Math 19 - 01
Homework #6
10/11/2019

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1 (Average Value Theorem). In HW 3 you showed that if f is a positive continuous function on $[a, b]$ then there exists some $c \in (a, b)$ such that the area A under the graph of f satisfies $A = f(c)(b - a)$. Apply the Mean Value Theorem to the function $F(x) = \int_a^x f(t)\, dt$ to show that this property holds for any continuous function f (not just positive) and that, in fact, the value $f(c)$ is given by

$$f(c) = \frac{1}{b-a} \int_a^b f(t)\, dt. \tag{1}$$

Note: The value $f(c)$ in (1) is called *the average value of f on the interval $[a, b]$*. For a discrete set of values their average is the sum of values divided by the total number of values; here we have a continuum of values $f(x)$ for $x \in [a, b]$ and their average is the continuous sum of values $f(x)$ for $x \in [a, b]$ ($\int_a^b f(t)\, dt$) divided by the total number of x's $(b - a)$.

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(a) Determine $F'\left(\sqrt{\frac{\pi}{4}}\right)$ if $F(x) = \displaystyle\int_0^{\sin x^2} \frac{1}{t}\, dt$.

(b) Determine $f(4)$ if $\displaystyle\int_0^x f(t)\, dt = x\cos(\pi x)$.

Hint for (b): The answer is not $\displaystyle\int_0^4 f(t)\, dt$.

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3 Consider $f(x) = \dfrac{\ln x}{x}$ for $x > 0$.

(a) Show f has a global maximum at e. What is the maximum value of f?

(b) Sketch a graph of the function f.

(c) Since $e < \pi$ it follows from (a) that $\frac{\ln \pi}{\pi} < \frac{\ln e}{e}$. Use this to show $\pi^e < e^\pi$.

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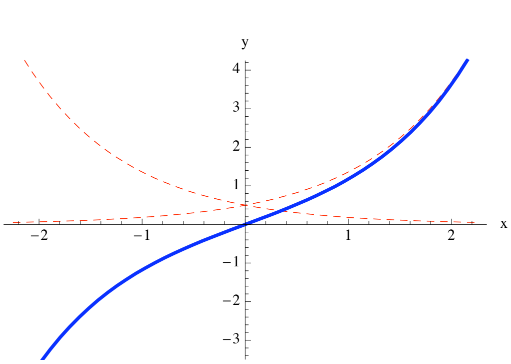
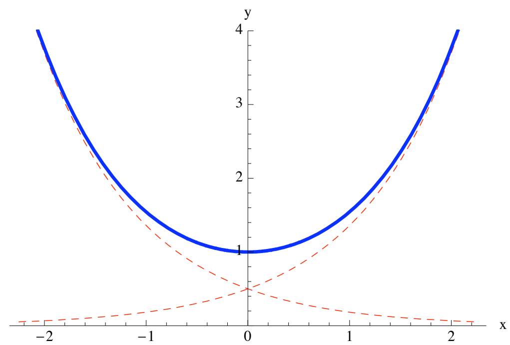
4 (**Hyperbolic Trig functions**.) The hyperbolic sine and cosine are important functions in many areas of science and engineering. They are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$



$$\sinh x = \frac{e^x}{2} - \frac{e^{-x}}{2}$$



$$\cosh x = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

(dashed graphs are of $e^x/2$ and $e^{-x}/2$, $\sinh x$ is their difference, $\cosh x$ is their sum). The hyperbolic trig functions are not periodic, but they share many similar properties to their periodic cousins $\sin x$ and $\cos x$.

(a) Show that $\sinh 0 = 0$ and $\cosh 0 = 1$.

(b) Show that $\sinh x$ is an odd function and $\cosh x$ is an even function.

(c) Show that $\cosh^2 x - \sinh^2 x = 1$.

(d) Part (c) implies the set of points $(x, y) = (\cosh t, \sinh t)$ satisfy $x^2 - y^2 = 1$, an equation whose graph is a hyperbola. Do the points $(\cosh t, \sinh t)$ cover the entire hyperbola $x^2 - y^2 = 1$? Explain.

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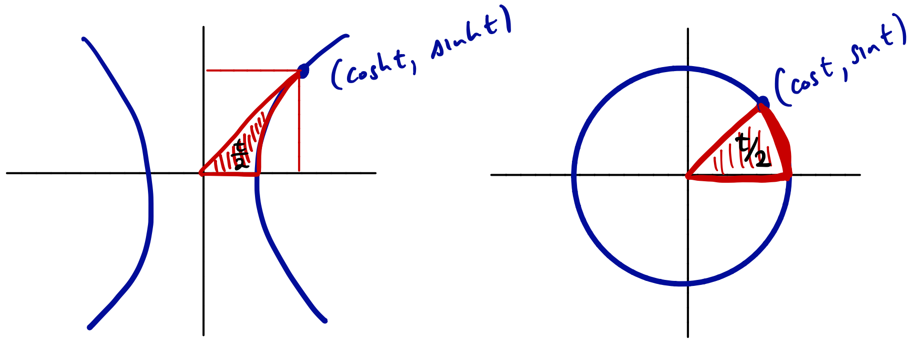
5 The following figure illustrates another interesting connection between the circular trig functions sine and cosine and the hyperbolic trig functions sinh and cosh:



In each case the area of the shaded region defined by t is $t/2$. For the circle, the entire area is π and the wedge has radian angle t, so the area is $A = \frac{t}{2\pi} \cdot \pi = \frac{t}{2}$. For the hyperbola, the shaded area can be realized as the area of the triangle defined by the points $(0,0)$, $(\cosh t, 0)$, and $(\cosh t, \sinh t)$, less the area under the hyperbola from $x = 1$ to $x = \cosh t$:

$$A(t) = \frac{1}{2}\cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

(a) Instead of computing this unfriendly integral for $A(t)$, first show that $A'(t) = \frac{1}{2}$.

(b) Explain why part (a) implies $A(t) = \frac{t}{2} + C$ for some constant C.

(c) Use the known value $A(0)$ to determine C and therefore verify $A(t) = \frac{t}{2}$, as desired.

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6 (Derivatives of Hyperbolic Trig Functions)

(a) Show that

$$\frac{d}{dx}(\,\sinh x\,) = \cosh x \qquad \frac{d}{dx}(\,\cosh x\,) = \sinh x$$

(b) Use the result of (a) to verify that $y = \sinh x$ and $y = \cosh x$ each satisfy the differential equation

$$y'' = y. \tag{2}$$

Similarly, verify that $y = \sin x$ and $y = \cos x$ each satisfy the differential equation

$$y'' = -y. \tag{3}$$

We will see in Math 82 that, in fact, *any* solution of (2) has the form $y = c_1 \sinh x + c_2 \cosh x$ for some constants c_1 and c_2, and *any* solution of (3) has the form $y = c_1 \sin x + c_2 \cos x$ for some constants c_1 and c_2. We say these functions form a "basis" for the set of all solutions.

(c) One defines the hyperbolic tangent (read "tanch") and hyperbolic secant (read "setch") by

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \text{and} \qquad \operatorname{sech} x = \frac{1}{\cosh x}.$$

Show that

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x.$$

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