Quiver Varieties and Symmetric Pairs

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REPRESENTATION THEORY
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Overview:

- Motivations: Schur dualities
  —based on joint works with H. Bao, J. Kujawa and W. Wang.

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  —based on joint works with H. Bao, J. Kujawa and W. Wang.
- Results on nilpotent Slodowy slices
- The construction of iQuiver Varieties (iQV)
- Connection with real classical groups

Background: Schur duality and beyond

Schur duality and its generalizations.

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- iSchur duality: $U_q^\sigma(\mathfrak{gl}_n) \curvearrowright T_{n,d} \curvearrowleft \mathbf{H}_d^{B/C/D}$. (Green 1997, Bao-Wang 2013)

Compatibility and quantum symmetric pairs

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U_q^\sigma(\mathfrak{gl}_n) \arrow[r, bend left] \arrow[d, hook] & T_{n,d} \arrow[d, equal] & \mathbf{H}^{B/C/D} \arrow[l, bend right] & \text{iSchur} \\
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These algebras were studied previously by Noumi, Letzter, Kolb, etc.

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- A symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ is a complex semisimple Lie algebra $\mathfrak{g}$ together with its fixed-point subalgebra $\mathfrak{g}^\theta$ under an involution θ.

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- When specialized to $q = 1$, the pair $(U_q(\mathfrak{gl}_n), U_q^\sigma(\mathfrak{gl}_n))$ is of type $Aiii/Aiv$, with Satake diagram (all vertices are white):

$$\theta: \quad \circ \text{———} \circ \text{———} \cdots \text{———} \circ \text{———} \circ$$

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- Caution: $U_q^\sigma(\mathfrak{gl}_n)$ NOT a fixed-point subalgebra of $U_q(\mathfrak{gl}_n)$.

Applications/Why do we care?

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- Bao-Wang's work contains an icanonical basis (iCB for short) for representations of coideal subalgebras of quantum $\mathfrak{sl}_n$.
- Bao-Wang's work on iCB has been extended by themselves to any coideal subalgebras of quantum groups of finite type.

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- BKLW obtained an iCB for quantum symmetric pairs of type Aiii/iv. A slight modification leads to remarkable positivity (L.-Wang, Fan-L.). (weak categorification)
- Genuine categorification of (part of) BKLW's work has been done by H. Bao, P. Shan, W. Wang and B. Webster.

There is a ‘type A’ line of research:

$$\begin{array}{lll} \mathcal{F}_{n,d} \rightsquigarrow & T^*\mathcal{F}_{n,d} \rightsquigarrow & \text{Nakajima varieties} \\ \mathfrak{sl}_n & \text{dual} & \mathfrak{g}_{ADE} \end{array}$$

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In light of the previous work, We should have a line of research, based on "classical type" geometry:

$$\begin{array}{ccc} \mathcal{F}^{\sigma}_{n,d} \rightsquigarrow & T^*\mathcal{F}^{\sigma}_{n,d} \rightsquigarrow & \text{iQV????} \\ \mathfrak{sl}^{\sigma}_n & \text{dual} & \mathfrak{g}^{\theta}_{ADE} \end{array}$$

The existence of iQV is also conjectured by Weiqiang Wang.

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Analogue

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iQV?? $\quad \rightsquigarrow \quad \mathfrak{g}^\theta$, for $\theta \in \mathrm{Aut}(\mathfrak{g})$

Answer: $iQV = \mathfrak{M}_\zeta(\mathbf{w})^\sigma$: the fixed point locus of $\mathfrak{M}_\zeta(\mathbf{w})$ under an σ.

Comparison, I

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Nakajima varieties=QV	iQV
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Unitary instantons	Sp/SO instantons (Nakajima)

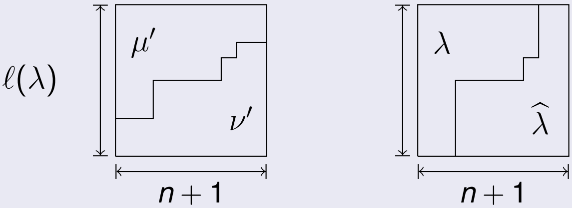
Caution: the automorphism σ is not always an involution. For the Weyl group action of type G_2, σ is of order 6.

Application I: Rectangular symmetry

One of the rectangular symmetries reads as follows

$$\begin{tikzcd}
\widetilde{S}^{\mathfrak{sp}}_{\mu',\lambda} \arrow[r, "\cong"] \arrow[d, "\pi^{\sigma}"'] & \widetilde{S}^{\mathfrak{o}}_{\nu',\widehat{\lambda}} \arrow[d, "\pi^{\widetilde{\sigma}}"] \\
S^{\mathfrak{sp}}_{\mu',\lambda} \arrow[r, "\cong"] & S^{\mathfrak{o}}_{\nu',\widehat{\lambda}},
\end{tikzcd}$$

where each pair $(\mu', \tilde{\mu}')$ and $(\lambda, \tilde{\lambda})$ can be fit into a rectangle:



Special case: two-row Slodowy slices

As a special case of the rectangular symmetry, we recover

Henderson-Licata, 2013

$\mathcal{S}^{\mathfrak{sp}_n}_{n^1, k^1(n-k)^1} \cong \mathcal{S}^{\mathfrak{o}_{n+2}}_{1^1(n+1)^1, (k+1)^1(n+1-k)^1}$.

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Wilbert, 2015; Ehrig-Stroppel, 2013

$\mathcal{B}^{\mathfrak{sp}_n}_{e_0} \cong \mathcal{B}^{\mathfrak{so}_{n+2}}_{\widehat{e}_o}$ — Springer fibers of the associated Slodowy slices.

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And we solve a conjecture of Henderson-Licata for free:

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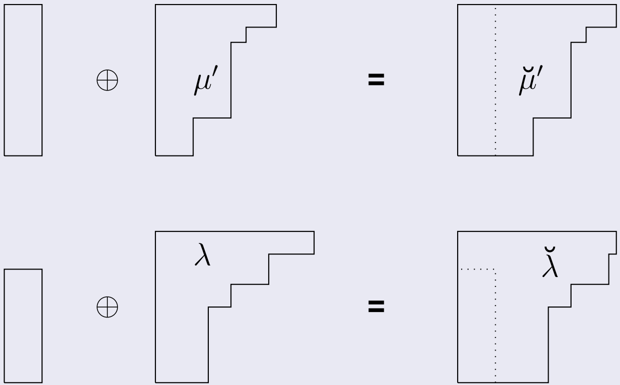
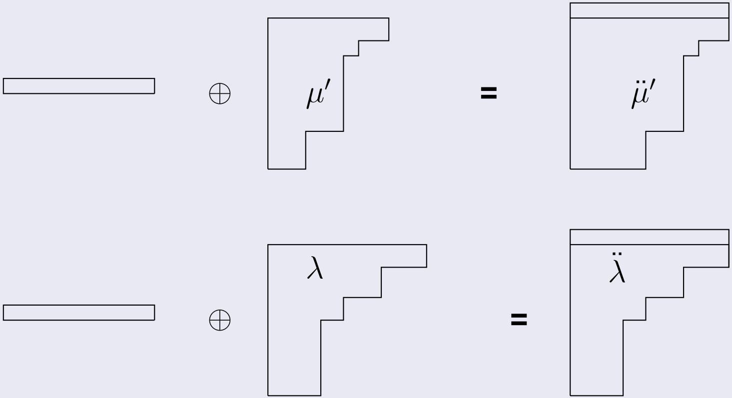
Enhancement of Kraft-Procesi's column removal reduction

We have $\mathcal{S}^{\mathfrak{sp}_{|\breve{\lambda}|}}_{\breve{\mu}',\breve{\lambda}} \cong \mathcal{S}^{\mathfrak{o}_{|\lambda|}}_{\mu',\lambda}$, if the partitions are related as follows.



Application II: Row removal reduction

We have $\mathcal{S}^{\mathfrak{sp}|\ddot{\lambda}|}_{\ddot{\mu}',\ddot{\lambda}} \cong \mathcal{S}^{\mathfrak{sp}|\lambda|}_{\mu',\lambda}$, if



One starts with a quiver Q with underlying graph Γ, and its framed version Q^{f} by adding an extra copy of vertex set and arrow connecting to the original vertices. For example

$$Q = \bullet \to \bullet, \quad Q^{\mathrm{f}} = \begin{array}{ccc} \spadesuit & & \spadesuit \\ \Big\downarrow & & \Big\downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

Consider the geometries:

$$
\begin{tikzcd}
 & T^*\mathrm{rep}Q \arrow[dr] & & \\
\mathrm{rep}Q \arrow[ur] \arrow[dr] & & T^*\mathrm{rep}Q^{\mathrm{f}} \arrow[r] & \text{Hamilt. reduction } T^*\mathrm{rep}Q^{\mathrm{f}}/G. \\
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Each step yields rich geometries and contains much representation theoretical information for the Lie algebra $\mathfrak{g}_\Gamma$ associated to Γ.

Hamiltonian reduction

Specifically, consider the cotangent space $T^*\mathrm{rep}Q^{\mathrm{f}}_{\mathbf{v},\mathbf{w}}$ of representations of $Q^{\mathrm{f}}_{\mathbf{v},\mathbf{w}}$ of fixed dimension vectors $\mathbf{v}, \mathbf{w}$. There is a (reductive/gauge) group $G_{\mathbf{v}}$ acts nicely on $T^*\mathrm{rep}Q^{\mathrm{f}}_{\mathbf{v},\mathbf{w}}$. General machinery in symplectic geometry says that there is a moment map

$$\mu : T^*\mathrm{rep}Q^{\mathrm{f}}_{\mathbf{v},\mathbf{w}} \to (\mathrm{Lie}G_{\mathbf{v}})^*.$$

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Nakajima (quiver) variety is defined to be the Hamiltonian reduction

$$\mathfrak{M}_{\zeta}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(\zeta_{\mathbb{C}})///_{\zeta_{\mathbb{R}}} G_{\mathbf{v}}, \quad \zeta = (\zeta_{\mathbb{C}}, \zeta_{\mathbb{R}}).$$

Rank one: $\zeta = (0, 1)$ or $(0, 0)$

In rank one case,

$$T^* \mathrm{rep} Q^{\mathrm{f}}_{\mathbf{v},\mathbf{w}} = \mathrm{Hom}(\mathbb{C}^{\mathbf{w}}, \mathbb{C}^{\mathbf{v}}) \oplus \mathrm{Hom}(\mathbb{C}^{\mathbf{v}}, \mathbb{C}^{\mathbf{w}})$$

and the fiber at 0 of the moment map is given by

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Rank one Nakajima variety

Nakajima varieties are

$$\mathfrak{M}_{(0,1)}(\mathbf{v}, \mathbf{w}) = \{(p, q) \in \mu^{-1}(0) | q \text{ injective}\}/GL(\mathbb{C}^{\mathbf{v}}), \text{(GIT quotient)}$$
$$\mathfrak{M}_{(0,0)}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)//GL(\mathbb{C}^{\mathbf{v}}) \quad \text{(categorical quotient)}$$

Rank one: Cotangent bundle of Grassmannian (Nakajima)

Nakajima varieties and cotangent bundle of Grassmannian

The assignment $(p, q) \mapsto (qp, \mathrm{im}(q))$ identifies Nakajima varieties with the cotangent bundle of Grassmannian and its affinization.

$$\begin{tikzcd}
\mathfrak{M}_{(0,1)}(\mathbf{v}, \mathbf{w}) \arrow[r, "\cong"] \arrow[d] & T^*\mathrm{Gr}(\mathbf{v}, \mathbf{w}) \arrow[d, "\text{Springer resolution}"] \\
\mathfrak{M}_{(0,0)}(\mathbf{v}, \mathbf{w}) \arrow[r] & \{x \in \mathrm{End}(\mathbb{C}^{\mathbf{w}}) | x^2 = 0, \ldots\}
\end{tikzcd}$$

Ginzburg's setting

In general, the cotangent bundle $T^*\mathcal{F}_{n,d}$ used in Ginzburg's construction is a very special case of Nakajima varieties of type A.

Isomorphisms on quiver varieties

Let $a : \Gamma \to \Gamma$ be a diagram automorphism.

Naive diagram isoomorphism

$a : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{a(\zeta)}(a(\mathbf{v}), a(\mathbf{w}))$.

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$\tau_\zeta : (p, q) \mapsto (-q^*, p^*)$ (modulo $G_\mathbf{v}$).

Isomorphisms cont'd

Recall $\mathcal{W}_\Gamma$ be the Weyl group of Γ.

Reflection functors S_ω of Nakajima, Lusztig and Maffei

$S_\omega : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{\omega(\zeta)}(\omega *_{\mathbf{w}} \mathbf{v}, \mathbf{w}), \forall \omega \in \mathcal{W}_\Gamma$.

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Rank one

$S_i : (p, q) \mapsto (p', q')$, $\mathbb{C}^{\mathbf{v}} \overset{q}{\hookrightarrow} \mathbb{C}^{\mathbf{w}} \overset{p'}{\twoheadrightarrow} \mathbb{C}^{\mathbf{w}-\mathbf{v}}$ is exact and $qp = q'p'$.

iQV

Taking the composition of the above three isomorphisms yields:

Isomorphism σ

$\sigma \equiv \sigma_{a,\zeta,\omega} := aS_\omega\tau_\zeta : \mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w}) \to \mathfrak{M}_{-a\omega(\zeta)}(a(\omega *_\mathbf{w} \mathbf{v}), a\mathbf{w})$.

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Definition of iQV

$\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})^\sigma$, if $\zeta = -a\omega(\zeta)$, $\mathbf{v} = a(\omega *_{\mathbf{w}} \mathbf{v})$ and $\mathbf{w} = a\mathbf{w}$.

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Quiver variety is an iQV

$\mathfrak{M}_\zeta^{\Gamma\times\Gamma}(\mathbf{w})^\sigma \cong \mathfrak{M}_\zeta(\mathbf{w})$ for $\sigma = a\tau_\zeta$.

Rank one

Rank one iQV

The assignment $(p, q) \mapsto (qp, \mathrm{im}(q))$ identifies iQV with the cotangent bundles of maximal isotropic Grassmannians.

$$\begin{tikzcd}
\mathfrak{M}_{(0,1)}(\mathbf{v}, \mathbf{w})^{\sigma} \arrow[r, "\cong"] \arrow[d] & T^*\mathrm{Gr}(\mathbf{v}, \mathbf{w})^{\sigma'} \arrow[d] \\
\mathfrak{M}_{(0,0)}(\mathbf{v}, \mathbf{w})^{\sigma} \arrow[r] & \{x \in \mathrm{End}(\mathbb{C}^{\mathbf{w}}) | x^2 = 0, \ldots\}^{\sigma'}
\end{tikzcd}$$

here σ' depends on the form on $\mathbb{C}^{\mathbf{w}}$.

Proof

The action of σ on (p, q) is $(p, q) \overset{\tau_\zeta}{\mapsto} (-q^*, p^*) \overset{S_i}{\mapsto} (-(q^*)', (p^*)')$.

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$$\mathrm{im}q \mapsto \mathrm{im}(p^*)' = \mathrm{ker}q^* \longleftrightarrow F \mapsto F^\perp.$$

$\perp$ is taken with respect to the form on $\mathbb{C}^{\mathbf{w}}$.

Fixed points vs Kraft-Procesi

Kraft-Procesi considered an A_n quiver with alternating forms, say:

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\begin{tikzcd}
\boxed{Sp_{\mathbf{w}_1}} \arrow[d, dash] & & & \\
O_{\mathbf{v}_1} \arrow[r, dash] & Sp_{\mathbf{v}_2} \arrow[r, dash] & O_{\mathbf{v}_3} \arrow[r, dash] & \cdots .
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Consider the fixed-points $\mu^{-1}(0)^\sigma$, $G_{\mathbf{v}}^\sigma = O_{\mathbf{v}_1} \times Sp_{\mathbf{v}_2} \times O_{\mathbf{v}_3} \times \cdots$.

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Theorem (Kraft-Procesi, 1982): classical nilpotent orbits

$\mu^{-1}(0)^\sigma // G_{\mathbf{v}}^\sigma \cong \overline{\mathcal{O}_{\mu'}}^{\mathfrak{sp}_{\mathbf{w}_1}}$, for certain $\mathbf{v}$, $\mathbf{w}$ and the associated μ'.

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Theorem: iQV and Kraft-Procesi

In Kraft-Procesi's setting, $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})^\sigma \cong \mu^{-1}(0)^\sigma // G_{\mathbf{v}}^\sigma$, for $a = 1$

GIT in Kraft-Procesi's approach (one of my mental blocks)

Geometric invariant theory does not seem to apply to Kraft-Procesi's approach: essentially no non-trivial character of $G_{\mathbf{v}}^{\sigma}$.

Cotangent bundles of flag varieties of classical groups

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Theorem: iAnalogue of Ginzburg: Cotangent bundles of isotropic flag varieties

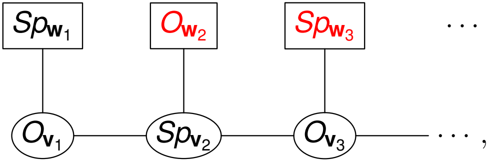
In Kraft-Procesi's setting, we get

$$\mathfrak{M}_{\zeta}(\mathbf{v},\mathbf{w})^{\sigma} \cong T^{*}\mathcal{F}_{\mathbf{v},\mathbf{w}}^{\mathfrak{sp}_{\mathbf{w}_1}}, \quad \text{if } \zeta=(1,0), \boldsymbol{\omega}=\omega_0.$$

where ω_0 is the longest element in $\mathcal{W}_{\Gamma}$ and $a=1$.

Nakajima's generalization: nilpotent Slodowy slices

In more general type-A setting of Kraft-Procesi, say:



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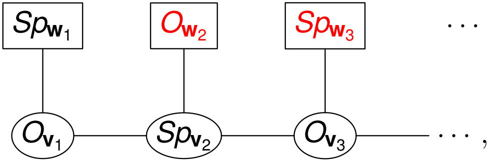

Nakajima asserted at several places that

$$\mu^{-1}(0)^{\sigma}//G_{\mathbf{v}}^{\sigma} \rightsquigarrow S^{\mathfrak{sp}}_{\mu',\lambda} \text{ or } S^{\mathfrak{o}}_{\mu',\lambda},$$

where $S^{\mathfrak{sp}}_{\mu',\lambda} = \overline{\mathcal{O}_{\mu'}} \cap S_\lambda \cap \mathfrak{sp}_{\widetilde{\mathbf{w}}_1}$ is a nilpotent Slodowy slice in $\mathfrak{sp}_{\widetilde{\mathbf{w}}_1}$.

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Proposition

In the above setting, there is a closed immersion (isomorphism expected):

$$\mu^{-1}(0)^{\sigma}//G_{\mathbf{v}}^{\sigma} \hookrightarrow S^{\mathfrak{sp}}_{\mu',\lambda},$$

which relies on a result of quiver-analogue of classical invariants.

Partial Resolutions of nilpotent Slodowy slices

Theorem: Partial Resolutions of nilpotent Slodowy slices

Moreover, in the above type A Dynkin diagram setting, we have

$$\begin{tikzcd}
\mathfrak{M}_{\zeta}(\mathbf{v}, \mathbf{w})^{\sigma} \arrow[r, "\cong"] \arrow[d, "\pi^{\sigma}"] & \widetilde{S}^{\mathfrak{sp}}_{\mu', \lambda} \arrow[r, hook] \arrow[d, "\Pi"] & T^{*}\mathcal{F}^{\mathfrak{sp}}_{\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}} \arrow[d, "\Pi"] \\
\mathrm{im}(\pi^{\sigma}) \arrow[r, "\cong"] & S^{\mathfrak{sp}}_{\mu', \lambda} \arrow[r, hook] & \mathfrak{sp}_{\widetilde{\mathbf{w}}}
\end{tikzcd}$$

(where $\widetilde{\mathbf{w}} = \sum_{i \in I} i\mathbf{w}_i$, $\widetilde{\mathbf{v}} = \mathbf{v}_i + \sum_{j \geq i}(j - i)\mathbf{w}_j$.)

This result leads to previous applications in classical geometries.

Instantons on ALE space and Nakajima varieties of type A, due to Nakajima

Unitary

Regular part of Nakajima varieties = unitary intantons on ALE spaces.

It is known to Nakajima that

Classical type

Regular part of iQV (of some σ) = SP/SO instantons on ALE spaces. (arxiv: 1801.06286.)

General properties of iQV

Now we return to the general setting:

Proposition: Independence of forms on V

The $\mathfrak{M}_\zeta(\mathbf{v}, \mathbf{w})^\sigma$ is independent of choices of forms on V.

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Proposition: Weyl group action

Let $\mathcal{W}_\Gamma^{a\omega} = \{x \in \mathcal{W}_\Gamma | x\omega = \omega x, a(x) = x\}$. There exists a $\mathcal{W}_\Gamma^{a\omega}$-action:

$$\mathcal{W}_\Gamma^{a\omega} \curvearrowright H^*(\mathfrak{M}_\zeta(\mathbf{v},\mathbf{w})^\sigma), \quad \mathbf{w} - \mathbf{C}_\Gamma \mathbf{v} = 0.$$

$\mathcal{W}_\Gamma^{a\omega}$ includes Weyl groups of $B_\ell/C_\ell/F_4/G_2$ types.

Conjecture

There is an action

$$\mathfrak{g}^\theta \curvearrowright H^*(\mathfrak{M}_\zeta(\mathbf{w})^\sigma), \quad (\zeta \text{ generic})$$

where $(\mathfrak{g}, \mathfrak{g}^\theta)$ for a symmetric pair of type *Ai*, *Aiii*, *Di*, *Ei*, *Eii*, *Ev*, *Eviii*, whose Satake diagram has no black dots. Note $\mathfrak{g}^\theta$ of type *Ai* is $\mathfrak{so}_n$.

It holds for *Aiii*/*Aiv*. There are several supporting evidence.

Symmetric pairs have been pervasive in the study of representations of real simple/reductive groups.

QV and real simple groups

Does QV/iQV have more direct connections with real classical groups?

Cartan decomposition for quiver varieties

To any symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$, it yields a complex Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{p},$$

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Quiver model of $\mathcal{N}(\mathfrak{p})$: Lagrangian version of iQV

The anti-symplectic version, say $\mathfrak{M}_\zeta(\mathbf{w})^{\hat{\sigma}}$, of iQV yields a quiver/linear model of $\mathcal{N}(\mathfrak{p})$ and associated Slodowy slices. Almost all results from symplectic version have an anti-symplectic counterpart, such as rectangular symmetry etc. (But not the semismallness of projection from π.)

Cartan decomposition for quiver varieties

A correspondence

$\mathfrak{g}$	$\mathfrak{g}^\theta$	$\mathfrak{p}$
$\updownarrow$	$\updownarrow$	$\updownarrow$
Nakajima varieties	Symplectic subvarieties	Lagrangian subvarieties

This correspondence works for all symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\theta)$ of classical type.

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This correspondence works for all symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\theta)$ of classical type. Note that $\mathcal{N}(\mathfrak{p}) \cong \mathcal{N}(G_{\mathbb{R}})$, the Kostant-Sekiguchi homeomorphism of Chen-Nadler. It is reasonable to expect the same holds in quiver setting.

Kostant-Sekiguchi correspondence for quiver varieties

There should be a homeomorphism $\mathfrak{M}_0(\mathbf{w})_{\mathbb{R}} \cong \mathfrak{M}_0(\mathbf{w})^{\hat{\sigma}}$.

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A shadow of the $(\mathfrak{g}, K)$-module: Harish-Chandra $(\mathfrak{g}, \mathrm{Lie} K)$-modules. $\mathrm{Lie} K = \mathfrak{g}^{\theta}$ for some θ, studied by Diximier, Lepowsky, Zuckerman, etc. On the other hand, there is a Yangian $\mathcal{Y} \equiv \mathcal{Y}(\mathfrak{g}_{\Gamma})$ action on the torus equivariant cohomology $H^*_{\mathbb{T}}(\mathfrak{M}_{\zeta}(\mathbf{w}))$ of Nakajima variety, due to Varagnolo (via correspondence), Maulik-Okounkov (via R-matrix).

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$(\mathfrak{g}, \mathrm{Lie} K)$-structure in Nakajima varieties

There is an action of an (affine) symmetric pair $(\mathcal{Y}, \mathcal{Y}_{\sigma}) \curvearrowright H^*_{\mathbb{T}}(\mathfrak{M}_{\zeta}(\mathbf{w}))$, where $\mathcal{Y}_{\sigma}$ is a twisted Yangian, i.e., an affinization of $U(\mathfrak{g}_{\Gamma}^{\theta})$.

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How to lift $(\mathfrak{g}, \mathrm{Lie} K)$-structure to a $(\mathfrak{g}, K)$-structure remains to be done.

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- Stable envelopes exist.
- The R-matrix (or rather K-matrix) satisfies the reflection equation $RKRK = KRKR$.
- The twisted Yangian $\mathcal{Y}_\sigma$ is then constructed using K-matrix via Faddeev, Reshetikhin, and Takhtajan's construction.

Comparison, II

Nakajima varieties	iQV / lgrngn version
symp. resolution	symp. partial resolution/ lgrngn
π semismall	π^{σ} semismall / proper
Slodowy slices of type GL_n	symmetric pairs of classical type
Weyl groups action of type ADE	ADE, $B_\ell/C_\ell/F_4/G_2$
Rectangular symmetry of GL_n	symmetric pairs of classical type
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$\mathfrak{g} \curvearrowright H_{top}(\mathfrak{M}_\zeta(\mathbf{w}))$	$\mathfrak{g}^\theta \curvearrowright H^*(\mathfrak{M}_\zeta(\mathbf{w})^\sigma)$ (conj.)

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Geometric $\mathrm{Rep}(\mathfrak{g})$	Geometric $\mathrm{Rep}(\mathfrak{g}^\theta)$ (conj.)

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Yang-Baxter equation	Reflection equation

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Maulik-Okounkov $\mathcal{R}$-matrix	$\mathcal{K}$-matrix
Yang-Baxter equation	Reflection equation
RTT formalism of Yangian	Twisted Yangian
$(\mathcal{Y}, \mathcal{Y}_\sigma) \curvearrowright H^*_{\mathbb{T}}(\mathfrak{M}_\zeta(\mathbf{w}))$	
Kostant-Sekiguchi homeomorphism in quiver varieties (conj.)	

Thank you very much!