Constructive Torelli Theorem for Regular Matroids

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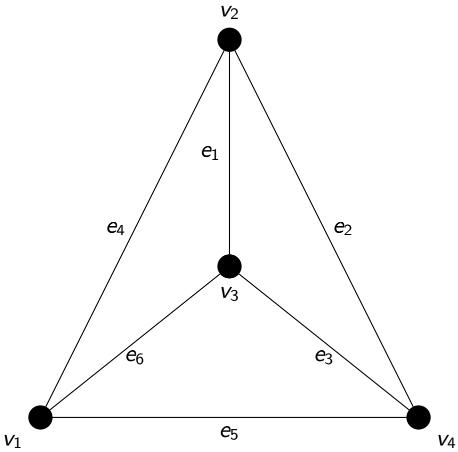
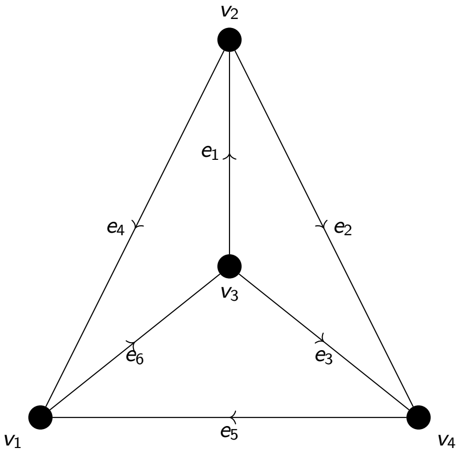
A **graph** $G = (V, E)$ consists of **vertices** $v_i \in V$ and **edges** $e_i = \{v_j, v_k\} \in E$.



A graph is **connected** if there is a path between any two vertices.

It can be **oriented**, edges become ordered pairs $e_i = (v_j, v_k)$.



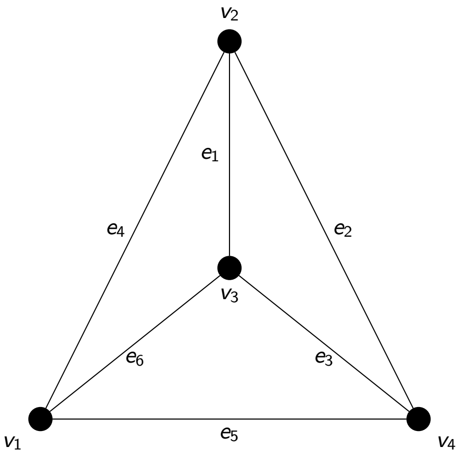
An oriented graph is **strongly connected** if there is an oriented path between any two vertices.

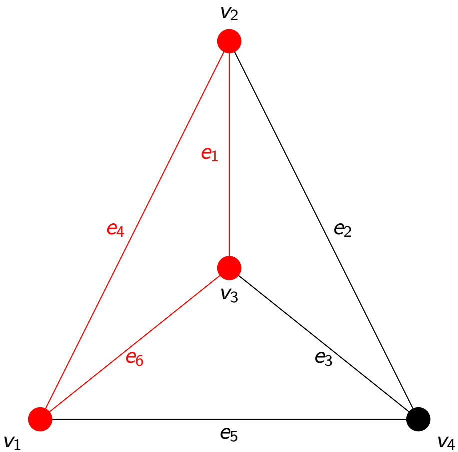
A graph is n-**connected** if upon removing $n-1$ edges it remains connected.

A **cycle** in a graph is a subset $C \subseteq E$ which is a minimal closed walk.

A graph is 2-connected if every edge participates in a cycle.

Cycles

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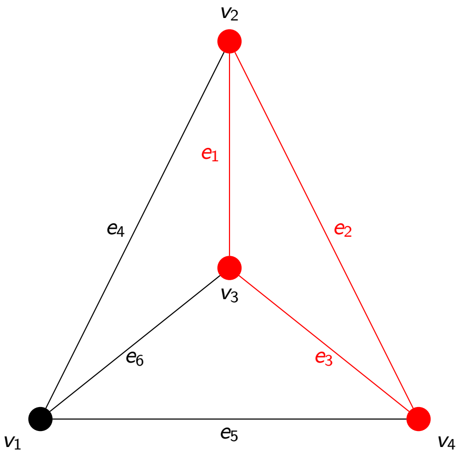

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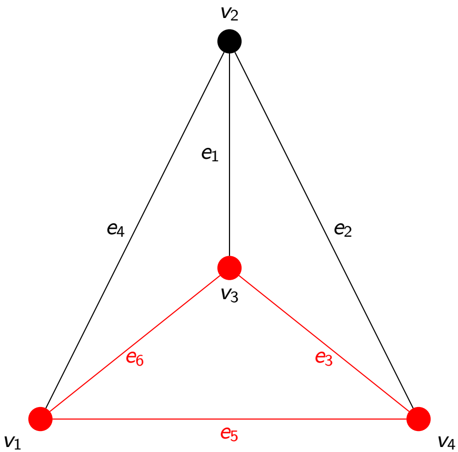
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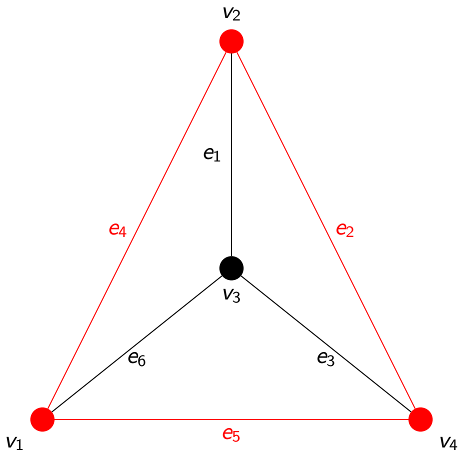
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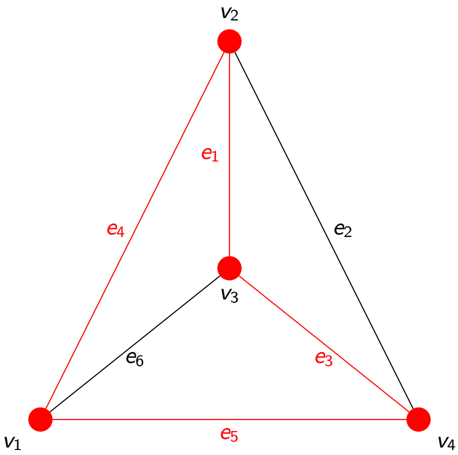

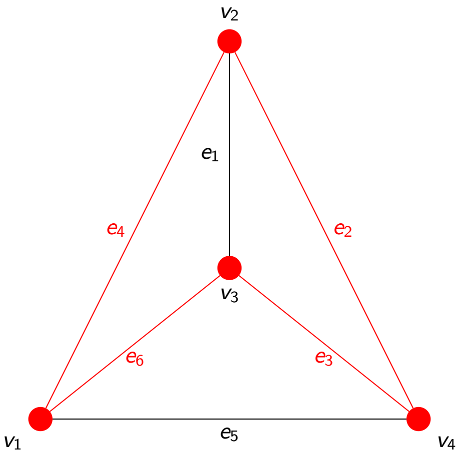
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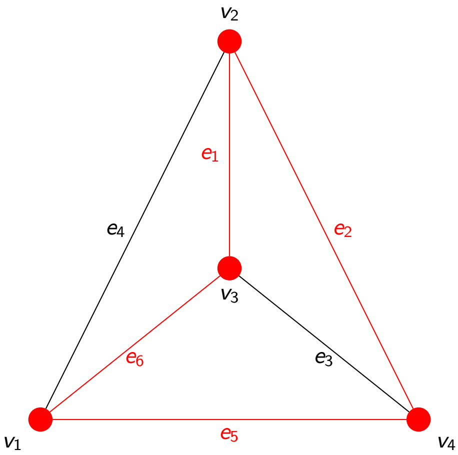

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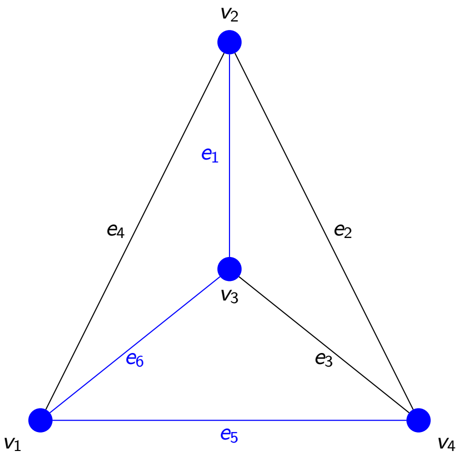
A **spanning tree** is a subset $T \subseteq E$ which meets every vertex but contains no cycles.

A **cycle basis** is a set of cycles which cover every edge (and are linearly independent) – these only exist for 2-connected graphs.

One can be found by considering the **fundamental cycles** with respect to a spanning tree.



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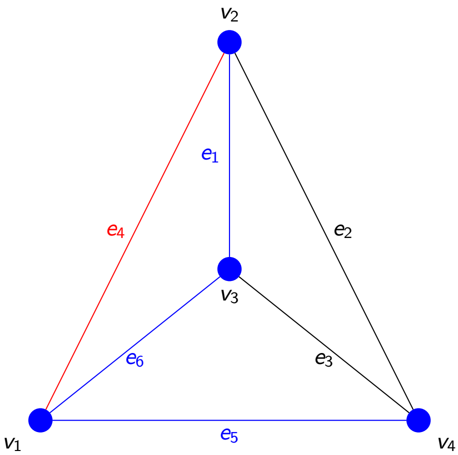
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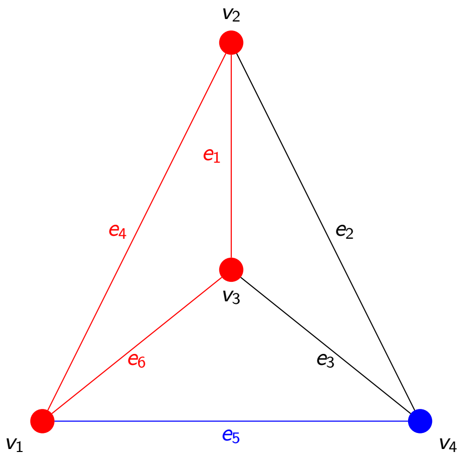

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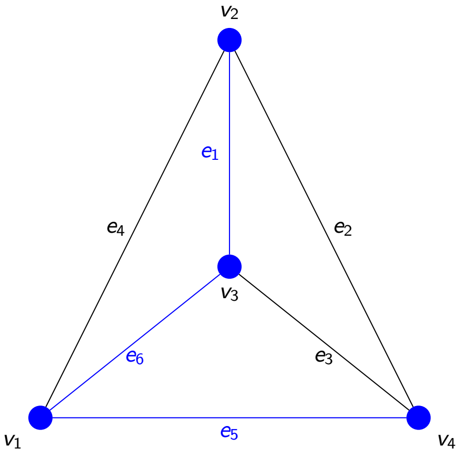

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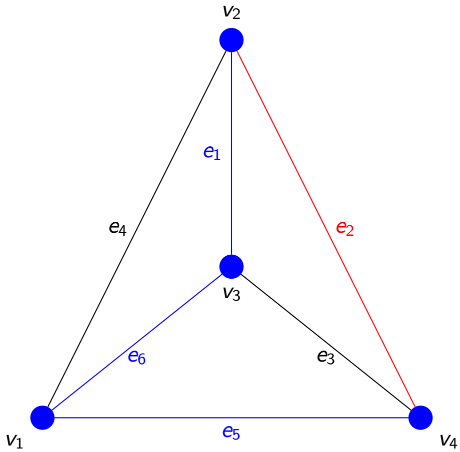

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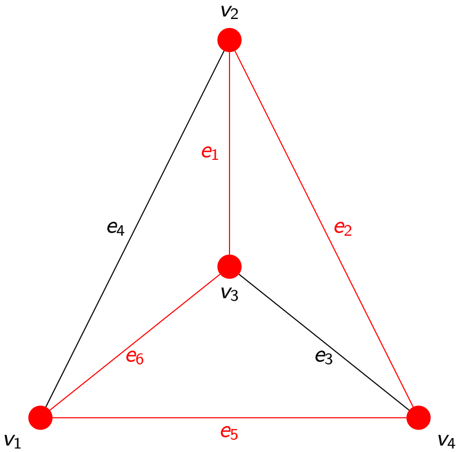

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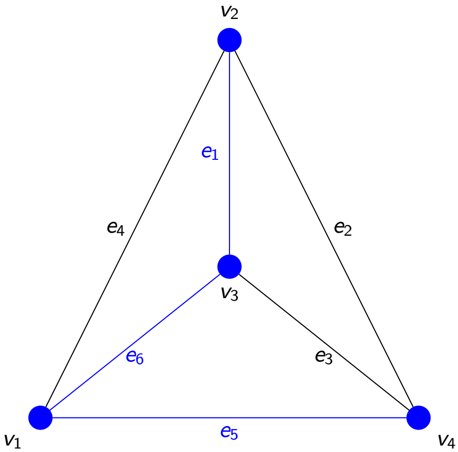

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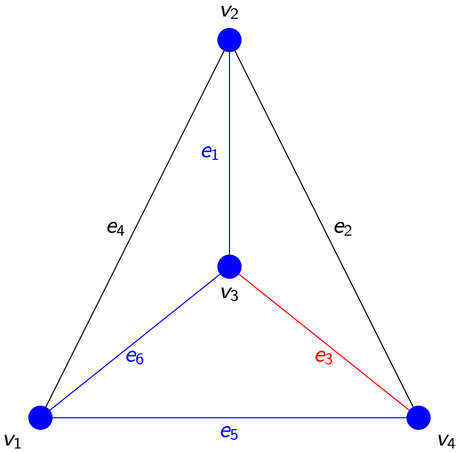

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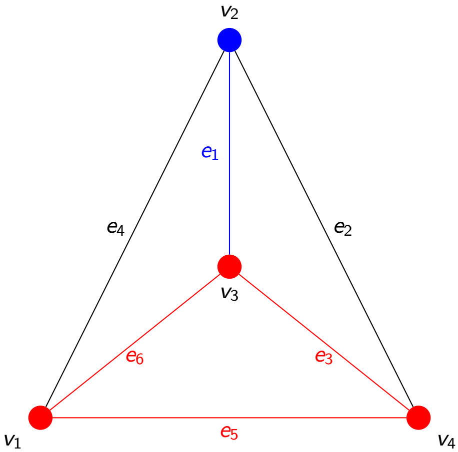

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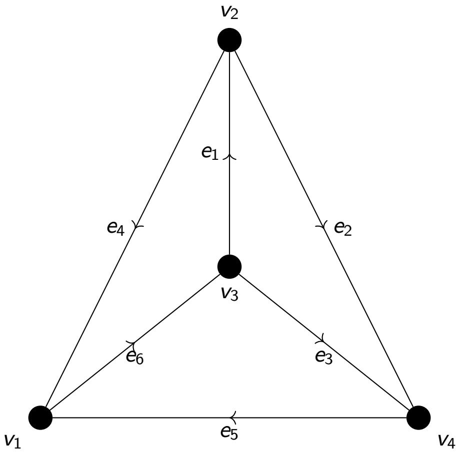
An orientation on a graph causes cycles to become **oriented cycles**.

Start following an edge clockwise (or counterclockwise), if an edges orientation agrees along the cycle it gets a positive coefficient, otherwise negative.

We call a cycle **positive** (or **negative**) if the coefficients are all positive (or negative).

A **positive cycle basis** consists of positive cycles.



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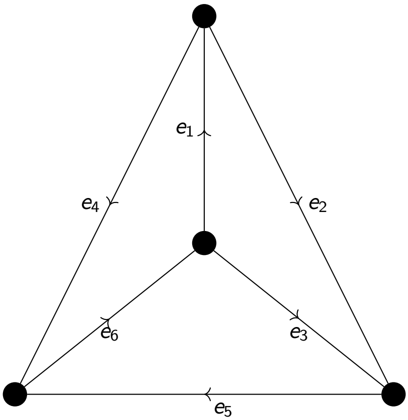
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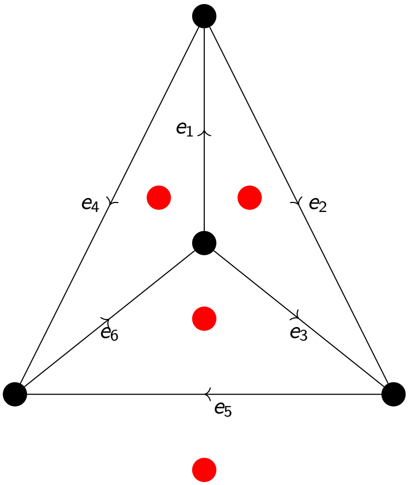
Dual (Planar) Graph



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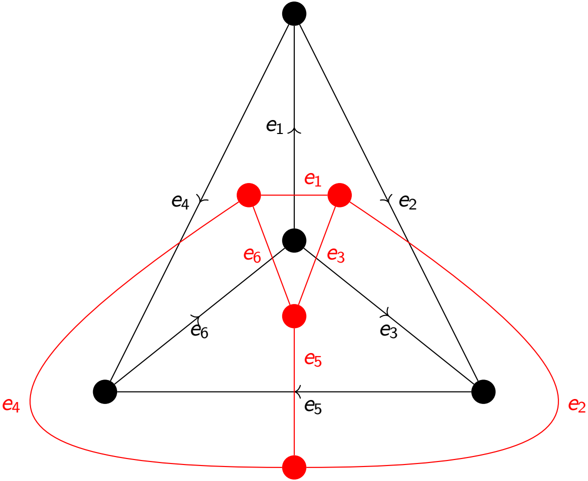

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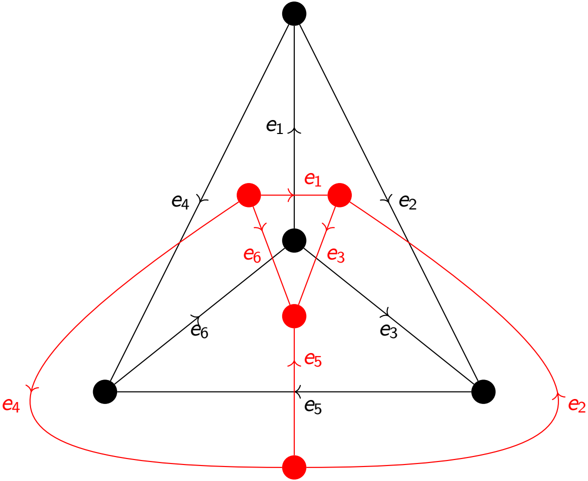

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- Regular matroids – Su and Wagner (2010)

Matroid

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A (finite) **matroid** $\mathcal{M} = (E, \mathcal{I})$ is an ordered pair of a finite set E, called the **base set**, and $\mathcal{I}$, called the "family of independent subsets of E" which satisfies the following axioms:

1. $\emptyset \in \mathcal{I}$.
2. For any $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.
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Any of the above can define a matroid, they each come with a set of axioms.

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A matroid which is representable over any field is called a **regular matroid**. Any graphical matroid is regular by finding the graphs signed adjacency matrix.

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In fact regular matroids are closed under duality, and graphical matroids and their duals (almost) generate all regular matroids.

Given a regular matroid $\mathcal{M}$ with representing matrix M, the lattice of integer flows is

$$\mathcal{F}(\mathcal{M}) = \ker(M) \cap \mathbb{Z}^E.$$

This coincides for the signed incidence matrix of a graph – the natural representing matrix for a graphical matroid.

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We provide a constructive algorithm for recovering the matroid – a *constructive Torelli theorem*. The first step is strengthening the following theorem.

Theorem 1: *[Su-Wagner] Let $\mathcal{M}$ and $\mathcal{N}$ be 2-connected regular matroids. Then $\mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{N})$ if and only if $\mathcal{M} \cong \mathcal{N}$.*

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Proposition: A positive circuit basis exists for any 2-connected regular matroid.

Proposition: An isomorphism of lattices $\varphi : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{N})$ lifts to an isomorphism of Euclidean lattices $\Phi : \mathbb{Z}^{E(\mathcal{M})} \to \mathbb{Z}^{E(\mathcal{N})}$.

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Proof:

1. Use Greene's rigid embedding theorem to show any automorphism of $\mathcal{F}(\mathcal{M})$ lifts – relies on the existance of a positive circuit basis of $\mathcal{M}$.

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\mathbb{Z}^{E(\mathcal{M})} \arrow[r, "\Psi"] & \mathbb{Z}^{E(\mathcal{N})} & \mathbb{Z}^{E(\mathcal{M})} \arrow[l, dashed, "\Phi"'] \\
\mathcal{F}(\mathcal{M}) \arrow[r, "\psi"'] & \mathcal{F}(\mathcal{N}) & \mathcal{F}(\mathcal{M}) \arrow[l, "\varphi"] \arrow[ll, bend left, "\psi^{-1} \circ \varphi"]
\end{tikzcd}$$

Lifting Isomorphisms

Proposition: An isomorphism of lattices $\varphi : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{N})$ lifts to an isomorphism of Euclidean lattices $\Phi : \mathbb{Z}^{E(\mathcal{M})} \to \mathbb{Z}^{E(\mathcal{N})}$.

Proof:

1. Use Greene's rigid embedding theorem to show any automorphism of $\mathcal{F}(\mathcal{M})$ lifts – relies on the existance of a positive circuit basis of $\mathcal{M}$.
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\\
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\mathcal{F}(\mathcal{M}) \arrow[u, hook] \arrow[r, "\psi"'] & \mathcal{F}(\mathcal{N}) \arrow[u, hook] & \mathcal{F}(\mathcal{M}) \arrow[u, hook] \arrow[l, "\varphi"] \arrow[ll, bend left, "\psi^{-1} \circ \varphi"]
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■

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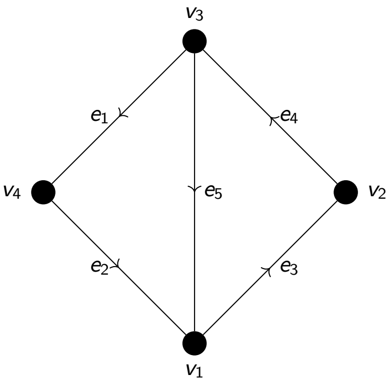

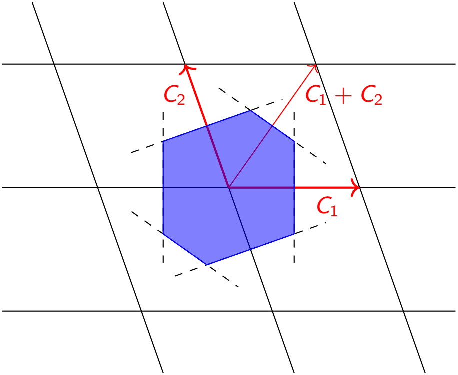

$C_1 = \{+e_1, +e_2, -e_5\}$, $C_2 = \{+e_3, +e_4, +e_5\}$, $C_1 + C_2 = \{+e_1, +e_2, +e_3, +e_4\}$.

Faces of the Voronoi cell form a partially ordered set (poset) $\mathcal{FP}(\mathcal{F}(\mathcal{M}))$ with inclusion given by dimension.

Oriented submatroids $(\mathcal{M}, \omega_{\mathcal{M}})$ which are strongly connected form a poset $\mathcal{SC}(\mathcal{M})$ with inclusion $(\mathcal{M}, \omega_{\mathcal{M}}) \leq (\mathcal{N}, \omega_{\mathcal{N}})$ if and only if $\mathcal{N}$ is a submatroid of $\mathcal{M}$ and $\omega_{\mathcal{M}}$ restricted to $\mathcal{N}$ is $\omega_{\mathcal{N}}$.

Theorem 2: *For a finite regular matroid $\mathcal{M}$, $\mathcal{FP}(\mathcal{F}(\mathcal{M})) \cong \mathcal{SC}(\mathcal{M})$ as graded posets.*

Example: Codimension one faces of the Voronoi cell correspond to circuits in $\mathcal{M}$. Edges of the Voronoi cell correspond to maximal strongly orientable submatroids of $\mathcal{M}$.

The parallel faces of the Voronoi cell correspond to the different strongly connected orientations of the same underlying matroid.

For a face F, we consider $[F]$ the **equivalence class of faces that are parallel to** F.

For a subset $A \subseteq E(\mathcal{M})$ we denote $[F_A]$ to be the face(s) which corresponds to $\mathcal{M} \setminus A$ (provided it is strongly orientable).

A matroid is 3-connected if $\mathcal{M} \setminus \{e\}$ is 2-connected for all $e \in E(\mathcal{M})$. Furthermore, a 2-connected matroid can always be strongly oriented as it always has a positive circuit basis.

Reconstructing 3-connected matroids:

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For general 2-connected matroids steps (2) and (3) remain the same, but step (1) requires much more work, as maximal strongly connected submatroids may be of the form $\mathcal{M} \setminus S$ for (possibly different sized) $|S| > 1$.

The key to the reconstruction for 2-connected matroids are **2-cut blocks**.

These are the equivalence classes of the equivalence relation $e \sim f$ if and only if $e = f$ or $\{e, f\}$ is a cocircuit (a circuit in the dual matroid).

These 2-cut blocks are precisely those sets for which $\mathcal{M} \setminus S$ is a maximal strongly connected submatroid, so denote $S_{[\epsilon]}$ for the 2-cut block which corresponds to the parallel class of edges $[\epsilon]$.

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3. For each basis circuit $C_i, i = 1, \dots, r$, write the equation

$$\sum_{[\epsilon] \notin F_{C_i}} \left|S_{[\epsilon]}\right| = \langle C_i, C_i\rangle,$$

and for each pair of basis circuits $\{\{C_i, C_j\}, i, j = 1, \dots, r, i \neq j\}$ write the equation

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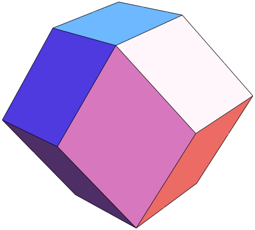
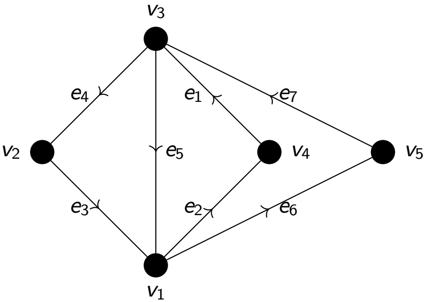
5. $E(\mathcal{M})$ is the disjoint union of the sets $\{S_{[\epsilon]}\}$.
6. Again, the element e belongs to a circuit C if and only if no member of the corresponding edge parallel class $[\epsilon]$ belongs to the face $[F_C]$.

Reconstructing





$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix} \end{array}$$

$$\begin{aligned} C_1 &= \{+e_1, +e_2, +e_5\}, \\ C_2 &= \{+e_5, +e_6, +e_7\}, \\ C_3 &= \{+e_1, +e_2, +e_3, +e_4\}. \end{aligned}$$

Greene showed that:

Theorem 3: *The d-invariant of the lattice of integer flows of a 2-connected graph determines its stable isomorphism type.*

Then naturally conjectured:

Conjecture 1: *The d-invariant of the lattice of integer flows of a 2-connected regular matroid determines its stable isomorphism type.*

The approach we used shows most of what is required for this conjecture to be true.