

## Homework #5

**1** Let  $f(x) = \alpha x^2 + \beta x + \gamma$  be an arbitrary quadratic function. Show that the number  $c$  that satisfies the conclusion of the Mean Value Theorem on the interval  $[a, b]$  is given by the midpoint  $c = \frac{a+b}{2}$ .

(Curiously, the value of  $c$  is independent of the quadratic's coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$ !)

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**2** Set  $f(x) = x^{-1}$ ,  $a = -1$ ,  $b = 1$ . Verify that there is no number  $c$  for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Explain how this does not violate the Mean Value Theorem.

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**3** Find, among all right circular cylinders of fixed volume  $V$ , the one with smallest surface area (counting the areas of the faces at the top and bottom).

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4 Show that the sum of a positive number and its reciprocal is at least 2.

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5 The area between two varying concentric circles is at all times  $9\pi^2$ . The rate of change of the area of the larger circle is  $10\pi^2$  in<sup>2</sup>/sec. How fast is the circumference of the smaller circle changing when it has area  $16\pi^2$  in<sup>2</sup>?

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6 Suppose that  $f$  and  $g$  are unknown differentiable functions that satisfy the relationships

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x) \quad \text{for all } x \quad (1)$$

- (a) Show that  $f^2(x) + g^2(x) = C$  for some constant  $C$ .
- (b) Determine the value of  $C$  if we also know  $f(0) = 0$  and  $g(0) = 1$ .
- (c) Differentiate the equations for  $f'$  and  $g'$  in (1) to show that  $f$  and  $g$  must also solve the differential equations

$$f''(x) + f(x) = 0 \quad \text{and} \quad g''(x) + g(x) = 0 \quad \text{for all } x.$$

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