Derivation of the Equations of Motion of a Spring-Mass Apparatus

1 Problem Introduction

For this specific system, we are given two masses and three springs connected as shown in Figure 1 below. The masses are connected by the springs to walls on either side of the apparatus. The mass and spring constant values are given: $m1 = 40kg$, $m2 = 20kg$, $k1 = 200N/m$, $k2 = 100N/m$, and $k3 = 250N/m$. The system is initially at rest in equilibrium and then set in motion by displacing the masses arbitrarily where the constants $\alpha = 0.6$ and $\beta = 0.4$ will be the arbitrary percent of which each mode is used respectively in the combined motion of all (both) of the modes together. This will be further elaborated on later in the report. We will now proceed by characterizing this spring-mass system using an eigenvalue/eigenvector analysis in order to implement and plot the equations of motion for the masses using MATLAB.

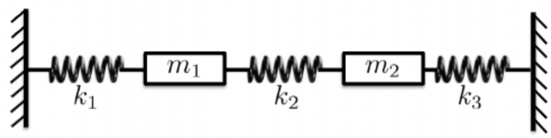

Figure 1: The spring-mass apparatus

2 Deriving the Equations of Motion

2.1 Using Newton's Second Law of Motion

We can begin by using Newton's second law of motion which states:

$$F = ma \tag{1}$$

In the case of our given system, it would be beneficial for us to rewrite the force as the second derivative of position with respect to time, and the force acting on each mass as the combined force of both springs attached to each respective mass. This gives us the equations:

Mass 1:

$$-k_1x_1 - k_2(x_1 - x_2) = m_1\frac{dx_1^2}{d^2t} \tag{2}$$

Mass 2:

$$k_2(x_1 - x_2) - k_3x_2 = m_2\frac{dx_2^2}{d^2t} \tag{3}$$

2.2 Substituting in the Given Displacement Equation

Now, the problems tell us that the displacement of any mass x_i is given as:

$$x_i(t) = A_i \sin(\omega t) \tag{4}$$

Where A_i is the maximum amplitude of displacement of m_i (measured in meters) and ω is the natural frequency (measured in radians per second).

We can now take the second derivative of this equation with respect to time to be used in our previous equation of Newton's second law in the place of acceleration.

First Derivative:

$$\dot{x_i}(t) = A_i \omega \cos(\omega t) \tag{5}$$

Second Derivative:

$$\ddot{x_i}(t) = -A_i \omega^2 sin(\omega t) \tag{6}$$

2.3 Substituting in equation of Acceleration

When we substitute in our new equation (6) of acceleration we are given:

Mass 1:

$$-k_1 A_1 \sin(\omega t) - k_2[A_1 \sin(\omega t) - A_2 \sin(\omega t)] = m_1[-A_1 \omega^2 \sin(\omega t)] \tag{7}$$

Mass 2:

$$k_2[A_1 \sin(\omega t) - A_2 \sin(\omega t)] - k_3 A_2 \sin(\omega t) = m_2[-A_2 \omega^2 \sin(\omega t)] \tag{8}$$

Now, we can simplify these equations into the following:

Mass 1:

$$m_1 A_1 \omega^2 = k_1 A_1 + k_2 A_1 - k_2 A_2 \tag{9}$$

Mass 2:

$$m_2 A_2 \omega^2 = -k_2 A_1 + k_2 A_2 + k_3 A_2 \tag{10}$$

2.4 Express in Matrix Format and Substitute in Given Constants

With our equations of forces fully defined in terms of the unknown amplitudes and radial wave frequencies, we are able to create an eigenvalue formulation. We will first divide both sides by the respective mass and then use the general matrix equation $A\vec{V} = \lambda\vec{V}$, where A is the matrix of amplitudes, the eigenvalues are directly related to the radial frequencies of the waves, and the eigenvectors are directly related to the relative amplitudes of the waves:

$$\begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} \end{bmatrix} \begin{bmatrix} A_1 \\ A2 \end{bmatrix} = \omega^2 \begin{bmatrix} A_1 \\ A2 \end{bmatrix} \tag{11}$$

As previously mentioned, the given constants are $m1 = 40kg$, $m2 = 20kg$, $k1 = 200N/m$, $k2 = 100N/m$, and $k3 = 250N/m$. We will substitute these values into our matrix and receive the corresponding eigenvalues and eigenvectors through linear computation (which can be done either on paper or with MATLAB):

$$\begin{bmatrix} 7.5 & -2.5 \\ -5 & 17.5 \end{bmatrix} \begin{bmatrix} A_1 \\ A2 \end{bmatrix} = \omega^2 \begin{bmatrix} A_1 \\ A2 \end{bmatrix} \tag{12}$$

This yields the following eigenvalues and eigenvectors:

$ev_1 = 6.3763$

$$EV_1 = \begin{bmatrix} -0.9121 \\ -0.41 \end{bmatrix} \tag{13}$$

$ev_2 = 18.6237$

$$EV_2 = \begin{bmatrix} 0.2193 \\ -0.9757 \end{bmatrix} \tag{14}$$

2.5 Gaining a Better Understanding

We have now finished characterizing our equations of forces by using an eigenvalue and eigenvector technique. This has yielded two eigenvalues and two eigenvectors. Since our eigenvalues were characterized by the equation $\lambda = \omega^2$, the natural resonant frequencies of our system are given by the equation $\omega = \sqrt{\lambda}$. These resonant frequencies or eigenvalues each provide their corresponding eigenvectors which are used to characterize the resonant amplitudes of said frequencies. For example, the first fundamental frequency is going to be the square root of eigenvalue 1 ($\omega^{(1)} = 2.51 \frac{rad}{s}$), and the corresponding maximum amplitudes of mass 1 and mass 2 at this frequency will be the first and second-row value of eigenvector 1 respectively ($A_1^{(1)} = -0.9121m, A_2^{(1)} = -0.41m$). The derived equations of motion for each mode are as follows:

Mode 1:

Mass 1:

$$x_1(t) - 0.9121 \sin(2.5251t) \tag{15}$$

Mass 2:

$$x_2(t) - 0.41 \sin(2.5251t) \tag{16}$$

Mode 2:

Mass 1:

$$x_1(t) 0.2193 \sin(4.3155t) \tag{17}$$

Mass 2:

$$x_2(t) - 0.9757 \sin(4.3155t) \tag{18}$$

When we possess the resonant frequencies and amplitudes of the system in eigenvalue and eigenvector form, we are able to generate any variation of vibration for the given system because we possess the fundamental basis of motion for each mass being vibrated in the apparatus. This means that we can superpose the two basis in order to plot any frequency of vibration and we will later plot this with a previously mentioned arbitrary combination of the two modes where $\alpha = 0.6$ and $\beta = 0.4$ in the superposition equations given as:

Mass 1:
$$x_1(t) = \alpha A_1^{(1)} \sin(\omega^{(1)} t) + \beta A_1^{(2)} \sin(\omega^{(2)} t) \tag{19}$$

Mass 2:
$$x_2(t) = \alpha A_2^{(1)} \sin(\omega^{(1)} t) + \beta A_2^{(2)} \sin(\omega^{(2)} t) \tag{20}$$

Now we can substitute in all of the values we have previously solved for:

Mass 1:
$$x_1(t) = 0.6(-0.9121 \sin(2.5251t)) + 0.4(0.2193 \sin(4.3155t)) \tag{21}$$

Mass 2:
$$x_2(t) = 0.6(-0.41 \sin(2.5251t)) + 0.4(-0.9757 \sin(4.3155t)) \tag{22}$$

3 Results

After implementing the previously stated calculations into MATLAB, we are able to plot the motion of the masses for each mode and for an example of the superposition of both modes.

3.1 Mode 1

For the first frequency, we can observe that the maximum amplitude of mass 1 is much greater than that of mass 2. We are also able to see that the periods of both masses are visually the same and approximately 2.5 seconds.

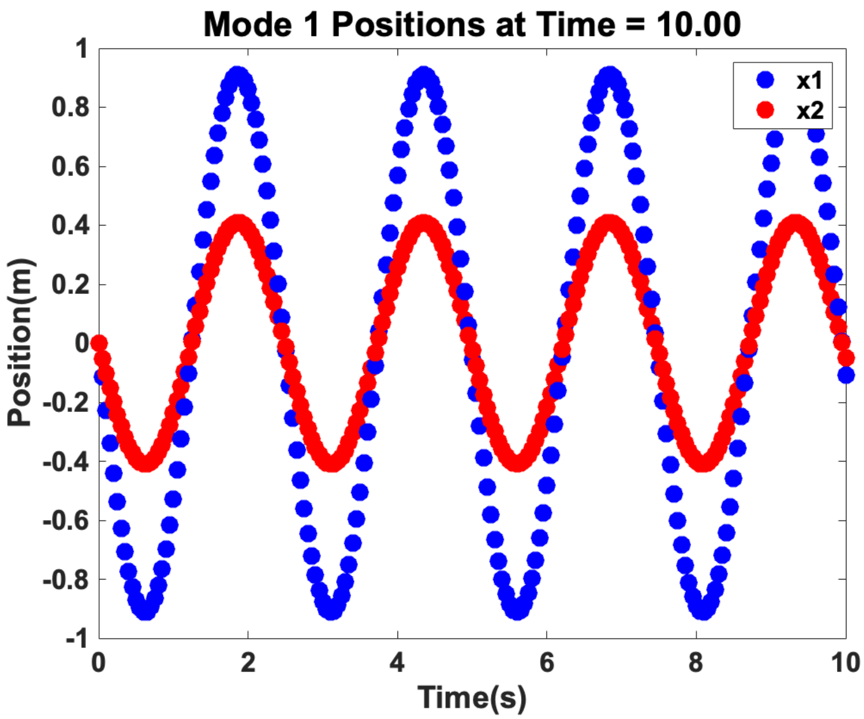

Figure 2: Motion of mass 1 and mass 2 when $\omega^{(1)} = 2.53\frac{rad}{s}$

3.2 Mode 2

For the second frequency, we can observe that now the maximum amplitude of mass 1 is much less than that of mass 2. We are also able to see that, again, the periods of both masses are visually the same and approximately 1.5 seconds.

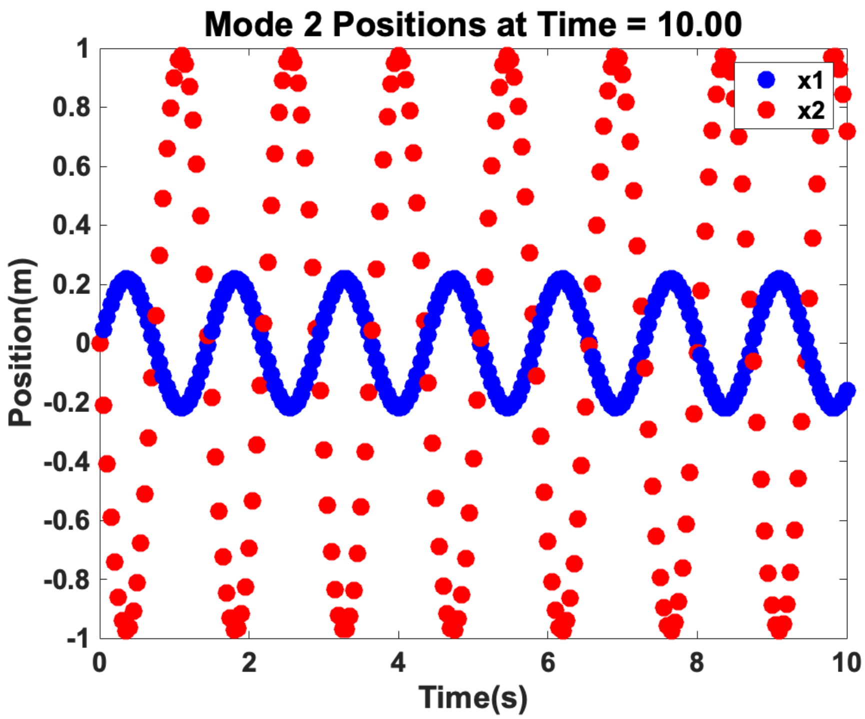

Figure 3: Motion of mass 1 and mass 2 when $\omega^{(2)} = 4.32\frac{rad}{s}$

3.3 Superposition of mode 1 and mode 2

The arbitrary superposition of the two modes is shown in the following plot and is much harder to decipher visually. The maximum amplitudes are now, not only much closer in value, but also not constant throughout the plot. The two masses seem to be affecting each other's movement in a more realistic situation and the motion is more chaotic. This is a fair representation of how the bases can be used to visualize countless different frequencies by the combination of modes.

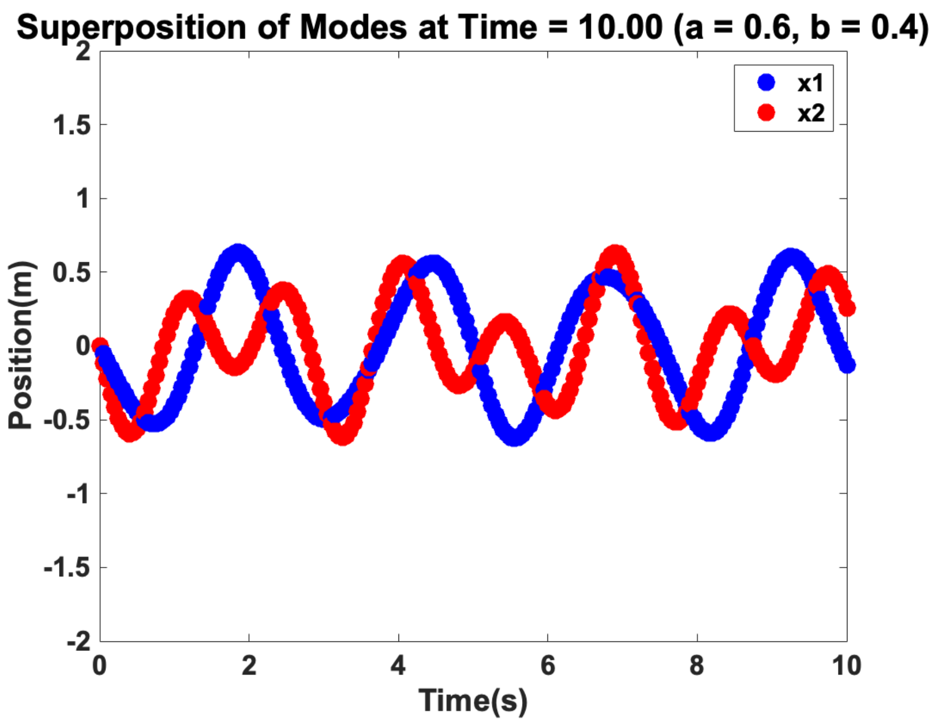

Figure 4: Motion of mass 1 and mass 2 with superposition of modes

4 Conclusion

In conclusion, we were given a spring-mass apparatus attached between two parallel walls, and arbitrary initial displacement was enacted on the system. We were able to use a combination of physical free-body analysis and eigenvalue formation to model the fundamental frequencies of this system into two modes and 4 corresponding equations of motion for the two masses. We were able to implement these computations in MATLAB in order to plot the various motions of each mass as a function of time. This simulation was one useful example of countless different real-world situations that would require the modeling of different motions and forces of mass. For our apparatus and simulation specifically, the values and modeling could be slightly altered in order to model the effects of earthquakes on different floors of a building. This computation is done by countless engineers every day all over the world in order to create structurally sound infrastructure for people to live in and use.