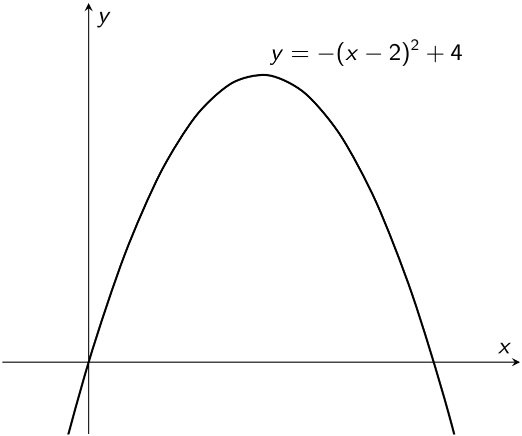
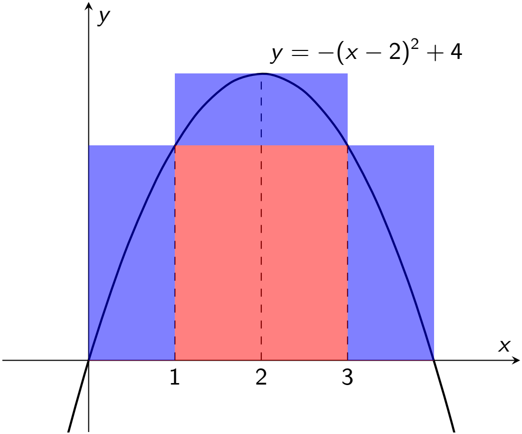
Diamond Duplication Glitch (Working 22/05/2025)

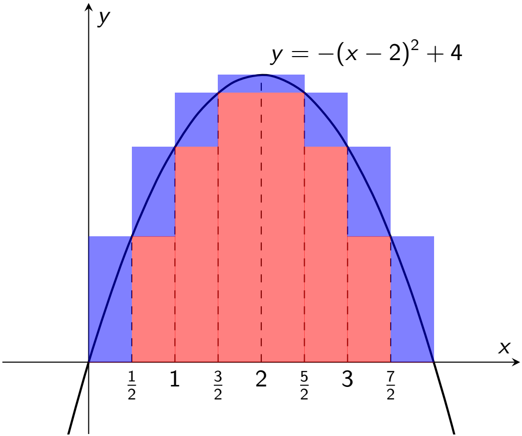
Alec Elhindi

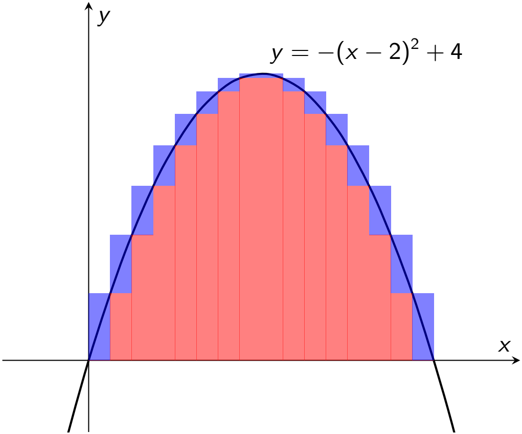
22 May 2025


y
$y = -(x-2)^2 + 4$
x



y
$y = -(x-2)^2 + 4$
x
1
2
3


y
$y = -(x-2)^2 + 4$
x
$\frac{1}{2}$
1
$\frac{3}{2}$
2
$\frac{5}{2}$
3
$\frac{7}{2}$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We can integrate 'nice' functions. What about

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Can we even do this?

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So

$$\int_{-\infty}^{\infty} 1\, dx = \int_{-\infty}^{\infty} 1_{\mathbb{Q}} + 1_{\mathbb{R}\setminus\mathbb{Q}}\, dx = \int_{-\infty}^{\infty} 1_{\mathbb{Q}}\, dx + \int_{-\infty}^{\infty} 1_{\mathbb{R}\setminus\mathbb{Q}}\, dx = 0$$

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Using Reimann integration, we are assigning the set

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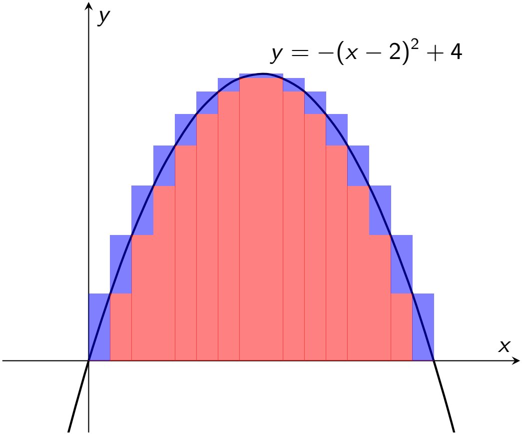
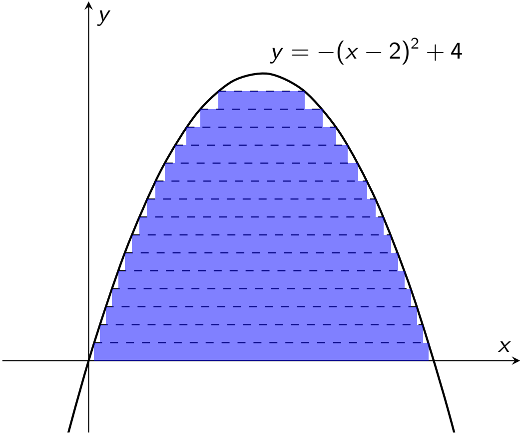
Lebesgue wanted to know: can we assign a notion of volume to **all sets**?



The volumes of the rectangles are $f(x) \cdot \Delta x$.



The volumes of the rectangles are $y \cdot m(f^{-1}(y))$. When is this useful?

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1. $m([a, b]) = b - a$,
2. Translation invariance $m(E) = m(E + t)$.

Again taking

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the integral under the Lebesgue measure can easily be calculated as

$$\int_{\mathbb{R}} 1_{\mathbb{Q}} \, dm = 0 \cdot m(1_{\mathbb{Q}}^{-1}(0)) + 1 \cdot m(1_{\mathbb{Q}}^{-1}(1))$$

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Similarly (with a slight lie),

$$\int_{\mathbb{R}} 1_{\mathbb{R}\setminus\mathbb{Q}} \, dm = m(\mathbb{R} \setminus \mathbb{Q}) = m(\mathbb{R}) - m(\mathbb{Q}) = \infty.$$

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Theorem 4: **Fubini/Tonelli's Theorem** *If* $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *is measurable and one of the following are finite:*
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each of which have Riemann integral 0. f_n converges to $1_{\mathbb{Q}\cap[0,1]}$, but this is not Riemann integrable, the MCT fails in Riemann integration.

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The integrals can be swapped on a rectangle $[a,b] \times [c,d]$ if $f(x,y)$ is continuous, that is

$$\int_a^b \left(\int_c^d f(x,y)\, dx\right)\, dy = \int_c^d \left(\int_a^b f(x,y)\, dy\right)\, dx.$$

Theorem 4: **Fubini/Tonelli's Theorem** *If* $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *is measurable and one of the following are finite:*
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$$\int_{\mathbb{R}^{n\times m}} f(x,y)\ d(x,y) = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\ dx\right)\ dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\ dy\right)\ dx.$$

The integrals can be swapped on a rectangle $[a,b] \times [c,d]$ if $f(x,y)$ is continuous, that is

$$\int_a^b \left(\int_c^d f(x,y)\ dx\right)\ dy = \int_c^d \left(\int_a^b f(x,y)\ dy\right)\ dx.$$

Fubini/Tonelli's theorem is a lot less restrictive!

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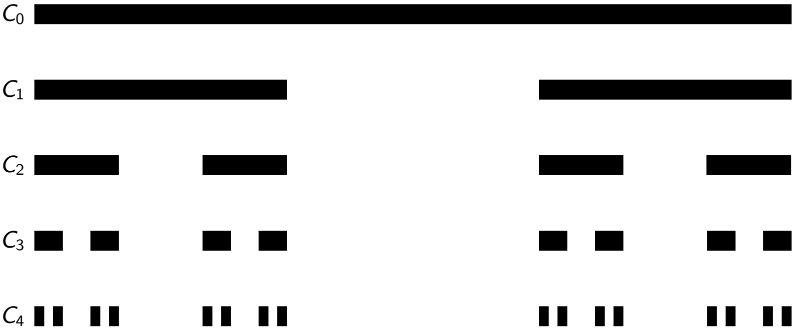

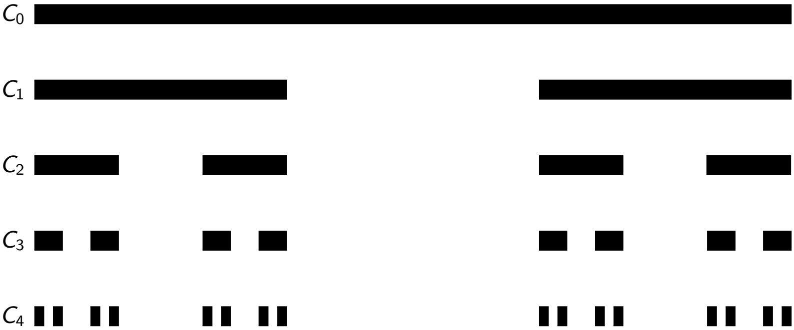

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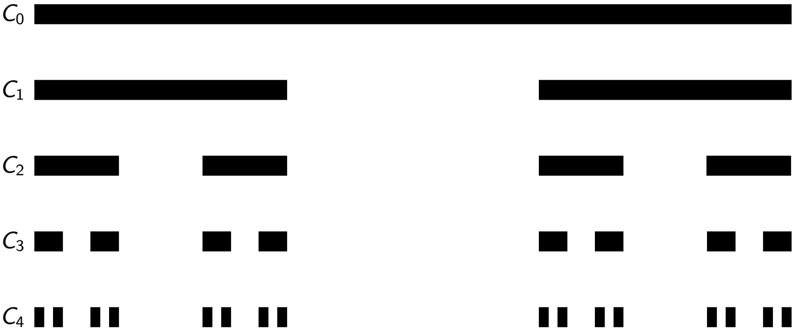

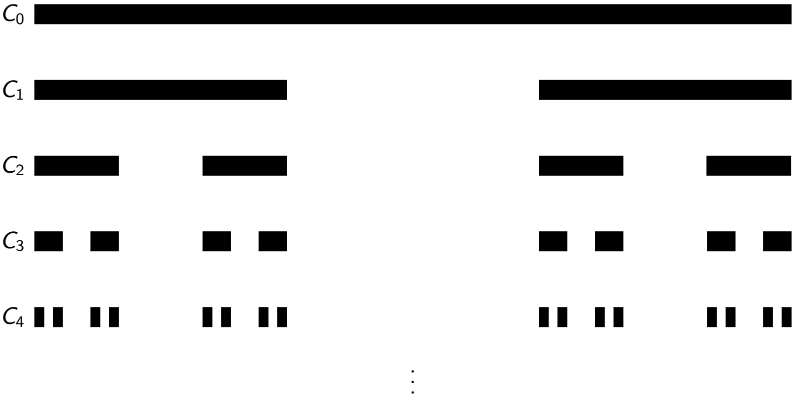

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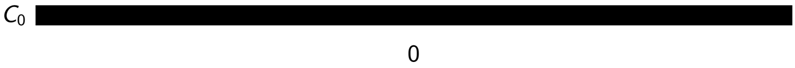

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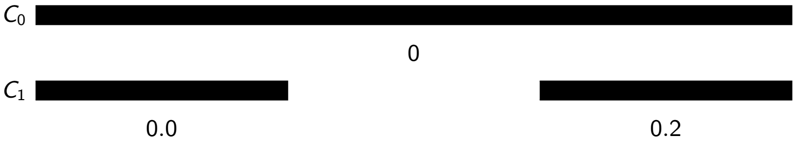

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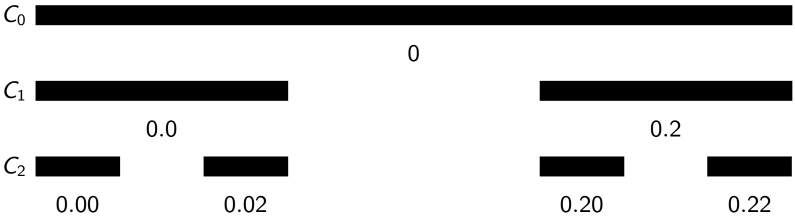

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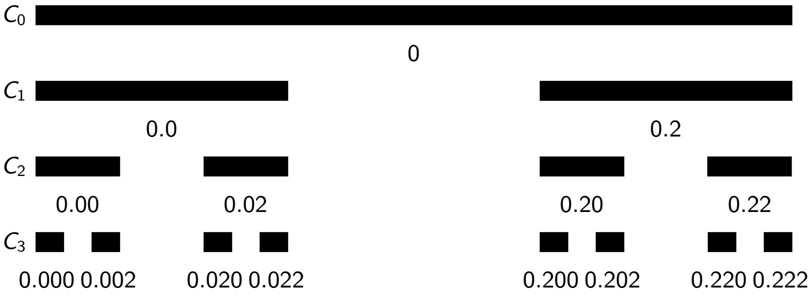

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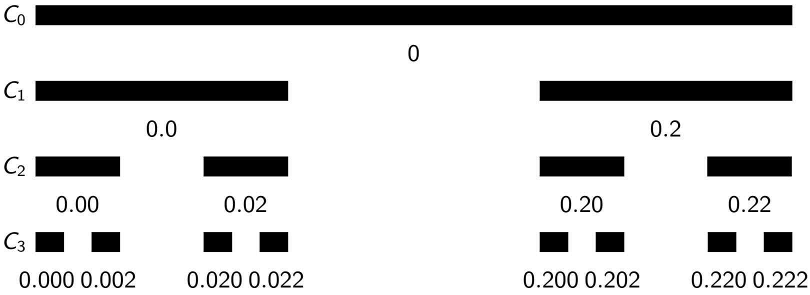

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Cantor set: $C = \{\text{all infinite decimal expansions consisting of 0's and 2's}\}$

These are called **ternary expansions**, i.e. writing the decimal base 3. So the cantor set is precisely the ternary expansions in $[0, 1]$ which do not contain 1.

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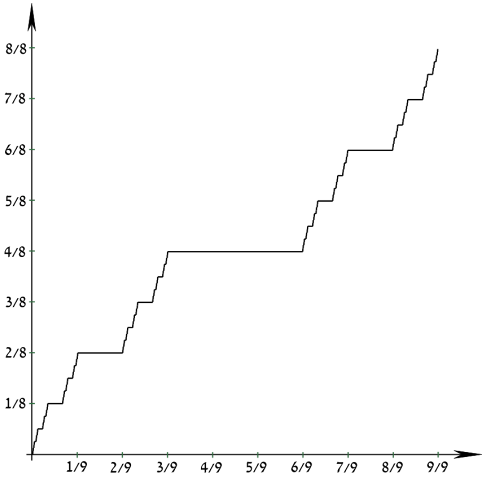
$$m(C_n) = \left(\frac{2}{3}\right)^n \Rightarrow m(C) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$



c is continuous, but not absolutely continuous. It is not differentiable on $\mathbb{R}$-INFINITELY many points, but has zero derivative almost everywhere.

Can All Sets be (Lebesgue) Measured?

Start with the unit interval $[0, 1]$ and partition it into equivalence classes where:

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}$$

so for example, $\frac{\pi}{4} \sim \frac{\pi}{4} + \frac{1}{123} \sim \frac{\pi}{4} + \frac{1}{2} \sim \ldots$.

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$$a + \mathbb{Q} = \{a + q \mid q \in \mathbb{Q}\},$$

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We have $\mathbb{R}$-INFINITELY many of these sets (one for each irrational), and each of them contains $\mathbb{Q}$-infinitely many numbers.

Moreover they are all disjoint: if there are $[a], [a']$ with some $a + q = a' + q'$ then $a - a' = q' - q \in \mathbb{Q}$ and so we must have $[a] = [a']$.

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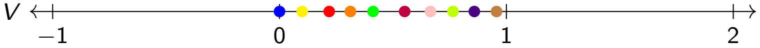
Using the axiom of choice, and our $\mathbb{R}$-INFINITELY many equivalence classes $[a]$, we can choose one from each to form a new set V called the **Vitali set**.

We now want to consider the set $V_q := V + q$ for all $q \in \mathbb{Q} \cap [-1, 1]$ which translates all elements in V by q.



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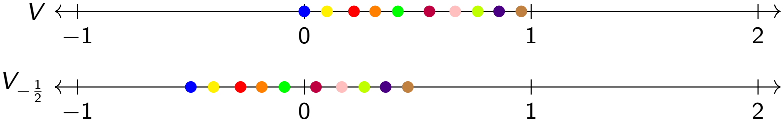

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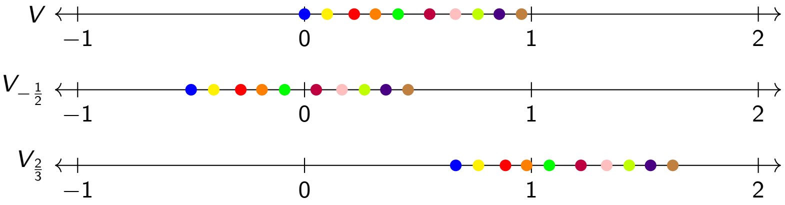

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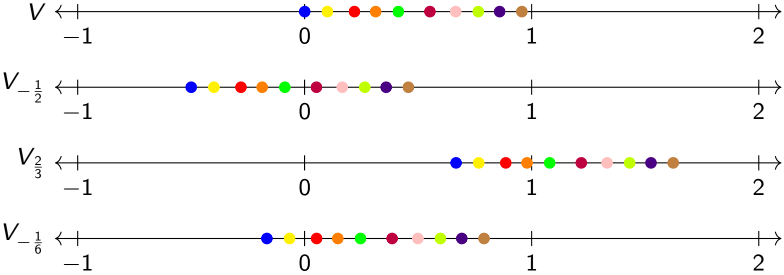

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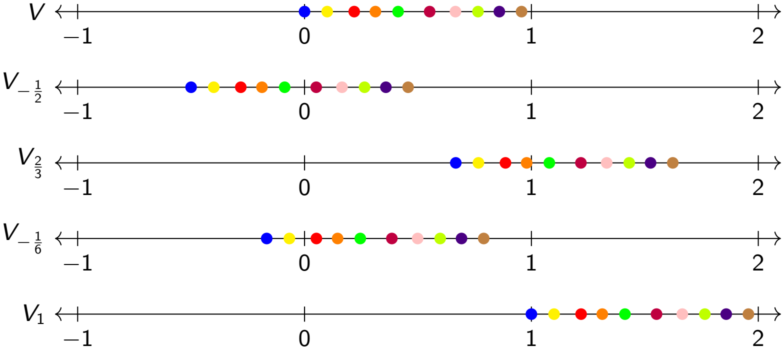

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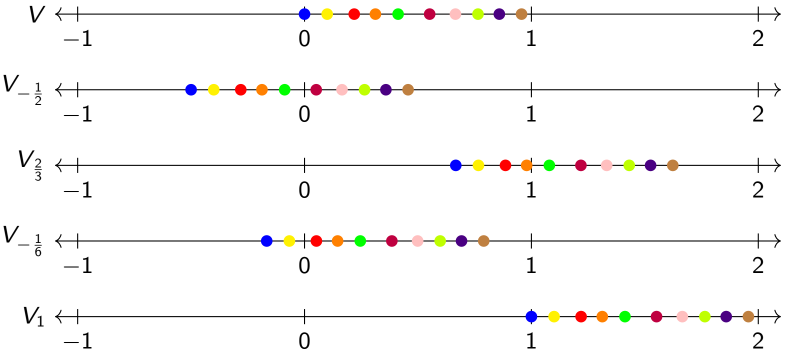

There are $\mathbb{Q}$-infinitely many V_q's, and they are disjoint for the same reason as the $[a]$'s.

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Finally, remember that $m(V_q) = m(V)$ by translation invariance.

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Logical conclusion? The Vitali set V **cannot be measured**.

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and so if we rotate the entire set $S(X)$ by X^{-1}, we get

$$X^{-1}S(X) = \{1\} \sqcup S(X) \sqcup S(Y) \sqcup S(Y^{-1}).$$

Not only have we got $S(X)$ in this set, but we have all of $S(Y)$ and $S(Y^{-1})$, and this is in bijection with just $S(X)$... a dupe?

$$\{1\} \sqcup S(X) \sqcup S(X^{-1}) \sqcup S(Y) \sqcup S(Y^{-1}) = F(X, Y)$$

Consider the set $X^{-1}S(X)$, it is made up of elements of the form:

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- $X \ldots,$
- $Y \ldots,$
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$$Y^{-1}S(Y) = \{1\} \sqcup S(Y) \sqcup S(X) \sqcup S(X^{-1}).$$

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Adding the remaining pieces $S(X^{-1})$ and $S(Y^{-1})$ we get that

$$X^{-1}S(X) \sqcup S(X^{-1}) = F(X,Y) \qquad Y^{-1}S(Y^{-1}) \sqcup S(Y^{-1}) = F(X,Y)$$

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and we have split $F(X, Y)$ into two copies of $F(X, Y)$.

Given a point on a sphere p_1, and letting $F(X, Y)$ be all the $\mathbb{Q}$-infinitely points attainable by rotations, called the **orbit** of p_1 and denoted O_{p_1}, we can duplicate all those points.

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We are not done, the sphere has $\mathbb{R}$-INFINITELY many points. So take $S^3 \setminus O_{p_1}$ and again take a point p_2 and its orbit O_{p_2} and duplicate that. Again with $S^3 \setminus (O_{p_1} \cup O_{p_2})$ and so on.

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We can make this CHOICE because of the axiom of choice, giving $\mathbb{R}$-INFINITELY many orbits O_{p_i} which together cover the sphere, and each of which we can duplicate. So

$$S^3 = S^3 \cup S^3 = \cup_{i=1}^{\infty} S^3$$

we have infinite spheres.

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So... infinite diamonds? When we split the sphere into its orbits, these are non-measurable sets, and so duplicating them turns your object with volume into two or more that have no notion of volume to measure.

Your one diamond has turned into infinite which cannot be moved, dropped or crafted with and vanish when you log out.

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6. $\mathbb{R}$ could be the union of two disjoint sets with SMALLER cardinality than $\mathbb{R}$. Or equivalently, there is a surjective function $f : \mathbb{R} \to S$ where S has LARGER cardinality than $\mathbb{R}$.
7. All subsets of $\mathbb{R}$ are Lebesgue measurable. As a consequence, it is possible to partition 2^ω into more than 2^ω many pairwise disjoint nonempty sets.

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"I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral."

-Henri Lebesgue