Constructive Torelli Theorem for Regular Matroids

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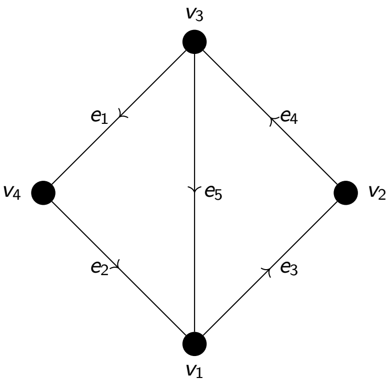

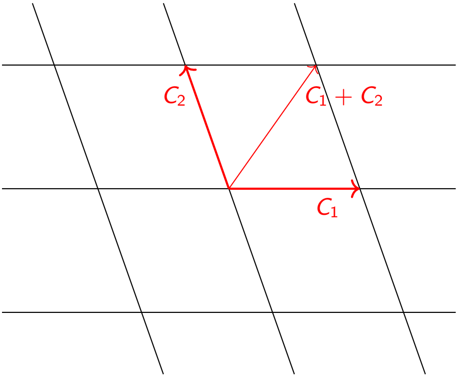

$$C_1 = +e_1 + e_2 - e_5, \quad C_2 = +e_3 + e_4 + e_5, \quad C_1 + C_2 = +e_1 + e_2 + e_3 + e_4.$$

Given a regular matroid $\mathcal{M}$ with representing matrix M, the lattice of integer flows is

$$\mathcal{F}(\mathcal{M}) = \ker(M) \cap \mathbb{Z}^E.$$

This coincides for the signed incidence matrix of a graph – the natural representing matrix for a graphical matroid.

Theorem: [Su–Wagner] Let $\mathcal{M}$ and $\mathcal{N}$ be matroids of cogirth ≥ 2. Then $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$ if and only if $\mathcal{M} \cong \mathcal{N}$.

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Assume all matroids are of cogirth ≥ 2. Note that a matroid has cogirth ≥ 2 if and only if it can be oriented totally cyclically.

Recovering $\mathcal{M}$ comes down to: detecting which elements in $\mathcal{F}(\mathcal{M})$ are circuit elements, determining the base set of $\mathcal{M}$, and determining which elements are in each circuit.

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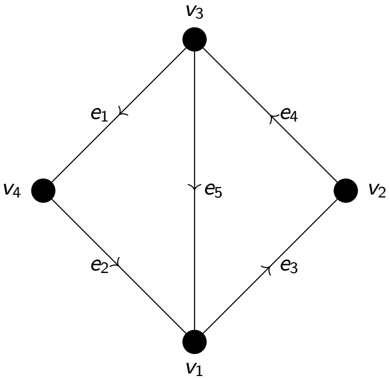
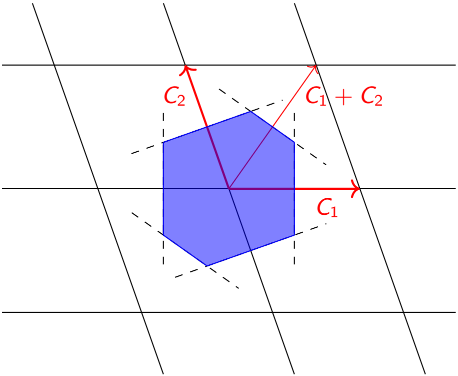
The **Voronoi cell** $\mathcal{V}(\Lambda)$ of a lattice Λ is the collection of points in space (i.e. in $\Lambda \otimes \mathbb{R}$) closer to the origin than any other lattice point in Λ.





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Faces of the Voronoi cell form a poset $\mathcal{FP}(\mathcal{F}(\mathcal{M}))$ with inclusion given by dimension.

Orientations on $\mathcal{M}$ and its submatroids which are totally cyclic form a poset $\mathcal{SC}(\mathcal{M})$ where $(\mathcal{N}', \omega_{\mathcal{N}'}) \leq (\mathcal{N}, \omega_{\mathcal{N}})$ if and only if $\mathcal{N}$ is a submatroid of $\mathcal{N}'$ and $\omega_{\mathcal{N}'}$ restricted to $\mathcal{N}$ is $\omega_{\mathcal{N}}$.

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Theorem: [Amini-Dancso-Lim] For a finite regular matroid $\mathcal{M}$, $\mathcal{FP}(\mathcal{F}(\mathcal{M})) \cong \mathcal{SC}(\mathcal{M})$ as graded posets.

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The parallel faces of the Voronoi cell correspond to the different totally cyclic orientations of the same underlying matroid, so denote the **equivalence class of faces that are parallel to** F by $[F]$.

For a subset $A \subseteq E(\mathcal{M})$ we denote $[F_A]$ to be the face(s) which corresponds to $\mathcal{M} \setminus A$ (provided it is totally cyclically orientable).

For all $e \in E(\mathcal{M})$, the submatroid $\mathcal{M} \setminus \{e\}$ has cogirth ≥ 2, so can be totally cyclically oriented.

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For general matroids of cogirth ≥ 2 steps (2) and (3) remain the same, but step (1) requires much more work, as maximal totally cyclic submatroids may be of the form $\mathcal{M} \setminus S$ for (possibly different sized) $|S| > 1$.

2-Cut Blocks

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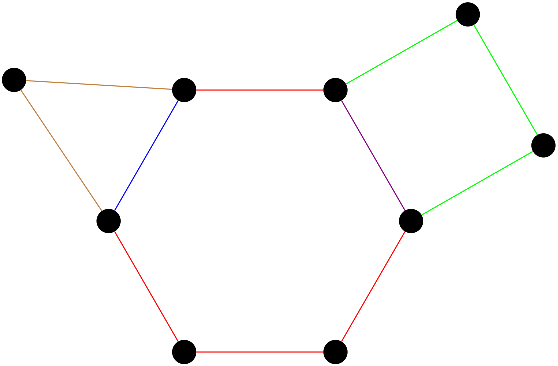

Proposition: [Dancso–E.–Garoufalidis] Given a circuit C and a 2-cut block S, either $S \cap C = \emptyset$ or $S \subseteq C$.

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We need to detect the sizes of $S_{[\epsilon]}$ from the sizes of circuits. This can be done using the pairwise inner product of circuit basis elements, but we want to choose a clever basis.

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3. Neither of the above is true, that is, C_i and C_j cannot be compatibly oriented. This is the case if and only if the faces $\{F_{C_i}, F_{C_j}, F_{-C_i}, F_{-C_j}\}$ are pairwise disjoint.

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Amini's theorem tells us that vertices correspond to totally cyclic orientations. Take any vertex, then the codimension one faces that intersect at that vertex form a compatibly oriented basis.

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5. $E(\mathcal{M})$ is the disjoint union of the sets $\{S_{[\epsilon]}\}$.
6. Again, the element $e \in S_{[\epsilon]}$ belongs to a circuit C if and only if no member of the corresponding edge parallel class $[\epsilon]$ belongs to the face $[F_C]$.

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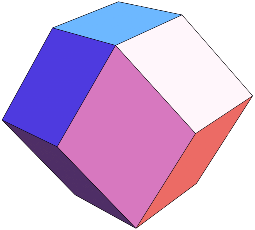
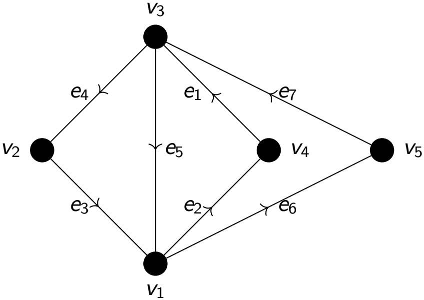
- Apply contractions and subdivisions to $\mathcal{M}$ until these integer solutions match.
- The circuits are in correspondence, which is an isomorphism of lattices of integer flows, thus lifts to an isomorphism of euclidean lattices, inducing a matroid isomorphism of $\mathcal{M}$.

$\blacksquare$

Reconstructing





$$\begin{array}{cc} & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \end{array} & \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix} \end{array}$$

$$\begin{aligned} C_1 &= +e_1 + e_2 + e_5, \\ C_2 &= +e_5 + e_6 + e_7, \\ C_3 &= +e_1 + e_2 + e_3 + e_4. \end{aligned}$$

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Proposition: [Dancso–E.–Garoufalidis] An isomorphism of lattices $\varphi : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{N})$ lifts to an isomorphism $\Phi : \mathbb{Z}^{E(\mathcal{M})} \to \mathbb{Z}^{E(\mathcal{N})}$.

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\mathcal{F}(\mathcal{M}) \arrow[u, hook] \arrow[r, "\psi"'] & \mathcal{F}(\mathcal{N}) \arrow[u, hook] & \mathcal{F}(\mathcal{M}) \arrow[u, hook] \arrow[l, "\varphi"] \arrow[ll, bend left, "\psi^{-1} \circ \varphi"]
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∎