Quantum category $\mathcal{O}$, I

1) Quantum groups & their forms

2) Quantum category $\mathcal{O}$.

1) Quantum groups & their forms

Notation: $\mathfrak{g}$ simple Lie algebra/$\mathbb{C}$, simply laced (ADE)
- to simplify the exposition.

$\mathfrak{h}\subset\mathfrak{g}$ Cartan

$\Lambda_0\subset\Lambda\subset\mathfrak{h}^*$ root & weight lattice

α_i $(i\in I)$ simple roots, $(a_{ij})_{i,j\in I}$ - Cartan matrix.

v: indeterminate $\rightsquigarrow$ field $\mathbb{C}(v)$

$(\cdot,\cdot)$: invariant form on $\mathfrak{h}^*$ normalized by $(\alpha_i,\alpha_i)=2$.

1.1) Generic version

Definition: Quantum group $\mathcal{U}_v(\mathfrak{g})$ is $\mathbb{C}(v)$-algebra w.
generators E_i, F_i $(i\in I)$, K_ν $(\nu\in\Lambda)$
& relations: • $K_\nu K_{\nu'}=K_{\nu+\nu'}$, $K_0=1$

• $K_\nu E_i K_\nu^{-1}=v^{(\nu,\alpha_i)}E_i$, $K_\nu F_i K_\nu^{-1}=v^{-(\nu,\alpha_i)}F_i$

- $[E_i, F_j] = \delta_{i,j} \dfrac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{v - v^{-1}}$
- quantum Serre relations: $E_i E_j = E_j E_i$ if $a_{ij} = 0$

$$E_i E_j^2 + E_j^2 E_i = (v + v^{-1}) E_j E_i E_j \text{ if } a_{ij} = -1$$

& same for F's

2.2) Forms & specializations

We want to consider $\mathbb{C}[v^{\pm 1}]$-forms of $\mathcal{U}_v(\mathfrak{g})$ i.e. $\mathbb{C}[v^{\pm 1}]$-subalgebras $A \subset \mathcal{U}_v(\mathfrak{g})$ w. $\mathbb{C}(v) \otimes_{\mathbb{C}[v^{\pm 1}]} A \xrightarrow{\sim} \mathcal{U}_v(\mathfrak{g})$

To define the forms we need we introduce some notation.

$n \in \mathbb{Z}_{>0} \rightsquigarrow [n]_v = v^{n-1} + v^{n-3} + \ldots + v^{1-n}$, a quantum integer

$$[n]_v! = \prod_{i=1}^{n} [i]_v$$

$E_i^{(n)} = E_i^n / [n]_v!$ (divided power), same for $F_i^{(n)}$.

Definitions:

- The De Concini-Kac form $\mathcal{U}_v^{DK}(\mathfrak{g})$ is the $\mathbb{C}[v^{\pm 1}]$-subalgebra generated by E_i's, F_i's & K_λ's (the most obvious form).

- The Lusztig form $\mathcal{U}_v^L$: by $E_i^{(n)}, F_i^{(n)}, K_\lambda$ ($\forall i \in I, \lambda \in \Lambda, n \in \mathbb{Z}_{>0}$)

· The mixed form $\mathcal{U}_v^{mix}$: $E_i^{(n)}, F_i, K_\lambda$ $(\forall i\in I, \lambda\in\Lambda, n\in\mathbb{Z}_{\geq 0})$

The point of considering the forms is that one can specialize v to $q\in\mathbb{C}^\times$ getting algebras over $\mathbb{C}$, e.g. $\mathcal{U}_q^{mix} = \mathcal{U}_v^{mix}\otimes_{\mathbb{C}[v^{\pm 1}]}\mathbb{C}_q$ ($\mathbb{C}_q$ is 1-dim. $\mathbb{C}[v^{\pm1}]$-algebra w. $v\mapsto q$). When q is not a root of 1, $[n]_q\neq 0$ $\forall n \Rightarrow \mathcal{U}_q^{DK} = \mathcal{U}_q^{mix} = \mathcal{U}_q^{L}$. But when q is primitive $\sqrt[d]{1}$ we have $[d]_q = 0$ (if d is even, $[d/2]_q = 0$) & the specializations behave differently from each other.

There are several reasons to be interested in the specializations to $q=\sqrt[d]{1}$ from a representation-theoretic perspective. Historically first was a connection to modular (= mod p) representation theory of $U(\mathfrak{g})$ & the corresponding algebraic group G.

The form whose representation theory resembles that of $G/\overline{\mathbb{F}}_p$ is $\mathcal{U}_q^L$, it's also the best understood (since the 90's). The form whose representations resemble those of $U(\mathfrak{g}_{\overline{\mathbb{F}}_p})$ is $\mathcal{U}_q^{DK}$, it's least understood. For these lectures we'll concentrate on the intermediate form, $\mathcal{U}_q^{mix}$.

2) Quantum category $\mathcal{O}$.

2.1) Definition

We fix an odd positive integer d. Let ε be a primitive $\sqrt[d]{1}$.

Note that $\mathcal{U}_v^{mix}$ & $\mathcal{U}_\varepsilon^{mix}$ are graded by Λ_0, $\mathcal{U}_\varepsilon^{mix} = \bigoplus_{\lambda\in\Lambda_0} \mathcal{U}_{\varepsilon,\lambda}^{mix}$

$$\deg F_i = -\alpha_i, \deg K_\nu = 0, \deg E_i^{(\ell)} = \ell\alpha_i$$

so it makes sense to speak about the category of graded $\mathcal{U}_\varepsilon^{mix}$-modules:

- the objects are $\mathcal{U}_\varepsilon^{mix}$-modules M w. direct sum decomposition $M = \bigoplus_{\mu\in\Lambda_0} M_\mu$ w. $\mathcal{U}_{\varepsilon,\lambda}^{mix} M_\mu \subset M_{\lambda+\mu}$ $\forall\ \lambda,\mu\in\Lambda_0$
- morphisms are $\mathcal{U}_\varepsilon^{mix}$-linear maps $\varphi: M \to N$ w.

$$\varphi(M_\mu)\subset N_\mu \quad \forall\ \mu.$$

Definition: The quantum category $\mathcal{O}$ to be denoted by $\mathcal{O}_\varepsilon^{mix}$ is the full subcategory in the category of graded $\mathcal{U}_\varepsilon^{mix}$-modules consisting of all M s.t.

(i) M is finitely generated over $\mathcal{U}_\varepsilon^{mix}$,

(ii) $K_\nu|_{M_\mu} = \varepsilon^{(\mu,\nu)} id_{M_\mu}$, where $(\cdot,\cdot)$ is a W-invariant form on $\mathfrak{h}^*$ normalized so that $(\alpha,\alpha)=2$ $\forall$ root α - recall that we assume that $\mathfrak{g}$ is simply laced for simplicity.

(iii) $\{\mu \mid M_\mu \neq 0\}$ is bounded from above: $\exists\, \mu_M \in \Lambda_0$ s.t. $M_\mu \neq 0 \Rightarrow \mu_M - \mu \in \mathrm{Span}_{\mathbb{Z}_{\geq 0}}(\alpha_i \mid i=1,\dots,r)$.

Example (of object - Verma module)

$\lambda \in \Lambda_0 \rightsquigarrow \Delta_\varepsilon^{mix}(\lambda) = \mathcal{U}_\varepsilon^{mix} / \mathcal{U}_\varepsilon^{mix}(K_\nu - \varepsilon^{(\nu,\lambda)}, E_i^{(\ell)} \mid \forall\, i, \nu, \ell)$ w. class of 1 in $\deg = \lambda$

2.2) Remarks

1) Note that the definition makes perfect sense when ε is not a root of 1. In this case (ii) can be made into the definition of M_μ: $\varepsilon^{(\nu,\mu_1)} = \varepsilon^{(\nu,\mu_2)}\ \forall\, \nu \Rightarrow \mu_1 = \mu_2$. So the category $\mathcal{O}_\varepsilon$ (we omit the superscript "mix" b/c there's no difference between the three specializations) is the complete analog of the classical BGG category $\mathcal{O}$ of representations of simple Lie algebras.

2) A counterpart of $\mathcal{O}_\varepsilon^{mix}$ for modular representations is

as follows. Let $\mathbb{F} = \overline{\mathbb{F}}_p$, $B_{\mathbb{F}} \subset G_{\mathbb{F}}$ be a Borel subgroup. We can talk about $(\mathfrak{g}_{\mathbb{F}}, B_{\mathbb{F}})$-modules: $\mathcal{U}(\mathfrak{g}_{\mathbb{F}})$-modules M equipped in addition w. a rational representation of $B_{\mathbb{F}}$ s.t. the action map $\mathcal{U}(\mathfrak{g}_{\mathbb{F}}) \otimes M \to M$ is $B_{\mathbb{F}}$-linear & the differential of the $B_{\mathbb{F}}$-action coincides w. the action of $\mathfrak{b}_{\mathbb{F}} \subset \mathfrak{g}_{\mathbb{F}}$. The modular analog of $\mathcal{O}_\varepsilon^{mix}$ consists of all $(\mathfrak{g}_{\mathbb{F}}, B_{\mathbb{F}})$-modules finitely generated over $\mathcal{U}(\mathfrak{g}_{\mathbb{F}})$.

3) As usual, each Verma $\Delta_\varepsilon^{mix}(\lambda)$ has a unique irreducible quotient, to be denoted $L_\varepsilon^{mix}(\lambda)$ & $\lambda \mapsto L_\varepsilon^{mix}(\lambda): \Lambda_0 \xrightarrow{\sim} Irr(\mathcal{O}_\varepsilon^{mix})$ One can show that $\dim L_\varepsilon^{mix}(\lambda) < \infty$. On the other hand, $\Delta_\varepsilon^{mix}(\lambda)$ is infinite dimensional (in fact it's formal character is the same as of the usual Verma). It follows that objects in $\mathcal{O}_\varepsilon^{mix}$ may have infinite length. But they are Noetherian - in fact, each object in $\mathcal{O}_\varepsilon^{mix}$ admits a finite filtration by quotients of Verma modules.

2.3) Block decomposition of $\mathcal{O}_\varepsilon^{mix}$

It turns out that the category $\mathcal{O}_\varepsilon^{mix}$ introduced last time splits into the direct sum of subcategories called blocks (a more precise name would be "infinitesimal blocks"). To define the blocks we'll need an affine Weyl group W^a and its suitable action on Λ_0, the root lattice of $\mathfrak{g}$.

We set $W^a = W \ltimes \Lambda_0$. We write t_μ for $\mu \in \Lambda_0$ viewed as an element of W^a. Define an action of W^a on Λ_0 by

$$w \cdot \lambda := w(\lambda+\rho)-\rho \quad (\text{as usual } \rho = \frac{1}{2}\sum_{\alpha>0}\alpha)$$

$$t_\mu \cdot \lambda := \lambda + d\mu \quad (\text{where } d \text{ is the order of } \varepsilon)$$

Definition: Let Θ be a W^a-orbit in Λ_0. Define the full subcategory $\mathcal{O}_\varepsilon^{\Theta} \subset \mathcal{O}_\varepsilon^{mix}$ as consisting of all objects M that admit a (finite) filtration by quotients of $\Delta_\varepsilon(\lambda)$, $\lambda \in \Theta$.

Proposition: $\mathcal{O}_\varepsilon^{mix} = \bigoplus_{\Theta} \mathcal{O}_\varepsilon^{\Theta}$.

Sketch of proof:

Recall the form $U_v^{DK} \subset U_v^{mix} \subset U_v(\mathfrak{g})$.

Exercise: The center Z_v^{DK} of U_v^{DK} is central in U_v^{mix} & consists of deg 0 (w.r.t. the Λ_0-grading on U_v^{mix}) elements.

So Z_v^{DK} acts on any object in $\mathcal{O}_\varepsilon^{mix}$ by endomorphisms & functorially, leading to the decomposition of this category into the direct sum indexed by homomorphisms $Z_v^{DK} \to \mathbb{C}$. The following was essentially established by De Concini & Kac.

Fact: $Z_v^{DK} \xrightarrow{\sim} \mathrm{Span}_{\mathbb{Z}[v^{\pm 1}]}(K_{2\lambda} \mid \lambda \in \Lambda)^{(W,\cdot)}$ & under this identification, $\sum_{w\in W} K_{2w\cdot\lambda}$ acts on $\Delta(\lambda)$ by $\sum_{w \in W} \varepsilon^{2(w\cdot\lambda, \lambda)}$

Exercise: Deduce Proposition from Fact.