## Topology General Exam August 16, 2010

Name:	
${\bf Instructions:}$	This is a four hour exam and 'closed book'. There are eight problems.

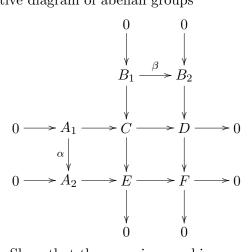
- 1. (a) Let  $T \subset \mathbb{R}^5$  be a closed subspace homeomorphic to  $\mathbb{R}^2$ . Explain why T will be a retract of  $\mathbb{R}^5$ .
- (b) View  $S^n$  as  $\mathbb{R}^n \cup \{\infty\}$ , so that the open subsets of  $S^n$  containing  $\infty$  are precisely the complements of compact subsets of  $\mathbb{R}^n$ . Recall that a continuous function  $f: \mathbb{R}^m \to \mathbb{R}^n$  is called *proper* if  $f^{-1}(C)$  is compact in  $\mathbb{R}^m$  whenever C is compact in  $\mathbb{R}^n$ . Show that such a proper map extends uniquely to a continuous function  $\bar{f}: S^m \to S^n$ .
- (c) With T as in part (a), check that the inclusion  $i: T \hookrightarrow \mathbb{R}^5$  is proper. By contrast, show that no retraction  $r: \mathbb{R}^5 \to T$  can be proper. (Hint: start by using part (b).)

**2.** Let  $\mathbb{R}^{\infty}$  denote the union  $\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \ldots$ , with the union topology, i.e.  $U \subset \mathbb{R}^{\infty}$  is open iff  $U \cap \mathbb{R}^n$  is open in  $\mathbb{R}^n$  for all n. Let  $\mathbb{R}^{\omega}$  denote the product of a countable number of copies of  $\mathbb{R}$ , with the product topology. Check that the evident set theoretic inclusion  $i : \mathbb{R}^{\infty} \to \mathbb{R}^{\omega}$  is continuous, but is *not* a homeomorphism onto its image.

- **3.** (a) Describe a connected double cover of  $\mathbb{R}P^2 \vee \mathbb{R}P^2$ . (There is more than one correct answer.)
- (b) What are the homology groups of your double cover?
- (c) What is the fundamental group of your double cover?

4. Let M be the compact surface with boundary circle C as pictured:
(a) Explain why M is homotopy equivalent to a figure eight. (Hint: M is the torus with a disk removed, and the torus is often represented as a square with opposite edges identified.)
(b) Explain why the inclusion i: C → M induces the zero homomorphism from H₁(C) to H₁(M).
(c) By contrast, explain why i is not null homotopic.

5. Suppose given a commutative diagram of abelian groups



with exact rows and columns. Show that there are isomorphisms  $\ker\alpha\simeq\ker\beta\quad\text{and}\quad\operatorname{coker}\alpha\simeq\operatorname{coker}\beta.$ 

**6.** Recall that an n-dimensional manifold is a Hausdorff topological space M that can be covered by open sets homeomorphic to open sets in  $\mathbb{R}^n$ . Prove that a compact n-dimensional manifold can be embedded in (i.e. is homeomorphic to a subset of)  $\mathbb{R}^N$  for large enough N. (Hint: use a partition of unity associated to a finite open cover  $U_1, \ldots, U_k$  of M equipped with embeddings  $f_i: U_i \to \mathbb{R}^n$ .)

7. Let  $C \subset \mathbb{R}^3$  be the union of the x-axis and the y-axis. Compute  $H_*(\mathbb{R}^3 - C)$ . (Hint: note that  $\mathbb{R}^3 - C = (\mathbb{R}^3 - x$ -axis)  $\cap (\mathbb{R}^3 - y$ -axis).)

- **8.** Let X be a Hausdorff space, and  $f: X \to X$  a continuous function such that
  - $f(x) \neq x$  for all  $x \in X$ , and
  - $f \circ f$  is the identity.
- (a) Show that every  $x \in X$  has an open neighborhood  $W_x$  satisfying  $f(W_x) \cap W_x = \emptyset$ .
- (b) Let  $\bar{X} = X/(x \sim f(x))$ , with the quotient space topology. Show that the quotient map  $q: X \to \bar{X}$  is a covering map.