i,d} \leq x_d\}$,

$$\hat{F}_{n,j}(x) = \frac{\sum_{i=1}^n \mathbb{1}_{\{\tilde{X}_{i,j} \leq x\}} I_{i,j}}{\sum_{i=1}^n I_{i,j}}, \ \forall x \in \mathbb{R}, \quad \hat{F}_n(\mathbf{x}) = \frac{\sum_{i=1}^n \mathbb{1}_{\{\tilde{\mathbf{X}}_i \leq \mathbf{x}\}} \Pi_{j=1}^d I_{i,j}}{\sum_{i=1}^n \Pi_{j=1}^d I_{i,j}}, \ \forall \mathbf{x} \in \mathbb{R}^d. \tag{7}$$

The idea raised here is to estimate non parametrically the margins using all available data of the corresponding series. We recall the definition of the *hybrid copula estimator* introduced by [22]

$$\hat{C}_n^{\mathcal{H}}(\mathbf{u}) = \hat{F}_n(\hat{F}_{n,1}^{\leftarrow}(u_1), \ldots, \hat{F}_{n,d}^{\leftarrow}(u_d)), \quad \mathbf{u} \in [0,1]^d,$$

where $F^{\leftarrow}$ denotes the generalized inverse function of F, *i.e.* $F^{\leftarrow}(u) = \inf\{x \in \mathbb{R} | F(x) \geq u\}$ with $0 < u < 1$. The normalized estimation error of the hybrid copula estimator is

$$\mathbb{C}_n^{\mathcal{H}}(\mathbf{u}) = \sqrt{n}\left(\hat{C}_n^{\mathcal{H}}(\mathbf{u}) - C(\mathbf{u})\right), \quad \mathbf{u} \in [0,1]^d. \tag{8}$$

On the condition that the first-order partial derivatives of the copula function C exists and are continuous on a subset of the unit hypercube, [21] obtained weak convergence of the normalized estimation error of the classical empirical copula process (see [2]). To satisfy this condition, we introduce the following assumption as suggested in [21] (see Example 5.3).

Assumption A.

1. The distribution function F has continuous margins $F_1, \ldots, F_d$.
2. For every $j \in \{1, \ldots, d\}$, the first-order partial derivative $\dot{\ell}_j$ of ℓ in Equation (2) with respect to x_j exists and is continuous on the set $\{x \in [0,\infty)^d : x_j > 0\}$.

The Assumption A1 guarantees that the representation $F(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d))$ is unique on the range of $(F_1, \ldots, F_d)$. Under the Assumption A2, the first-order partial derivatives of C with respect to u_j denoted as $\dot{C}_j$ exists and are continuous on the set $\{\mathbf{u} \in [0,1]^d : 0 < u_j < 1\}$. We now propose an estimator of the $\mathbf{w}$-madogram defined in Equation (4) under a general context (with possible missing data).

Definition 1. Let $(\mathbf{I}_i, \tilde{\mathbf{X}}_i)_{i=1}^n$ be a sample given by Equation (6), we define the hybrid nonparametric estimator of the $\mathbf{w}$-madogram in Equation (4) by

$$\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) = \frac{1}{\sum_{i=1}^n \Pi_{j=1}^d I_{i,j}} \sum_{i=1}^n \left[\left(\bigvee_{j=1}^d \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^d \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j}\right) \Pi_{j=1}^d I_{i,j}\right], \tag{9}$$

where $\hat{F}_{n,j}(x)$ are defined in Equation (7).

The intuitive idea here is to estimate the margins using all available data from the corresponding variables and estimate $\nu(\mathbf{w})$ using only the overlapping data. One may verify that in the complete data framework, *i.e.* when $p = 1$ we retrieve the $\mathbf{w}$-madogram such as defined in [16], namely

$$\hat{\nu}_n(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^n \left[\bigvee_{j=1}^d \left\{\hat{F}_{n,j}(X_{i,j})\right\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^d \left\{\hat{F}_{n,j}(X_{i,j})\right\}^{1/w_j}\right],$$

with $\hat{F}_{n,j}(x)$ the empirical cumulative distribution function of X_j.

Note that the theoretical quantity defined in (4) does verify endpoint constraints, *i.e.* $\nu(\mathbf{e}_j) = (d-1)/2d$ for all $j \in \{1,\dots,d\}$ where $\mathbf{e}_j$ is the jth vector of the canonical basis.

Remark 1. Unlike ν, our estimator defined in (9) does not verify the endpoints constraints. In addition, the variance at $\mathbf{e}_j$ does not equal 0. Indeed, suppose that we evaluate this statistic at $\mathbf{w} = \mathbf{e}_j$, we thus obtain the following quantity :

$$\hat{\nu}_n^{\mathcal{H}}(\mathbf{e}_j) = \frac{1}{\sum_{i=1}^n \Pi_{j=1}^d I_{i,j}} \sum_{i=1}^n \left[\hat{F}_{n,j}(\tilde{X}_{i,j}) - \frac{1}{d}\hat{F}_{n,j}(\tilde{X}_{i,j}) \right] \Pi_{j=1}^d I_{i,j}.$$

In this situation, the sample $(\tilde{X}_{i,1},\dots,\tilde{X}_{i,j-1},\tilde{X}_{i,j+1},\dots,\tilde{X}_{i,d})_{i=1}^n$ is taken into account through the indicators sequence $(I_{i,1},\dots,I_{i,j-1},I_{i,j+1},\dots,I_{i,d})_{i=1}^n$ and induce a supplementary variance when estimating.

As in [17], we propose a slightly modified estimator which satisfies the endpoint constraints. This can be imposed as follows.

Definition 2. Let $(\mathbf{I}_i, \tilde{\mathbf{X}}_i)_{i=1}^n$ be a sample given by Equation (6) and $\hat{\nu}_n^{\mathcal{H}}(\mathbf{w})$ be as in (9). Given continuous functions $\lambda_1,\dots,\lambda_d : \Delta^{d-1} \to \mathbb{R}$ verifying $\lambda_j(\mathbf{e}_k) = \delta_{jk}$ (the Kronecker delta) for $j,k \in \{1,\dots,d\}$, we define the hybrid corrected estimator of the $\mathbf{w}$-madogram by

$$\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) = \hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \sum_{j=1}^d \frac{\lambda_j(\mathbf{w})(d-1)}{d} \left[\frac{1}{\sum_{i=1}^n \Pi_{j=1}^d I_{i,j}} \sum_{i=1}^n \left(\left\{ \hat{F}_{n,j}(\tilde{X}_{i,j}) \right\}^{1/w_j} \Pi_{j=1}^d I_{i,j} \right) - \frac{w_j}{1+w_j} \right]. \tag{10}$$

Remark 2. One has often that endpoint corrections do not have an impact to the asymptotic behavior with complete data framework and unknown margins (see Section 2.3 and 2.4 of [6]). That is not always the case in the missing data framework and this feature is of interest as discussed in Remark 1.

We present with Theorem 1 in this Section a functional central limit theorem concerning the weak convergence of the following processes

$$\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right), \quad \sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right). \tag{11}$$

Before presenting this result, we introduce a specific assumption on the missing mechanism as detailed below.

Assumption B. We suppose that for all $i \in \{1,\dots,n\}$, the vector $\mathbf{I}_i$ and $\mathbf{X}_i$ are independent, i.e. the data are missing completely at random (**MCAR**).

Without missing data, the weak convergence of the normalized estimation error of the empirical copula process has been proved by [4] under a more restrictive condition than Assumption A. With the hybrid copula estimator, we need a technical assumption in order to guarantee the weak convergence of the process $\mathbb{C}_n^{\mathcal{H}}$ in (8) (see [22]). We note for convenience marginal distributions and quantile functions into vector valued functions $\mathbf{F}_d$ and $\mathbf{F}_d^{\leftarrow}$:

$$\mathbf{F}_d(\mathbf{x}) = (F_1(x_1),\dots,F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d, \qquad \mathbf{F}_d^{\leftarrow}(\mathbf{u}) = (F_1^{\leftarrow}(u_1),\dots,F_d^{\leftarrow}(u_d)), \quad \mathbf{u} \in [0,1]^d.$$

Assumption C. In the space $\ell^\infty(\mathbb{R}^d) \otimes (\ell^\infty(\mathbb{R}),\dots,\ell^\infty(\mathbb{R}))$ equipped with the topology of uniform convergence, we have the joint weak convergence

$$\left(\sqrt{n}(\hat{F}_n - F);\, \sqrt{n}(\hat{F}_{n,1} - F_1),\dots,\sqrt{n}(\hat{F}_{n,d} - F_d)\right) \rightsquigarrow (\alpha \circ \mathbf{F}_d, \beta_1 \circ F_1, \dots, \beta_d \circ F_d),$$

where the stochastic processes α and β_j, $j \in \{1,\dots,d\}$ take values in $\ell^\infty([0,1]^d)$ and $\ell^\infty([0,1])$ respectively, and are such that $\alpha \circ F$ and $\beta_j \circ F_j$ have continuous trajectories on $[-\infty,\infty]^d$ and $[-\infty,\infty]$ almost surely.

Under Assumptions A and C, the stochastic process $\mathbb{C}_n^{\mathcal{H}}$ in (8) converges weakly to the tight Gaussian process S_C defined by

$$S_C(\mathbf{u}) = \alpha(\mathbf{u}) - \sum_{j=1}^d \dot{C}_j(\mathbf{u})\beta_j(u_j), \quad \forall \mathbf{u} \in [0,1]^d.$$

We have all tools in hand to consider the weak convergence of the stochastic processes in Equation (11). We note by $\{\mathbf{X} \leq \mathbf{F}_d^{\leftarrow}(\mathbf{u})\} = \{X_1 \leq F_1^{\leftarrow}(u_1),\dots,X_d \leq F_d^{\leftarrow}(u_d)\}$.

Theorem 1. *Let $\mathbb{G}$ a tight Gaussian process. If C is an extreme value copula with Pickands dependence function A and under Assumptions A, B, C we have the weak convergence in $\ell^\infty(\Delta^{d-1})$ for hybrid estimators defined in Equations (9) and (10), as $n \to \infty$,*

$$\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right) \rightsquigarrow \left(\frac{1}{d}\sum_{j=1}^d \int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx - \int_{[0,1]} S_C(x^{w_1}, \dots, x^{w_d})dx\right)_{\mathbf{w}\in\Delta^{d-1}},$$

$$\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right) \rightsquigarrow \left(\frac{1}{d}\sum_{j=1}^d (1 + \lambda_j(\mathbf{w})(d-1)) \int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx - \int_{[0,1]} S_C(x^{w_1}, \dots, x^{w_d})dx\right)_{\mathbf{w}\in\Delta^{d-1}},$$

where $S_C(\mathbf{u}) = \alpha(\mathbf{u}) - \sum_{j=1}^d \dot{C}_j(\mathbf{u})\beta_j(u_j)$, $\alpha(\mathbf{u}) = p^{-1}\mathbb{G}(\mathbb{1}_{\{\mathbf{X}\leq \mathbf{F}_d^{\leftarrow}(\mathbf{u}), \boldsymbol{I}=\boldsymbol{1}\}} - C(\mathbf{u})\mathbb{1}_{\{\boldsymbol{I}=\boldsymbol{1}\}})$ and $\beta_j(u_j) = p_j^{-1}\mathbb{G}(\mathbb{1}_{\{X_j\leq F_j^{\leftarrow}(u_j), I_j=1\}} - u_j\mathbb{1}_{\{I_j=1\}})$ for $j \in \{1,\dots,d\}$ and $\mathbf{u} \in [0,1]^d$. For $(\mathbf{u}, \mathbf{v}, v_k) \in [0,1]^{2d+1}$, for $j \in \{1, \dots, d\}$ and $j < k$ the covariance functions of the processes α and β_j are given by

$$\begin{aligned} cov\left(\beta_j(u_j), \beta_j(v_j)\right) &= p_j^{-1}\left(u_j \wedge v_j - u_j v_j\right), \\ cov\left(\beta_j(u_j), \beta_k(v_k)\right) &= \frac{p_{jk}}{p_j p_k}\left(C(\mathbf{1}_{j,k}(u_j, v_k)) - u_j v_k\right), \end{aligned}$$

and

$$\begin{aligned} cov\left(\alpha(\mathbf{u}), \alpha(\mathbf{v})\right) &= p^{-1}\left(C(\mathbf{u}\wedge\mathbf{v}) - C(\mathbf{u})C(\mathbf{u})\right), \\ cov\left(\alpha(\mathbf{u}), \beta_j(v_j)\right) &= p_j^{-1}\left(C(\mathbf{u}_j(u_j \wedge v_j)) - C(\mathbf{u})v_j\right), \end{aligned}$$

where $\mathbf{u}\wedge\mathbf{v}$ denotes the vector of componentwise minima and $p_{jk} = \mathbb{P}(I_j = 1, I_k = 1)$.

We use empirical process arguments formulated in [26] to establish such a result. Details can be found in Section 4.1. The following proposition states the asymptotic distribution of the estimators and gives explicit formula for the asymptotic variances.

Proposition 1. *Let $\mathbf{p} = (p_1, \dots, p_d, p)$ and given continuous functions $\lambda_1, \dots, \lambda_d : \Delta^{d-1} \to \mathbb{R}$ verifying $\lambda_j(\boldsymbol{e}_k) = \delta_{jk}$. For $\mathbf{w} \in \Delta^{d-1}$ and under the framework of Theorem 1, we have*

$$\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right) \underset{n\to\infty}{\overset{d}{\to}} \mathcal{N}\left(0, \mathcal{S}^{\mathcal{H}}(\mathbf{p}, \mathbf{w})\right), \quad \sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right) \underset{n\to\infty}{\overset{d}{\to}} \mathcal{N}\left(0, \mathcal{S}^{\mathcal{H}*}(\mathbf{p}, \mathbf{w})\right).$$

Moreover the asymptotic variances of the random variables $\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right)$ and $\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right)$ are given by

$$\begin{aligned} \mathcal{S}^{\mathcal{H}}(\mathbf{p}, \mathbf{w}) =& \frac{1}{d^2}\sum_{j=1}^d (p^{-1} - p_j^{-1})\sigma_j^2(\mathbf{w}) + \sigma_{d+1}^2(\mathbf{p}, \mathbf{w}) + \frac{2}{d^2}\sum_{j<k}\left(p^{-1} - p_j^{-1} - p_k^{-1} + \frac{p_{jk}}{p_j p_k}\right)\sigma_{jk}(\mathbf{w}) \\ & - \frac{2}{d}\sum_{j=1}^d (p^{-1} - p_j^{-1})\sigma_j^{(1)}(\mathbf{w}) + \frac{2}{d}\sum_{j=1}^d\sum_{k=1}^d \left(p_k^{-1} - \frac{p_{jk}}{p_j p_k}\right)\sigma_{jk}^{(2)}(\mathbf{w}), \end{aligned}$$

$$\begin{aligned} \mathcal{S}^{\mathcal{H}*}(\mathbf{p}, \mathbf{w}) =& \frac{1}{d^2}\sum_{j=1}^d (p^{-1} - p_j^{-1})(1 + \lambda_j(\mathbf{w})(d-1))^2\sigma_j^2(\mathbf{w}) + \sigma_{d+1}^2(\mathbf{p}, \mathbf{w}) \\ & + \frac{2}{d^2}\sum_{j<k}\left(p^{-1} - p_j^{-1} - p_k^{-1} + \frac{p_{jk}}{p_j p_k}\right)(1 + \lambda_j(\mathbf{w})(d-1))(1 + \lambda_k(\mathbf{w})(d-1))\sigma_{jk}(\mathbf{w}) \\ & - \frac{2}{d}\sum_{j=1}^d (p^{-1} - p_j^{-1})(1 + \lambda_j(\mathbf{w})(d-1))\sigma_j^{(1)}(\mathbf{w}) \\ & + \frac{2}{d}\sum_{j=1}^d\sum_{k=1}^d \left(p_k^{-1} - \frac{p_{jk}}{p_j p_k}\right)(1 + \lambda_j(\mathbf{w})(d-1))\sigma_{jk}^{(2)}(\mathbf{w}). \end{aligned}$$

where explicit expressions of the functions σ_j^2 for $j \in \{1, \dots, d\}$, σ_{d+1}^2, σ_{jk} with $j < k$, $\sigma_j^{(1)}$ with $j \in \{1, \dots, d\}$, $\sigma_{jk}^{(2)}$ for $j, k \in \{1, \dots, d\}$ are postponed to Section 4.1 for the sake of readibility.

Technical details are available on Section 4.1. Considering the special case of independent copula, Corollary 1 below gives a closed form of the limit variance which no longer depends as a functional of the Pickands dependence function.

Corollary 1. *In the framework of Theorem 1 and if $C(\boldsymbol{u}) = \Pi_{j=1}^d u_j$, then the functions σ_{d+1}^2, $\sigma_j^{(1)}$ with $j \in \{1, \dots, d\}$, have the following forms, for $\mathbf{w} \in \Delta^{d-1}$:*

$$\sigma_{d+1}^2(\mathbf{p}, \mathbf{w}) = \frac{1}{4}\left(\frac{1}{3p} - \sum_{j=1}^d p_j^{-1} \frac{w_j}{4 - w_j}\right),$$

$$\sigma_j^{(1)}(\mathbf{w}) = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{1 + w_j}\right] + \frac{w_j}{3(1 + w_j)(3 + w_j)},$$

and σ_{jk} for $j < k$, $\sigma_{jk}^{(2)}$ for $j < k$ and $\sigma_{kj}^{(2)}$ with $k < j$ are constants and equal to 0.

Remark 3. From our knowledge, only [9] detailed the variance for the madogram of a bivariate random vector while taking the independent copula and found 1/90. The result stated in Corollary 1 is not an extension of this result because the hypothesis $\mathbf{w} \in \Delta^{d-1}$ is crucial. Nevertheless, the same techniques used to prove Proposition 1 could be of interest to show a similar explicit formula of the asymptotic variance for an extension of the madogram with $d \geq 2$.

Weak consistency of our estimators directly comes down from Proposition 1. We are nonetheless able to state the strong consistency with only the help of Assumption B.

Proposition 2 (Strong consistency). *Let $(\boldsymbol{I}_i, \tilde{\boldsymbol{X}}_i)_{i=1}^n$ a i.i.d sample given by Equation (6). We have, under Assumption B for a fixed $\mathbf{w} \in \Delta^{d-1}$*

$$\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) \underset{n \to \infty}{\overset{a.s.}{\to}} \nu(\mathbf{w}), \quad \hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) \underset{n \to \infty}{\overset{a.s.}{\to}} \nu(\mathbf{w}).$$

Details of the proof are given in Section 4.1. For the rest of this section, we use all of our results to state some considering the Pickands estimator in the missing data framework.

It is a common knowledge that the **w**-madogram is of main interest to construct of the Pickands dependence function. Indeed, given Equation (5), one can define an estimator of the Pickands dependence function by estimating the **w**-madogram and using it as a plug-in estimator. Most interesting properties of the **w**-madogram such as strong consistency and the weak convergence are thus translated for the Pickands estimator using continuous mapping theorem and the Delta method. In the missing data framework we define the following estimator.

Definition 3. Let $(\mathbf{I}_i, \tilde{\mathbf{X}}_i)_{i=1}^n$ be a samble given by (6), the hybrid nonparametric estimator of the Pickands dependence function is defined as

$$\hat{A}_n^{\mathcal{H}*}(\mathbf{w}) = \frac{\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) + c(\mathbf{w})}{1 - \hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - c(\mathbf{w})}, \tag{12}$$

where $\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w})$ defined in Equation (10) and $c(\mathbf{w}) = d^{-1} \sum_{j=1}^d w_j/(1 + w_j)$.

Using the results of [16] (namely, Theorem 2.4) and Proposition 1 and Proposition 2 of this paper, we state the following corollary.

Corollary 2. *Let $(\boldsymbol{I}_i, \tilde{\boldsymbol{X}}_i)_{i=1}^n$ be a samble given by* (6)*. For $\mathbf{w} \in \Delta^{d-1}$, if C is an extreme value copula with Pickands dependence function, we have under Assumption B*

$$\hat{A}_n^{\mathcal{H}*}(\mathbf{w}) \underset{n \to \infty}{\overset{a.s.}{\to}} A(\boldsymbol{w}).$$

Furthermore, if $(\boldsymbol{I}_i, \tilde{\boldsymbol{X}}_i)_{i=1}^n$ additionally verifies assumptions A and C, we obtain

$$\sqrt{n}\left(\hat{A}_n^{\mathcal{H}*}(\mathbf{w}) - A(\mathbf{w})\right) \underset{n \to \infty}{\overset{d}{\to}} \mathcal{N}\left(0, \mathcal{V}(\mathbf{p}, \mathbf{w})\right),$$

where $\mathcal{V}(\mathbf{p}, \mathbf{w})$ is the asymptoptic variance and its closed formula is given by

$$\mathcal{V}(\mathbf{p}, \mathbf{w}) = (1 + A(\mathbf{w}))^4 \mathcal{S}^{\mathcal{H}*}(\mathbf{p}, \mathbf{w}).$$

Proof of this result can be find in Section 4.1.

3. Numerical results

We verify our findings concerning the closed formula of the asymptotic variance through a simulation study. To do so, we compare an empirical counterpart of the asymptotic variance computed out with Monte-Carlo simulation with the explicit asymptotic variance given by Proposition 1. Before going into details, we present several models that will be used in the simulation section. The Gumbel and the asymmetric logistic models are exposed to their d-dimensional version (see Model 1 and Model 2 below). Multivariate extensions of other bivariate models are more complex whence the asymmetric negative logistic model and the asymmetric mixed are given for the bivariate case (Model 3 and Model 4).

1. The symmetric logistic model [11] defined by the following dependence function

$$A(w_1, \dots, w_d) = \left(\sum_{j=1}^{d} w_j^{\theta} \right)^{1/\theta},$$

   with $\theta \in [1, \infty)$. In the Gumbel model, we retrieve the independent case when $\theta = 1$, the dependence between the variables is stronger as θ goes to infinity. The restriction to $d = 2$ is immediate from the definition.

2. Let B be the set of all nonempty subsets of $\{1, \dots, d\}$ and $B_1 = \{b \in B, |b| = 1\}$, where $|b|$ denotes the number of elements in the set b. The asymmetric logistic model [25] is defined by the following dependence function

$$A(w_1, \dots, w_d) = \sum_{b \in B} \left(\sum_{j \in b} (\theta_{j,b} w_j)^{\theta_b} \right)^{1/\theta_b},$$

   where $\theta_b \in [1, \infty)$ for all $b \in B \setminus B_1$, and the asymmetry parameters $\theta_{j,b} \in [0, 1]$ for all $b \in B$ and $j \in b$. The model should verify the following constrains $\sum_{b \in B(j)} \theta_{j,b} = 1$ for $j \in \{1, \dots, d\}$ where $B_{(j)} = \{b \in B, j \in b\}$ and if $\theta_b = 1$ for every $b \in B \setminus B_1$, then $\theta_{j,b} = 0 \; \forall j \in b$. The model contains $2^d - d - 1$ dependence parameters and $d(2^{d-1} - 1)$ asymmetry parameters. In case of $d = 2$, we go back to the asymmetric logistic model [24], namely

$$A(w) = (1 - \psi_1)w + (1 - \psi_2)(1 - w) + \left[(\psi_1 w)^{\theta} + (\psi_2 (1 - w))^{\theta} \right]^{1/\theta},$$

   with $\theta \in [1, \infty)$, $\psi_1, \psi_2 \in [0, 1]$. For $d = 3$, the Pickands is expressed as

$$\begin{aligned} A(\mathbf{w}) = & \alpha_1 w_1 + \psi_1 w_2 + \phi_1 w_3 + \left((\alpha_2 w_1)^{\theta_1} + (\psi_2 w_2)^{\theta_1} \right)^{1/\theta_1} + \left((\alpha_3 w_2)^{\theta_2} + (\phi_2 w_3)^{\theta_2} \right)^{1/\theta_2} + \left((\psi_3 w_2)^{\theta_3} + (\phi_3 w_3)^{\theta_3} \right)^{1/\theta_3} \\ & + \left((\alpha_4 w_1)^{\theta_4} + (\psi_4 w_2)^{\theta_4} + (\phi_4 w_3)^{\theta_4} \right)^{1/\theta_4}, \end{aligned}$$

   where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_4)$, $\boldsymbol{\psi} = (\psi_1, \dots, \psi_4)$, $\boldsymbol{\phi} = (\phi_1, \dots, \phi_4)$ are all elements of Δ^3.

3. The asymmetric negative logistic model [14] as

$$A(w) = 1 - \left[(\psi_1 (1 - w))^{-\theta} + (\psi_2 w)^{-\theta} \right]^{-1/\theta},$$

   with parameters $\theta \in (0, \infty)$, $\psi_1, \psi_2 \in (0, 1]$. The secial case $\psi_1 = \psi_2 = 1$ returns the Galambos model [18].

4. The asymmetric mixed model [24] :

$$A(w) = 1 - (\theta + \kappa)w + \theta w^2 + \kappa w^3,$$

   with parameters θ and κ satisfying $\theta \geq 0$, $\theta + 3\kappa \geq 0$, $\theta + \kappa \leq 1$, $\theta + 2\kappa \leq 1$. The special case $\kappa = 0$ and $\theta \in [0, 1]$ yields the symmetric mixed model. In the symmetric mixed model, when $\theta = 0$, we recover the independent copula.

For each numerical experiment, the endpoint-corrected $\mathbf{w}$-madogram estimator in (10) is computed using $\lambda_j(\mathbf{w}) = w_j$. The study consists of three experiments. For each experiment, the empirical counterpart of the asymptotic variance given by Proposition 1 is computed out through a given grid of the simplex Δ^{d-1}, for each element $\mathbf{w}$ of this grid 100 simulated random sample of size $n = 512$ are simulated from the models given above under which we compute an estimator of the $\mathbf{w}$-madogram. We thus compute the empirical variance of the normalized estimation error namely,

$$\mathcal{E}_n^{\mathcal{H}}(\mathbf{w}) := \widehat{Var}\left(\sqrt{n}\left(\hat{\boldsymbol{\nu}}_{\boldsymbol{n}}^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right)\right), \quad \mathcal{E}_n^{\mathcal{H}*}(\mathbf{w}) := \widehat{Var}\left(\sqrt{n}\left(\hat{\boldsymbol{\nu}}_{\boldsymbol{n}}^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right)\right), \tag{13}$$

where $\hat{\boldsymbol{\nu}}_{\boldsymbol{n}}^{\mathcal{H}}$ and $\hat{\boldsymbol{\nu}}_{\boldsymbol{n}}^{\mathcal{H}*}$ are the vectors composed out of the n_{iter} estimator of the hybrid and the corrected estimator of the $\mathbf{w}$-madogram with $n_{iter} \in \mathbb{N} \setminus \{0\}$, respectively. Note that for the first and the second experiment, the missing mechanism is such as $I_1, \dots, I_d$ are pairwise independent and $p_j = p_1, \forall j \in \{1, \dots, d\}$. The independence setup corresponds to the worst scenario where the missigness of one variable does not influence the missingness of the other variables. *A contrario*, if we suppose that $I_1, \dots, I_d$ are strongly dependent, *i.e.* none or all entries are missing. We thus estimate a statistic on a sample of average length $p \times n$ and we are turning back to inference in a complete data framework with a reduced sample size. This is also readily seen from the closed formula in Proposition 1, indeed in a strongly dependent setup we have $p = p_1$, so the asymptotic variance is reduced to the complete data framework up to a multiplicative factor.

First experiment : We set $d = 2$. A Monte Carlo study is implemented here to illustrate Proposition 1 in finite-sample setting with missing data where we fix $n_{iter} = 500$. We consider the Galambos, the asymmetric negative logistic, the asymmetric logistic and the asymmetric mixed models. The chosen grid is $\{1/200, \dots, 199/200\}$ and we take $p_1 = p_2 = 0.75$.

Second experiment : We fix $d = 3$ and we consider the independent, the symmetric logistic and the asymmetric logistic models given in Model 1 and 2 with $n_{iter} = 100$. For the two first models, we set the dependence parameter in Model 1 as $\theta = 1$ and $\theta = 2$. We grid the $[0,1] \times [0,1]$ cube into 10000 points at same distance from each other and we only keep those with $w_2 + w_3 < 1.0$ where w_2 and w_3 are in the grid of the cube, we set $w_1 = 1 - w_2 - w_3$. In this experiment, we took $\boldsymbol{\alpha} = (0.4, 0.3, 0.1, 0.2)$, $\boldsymbol{\psi} = (0.1, 0.2, 0.4, 0.3)$, $\boldsymbol{\phi} = (0.6, 0.1, 0.1, 0.2)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_4) = (0.6, 0.5, 0.8, 0.3)$ as the dependence parameter. We take $p_1 = p_2 = p_3 = 0.9$ and thus $p = 0.729$, $p_{ij} = 0.81$ with $i, j \in \{1, 2, 3\}$ and $i < j$.

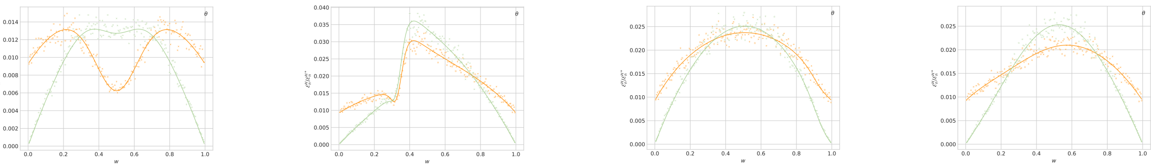
Third experiment : In this technical experiment, we want to show that our conclusions can be verified in a high dimension setting. In other words, we verify that our program is able to compute the asymptotic variance for a varying dimension and we also check with Monte-Carlo replications that this quantity is close to the empirical counterpart. We also set that there is no missing data as it is accessory of the purpose of this experiment. We consider the symmetric logistic model with dependence parameter $\theta = 2$. We sample 100 points from the unit simplex Δ^{d-1} and we compute the following quantity

$$\delta_n^{\mathcal{H}}(\mathbf{w}) \triangleq \frac{\left|\mathcal{E}_n^{\mathcal{H}}(\mathbf{w}) - \mathcal{S}^{\mathcal{H}}(\mathbf{1}, \mathbf{w})\right|}{\mathcal{S}^{\mathcal{H}}(\mathbf{1}, \mathbf{w})}, \tag{14}$$

where $\mathcal{E}_n^{\mathcal{H}}$ is computed from $n_{iter} = 100$ estimators of the $\mathbf{w}$-madogram with sample size $n = 512$. The results are collected for some values of $d \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$.

Results of the first experiment are depicted in Fig. 1. For all panels, empirical counterparts given by (13) depicted as points fits the theoretical values exhibited from Proposition 1 depicted as solid lines. For the hybrid estimator, as discussed in Remark 1, both empirical and theoretical values of the asymptotic variance are different from zero for each $w \in \{\{0\}, \{1\}\}$. The corrected version gives back this feature and also modify the shape of the curve similar to what can be found in the case of complete data. Notice that, in terms of variance, we do not have a strict dominance from one estimator to another.

Results for the second experiment are depicted in Fig. 2 and Fig. 3. In the first figure, empirical counterparts given by Equation (13) are depicted with points and closed expression of the asymptotic variance given by Proposition 1 is drawn by a surface. As in the first experiment, empirical counterparts given by the points fits the surface. Also, for the first row, we see that if $\mathbf{w} \in \{\{\mathbf{e}_1\}, \{\mathbf{e}_2\}, \{\mathbf{e}_3\}\}$, thus both theoretical and empirical counterparts are different from zero while this feature is erased, in the second row, with the introduction of the corrected version. For the second figure, levelsets of panels of the first figure are depicted. The conclusion remains, levelsets of empirical counterparts are closed to those drawn by the theoretical ones.



(a) Galambos ($\theta = 2.5$) **(b) Neg. Log.** ($\theta = 10, \psi_1 = .5, \psi_2 = 1$) **(c) Asy. Log.** ($\theta = \frac{5}{2}, \psi_1 = .1, \psi_2 = 1$) **(d) Asy. Mixed** ($\theta = \frac{4}{3}, \kappa = -\frac{1}{3}$)

Fig. 1: $\mathcal{E}_n^{\mathcal{H}}$ in red and $\mathcal{E}_n^{\mathcal{H}*}$ in green for the corrected version of the estimator, as a function of w, of the asymptotic variances of the estimators of the $\mathbf{w}$-FMadogram for four extreme-value copula models. The empirical variances are based on 500 samples of size $n = 512$. Solid lines are the theoretical value given by Proposition 1.

(a) Independent **(b) Gumbel** **(c) Asymmetric negative logistic**

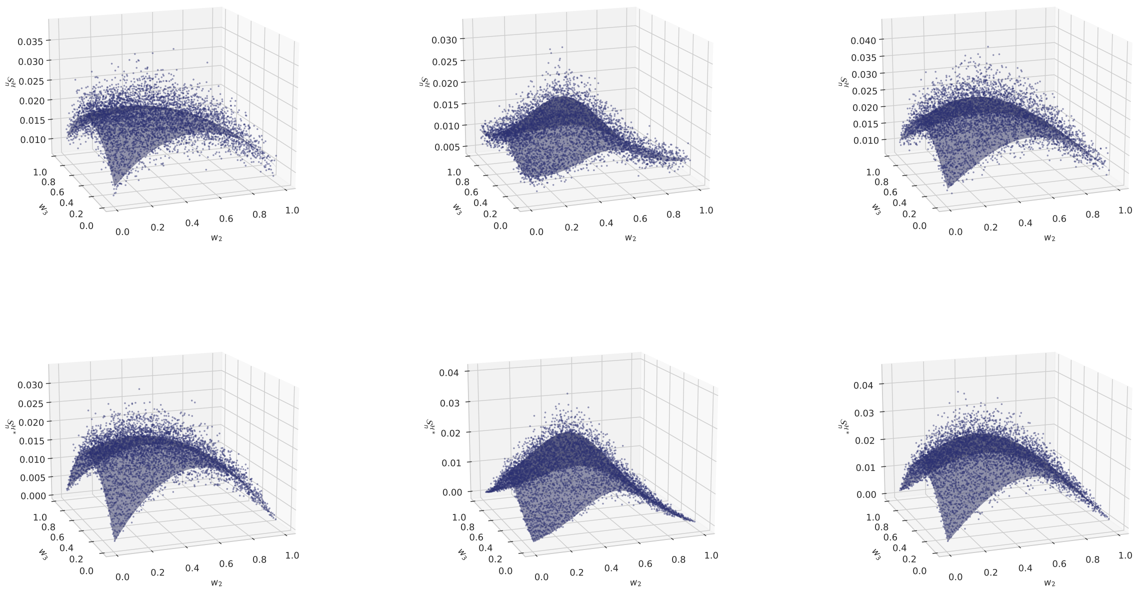


Fig. 2: $\mathcal{E}_n^{\mathcal{H}}$ and $\mathcal{E}_n^{\mathcal{H}*}$, as a function of $\mathbf{w}$, of the asymptotic variances of the estimators of the $\mathbf{w}$-madogram for independent, symmetric logistic and asymmetric logistic model (resp. first, second and third columns). Comparison with the hybrid estimator (first row) with the corrected estimator (second row). The empirical variances are based on 100 samples of size $n = 512$. Empirical counterparts are represented with points and theoretical values given by Proposition 1 are drawn by a surface.

(a) Independent **(b) Gumbel** **(c) Asymmetric negative logistic**

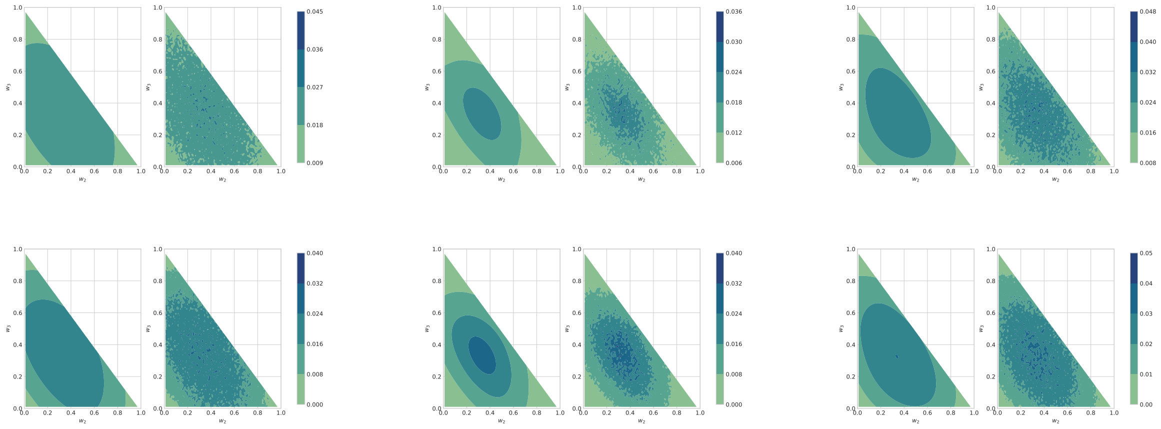


Fig. 3: levelset of $\mathcal{E}_n^{\mathcal{H}}$ and $\mathcal{E}_n^{\mathcal{H}*}$, as a function of $\mathbf{w}$, of the asymptotic variances of the estimators of the $\mathbf{w}$-madogram for independent, symmetric logistic and asymmetric logistic model (resp. first, second and third columns). We present the levelset of the corresponding figures above (see Fig 2). On the left panel is represented the theoretical value given by Proposition 1 while on the right the empirical counterpart is given.

Fig. 4 illustrate the results of the third experiment where we have drawn the empirical confidence for Equation (14) of order 0.95. We observe that the length of the empirical confidence interval increases with d. This behavior comes from the fact that, as d increases, the theoretical value of the asymptotic variance grows.

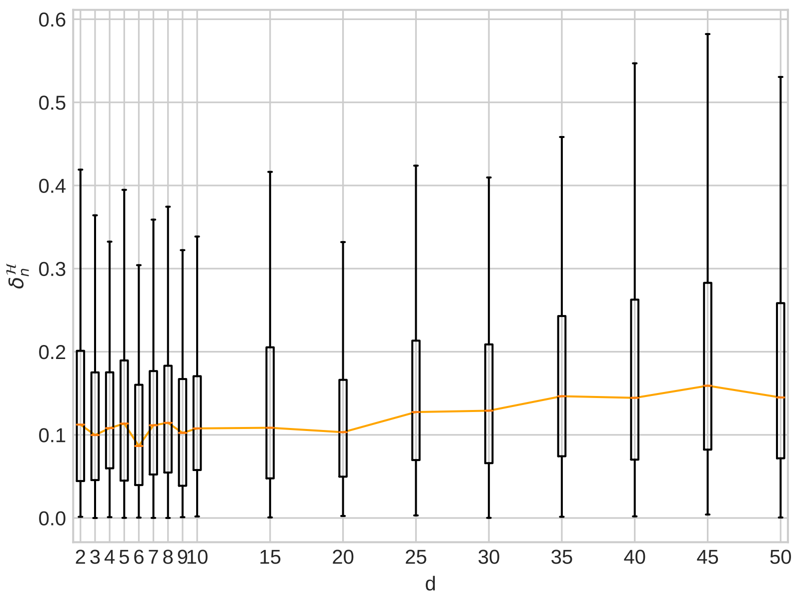


Fig. 4: Empirical confidence intervals for $\delta_n^{\mathcal{H}}$ of order 0.95 for different values of d. The solid line in orange correspond to the median.

4. Proof

4.1. Proof of the main results

For the rest of this section, we will write, for notational convenience, $n_i = \Pi_{j=1}^d I_{i,j}$ and $N = \sum_{i=1}^n n_i$. The following proof gives arguments used to establish the functional central limit theorem of our processes defined in Equation (11). Before going into the details, we need a intermediary lemma to assert that the empirical cumulative distribution functions in case of missing data does verify Assumption C and give covariance functions of the asymptotic processes α and β_j with $j \in \{1, \dots, d\}$. This result comes down from [22] (see Example 3.5) where the result was proved for bivariate random variables but the higher dimension is directly obtained using same arguments.

Lemma 1. *The vector* $(\sqrt{n}(\hat{F}_n - F);\ \sqrt{n}(\hat{F}_{n,1} - F_1), \dots, \sqrt{n}(\hat{F}_{n,d} - F_d))$ *where* $\hat{F}_n$ *and* $\hat{F}_{n,j}$ *for* $j \in \{1, \dots, d\}$ *are defined in* (7) *does verify Assumption C with*

$$\begin{aligned}
\beta_j(u_j) &= p_j^{-1} \mathbb{G}\left(\mathbb{1}_{\{X_j \leq F_j^{\leftarrow}(u_j), I_j = 1\}} - u_j \mathbb{1}_{\{I_j = 1\}}\right), \quad j \in \{1, \dots, d\}, \\
\alpha(\boldsymbol{u}) &= p^{-1} \mathbb{G}\left(\mathbb{1}_{\{\mathbf{X} \leq \mathbf{F}_d^{\leftarrow}(\mathbf{u}), \mathbf{I} = \mathbf{1}\}} - C(\mathbf{u}) \mathbb{1}_{\{\mathbf{I} = \mathbf{1}\}}\right),
\end{aligned}$$

where $\mathbb{G}$ *is a tight Gaussian process. Furthermore the covariance functions of the processes* $\beta_j(u_j)$, $\alpha(\boldsymbol{u})$ *are, for* $(\mathbf{u}, \mathbf{v}, v_k) \in [0,1]^{2d+1}$, *and for* $j \in \{1, \dots, d\}$ *and* $j < k$

$$\begin{aligned}
cov\left(\beta_j(u_j), \beta_j(v_j)\right) &= p_j^{-1}\left(u_j \wedge v_j - u_j v_j\right), \\
cov\left(\beta_j(u_j), \beta_k(v_k)\right) &= \frac{p_{jk}}{p_j p_k}\left(C(\mathbf{1}_{j,k}(u_j, v_k)) - u_j v_k\right),
\end{aligned}$$

and

$$\begin{aligned}
cov\left(\alpha(\mathbf{u}), \alpha(\mathbf{v})\right) &= p^{-1}\left(C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{u})\right), \\
cov\left(\alpha(\mathbf{u}), \beta_j(v_j)\right) &= p_j^{-1}\left(C(\mathbf{u}_j(u_j \wedge v_j)) - C(\mathbf{u}) v_j\right),
\end{aligned}$$

where $\mathbf{u} \wedge \mathbf{v}$ *denotes the vector of componentwise minima and* $p_{jk} = \mathbb{P}(I_j = 1, I_k = 1)$.

Proof of Lemma 1 is postponed to Section 4.2.

Proof of Theorem 1. Details for the proof are given solely for the estimators $\hat{\nu}_n^{\mathcal{H}*}$ as the proof for $\hat{\nu}_n^{\mathcal{H}}$ is similar. Using that $\mathbb{E}[F_j(X_j)^\alpha] = (1+\alpha)^{-1}$ for $\alpha \neq 1$, we can write $\nu(\mathbf{w})$ as :

$$\begin{aligned}
\nu(\mathbf{w}) =& \mathbb{E}\left[\bigvee_{j=1}^d \left\{F_j(X_j)\right\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^d \left\{F_j(X_j)\right\}^{1/w_j}\right] + \sum_{j=1}^d \frac{\lambda_j(\mathbf{w})(d-1)}{d}\left(\frac{w_j}{1+w_j} - \mathbb{E}\left[\left\{F_j(X_j)\right\}^{1/w_j}\right]\right), \\
=& \mathbb{E}\left[\bigvee_{j=1}^d \left\{F_j(X_j)\right\}^{1/w_j}\right] - \frac{1}{d}\sum_{j=1}^d (1 + \lambda_j(\mathbf{w})(d-1)) \mathbb{E}\left[\left\{F_j(X_j)\right\}^{1/w_j}\right] + a(\mathbf{w}),
\end{aligned}$$

with $a(\mathbf{w}) = (d-1)d^{-1}\sum_{j=1}^d \lambda_j(\mathbf{w}) w_j/(1+w_j)$. Let us note by $g_{\mathbf{w}}$ the function defined as

$$g_{\mathbf{w}} : [0,1]^d \to [0,1], \quad \mathbf{u} \mapsto \bigvee_{j=1}^d u_j^{1/w_j} - \frac{1}{d}\sum_{j=1}^d (1 + \lambda_j(\mathbf{w})(d-1)) u_j^{1/w_j}.$$

One can write our estimator of the $\mathbf{w}$-madogram and the theoretical $\mathbf{w}$-madogram in missing data framework as an integral with respect to the hybrid copula estimator and the copula function. We thus have:

$$\begin{aligned}
\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) &= \frac{1}{N}\sum_{i=1}^n g_{\mathbf{w}}\left(\hat{\mathbf{F}}_n(\tilde{\mathbf{X}}_i)\right) n_i + a(\mathbf{w}) = \int_{[0,1]^d} g_{\mathbf{w}}(\mathbf{u})\, d\hat{C}_n^{\mathcal{H}}(\mathbf{u}) + a(\mathbf{w}), \\
\nu(\mathbf{w}) &= \int_{[0,1]^d} g_{\mathbf{w}}(\mathbf{u})\, dC(\mathbf{u}) + a(\mathbf{w}),
\end{aligned}$$

where $\hat{\mathbf{F}}_n(\tilde{\mathbf{X}}_i) = (\hat{F}_{n,1}(\tilde{X}_{i,1}), \dots, \hat{F}_{n,d}(\tilde{X}_{i,d}))$. We obtain, proceeding as in Theorem 2.4 of [16] :

$$\sqrt{n}\left(\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w}) - \nu(\mathbf{w})\right) = \frac{1}{d}\sum_{j=1}^{d}\left(1 + \lambda_j(\mathbf{w})(d-1)\right)\int_{[0,1]} \mathbb{C}_n^{\mathcal{H}}(\mathbf{1}_j(x^{w_j}))dx - \int_{[0,1]} \mathbb{C}_n^{\mathcal{H}}\left(x^{w_1}, \dots, x^{w_d}\right) dx,$$

where $\mathbf{1}_j(u)$ denotes the vector composed out of 1 except for the jth component where u does stand and with $\mathbb{C}_n^{\mathcal{H}}$ in (8). Consider the function $\phi : \ell^\infty([0,1]^d) \to \ell^\infty(\Delta^{d-1}), f \mapsto \phi(f)$, defined by

$$(\phi)(f)(\mathbf{w}) = \frac{1}{d}\sum_{j=1}^{d}(1 + \lambda_j(\mathbf{w})(d-1))\int_{[0,1]} f(\mathbf{1}_j(x^{w_j}))dx - \int_{[0,1]} f(x^{w_1}, \dots, x^{w_d})dx.$$

This function is linear and bounded thus continuous. The continous mapping theorem (Theorem 1.3.6 of [26]) implies, as $n \to \infty$

$$\sqrt{n}(\hat{\nu}_n^{\mathcal{H}*} - \nu) = \phi(\mathbb{C}_n^{\mathcal{H}}) \rightsquigarrow \phi(S_C),$$

in $\ell^\infty(\Delta^{d-1})$. Recall that S_C is the asymptotic process where $\mathbb{C}_n^{\mathcal{H}}$ does converge in the sense of the weak convergence in $\ell^\infty(\Delta^{d-1})$ and is defined by $S_C(\mathbf{u}) = \alpha(\mathbf{u}) - \sum_{j=1}^{d}\beta_j(u_j)\dot{C}_j(\mathbf{u})$ with $\mathbf{u} \in [0,1]^d$ and α and β_j are processes defined in Lemma 1. We note that $S_C(\mathbf{1}_j(x^{w_j})) = \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(u_j)$ and we obtain our statement. □

The asymptotic normality of our estimators directly comes down from being a linear transformation of a tight Gaussian process for $\mathbf{w} \in \Delta^{d-1}$. The proof below use technical arguments to exhibits the closed expressions of the asymptotic variances of the Gaussians limit law of our estimators defined in Equation (9) and (10). Two tools make the computation feasible, the first one is the form exhibited by Equation (2) which transform a double integrals with respect to the trajectory of the copula function as the double integrals of a power function. When this trick is not possible, again the expression of the extreme value copula with respect to the Pickands dependence function is of main interest. Indeed, with some substitutions, we are able to express the double integrals as the integral with respect to the Pickands dependence function using the following equality :

$$-\int_{[0,1]} w^\alpha ln(w)dw = \frac{1}{(\alpha+1)^2},$$

where $\alpha \neq 1$.

Proof of Proposition 1. By definition the asymptotic variance $\mathcal{S}^{\mathcal{H}}(\mathbf{p}, \mathbf{w})$ is given for a fixed $\mathbf{w} \in \Delta^{d-1}$ by

$$\mathcal{S}^{\mathcal{H}}(\mathbf{p}, \mathbf{w}) := Var\left(\frac{1}{d}\sum_{j=1}^{d}\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx - \int_{[0,1]} S_C(x^{w_1}, \dots, x^{w_d})dx\right).$$

Using the property of the variance, we thus obtain

$$\begin{aligned}
\mathcal{S}^{\mathcal{H}}(\mathbf{p}, \mathbf{w}) = & \frac{1}{d^2}\sum_{j=1}^{d} Var\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx\right) + Var\left(\int_{[0,1]} S_C(x^{w_1}, \dots, x^{w_d})dx\right) \\
& + \frac{2}{d^2}\sum_{j<k} cov\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx, \int_{[0,1]} \alpha(\mathbf{1}_k(x^{w_k})) - \beta_k(x^{w_k})dx\right) \\
& - \frac{2}{d}\sum_{j=1}^{d} cov\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx, \int_{[0,1]} \alpha(x^{w_1}, \dots, x^{w_d})dx\right) \\
& + \frac{2}{d}\sum_{j=1}^{d}\sum_{k=1}^{d} cov\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx, \int_{[0,1]} \beta_k(x^{w_k})\dot{C}_k(x^{w_1}, \dots, x^{w_d})dx\right).
\end{aligned}$$

By order, we write

$$Var\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx\right) = \left(p^{-1} - p_j^{-1}\right)\sigma_j^2(\mathbf{w}),$$
$$Var\left(\int_{[0,1]} S_C(x^{w_1},\dots,x^{w_d})dx\right) = \sigma_{d+1}^2(\mathbf{p},\mathbf{w}),$$
$$cov\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx, \int_{[0,1]} \alpha(\mathbf{1}_k(x^{w_k})) - \beta_k(x^{w_k})dx\right) = \left(p^{-1} - p_j^{-1} - p_k^{-1} + \frac{p_{jk}}{p_j p_k}\right)\sigma_{jk}(\mathbf{w}),$$
$$cov\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx, \int_{[0,1]} \alpha(x^{w_1},\dots,x^{w_d})dx\right) = \left(p^{-1} - p_j^{-1}\right)\sigma_j^{(1)}(\mathbf{w}),$$
$$cov\left(\int_{[0,1]} \alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx, \int_{[0,1]} \beta_k(x^{w_k})\dot{C}_k(x^{w_1},\dots,x^{w_d})dx\right) = \left(p_k^{-1} - \frac{p_{jk}}{p_j p_k}\right)\sigma_{jk}^{(2)}(\mathbf{w}).$$

We first show in details the closed form for σ_{d+1}^2, the other forms are given without explanations as the technical tools used are those used for σ_{d+1}^2. Proceding as before, we decompose this quantity as its sum of the variance (the squared term γ_1^2 and γ_j^2 for $j \in \{1,\dots,d\}$) and the covariance terms (γ_{1j} and τ_{jk}). The explicit formula of these quantities will be defined below. We thus explicit σ_{d+1}^2 as a linear combination of the probabilities of observing and the variance and covariances terms such as :

$$\sigma_{d+1}^2(\mathbf{p},\mathbf{w}) = p^{-1}\gamma_1^2(\mathbf{w}) + \sum_{j=1}^d p_j^{-1}\gamma_j^2(\mathbf{w}) - 2\sum_{j=1}^d p_j^{-1}\gamma_{1j}(\mathbf{w}) + 2\sum_{j<k}\frac{p_{jk}}{p_j p_k}\tau_{jk}(\mathbf{w}). \tag{15}$$

Let us before exhibit a useful form of the partial derivatives of the extreme value copula. We have $\forall j \in \{1,\dots,d\}$:

$$\dot{C}_j(\mathbf{u}) = \frac{C(\mathbf{u})}{u_j}\dot{\ell}_j(-ln(u_1),\dots,-ln(u_d)).$$

Furthermore, as $\ell(x_1,\dots,x_d)$ is homogeneous of degree 1, the partial derivative $\dot{\ell}_j(x_1,\dots,x_d)$ is homogeneous of degree 0 for $j \in \{1,\dots,d\}$, we thus obtain a suitable form of the partial derivatives of the extreme value copula for $u \in]0,1[$ and $\mathbf{w} \in \Delta^{d-1}$:

$$\dot{C}_j(u^{w_1},\dots,u^{w_d}) = \frac{u^{A(\mathbf{w})}}{u^{w_j}}\dot{\ell}_j(-w_1 ln(u),\dots,-w_d ln(u)) = \frac{u^{A(\mathbf{w})}}{u^{w_j}}\dot{\ell}_j(-w_1,\dots,-w_d) = \frac{u^{A(\mathbf{w})}}{u^{w_j}}\mu_j(\mathbf{w}).$$

Now, using linearity of the integral and the definition of the covariance function of α, we obtain

$$\begin{aligned} p^{-1}\gamma_1^2(\mathbf{w}) &\triangleq \mathbb{E}\left[\int_{[0,1]} \alpha(u^{w_1},\dots,u^{w_d})du \int_{[0,1]} \alpha(v^{w_1},\dots,v^{w_d})dv\right], \\ &= \frac{2}{p}\int_{[0,1]}\int_{[0,v]} u^{A(\mathbf{w})}(1 - v^{A(\mathbf{w})})dudv. \end{aligned}$$

let us compute

$$\gamma_1^2(\mathbf{w}) = 2\int_{[0,1]}\int_{[0,v]} u^{A(\mathbf{w})}(1 - v^{A(\mathbf{w})})dudv = \frac{1}{(1+A(\mathbf{w}))^2}\frac{A(\mathbf{w})}{2+A(\mathbf{w})}.$$

The quantity $\gamma_j^2(\mathbf{w})$ is defined by the following

$$\begin{aligned} p_j^{-1}\gamma_j^2(\mathbf{w}) &\triangleq \mathbb{E}\Big[\int_{[0,1]} \beta_j(u^{w_j})\dot{C}_j(u^{w_1},\dots,u^{w_d})du \int_{[0,1]} \beta_j(u^{w_j})\dot{C}_j(v^{w_1},\dots,v^{w_d})dv\Big], \\ &= \frac{2}{p_j}\int_{[0,1]}\int_{[0,v]} u^{w_j}(1 - v^{w_j})\mu_j(\mathbf{w})\mu_j(\mathbf{w})u^{A(\mathbf{w})-w_j}v^{A(\mathbf{w})-w_j}dudv. \end{aligned}$$

It is clear that

$$\gamma_j^2(\mathbf{w}) = 2\int_{[0,1]}\int_{[0,v]} u^{w_j}(1-v^{w_j})\mu_j(\mathbf{w})\mu_j(\mathbf{w})u^{A(\mathbf{w})-w_j}v^{A(\mathbf{w})-w_j}duv = \left(\frac{\mu_j(\mathbf{w})}{1+A(\mathbf{w})}\right)^2 \frac{w_j}{2A(\mathbf{w})+1+1-w_j}.$$

We now deal with cross product terms, the first we define is

$$\begin{aligned} p_j^{-1}\gamma_{1j}(\mathbf{w}) &\triangleq \mathbb{E}\Big[\int_{[0,1]}\alpha(u^{w_1},\dots,u^{w_d})du\int_{[0,1]}\beta_j(v^{w_j})\dot{C}_j(v^{w_1},\dots,v^{w_d})dv\Big], \\ &= p_j^{-1}\int_{[0,1]^2}\left(C(u^{w_1},\dots,(u\wedge v)^{w_j},\dots,u^{w_d}) - u^{A(\mathbf{w})}v^{w_j}\right)\dot{C}_j(v^{w_1},\dots,v^{w_d})duv. \end{aligned}$$

Under the rectangle $[0,1]\times[0,v]$, we have

$$\begin{aligned} \gamma_{1j}(\mathbf{w}) &= \int_{[0,1]\times[0,v]}\left(C(u^{w_1},\dots,u^{w_j},\dots,u^{w_d}) - u^{A(\mathbf{w})}v^{w_j}\right)\dot{C}_j(v^{w_1},\dots,v^{w_d})duv, \\ &= \int_{[0,1]\times[0,v]} u^{A(\mathbf{w})}(1-v^{w_j})v^{A(\mathbf{w})-w_j}\mu_j(\mathbf{w})duv = \frac{\mu_j(\mathbf{w})}{2(1+A(\mathbf{w}))^2}\frac{w_j}{2A(\mathbf{w})+1+(1-w_j)}. \end{aligned}$$

Under the rectangle $[0,1]\times[0,u]$, we have for the right term

$$\int_{[0,1]\times[0,u]} u^{A(\mathbf{w})}v^{w_j}v^{A(\mathbf{w})-w_j}\mu_j(\mathbf{w})dvu = \frac{\mu_j(\mathbf{w})}{2(1+A(\mathbf{w}))^2}.$$

For the left term, by definition, we have

$$\int_{[0,1]\times[0,u]} C(u^{w_1},\dots,v^{w_j},\dots,u^{w_d})\dot{C}_j(v^{w_1},\dots,v^{w_d})dvu.$$

Let us consider the substitution $x = v^{w_j}$ and $y = u^{1-w_j}$, we obtain

$$\frac{1}{w_j(1-w_j)}\int_{[0,1]}\int_{[0,y^{w_j/(1-w_j)}]} C\left(y^{w_1/(1-w_j)},\dots,x,\dots,y^{w_d/(1-w_j)}\right)\dot{C}_j\left(x^{w_1/w_j},\dots,x^{w_d/w_j}\right)x^{(1-w_j)/w_j}y^{w_j/(1-w_j)}dxy.$$

Let us compute the quantity

$$\dot{C}_j(x^{w_1/w_j},\dots,x^{w_d/w_j}) = \frac{C(x^{w_1/w_j},\dots,x^{w_d/w_j})}{x}\mu_j(\mathbf{w}).$$

Using Equation (1), we have

$$C(x^{w_1/w_j},\dots,x^{w_d/w_j}) = exp\left(-\ell\left(-\frac{ln(x)}{w_j}w_1,\dots,\frac{ln(x)}{w_j}w_d\right)\right) = exp\left(-\frac{ln(x)}{w_j}\ell\left(-w_1,\dots,-w_d\right)\right) = x^{A(\mathbf{w})/w_j} = x^{A_j(\mathbf{w})}.$$

Where we use the homogeneity of order one of ℓ and that $-\ell(-w_1,\dots,-w_d) = A(\mathbf{w})$ because of Equation (2) and that $\mathbf{w}\in\Delta^{d-1}$. Now, consider the substitution $x = w^{1-s}$ and $y = w^s$, the jacobian of this transformation is given by $-ln(w)$, we have

$$-\frac{\mu_j(\mathbf{w})}{w_j(1-w_j)}\int_{[0,1]}\int_{[0,1-w_j]} C\left(w^{sw_1/(1-w_j)},\dots,w^{1-s},\dots,w^{sw_d/(1-w_j)}\right)w^{(1-s)\left[A_j(\mathbf{w})+\frac{1-w_j}{w_j}-1\right]+s\frac{w_j}{1-w_j}}ln(w)dsw,$$

Where we note by $A_j(\mathbf{w}) := A(\mathbf{w})/w_j$ with $j\in\{1,\dots,d\}$. We now compute the quantity

$$C\left(w^{sw_1/(1-w_j)},\dots,w^{1-s},\dots,w^{sw_d/(1-w_j)}\right).$$

Using the same techniques as above, we have

$$C\left(w^{sw_1/(1-w_j)}, \dots, w^{1-s}, \dots, w^{sw_d/(1-w_j)}\right) = exp\left(-\ell\left(-\frac{sw_1}{1-w_j}ln(w), \dots, -(1-s)ln(w), \dots, -\frac{sw_d}{1-w_j}ln(w)\right)\right),$$
$$= exp\left(-ln(w)\ell\left(-\frac{sw_1}{1-w_j}, \dots, -(1-s), \dots, -\frac{sw_d}{1-w_j}\right)\right).$$

Now, using that $\mathbf{w} \in \Delta^{d-1}$, remark that $s\sum_{i\neq j} w_i/(1-w_j) = s$, we have, using Equation (2)

$$-\ell\left(-\frac{sw_1}{1-w_j}, \dots, -(1-s), \dots, -\frac{sw_d}{1-w_j}\right) = A\left(\mathbf{z}_j(1-s)\right),$$

where $\mathbf{z} = (sw_1/(1-w_j), \dots, sw_d/(1-w_j))$. So we have

$$\gamma_{1j}(\mathbf{w}) = -\frac{\mu_j(\mathbf{w})}{w_j(1-w_j)}\int_{[0,1-w_j]}\int_{[0,1]} w^{A(\mathbf{z}_j(1-s))+(1-s)\left(A_j(\mathbf{w})+\frac{1-w_j}{w_j}-1\right)+s\frac{w_j}{1-w_j}} ln(w)dws,$$
$$= \frac{\mu_j(\mathbf{w})}{w_j(1-w_j)}\int_{[0,1-w_j]}\left[A\left(\mathbf{z}_j(1-s)\right) + (1-s)\left(A_j(\mathbf{w}) + \frac{1-w_j}{w_j} - 1\right) + s\frac{w_j}{1-w_j} + 1\right]^{-2} ds.$$

No further simplifications can be obtained. For $j < k$, let us define the quantity τ_{jk} such as

$$\frac{p_{jk}}{p_j p_k}\tau_{jk}(\mathbf{w}) \triangleq \mathbb{E}\Big[\int_{[0,1]} \beta_j(u^{w_j})\dot{C}_j(u^{w_1}, \dots, u^{w_d})du \int_{[0,1]} \beta_k(v^{w_k})\dot{C}_k(v^{w_1}, \dots, v^{w_d})dv\Big]. \tag{16}$$

Again, we have

$$\tau_{jk}(\mathbf{w}) = \int_{[0,1]^2}\left(C(\mathbf{1}_{jk}(u^{w_j}, v^{w_j})) - u^{w_j}v^{w_j}\right)\dot{C}_j(u^{w_1}, \dots, u^{w_d})\dot{C}_k(v^{w_1}, \dots, v^{w_d})duv.$$

We set $x = u^{w_j}$ and $y = v^{w_k}$, the left side become

$$\tau_{jk}(\mathbf{w}) = \frac{1}{w_j w_k}\int_{[0,1]^2} C(\mathbf{1}_{jk}(x,y))\dot{C}_j(x^{w_1/w_j}, \dots, x^{w_d/w_j})\dot{C}_k(y^{w_1/w_k}, \dots, y^{w_d/w_k})x^{(1-w_j)/w_j}y^{(1-w_k)/w_k}dxy,$$
$$= \frac{\mu_j(\mathbf{w})\mu_k(\mathbf{w})}{w_j w_k}\int_{[0,1]^2} C(\mathbf{1}_{jk}(x,y))x^{A_j(\mathbf{w})+(1-w_j)/w_j-1}y^{A_k(\mathbf{w})+(1-w_k)/w_k-1}dxy.$$

Now, we set $x = w^{1-s}$ and $y = w^s$ and we obtain

$$\tau_{jk}(\mathbf{w}) = \frac{\mu_j(\mathbf{w})\mu_k(\mathbf{w})}{w_j w_k}\int_{[0,1]}\left[A(\mathbf{0}_{jk}(1-s,s)) + (1-s)\left(A_j(\mathbf{w}) + \frac{1-w_j}{w_j} - 1\right) + s\left(A_k(\mathbf{w}) + \frac{1-w_k}{w_k} - 1\right) + 1\right]^{-2} ds.$$

The right side of Equation (16) is given by

$$\int_{[0,1]^2} u^{w_j}v^{w_k}\dot{C}_j(u^{w_1}, \dots, u^{w_d})\dot{C}_k(v^{w_1}, \dots, v^{w_d})duv = \frac{\mu_j(\mathbf{w})\mu_k(\mathbf{w})}{(1+A(\mathbf{w}))^2}.$$

Hence the result for $\sigma^2_{d+1}(\mathbf{w})$. Using the same techniques, we show that for $j \in \{1, \dots, d\}$

$$\sigma_j^2(\mathbf{w}) = \int_{[0,1]^2}(u \wedge v)^{w_j} - u^{w_j}v^{w_j}duv = \frac{1}{(1+w_j)^2}\frac{w_j}{2+w_j}.$$

For $j < k$, we compute

$$\sigma_{jk}(\mathbf{w}) = \int_{[0,1]^2} C(1_{jk}(u^{w_j}, v^{w_k})) - u^{w_j}v^{w_k}duv,$$
$$= \frac{1}{w_j w_k}\int_{[0,1]}\left[A(0_{jk}(1-s,s)) + (1-s)\frac{1-w_j}{w_j} + s\frac{1-w_k}{w_k} + 1\right]^{-2} ds - \frac{1}{1+w_j}\frac{1}{1+w_k}.$$

Let $j \in \{1, \dots, d\}$, thus

$$\sigma_j^{(1)}(\mathbf{w}) = \int_{[0,1]^2} C\left(u^{w_1}, \dots, (u \wedge v)^{w_j}, \dots, u^{w_d}\right) - C(u^{w_1}, \dots, u^{w_d}) v^{w_j} ds,$$
$$= \frac{1}{w_j(1-w_j)} \int_{[0,1]} \left[A(\mathbf{z}_j(1-s) + (1-s)\frac{1-w_j}{w_j} + s\frac{w_j}{1-w_j} + 1 \right]^{-2} ds + \frac{1}{1+A(\mathbf{w})}\left[\frac{1}{2+A(\mathbf{w})} - \frac{1}{1+w_j}\right].$$

Now, for $\sigma_{jk}^{(2)}$, we have two distinguish between three cases :

- if $j = k$, we have
$$\sigma_{jk}^{(2)}(\mathbf{w}) = 0.$$
- if $j < k$, we obtain :
$$\sigma_{jk}^{(2)}(\mathbf{w}) = \frac{\mu_k(\mathbf{w})}{w_j w_k} \int_{[0,1]} \left[A(\mathbf{0}_{jk}(1-s,s)) + (1-s)\frac{1-w_j}{w_j} + s\left(A_k(\mathbf{w}) + \frac{1-w_k}{w_k} - 1\right) + 1\right]^{-2} ds - \frac{\mu_k(\mathbf{w})}{1+A(\mathbf{w})}\frac{1}{1+w_j}.$$
- Lastly, if $j > k$, we have
$$\sigma_{jk}^{(2)}(\mathbf{w}) = \frac{\mu_k(\mathbf{w})}{w_j w_k} \int_{[0,1]} \left[A(\mathbf{0}_{kj}(1-s,s)) + s\frac{1-w_j}{w_j} + (1-s)\left(A_k(\mathbf{w}) + \frac{1-w_k}{w_k} - 1\right) + 1\right]^{-2} ds - \frac{\mu_k(\mathbf{w})}{1+A(\mathbf{w})}\frac{1}{1+w_j}.$$

Hence the statement. □

The following lines will gives some details to establish the explicit formula of the asymptotic variance when we suppose that each components of the random vector $\mathbf{X}$ are independent. In this framework, we have that $\mu_j(\mathbf{w}) = 1$ for every $j \in \{1, \dots, d\}$ and thus $\dot{C}_j(u^{w_1}, \dots, u^{w_d}) = u^{1-w_j}$. Furthermore, in the independent case, most of the integrals are reduced to zero.

Proof of Corollary 1. In the term σ_{d+1}^2 given in (15), only the terms γ_1^2, γ_j^2 and γ_{1j} matters because, in the independent case :

$$\tau_{jk}(\mathbf{w}) = \int_{[0,1]^2} (u^{w_j} v^{w_k} - u^{w_j} v^{w_k})\, \dot{C}_j(u^{w_1}, \dots, u^{w_d}) \dot{C}_k(v^{w_1}, \dots, v^{w_d}) du v = 0.$$

For γ_{1j}, we have to compute

$$\gamma_{1j}(\mathbf{w}) = 2\int_{[0,1]\times[0,v]} u(1-v^{w_j})v^{1-w_j} duv = \frac{1}{4}\frac{w_j}{4-w_j}.$$

For γ_1^2 and γ_j^2, we just have to set $A(\mathbf{w})$ in their expressions, we thus have

$$\gamma_1^2(\mathbf{w}) = \frac{1}{12}, \quad \gamma_j^2 = \frac{1}{4}\frac{w_j}{4-w_j}.$$

We thus have

$$\sigma_{d+1}^2(\mathbf{p}, \mathbf{w}) = \frac{1}{4}\left(\frac{1}{3p} - \sum_{j=1}^{d} p_j^{-1} \frac{w_j}{4-w_j}\right).$$

The rest is left to the reader as the other computations follows from the same arguments. □

We are now going to prove Proposition 2. The strong consistency of the our estimators will be established in a two-step process : first, we prove the strong consistency of the estimator $\nu_n(\mathbf{w})$ which is the nonparametric estimator of the $\mathbf{w}$-madogram with known margins and, second, we show that the limit of

$$\sup_{j\in\{1,\dots,d\}} \sup_{i\in\{1,\dots,n\}} \left| \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \left\{F_j(\tilde{X}_{i,j})\right\}^{1/w_j} \right|,$$

is zero almost surely. Before going into the main arguments, we need the following lemma

Lemma 2. *We have, $\forall i \in \{1, \ldots, n\}$*

$$\left|\bigvee_{j=1}^{d} \{\hat{F}_{n,j}(\tilde{X}_{i,j})\}^{1/w_j} - \bigvee_{j=1}^{d} \{F_j(X_j)\}^{1/w_j}\right| \leq \sup_{j\in\{1,\ldots,d\}} \left|\{\hat{F}_{n,j}(\tilde{X}_{i,j})\}^{1/w_j} - \{F_j(X_j)\}^{1/w_j}\right|.$$

The proof of Lemma 2 can be find in Section 4.2.

Proof of Proposition 2. We prove it for $\hat{\nu}_n^{\mathcal{H}}(\mathbf{w})$ as the strong consistency for $\hat{\nu}_n^{\mathcal{H}*}(\mathbf{w})$ use the same arguments. The estimator $\hat{\nu}_n^{\mathcal{H}}(\mathbf{w})$ in (9) is strongly consistent since it holds

$$\left|\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right| = \left|\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu_n(\mathbf{w}) + \nu_n(\mathbf{w}) - \nu(\mathbf{w})\right| \leq \left|\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu_n(\mathbf{w})\right| + |\nu_n(\mathbf{w}) - \nu(\mathbf{w})|,$$

where

$$\nu_n(\mathbf{w}) = \frac{1}{N}\sum_{i=1}^{n}\left[\left(\bigvee_{j=1}^{d}\left\{F_j(\tilde{X}_{i,j})\right\}^{1/w_j} - \frac{1}{d}\sum_{j=1}^{d}\left\{F_j(\tilde{X}_{i,j})\right\}^{1/w_j}\right)n_i\right],$$

By direct application of Assumption B and the law of large number, we have that

$$|\nu_n(\mathbf{w}) - \nu(\mathbf{w})| \underset{n\to\infty}{\overset{a.s.}{\to}} 0$$

For the second term, we write :

$$\begin{aligned}
\left|\hat{\nu}_n^{\mathcal{H}}(\mathbf{w}) - \nu(\mathbf{w})\right| \leq & \frac{1}{N}\sum_{i=1}^{n}\left|\bigvee_{j=1}^{d}\left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \bigvee_{j=1}^{d}\left\{F_j(X_j)\right\}^{1/w_j}\right| n_i + \frac{1}{Nd}\sum_{i=1}^{n}\sum_{j=1}^{d}\left|\left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \left\{F_j(\tilde{X}_{i,j})\right\}^{1/w_j}\right| n_i \\
\leq & 2 \sup_{j\in\{1,\ldots,d\}}\sup_{i\in\{1,\ldots,n\}}\left|\left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \left\{F_j(\tilde{X}_{i,j})\right\}^{1/w_j}\right|,
\end{aligned}$$

where we used Lemma 2 to obtain the second inequality. The right term converges almost surely to zero by Glivencko-Cantelli Theorem and the uniform continuity of $x \mapsto x^{1/w_j}$ on $[0, 1]$. □

Finally, we give some elements to establish Corollary 2. The strong consistency follows directly from the stability of the almost surely convergence through a continuous fuction. The weak convergence comes down from the functional Delta method (Theorem 3.9.4 of [26]) and the result establish in Proposition 1.

Proof of Corollary 2. Applying the functional Delta method, we have as $n \to \infty$,

$$\begin{aligned}
\sqrt{n}\left(\hat{A}_n^{\mathcal{H}*}(\mathbf{w}) - A(\mathbf{w})\right) \rightsquigarrow & -(1 + A(\mathbf{w}))^2\left\{\frac{1}{d}\sum_{j=1}^{d}\left(1 + \lambda_j(\mathbf{w})(d-1)\right)\int_{[0,1]}\alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx\right. \\
& \left. - \int_{[0,1]} S_C(x^{w_1}, \ldots, x^{w_d}dx)\right\}_{\mathbf{w}\in\Delta^{d-1}}.
\end{aligned}$$

For a fixed $\mathbf{w} \in \Delta^{d-1}$, as a linear transformation of a tight Gaussian process, it follows that

$$\sqrt{n}\left(\hat{A}_n^{\mathcal{H}*}(\mathbf{w}) - A(\mathbf{w})\right) \underset{n\to\infty}{\overset{d}{\to}} \mathcal{N}(0, \mathcal{V}(\mathbf{p}, \mathbf{w})),$$

where, by definition :

$$\begin{aligned}
\mathcal{V}(\mathbf{p}, \mathbf{w}) \triangleq & Var\left(-(1 + A(\mathbf{w}))^2\left\{\frac{1}{d}\sum_{j=1}^{d}\left(1 + \lambda_j(\mathbf{w})(d-1)\right)\int_{[0,1]}\alpha(\mathbf{1}_j(x^{w_j})) - \beta_j(x^{w_j})dx - \int_{[0,1]} S_C(x^{w_1}, \ldots, x^{w_d})\right\}dx\right) \\
= & (1 + A(\mathbf{w}))^4 \mathcal{S}^{\mathcal{H}*}(\mathbf{p}, \mathbf{w}),
\end{aligned}$$

where we used Proposition 1 to conclude. □

4.2. Proof of auxiliary results

Proof of Lemma 1. Following [22] Example 3.5, we consider the function from $\{0,1\}^d \times \mathbb{R}^d$ into $\mathbb{R}$: for $\mathbf{x} \in \mathbb{R}^d$, and $j \in \{1,\dots,d\}$

$$f_j(\mathbf{I},\mathbf{X}) = \mathbb{1}_{\{I_j=1\}}, \quad g_{j,x_j}(\mathbf{I},\mathbf{X}) = \mathbb{1}_{\{X_j \le x_j, I_j=1\}},$$
$$f_{d+1} = \Pi_{j=1}^d f_j, \quad g_{d+1,\mathbf{x}} = \Pi_{j=1}^d g_{j,x_j}.$$

Let P denote the common distribution of the tuple $(\mathbf{I},\mathbf{X})$. The collection of functions

$$\mathcal{F} = \{f_1,\dots,f_d,f_{d+1}\} \cup \bigcup_{j=1}^d \{g_{j,x_j}, x_j \in \mathbb{R}\} \cup \{g_{d+1,\mathbf{x}}, \mathbf{x} \in \mathbb{R}^d\}$$

is a finite union of VC-classes and thus P-Donsker (for more information, see Chapter 2.6 of [26]). The empirical process $\mathbb{G}_n$ defined by

$$\mathbb{G}_n(f) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n f(\mathbf{I}_i,\tilde{\mathbf{X}}_i) - \mathbb{E}[f(\mathbf{I}_i,\tilde{\mathbf{X}}_i)]\right), \quad f \in \mathcal{F},$$

converges in $\ell^\infty(\mathcal{F})$ to a P-browian bride $\mathbb{G}$. For $\mathbf{x} \in \mathbb{R}^d$,

$$\hat{F}_{n,j}(x_j) = \frac{p_j F_j(x_j) + n^{-1/2}\mathbb{G}_n g_{j,x_j}}{p_j + n^{-1/2}\mathbb{G}_n f_j},$$
$$\hat{F}_n(\mathbf{x}) = \frac{pF(\mathbf{x}) + n^{-1/2}\mathbb{G}_n g_{d+1,\mathbf{x}}}{p + n^{-1/2}\mathbb{G}_n f_{d+1}}.$$

We obtain for the second one

$$p\left(\hat{F}_n(\mathbf{x}) - F(\mathbf{x})\right) = n^{-1/2}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}}) - \hat{F}_n(\mathbf{x})\mathbb{G}_n(f_{d+1})\right),$$
$$= n^{-1/2}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}} - F(\mathbf{x})f_{d+1})\right) - n^{-1/2}\mathbb{G}_n(f_{d+1})(\hat{F}_n(\mathbf{x}) - F(\mathbf{x})).$$

We thus have

$$\sqrt{n}\left(\hat{F}_n(\mathbf{x}) - F(\mathbf{x})\right) = p^{-1}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}} - F(\mathbf{x})f_{d+1})\right) - p^{-1}\mathbb{G}_n(f_{d+1})(\hat{F}_n(\mathbf{x}) - F(\mathbf{x})).$$

Applying the central limit theorem and Condition B gives that $\mathbb{G}_n(f_{d+1}) \xrightarrow{d} \mathcal{N}(0, \mathbb{P}(f_{d+1} - \mathbb{P}f_{d+1})^2)$, the law of large numbers gives also $\hat{F}_n(\mathbf{x}) - F(\mathbf{x}) = o_{\mathbb{P}}(1)$. Using Slutsky's lemma gives us

$$\sqrt{n}\left(\hat{F}_n(\mathbf{x}) - F(\mathbf{x})\right) = p^{-1}\left(\mathbb{G}_n(g_{d+1,\mathbf{x}} - F(\mathbf{x})f_{d+1})\right) + o_{\mathbb{P}}(1).$$

Similar reasoning might be applied to the margins, as a consequence, Condition C is fulfilled with for $\mathbf{u} \in [0,1]^d$,

$$\beta_j(u_j) = p_j^{-1}\mathbb{G}\left(g_{j,F_j^{\leftarrow}(u_j)} - u_j f_j\right),$$
$$\alpha(\mathbf{u}) = p^{-1}\mathbb{G}\left(g_{d+1,\mathbf{F}_d^{\leftarrow}(\mathbf{u})} - C(\mathbf{u})f_{d+1}\right).$$

Let us compute one covariance function, the method still the same for the others, without loss of generality, suppose that $j < k$, we have for $u_j, v_k \in [0,1]$

$$\begin{aligned}
cov(\beta_j(u_j),\beta_k(v_k)) &= \mathbb{E}\left[p_j^{-1}\mathbb{G}\left(g_{j,F_j^{\leftarrow}(u_j)} - u_j f_j\right) p_k^{-1}\mathbb{G}\left(g_{k,F_k^{\leftarrow}(v_k)} - v_k f_k\right)\right],\\
&= \frac{1}{p_j p_k}\mathbb{E}\left[\mathbb{G}\left(g_{j,F_j^{\leftarrow}(u_i)} - u_j f_j\right)\mathbb{G}\left(g_{k,F_k^{\leftarrow}(v_j)} - v_k f_k\right)\right],\\
&= \frac{1}{p_j p_k}\mathbb{P}\left\{X_j \le F_j^{\leftarrow}(u_j), X_k \le F_k^{\leftarrow}(v_k), I_j = 1, I_k = 1\right\} - \frac{p_{jk}}{p_j p_k}u_j v_k,\\
&= \frac{1}{p_j p_k}\mathbb{P}\left\{X_j \le F_j^{\leftarrow}(u_j), X_k \le F_k^{\leftarrow}(v_k)\right\}\mathbb{P}\left\{I_j = 1, I_k = 1\right\} - \frac{p_{jk}}{p_j p_k}u_j v_k,\\
&= \frac{p_{jk}}{p_j p_k}\left(C(\mathbf{1}_{jk}(u_j,v_k)) - u_j v_k\right).
\end{aligned}$$

Hence the result. □

Proof of Lemma 2. The lemma becomes trivial once we write, $\forall i \in \{1,\dots,n\}$ and $j \in \{1,\dots,d\}$

$$\begin{aligned}
\left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} &= \left\{F_j(X_j)\right\}^{1/w_j} + \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \left\{F_j(X_j)\right\}^{1/w_j}, \\
&\leq \left\{F_j(X_j)\right\}^{1/w_j} + \sup_{j\in\{1,\dots,d\}} \left| \left\{\hat{F}_{n,j}\right\}(\tilde{X}_{i,j})^{1/w_j} - \left\{F_j(X_j)\right\}^{1/w_j} \right|, \\
&\leq \bigvee_{j=1}^{d} \left\{F_j(X_j)\right\}^{1/w_j} + \sup_{j\in\{1,\dots,d\}} \left| \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \left\{F_j(X_j)\right\}^{1/w_j} \right|.
\end{aligned}$$

Taking the max over $j \in \{1,\dots,d\}$ gives

$$\bigvee_{j=1}^{d} \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \bigvee_{j=1}^{d} \left\{F_j(X_j)\right\}^{1/w_j} \leq \sup_{j\in\{1,\dots,d\}} \left| \left\{\hat{F}_{n,j}(\tilde{X}_{i,j})\right\}^{1/w_j} - \left\{F_j(X_j)\right\}^{1/w_j} \right|.$$

Moreover, by symmetry of $\hat{F}_{n,j}$ and F_j, the second ones follows similarly. □

5. Conclusions

Here are our conclusions.

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Appendix

Essential details needed to make the paper reasonably self contained can be presented here.

References

[1] P. Capéraà, A.-L. Fougères, C. Genest, A nonparametric estimation procedure for bivariate extreme value copulas, Biometrika 84 (1997) 567–577.
[2] P. Deheuvels, La fonction de dépendance empirique et ses propriétés. un test non paramétrique d'indépendance, Bulletin Royal Belge de L'Académie des sciences, 65, 274-292 (1979).
[3] P. Deheuvels, On the limiting behavior of the pickands estimator for bivariate extreme-value distributions, Statistics & Probability Letters 12 (1991) 429–439.
[4] J.-D. Fermanian, D. Radulović, M. Wegkamp, Weak convergence of empirical copula processes, Bernoulli 10 (2004) 847–860.
[5] H. Ferreira, A. Martins, M. Temido, Extremal behaviour of a periodically controlled sequence with imputed values, Statistical Papers 62 (2021) 1–23.
[6] C. Genest, J. Segers, Rank-based inference for bivariate extreme-value copulas, The Annals of Statistics 37 (2009) 2990 – 3022.
[7] G. Gudendorf, J. Segers, Extreme-value copulas, in: Copula theory and its applications, volume 198 of Lect. Notes Stat. Proc., Springer, Heidelberg, 2010, pp. 127–145.
[8] G. Gudendorf, J. Segers, Nonparametric estimation of multivariate extreme-value copulas, Journal of Statistical Planning and Inference 142 (2012) 3073–3085.
[9] A. Guillou, P. Naveau, A. Schorgen, Madogram and asymptotic independence among maxima, REVSTAT 12 (2014) 119–134.
[10] A. Guillou, S. A. Padoan, S. Rizzelli, Inference for asymptotically independent samples of extremes, Journal of Multivariate Analysis 167 (2018) 114–135.
[11] E. Gumbel, Distributions de valeurs extrêmes en plusieurs dimensions., Publications de l'institut de Statistique de l'Université de Paris 9 (1960) 171–173.
[12] A. Hall, M. Scotto, On the extremes of randomly sub-sampled time series, REVSTAT – Statistical Journal Volume 6 (2008) 151–164.
[13] P. Hall, N. Tajvidi, Distribution and dependence-function estimation for bivariate extreme-value distributions, Bernoulli 6 (2000) 835–844.
[14] H. Joe, Families of min-stable multivariate exponential and multivariate extreme value distributions, Statistics & Probability Letters 9 (1990) 75–81.
[15] J. Josse, N. Prost, E. Scornet, G. Varoquaux, On the consistency of supervised learning with missing values, 2020.
[16] G. Marcon, S. Padoan, P. Naveau, P. Muliere, J. Segers, Multivariate nonparametric estimation of the pickands dependence function using bernstein polynomials, Journal of Statistical Planning and Inference 183 (2017) 1–17.

[17] P. Naveau, A. Guillou, D. Cooley, J. Diebolt, Modelling pairwise dependence of maxima in space, Biometrika 96 (2009) 1–17.
[18] J. D. T. Oliveira, J. Galambos, The asymptotic theory of extreme order statistics, International Statistical Review 47 (1977) 230.
[19] A. J. Patton, J. Wiley, Estimation of multivariate models for time series of possibly different lenghts, Journal of Applied Econometrics (2006) 147–173.
[20] J. Pickands, Multivariate extreme value distribution, Proceedings 43th, Session of International Statistical Institution, 1981 (1981).
[21] J. Segers, Asymptotics of empirical copula processes under non-restrictive smoothness assumptions, Bernoulli 18 (2012) 764 – 782.
[22] J. Segers, Hybrid copula estimators, J. Statist. Plann. Inference 160 (2015) 23–34.
[23] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publications de l'Institut de Statistique de l'Université de Paris 8 (1959) 229–231.
[24] J. A. Tawn, Bivariate extreme value theory: Models and estimation, Biometrika 75 (1988) 397–415.
[25] J. A. Tawn, Modelling multivariate extreme value distributions, Biometrika 77 (1990) 245–253.
[26] A. W. van der Vaart, J. A. Wellner, Weak Convergence and Empirical Process: With Applications to Statistics, Springer, 1996.
[27] Y. Xia, P. Fabian, A. Stohl, M. Winterhalter, Forest climatology: estimation of missing values for bavaria, germany, Agricultural and Forest Meteorology 96 (1999) 131–144.