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04/20/2020

Homework 1

CSCI 405

1

1.1 a.

We know that f and g are positive and define $\mathbb{R}^+ = [0, \infty)$. Suppose $f(n_0)$ is a maximum for some $n_0 \in \mathbb{N}$. Then $g(n_0) \leq f(n_0) \implies 2g(n_0) \leq 2f(n_0)$. Since $f(n_0)$ is a maximum, $f(n_0) + g(n_0) \leq f(n_0) + f(n_0) \iff f(n_0) + g(n_0) \leq 2f(n_0)$. This logic follows if instead we have $g(n_0)$ as a maximum. Therefore, $f(n) + g(n) \leq 2max\{f(n), g(n)\} \forall n \in \mathbb{N}$.

1.2 b.

We know that f and g are positive and define $\mathbb{R}^+ = [0, \infty)$. Then $\forall k \in \mathbb{R}^+$ and $\forall n \in \mathbb{N}$ we have $f(n) + k \geq f(n)$ and $g(n) + k \geq g(n)$. Now suppose $f(n_0) > g(n_0)$ for some $n_0 \in \mathbb{N}$. Since $g(n_0) \geq 0$, we know from above that $f(n_0) + g(n_0) \geq f(n_0)$. This implies that if $f(n)$ is a maximum, $f(n) + g(n) \geq f(n)$. The same logic follows if instead we have $g(n)$ as a maximum. Thus, $f(n) + g(n) \geq max\{f(n), g(n)\} \forall n \in \mathbb{N}$.

2

We have

$$\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=0}^{n} (i+1)^4 = \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

We can subtract $\sum_{i=1}^{n} i^4$ from both sides to get

$$(n+1)^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$$

Since we know all the sums except for $\sum_{i=1}^{n} i^3$, we can expand them out and subtract them from $(n+1)^4$ to get the closed form of the unknown sum:

$$(n+1)^4 = 4\sum_{i=1}^{n} i^3 + \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n + 1 = 4\sum_{i=1}^{n} i^3 + 6n(n+1)(2n+1) + 2n(n+1) + n + 1$$

Now we expand $(n+1)^4$, simplify the RHS and subtract it from the LHS:

$$n^4 + 4n^3 + 6n^2 + 4n + 1 - 2n^3 - 5n^2 - 4n - 1 = 4\sum_{i=1}^{n} i^3$$

$$\frac{n^4 + 2n^3 + n^2}{4} = \sum_{i=1}^{n} i^3$$

We can simplify further:

$$\frac{n^2(n^2+2n+1)}{4} = \frac{n^2(n+1)(n+1)}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

This gives a simplified closed form

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

3

Given

$$\sum_{k=1}^{n} k^2$$

We will let $a_k = b_k = k$ and get the following first step of Abel's formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2}n + \sum_{k=1}^{n-1} \frac{k(k+1)}{2}(k-(k+1))$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2}n - \sum_{k=1}^{n-1} \frac{k(k+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2}n - \frac{1}{2}\sum_{k=1}^{n-1} k - \frac{1}{2}\sum_{k=1}^{n-1} k^2$$

Looking at the left hand side, we can see that

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 - n^2$$

We can add $\frac{1}{2}\sum_{k=1}^{n-1} k^2$ to both sides to get

$$\frac{1}{2}\sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} k^2 - n^2 = \frac{n(n+1)}{2}n - \frac{1}{2}\sum_{k=1}^{n-1} k$$

$$\iff \frac{3}{2}\sum_{k=1}^{n} k^2 - \frac{n^2}{2} = \frac{n(n+1)}{2}n - \frac{1}{2}\sum_{k=1}^{n-1} k$$

Note that we get the sum on the left hand side by artificially adding the term $\frac{n^2}{2}$ to the combined sum to get it in the form we want. Doing this forces us to subtract this term at the end to maintain the equality. From here, we can evaluate the last term on the right hand side and solve algebraically.

$$\frac{3}{2}\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2}n - \frac{(n-1)n}{4} + \frac{n^2}{2}$$
$$3\sum_{k=1}^{n} k^2 = n^2(n+1) - \frac{(n-1)n}{2} + n^2$$
$$3\sum_{k=1}^{n} k^2 = \frac{2n^2(n+1) - n^2 + n + n^2}{2}$$
$$3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

4

First, we must show that $T(1) = 1$. In general we have,

$$T(n) = 2T(n-1) + n = 2(2^{(n-1)+1} - (n-1) - 2) + n$$

So

$$T(1) = 2(2^{(1-1)+1} - (1-1) - 2) + 1 = 2(2^1 - 2) + 1 = 1$$

Therefore, we have $T(1) = 1$. Now, to show that $T(n) = 2(2^{(n-1)+1} - (n-1) - 2) + n = 2^{n+1} - n - 2\ \forall n \geq 1$, we let $n = k$ for some arbitrary $k \geq 1$. We then plug in k for both equations to show they give the same result:

$$T(k) = 2(2^{(k-1)+1} - (k-1) - 2) + k = 2^{k+1} - k - 2$$
$$\iff 2(2^k - k + 1 - 2) + k = 2(2^k) - k - 2$$
$$\iff 2(2^k) - k - 2 = 2(2^k) - k - 2$$

Therefore, $2^{n+1} - n - 2$ is an exact solution to the recurrence $T(n) = 2T(n-1) + n$ $\forall n \geq 1$.

5 Q: 5,6

The first for loop iterates n times followed by the second for loop that iterates $n-1$ times. This gives the sum $\sum_{i=1}^{n}(n-i)$. To evaluate this, we can note that this sum follows the same idea as the arithmetic series where we iterate $n + (n-1) + (n-2) + ... + 1$ times. The only difference here is that we are counting down from n rather than counting up from 1. This allows us to rewrite the sum as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.