Delooping the Connecting Maps in the Taylor Tower

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I. Given X, get a tower of fibrations. Gives a way to pass from stable to unstable information

$$\begin{tikzcd} & \vdots \arrow[d] \\ X \arrow[r] \arrow[dr] \arrow[ddr] & P_3(X) \arrow[d] \\ & P_2(X) \arrow[d] \\ & QX = \Omega^\infty \Sigma^\infty X \end{tikzcd}$$

Taylor Tower of Identity

$\varprojlim P_n(X) = X$ (good spaces)

Know a lot about this: $D_n(X) = \text{Fiber}(P_n(X) \to P_{n-1}(X))$

$\simeq \Omega^\infty((W_n \wedge X^n)_{h\Sigma_n})$, so just like

derivative $\quad X^n \quad \frac{1}{n!}$

the Taylor series. Since these are ∞ loop spaces, have maps

$$\begin{tikzcd} & P_n(X) \arrow[d] & D_n(X) \arrow[l] \\ BD_n(X) & P_{n-1}(X) \arrow[l] & \end{tikzcd}$$

$\Rightarrow$ connecting (k-invariant style) maps $\quad D_{n-1}(X) \to BD_n(X)$

This gives the d_1 differential in the SS for $\pi_*(X)$, using the tower.

Fix a prime p & let $X = S^m$, m odd, $p>2$, m anything, $p=2$.

<u>Thm</u> In this case, $D_n(X) \simeq *$ unless $n = p^k$.

We can then regrade the tower

$DI_k = D_{p^k}(X)$

$$\begin{tikzcd} \vdots \arrow[d] & \\ P_{p^2}(X) \arrow[d] & DI_2 \arrow[l] \\ P_p(X) \arrow[d] & DI_1 \arrow[l] \\ P_1(X) = QX & \end{tikzcd}$$

Want a connecting map, and since

$$\begin{tikzcd} P_{p^k-1}(X) \arrow[r, "\simeq"] \arrow[d] & P_{p^{k-1}} \\ BDI_k & DI_{k-1} \arrow[l, dashed, "\alpha_{k-1}"'] \arrow[u] \end{tikzcd}$$

this is our connecting map

Have another functor

$V \longmapsto BU(V)$ $\qquad$ Have a Weiss calculus starting with

$\mathbb{C}\text{-vect} \longrightarrow \text{Spaces}$ $\qquad$ $BU = \varinjlim BU(V)$

inner product

} after p-completing, again have faster

collapse

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
Bu(V) \arrow[r] \arrow[dr] \arrow[ddr] & T_p(V) \arrow[d] \\
 & T_{p^0}(V) \arrow[d] \\
\text{constant term} \arrow[r] & T_0(V) = BU
\end{tikzcd}
$$

w/ derivatives $DU_k(V) \simeq \Omega^\infty \left(\underline{W}_{p^k} \wedge S^{p^k V} \right)_{hU(p^k)}$

$(\mathbb{C}^{p^k} \otimes V)^+$

$U(p^k)$-spectrum

Again get connecting maps $DU_k \xrightarrow{\beta_k} BDU_{k+1}$

These are not infinite loop maps but

<u>Thm</u> α_k & β_k are k-fold loop maps.

i.e. $\exists\, \tilde{\alpha}_k : B^k DI_k \longrightarrow B^{k+1} DI_{k+1}$ w/ $\alpha_k \simeq \Omega^k \tilde{\alpha}_k$

<u>Example</u> S^1 (even for this, α_k not ∞-loop map)

$$
\begin{tikzcd}
S^1 \arrow[dr] \arrow[ddr] & & \\
 & P_p(S^1) \arrow[d] & DI_1 \arrow[l] \\
BDI_1 & QS^1 \arrow[l] &
\end{tikzcd}
$$

Have a sequence of spaces using the deloopings:

$$
\begin{tikzcd}
S^1 \arrow[r] & QS^1 \arrow[r] \arrow[l, dashed, bend left] & BDI_1 \arrow[r] \arrow[l, dashed, bend left] & B^2 DI_2 \arrow[r] \arrow[l, dashed, bend left] & \cdots
\end{tikzcd}
$$

$\xrightarrow{\alpha_k}$ = not ∞ loop maps

f_k: $\dashleftarrow$ = ∞ loop map coming from $Sp^\infty(X) \simeq K(H_* X) \leadsto$

$S^0 \hookrightarrow Sp^2(S^0) \hookrightarrow \cdots \hookrightarrow Sp^\infty(S^0) = H\mathbb{Z}$

gives a filtration of $H\mathbb{Z}$. Also have (p-locally) $Sp^n(S^0)/Sp^{n-1}(S^0) \simeq *$ unless $n = p^k$. Also get $H\mathbb{Z} \leftarrow S^0 \leftarrow \Sigma^{-1} Sp^p(S^0)/Sp^1(S^0) \leftarrow \cdots$

Looping other sequence gives

$$
\mathbb{Z} \leftarrow QS^0 \leftarrow \underset{\underset{\displaystyle QB\mathbb{Z}_p}{\|}}{\Omega BDI_1} \leftarrow \cdots
$$

$= \Omega^\infty(\text{above tower})!$

Have an analogous filtration in the orthogonal case.

$bu \to A_1 \to \cdots \to H\mathbb{Z}$

II.

The Weiss calculus gives

$$\begin{array}{ccccccccc} BU(1) & \to & BU & \to & BDU_0 & \to & B^2DU_1 & \to & \cdots \\ \| & & & & \| & & & & \\ \mathbb{C}P^\infty & & & & \Omega^2 Q\mathbb{C}P^\infty & & & & \end{array}$$

taking Ω^2, get

$$\begin{tikzcd} \mathbb{Z} \arrow[r, shift left] & \mathbb{Z}\times BU \arrow[l, dashed, shift left] \arrow[r, shift left] & Q\mathbb{C}P^\infty \arrow[l, dashed, shift left] \arrow[r, shift left] & \Omega^2 B^2 DU_1 \arrow[l, dashed, shift left] \arrow[r] & \cdots \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r, shift left, "f_k"] & {} \arrow[l, dashed, shift left, "g_k"] \end{tikzcd}$$

same as the filtration from above

<u>Conjecture</u>: There exist deloopings α_k, β_k that act as a contracting homotopy

ie $f_{k_0} \circ \alpha_k + \alpha_{k+1} \circ f_{k+1} \simeq Id$

$\Rightarrow$ The "chain complexes" are "exact"

$\Rightarrow$ homotopy SS collapses at E_2.

Idea of proof: How can we show a map is a k-fold loop map?

$\Omega^k X \to \Omega^k Y$! find the image of $\Omega^k: Map(X,Y) \to Map(\Omega^k X, \Omega^k Y)$

These are too big. Instead look at $Nat(B^k(-), B^{k+1}(-)) \xrightarrow{\Omega^k} Nat(DI_k(\ ), BDI_{k+1}(-))$

(really consider $V \mapsto S^1 \wedge S^V$ in Weiss story!)

Care about $\Omega^d \Omega^\infty \Sigma^\infty X^{\wedge 2^k}_{h\Sigma^k \Sigma_2} \to \Omega^d Q X^{\wedge 2^{k+1}}_{h\Sigma^{k+1}\Sigma_2}$.

III.