Nick Kuhn - Some Topological Non-Realizability Results

Note Title

Given X, $H^*(X)$ is a graded commutative ring. ie

$H^*(-)$ is a functor

$$\text{HoTop} \longrightarrow \text{Gr Com Ring}$$

Question: What graded rings can be topologically realized?

Ie given R^*, is there X s.t. $H^*(X) = R^*$?

Most famous version:

Steenrod Problem [1961] If $H^*(X) = \mathbb{Z}[y_{2d_1}, \dots, y_{2d_n}]$,

what can $d_1, \dots, d_n$ be?

Ex: $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x_2]$

$H^*(BSU(n)) = \mathbb{Z}[x_4, x_6, \dots, x_{2n}]$ $\qquad x_{2i} = c_i$

$H^*(BSp(n)) = \mathbb{Z}[x_4, x_8, \dots, x_{4n}]$ $\qquad x_{4i} = p_i$

Thm (Anderson - Grodal 2007)

If $H^*(X)$ is polynomial, it is iso to a product of these

Closely related to theory of finite loop-spaces.

Can pose the questions with coefs in $\mathbb{F}$. If $\mathbb{F}$ has char 0, then

$H^*(\Omega S^{2n+1}; \mathbb{Q}) = \mathbb{Q}[x_{2n}]$, so any polynomial algebra over $\mathbb{Q}$ works.

Char $\mathbb{F} = p > 0$ is much more exciting: mod p cohomology has extra structure: module over $\mathcal{A}$, the Steenrod algebra

$p=2$: $\exists$ $Sq^k : H^*(\ ) \to H^{*+k}(\ )$ natural group hom

$\mathcal{A}$ is the algebra generated by Sq^k, $0 \le k < \infty$

subject to the Adem Relations. Same size as a poly alg on gen in degree $2^k - 1$, $k \ge 1$

Then $H^*(X; \mathbb{F}_2)$ is an A-module, satisfying the unstable condition:

$$x \in H^n(X;\mathbb{F}_2), \quad \text{then} \quad Sq^i x = 0 \quad \text{for } i > n.$$

Ie $\quad H^*(X) \in \mathcal{U} =$ category of unstable $\mathcal{A}$-modules

So $\quad H^*(\;): \mathrm{HoTop} \longrightarrow \mathcal{U}$ $\quad \leftarrow$ These interact:

$\qquad\qquad\qquad\qquad \searrow$ Gr Com Alg

Q2: What unstable $\mathcal{A}$-modules can be realized?

① Cartan formula: $Sq^k(x \smile y) = \sum_{i=0}^{k} Sq^i(x) \smile Sq^{k-i}(x)$

② Restriction Axiom: If $x \in H^k(X)$, then $Sq^k(x) = x \smile x$ $\leftarrow$ like restricted Lie alg

(Steenrod called these the "reduced squares")

Ex: $H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[t] \qquad t \in H^1$

$$Sq^k(t^n) = \binom{n}{k} t^{n+k}$$

"Unstable algebras" = Algebra that is an unstable $\mathcal{A}$-mod satisfying ① & ②

$\mathcal{K}$ = cat of unstable $\mathcal{A}$-alg $\quad$ so

$$H^*(\;;\mathbb{F}_2): \mathrm{HoTop} \longrightarrow \mathcal{K}$$

The $\mathcal{A}$-mod structure restricts the alg structure

Ex There does not exist X s.t. $H^*(X) = \mathbb{F}_2[x_3]$.

$$\underset{\shortparallel}{Sq^3}\, x_3 = x_3^2 \neq 0$$

$$Sq^1 Sq^2 x_3 = Sq^1(0) \qquad \text{for degree reasons.}$$

The algebra restricts the module structure

Ex $\quad \underset{\shortparallel}{\mathcal{A}} \cdot t \subseteq \mathbb{F}_2[t]$

$$\phi = \langle t \underbrace{,}_{Sq^1} t^2 \underbrace{,}_{Sq^2} t^4 \underbrace{,}_{Sq^4} t^8, \dots \rangle$$

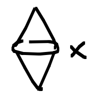
Is ϕ the cohomology of a space? No.

$H^*(X) = x_1, x_2, x_4, \dots$

$$\left.\begin{aligned} Sq^1 x_1 &= x_2 \\ Sq^2 x_2 &= x_4 \end{aligned}\right\} \Rightarrow \begin{aligned} x_2 &= x_1^2 \\ x_4 &= x_2^2 = x_1^4 \end{aligned} \qquad \text{but have no } x_1^3!$$

<u>Conjecture</u>: If M is a f.g. unstable $\mathcal{A}$-module, but ∞ dim over $\mathbb{F}_2$, then $\nexists$ X s.t. $\tilde{H}^*(X) \cong M$

Thm (Kuhn/Schwartz) This is true

The suspension of a space: $\Sigma X =$ 

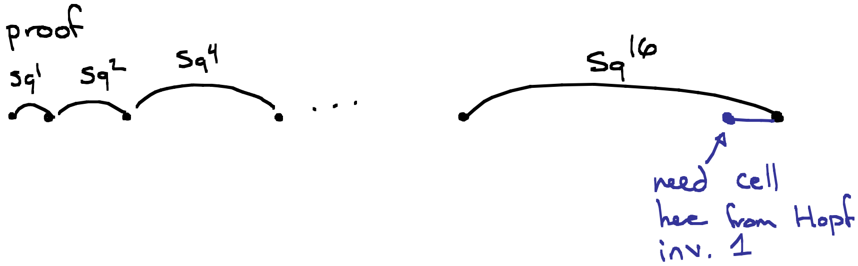
there is a suspension endofunctor of $\mathcal{U}$

$$(\Sigma M)^n = M^{n-1}$$

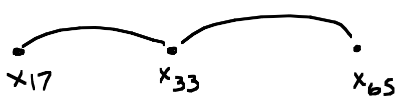
$$\begin{cases} H^*(\Sigma X) = \Sigma H^*(X) \\ \text{All cup products vanish} \end{cases}$$

Does there exist a space w/ $H^*(X) = \Sigma \phi$? NO!

① Hard proof



② Lionell shows that a 3 stage part can't exist

Z 

Try to compute $H^*(\Omega Z)$. Can get enough from this to get a contradiction.

Then you replace $\mathbb{Z}/2$ w/ M

You need lots of loops, depending on dim M, etc, and recent paper allows easier argument with $Z \to \Sigma^\infty \Omega^n Z$.