EMBEDDING THE HILBERT SPACE INTO BANACH SPACES

What we will call the Hilbert space in this talk is $\ell_2 = \{x = (x_n)_{n=1}^{\infty},\ \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ equipped with its natural norm. Some classical Banach spaces such as $L_p(\mathbb{R})$, for $1 \leq p \leq \infty$, or $C([0,1])$ contain a subspace which is linearly isometric to ℓ_2. Some others such as c_0 or ℓ_p (for $1 \leq p < \infty,\ p \neq 2$) do not contain any subspace that is linearly homeomorphic to ℓ_2. However, any infinite dimensional Banach space contains subspaces that are "uniformly" isomorphic to the finite dimensional Hilbert spaces (a celebrated result due to Dvoretsky).

After explaining these results on linear embeddings, we will concentrate on a much weaker notion of embedding. A coarse embedding between metric spaces is, intuitively speaking, a map which preserves, in a weak but uniform sense, the geometry at large distances. More precisely, if (M, d) and (N, δ) are metric spaces, $f : M \to N$ is a coarse embedding if there exist maps $\rho, \omega : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to\infty} \rho(t) = \infty$ and for all x, y in M : $\rho(d(x, y)) \leq \delta(f(x), f(y)) \leq \omega(d(x, y))$. Note that no linearity is involved in this definition.

The question of coarsely embedding metric spaces into "nice" Banach spaces has gathered researchers coming from very diverse origins. The starting point is to try to get a better understanding of complicated metric spaces, such as graphs with many vertices, by embedding them into well understood Banach spaces, like the Hilbert space. However many results, including Dvoretsky's theorem, indicate that the Hilbert space is the most difficult Banach space to embed into. For the end of this talk, we will investigate to what extent this impression is true.