Math225 TOTC

> MATH225 - 2025-01-06
>
> #notes #math225 #math
>
> ## general housekeeping
>
> - Dr. Hurt
> - Local Mockheed Lartin worker
> - Office hours are Engineering Annex 132 MWF from 3:00 to 4:00, or by appointment
> - Strictly speaking the only requirement is like calc 2, the only thing we need from calc 3 is partials, which, w/e
> - WebAssigns are generally due Monday, with worksheets due Wednesday
>
> # "Things we're going to need to know how to do"
>
> - I just would like to note 14 minutes to content.
> - Integration!
>   - I mean, huge shocker there.
>   - This is not a course on integration, so we aren't going to go super crazy, but there are a few things we need to know
>   - u-sub comes up a good bit
>   - As an example, $\int \frac{dx}{x+5}$, if we were to be complete and use the whole thing, we'd say $u = x + 5$, which makes $du = dx$, so then the integral is just $\int \frac{du}{u} = \ln(|u|)$ that absolute value is kind of a pain $+C$, so then you sub back out and it's just $\ln(|x+5|) + C$, and we're so good.
>   - Versions of the integral $\int \sin(2\pi\nu x + \phi)dx$
>   - That's a ν nu smuggled in there, which tends to be frequency in physics, and ϕ is a phase shift, as a physicist writes things
>   - Sixteen minutes for me to learn a new greek letter.
>   - You'd let $u = 2\pi\nu x + \phi$, where then $du = 2\pi\nu dx$
>   - Which, if you actually did your integral, becomes $-\frac{1}{2\pi\nu}\cos(2\pi\nu x + \phi)$
>   - Also comes up a good bit is integration by parts
>   - Y'know, for a course not on integration, we are hitting the highlights.
>   - Example integral of $\int xe^x dx$

- In this case, let $u = x$ and $dv = e^x dx$
    - With those, $du = dx$, and $v = e^x$
- Integration by parts just says that $uv - \int vdu = xe^x - \int e^x dx$, which is then $xe^x - e^x + C_1$
- We don't really need trig sub! Yippee!
- This is not a technique, but useful regardless
    - $\int \frac{dx}{x^2+a^2}$
        - Strictly speaking, you should probably do a tangent substitution here. It's perfectly chill if you just look then up in an integral table.
    - So like, this one becomes $\frac{1}{a}\tan^{-1}(\frac{x}{a}) + C_1$
- Not really integration, so new heading - Partial fractions!
    - Where you decompose fraction functions and get quirky with it.
    - There are a couple different cases that pop up, such as:
        1. Simple Linear Factors
            - $\frac{1}{x^2-1}$ can be pretty easily factored into $\frac{1}{(x+1)(x-1)}$, which you would then decompose into $\frac{A}{x+1} + \frac{B}{x-1}$
            - So you do a little multiplication to get $\frac{1}{(x+1)(x-1)} = \frac{A(x-1)}{(x+1)(x-1)} + \frac{B(x+1)}{(x+1)(x-1)} = \frac{Ax-A+Bx+B}{(x-1)(x+1)}$
            - We want to make an always true expression, so $Ax - A + Bx + B = 1$ has to always be true, so get grouping by powers of x
                - $A + B = 0$
                - $B - A = 1$
                - Which then gives that $B = \frac{1}{2}$ and $A = \frac{-1}{2}$
            - So fully written out this becomes $\frac{1}{x^2-1} = \frac{-1}{2} * \frac{1}{x+1} + \frac{1}{2} * \frac{1}{x-1}$
        2. Repeated Factors
            - This gives us factors raised to some power, generally of the form $(x-r)^n$
            - The partial fraction decomposition includes a separate term for each power of the repeated term
            - An example being $\frac{1}{x^2(x+1)}$
                - In this case, x^2 is our repeated, which does fit the general form - r is just 0 and n is 2.
                - We're going to have one term that's $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$
                - Exact same process as before, just longer, so

$$\frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

                - Which then becomes $Ax^2 + Ax + Bx + B + Cx^2$

- Grouping our terms like before
  - $A+C=0$
  - $A+B=0$
  - $B=1$
  - Which, you can work your way back, $A=-1$, then then makes $C=1$

3. You run into one last type - Irreducible Quadratics
   - With something like $\frac{1}{x(x^2+1)}$
   - Which then produces $\frac{A}{x}+\frac{Bx+C}{x^2+1}$
   -

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Right back to our review

$$\frac{x^2}{(x+2)(x^2+1)}$$

- Just as an example piece. That second term (x^2+1) is an irreducible quadratic, so then expanding this all would look something like

$$\frac{A}{(x+2)}+\frac{Bx+C}{(x^2+1)}$$

- If we had to integrate this, you'd be integrating the two different fraction chunks, probably do a u sub or look something up, a little $\ln$, all such fun

Implicit Differentiation

$$y\sin x = x+y$$

- Let's say we want to figure out $\frac{dy}{dx}$ for that
  - If you do $\frac{d}{dx}$ for both sides (ie)

$$\frac{d}{dx}(y\sin x)=\frac{d}{dx}(x+y)$$

$$y\cos x+\frac{dy}{dx}\sin x = 1+\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - y\cos x}{\sin x - 1}$$

Last up is Log/Exponent rules

$$\ln(xy) = \ln x + \ln y$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$\ln(x^p) = p\ln x$$

$$e^{x+y} = e^x * e^y$$

$$(e^x)^p = e^{px}$$

$$\ln(e^x) = x$$

$$e^{\ln x} = x$$

$$\ln(e) = 1$$

Ok lasty of last up is absolute value

$$|x| = \{ {x \text{ if } x \geq 0 \atop -x \text{ if } x < 0} \}$$

- Just to have a formal definition of absolute value up on the board

Now for differential equations (and why) y'know, we have a whole course about it

- A differential equation is an equation that has derivatives in it (wow!)
- So, an example might be

$$m\frac{d^2x}{dt^2} = -mg - b\frac{dx}{dt}$$

> - $-b\frac{dx}{dt}$ is a resistive force we smacked on, proportional to the velocity (because velocity is the first derivative of position with respect to time)
>   - Cheap and dirty way of showing fluid/air resistance.
>   - If you go into aerospace, shockingly, this doesn't work. But it gets the vibe across.

- Another example is something like

$$\frac{dN}{dt} = kN$$

> - This can represent an awful lot of things, but the vibe here is a population, where this works as a simple model.

- Yet another example

$$a\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

> - This is the wave equation, represents the propagation of a wave

- Fun note: Differential equations can't always be solved! Or have some other weird bullshit going on in the answer that just doesn't really work! Or maybe they represent unnamed functions that don't really exist! Shit's weird.

How do we classify differential equations, anyway?

Type

- Type can be either
  - Ordinary
    - This is $\frac{dx}{dt}$, $\frac{d^2x}{dt}$, etc etc
  - Partial
    - If there are partials, ($\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$), that makes it partial
  - This is a course on ordinary derivatives, not partials.

Order

- Refers to the order of the highest derivative in the equation
  - Quick aside on how we can represent derivatives (review)
  - Can have y',y'', this becomes rapidly impractical if you're writing something like y'''''', so you can also just put $y^{(6)}$ to represent the 6th derivative
  - Can get leibniz-y with it, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$
  - Especially with time derivatives, $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$

Examples

- So, if we were to classify something

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 + 3 = 0$$

- So that's ordinary, because there's no partials
- Second order, because there's that d^2y bit hanging out - that ^{3} is just there as bait on the $\frac{dy}{dx}$, that's just a first order derivative, cubed.

$$\frac{dy}{dt} + \frac{d^3y}{dt^3} + t = 0$$

- This is a 3rd order, ordinary differential equation.

Linearity and Such

- We can express an nth-order ODE as $F(x, y, y' \dots y^{(n)}) = 0$
    - Here x is the independent variable and y is the dependent variable.
    - It is customary to separate out the highest derivative (or otherwise solve for it), and then you're left with something like

    $$y^{(n)} = (x, y, \dots . y^{(n-1)})$$

        - This is referred to as "normal form"
    - For a first order equation, $\frac{dy}{dx} = f(x, y)$
    - Second order (like Newton's/Kirchoff's Laws) $\frac{d^2y}{dx^2} = g(x, y, y')$
- Ok actually getting back to linearity and what we mean by that
    - $F(x, y, y' \dots y^{(n)}) = 0$ is linear if $_{\text{textbook quote}}$ "F is linear in $y, y', \dots . y^{(n)}$"
        - Wow, it's linear if it's linear. Fuck you too.
    - Alright, what does this actually mean?
        1. Dependent variable y and all of its derivatives occur only to the first power.
            1. y, y', zy'' etc are all linear
            2. $y^2, \sqrt{y}, (y')^2$ are all examples of nonlinear terms
        2. The coefficients that multiply the dependent variable and its derivatives are constants or functions of the independent variable (x, in our running example)
            1. $2y, 3y', xy$ are all examples of linear (even that cheeky little x). Hell, even $\sqrt{x}y$ is fine, because that x is a function of the independent variable
            2. $y * y'$, however, is nonlinear. $\sqrt{y}y'$ is nonlinear, all those things with multiple y's lobbing themselves around is nonlinear

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$$x\frac{d^2y}{dx^2}+y^2=0$$

- That's an ordinary 2nd order equation, that is nonlinear because of that y^2 term kicking it around
  - If there were any other y terms not just as normal, we would be nonlinear

$$x\left(\frac{dy}{dx}\right)^2+y=0$$

- Is just fine, that's ordinary (no partials), first order (no powers on differential), linear (no bonus powers)

$$\sin\theta*\frac{d^2y}{d\theta^2}+\cos\theta*y=0$$

- Still looks ordinary to me, 2nd order (from the y), sure looks linear to me

Solutions

- A solution to an ordinary differential equation is some function ϕ, such that ϕ and its derivatives exist over some interval $\alpha < x < \beta$ and ϕ satisfies the D.E.
- In general, our solution might comprise a family of curves (more than one function that satisfies the differential equation)
  - Consider $\frac{dy}{dx}=x$
    - I mean, you just integrate the thing. $y=\frac{x^2}{2}+C_1$ works just fine for me.
      - That C_1 means you have a whole family of curves you can shift around
- Often, we'll be given extra information that allows us to determine the constant (ie, to narrow which of the family of curves we actually care about)
  - This information might be the value of y at some x, typically $x=0$
    - That's an initial value problem right there
      - A differential equation along with the initial conditions (ICs) is called, well, that. An initial value problem

Example time

firstsies

$$y' + 2xy^2 = 0$$

- We assert that $y = \frac{1}{x^2+C_1}$ is a solution to this problem
- Show that that is a solution to this problem

$$y' = \frac{-2x}{(x^2+C_1)^2}$$

We plug in!

$$\frac{-2x}{(x^2+C_1)^2} + \frac{2x}{(x^2+C_1)^2} = 0$$

- That sure looks correct to me. $0 = 0$ checks out just fine.
- Oh, let's say we get some initial value, like $y(2) = \frac{1}{3}$
    - We can go around plugging this in to our $y(x)$ equation, or $\frac{1}{3} = \frac{1}{4+C_1}$, so $C_1 = -1$
- So, the solution to this IVP is $y = \frac{1}{x^2-1}$
    - Which, of note, doesn't exist at $x = \pm 1$
    - Also taking a peeksie at $y' = \frac{-2x}{(x^2-1)^2}$, which also does not exist at the same places, $x = \pm 1$
    - Possible intervals are $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$, essentially anything *not* including ± 1
        - Which is all, like, standard algebra stuff, just bears writing out

we go again

$$x'' + x = 0$$

- This has the solution $x(t) = C_1 \cos t + C_2 \sin t$
- We are told that:
    - $x(0) = -1$ (the initial position)
    - $x'(0) = 8$ (the initial velocity)
- So just plugging in to the equation
    - $x(0) = -1 = C_1$
    - $x' = -C_1 \sin t + C_2 \cos t$
    - $x'(0) = 8 = c_2$
- So then the solution to this just becomes
    - $x(t) = -\cos t + 8 \sin t$
- We'll get to solving equations... someday! We'll get there. I swear.

Existence and Uniqueness Theorem (what a name)

- If $f(t,y)$ and the partial $\frac{\partial f}{\partial y}(t,y)$ are continuous at (t_0, y_0) then the IVP $y' = f(t,y)$ with the initial condition $y(t_0) = y_0$ has a unique solution in the neighborhood of (t_0, y_0)
  - If these conditions aren't met, there *could* be a solution. We have no fucking idea, and there's no theorem to help us. Godspeed, brave soldier.
- Aside: partials exist
  - ie, the partial $\frac{\partial f}{\partial y}$, if $f(x,y)$ is a thing, you just do the derivative with respect to y. It's a partial. It exists. You do it. Treat the bits you don't care about as constant, have fun.
    - ie ie, if the function is x^2y, the partial with respect to y is x^2

Exampling

$$y' = t\sqrt{y},\ t(0) = 0$$

Ok, so you ask your questions, do as you do

$$\frac{\partial f}{\partial y} = \frac{t}{2\sqrt{y}}$$

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examples from our theorem adventure last time

$$y' = t\sqrt{y},\ y(0) = 0$$

- So in the terminology of our original theorem, $f(t,y) = t\sqrt{y}$
  - Continuous at $(0,0)$
    - Function is doing just fine, no shenanigans, we're good
- Our other check was $\frac{\partial f}{\partial y} = \frac{\frac{1}{2}t}{\sqrt{y}}$
  - That, uh, doesn't exist at $y = 0$. So is therefore not continuous.
- This means that our theorem is not met, so we have no fucking idea what happens.
  - There could be a unique solution! Could not be! Couldn't tell ya.

$$y' = \sqrt{y^2 - 9}$$

- $f(x,y) = \sqrt{y^2 - 9}$, which has a problem on the interval $-3 < y < 3$
  - Which is a problem in and of itself, but regardless
- $\frac{\partial f}{\partial y} = \frac{2y}{2\sqrt{y^2-9}} = \frac{y}{\sqrt{y^2-9}}$
  - Noooow we have a problem on the interval $-3 \leq y \leq 3$
    - Which causes that whole thing to blow up / not exist, and generally is a bad time
- Unique solutions are guaranteed everywhere except $-3 \leq y \leq 3$, any other initial condition, existence is guaranteed (yippee!)
  - So if we were given $(1, 4)$, or $y(1) = 4$, that works out just fine. It's not in our bounds.
  - On the other hand, if we were given $y(0) = 2$, we have no idea what happens. Could be a solution! Couldn't tell ya.

Show that $y = \frac{1}{3}e^x + c_1 e^{-2x}$ is a solution of $y' + 2y = e^x$

- So just, take the derivative

$$y' = \frac{1}{3}e^x - 2c_1 e^{-2x}$$

- If you go and plug in

$$\frac{1}{3}e^x - 2c_1 e^{-2x} + 2(\frac{1}{3}e^x + c_1 e^{-2x}) = e^x$$

$$e^x - 2c_1 e^{-2x} + 2c_1 e^{-2x} = e^x, e^x = e^x$$

- Suppose we have some initial condition $y_0 = 1$, what is the particular solution?

$$1 = \frac{1}{3} + c_1$$

$$c_1 = \frac{2}{3}$$

- So, fully written out with the initial condition, the solution becomes

$$y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}$$

$$y = \frac{1}{1 - e^{-t}}$$

- is a solution to some differential equation.
  - This solution does not exist when $e^{-t} = 1$, or when $t = 0$,
- We can define intervals of solution, for instance $(-\infty, 0)$, or $(0, \infty)$

moving on to chapter 2 (eat it, 1.3)

- Quirky little reminder that $\frac{dy}{dx}$ is a slope - the slope of a tangent line at some given point, but a slope nonetheless
  - If we evaluate $\frac{dy}{dx} = f(x, y)$, we can determine the slope as we move around the x-y plane (this is called a slope or direction field)

Autonomous Differential Equations

- This is when the independent variable does not explicitly occur
  - ie, $\frac{dy}{dx} = f(y)$
- The points, c, at which $f(y) = 0$, are called fixed points, **critical points**, stationary points, equilibrium points, the list goes on
- If c is a critical point, then the function $y = c$ is a solution to the differential equation

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Previously on: Autonomous Equations

- $\frac{dy}{dx} = f(y)$, we've got critical points where $f(y) = 0$, yadayadayada

New things

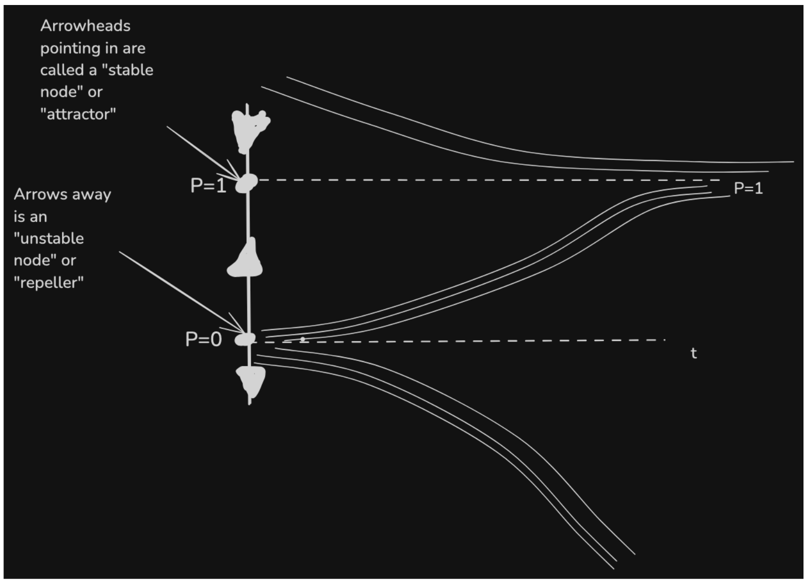
Construct a Phase Portrait (or phase line)

1. Find critical points
2. Divide the real number line into regions according to the critical points
3. Sketch whether f is positive or negative
   - Call me crazy, but we did this in Calc 1 in like Math111 - 2023-09-29

$$\frac{dp}{dt} = p - p^2$$

$$p(1-p) = 0$$

- This has two critical points, 1 and 0
- Also, this has a special name, the logistic equations
- I'm feeling a bit lazy, so don't really want to do the sign chart
  - But working it out logistically, below 0, it's going to be negative
  - Above 1, it's going to be negative (the $1-p$) is up to some shenanigans
  - Betwixt the two, it's positive



- If the arrows are kind of towards/away, that's a semistable node, just for the record
- Couple of fun bobs about phase portraits:
  1. Solution curves stay within a given region (They don't cross the critical lines)
     1. Crossing of lines is banned by the uniqueness theorem
  2. The critical points, ie $P = 0$, are solutions to the differential equation
  3. Solutions either increase or decrease within a region (not both! no extrema!)

$$\frac{dy}{dx} = y^2 - y^3$$

$$= y^2(1-y)$$

- Which gives critical points of, once again, 0 and 1.
- Assess the sign of dy/dx in each region
  - When $y < 0$, it sure looks positive to me
  - When $y > 1$, it sure looks negative to me.

    - When in between the two equations, still appears positive to me.
- Theoretically speaking, go crazy and draw another phase portrait. I, dear reader, no no wanna.
- We may be told to sketch a solution that satisfies the initial condition $y(0) = \frac{1}{2}$, or something
    - You just sketch something that goes through the point at $\frac{1}{2}$, and bob's your uncle
- For the solution that has $y(0) = \frac{1}{2}$, what is $\lim_{x \to \infty} y(x)$?
    - For the one we've been working with, that'll be 1 (it approaches the critical point)
- Obligatory reminder: critical points are solutions!
-

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more on portraits of phase

- Last time we had $\frac{dP}{dt} = P(1 - P)$, which was the logistic equation, or put in a more general form with parameters, we have $\frac{dP}{dt} = P(a - bP)$
    - We got two critical points! And that's it! $P = 0$, and $P = \frac{a}{b}$
        - Stuff is happening between the points, you can do the pos/neg logic
            - If you start with something like $P < 0$ or $bP < 0$, and multiply by -1, the inequality flips (and picks up an equal? right? not mentioned but I swear that happens)
                - $-p > 0$
                - $-bP > 0$

"now let's actually learn how to solve a differential equation"

Separable Equations

- If the function $f(x, y)$ in $\frac{dy}{dx} = f(x, y)$ can be written as $f(x, y)$ as some $g(x) * h(y)$, we call this separable
- We can then write $\frac{dy}{dx} = g(x) * h(y)$, or could rearrange into a vibe like $\frac{dy}{h(y)} = g(x)dx$, and from there you can just smack an $\int$ on and go wild.
- The solution just kind of pops out from doing the integrals of like $\int \frac{dy}{h(y)} = \int g(x)dx$

- For instance, $\frac{dy}{dx} = xy$ is perfectly separable
  - $\frac{dy}{dx} = \cos(xy)$ is not happily separable.
  - $\frac{dy}{dx} = f(x)$ is perfectly separable! The y component is just $*1$, but that doesn't mean it's not separable

example time

$$\frac{dy}{dx} = (x+1)^2$$

- This sure looks separable to me.

$$dy = (x+1)^2 dx$$

- Once rearranged, go about integrating both sides

$$y + c_1 = \frac{(x+1)^3}{3} + c_2$$

$$y = \frac{(x+1)^3}{3} + (c_2 - c_1) = \frac{(x+1)^3}{3} + c_3$$

- Constant minus a constant is still...... some constant.

anotha one

$$\frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx$$

$$\int \frac{dy}{y} = \int dx$$

$$\ln(|y|) = x + c_3$$

- This is an implicit solution because y is inside some function. An *explicit* solution would have $y = g(x)$
  - so like, exponentiate both sides, do

$$e^{\ln|y|} = e^{x+c_1}, \text{ which is } |y| = e^{c_1}e^x$$

  - Oh hey, that's a constant out front, so $|y| = c_2 e^x$

- Really, if we're being honest, the absolute value is just the *possibility* that y is $-y$ if it's negative, and $-y$ is really just y with a constant attached, so we can suction all the constants into the constant, leaving $y = c_3 e^x$

y'know do it again

- Our good friend the logistic equation, $\frac{dP}{dt} = P(1-P)$, is separable
- So $\int \frac{dP}{P(1-P)} = \int dT$

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$$1 = A - AP + BP$$

So group like powers, solve, get a little jiggy with it

$$\frac{1}{P} + \frac{1}{1-P}$$

- So if we hop back to our integrals we end up with

$$\int \frac{dP}{P} + \int \frac{dP}{1-P} = t + c_1$$

$$\ln|P| - \ln|1-P| = t + c_1$$

$$\ln|\frac{P}{1-P}| = t + c_1$$

- Quite unfortunately our P appears to be wrapped up quite completely inside that natural log, so if we exponentiate both sides

$$|\frac{P}{1-P}| = c_2 e^t$$

- Siphon the absolute value into the constant, leaving

$$\frac{P}{1-P} = c_3 e^t$$

$$P = c_3 e^t - c_3 P e^t$$

- Do some more algebra and a little factors

$$P = \frac{c_3 e^t}{1 + c_3 e^t}$$

- If you really wanted to keep going and doing more algebra, you could make this into

$$P = \frac{1}{1 + c_4 e^{-t}}$$

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straight back into separation of examples

$$\frac{dy}{dt} + 2y = 1, y(0) = \frac{5}{2}$$

$$\frac{dy}{dt} = 1 - 2y$$

$$\frac{dy}{1 - 2y} = dt$$

$$u = 1 - 2y, du = -2dy$$

$$\int \frac{dy}{1 - 2y} = \frac{-1}{2} \int \frac{du}{u} = \frac{-1}{2} \ln |u|$$

$$= \frac{-1}{2} \ln |1 - 2y|$$

$$-\frac{1}{2} \ln |1 - 2y| = t + c_1$$

$$\ln |1 - 2y| = -2t + c_2$$

$$|1 - 2y| = c_3 * e^{-2t}$$

$$1 - 2y = c_4 * e^{-2t}$$

$$2y = 1 - c_4 * e^{-2t}$$

$$y = -\frac{1}{2} - c_5 e^{-2t}$$

- Oh hey, do remember we had some initial condition

$$y(0) = \frac{5}{2}, \text{sooo } \frac{5}{2} = \frac{1}{2} + c_5, c_5 = 2$$

integrating factors

$$\frac{dy}{dx} + p(x)y = f(x)$$

- This method only works for linear equations - unlike separation of variables, which can work for nonlinear equations
- We define an integrating factor $\mu = e^{\int p(x)dx}$
  - We then multiply our equation by this factor μ

example

$$\frac{dy}{dx} + 4y = \frac{4}{3}$$

$$\mu = e^{\int 4dx} = e^{4x}$$

$$e^{4x}y' + 4e^{4x}y = \frac{4}{3}e^{4x}$$

- The left hand side is now a full derivative

$$[e^{4x} * y]' = \frac{4}{3}e^{4x}$$

- Integrate both sides, go crazy

$$e^{4x} * y = \frac{1}{3}e^{4x} + c_1$$

$$y = \frac{1}{3} + c_1e^{-4x}$$

- And that pops out an explicit solution of the differential equation

---

$$x\frac{dy}{dx} + y = 4x + 1, \text{ with } y(1) = 8$$

- This isn't quite in standard form, we need to multiply by $\frac{1}{x}$ to get there
- $$\frac{dy}{dx} + \frac{1}{x}y = 4 + \frac{1}{x}$$
- We're going to use integrating factors here (I mean, you could factor it? no?)

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

$$x\frac{dy}{dx} + y = 4x + 1$$

$$\int [xy]' = \int (4x+1)dx$$

$$xy = 2x^2 + x + c_1$$

$$y = 2x + 1 + \frac{c_1}{x}$$

$$y(1) = 8, 8 = 2 + 1 + c_1, c_1 = 5$$

$$\frac{dy}{dt} + 2y = 1$$

- We initially did this with separation of variables, (see straight back into separation of examples)
  - oooor, we could use integrating factors, and say $\mu = e^{2t}$

$$[e^{2t}y]' = e^{2t}$$

$$e^{2t}y = \frac{1}{2}e^{2t} + c_1$$

$$y = \frac{1}{2}c_1e^{-2t}$$

---

back to our ol pal the logistic equation

$$\frac{dP}{dt} = P - P^2$$

- unfortunately, we gain absolutely nothing from learning about integrating factors, since that only works with linear equations. womp. womp.

Method of Undetermined Coefficients

- Also known as muc
- This technique works for the following form

$$y' + a * y = f(t)$$

  - that constant a hanging out, and the right hand side is some function of t
- this only works for a certain set of functions for $f(t)$
  - this can be a polynomial ($f(t) = t^2 + 1$ is perfectly polynomial, for instance)

- exponential ($f(t) = 2e^t$, for instance)
- can be sine or cosine
- Combinations of any of those

MATH225 - 2025-01-24

#notes #math225 #math

getting muc-y with it

$$y' + ay = f(t)$$

- this only works for a certain set of functions for $f(t)$
  - this can be a polynomial ($f(t) = t^2 + 1$ is perfectly polynomial, for instance)
  - exponential ($f(t) = 2e^t$, for instance)
  - can be sine or cosine
  - Combinations of any of those

General Solution

$$y = y_c + y_p$$

- The y_c is the complementary, or homogenous solution
- The y_p is the particular solution

Homogenous equation

- Has a right hand side = 0 so

$$y' + ay = 0$$

- You can almost certainly separation of variables that, so

$$\ln(y) - at + c_1$$

$$y = c_1 e^{-at}$$

- Which means the original form is going to be

$$y = c_1 e^{-at} + y_p$$

The particular part

- New problem: finding y_p
  - Solution: just fucking guess.

$$2y' + 6y = e^{-t}$$

$$-> y' + 3y = \frac{1}{2}e^{-t}$$

- We need some y_p that produces $\frac{1}{2}e^{-t}$ when plugged in to the left hand side. Given that the only damn thing that derives itself into an exponential is an exponential, we're just going to guess

$$y_p = Ae^{-t}$$

- The A is the undetermined coefficient in question (bum bum buuuum)
  - If we're calling that y_p, then $y_p' = -Ae^{-t}$

$$2Ae^{-t} + 6Ae^{-t} = e^{-t} -> 4Ae^{-t} = e^{-t}$$

- Which just gives you $4A = 1$, so $A = \frac{1}{4}$
- Our overall answer is then smacking them together

$$y = c_1 e^{-3t} + \frac{1}{4}e^{-t}$$

$f(t)$	**Guess**
e^{2t}	Ae^{2t}
$\sin(3t)$	$A\sin(3t) + B\cos(3t)$
$\cos(3t)$	$A\sin(3t) + B\cos(3t)$
t^3	$At^3 + Bt^2 + Ct + D$
t^2e^t	$(At^2 + Bt + C)e^{-t}$
$t^2\sin(4t)$	$(At^2 + Bt + C)(\sin(4t)) + (Dt^2 + Et + F)(\cos(4t))$
$e^{-t} + t$	$Ae^{-t} + Bt + C$

example time

$$\frac{dy}{dx} - y = 1$$

- I mean, sure looks separable to me

$$\frac{dy}{y+1} = dx$$

$$\ln(y+1) = x + c$$

$$y + 1 = c_1 e^x$$

$$y = c_1 e^x - 1$$

- I mean, y'know, when in Rome, sure looks integrating factor-able to me

$$\mu = e^{\int(-1)dx} = e^{-x}$$

$$(e^{-x}y)' = e^{-x}$$

$$e^{-x}y = -e^{-x} + c_1$$

$$y = \frac{c_1}{e^{-x}} - 1$$

- I mean, y'know, when in Rome, sure looks muc-able to me.

$$y = y_c + y_p$$

$$y_c = c_1 e^x + y_p$$

- We're just going to guess that y_p is a constant, since the right side is basically just a low order polynomial (also, $y_p' = 0$)
  - lob that back in to our differential equation
  - $-A = 1$, $A = -1$

$$y = c_1 e^x - 1$$

new example

$$y' - y = \cos(2x)$$

$$\frac{dy}{dx} - y = \cos(2x)$$

- That sure doesn't look separable, you coooould integrating factor, but then you'd have to (probably repeated) integration by parts, and.... we don't wanna
- So instead, muc!

$$y = y_c + y_p$$

$$c_1 e^x + y_p$$

- Oh hey, we have a cosine lying around, so we're going to guess those are hopping around

$$y_p = A\cos(2x) + B\sin(2x)$$

$$y_p' = -2A\sin(2x) + 2B\cos(2x)$$

- Plug that back in to the original differential equation

$$-2A\sin(2x) + 2B\cos(2x) + A\cos(2x) + B\sin(2x) = \cos(2x)$$

$$(-2A - B)(\sin(2x)) = 0$$

$$(2B - A)(\cos(2x)) = \cos(2x)$$

$$2B - A = 1$$

$$-2A - B = 0$$

$$B = -2A$$

$$-4A - A = 1$$

$$-5A = 1, A = -\frac{1}{5}$$

$$B = \frac{2}{5}$$

- So, overall becomes

$$c_1e^x - \frac{1}{5}\cos(2x) + \frac{2}{5}\sin(2x)$$

MATH225 - 2025-01-27

#notes #math225 #math

when muc fails (or is at least weird)

- This is when the homogeneous solution includes our guess for y_p

$$y' + 2y = 3e^{-2t}$$

$$y = y_c + y_p$$

- y_c is the homogeneous solution, which solves

$$y' + 2y = 0$$

- Which, in this case, is $y_c = c_1e^{-2t}$
- Our usual guess is $y_p = Ae^{-2t}$

- uhhhhhhh, minor problem, it's looking an awful lot like we're going to get ourselves a zero if we keep barking down this tree
- In this case, we're just going to throw an extra t in there, so our guess is going to become $y_p = Ate^{-2t}$

$$y_p = Ate^{-2t}$$

$$y_p' = -2Ate^{-2t} + Ae^{-2t}$$

- So we then lob back in to the original differential equation

$$-2Ate^{-2t} + Ae^{-2t} + 2Ate^{-2t} = 3e^{-2t}$$

- Group like powers of t
  - I mean, we can just divide fucking everything by e^{-2t}

$$-2At + A + 2At = 3$$

$$A = 3$$

- So then our general solution just becomes

$$y = c_1 e^{-2t} + 3te^{-2t}$$

- So just, as a rule of thumb, always check that your guess is not already included in the homogeneous solution.
  - If things go wrong around the solving for A section, you probably forgot to check and should double back

transience

- Sometimes, we might get solutions that look like

$$y = c_1 e^{-t} + \cos(t)$$

- $c_1 e^{-t}$ is referred to as transient, since it does something we care about for a limited time (I love act 3 slay the spire)
- $\cos(t)$ is called the steady state, since it's what happens in the long term
- We'll eventually get back here with higher order equations

other such fun things

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

$$x(\frac{dy}{dx})+2y=x^2$$

$$x(\frac{dy}{dx}-x)=-2y$$

- I swear this is separable, but, regardless, we use integrating factors
  - Did I do legal math up there? Is that separate? the dy/dx is perpetually a bit confusing to me

$$e^{\int p(x)dx}=e^{\int \frac{2}{x}dx}=e^{2\ln(x)}=e^{\ln(x^2)}=x^2$$

$$(x^2y)'=x^3, x^2y=\frac{x^4}{4}+c_1$$

$$y=\frac{x^2}{4}+\frac{c_1}{x^2}$$

now we get another one (substitution method)

- Applies to Bernoulli Equation(s)

$$\frac{dy}{dx}+p(x)y=f(x)y^n$$

- We're going to use a substitution of $u=y^{1-n}$. This will produce a linear equation in u
  - This is, of note, only really relevant for $y\neq 0$ & $y\neq 1$, since like, yeaaaah.
- $$x\frac{dy}{dx}+y=\frac{1}{y^2}$$
- $$\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{x}*\frac{1}{y^2}$$
- This is now a Bernoulli equation with $n=-2$
- We let $u=y^1--2=y^3$, or $y=u^{\frac{1}{3}}$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

- You always end up having to do this funky conversion whenever you're doing a Bernoulli equation

$$\frac{dy}{du}=\frac{1}{3}u^{\frac{-2}{3}}$$

$$\frac{dy}{dx}=\frac{1}{3}u^{\frac{-2}{3}}\frac{du}{dx}$$

$$\frac{1}{3}u^{\frac{-2}{3}}\frac{du}{dx} + \frac{1}{x}u^{\frac{1}{3}} = \frac{1}{x}u^{\frac{-2}{3}}$$

- Multiply by $3u^{2/3}$ to hopefully clear out a whole bunch of things

$$u' + \frac{3}{x}u + \frac{3}{x}$$

- We are, quite unfortunately, in a rut of having to do some integrating factors

$$u = e^{\int \frac{3}{x}dx} = x^3$$

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x} \to (x^3u)' = 3x^2$$

$$x^3u = x^3 + c_1$$

$$u = 1 + \frac{c_1x}{3}$$

buuut $u = y^3$
so

$$y^3 = 1 + \frac{c_1}{x^3}$$

- If you really want to make this explicit, you're going to have to cube root this entire thing, but like, that's a lotta work

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x}$$

$$\frac{du}{dx} = \frac{3}{x}(1-u)$$

$$\frac{du}{1-u} = \frac{3}{x}dx$$

$$-\ln(1-u) = 3\ln(x) + c_1$$

- That's separation, presumably if you keep cooking you'll get somewhere reasonable
- And that's presumably it for linear equations, now we're going to get into physical models to actually use differential equations
-

MATH225 - 2025-01-31

#notes #math225 #math

- Of note, was rather busy working on getting employed between here and the previous note. So, there's something I missed. I could email about it, but that's a *lot* of work, so I'll probably just be ignoring it.

Interval of Solution

An interval in which the following are true:

1. The DE (ie, the derivative) is defined

- General form for a linear differential equation looks something like

$$y' + p(x)y = f(x)$$

- What we *really* mean is that $p(x)$ and $f(x)$ are defined

2. Include the initial condition at x_0.
3. The solution ϕ is defined.

example

$$y' = y \quad y(0) = \quad y' - y = 0$$

- The D.E. has an acceptable interval I : $(-\infty, \infty)$
- The IC is at $x_0 = 0$
- We might move on and solve, where $y = c_1 e^x$
- So the interval for this particular IVP is any real number

example 2

$$y' = y^2 \quad y(0) = 1$$

- The differential equation is still defined for any value of x - there's nothing you can put in for x anywhere that causes problems (notably, there is no x, but that's besides the point)
- The IC is at $x = 0$

$$\frac{dy}{y^2} = dx$$

$$\frac{-1}{y} = x + c_1$$

$$y = \frac{1}{c_2 - x}$$

- Minor problem: this does not exist at $x = c_2$
  - Solving with our initial condition gives us that $c_2 = 1$
  - Which means that the solution is

$$\frac{1}{1-x} = y$$

  - Which, again, does not exist at $x = 1$
  Valid intervals are

$$(-\infty, 1) \text{ or } (1, \infty)$$

yet another example

$$y' + \frac{3}{x+1}y = (x+1)^{\frac{1}{2}}$$

$$y(0) = \frac{11}{9}$$

- Here we have a problem with our initial D.E., where we're undefined at $x = -1$
  - Also, if we look at the right side, we're also not defined literally any time where $x + 1$ is < 0 (or when $x < -1$)
- So out initial possible intervals just from the left is $\cancel{(\infty, -1)}$ and $(-1, \infty)$, smacking on the right side constraint we can just cross that one out, and we need our initial condition, so we're sticking with $(-1, \infty)$

Reportedly, dealing with population growth and such

- Our textbook rewrites the logistic equation as

$$\frac{dP}{dt} = P(a - bP)$$

  How do we solve these things?

1. Separation of variables
2. ~~Integrating Factors~~
3. ~~Method of Undetermined Coefficients~~
4. Bernoulli
   Welp, this isn't linear, soooo

- Looks separable, and happens to be a Bernoulli equation, soooo

$$\int \frac{dP}{P(a-bP)} = \int dt$$

- mmmm. Partial fractions.

$$\int \frac{A}{P} + \frac{B}{a-bP} = \int dt$$

$$A(a-bP) + B(P) = 1$$

$$aA - bAP + BP = 1$$

- Regular ol' equate like powers of P

$$-bA + B = 0$$

$$A = \frac{B}{b}$$

$$aA = 1$$

$$A = \frac{1}{a}$$

$$\frac{B}{b} = \frac{1}{a}$$

$$B = \frac{b}{a}$$

$$= \frac{1}{a} * \frac{1}{P} + \frac{b}{a} \frac{1}{a-bp}$$

$$\frac{1}{a} \int \frac{dP}{P} + \frac{b}{a} \int \frac{dP}{a-bp} = t + c_1$$

$$\frac{1}{a} \ln P + (\frac{b}{a}) \left(\frac{-1}{b}\right) \ln(a-bP) = t + c_1$$

$$\ln(P) - \ln(a-bP) = at + c_2$$

$$\ln\left(\frac{P}{a-bP}\right) = at + c_2$$

$$\frac{P}{a-bP} = e^{at+c_2}$$

$$\frac{P}{a-bP} = c_3 e^{at}$$

$$P = c_3 a e^{at} - c_{3bPe^{at}}$$

$$P(1 + c_4 b e^{at}) = c_4 a e^{at}$$

$$P=\frac{c_4ae^{at}}{1+c_4be^{at}}$$

- It's not uncommon to attempt to clear out the exponential by dividing by e^{at}

$$P=\frac{ac_4}{bc_4+e^{-at}}$$

- We usually define $P(0)=P_0$

$$P(0)=\frac{ac_4}{b_{c4}+1}$$

- You can go ahead and solve that for c_4 if you're bored, which spits out

$$c_4=\frac{P_0}{a-bP_0}$$

- This c_4 can then be plugged back in to our solution to get

$$P=\frac{aP_0}{bP_0+(a-bP_0)e^{-at}}$$

$$P(0)=\frac{aP_0}{bP_0+(a-bP_0)}=P_0$$

$$\lim_{t\to\infty}P(t)=\frac{a}{b}$$

- Which is then the carrying capacity of our population
-

MATH225 - 2025-02-03

#notes #math225 #math

more fun with physical systems

radioactive decay!

$$\frac{dN}{dt}=-kN$$

- When we're writing this equation for radioactive decay, N is the number of atoms (or particles) of the isotope, and k is the proportionality constant (commonly referred to as a decay constant, in this particularly radioactive case)
- When we write it this way (with the explicit negative sign), k is meant to be a positive number
  - It could very easily be a negative constant and have that just be that.
- If we have some isotope like $I - 131$:
  - Half-life (λ jumpscare) of about eight days
    - Time it takes for half of the original sample to be left
- Just as a note, our solution to our differential equation would be

$$N = N_0 e^{-kt}$$

- So, our half life is where $\frac{1}{2} N_0$ is equal to N
- $$\frac{1}{2} = e^{-kt}$$
  $$\frac{1}{2} = e^{-kt_{\frac{1}{2}}}$$
- $$\ln\left(\frac{1}{2}\right) = -kt_{\frac{1}{2}}, \ln(2^{-1}) = -\ln(2) = kt_{\frac{1}{2}}$$
  $$t_{\frac{1}{2}} = \ln\frac{2}{k}$$
- If you put in an eight day half life, you can solve for k and end up with $k = 0.0866 \text{ day}^{-1}$

Example work question

- How long can we keep this sample of iodine before we have only 10% of the original radioactive quantity?

$$N = N_0 e^{-kt}$$

- Remains our governing equation
- We may also need to solve for k, or be given a half life, or something along those lines and you then have to, as they say, get jiggy with it

$$\frac{1}{10} N_0 = N_0 e^{-kt}$$
$$\frac{1}{10} = e^{-kt}$$

$$k = 0.0866 \text{ day}^{-1}$$

$$\ln\left(\frac{1}{10}\right) = -(0.0866)t$$

$$t = \frac{\ln \frac{1}{10}}{-0.0866} = 26.6 \text{ days}$$

Newton's Law of Cooling

- The temperature T of an object in some surrounding medium with temperature T_m is given by

$$\frac{dT}{dt} = k(T - T_m)$$

> - This is, of note, assuming that the medium is constant temperature

- Mad shoutout that he was doing Lockheed work earlier today before teaching

Cooling Example

- Suppose we have some cup of coffee at $170^\circ F$. We put this cup in a room at $70^\circ F$.
  - Write the initial value problem given by this problem statement.
  - Solve IVP for T

$$\frac{dT}{dt} = k(T - 70),\ T_0 = 170$$

- We uh, need a bit more detail to finish

MATH225 - 2025-02-05

#notes #math225 #math

still on that cooling type beat

$$\frac{dT}{dt} = k(T - T_m)$$

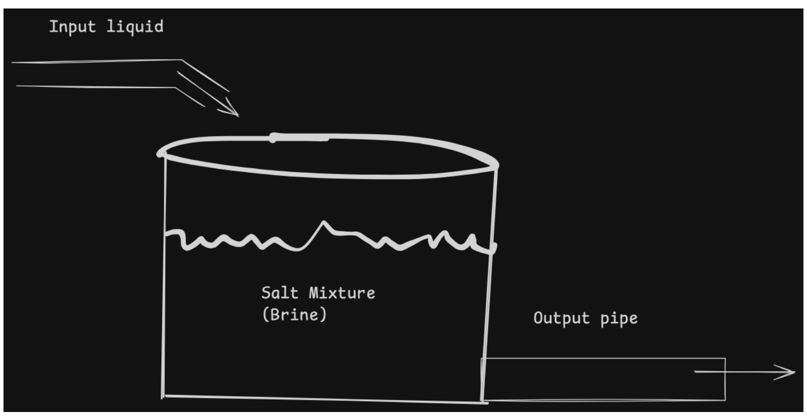
$$\frac{dT}{T - T_m} = kdt$$

$$\ln(T - T_m) = kt + C_1$$

$$T = T_m + c_1 e^{kt}$$

- Of note, we have two constants floating around, so we'd need some more information to get going into properly solving this
- Anyways, back to our 170° coffee cup in the 70° room. Further assume that after 5 minutes, the coffee is 110°. Write the equation for temperature as a function of time
    - So we have our $T = T_m + c_1 e^{kt}$ as our general solution, and now we need to start plugging stuff in and going crazy.
    - $T(0) = 170 = T_m + c_1$
        - T_m represents temperature of the surroundings, so that'll be 70, so $c_1 = 100$
    - $T = 70 + 100e^{kt}$
    - $T(5) = 110 = 70 + 100e^{k(5)}$
        - So $40 = 100e^{k5}$
            - $0.4 = e^{5k}$
            - $k = \frac{\ln(0.4)}{5} \approx -0.18$ min^{-1}
    - $T = 70 + 100e^{-0.18t}$

mixing problems



- We start with one input, one output, one tank, easy as can be
- $A(t)$ is the amount of salt in the tank (for some reason we prefer imperial units, so, lbs)
- $R_{in} =$ input rate of salt per unit time to the tank (lbs/min)
- $R_{out} =$ output rate of salt per unit time from the tank (lbs/min
- $c(t) =$ concentration of the liquid in the tank
    - $c(t) = \frac{A(t)}{\text{liquid in tank}}$
    - Which has units of lbs/gallon

Example

- Tank initially has 300 gallons of saltwater, which has 50 lbs of salt dissolved within. (concentration of $\frac{1}{6}$ lb/gallon). Another brine solution with a concentration of 2 lbs/gallon is added at 3 gallons per minute. (Also, we assume that this is all instantly mixed and so on and so forth). Liquid is pumped out of the tank at 3 gallons per minute.
  - For the record, the input brine is roughly ~6.8x as salty as seawater. Quirky!

$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$R_{in} = 2\frac{\text{lb}}{\text{gal}} * 3\frac{\text{gal}}{\text{min}} = 6\frac{\text{lb}}{\text{min}}$$

$$R_{out} = \left(\frac{A(t)}{\text{Liquid in tank}}\right)_{\text{(Concentration in tank)}} * 3\frac{\text{gallon}}{\text{minute}} = \frac{A(t)}{300} * 3 = \frac{A(t)}{100} = R_{out}$$

$$\frac{dA}{dt} = 6 - \frac{A(t)}{100}, A(0) = 50$$

- Ok, we're going to use integrating factors here

$$\frac{dA}{dt} + \frac{1}{100}A = 6$$

- use an integrating factor of

$$\mu = e^{\int \frac{1}{100}dt} = e^{\frac{t}{100}}$$

- $$e^{\frac{t}{100}} * A = 6e^{t/100}$$

- $$e^{\frac{t}{100}}A = 600e^{t/100} + C$$

- $$A = 600 + \frac{C}{e^{\frac{t}{100}}}$$

$$A(0) = 50 = 600 + C_1, C_1 = -550$$

- $$A(t) = 600 - 550e^{-t/100}$$

If we do another example problem

- Same input and output setup
  - Start with 300 gallons in the tank, $A(0) = 50$lb
  - Still bringing in 2 lb/gal salt at 3 gal/min
  - Quirky bit - we're only outputting at 2 gal/min

- Which makes this particular tank fill up over time
- $V(t) = 300 + (3-2)t$

$$\frac{dA}{dt} = R_{in} - R_{out}$$

- So we use the same governing principle here to setup our equation

$$\frac{dA}{dt} = 6 - \frac{2A(t)}{300+t}, A(0) = 50$$

- That whole R_{out} term has gotten noticeably more funky, we still have a changing quantity of salt and now we also having a changing quantity of volume

$$\frac{dA}{dt} = \frac{2}{300+t} * A = 6$$

- We, once again, integrate our factors

$$\mu = e^{\int \frac{2}{300+t} dt}$$

$$\mu = e^{2\ln(300+t)} = (300+t)^2$$

MATH225 - 2025-02-07

#notes #math225 #math

just going to copy over the example we were working on last time, see if we go back to it

If we do another example problem

- Same input and output setup
  - Start with 300 gallons in the tank, $A(0) = 50$lb
  - Still bringing in 2 lb/gal salt at 3 gal/min
  - Quirky bit - we're only outputting at 2 gal/min
  - Which makes this particular tank fill up over time
  - $V(t) = 300 + (3-2)t$

$$\frac{dA}{dt} = R_{in} - R_{out}$$

- So we use the same governing principle here to setup our equation

$$\frac{dA}{dt} = 6 - \frac{2A(t)}{300+t}, A(0) = 50$$

- That whole R_{out} term has gotten noticeably more funky, we still have a changing quantity of salt and now we also having a changing quantity of volume

$$\frac{dA}{dt} + \frac{2A}{300+t} = 6$$

- We, once again, integrate our factors

$$\mu = e^{\int \frac{2}{300+t} dt}$$

$$\mu = e^{2\ln(300+t)} = (300+t)^2$$

$$(300+t)^2 \frac{dA}{dt} + 2(300+t)A = 6(300+t)^2$$

$$(300+t)^2 A = 6\int (300+t)^2$$

$$(300+t)^2 A = 2(300+t)^3 + C_1$$

$$A = 2(300+t) + \frac{C_1}{(300+t)^2}$$

$$A(0) = 50$$

$$50 = 600 + \frac{C_1}{90000}$$

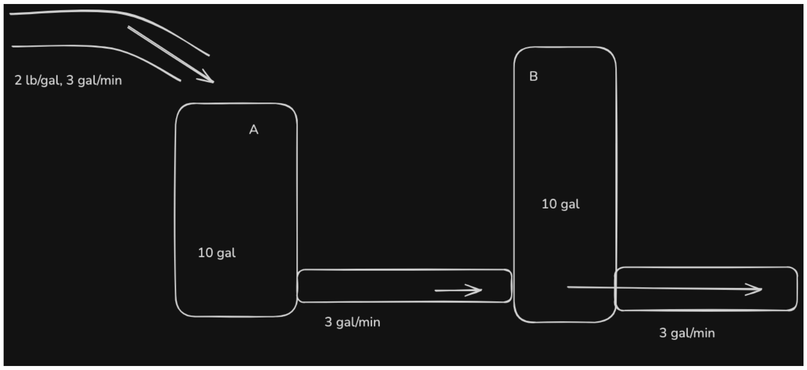
$$C_1 = (90,000)(-550) = -4.95e^7$$

$$A(t) = 2(300+t) - \frac{4.95 * 10^7}{(300+t)^2}$$

- Hey, what's the $\lim\limits_{t \to \infty}$ of that look like?
  - Infinity. Infinitely big tank, infinitely large amount of salt. This checks out.

Yet another type of tank problem

- In this kind, we've got two tanks!
  - We're practically WW1 up in here.



- Ok, so, initial conditions and other setup things:
  - $A(0) = 5$
  - $B(0) = 2$
  - $A(t)$ and $B(t)$ are the amount of salt in their respective tanks
- Just writing things out condition wise so I don't have to check the drawing:
  - $A_{in} = 3\frac{gal}{min}, C_{A_{in}} = 2$
  - $A_{out} = 3\frac{gal}{min}$
  - $B_{in} = A_{out}$
  - $B_{out} = B_{in} = A_{out}$
- Okie dokies, so what's going on with tank A?
  - $R_{in} = 6$
  - $R_{out} = \frac{A}{10} * 3$
  - $\frac{dA}{dt} = 6 - 0.3A$
- And with tank B:
  - $R_{in} = \frac{\text{Salt in A}}{\text{Volume of a}} * \text{Flow from } a \to b$
  - $= \frac{3A}{10} = 0.3A$
  - $R_{out} = \frac{\text{Salt in B}}{\text{Volume of B}} * \text{Flow rate out of } b$
  - $= \frac{B}{10} * 3 = 0.3B$
  - $\frac{dB}{dt} = 0.3A - 0.3B$

$$\frac{dA}{dt} = 6 - 0.3A$$

$$\frac{dB}{dt} = 0.3A - 0.3B$$

- This is an example (our first) of a coupled differential equation, where one depends on the other. How silly. How quirky.
  - This is solvable because A depends only on itself

$$\frac{dA}{dt} + 0.3A = 6$$

$$\mu = e^{\int(0.3)} = e^{0.3t}$$

$$e^{0.3t}A = \int 6e^{0.3t}$$

$$e^{0.3t}A = \frac{6}{0.3}e^{0.3t} + C_1$$

$$A = 20 + C_1e^{-0.3t}$$

$$A(0) = 5$$

$$20 + C_1 = 5, C_1 = -15$$

$$A(t) = 20 - 15e^{-0.3t}$$

- Now we bounce back over to our $\frac{dB}{dt}$

$$\frac{dB}{dt} = 0.3(20 - 15e^{-0.3t}) - 0.3B$$

$$\frac{dB}{dt} + 0.3B = 6 - 4.5e^{-0.3t}$$

$$\mu = e^{0.3t}$$

$$e^{0.3t}B = \int 6e^{0.3t} - 4.5$$

$$e^{0.3t}B = 20e^{0.3t} - 4.5t + C_1$$

$$B = 20 - \frac{4.5t + C_1}{e^{0.3t}}$$

$$B(0) = 2 = 20 - C_1, \;\; C_1 = -18$$

$$\frac{dB}{dt} = 20 - \frac{4.5t - 18}{e^{0.3t}}$$

MATH225 - 2025-02-10

#notes #math225 #math

Quick continuation of example from Year 2/Semester 2/MATH225/MATH225 - 2025-02-07

$$\frac{dB}{dt} + 0.3B = 6 - 4.5e^{-0.3}$$

- Which we had solved with an integrating factor and had a jolly ol' time

- However, MUC is also perfectly legal here

$$B_c = c_1 e^{-0.3t}$$

- Super easy peasy, job done

$$B_p = C + De^{-0.3t}$$

- Is our guess here, where the C is basically a 0th order polynomial
- Minor wicket: $e^{-0.3t}$ already occurs in our homogeneous solution, so we're going to have to get funky.

$$B_p = C + Dte^{-0.3t}$$

And we go again.

- We have two tanks, A and B. They are connected by a pipe, with flow on the bottom of A and B being {blank}. Tank B has an output pipe into the ether, as well as a pipe with flow from B to A. Additionally, there is an input pipe from the ether into tank A.
  - Input to A from ether is pure, unsalinated water at 3 gal/minute
  - Transfer from A to B is 4 gal/min
  - Transfer from B to A is 1 gal/minute
  - Transfer from B to ether is 3 gal/minute
  - At time zero:
    - Tank A has 50 gallons of brine with 25 lbs of salt
      - $A(0) = 25$
    - Tank B has 50 gallons of pure water (zero lbs of salt)
      - This means $B(0)$ is 0, headass.
- Both tanks have a constant fluid level, which is simply delightful.
- Soooo, for tank A:

$$R_{in} = 0 + \frac{B}{50}(10)$$

$$R_{out} = \frac{A}{50} * 4$$

$$\frac{dA}{dt} = \frac{B}{50} - \frac{4A}{50}$$

- We, quite sadly, do not have enough information to solve here. The good news is, we have a second tank.

$$\frac{dB}{dt} = (\frac{4A}{50}) - \frac{3B}{50} - \frac{B}{50}$$

$$\frac{dB}{dt} = \frac{4A}{50} - \frac{4B}{50}$$

- Yet another wicket: We can't solve this yet! (I mean, I could probably fling darts and one of them would stick, but) We're going to be expected to set this up, not to solve.

AGAIN!

- Tank A has a capacity of 500 gal. At $t = 0$, it contains 100 gal of brine with 10 lb of salt.
- Tank A has an input of 6 gal/minute at a concentration of 4 lb/gallon, and an output of 1 gal/minute.
- Ok, so just kind of thinking about what's happening here, overall liquid level is increasing by five gallons / minute.
- When does the tank begin to overflow?

$$500 = 100 + 5t$$

- Which should just happen at $t = 80$, easy peasy as can be.

$$\frac{dA}{dt} = 24 - \frac{A}{100 + 5t}(1)$$
$$A(0) = 10$$

MATH225 - 2025-02-12

#notes #math225 #math

Housekeeping

- There are office hours Friday 1:15 to 3:5 in Engineering Annex, not the usual 3-4.
- Topic list coming out soon, previous exams are somewhere under Files.

The Great Lake of Acid

- If you end up doing the problem and getting something like

$$\frac{(2000)^{51}}{(2001)^{50}} = \frac{(2000)^{51} * 2001}{(2001)^{50} * 2001} = \left(\frac{2000}{2001}\right)^{51} * 2001$$

- You could also binomial expansion this if you're feeling quirked up, but that's a *lot* of work.

Some Review

Existence + Uniqueness

$$\frac{dy}{dx} = f(x,y) \quad y(x_0) = y_0$$

- This IVP is guaranteed a unique solution when both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous
- Section 1.2 Problems 25-28

$$\frac{dy}{dx} = (y^2 - 9)^{\frac{1}{2}}$$

$$f(x,y) = (y^2 - 9)^{\frac{1}{2}}$$

- This function is independent of x, so it'll be continuous for any value of x.
- f is defined when $y \geq 3$ or when $y \leq -3$. It'll be defined at $y = \pm 3$, but it definitely won't be continuous.

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 9}}$$

- This function $\frac{\partial f}{\partial y}$ is defined for $y > 3$ and $y < -3$
- The general takeaway is any x value, but y has to be greater than 3 or less than negative three.
    - Is there a unique solution at $(1,4)$? Yep, sure seems like it.
    - Is there a unique solution at $(5,3)$? Nope. The partial isn't continuous in there, so absolutely not.
    - Is there a unique solution at $(2,-3)$? Nope. The partial doesn't exist there, so absolutely not.
    - Is there a unique solution at $(-1,1)$? Nope. That shit's imaginary.

Worksheet 3, Problem 3D

$$\frac{dy}{dx} = y^2(4 - y^2) \qquad y(0) = 2$$

$$\frac{dy}{4y^2 - y^4} = dx$$

- What is the solution that goes through that initial condition (Note: you don't really have to solve anything here.)
- This differential equation has critical points at $y = 0, -2, 2$, you figure out if the slope is positive or negative in each of those regions, but remember, the critical points themselves are solutions! So we're just saying that $\frac{dy}{dx} = 0$, which works for literally any constant, but the IC was $y(0) = 2$ so $y = 2$ is the solution for $y = 2$. Shit crazy.
- The limit for that solution? 2. $y = 2$. Don't get futsy.

$$\int xe^x dx$$

- Spits us out an integration by parts

$$u = x, dv = e^x dx$$

$$\int udv$$

$$\int x \sin(x) dx$$

$$u = x, dv = \sin(x) dx$$

$$\int x \ln(x) dx$$

- You'd at least try parts

$$u = \ln(x), dv = xdx$$

$$\frac{dy}{dx} = y^2$$

$y = 0$, the critical point, is semi stable, because the solutions near it the bottom is approaching and top is goin away

MATH225 - 2025-02-24

#notes #math225 #math

Determining Linear Independence

- Or something.
- A set of functions $f_1(x), f_2(x)... f_n(x)$ is called linearly dependent if there exists constants $c_1, c_2, ... c_n$ (that are not all 0) such that $c_1 f_1(x) + c_2 f_2(x)... c_n f_n(x) = 0$ for every and any value of x.
- For two functions, f_1 and f_2, these are dependent if when added together, $c_1 f_1 + c_2 f_2 = 0$
- Which just means $c_1 f_1 = -c_2 f_2$, or, daresay, $f_1 = c_3 f_2$
  - For just two functions, dependence does in fact mean they're a constant multiple of each other.

The Wronskian

- The Wronskian, W, takes $W(F_1, F_{2,} ... F_n)$ (functions as an input)

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ .. & .. & .. \\ f_1^{n-1} & f_2^{n-1} & f_3^{n-1} \end{vmatrix}$$

- Holy shit, I finally rendered a determinant properly. Goooood lord.
  - We only really worry about 2x2 determinants here again.
- The set of functions $(f_1, f_2 ... f_n)$ is linearly independent if and only if the Wronskian is $\neq$ to 0 for every x in the interval we're considering.

Wronskian Examples

$$\text{Consider } W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^x e^{2x} - e^x e^{2x} = e^{3x} \neq$$

$$\text{Consider } W(\cos(x), \sin(x)) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1 \neq 0 \therefore \text{ Independent}$$

- Consider a homogeneous, linear, nth order ODE has a set of linearly independent solutions which can be written $y = c_1 y_1(x) + c_2 y_2(x) \ldots c_n y_n(x)$
- Example:

$$y'' - y' - 12y = 0$$

- Assert that e^{-3x} and e^{4x} solve this.
- $y_1 = e^{-3x}$, $y' = -3e^{-3x}$, $y'' = 9e^{-3x}$
- So we go and plug in, and we get $9e^{-3x} + 3e^{-3x} - 12e^{-3x} \stackrel{?}{=} 0$

- $0 = 0$, yep we're good.
- Alrighty, So that works as a solution, now let's Wronskian this bad boy

$$W(e^{-3x}, e^{4x}) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^{4x}e^{-3x} + 3e^{4x}e^{-3x} = 7e^{x} \neq 0$$

- Which then means these are independent, and our general solution is $y = c_1 e^{-3x} + c_2 e^{4x}$

Complex Numbers

- $i = \sqrt{-1}$
  - EE tend to use j since they use i for current, but we're going to use i
- A complex number generally has the form $z = \overbrace{a}^{\text{Re}} + \overbrace{bi}^{\text{Im}}$, which any complex number can be written as. $\text{Re}(z) = a, \ \text{Im}(z) = b$,
  - Look at that fancy overbrace! Isn't that *fucking awesome?* I love reading documentation.

Complex Conjugate

- Often $\bar{z}$, sometimes z^*
- If $z = a + bi, \bar{z} = a - bi$

Modulus

- Modulus of z is $|z| = \sqrt{z \cdot \bar{z}}$

Ex

- Consider $z = 1 + i$
- $\bar{z} = 1 - i, \ |z| = \sqrt{(1+i)(1-i)} = \sqrt{2}$

Euler's Formula

- $e^{ix} = \cos(x) + i\sin(x)$
- Fun fact: It's really easy to derive trig identities from here

$$(e^{ix})^2 = [\cos x + i \sin x]^2 = \cos^2 x - \sin^2 x + 2i \cos x \sin x$$

$$(e^{ix})^2 = e^{i(2x)} = \cos(2x) + i\sin(2x)$$

- We can pretty easily equate real and imaginary parts, ie,

$$\text{Real} : \cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\text{Imag} : \sin(2x) = 2\sin(x)\cos(x)$$

Author's Note

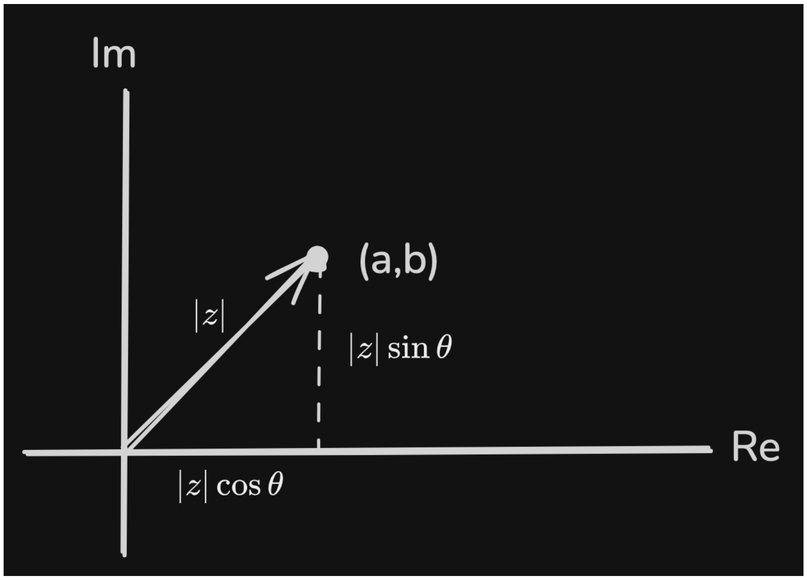
- During this note is where I swapped over from using Quick LaTeX to Latex Suite, which I had been meaning to do for a while. I might go back, I had it working pretty well, but I like the vibe of snippets and it's maintained.

MATH225 - 2025-02-26

#notes #math225 #math

Previously On

- We were dealing with complex numbers ($z = a + bi$), the conjugates of a complex number being $\bar{z} = a - bi$, the modulus of a complex number being $|z| = \sqrt{z * \bar{z}}$
- We can also write a complex number as the equivalent from $z = re^{i\theta}$
  - Note that this way also takes two constants to define, they're just r and θ instead of a, b
  - $z = re^{i\theta} = r\cos\theta + ir\sin\theta$
  - $\bar{z} = r\cos\theta - ir\sin\theta = re^{-i\theta}$
    - If you have $re^{-i\theta} = re^{i(-\theta)} = r(\cos(-\theta) + i\sin(-\theta)) = r(\cos\theta - i\sin(\theta))$
  - So if we use $z = re^{i\theta}$, we've just proven that its conjugate, $|z| = re^{-i\theta}$
- Fun fact, you can represent a complex number as a vector in two dimensions (isn't it neat how it also takes two number inputs, just like a 2-d vector)



- By and large differential equations isn't a vector analysis class, so we don't really care too much.
  - However, this shows up all the damn time in all kinds of other situations.

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

- We do this through the fact that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

$$z\cos\theta = e^{i\theta} + e^{-i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{z}$$

$$\sin\theta = \frac{1}{2i}[e^{i\theta} - e^{-i\theta}]$$

Quick Aside - Roots of i

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i = i$$

So $\sqrt{i} = i^{1/2} = (e^{i\,\pi/2})^{1/2} = e^{i\,\pi/4}$
Which is then $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

- To get the cube root of i, you end up with $\frac{\pi}{6}$, and have to do the same kind of calculations.

How to Solve Homogeneous Second Order Differential Equations with Constant Coefficients

$$\text{General form of: } ay'' + by' + cy = 0$$

- a, b, c are all just numbers, not functions or any such nonsense.
- We're going to be here a while.
  - Pops up all the time in Newton's Second Law (with acceleration/velocity/position), EE type shit with Kirchoff's Law, etc, etc
- At some point by which I mean yesterday, we wrote the equation $y'' - y' - 12y = 0$
  - And we asserted that e^{-3x} and e^{4x} are solutions, which we proved and got to work, and ended up with the general solution $y = c_1 e^{-3x} + c_2 e^{4x}$
- We investigate whether solutions can always be written as e^{mx}
  - If we assume the solution is $y = e^{mx}$, then $y' = me^{mx}$, and $y'' = m^2 e^{mx}$
  - Then we have $em^2 e^{mx} + bme^{mx} + ce^{mx} = 0$
    - Factor a smidge, you get $e^{mx}(am^2 + bm + c) = 0$
      - e^{mx} is never zero, so you can divide by it, so $am^2 + bm + c = 0$
      - This neat little equation we've made is called the auxiliary equation, and occasionally the characteristic equation
  - We then solve for m, then the solution to the DE is e^{mx}
- The characteristic equation can have
  1. Two real roots
  2. Repeated real roots
  3. Two complex conjugate roots

Alright, enough talk. Example time.

$$y'' - y' - 6y = 0$$

- That there is our differential equation.
- First up, write the characteristic equation, which will be $m^2 - m - 6 = 0$
  - $(m-3)(m+2) = 0$
  - $m = 3$ and $m = -2$ are the roots of this equation
  - Which sure look like two real, not repeated roots
- So the solution is

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

$$y'' - 10y' + 25y = 0$$

- Characteristic equation becomes $m^2 - 10m + 25 = 0$, or $(m-5)^2 = 0$, which is $m = 5$ as a repeated real root
- One solution is e^{5x}, however we need another linearly independent solution because of how second order works
- The second solution, however, is actually just xe^{5x}, so you're really just slapping an x on.

$$y = xe^{5x}, y' = 5xe^{5x} + e^{5x}, y'' = 25xe^{5x} + 5e^{5x} + 5e^{5x}$$
$$25xe^{5x} + 10e^{5x} - 50xe^{5x} - 10e^{5x} + 25xe^{5x} = 0$$

- That's a lotta e^{5x}

MATH225 - 2025-02-28

#notes #math225 #math

Still doing second order homogeneous

- Which were, to recap, of the form

$$ay'' + by' + cy = 0$$

- From the characteristic equation, we can end up with two real roots m_1 and m_2, which gives you the general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
  - You can also end up with one real (repeated) root, m_1, which pops out the general solution $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$
- We can also have complex conjugate roots!

Leading in with an example

$$y'' + 4y' + 7y = 0$$

- Soooo... we're going to use the characteristic equation.

$$m^2 + 4m + 7$$

- That looks rather the far side of factorable, so we're going to pop in ol tried and true, the quadratic formula

$$m = -\frac{4 \pm \sqrt{16-28}}{2} = \frac{-4 \pm \sqrt{-12}}{3} = -\frac{4 \pm \sqrt{-(4)(3)}}{2} = -\frac{-4 \pm 2\sqrt{-3}}{2} = -2 \pm i\sqrt{3}$$

- So our solution then becomes the properly ugly

$$y = c_1 e^{(-2+i\sqrt{3})x} + c_2 e^{(-2-i\sqrt{3})x}$$

- This is certainly an answer of all time. However, we won't write our answers with these complex shenanigans.

$$y = c_1 e^{-2x} e^{-\sqrt{3}x} + c_2 e^{-2x} e^{-i\sqrt{3}x}$$

- If you suppose the characteristic equation is $a \pm bi$, then

$$y = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}$$

- Since these two functions are linearly independent, sums and/or differences of these terms are also solutions to the differential equation.
  - For example, $y_1 + y_2$ is a solution that gives you $e^{(a+ib)x} + e^{(a-ib)x}$
    - That's, believe it or not, a form of cosine. Go tap ol' Euler in [Year 2/Semester 2/MATH225/MATH225 - 2025-02-24]()
      - You get things like $\cos(x) = \frac{e^{-ix}+e^{ix}}{2}$
    - So this becomes $2e^{ax}(\cos(bx))$
  - So if we then have $e^{ax}[e^{ibx} - e^{-ibx}]$
    - and then $\sin(x) = \frac{1}{2i}[e^{ix} - e^{-ix}]$
    - Which then gives you $2ie^{ax}\sin(bx)$
- So then combing together the general solution is going to be

$$y = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$$

- Complex conjugate roots $m = a \pm ib$ will be written as

$$y = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$$

Brief Aside

- We can also write the solution as $y = c_1 e^{ax}\cos(bx - \phi)$
  - This is just trig shenanigans because sine is just cosine with a phase shift, so this is collapsing it down to c_1 and ϕ as our constants.

More Example

$$\frac{d^2y}{d\theta^2} + y = 0$$

$$y\left(\frac{\pi}{3}\right) = 0, y'\left(\frac{\pi}{3}\right) = 2$$

- Character equation is going to be $m^2 + 1 = 0 \implies m = \pm i$
  - Boy howdy, that sure looks like a complex conjugate root.
  - These roots have no real part, or $a = 0$

$$y = c_1\cos(\theta) + c_2\sin(\theta)$$

$$y\left(\frac{\pi}{3}\right) = 0 = c_1\cos\left(\frac{\pi}{3}\right) + c_2\sin\left(\frac{\pi}{3}\right)$$

$$0 = \frac{1}{2}c_1 + c_2\frac{\sqrt{3}}{2}$$

$$y' = -c_1\sin\theta + c_2\cos(\theta)$$

$$y'\left(\frac{\pi}{3}\right) = c_1\sin\left(\frac{\pi}{3}\right) + c_2\cos\left(\frac{\pi}{3}\right)$$

$$2 = \frac{c_1\sqrt{B}}{2} + \frac{1}{2}c_2$$

$$0 = \frac{1}{2}c_1 + c_2\frac{\sqrt{3}}{2}$$

$$2 = -\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2$$

- Do a little simplifying

$$0 = c_1 + \sqrt{3}c_2$$

$$4 = -\sqrt{3}c_1 + c_2$$

$$c_1 = -\sqrt{3}\,c_2$$

$$4 = 3c_2 + c_2, 4 = 4c_2, \boxed{c_2 = 1}$$

$$\boxed{c_1 = -\sqrt{3}}$$

- So the solution to the IVP is going to be $y = -\sqrt{3}\cos(\theta) + \sin(\theta)$

- Fun aside, we can use the characteristic equation with higher order ODES, but you need to solve cubics or quartics, which, been a minute.

MATH225 - 2025-03-03

#notes #math225 #math

In homogenous equations,

$$a_2y'' + a_1' + a_0y = f(x)$$

- We will limit $f(x)$ to be what MUC works with, so exponentials, polynomials, sines and cosines, and combinations of those
- The general solution is

  $$y = y_c(x) + y_p(x)$$

  Test dummy line

  $$y = \underbrace{c_1y_1(x) + c_2y_2(x)}_{\text{Complementary term}} + y_p(x)$$

Example already!

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$

- First things first, ~~I'm gonna eat your brains~~ solve the homogeneous equation

$$y'' + 4y' - 2y = 0$$

$$m^2 + 4m - 2 = 0$$

$$m = -\frac{4 \pm \sqrt{16+8}}{2} = -\frac{4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$

$$y_c = c_1e^{(-2+\sqrt{6})x} + c_2e^{(-2-\sqrt{6})x}$$

- Now for the fun part (the fart, if you will) - we gotta actually do muc$_{\text{which I really don't like}}$ for the particular solution.

$$y_p = Ax^2 + Bx + C$$

- We, also, remember, need to check if our guess occurs in y_c

- We plug the guess into the ODE to get equations for our coefficients A,B, and C.

$$y_p' = 2Ax + B, y_p'' = 2A$$

- So you plug back in, aaaand you get

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

$$2A + 8Ax + 4B - 2Ax^2 + 2Bx + 2C = 2x^2 - 3x + 6$$

$$-2A = 2, A = -1$$

$$8A + 2B = -3$$

$$-8 - 2B = -3, \quad 2B = 5, B = -2.5$$

$$2A + 4B + 2C = 6$$

$$-2 - 10 + 2C = 6, -18 = 2C, C = -9$$

- Having determined the undetermined coefficients, we have

$$y = y_c + y_p = c_1 e^{(-2+\sqrt{6}))x} + c_2 e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$$

And now for something completely different (oscillations)

- Spring Force, governed by Hooke's Law, says that the restoring force is proportional to the displacement and some k constant. Or, in short,

$$F = -kx$$

- If you hang a spring from the ceiling, it has where it wants to be (distance ℓ), attach another weight, there'll be some new distance (distance s)
    - That new distance is in equilibrium, or balanced, or not moving, etc, where $mg = ks$
- If you then stretch the spring some distance x from that position, we want to know what the motion of the mass looks like.
- In this particular case, it's given by Newton's Second Law, which says

$m\ddot{x} = -kx - \overbrace{ks + mg}^{\text{This is zero.}}$

    - Motion only depends on how far it's stretched from the new equilibrium position.
- We now have $m\ddot{x} + kx = 0$
  - Spring constant and mass are both constants, so this is just a second order, homogeneous differential equation.
  - This is what the characteristic equation is *built* for.
  - On that note - we're going to use r for the characteristic equation while we're in spring

land, because, y'know, m is a character we're already using for mass.
- Strictly speaking, $\ddot{x} + \frac{k}{m}x = 0$

$$r^2 + \frac{k}{m} = 0$$

- This has roots of $r = \pm\sqrt{\frac{k}{m}}i$
  - Those sure look like complex conjugates to me.
- Our solution then becomes

$$x = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

- We define $\omega_o = \sqrt{\frac{k}{m}}$
  - Aaaaaand we getting into what that actually means later.

---

unrelated to diffEq toDo - make a new snippet for pm instead of +-

MATH225 - 2025-03-05

#notes #math225 #math

Previously talking about Simple Harmonic Motion

- Also known as free undamped motion
  - Essentially, there's no driving force
    - The alternative being driven motion, which we'll get to soon™
  - Undamped means there's no friction, resistance, etc.
    - No term with $-\dot{x}$, which is the next thing we're going to deal with.
- Our simple equation was $\ddot{x} + \frac{k}{m}x = 0$
  - Which then had the simple answer of $x(t) = c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t)$
    - $\omega_o = \sqrt{\frac{k}{m}} \equiv$ natural angular frequency
      - Units of rad/s, just for record's sake
        - Radians are really unitless if you get into it, so this is really just $\frac{1}{s}$ or s^{-1}

- $\nu = \frac{\omega}{2\pi}$ $\equiv$frequency, which like really really has units of $\frac{1}{s}$, or... {physics drumroll} Hertz
- $P|T = \frac{1}{\nu} \equiv$ Period, which is $\frac{1}{\text{frequency}}$, which is the time it takes to undergo one full cycle.

text example 5.1.3

- A mass weighing 24lb hanging from a spring and stretches it 4in. If it is released from a point 3in above the equilibrium position, what is the equation of motion?
  - ie, what is $x(t)$
  - We were in fact given a force of 24lb, so we gotta use the conversion $F = mg$ where $g = 32\ \text{ft} \cdot \text{s}^{-2}$
  - Our whole book apparently uses english units, which is awful.
  - $m = \frac{24\text{lb}}{32\ \text{ft}\cdot\text{s}^{-2}} = \frac{3}{4}\text{lb} * \text{ft}^{-1} \cdot \text{s}^2 = 2\ \text{slug}$
  - We also need the spring constant, which we can derive from Hooke's Law, and pops out to be $72\ \text{lb} \cdot \text{ft}^{-1}$
  - So, plugging all of our solving in

$$x(t) = c_1 \cos(\omega_o t) + c_2 \sin(\omega_o t)$$

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{72}{\frac{3}{4}}} = \sqrt{96}\ \text{s}^{-1}$$

$$= c_1 \cos(\sqrt{96}t) + c_2 \sin(\sqrt{96}t)$$

- Now, we were given some initial conditions, noticeably that we start at rest 3in above equilibrium
  - Generally the way the book and such writes solutions is such that down is positive.
  - This tells us that $\dot{x}(0) = 0$, and that $x(0) = -3in = -\frac{1}{4}ft$
  - After plugging these in, we get $x(t) = -\frac{1}{4}ft \cos(\sqrt{96}t)$

Damped Motion

- Now we're dunking our poor spring into a pool of viscous liquid, and it's going to try and oscillate and have to move fluid out of the way.
- So now we'll include a damping term which is proportional to velocity and opposes the motion.

$$m\ddot{x} = \overbrace{-kx}^{\text{Hooke's Law}} - \underbrace{\beta}_{\text{Resistance}} \dot{x}$$

```
- That $\beta$ resistance constant is constant for a given object in a given medium, particularly in this not real physics class. It's dependent on like, geometry, viscosity, lots of things we don't care about
```

$$m\ddot{x} + \beta\dot{x} + kx = 0$$

$$\ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = 0$$

- Soooo, we characteristic equation that bad boy

$$r^2 + \frac{\beta}{m}r + \frac{k}{m} = 0$$

$$r = \frac{-\frac{\beta}{m} \pm \sqrt{\frac{\beta^2}{m^2} - \frac{4k}{m}}}{2}$$

$$r = -\frac{\frac{\beta}{m} \pm \sqrt{\frac{\beta^2 - 4km}{m^2}}}{2}$$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

- Dynamics will depend on the sign of $\beta^2 - 4km$
  - We've got a couple of cases for that, actually
    1. If $\beta^2 - 4km > 0$ The system is overdamped
       - This means we have two real, distinct roots
       - The mass returns smoothly and exponentially to equilibrium without oscillating
    2. If $\beta^2 - 4km = 0$, The system is critically damped
       - This has a repeated real root
         - General solution of the form $x(t) = c_1e^{(-\beta/2m)t} + c_2te^{-(\beta/2m)t}$
       - Also doesn't oscillate (those are just decaying exponentials sitting up there), but this gets us back to equilibrium real quick.
       - The graphs of case 1 and case 2 look pretty darn similar.
    3. $\beta^2 - 4km < 0$ is underdamped
       - This one then looks like $x(t) = e^{(-\beta/2m)t}[c_1\cos(\sqrt{4km - \beta^2}t) + c_2\sin(\sqrt{4km - \beta^2}t)]$

MATH225 - 2025-03-10

#notes #math225 #math

Driven Oscillations

- Driving with the sinusoid $\cos(\omega t)$
  - First case we looked at was where $\omega \neq \omega_0$ where driving frequency ω is not equal to natural frequency ω_0
  - Which gives you $m\ddot{x} + kx = F\cos(\omega t)$
    - $\omega_0 = \sqrt{\frac{k}{m}}$
    - We have the homogeneous solution $x_c(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$
    - $x_p = A\sin(\omega t) + B\cos(\omega t)$
- We're going to worry about damping later
- Other case is $\omega = \omega_0$, which is when the driving frequency is exactly the same as the natural frequency
  - Which then pops out $m\ddot{x} + kx = F\cos(\omega_0 t)$
  - Complementary solution is the same $x_c(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$
  - For particular, we start with the same guess, just with $x_p(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$, which rapidly raises the problem of our guess existing in the other solution
    - So, you slap a t on, you get $x_p(t) = At\cos(\omega_0 t) + B\sin(\omega_0 t)$
      - That t means we're going to grow with time in resonance, so we increase with time without bound

Now we bring in that damn Frenchman (Laplace)

- One of the great big benefits of Laplace is the ability to deal with systems that aren't persistent for all infinity, that you turn on and off - so it pops up all the damn time in electronics

Laplace Transforms

- Are a kind of integral transform
- The (Laplace) transform of a function $f(t)$ is given by:

$$F(s) = \int_0^\infty e^{-st} f(t)dt \text{ for } t \geq 0$$

- We can represent the Laplace transform as $\mathcal{L}\{f(t)\}$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_{t=0}^{t=\infty} = \frac{1}{s}$$

- You get this table on the test and on the final, but be familiar with it

Differential Equations MATH 225
TABLE OF LAPLACE TRANSFORMS

No.	Function $f(t)$	Laplace Transform $F(s)$	No.	Function $f(t)$	Laplace Transform $F(s)$
1.	$f'(t)$	$sF(s)-f(0)$	12.	e^{at}	$\frac{1}{s-a}$
2.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	13.	$\cos kt$	$\frac{s}{s^2+k^2}$
3.	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	14.	$\sin kt$	$\frac{k}{s^2+k^2}$
4.	$e^{at}f(t)$	$F(s-a)$	15.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
5.	$\mathcal{U}(t-a)f(t-a)$	$e^{-as}F(s)$	16.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
6.	f(t)*g(t) $= \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	17.	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
7.	1	$\frac{1}{s}$	18.	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
8.	t	$\frac{1}{s^2}$	19.	$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
9.	t^n	$\frac{n!}{s^{n+1}}$	20.	$\delta(t-t_0)$	$e^{-s\,t_0}$
10.	$\cosh kt$	$\frac{s}{s^2-k^2}$	21.	$\sinh kt$	$\frac{k}{s^2-k^2}$
11.	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$	22.	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$

Four useful formulas

$$\int_0^\infty \delta(t-t_0)f(t)\,dt = f(t_0)$$

$$\int_0^t \delta(\tau-t_0)g(t-\tau)\,d\tau = \mathcal{U}(t-t_0)g(t-t_0)$$

$$F(s)=\mathcal{L}\{f(t)\} = \frac{1}{1-\mathrm{e}^{-sT}}\int_0^T \mathrm{e}^{-st}f(t)dt \quad \text{if } f(t+T)=f(t)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\} \text{ (Alternative Form of 5.)}$$

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = -\frac{1}{s}te^{-st}\Big|_{t=0}^{t=\infty} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=\infty}$$

- We do this integral with integration by parts, just for record's sake.
- $\lim_{t\to\infty} te^{-st} = 0$ by L'Hôpital's rule (mmm. Hospital.)
- $\lim_{t\to\infty} e^{-st} = 0$, so we eventually end up with the transform being $\frac{1}{s^2}$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

- Funny bit about Laplace transforms - they're a linear operation
- The fuck does that mean? So glad you asked.
  - Suppose we have a number a, and we look at the transformation $\mathcal{L}\{af(t)\}$
  - Going to look like $\int_0^\infty a \cdot f(t) \cdot e^{-st} dt$
    - a is just some number, so we can pop that out, so we have $a \int_0^\infty f(t) \cdot e^{-st} dt$
    - This now just looks like $a * \mathcal{L}\{f(t)\}$
      - This is like, half of what we mean by linearity.
- For instance, our good friend formula seven, $\mathcal{L}\{1\} = \frac{1}{s}$
  - We can apply this linearity when we do something like $\mathcal{L}\{2\}$, which then just becomes $2 * \mathcal{L}\{1\} = \frac{2}{s}$
- Formula 8 told us that $\mathcal{L}\{t\} = \frac{1}{s^2}$, so $\mathcal{L}\{7t\} = \frac{7}{s^2}$

- Note for the road

$$\mathcal{L}\{f(t) + g(t)\} = \int_0^\infty e^{-st}[f(t) + g(t)]dt$$

$$= \int_0^\infty e^{-st} f(t) dt + \int_0^\infty g(t) e^{-st} dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

MATH225 - 2025-03-12

#notes #math225 #math

Still in Laplace land

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$$

- Is what we were doing last time, generally covering linearity and such where constants pop out and you can add functions together in integral shenanigans.

Shift

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt$$

$$= \int_0^\infty e^{(a-s)t}f(t)dt$$

$$w = s-a, dw = ds$$

$$= \int_0^\infty e^{-wt}f(t)dt$$

- That sure seems to just deadass look like the definition of a Laplace transform, where we use w instead of s

$$\int_0^\infty e^{-wt}f(t)dt = F(w) = F(s-a)$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

- This is one of two shift properties of (Laplace, but also Fourier) integral transforms

Inverse Transform

- if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

- We go look at the formula sheet, so the inverse of $\frac{1}{s^2}$ is t

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

$$e^{at} \overset{\mathcal{L}\{\}}{\Longrightarrow} = \frac{1}{s-a}$$

- Sometimes, you unfortunately have to do some algebra to get our function into a form on the sheet
  - Sometimes it's really simple and you just multiply and it's great, but otherwise...
  1. Sometimes you have to do partial fractions
  2. Sometimes you must complete the square

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s+1)}\right\}$$

$$\frac{s}{(s+3)(s-1)} = \frac{A}{(s+3)} + \frac{B}{(s-1)}$$

$$As - A + Bs + 3B = s$$

$$A + B = 1, -A + 3B = 0$$

$$B = \frac{1}{4}, A = \frac{3}{4}$$

$$\frac{3}{4} * \frac{1}{s+3} + \frac{1}{4} * \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\{\} = \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9} - \frac{6}{s^2+9}\right\}$$

- We've got a couple interesting formulas, notably 13, which has $\cos kt \implies \frac{s}{s^2+k^2}$, and 14 has $\sin kt \implies \frac{k}{s^2+k^2}$

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = 2\cos(3t)$$

$$\mathcal{L}^{-1}\left\{-\frac{6}{s^2+9}\right\} = -6\mathcal{L}^{-1}\left\{\frac{1}{s^2+9} * \frac{3}{3}\right\} = -\frac{6}{3}\{\} = -2\left\{\frac{3}{s^2+9}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2\cos(3t) - 2\sin(3t)$$

$$\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2-6s+18}\right\}$$

- None of the really obvious options are options here, so we're completing the square in the denominator (divide middle by 2 and square it)

$$\frac{3s+2}{s^2-6s+9-9+18}$$

$$\frac{3s+2}{(s-3)^2+9}$$

- Some fun formulae from the sheet:
  17. $e^{at}\cos kt \implies \frac{s-a}{(s-a)^2+k^2}$
  18. $e^{at}\sin kt \implies \frac{k}{(s-a)^2+k^2}$

$$3\left[\frac{s}{(s-3)^2+9}\right]+\frac{2}{(s-3)^2+9}=3\left[\frac{s}{(s-3)^2+9}-\frac{3}{(s-3)^2+9}+\frac{3}{(s-3)^2+9}\right]+\frac{}{(s -}$$

$$3\left[\frac{s-3}{(s-3)^2+9}\right]+\frac{1}{(s-3)^2+9}*\left(\frac{3}{3)}\right)=3e^{3t}\cos(3t)+\frac{11}{3}e^{3t}\sin(3t)$$

MATH225 - 2025-03-14

#notes #math225 #math

Laplace Transform of a Derivative

$$\mathcal{L}\{f'(t)\}=\int_0^\infty f'(t)e^{-st}dt$$

$$\text{Consider}\frac{d}{dt\{e^{-st}f(t)\}}=e^{-st}f'(t)+f(t)-se^{-st}f'(t)$$

$$f'(t)e^{-st}=\frac{d}{dt}\{e^{-st}f(t)\}+se^{-st}f(t)$$

$$\int_0^\infty\frac{d}{dt}\{e^{-st}f(t)\}dt+\int_0^\infty se^{-st}f(t)$$

$$e^{-st}f(t)\Big|_0^\infty+s\int_0^\infty f(t)e^{-st}dt$$

- We impose a condition that $\lim_{t\to\infty}f(t)e^{-st}=0$
  - This is a condition we're putting on $f(t)$, which isn't a particularly strong condition and is generally how most things work

$$-f(0)+sF(s)=\mathcal{L}\{f'(t)\}$$

- Y'know, while we're in derivative land, we might as well go pop a peek at what's going on with the second derivative
  - We can use this formula to calculate $\mathcal{L}\{f''\}$
- We're going to let $g=f'$

$$\mathcal{L}\{f''\}=\mathcal{L}\{g'\}=sG(s)-g(0)=sG-f'(0)$$

$$\mathcal{L}\{g\}=G=\mathcal{L}\{f'\}=-f(0)+sF(s)$$

$$\mathcal{L}\{f''\} = -sf(0) + s^2F(s) - f'(0)$$

- Oh hey howdy, there's formula two.

Where are we going!

- So glad you asked. We can use Laplace to solve IVPs. Let's do that, like, now.

$$y' - 5y = 0, \quad y(0) = 2$$

- Essentially what you do is the laplace transform of both sides

$$\mathcal{L}\{y' - 5y\} = \mathcal{L}\{0\}$$

- We use y for the time domain, and for laplace use Y, aka $y \stackrel{\mathcal{L}}{=} Y$
- Transforms are linear, so we do

$$\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

- Just for records sake, $\mathcal{L}\{0\} = 0$

$$sY - y(0) - 5Y = 0$$

$$y(0) = 2$$

$$Y(s-5) - 2 = 0$$

- Oh hey, now we have algebra to get Y

$$Y = \frac{2}{s-5}$$

- Quite unfortunately, this is the answer in terms of the Laplace domain, not the time domain

$$\mathcal{L}^{-1}\left\{Y = \frac{2}{s-5}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-5}\right\}$$

$$y(t) = 2e^{5t}$$

Anotha one

$$y'' - 6y' + 9y = t, \quad y(0) = 0, y'(0) = 1$$

- Transform the equation time

$$\mathcal{L}\{f''\} = s^2F - sf(0) - f'(0)$$
$$\mathcal{L}\{f'\} = sF - f(0)$$

- (equations one and two off the sheet, to recap)

$$\overbrace{s^2Y - sy(0) - y'(0)}^{\text{Eq 2}} - \underbrace{6[sY - y(0)]}_{\text{Eq 1}} + 9Y = \overbrace{\frac{1}{s^2}U}^{\text{Eq 9}}$$

$$s^2Y - 0 - 1 - 6[sY - 0] + 9Y = \frac{1}{s^2}$$

$$s^2Y - 1 - 6sY + 9Y = \frac{1}{s^2}$$

$$Y(s^2 - 6s + 9) = \frac{1}{s^2} + 1$$

$$Y(s-3)^2 = \frac{1}{s^2} + 1$$

$$Y = \frac{s^2+1}{s^2(s-3)^2}$$

- That's our answer in Laplace, now we need to pip pop our way back into time
  - Yeaaaaaah we're going to need to do partial fractions. I hate it here.

$$\frac{s^2+1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$
$$(s^2+1) = A(s)(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + D(s^2)$$
$$(s^2+1) = As^3 - 6As^2 + 9As + Bs^2 - 6Bs + 9B + Cs^3 - 3Cs^2 + Ds^2$$

$$A + C = 0$$
$$-6A + B - 3C + D = 1$$
$$9A - 6B = 0$$
$$9B = 1$$

- Is all those equations extracted

$$A = \frac{6}{8}, B = \frac{1}{9}, C = -\frac{6}{81}, D = \frac{90}{81}$$

$$\frac{6}{81} * \frac{1}{s} + \frac{1}{9} * \frac{1}{s^2} - \frac{6}{81} * \frac{1}{s-3} + \frac{90}{81} * \frac{1}{(s-3)^2}$$

- Ok now do the inverse laplace transform of like all of those. Dance, monkey, dance.

$$y(t) = \frac{6}{81}(1) + \frac{1}{9}t - \frac{6}{81}e^{3t} + \frac{90}{81}te^{3t}$$

MATH225 - 2025-03-24

#notes #math225 #math

Transforming Differential Equations

$$\ddot{y} + y = \sin(2t)$$

$$y(0) = \dot{y}(0) = 0$$

- Transform boths ides of this equation

$$s^2Y - sy(0) - y'(0) + Y = \frac{2}{s^2+4}$$

$$Y(s^2+1) = \frac{2}{s^2+4}$$

- Which then pops into

$$Y = \frac{2}{(s^2+1)(s^2+4)}$$

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$A = 0, C = 0, B = \frac{2}{3}, D = -\frac{2}{3}$$

$$Y = \frac{2}{3} * \frac{1}{s^2+1} - \frac{2}{3} * \frac{1}{s^2+4}$$

$$\sin(kt) \overset{\mathcal{L}}{\Longrightarrow} \frac{k}{s^2+k^2}$$

$$y(t) = \frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)$$

- On the next homework, hyperbolic trig functions pop up, like $\sinh$ and $\cosh$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

- This really deeply doesn't matter, but they are on the sheet and are therefore relevant. Have fun?

$$\mathcal{L}\{t\cos(2t)\}$$

$$t^n f(t) \overset{\mathcal{L}}{\Longrightarrow} (-1)^n F^{(n)}(s)$$

- In this case, $n = 1$ and $f(t) = \cos(2t)$

$$F(s) = \frac{s}{s^2+4} = s(s^2+4)^{-1}$$

$$F^{(1)}(s) = F' = -\frac{2s^2}{(s^2+4)^2} + \frac{1}{s^2+4} = \frac{-2s+s^2+4}{(s^2+4)^2}$$

$$\frac{s^2-4}{(s^2+4)^2}$$

- Is our final answer, honestly I was tuned out working on my schedule. Which, by the by, shit's fucked.

And now, for something completely different (Piecewise Functions)

- Voltage in a circuit as a function of time

$$v(t) = \begin{cases} 0 & : -\infty \leq t \leq 0 \\ 1 & : t > a \end{cases}$$

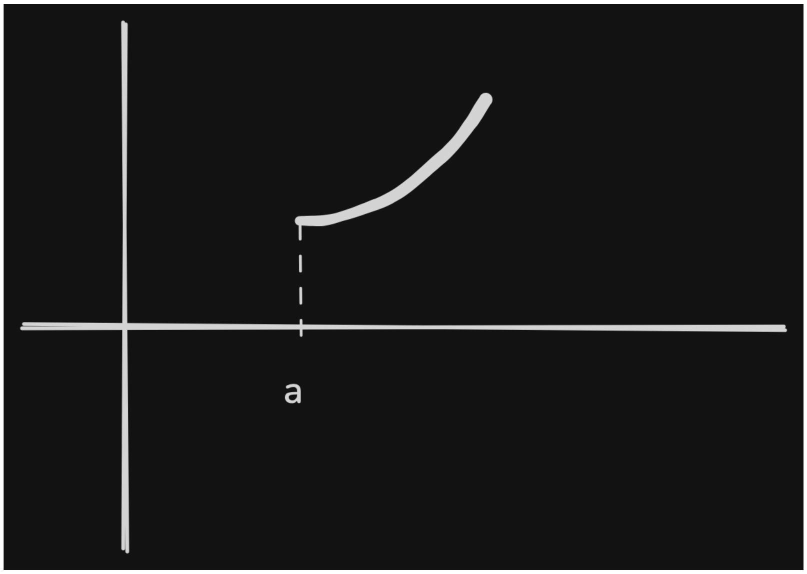
- This comes up so often that our boy Heaviside (famous for covering things up) has a Step Function

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$$

$$u(t-a) = \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ 1 & \text{if } t > a \end{cases}$$

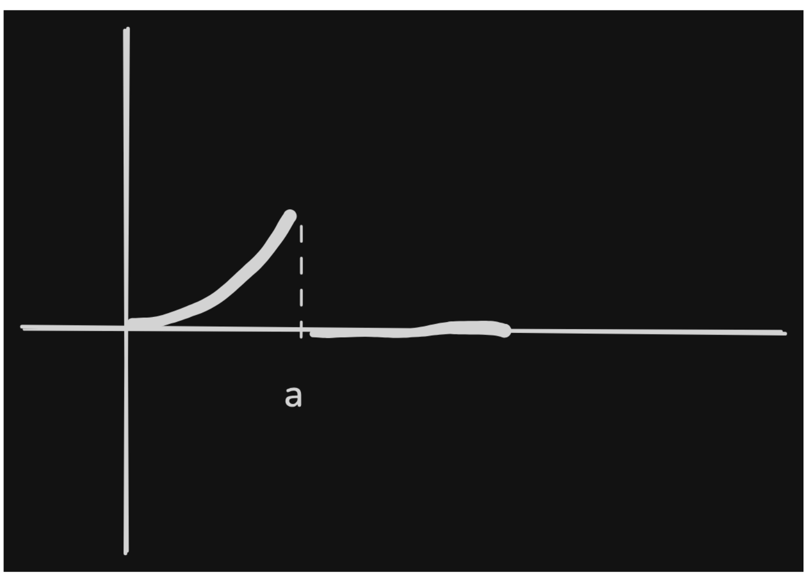
Very Important Thing™

- So we have some function t^2, and we already know that $u(t-a)$ is just a shifted step function
- $t^2 * u(t-a)$



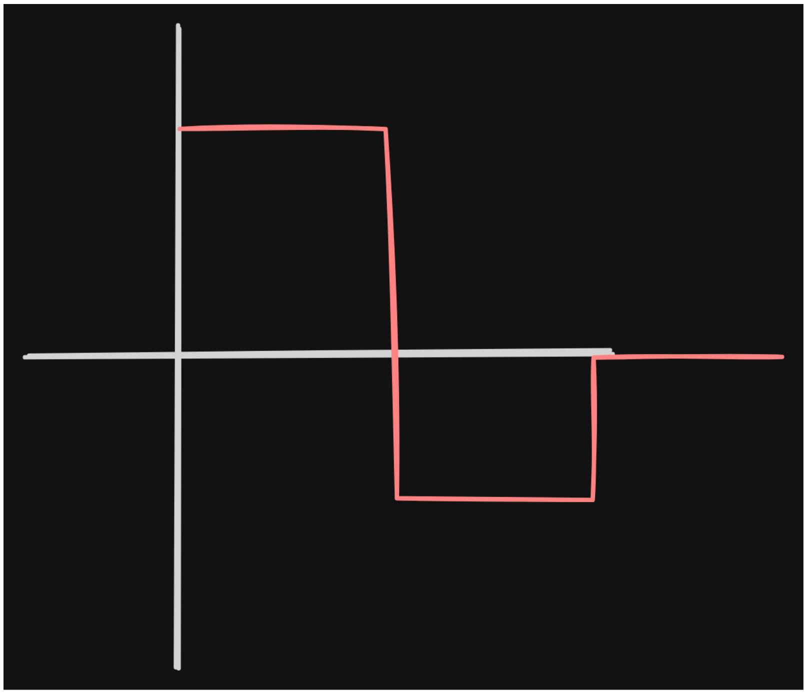
- Function essentially starts to exist at a, since that's where the Heaviside function flips from 1 to 0

$$f(t) = t^2 - t^2 * u$$



- Function essentially gets flipped off at a

$$f(t) = \begin{cases} 2 & 0 \leq t \leq_2 \\ -1 & 2 < t \leq_3 \\ 0 & t > 3 \end{cases}$$



$$f(t) = 2 - 3u(t-2) + u(t-3)$$

$$f(t) = \begin{cases} t^2 & 0 \leq t \leq 2 \\ t+2 & t > 2 \end{cases}$$

$$f(t) = t^2 - t^2 * u(t-2) + (t+2)u(t-2)$$

$$\mathcal{L}\{u(t-a) * f(t-a)\} = e^{-as}F(s)$$

- Which is nice if you happen to have a function in the form of $f(t-a)$, which we often don't, sooo

$$\mathcal{L}\{g(t) * u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

---

example

$$\mathcal{L}\{u(t-2) * \sin(t-2)\}$$

- This has $f(t) = \sin(t)$ and $f(t-2) = \sin(t-2)$
- Using our sheet to do some transforms

$$F(s) = \frac{1}{s^2+1}$$

- Now using formula 5 (alternate)

$$\frac{e^{-2s} * 1}{s^2 + 1}$$

MATH225 - 2025-03-26

#notes #math225 #math

$$f(t) = \begin{cases} t & 0 \leq t \leq 2 \\ 0 & t > 2 \end{cases} \implies f(t) = t - t * u(t-2)$$

- Again, the whole reason we're doing this is to do Laplace transforms, sooo

$$\mathcal{L}\{u(t-a) * f(t-a)\} = e^{-as}F(s)$$

- That's nice, but the alternate form of

$$\mathcal{L}\{g(t) * u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$\mathcal{L}\{u(t-2) * \sin(t-2)\}$$

- This just looks like formula 5 with $a = 2$ and $f(t) = \sin(t)$
- if $f(t) = \sin(t)$ then $F(s) = \frac{1}{s^2+1}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$$

- This sure looks like 5 with $a = 2$ and $F(s) = \frac{1}{s^3}$
- 5: $u(t-a)f(t-a) \stackrel{\mathcal{L}}{=} e^{-as}F(s)$
- If $F(s) = \frac{1}{s^3}$ then $f(t)$
- 9: $t^n \implies \frac{n!}{s^{n+1}}$ or $t^2 \implies \frac{2!}{s^3}$

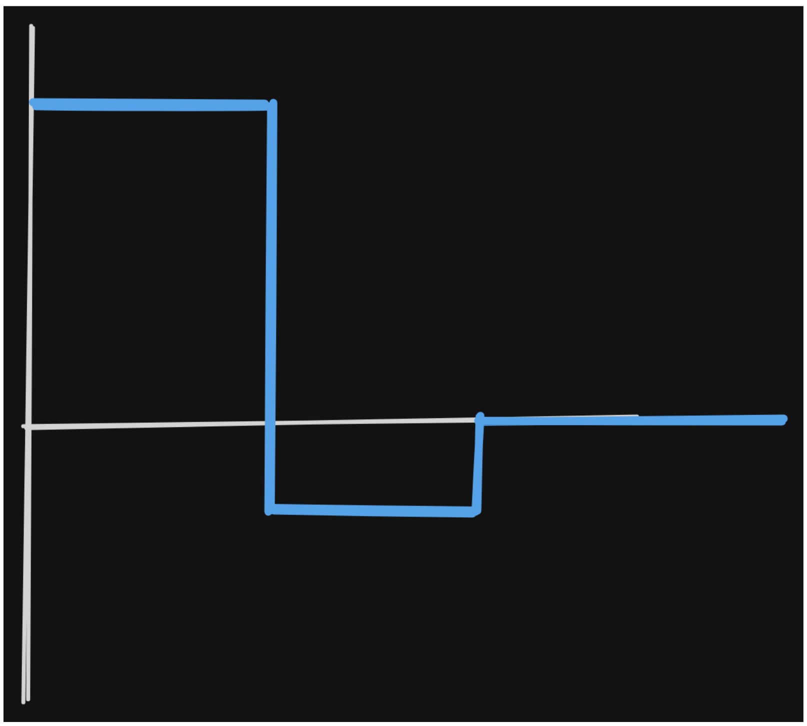
$$u(t-2) * \frac{1}{2}(t-2)^2$$

$$\mathcal{L}^{-1}\left\{e^{-s} * \frac{1}{(s+3)^2}\right\}$$

- This is, believe it or not, five again.
  - $a = 1, F(s) = \frac{1}{(s+3)^2}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

- We're going to pop formula sixteen in, which is that $t^n e^{at} \implies \frac{n!}{(s-a)^{n+1}}$
  - So therefore $te^{-3t} = \frac{1}{(s+3)^2}$
- Oookie dokies, so using 5 we then get
  - $u(t-1) * (t-)e^{-3(t-1)}$



- If we want to find $\mathcal{L}\{f(t)\}$ of that, first we need to write out function out

$$f(t) = \underbrace{2}_{7} - \underbrace{3u(t-2)}_{19} + \underbrace{u(t-3)}_{19}$$

$$y' + y = f(t), \;\; y(0) = 0$$

$$f(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ 5 & t > 1 \end{cases} = 5u(t-1)$$

- Get Laplace-y with it

$$\overbrace{sY - y(0)}^{F1} + Y = \overbrace{\frac{5 * e^{-s}}{s}}^{F19}$$

$$Y = \frac{5e^{-s}}{s(s+1)}$$

- That's an answer... which is unfortunately stuck in the wrong domain. Let's go transform it back

$$y(t) = \mathcal{L}^{-1}\left\{\frac{5e^{-s}}{s(s+1)}\right\}$$

$$\frac{5}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$As + A + B = 5$$

$$A = 5, B = -5$$

$$\mathcal{L}^{-1}\left\{e^{-s}\left[\frac{5}{s} - \frac{5}{s+1}\right]\right\}$$

- We pop equation 5 back in

$$\mathcal{L}^{-1}\left\{\frac{5e^{-s}}{s}\right\} = 5u(t-1)$$

- Believe it or not, we're using equation 5 again with a=1 and $F(s) = \frac{1}{s+1}$ which we then transform with 12 into e^{-t}

$$\mathcal{L}^{-1}\left\{-\frac{5e^{-s}}{s+1}\right\} = -5u(t-1)e^{-(t-1)}$$

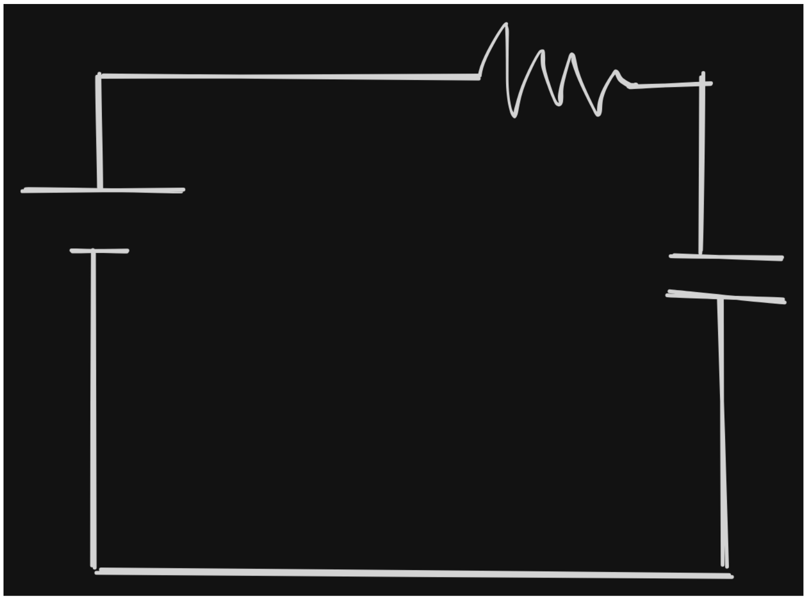
- So our final answer becomes

$$5u(t-1) - 5u(t-1)e^{-(t-1)}$$

$$y'' + 16y = 0$$

MATH225 - 2025-03-28

#notes #math225 #math



- Quick circuit.

$$\overbrace{\epsilon}^{Voltage} = \overbrace{R}^{\text{Resistance}} \frac{dq}{dt} + \frac{1}{\underbrace{c}_{\text{Capacitance}}} \overbrace{q}^{\text{Charge}}$$

- We can solve this ODE relatively normally, and we end up with

$$q(t) = c\epsilon(1 - e^{-t/RC})$$

- Let's suppose we have the switch as closed, and then we open it and turn it off. (ie, heaviside step function shenanigans)

So then

$$\epsilon = 1 - u(t - t_c)$$

- We can solve this ode with $\mathcal{L}\{\}$ tranasform if we let R=1 megaohm, and C = 1 microfarad, and RC = 1 sec

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{1}{R}[1 - u(t - tc)]$$

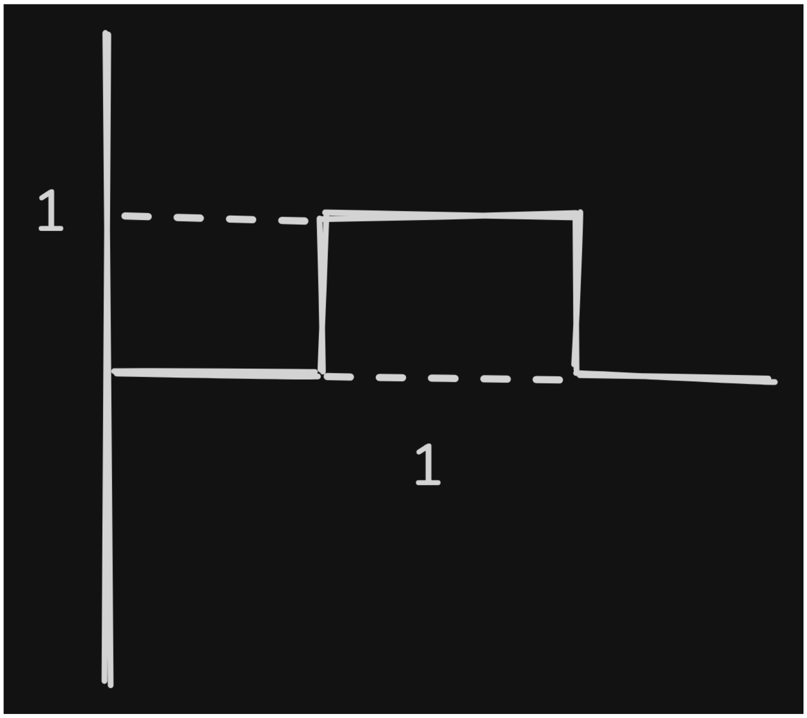
- Now for the actual Laplace step now that we've replaced everything

$$Y(s+1) = \frac{1e-6}{s}[1 - e^{-st_c}]$$

- This is using $\mathcal{L}\{q\} = Y$

$$q(t) = 1e^{-6}[1 - e^{-t} - u(t - t_c) + u(t - t_c)e^{-(t-t_c)}]$$

"The Last Possible Topic that can be on the exam"

- 

- This is called a "Unity Impulse," because $\int_{-\infty}^{\infty} f(t)dt = 1$
- Ok make this thing comically sharp. Probably d/dx sharp if I were to guess.
  - Either way, you make it sharper, and your integrals are always still one.
- In order to go infinitesimally small, we need a new function

Dirac Delta Function (δ)

- Is, strictly speaking, the limit of that other shenanigans.

$$\delta(x) = 0 \quad \text{if} \; \neq 0$$

- At $x = 0$, it's an infinitesimally wide spark.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

- (Still a unity impulse)
- Amusing aside:
  - Dirac didn't actually prove this, he just started using it and its properties

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

- We use δ-fn to represent a sharp impulse

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

Example

$$\underbrace{y'' + \omega_0^2 y}_{\text{Homogeneous for undamped oscillator}} = \overbrace{\delta(t-t_0)}^{\text{Sharp impulse}}$$

$$y(0) = y'(0) = 0$$

$$s^2Y - sy(0) - y'(0) + \omega_0^2 Y = e^{-st_0}$$

$$Y = \frac{e^{-st_0}}{s^2 + \omega_0^2}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-st_0}}{s^2 + \omega_0^2}\right\}$$

- Pop our good old friend friend formula 5 in.
  - $a = t_0$, $F(s) = \frac{1}{s^2+\omega_0^2}$
  - use formula 14 for that transform there on F(s)

$$\mathcal{L}^{-1}\left\{\frac{e^{-st_0}}{s^2 + \omega_0^2}\right\} = u(t-t_0) * \frac{1}{\omega_0} * \sin(\omega_0(t-t_0))$$

MATH225 - 2025-03-31

#notes #math225 #math

Housekeeping

- Exam is in Hill 202 like the last one

Examples and Practice and Such

$$y'' + 25y = 0$$

- This is undamped because there's no y' term.
  - Is also undriven/unforced because we weren't given a driving force, so it's just

> rockin along.

- Characteristic equation this bad boy

$$(r^2 + 25 = 0) \implies r = \pm 5i$$

$$y = c_1 \cos(5t) + c_2 \sin(5t)$$

- Now, if we were to change it up a bit and start driving

$$y'' + 2ty = 2\cos(5t)$$

```
- Still undamped, but now we have a driving force
```

$$y = y_c + y_p$$

$$y_c = c_1 \cos(5t) + c_2 \sin(5t)$$

- With something muc-y like this, we make a guess

$$y_p = A\cos(5t) + B\sin(5t)$$

- Problem, guess is exactly the same as y_c, so we need to shift the guess by slapping a t on repeated terms in the guess

$$y_p = At\cos(5t) + Bt\sin(5t)$$

- Because we've slapped our t on, we're now in resonance and therefore will be growing over time.
- Now, if we were to change it up a bit more

$$y'' + 2ty = \cos(4t)$$

$$y_p = A\cos(4t) + B\sin(4t)$$

- Which is then not present in y_c, so you can just generally slap them together.
  > - This leads to beats, where sometimes they spike due to happening to add together.

EXAM HINT

- THERE IS TOTALLY A QUESTION WHERE YOU MATCH GRAPHS OF SINUSOIDAL DIFFERENTIAL EQUATIONS THAT ARE OSCILLATING.
- Now, if we were to change it up yet a bit more

$$y'' + 2ty = u(t-3)$$

$$y(0) = y'(0) = 0$$

- We know at times between $0 < t < 3$, nothing is happening, because the driving force hasn't turned on yet.
- We know that this starts oscillating at 3 probably sinusoidish, but until then there is NOTHING
- This solution would pop out some $y(t) = u(t-3) * f(n)$

$$y'' + y' + 25y = 0$$

- This is damped because of the y' term, but undriven because still equal to zero.
  - Dynamics for this are determined by the sign of $\beta^2 - 4km$
    - That all comes from quadratic formula solving, the β term is inside the square root of the quadratic
      - If $\beta^2 - 4km > 0$, then taking the square root is perfectly fine and legal and we get an overdamped system.
      - If $\beta^2 = 4km$, we're critically damped
      - If it's negative, we're both in imaginary shenanigans and underdamped and that's how we get sines and cosines through Euler's formula
        - This is, of note, the only one that really oscillates ad infinitem.

$$f(t) = te^{-t}$$

- Fun with graphs.

Worksheet Stuff

- 1d

$$h(t) = 1 - u(t-2) + u(t-2)\cos(t-2)$$

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u(t-2)\} + \mathcal{L}\{u(t-2)\cos(t-2)\}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{u(t-2)\} = \frac{e^{-2s}}{s}$$

$$\mathcal{L}\{u(t-2)\cos(t-2)\} = \frac{e^{-2s}s}{s^2+4}$$

$$\mathcal{L}\{h(t)\} = \frac{1}{s} - \frac{e^{-2s}}{s} + \frac{se^{-2s}}{s^2+1}$$

MATH225 - 2025-04-02

#notes #math225 #math

More Examples and Review

$$\mathcal{L}\{f(t)\} = \frac{2}{s^2}$$

- What is the laplace transform of.

$$\mathcal{L}\{e^{2t}f(t)\}$$

$$f(t) = 2t$$

$$\mathcal{L}\{2te^{2t}\} = 2 * \left(\frac{1}{(2-a)}\right)$$

Change the problem!

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3}$$

$$t^2 = f(t)$$

$$\mathcal{L}\{t^2e^{2t}\} = \frac{2}{(s-2)^3}$$

- Formula six is NOT multiplication

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

- That $*$ is a convolution, NOT NOT NOT multiplication

$$y'' + 4y' + 8y = 0$$

$$y(0) = y'(0) = 0$$

- The solution here is just $y = 0$ since it's a physical system sitting at equilibrium as all of nothing happens.

$$s^2Y - sy(0) - y'(0) + 4(sY - y(0)) + 8Y = 0$$

$$s^2Y - 0 - 0 + 4sY + 8Y$$

$$Y(s^2 + 4s + 8) = 0$$

$$Y = 0$$

$$\mathcal{L}^{-1}\{Y\} = 0$$

- If we wiggle the initial conditions a bit

$$y(0) = 0, y'(0) = 2$$

- Keep the transform from earlier

$$s^2Y - sy(0) - 2 + 4(sY - y(0)) + 8Y = 0$$

$$s^2Y - 2 + 4(sY) + 8Y$$

$$Y(s^2 + 4s + 8) + 2 = 0$$

$$Y = -\frac{2}{s^2 + 4s + 8}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4s + 8}\right\}$$

- If we go and complete the square here (which I'm not great at), we end up with

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4s + 4 + 4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2 + 4}\right\}$$

- This is giving formula 18.

$$\frac{2}{(s - -2)^2 + 2^2}, a = -2, k = 2$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+4}\right\} = e^{-2t}\sin(2t)$$

- You could also characteristic equation this, you get

$$r = -\frac{4 \pm \sqrt{16-32}}{2} = -2 \pm_2 i$$

$$f(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t)$$

- Which gives you $c_1 = 0$ and $c_2 = 1$
- There might be a situation or two where you're forced to do characteristic, but the general vibe is do whatever.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t}$$

Differential equation with piecewise driving force

- We worked this one in class, so you get my class notes thrown in of an actual explanation. Merry Christmas?

$$y'' + 4y = f(t)$$

$$f(t) = \begin{cases} 0 & 0 \le t \le 4 \\ 10e^{t-4} & t > 4 \end{cases}$$

- We're going to write $f(t)$ in terms of Heaviside shenanigans

$$f(t) = 10e^{t-4}u(t-4)$$

- We're hitting this one with a transform

$$s^2Y - sy(0) - y'(0) + 4\,(Y) = \frac{10e^{-4s}}{s-1}$$

$$Y = \frac{10e^{-4s}}{(s^2+4)(s-1)}$$

- uh, this smells awfully of partial fractions

$$\frac{10}{(s^2+4)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}$$

$$(As+B)(s-1) + C(s^2+4) = 10$$

$$As^2 - As + Bs - B + Cs^2 + 4C = 10$$

$$A + C = 0$$

$$B - A = 0$$

$$-B + 4C = 10$$

$$\frac{10}{(s-1)(s^2+4)} = -\frac{2(s+1)}{s^2+4} + \frac{2}{s-1}$$

$$\mathcal{L}^{-1}\{-\frac{2(s+1)}{s^2+4} + \frac{2}{s-1}\} = 2e^t - 2(\cos(2t) + \sin(t)\cos(t))$$

$$f(t) = \begin{cases} 0 & t < 3 \\ 4 & t \geq 3 \end{cases}$$

$$2y'' + 4y' + 4y = 4u(t-3), \quad y'(0) = y(0) = 0$$

$$2(s^2Y(s) - sy(0) - y'(0)) + 4(2Y(s) - y(0)) + 4Y = \frac{e^{-4s}}{s}$$

$$2s^2Y(s) + 8(Y(s)) + 4Y(s) = \frac{e^{-4s}}{s}$$

$$Y(2s^2 + 12) = \frac{e^{-4s}}{s}$$

$$Y = \frac{e^{-4s}}{s(2s^2+12)}$$

$$y(t) = u(t-4)\mathcal{L}^{-1}\left\{\frac{1}{2s^3+12s}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2s^3+12s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2s(s^2+6)}\right\} = \frac{1}{12}(1 - \cos(\sqrt{6}t))$$

$$y(t) = u(t-4)\left(\frac{1}{12}\right)(1 - \cos(\sqrt{6}t))$$

MATH225 - 2025-04-07

#notes #math225 #math

Systems of Ordinary Differential Equations

$$\frac{dx_1}{dt} = x_1 + x_2$$

$$\frac{dx_2}{dt} = x_1 - x_2$$

- Dealing with "coupled" differential equations

Matrices!

- A matrix is a collection of numbers arranged in rows and columns
- Typical is to use a capital letter, often boldfaced in text
    - Texts will often use (), he uses [] so I'm using those for my mental sake of not worrying about brackets.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 2 & 1 & 8 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

- This matrix that we're looking at has 3 rows and 4 columns, we say it's a 3×4 matrix
- We can index elements in the matrix as $A_{i,j}$
    - $A_{1,2} = 2$
    - We're quite tragically not zero indexed. hate it here.
- There are a few important matrix operations we need to know how to do
  - Why am I in this class and not in linalg. i like matrices a lot.
  - For addition, you add the corresponding elements

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix}, A + B = \begin{bmatrix} 1 & 4 \\ 7 & 3 \end{bmatrix}$$

- For scalar multiplication, you just multiply everything.

$$c_1 \begin{bmatrix} 1 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} c_1 & 2c_1 \\ c_1 & 6c_1 \end{bmatrix} \text{ If c}_1\text{ is equal to 7, we'd have } \begin{bmatrix} 7 & 14 \\ 7 & 42 \end{bmatrix}$$

3. Transpose
   - Swap rows and columns, first row becomes first column, etc,etc
   - Denoted by A^T

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 6 \end{bmatrix}, A^T = \begin{bmatrix} 0 & 3 \\ 1 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

```
- If it wasn't already obvious, transpose can shit out a different
size (ie, a $2 \times 3$ turns into a $3 \times 2$)
```

- Identity matrix has 1's on the primary diagonal, and 0's everywhere else (represented by I)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Vectors are, essentially, matrices
  - Just with either one row/column
  - $\bar{V} =< 1, 3 >$ can be represented as either $\begin{bmatrix} \stackrel{1}{3} \end{bmatrix}$ or as $[1, 3]$
    - That doesn't quite look right with the stackrel but is the only way I could think to do it inline.
- Determinant of a matrix
  - Can be written as $\det(A)$ or as $|A|$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \det(A) = |A| = (1)(4) - (2)(3) = -2$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = ad - bc$$

```
- That's the determinant of a $2 \times 2$, which is nice and neat and
tidy. $3\times 3$s and above are icky stinky, however, we're only
really dealing with two equations and two unknowns, so we really
*don't care*
```

- Multiplication!

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 6 & 2 \\ 3 & 0 \end{bmatrix}$$

- A is an $m \times n$ matrix, and B is an $n \times p$ matrix
  - To multiply matrices, the number of columns in A must equal the number of rows in B

$$AB = \begin{bmatrix} 1 + 12 + 9 & 5 + 4 + 0 \\ 4 + 30 + 18 & 20 + 10 + 0 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ 52 & 30 \end{bmatrix}$$

$$BA = \begin{bmatrix} 21 & 22 & 9 \\ 14 & \dots & \dots \\ 3 & \dots & \dots \end{bmatrix}$$

```
- Matrix multiplication is NOT COMMUTATIVE! It **DOES MATTER** what order you put things in.
```

- Eigenvalues and eigenvectors

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

- This is a matrix and a vector multiplied, where the vector is represented as a column vector.
- The matrix is a transformation that changes a vector into a new vector.
    - There is a special transformation that changes the length of the vector but not its direction.
        - This type of vector is called an eigenvector of the matrix and the scale factor of how much its length changes is called the eigenvalue of the matrix
            - $A\bar{v} = \lambda\bar{v}$, where λ is some number that tells me how much the length of the vector changes
            - λ is known as an eigenvalue
            - To solve this equation, we need results from linear algebra, but we're just going to skip to the results

MATH225 - 2025-04-09

#notes #math225 #math

Previously on: matrices!

- Last time we made it as far as eigenvectors and eigenvalues being a thing that exists, but we didn't really talk about them
    - $A\bar{v} = \lambda\bar{v}$
        - Where $\bar{v}$ is an eigenvector and λ is an eigenvalue.
    - Also, they can have complex numbers in them, because fuck it, why not
- Teensy weensy question - how do we find eigen{thingys}
  - You could derive this equation if you were like in linear algebra, but we're not, so we don't.

$$\det(A-\lambda I)=0$$

- Where I is an identity matrix
- So what in the hell does doing this look like?

$$A=\begin{bmatrix}3 & -1\\ 4 & -2\end{bmatrix}$$

- We're trying to find eigenvalues

$$\det(A-\lambda I)=\begin{vmatrix}3-\lambda & -1\\ 4 & -2-\lambda\end{vmatrix}=0$$

$$0=(3-\lambda)(-2-\lambda)+4$$

$$0=-6-3\lambda+2\lambda+\lambda^2+4$$

$$0=\lambda^2-\lambda-2=(\lambda-2)(\lambda+1)$$

- Eigenvalues then become $\lambda=2$ and $\lambda=-1$
- Each eigenvalue has an associated eigenvector that we also need to find, because, why not

$$\lambda=2$$

$$A\bar{v}=2\bar{v}\implies\begin{bmatrix}3 & -1\\ 4 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=2\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

$$\begin{bmatrix}3v_1-v_2\\ 4v_1-2v_2\end{bmatrix}=\begin{bmatrix}2v_1\\ 2v_2\end{bmatrix}$$

$$3v_1-v_2=2v_1,\ \ v_1=v_2$$

$$4v_1-2v_2=2v_2,\ \ v_1=v_2$$

- Wow, that's crazy how remarkably unhelpful that was.
- We determine the eigenvector up to an overall multiplicative constant, ie, we're going to find *an* eigenvector, and then any constant you multiply by would spit out another perfectly valid eigenvector.
- So, just pick something that satisfies

$$\bar{v}=\begin{bmatrix}1\\ 1\end{bmatrix}$$

- That's one eigenvector for the eigenvalue $\lambda=2$
- We have another eigenvalue of $\lambda=-1$

$$\begin{bmatrix}3 & -1\\ 4 & -2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}-v_1\\ -v_2\end{bmatrix}\implies 3v_1-v_2=-v_1,\ \ v_2=4v_1,\bar{v}=\begin{bmatrix}1\\ 4\end{bmatrix}$$

- Consider the following

\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix}1 \\ 2

\end{bmatrix}
=\begin{bmatrix}
1 \
0
\end{bmatrix}

$$-If you were to graph that, the transformation would change the direction of the vecotr-$$

A=\begin{bmatrix}
3 & -2 \
4 & -1
\end{bmatrix}
\begin{vmatrix}
3-\lambda & -2 \
4 & -1-\lambda
\end{vmatrix}
=0

(3-\lambda)(1-\lambda)+8=0

\lambda^{2}-2\lambda+5=0

\lambda=\frac{2\pm \sqrt{ 4-20 }}{2}=\frac{2\pm 4i}{2}

\lambda=1\pm 2i

$$-You can go find some eigenvectors with those complex eigenvaleus$$

\begin{bmatrix}
3 & -2 \
4 & -1
\end{bmatrix}
\begin{bmatrix}
v*{1}* \
v{2}
\end{bmatrix}
=(1+2i)\begin{bmatrix}

v{1} \
v{2}
\end{bmatrix}

3v{1}-2v{2}=(1+2i)v_{1}

(1-i)v{1}=v{2}

$Sowegetthat\$\bar{v}=\begin{bmatrix}1\\1-i\end{bmatrix}\$-Ohhey, youcanembedmatricesininlinemath. Minewasjustbeingu$

\bar{v}=\begin{bmatrix}
1 \
1+i
\end{bmatrix}

$-That'sthecomplexconjugate, becausethat'showthisworks. -Generalpracticesaystocleanupy$

A=\begin{bmatrix}
\frac{1}{2} & 0 \
1 & -\frac{1}{2}
\end{bmatrix}, \lambda^{2}=\frac{1}{4}

\lambda= \pm \frac{1}{2}

$$-Ifit'splus\$\frac{1}{2}\$, theeigenvectorjustworksouttobe\$<1,1>\$-Minus\$\frac{1}{2}\$though$$

\begin{bmatrix}
\frac{1}{2} & 0 \
1 & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
v{1} \
v{2}
\end{bmatrix}
= \begin{bmatrix} -\frac{1}{2}v{1} \
-\frac{1}{2}v{2}

\end{bmatrix}

$-Topequationsaysthat\$v_1=0\$-Secondequationsaysthat\$v_1=0\$-Whichjusttellsusthatt$

\bar{v}=\begin{bmatrix}
0 \
1
\end{bmatrix}

----- ## Webassign Tips - Adjust the size of the matrix if you have to

MATH225 - 2025-04-23

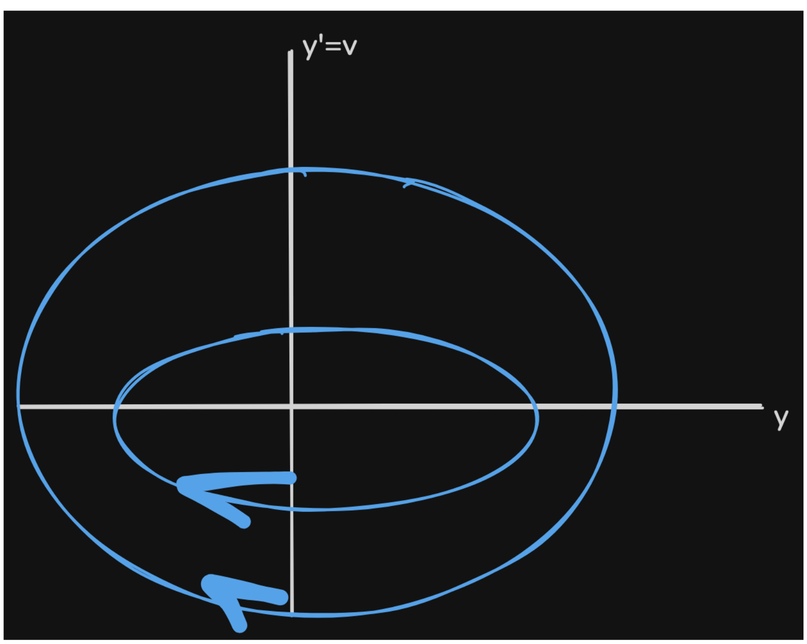
#notes #math225 #math

$$y'' + y = 0$$

- Is quite clearly an undamped oscillator
- Can be rewritten as

$$\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

- The eigenvalues for this work out to be $\lambda = \pm i$
  - When you have complex eigenvalues with nothin' real anchoring them, the stability is a 'center', which actually is stable.
  -



- To determine direction of arrows, pick a point, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- And then at that point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

\begin{bmatrix} y \\ v

\end{bmatrix}'= \begin{bmatrix}
0 & 1 \
-1 & 0
\end{bmatrix}\begin{bmatrix}
1 \
0
\end{bmatrix}
=\begin{bmatrix}
0 \
-1
\end{bmatrix}

- Which tells us that this goes clockwise - Velocity of this would be a nice delightful sinusoidal shape

y''+y'+y=0

$$-Weknowhowtosolvethatwithcharacteristic, normalworkasexpected, blibbityblah.$$

\begin{bmatrix} y \
v

\end{bmatrix}'= \begin{bmatrix}
0 & 1 \
-1 & -1
\end{bmatrix}\begin{bmatrix}
y \
v
\end{bmatrix}

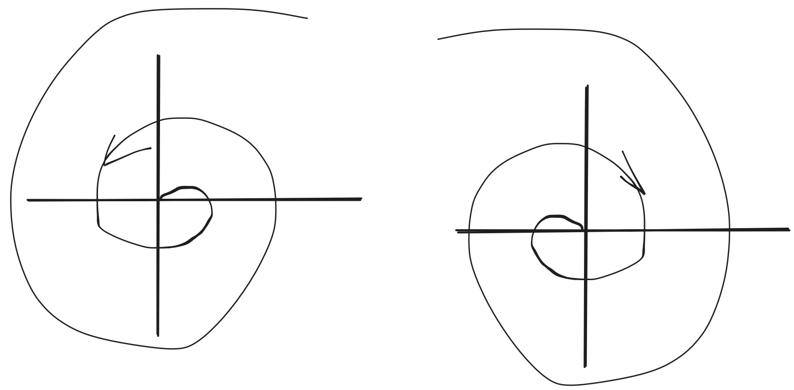
\begin{vmatrix}
0-\lambda & 1\
-1 & -1-\lambda
\end{vmatrix}=0 = \lambda+\lambda^{2}+1=0

\lambda = -\frac{1\pm \sqrt{ 1-4 }}{2}= -\frac{1}{2}
\pm \frac{\sqrt{ 3 }}{2}i$$

- The other way to find this is to calculate the trace of this matrix
    - Trace τ is -1
- Determinant $\Delta = 1$

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = -\frac{1}{2} \pm \sqrt{\frac{3}{2}} i$$

- Here we're looking at a complex conjugate with a negative real part
  - The negative real part makes a stable solution, and slapping the complex part on makes it oscillate, making what's known as a stable spiral.



- Pick some point to investigate the vibes of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

- Which means the arrow at $(1, 0)$ is going to point along the negative y axis, which means we're the first spiral

Nonlinear Systems

$$x' = P(x, y)$$

$$y' = Q(x, y)$$

- In general, P, Q can be nonlinear, investigate what happens near a critical point
- If you have some graph, generally, $f(x) = f(x_1) + f'(x_1)(x - x_1)$
  - That's actually just a Taylor series popping back out, which is kinda silly.
  - You can approximate a function but taking a known value, figuring out it's derivative, and adding the difference, which is neat and how math just works.
  - New problem though, we have functions of two variables

$$f(x, y) \approx f(x_1, y_1) + \frac{\partial f}{\partial x}(x_1, y_1)(x - x_1) + \frac{\partial f}{\partial y}(x_1, y_1)(y - y_1) + \dots$$

- That's what we in the business call a tangent plane. Damn, been a minute.
  - We actually don't really need this, it's just an important middleman step.

$$x' = P(x, y) = P(x_1, y_1) + \frac{\partial P}{\partial x}(x - x_1) + \frac{\partial P}{\partial y}(y - y_1)$$

$$\bar{x}' = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} (\bar{x} - \bar{x}_1)$$

- We can make a substitution where $\bar{H} = \bar{x} - \bar{x}_1$, $\bar{H}' = \bar{x}'$

$$\bar{H}' = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} \bar{H} = \bar{H}' = J\bar{H}$$

- The big old partial matrix is the Jacobian (flashbacks)
- Near a critical point, this system is going to act like a linear system with a matrix given by the Jacobian J

Example

$$x' = -(x-y)(1-x-y)$$

$$y' = x(2+y)$$

- Coupled system yadayadayada

$$P(x,y) = -(x-y)(1-x-y)$$

$$Q(x,y) = x(2+y)$$

- We need to find the critical points
  - Possible options from Q include $x = 0$ and $y = -2$
  - If $x = 0$, then $y = (1-y) = 0$, which gives us that $y = 0$ or $y = 1$
  - If $y = -2$, then $(x+2)(1-x+2) = 0$, which gives us that $x = -2$ or $x = 3$
  - Critical Points
  - $(0,0), (0,1), (-2,-2), (3,-2)$
  If you go expand P out $$
  P(x,y) = -[x-x^{2}-xy-y+xy+y^{2}]

  =x^{2}-y^{2}-x+y
  $$
- With that, $\frac{\partial P}{\partial x} = 2x - 1$
- And $\frac{\partial P}{\partial y} = -2y + 1$
- Do the same for Q, and shove em in the jacobian

$$J(x,y) = \begin{bmatrix} 2x-1 & -2y+1 \\ 2+y & x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

- We do our "usual" stability assessment of the system
  - Near a critical point, it should act like a linear system
- Trace here is just -1
- Determinant is negative two
- Eigenvalues just work out to be $\lambda = -2, \lambda = 1$
- This is a saddle point, which is inherently unstable

MATH225 - 2025-04-30

#notes #math225 #math

- No Bernoulli!
- Just as a reminder when dealing with something of the form $ay'' + by' + cy = 0$
  - This leads to the characteristic equation
    - Real distinct, real repeated, complex roots, they have their forms, it's about what you expect them to do.
      - Be familiar with those, how to write solutions, yadayada.
- 2nd-Order inhomogeneous either leads to doing MUC and having the basic forms of the guesses and whatever, or most of those you can just do with Laplace
  - There's definitely not *wink wink* going to be a delta and heaviside function on the exam
- From there we move into laplace

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1-1+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\}$$

$$e^{at}\sin kt = \frac{k}{(s-a)^2+k^2}$$

- So here we have that $a = -1$, and we're somehow going to want k to be equal to three - So to make that all matchy matchy, you just need to multiply it by three which is, as always, remarkably chill with laplace {chillguy.png} (as long as you put the $\frac{1}{3}$ out front)

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2+2s+10}\right\}$$

Systems of Differential Equations

$$\bar{x}' = A\bar{x}$$

- Where A is a matrix of coefficients

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

If we use $\bar{x}$ then

$$\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Then the system could be written as

$$x' = 2x + 3y$$

$$y' = 2x + y$$

- So now to start solving this, we need to find the eigenvalues of A

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = -1, 4$$

- Which means our solutions are going to involve an e^{-t} and e^{4t}

- $$A\bar{v} = \lambda\bar{v}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$2v_1 + 3v_2 = -v_1$$

$$3v_1 = -3v_1, v_2 = -v_1$$

- Which, for these eigenvalues, we get eigenvectors of $$
\begin{bmatrix}
1 \
-1
\end{bmatrix} \text{and } \begin{bmatrix}
3 \
2
\end{bmatrix}

- For $\lambda = 4$, we get $2v_1 + 3v_1 = 4v_1$, which tells us that $v_1 = \frac{3}{2}v_2$

$$\bar{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- Writing the solution as a single vector would give us

$$\bar{x}(t) = \begin{bmatrix} c_1 e^t + 3c_2 e^{4t} \\ -c_1 e^{-t} + 2c_2 e^{4t} \end{bmatrix}$$

- The matrix, A, we were given might represent

$$x' = 2x + 3y$$

$$y' = 2x + y$$

- The only critical point for ones with all those coefficients is going to be the origin
    - That's an unstable saddle, because.... we have one positive and one negative.
- When you're making a phase portrait, it's generally a good vibe to sketch your eigenvectors (rise over run)
- Eigenvector tied to the positive eigenvalue shoots itself away from the origin
- Eigenvector tied to a negative eigenvalue tucks into the origin
- Toss some curves in between the eigenvectors that generally follow the vibes of the eigenvectors they're near

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$$x' = ax + bxy$$

$$y' = cy - gxy$$

- This is a classic predator-prey relationship - the predator is x', because their population is helped by there being more prey
    - Double plus, where both are helped is cooperating
    - Double minus where both are hurt is competing
-

MATH225 - 2025-05-02

#notes #math225 #math

$$\lambda = 5 \pm \sqrt{25 - 9}$$

$$\lambda = 5 \pm \sqrt{16}$$

$$\lambda = 5 \pm 4$$

$$\lambda = 9, 1$$