Stability of the centers of group algebras of $GL_n(q)$

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Outline

Modified type

- Conjugacy classes of symmetric group $S_n \Leftrightarrow \mathrm{Par}_n = \{\text{partitions of } n\}$

$n = 6$. $\sigma = (1,3)(2,4,5,6) \rightsquigarrow$ type

$n = 7$. σ again, $\rightsquigarrow$ type

- Problem: same σ in S_n and S_{n+1}, different cycle type.
- Solution: delete the first (=green) column.
- Call the remaining partition, , the modified type of σ

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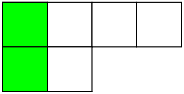
$n = 6$. $\sigma = (1,3)(2,4,5,6) \rightsquigarrow$ type 

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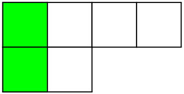
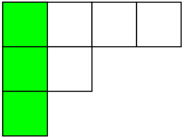
$n = 6$. $\sigma = (1,3)(2,4,5,6) \rightsquigarrow$ type 

$n = 7$. σ again, $\rightsquigarrow$ type 

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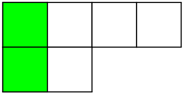
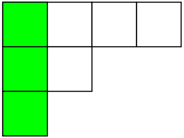
$n = 6$. $\sigma = (1,3)(2,4,5,6) \rightsquigarrow$ type 

$n = 7$. σ again, $\rightsquigarrow$ type 

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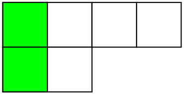
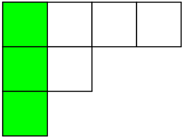
$n = 6$. $\sigma = (1,3)(2,4,5,6) \rightsquigarrow$ type 

$n = 7$. σ again, $\rightsquigarrow$ type 

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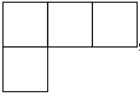
- Call the remaining partition, , the modified type of σ

Class sums

- σ has modified type $\lambda \Rightarrow |\lambda|$ =length $\ell\ell(\sigma) :=$ minimal length for σ as a product of transpositions.

- $\mathcal{C}_\lambda(n)$: conjugacy class of S_n of modified type λ (if $|\lambda| + \ell(\lambda) \leq n$)

- $c_\lambda(n)$: class sum of the class $\mathcal{C}_\lambda(n)$ (if $|\lambda| + \ell(\lambda) \leq n$); otherwise $= 0$.

- Center of the group algebra, $\mathcal{Z}(\mathbb{Z}S_n)$, has a $\mathbb{Z}$-basis $\{c_\lambda(n) \mid \lambda \in \mathrm{Par}\}\backslash\{0\}$.

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An example of structure constants

Write the multiplication in the center $\mathcal{Z}(\mathbb{Z}S_n)$ as

$c_\lambda(n)c_\mu(n) = \sum_\nu g^\nu_{\lambda\mu}(n)c_\nu(n)$, for $g^\nu_{\lambda\mu}(n) \in \mathbb{N}$.

Example

$c_{(1)}(n)$:= class sum of transpositions (=reflections) (i,j) in S_n.

$$c_{(1)}(n)\ c_{(1)}(n) = n(n-1)/2\ c_\emptyset(n) + ??\ c_{(1,1)}(n) + ???\ c_{(2)}(n)$$

(independent of n)

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Stable structure constants

Theorem (Farahat-Higman'59)

(0) *$g^{\nu}_{\lambda\mu}(n)$ is polynomial in n*

(1) *$g^{\nu}_{\lambda\mu}(n) = 0$ unless $|\nu| \leq |\lambda| + |\mu|$*

(2) *If $|\nu| = |\lambda| + |\mu|$, then $g^{\nu}_{\lambda\mu}(n) = g^{\nu}_{\lambda\mu}$ is independent of n*

- Application: modular representation theory of S_n
- Connections: Jucys-Murphy elements

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A stable ring

- S_n and the center $\mathcal{Z}(\mathbb{Z}S_n)$ admits a filtration by $\ell\ell(\sigma)$ (minimal length for σ as a product of transpositions), $\forall \sigma \in S_n$.

Theorem (Farahat-Higman reformulated)

1. *The associated graded $\mathcal{Z}^{gr}(\mathbb{Z}S_n)$ has structure constants independent of n:* $c_\lambda(n)c_\mu(n) = \sum_{|\nu|=|\lambda|+|\mu|} g^\nu_{\lambda\mu} c_\nu(n)$
2. $\exists$ *a stable center "$\mathcal{Z}^{gr}(\mathbb{Z}S_\infty)$" with basis $\{c_\lambda \mid \lambda \in Par\}$ and* $c_\lambda c_\mu = \sum_{|\nu|=|\lambda|+|\mu|} g^\nu_{\lambda\mu} c_\nu$
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  1. The stable center $\mathbb{Q} \otimes_{\mathbb{Z}}$ "$\mathcal{Z}^{gr}(\mathbb{Z}S_\infty)$" is a polynomial algebra in $c_{(r)}, r \geq 1$ and "$\mathcal{Z}^{gr}(\mathbb{Z}S_\infty)$" $\cong \Lambda$.
  2. $\mathcal{Z}^{gr,*}(\mathbb{Z}S_n) \cong H^{2*}(Hilb^n(\mathbb{C}^2), \mathbb{Z})$, cohomology ring of Hilbert scheme of n points on $\mathbb{C}^2$ [Lehn-Sorger, Vasserot]

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Wreath products

- Γ: a finite group
- $\Gamma_n := \Gamma^n \rtimes S_n$ – a wreath product
- [Wang'04]. Generalization of Farahat-Higman stability to Γ_n.
- Let $\Gamma \leq SL_2(\mathbb{C})$. $\mathcal{Z}^{\mathrm{gr},*}(\mathbb{Z}\Gamma_n) \cong H^{2*}(Hilb^n(\mathbb{C}^2//\Gamma))$, cohomology ring of Hilbert scheme of n points on the surfaces $\mathbb{C}^2//\Gamma$
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(i) cohomology ring of Hilbert scheme of n points of more general surfaces [Li-Qin-Wang'04]
(ii) Chen-Ruan orbifold cohomology of symmetric products [Qin-Wang'04]
- [Francis-Wang'09]. Analogous stability for Hecke algebra associated to S_n.

Reflection filtration on $GL_n(q)$

- $G_n := GL_n(q) = \{g \in \mathrm{Mat}_n(\mathbb{F}_q) \text{ invertible}\}$ acts on $V = \mathbb{F}_q^n$.
- Reflections on G_n: $g \in G_n$ such that $\operatorname{codim} V^g = 1$.

(i) $\operatorname{diag}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, I_{n-2}\right)$, or conjugates – (unipotent)

(ii) $\begin{bmatrix} \xi & 0 \\ 0 & T_{n-1} \end{bmatrix}$ with $\xi \in \mathbb{F}_q \backslash \{0, 1\}$, or conjugates – (semisimple)

- Fact. G_n is generated by reflections.

  Proof. Gaussian elimination (Linear Algebra)

- Assigning $\ell\ell(g)$= minimal length of $g \in G_n$ as products of reflections defines a filtered ring structure on G_n
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Modified type

- Let $g \in G_n$ be of type λ. Define modified type $\tilde{\lambda} = \left(\tilde{\lambda}(f)\right)_{f \in \Phi}$:

(i) $\tilde{\lambda}(f) = \lambda(f)$, for $f \neq t-1$

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- **Fact.** $\ell\ell(g) = \|\mu\|$, for g of modified type μ

Example

The lengths of the following matrices are d and $d-1$, respectively:

$$J_f = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_d \end{bmatrix}, \qquad J_m(t-1) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

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$$J_f = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_d \end{bmatrix}, \qquad J_m(t-1) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Modified type, II

- [Huang-Lewis-Reiner '17]

(i) $\ell\ell(g) = \text{codim } V^g$

(ii) Let λ, μ, ν be the modified types of g, h, gh. If $\|\lambda\| + \|\mu\| = \|\nu\|$, then $V^g \cap V^h = V^{gh}$ and $V = V^g + V^h$

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Stable structure constants

- $K_\lambda(n)$: class sum of elements in $GL_n(q)$ of modified type $\boldsymbol{\lambda}$ (if $\|\boldsymbol{\lambda}\| + \ell(\boldsymbol{\lambda}(t-1)) \leq n$); otherwise $= 0$.

- The multiplication in the center is $K_\lambda(n)K_\mu(n) = \sum_\nu a^\nu_{\lambda\mu}(n)K_\nu(n)$, for $a^\nu_{\lambda\mu}(n) \in \mathbb{N}$.

Theorem 1 (W-Wang'18)

(1) $a^\nu_{\lambda\mu}(n) = 0$ *unless* $\|\boldsymbol{\nu}\| \leq \|\boldsymbol{\lambda}\| + \|\boldsymbol{\mu}\|$

(2) *If* $\|\boldsymbol{\nu}\| = \|\boldsymbol{\lambda}\| + \|\boldsymbol{\mu}\|$, *then* $a^\nu_{\lambda\mu}(n) = a^\nu_{\lambda\mu}$ *is independent of* n

Proof uses a [remarkable] normal form for triples (g, h, gh) of modified type $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$ with $\|\boldsymbol{\nu}\| = \|\boldsymbol{\lambda}\| + \|\boldsymbol{\mu}\|$.

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Theorem 2 ([W-Wang], a reformulation)

1. *$\mathcal{Z}(\mathbb{Z}GL_n(q))$ is a filtered ring with $\ell\ell(K_\lambda(n)) = \|\boldsymbol{\lambda}\|$.*
2. *The associated graded $\mathcal{Z}^{gr}(\mathbb{Z}GL_n(q))$ has structure constants independent of n:*

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Example

1. *Computed $a^{\nu}_{\lambda\mu}$ completely when $\|\boldsymbol{\lambda}\| = \|\boldsymbol{\mu}\| = 1$, e.g.,*

$$a^{(2)_{t-\xi'}}_{(1)_{t-\xi}(1)_{t-\eta}} = q \text{ if } \xi' \notin \{\xi, \eta\};$$
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$\Rightarrow [[x]]\,[[y]] = 3[[h]] + \ldots$

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- Recall 2 types of reflections: semisimple or unipotent. The structure constants in Examples above ignore such differences.

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Conjecture I: a polynomial ring

Computations have suggested general patterns.
We shall present several conjectures and open problems.

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The stable center $\mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}(q)$ is a polynomial algebra generated by the single cycle class sums $K_{(r)_f}$, for all $r \geq 1$ and $f \in \Phi$.

(Analogous statements hold for S_n and wreath products.)

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- $\forall \boldsymbol{\lambda} \in \mathrm{Par}(\Phi)$, define its support $\Phi(\boldsymbol{\lambda}) = \{f \in \Phi \mid \boldsymbol{\lambda}(f) \neq \emptyset\}$
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The structure constants $a^{\nu}_{\lambda\mu}$ only depend on the configurations of $\{\lambda, \mu, \nu\}$, i.e., $a^{\nu}_{\lambda\mu} = a^{\tilde{\nu}}_{\tilde{\lambda}\tilde{\mu}}$.

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Generic/motivic structure constants

Question. How does $a^{\nu}_{\lambda\mu}$ depend on q?

- Write $\Phi_q = \Phi$ to indicate its dependence on q.
- $\Phi_{\mathbb{Z}}$: set of monic irreducible polynomials in $\mathbb{Z}[t]$ other than t.
- Any polynomial in $\mathbb{Z}[t]$ lies in $\mathbb{F}_q[t]$ by reduction modulo q.

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Conjecture III (Generic/motivic structure constants)

(1) Suppose $\lambda, \mu, \nu \in \mathcal{P}(\Phi_{\mathbb{Z}})$. Then $\exists\, A^{\nu}_{\lambda\mu}(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}]$ such that $a^{\nu}_{\lambda\mu} = A^{\nu}_{\lambda\mu}(q), \forall q$ with $\Phi_{\mathbb{Z}}(\lambda), \Phi_{\mathbb{Z}}(\mu), \Phi_{\mathbb{Z}}(\nu) \subset \Phi_q$.

(2) (Positivity) Let $B^{\nu}_{\lambda\mu} \in \mathbb{Z}[\mathbf{q}]$ be s.t. $B^{\nu}_{\lambda\mu}(\mathbf{q}) = A^{\nu}_{\lambda\mu}(\mathbf{q}+1)$. Then $B^{\nu}_{\lambda\mu} \in \mathbb{N}[\mathbf{q}]$.

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(2) (Positivity) Let $B^{\nu}_{\lambda\mu} \in \mathbb{Z}[\mathbf{q}]$ be s.t. $B^{\nu}_{\lambda\mu}(\mathbf{q}) = A^{\nu}_{\lambda\mu}(\mathbf{q}+1)$. Then $B^{\nu}_{\lambda\mu} \in \mathbb{N}[\mathbf{q}]$.

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Question. How does $a^{\nu}_{\lambda\mu}$ depend on q?

- Write $\Phi_q = \Phi$ to indicate its dependence on q.
- $\Phi_{\mathbb{Z}}$: set of monic irreducible polynomials in $\mathbb{Z}[t]$ other than t.
- Any polynomial in $\mathbb{Z}[t]$ lies in $\mathbb{F}_q[t]$ by reduction modulo q. ($\forall f(t) \in \Phi_{\mathbb{Z}}$, $f(t) \in \Phi_q$ for q any power of a large enough prime.)

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- [Méliot'14] $\exists$ polynomials $\tilde{p}^{\nu}_{\lambda\mu}(x)$ with rational coefficients such that $a^{\nu}_{\lambda\mu}(n) = \tilde{p}^{\nu}_{\lambda\mu}(q^n)$
- (Equiv.) $\exists$ a polynomial $\mathfrak{p}^{\nu}_{\lambda\mu}(x)$ with rational coefficients such that $a^{\nu}_{\lambda\mu}(n) = \mathfrak{p}^{\nu}_{\lambda\mu}([n]_q)$. (Use $q^n = (q-1)[n]_q + 1$)

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- Geometric interpretation,......
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References

[W-Wang'18] *Stability of the centers of group algebras of $GL_n(q)$*, arxiv:1805.08796

Thank you!