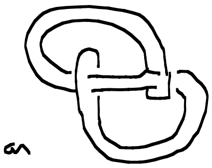
Josh Greene — Floer Homology and Knot Concordance

Note Title 4/17/2008

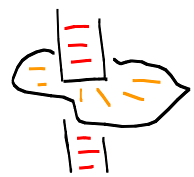
(Joint w/ Slaven Jabuka)

Consider the knot 6_1 :

This bounds a disk (but not an embedded one)



A "ribbon disk" is an immersed disk in S^3, all singularities look like



S^3 is the boundary of D^4, and can push in parts of a ribbon disk in S^3 into $\operatorname{int}(D^4)$ to get a smoothly embedded disk whose boundary is the knot you started with.

A knot $K \subset S^3$ which bounds a smoothly embedded disk in D^4 is called a <u>slice knot</u>.

<u>Classical Obstructions to Sliceness</u>:

If K is a slice knot

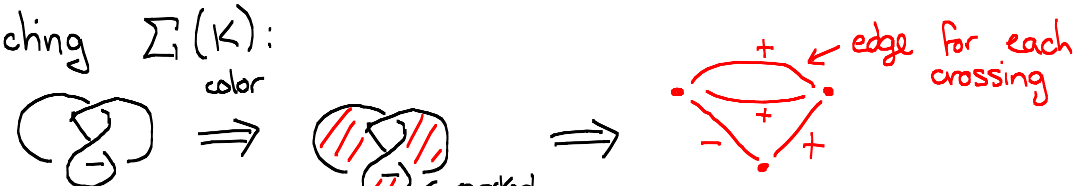
- $\Delta_K(t) = f(t) \cdot f(t^{-1})$ some $f(t) \in \mathbb{Z}[t]$
- $\det(K) = |\Delta_K(-1)| = f(-1)^2$
- $\sigma(K) = 0$
- $\tau(K) = 0$ (Knot Floër homology)
- $s(K) = 0$ (Khovanov homology)
- $\delta(K) = 0$ Floer homology of $\Sigma(K)$

<u>Thm</u> (Casson - Gordon)

If K is slice, then $\Sigma(K) = \partial W^4$, W^4 a $\mathbb{Q}$ homology ball.

<u>Def</u> $\Sigma(K)$ - branched double cover of S^3, branch locus maps to K.

Constructing $\Sigma(K)$:



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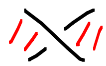

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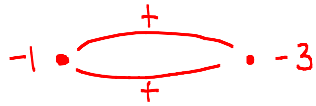
Throw away the marked vertex, and label vertices w/ − sum of "orientations":

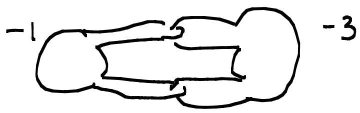


← Reduced red graph

Fact The reduced red graph encodes K (but not quite the diagram)

To get $\Sigma(K)$:

put an unknot at each vertex & a clasp for each edge:



$\Rightarrow$ framed link, surgery on which gives $\Sigma(K)$

From a red graph, get a matrix: diag entries: number at vertex

off diag: sum of edges

$$G_\Gamma = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

= linking matrix of associated link

= intersection matrix on X_Γ:

starting from a diagram for K, we have a 4-manifold X_Γ whose boundary is $\Sigma(K)$

If you start with an alternating diagram for K, G_Γ is neg def

so X_Γ is neg def $\Rightarrow$

<u>Thm</u> <u>A</u> (Donaldson)

If X is a closed, orientable, neg def, smooth 4 manifold then its intersection form is diagonalizable.

If K is slice, then $X_\Gamma \cup W^4$ is a closed, neg definite orient. 4-manifold!

This says that G_Γ is represented by $-I_r$, $r = \mathrm{rank}(G_\Gamma)$.

Conversely, this means $\exists A$ s.t. $G_\Gamma = -A \cdot A^T$, $A \in M_r(\mathbb{Z})$

So: if you can't write $G_\Gamma = -A \cdot A^T$, then K is not slice!

Using this, can see $K = P(-3,5,7)$ is <u>not</u> smoothly slice

OTOH: $\Delta_K = 1 \Rightarrow$ K is top. slice (Freedman)

More recently, used by Lisca '07 to classify slice, 2-bridge knots

Another Test: Heegard – Floer Homology

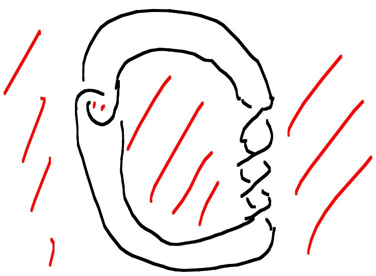
Ozsváth – Szabó defined a "d-invariant" $\quad d: H^2(Y^3) \to \mathbb{Q}$

and if $Y^3 = \Sigma_2(K)$, K slice, then d vanishes on a special s.g of H^2.

<u>Thm</u> K a slice knot, D diagram s.t. • alternating or • red, red graph a tree, and G neg def. Then

- $G_\Gamma = -A \cdot A^T$
- Every class in coker(A^T) has a rep all of whose entries are ± 1.

<u>Ex</u>



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6_1 !

$$\rightsquigarrow \quad -2 \overset{+}{\bullet\!\frown\!\bullet} -5 \quad \rightsquigarrow \quad G_\Gamma = \begin{bmatrix} -2 & 1 \\ 1 & -5 \end{bmatrix}$$

$$= - \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

& $\bar{o} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in Im(A^T)$,

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin$ coker while coker $= \mathbb{Z}/3$

(det = |coker| = 3).

If we have more twists (say $n \geq 5$)

$$\rightsquigarrow \quad -2 \overset{+}{\bullet\!\frown\!\bullet} -(n+1)$$

$G_\Gamma = \begin{bmatrix} -2 & 1 \\ 1 & -(n+1) \end{bmatrix} \Rightarrow \det(A^t) = \sqrt{2n+1}$, and if $n \geq 5$, this is ≥ 4 (the number of elements w/ ± 1)

$\Uparrow$
$\mathbb{Z}$

P(-3, 5, 7)