Quantum category $\mathcal{O}$, III

1) Images of Vermas

1.0) Recap & questions.

Last time we have stated the main equivalence between triangulated categories

$$D^b(\mathcal{O}_\varepsilon^\Theta) \xrightarrow{\sim} {}_J\mathcal{H} := K^b({}_J\underline{SMod}) \qquad (*)$$

Questions:

1) Where does $\mathcal{O}_\varepsilon^\Theta \subset D^b(\mathcal{O}_\varepsilon^\Theta)$ go under $(*)$?

2) Where do Verma modules $\Delta_\varepsilon(\lambda) \in \mathcal{O}_\varepsilon^\Theta$, $\lambda \in \Theta$, go?

1.1) From 2) to 1)

It turns out that we can, to some extent, answer 1) if we know an answer to 2). Define the following subcategories in $D^b(\mathcal{O}_\varepsilon^\Theta)$:

$$\mathcal{T}^{>0} = \{X \in D^b(\mathcal{O}_\varepsilon^\Theta) \mid \operatorname{Hom}_{D^b(\mathcal{O}_\varepsilon^\Theta)}(\Delta_\varepsilon(\lambda)[i], X) = 0, \ \forall \lambda \in \Theta, i \geqslant 0\}$$

$$\mathcal{T}^{\leqslant 0} = \{Y \in D^b(\mathcal{O}_\varepsilon^\Theta) \mid \operatorname{Hom}_{D^b(\mathcal{O}_\varepsilon^\Theta)}(Y, X) = 0 \ \forall X \in \mathcal{T}^{>0}\}$$

Exercise: Show that $\mathcal{O}_\varepsilon^\Theta = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{>0}[1]$ (hint: $\forall M \in \mathcal{O}_\varepsilon^\Theta$ $\exists \lambda \in \Theta$ s.t. $\mathrm{Hom}_{\mathcal{O}_\varepsilon^\Theta}(\Delta_\varepsilon(\lambda), M) \neq 0$).

1.2) Affine braid group action

Motivation: $\mathcal{O}_\varepsilon^\Theta$ has very easy categorical symmetry – the shift of grading by an element of $d\Lambda_0$ sending Vermas to Vermas. Can we see that for ${}_J\mathcal{H}$?

To W^a we can assign the corresponding affine braid group Br^a: it's generated by $T_s, s \in S$, with relations

$$\underbrace{T_s T_t T_s \ldots}_{m_{st}} = \underbrace{T_t T_s T_t \ldots}_{m_{st}} \text{ for } s \neq t,\ m_{st} = \text{order of } st.$$

We want to construct an action of Br^a on a deformed version of $\mathcal{H}$ by equivalences (of triangulated categories), which is more elementary than dealing w. ${}_J\mathcal{H}$ or $\mathcal{H}$.

- Deformed version: $\hat{R} = \varprojlim_n S(V)/(V)^n$, completion at 0

$SBim^\wedge$:= full subcategory in $\hat{R}$-bimod w. objects $B \otimes_R \hat{R} \simeq \hat{R} \otimes_R B$

w. $B \in \underline{SBim}$, monoidal additive category

$\rightsquigarrow \hat{\mathcal{H}} = K^b(\underline{SBim}^\wedge)$ w. monoidal structure denoted by $*$

$ev_0 : \underline{SBim}^\wedge \longrightarrow \underline{SMod}$, $B \mapsto B/BV \rightsquigarrow ev_0 : \hat{\mathcal{H}} \longrightarrow \mathcal{H}$

· Br^a-action (Rouquier): $\exists$ group homomorphism

$\mathcal{R}: Br^a \longrightarrow$ group of iso. classes of invertible objects in $\hat{\mathcal{H}}$

$$T_s \mapsto [0 \to \underset{\text{deg}\,-1}{\hat{R}} \xrightarrow[1 \mapsto \alpha_s \otimes 1 + 1 \otimes \alpha_s]{} \hat{R} \otimes_{\hat{R}^s} \hat{R} \to 0]$$

1.3) Objects $\hat{\Delta}(x)$ & $\Delta(x)$

$x \in W^a \rightsquigarrow T_x \in Br^a$ ($x = s_1 \dots s_e$, reduced, $\rightsquigarrow T_x := T_{s_1} \dots T_{s_e}$)

$\hat{\Delta}(x) := \mathcal{R}(T_{x^{-1}}^{-1}) \in \hat{\mathcal{H}}$, $\quad \Delta(x) := ev_0(\hat{\Delta}(x)) \in \mathcal{H}$.

Remark: One can play the same game as in Exercise in Sec 1.1 w. the objects $\Delta_x \in \mathcal{H}$ getting an abelian subcategory in $\mathcal{H}$ (the heart of a t-structure, in fact). In the constructible realization of $\mathcal{H}$, this is the category of perverse sheaves.

1.4) Lattice & stabilization

The embedding $\Lambda_0 \hookrightarrow W^a$ lifts to $\Lambda_0 \hookrightarrow Br^a$ uniquely so that for $\lambda \in \Lambda_0^+$ (dominant) $\lambda \mapsto T_{t_\lambda}$ (unique b/c $\forall$ element of Λ_0 is the difference of dominant ones).

Notation: $J_\mu :=$ image of $\mu \in \Lambda_0$ in Br^a.

Note that $\hat{\Delta}(xt_\mu) \not\simeq \hat{\Delta}(x) * J_\mu$ for <u>general</u> x, μ. But there's "stabilization."

Observation: $\lambda, \mu \in -\Lambda_0^+, w \in W, x = wt_\lambda \Rightarrow \ell(xt_\mu) = \ell(x) + \ell(t_\mu)$

$\Rightarrow T^{-1}_{(xt_{-\mu})^{-1}} J_{-\mu} = T^{-1}_{x^{-1}} \quad \Rightarrow \hat{\Delta}(xt_{-\mu}) = \hat{\Delta}(x) * J_{-\mu}$

Definition (of objects $\hat{\Delta}^{st}(x)$)/Lemma: $\forall x \in W^a \; \exists \mu_x \in \Lambda_0^+$ s.t. $\hat{\Delta}(xt_{-\mu}) * J_\mu$ doesn't depend on μ if $\mu \geqslant \mu_x$. Denote this object by $\hat{\Delta}^{st}(x)$ ("st" for stabilized)

Exercise: $\hat{\Delta}^{st}(xt_\mu) \simeq \hat{\Delta}^{st}(x) * J_\mu \; \forall x \in W^a, \mu \in \Lambda_0$.

$\Delta^{st}(x) := ev_0(\hat{\Delta}^{st}(x)) \in \mathcal{H}.$

Thm: Let $\Theta = W^a \cdot 0$. $\exists$ equivalence $D^b(\mathcal{O}_\varepsilon^{\Theta}) \xrightarrow{\sim} \mathcal{H}$ w. $\Delta_\varepsilon(x^{-1}\cdot(-2\rho)) \mapsto \Delta^{st}(x)$.

1.5) Singular blocks

$\mathcal{A} :=$ anti-dominant alcove in Λ_0, $\lambda^\circ \in \Theta \cap \mathcal{A} \rightsquigarrow J \subset S \rightsquigarrow$ $R^J \subset R \rightsquigarrow \pi_J : \underline{SMod} \longrightarrow {}_J\underline{SMod} \rightsquigarrow \pi_J : \mathcal{H} \longrightarrow {}_J\mathcal{H}$.

Observation: $\pi_J(\Delta(wx)) \simeq \pi_J(\Delta(x))$ $\forall w \in W_J$, $x \in W^a$

Compare to: Behavior of Vermas in the BGG category $\mathcal{O}$ under translations to the wall.

From Observation (& its deformed version): $\pi_J(\Delta^{st}(wx)) \simeq \pi_J(\Delta^{st}(x))$

Thm': Let $\Theta = W^a \cdot 0$. $\exists$ equivalence $D^b(\mathcal{O}_\varepsilon^{\Theta}) \xrightarrow{\sim} {}_J\mathcal{H}$ w. $\Delta_\varepsilon(x^{-1}\cdot\lambda^\circ) \mapsto \pi_J(\Delta^{st}(x))$.