

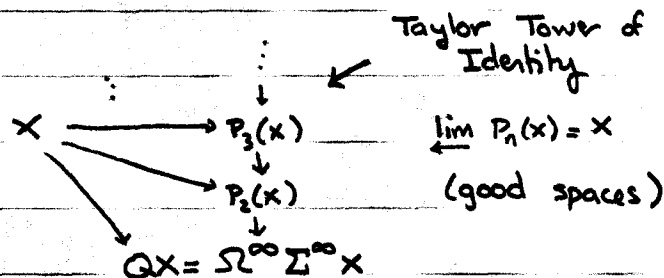
Delooping the Connecting Maps in the Taylor Tower

Greg Arone
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(Joint w/ W. Dwyer & K. Lesh)

I. Given X , get a tower of fibrations

Gives a way to pass from
stable to unstable information



Know a lot about this: $D_n(X) = \text{Fiber}(P_n(X) \rightarrow P_{n-1}(X))$

$$\Omega^\infty((W_n \wedge X^n)_{hZ_n}) \quad , \text{ so just like } \frac{1}{n!} \text{ derivative } x^n$$

the Taylor series. Since these are ∞ loop spaces, have maps

$$P_n(X) \leftarrow D_n(X) \\ \downarrow \\ BD_n(X) \leftarrow P_{n-1}(X)$$

\Rightarrow connecting (k-invariant style) maps $D_{n-1}(X) \rightarrow BD_n(X)$

This gives the d_i differential in the SS for $\pi_*(X)$, using the tower.

Fix a prime $p \nmid$ let $X = S^m$, m odd, $p > 2$, m anything, $p = 2$.

Thm In this case, $D_n(X) \simeq *$ unless $n = p^k$.

We can then regrade the tower

$$DI_k = D_{p^k}(X)$$

$$\begin{array}{c} \vdots \\ P_{p^2}(X) \leftarrow DI_2 \\ \downarrow \\ P_p(X) \leftarrow DI_1 \\ \downarrow \\ P_1(X) = QX \end{array}$$

Want a connecting map, and since

$$P_{p^k}(X) \xrightarrow{\sim} P_{p^{k-1}}(X) \\ \downarrow \quad \uparrow \\ BDI_k \leftarrow \alpha_{k-1} \leftarrow DI_{k-1}$$

this is our connecting map

Have another functor

$$V \mapsto BU(V)$$

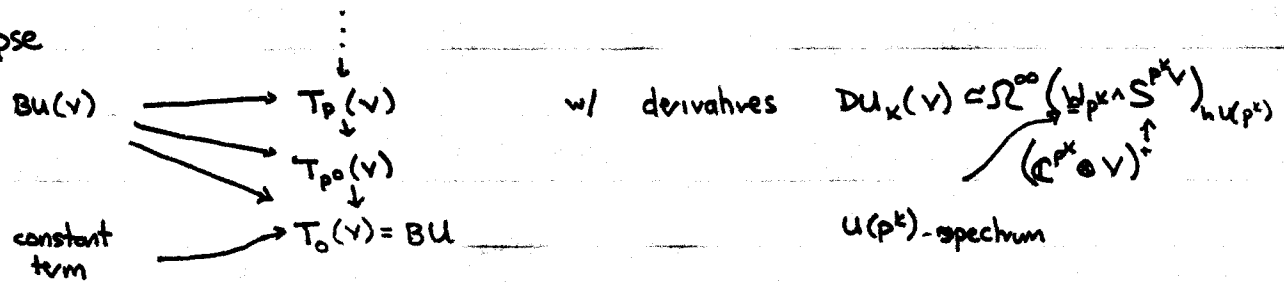
\mathbb{C} -vect \rightarrow Spaces
tensor product

Have a Weiss calculus starting with

$$BU = \varinjlim BU(V)$$

after p -completing, again have faster

collapse



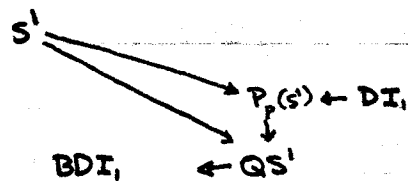
Again get connecting maps $DU_k \xrightarrow{\beta_k} BDU_{k+1}$

These are not infinite loop maps but

Thm $\alpha_k \neq \beta_k$ are k -fold loop maps.

i.e. $\exists \tilde{\alpha}_k: B^k DI_k \rightarrow B^{k+1} DI_{k+1}$ w/ $\alpha_k \simeq \Omega^k \tilde{\alpha}_k$

Example S^1 (even for this, α_k not ∞ -loop map)



Have a sequence of spaces using the deloopings:

$$S^1 \xrightarrow{\alpha_1} QS^1 \xrightarrow{\beta_1} BDI_1 \xrightarrow{\alpha_2} B^2 DI_2 \rightarrow \dots \quad \xrightarrow{\alpha_k} = \text{not } \infty \text{ loop maps}$$

$f_k: \leftarrow \dots = \infty$ loop map coming from $Sp^\infty(X) \simeq K(H_k X) \leadsto$

$$S^0 \hookrightarrow Sp^2(S^0) \hookrightarrow \dots \hookrightarrow Sp^\infty(S^0) = H\mathbb{Z}$$

gives a filtration of $H\mathbb{Z}$. Also have (p -locally) $Sp^n(S^0)/Sp^m(S^0) \simeq *$

unless $n = p^k$. Also get $H\mathbb{Z} \leftarrow S^0 \leftarrow \Sigma^{-1} Sp^p(S^0)/Sp^1(S^0) \leftarrow \dots$

Looping other sequence gives

$$\mathbb{Z} \leftarrow QS^0 \leftarrow \Omega BDI_1 \leftarrow \dots = \Omega^\infty (\text{above tower})!$$

$$QB\mathbb{Z}_p$$

Have an analogous filtration in the orthogonal case:

$$bu \rightarrow A_1 \rightarrow \dots \rightarrow H\mathbb{Z}$$

The Weiss calculus gives

$$\begin{array}{ccccccc} BU(1) & \rightarrow & BU & \rightarrow & BDU_0 & \rightarrow & B^2 DU_1 \rightarrow \dots \\ \parallel & & & & \parallel & & \\ \mathbb{C}P^\infty & & & & Q\mathbb{C}P^\infty & & \end{array}$$

taking Ω^2 , get

$$\mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \times BU \rightarrow Q\mathbb{C}P^\infty \rightarrow \Omega^2 B^2 DU_1 \rightarrow \dots \xrightarrow{f_k} \Omega^k$$

same as the filtration from above

Conjecture: There exist deloopings α_k, β_k that act as a contracting homotopy

$$f_k \circ \alpha_k + \alpha_{k+1} \circ f_{k+1} \simeq \text{Id}$$

\Rightarrow The "chain complexes" are "exact"

\Rightarrow homotopy SS collapses at E_2 .

Idea of proof: How can we show a map is a k -fold loop map?

$$\Omega^k X \rightarrow \Omega^k Y \quad ! \text{ find the image of } \Omega^k: \text{Map}(X, Y) \rightarrow \text{Map}(\Omega^k X, \Omega^k Y)$$

These are too big. Instead look at $\text{Nat}(B^k(-), B^{k+1}(-)) \xrightarrow{\Omega^k} \text{Nat}(DI_k(-), BDI_{k+1}(-))$

(really consider $V \mapsto S^1 \wedge S^V$ in Weiss story!)

$$\text{Care about } \Omega^d \Omega^\infty \Sigma^\infty X_{h\Omega^k \mathbb{Z}_2} \xrightarrow{\Omega^d} \Omega^d Q X_{h\Omega^{k+1} \mathbb{Z}_2}$$