Quantum category $\mathcal{O}$, II

1) Affine Hecke category.

We have seen that the affine Weyl group controls certain aspects of $\mathcal{O}_\varepsilon^{mix}$. In fact, much more is true: there are derived equivalences between $\mathcal{O}_\varepsilon^{\Theta}$'s & certain "standard" categories associated to W^a: (singular) blocks of the affine Hecke category.

1.1) Definition

Let W' be a (possibly infinite) reflection group acting on its reflection representation V.

Example: $W'=W^a$, $V=\mathfrak{h}\oplus\mathbb{C}\hbar$; $W^a \curvearrowright V$: $\hbar$ is W^a-invariant, W acts on $\mathfrak{h}$ as usual, and for $\lambda\in\Lambda_0$, have $t_\lambda x = x+\langle\lambda,x\rangle\hbar$, $x\in\mathfrak{h}$.

Let $S\subset W'$ be the subset of simple reflections in W'. Set $R=S(V)$, this is a graded algebra.

Definition: Let $s \in S$. Define the Bott-Samelson R-bimodule $B_s = R \otimes_{R^s} R$ (where R^s is the subalgebra of s-invariants in R). Note that B_s is graded as a bimodule.

Consider the categories R-grbimod & R-grmod of graded finitely generated R-bimodules & graded R-modules. For $j \in \mathbb{Z}$, let $\langle j \rangle$ denote the endofunctor of grading shift:

$$B\langle j\rangle_i = B_{i+j} \quad (\text{for } B \in R\text{-gr(bi)mod})$$

We consider the following (additive) categories:

- SBim (the category of Soergel <u>bimodules</u>): the full subcategory in R-grbimod consisting of all graded bimodules obtained from R & B_s, $s \in S$, by $\otimes$, $\oplus$, $\langle ? \rangle$ & taking direct summands.
- SMod (the category of Soergel <u>modules</u>), the full subcat $\{B/BV \in R\text{-grmod} \mid B \in \text{SBim}\}$. One can show it's closed under taking direct summands.
- <u>SMod</u> (the category of <u>ungraded</u> Soergel modules), a full subcategory in R-mod whose objects are the same as in

SMod (w. grading forgotten).

The categories SBim, SMod & $\underline{\text{SMod}}$ are "Krull-Schmidt": every object is isomorphic to a unique $\oplus$ of indecomposables. The Soergel categorification theorem describes the indecomposable objects in these categories, we will need a version for $\underline{\text{SMod}}$

Proposition: The indecomposable objects in $\underline{\text{SMod}}$ are in bijection w. W'

We are now in position to define the version of the affine Hecke category we need. Set $W' = W^a$ & $V = \mathfrak{h} \oplus \mathbb{C}\hbar$, see Example.

$\mathcal{H} := K^b(\underline{\text{SMod}})$, where K^b stands for the bounded homotopy category of complexes.

1.2) Derived equivalence, regular blocks

Thm 1 (I.L.23) Suppose Θ is a free W^a-orbit (no stabilizer). Then $D^b(\mathcal{O}_\varepsilon^\Theta) \xrightarrow{\sim} \mathcal{H}$.

A somewhat weaker version of Thm was also obtained by Q. Situ.

Remark: Such orbit exists $\Leftrightarrow d \geq h$ (= Coxeter number of G, e.g. for $G = SL_n$, $h = n$). In this case $\Theta := W^a \cdot 0$ is free, the corresponding block is called principal.

1.3) Derived equivalence, singular blocks.

Consider the general (not necessarily free) Θ. We want a description of $\mathcal{O}_\varepsilon^\Theta$ similar to that in Theorem 1. First, we'll choose a distinguished element $\lambda^\circ \in \Theta$. Set $\mathfrak{A} = \{\lambda \in \Lambda_0 \mid \langle \lambda+\rho, \alpha_i^\vee \rangle \leq 0 \ \& \ \langle \lambda+\rho, \delta^\vee \rangle \geq -d\}$, where $\delta^\vee$ is the maximal coroot. Then $\mathfrak{A}$ is a fundamental domain for $W^a \curvearrowright \Lambda_0$ (exercise) $\rightsquigarrow \lambda^\circ :=$ unique point in $\mathfrak{A} \cap \Theta$.

For S take the set of reflections about the walls of $\mathfrak{A}$. Set $J = \{s \in S \mid s \cdot \lambda^\circ = \lambda^\circ\} \rightsquigarrow R^J = \bigcap_{s \in J} R^s$, polynomial algebra.

Definition: The category ${}_J\underline{SMod}$ is the full subcategory in R^J-mod whose objects are direct summands of objects in $\underline{SMod}$ (viewed as modules over $R^J \subset R$). Set ${}_J H := K^b({}_J\underline{SMod})$

Thm (I.L. 23) $D^b(\mathcal{O}_\varepsilon^{\oplus}) \xrightarrow{\sim}_{\mathfrak{J}} \mathcal{H}$.

1.4) Motivations

$\mathcal{O}_\varepsilon^{mix}$ was popularized by Gaitsgory who wanted an extension of the famous Kazhdan-Lusztig equivalence between

(i) a certain parabolic category $\mathcal{O}$ for the affine Lie algebra $\hat{\mathfrak{g}}$

(ii) and the finite dimensional representations of $\mathcal{U}_\varepsilon^L$

He proposed to replace (i) w. the full cat. $\mathcal{O}$ & $\mathcal{U}_\varepsilon^L$-$\mathrm{mod}_{fd}$ w. $\mathcal{O}_\varepsilon^{mix}$. His conjectures essentially yield our main equivalence & were proved by his students Chen & Fu by methods very different from mine (& with different assumptions on d). The motivation came from Quantum Geometric Langlands.

Another motivation: one should be able to use a connection between $\mathcal{O}_\varepsilon^{mix}$ and the affine Hecke category to get character formulas in the category of finite dimensional of $\mathcal{U}_\varepsilon^{DK}$ with some (currently a bit mysterious) additional "equivariance" structure, similar in spirit to Bezrukavnikov-I.L. 20.

Finally, a far reaching motivation is to understand the affine analog of $\mathcal{O}_\varepsilon^{mix}$: where instead of $\mathcal{U}_\varepsilon^{mix}(\mathfrak{g})$ one has $\mathcal{U}_\varepsilon^{mix}(\hat{\mathfrak{g}})$ – understanding $\mathcal{O}_\varepsilon^{mix}$ is a necessary prerequisite.