Math 19 - 07
Homework #4
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# Homework #4

**1** Use the definition of derivative to determine if the following functions are differentiable at $x = 0$. If not, why not? If so, what is $f'(0)$?

(a)

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases}$$

(b)

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

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**2** Suppose $f$ is differentiable at $x_0$ and let $c \in \mathbb{R}$ be a constant. Use the definition of derivative to prove that the function $(cf)$ is differentiable at $x_0$ and

$$(cf)'(x_0) = cf'(x_0).$$

Note: The function $(cf)$ is defined by $(cf)(x) = cf(x)$.

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**3** Let $f$, $g$, and $h$ be differentiable functions. Use the product rule to show

$$(fgh)' = f'gh + fg'h + fgh'.$$

what about a product of $n$ functions $f_1 f_2 \dots f_n$? Prove your claim.

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**4** Use the rules of differentiation to calculate $f'$ for each of the following functions $f$ (don't worry about the domain of $f$ or $f'$; just obtain a formula for $f'$ that is valid when it makes sense).

(a) $f(x) = \dfrac{\sin(\cos x)}{x}$

(b) $f(x) = \left(x + \sin^5 x\right)^6$

(c) $f(x) = \sin\left(\dfrac{x}{x - \sin\left(\frac{x}{x - \sin x}\right)}\right)$

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**5** If $f$ is differentiable at a then the graph of $f$ has a well-defined tangent line at $(a, f(a))$ defined by

$$\ell(x) = f(a) + f'(a)(x - a).$$

For $x$ near $a$ we can use the tangent line to approximate the value $f(x)$:

$$\begin{aligned} f(x) &\approx \ell(x) \\ &= f(a) + f'(a)(x - a). \end{aligned}$$

The error in this approximation is the difference between $\ell(x)$ and the actual value $f(x)$:

$$\begin{aligned} e(x) &= f(x) - \ell(x) \\ &= f(x) - (f(a) + f'(a)(x - a)). \end{aligned}$$

Note that $e(a) = 0$ since the tangent line agrees with $f$ at $a$.

(a) Prove that if $f$ is differentiable at $a$ then the error $e(x)$ satisfies

$$\lim_{x \to a} \frac{e(x)}{x - a} = 0 \tag{1}$$

(b) Suppose $f$ is not necessarily differentiable at $a$, but has the property

$$f(x) = f(a) + M(x - a) + e(x) \tag{2}$$

for some constant $M$ and some function $e(x)$ that satisfies the limit (1). Prove that $f$ must be differentiable at $a$ and $f'(a) = M$. Hint: According to (2) what form must $\frac{f(x) - f(a)}{x - a}$ have? Now take limits.

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