MATH213 TOC

> MATH213 - 2024-08-19
>
> #notes #math213 #math #calc
>
> this is wrong. something is missing here.

> MATH213 - 2024-08-20
>
> #notes #math213 #math #calc
>
> ## we're doing more parameter stuff
>
> - ex 2: Parametrize a line segment
>   - $y = 9 - x^2$ from $-1 \leq x \leq 3$
>   - Let's see what happens if we let $x = t$
>   - $y = 9 - t^2$
>   - (from $-1 \leq t \leq 3$) because... we didn't actually change x at all in order to make this work
>   - Fun fact, parameterizations are not unique, so let's do another one
>   - $x = 1 - t$
>   - $y = 9 - (1 - t)^2$
>   - Bounds become $-2 \leq t \leq 2$
>
> ### General form!
>
> $x = x_0 + at$
> $y = y_0 + bt, -\infty < t < \infty$
>
> - Boy howdy, that there's a line . Crazy.
> - (The lack of bounds other than infinity make it just a line, instead of a segment)
> - Smack some more constraints on (ie, a,b are constants, a $\neq$ 0, with slope $\frac{b}{a}$, passing through (x_0, y_0)) and then we're in a good place
>   - (domain restrictions make this a line segment, as opposed to a whole line)

ex 3

- Find the parametric equations for the line segment from $P(4,7)$ to $Q(2,-3)$
    - Slope = $\frac{-10}{-2} = 5$ (which is distinctly not negative)
    -
    - We're going to assign $(x_0, y_0) = (4,7)$
    - $x = 4 + (-1)t$
    - $y = 7 + (-5)t$
    - So, with that whole $P(4,7)$ thing going on
        - $4 - t = 4$
        - $7 - 5t = 7$
        - $t = 0$
    - Aaand, with that $Q(2,-3)$ thing happening
        - $4 - t = 2$
        - $7 - 5t = -3 => t = 2$
    - So, for our final answer
        - $x = 4 - t$
        - $y = 7 - 5t$
        - $0 \leq t \leq 2$

aaand we're going to polar

- To convert polar to rectangular
    - $(r, \theta) -> (x, y)$
        - $x = r\cos(\theta)$
        - $y = r\sin(\theta)$
- To convert from rectangular to polar
    - $(x, y) -> (r, \theta)$
        - $r^2 = x^2 + y^2$
    - $\tan(\theta) = \frac{y}{x}$
- Example!
    - Convert $(2, \frac{3\pi}{4})$ into rectangular
        - $x = 2 * \cos(\frac{3\pi}{4})$
        - $y = 2 * \sin(\frac{3\pi}{4})$
        - = $(-\sqrt{2}, \sqrt{2})$
- Example... but more example-y. I swear.
    - Convert $(1, -1)$ to polar

    - $r^2 + 1^2 + (-1)^2 = 2 = r^2$
    - $r = \sqrt{2}$
    - $\tan(\theta) = \frac{-1}{1} = -1 = \tan(\frac{3\pi}{4})$
    - $= (\sqrt{2}, -\frac{\pi}{4})$

---

converting equations

Convert $r = 4\cos(\theta)$ to rectangular
Alright, so, in the toolbox

- $x = 4\cos(\theta)$
- $y = r\sin(\theta)$
- $r^2 = x^2 + y^2$
  So, uh, how can we solve with that

$r^2 = 4r\cos(\theta)$
$x^2 + y^2 = 4x$
So how do we go about making $x^2 - 4x + y^2 = 0$

$(x^2 - 4x + 4) + y^2 = 0 + 4$
$(x-2)^2 + y^2 = 4$

- This is now a circle centered at $(2, 0)$ with a radius of two (wowie)

Pretty graphs!

- A petal graph is given by $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ where n is an integer
    - if n is odd, you have n many petals
    - if n is even, you have $2n$ petals
    - For instance, $r = 3\sin(2\theta)$
        - This can get out of hand quickly, for instance, [this graph]
- Cardioid
    - $r = \alpha \pm \alpha\cos\theta$
    - Or, vertical uses sine, so, $r = \alpha \pm \alpha\sin\theta$
- Just make a table if you have to do weird shapes like this.

MATH213 - 2024-08-21

#notes #math213 #math #calc

vector time!

- Vectors are quantities that have directions and magnitude
- Vectors are equal if they have the same direction and magnitude

if we have two points, how do we even *make* one of these things?

- $\vec{PQ} =< x_2 - x_1, y_2 - y_1 >$
    - This can extend to three dimensional vectors as well

If we have a vector to start with, we can probably wiggle a magnitude out

- we call up our homeboy pythagoras (technically the distance formula, but like, you get pythagoras with it)
- $|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- When drawing vectors, you start the tail (P, in this case) and go to the head (Q). This helps you figure out direction and other such fun things
- You can multiply vectors by a scalar! That just makes them longer (or maybe even flips it, if there's a negative sign involved). Shit's crazy.

also happening is the zero vector

(which is a tad quirky)

1. Multiplying a vector by the scalar of 0 will pop out $\vec{0}$
2. 0 vector is parallel to all vectors
    1. again, quirked up
3. $\vec{0}$ also manages to be perpendicular to all vectors

- This is really really weird to think about graphically, but cross/dot products prove it algebraically fairly reasonably

vector operations!

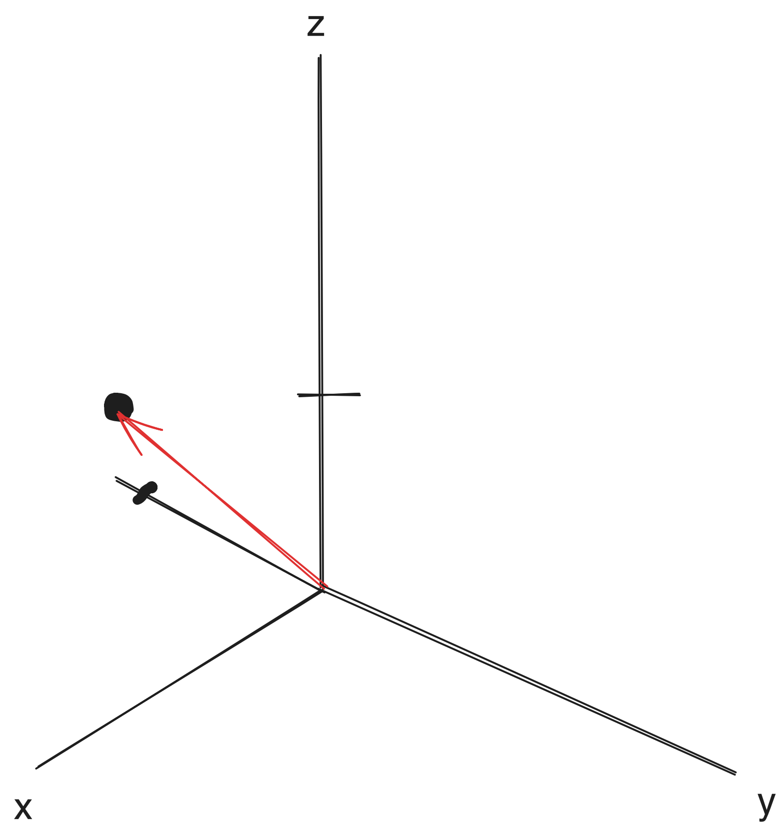
- Given $\vec{u}$, $\vec{v}$
- $\vec{u}+\vec{v} =< u_1+v_1, u_2+v_2 >$
  - This also works with subtraction
- $c\vec{u} =< cu_1, cu_2 >$
- This works for n many dimensions that you could have for vectors

a vector of length one is known as...

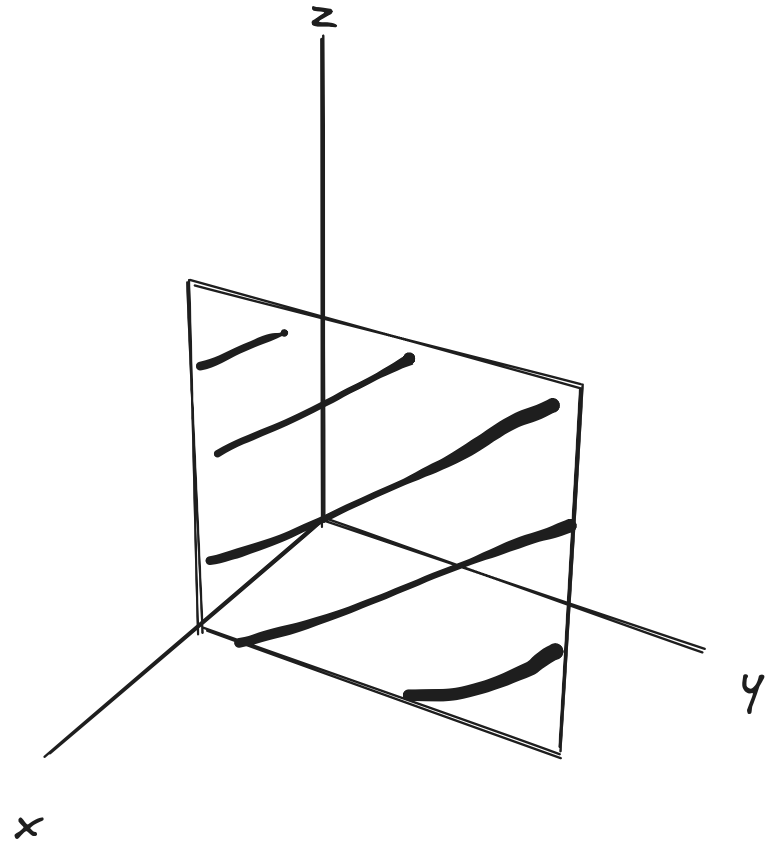
- a unit vector!
- We can fanagle a unit vector out of *any* vector by dividing by magnitude

example time

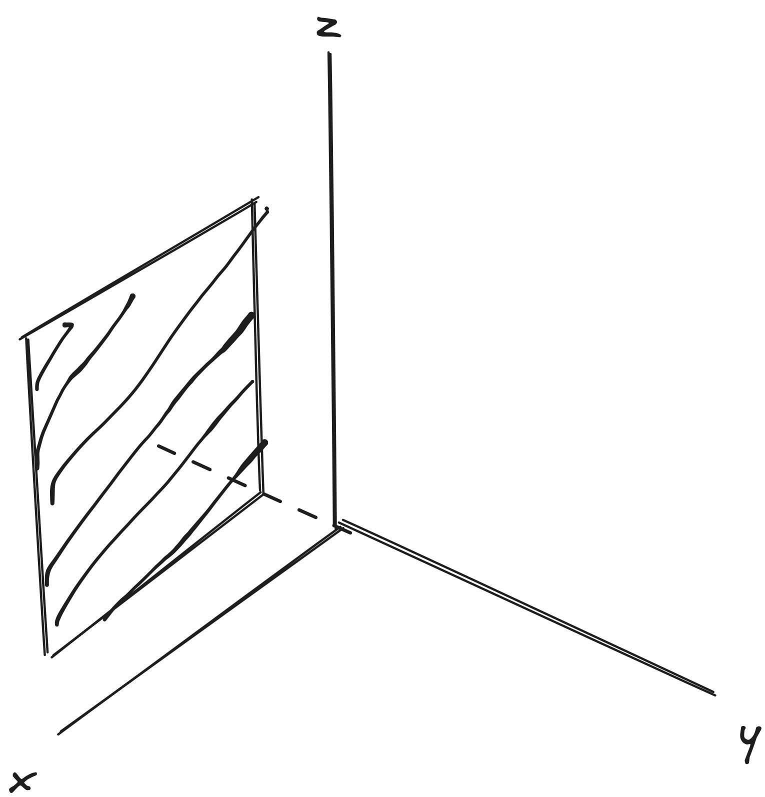
- Find $\vec{PQ}$ given $P(1,-2)$ & $Q(6,10)$
- $\vec{PQ} =< 5, 12 >$
- Find two unit vectors for $\vec{PQ}$
  - magnitude is $\sqrt{25+144} = 13$
- So we just scale the whole thing by $\frac{1}{13}$ and end up with
- $< \frac{5}{13}, \frac{12}{13} >$
  - We could also slap a negative sign on $< -\frac{5}{13}, -\frac{12}{13} >$
- Alternatively written as $\vec{v}_1 = \frac{5}{13}\hat{\imath} + \frac{12}{13}\hat{\jmath}$ (and the negative exists as well)
- Lets graph the vector $< 2, -4, 3 >$



- Alright, moving on to graphing of planes
- Graph the plane $x = 2$
    - in 2d space this is a vertical line
        - in 3d space, we have another axis worth of fun!

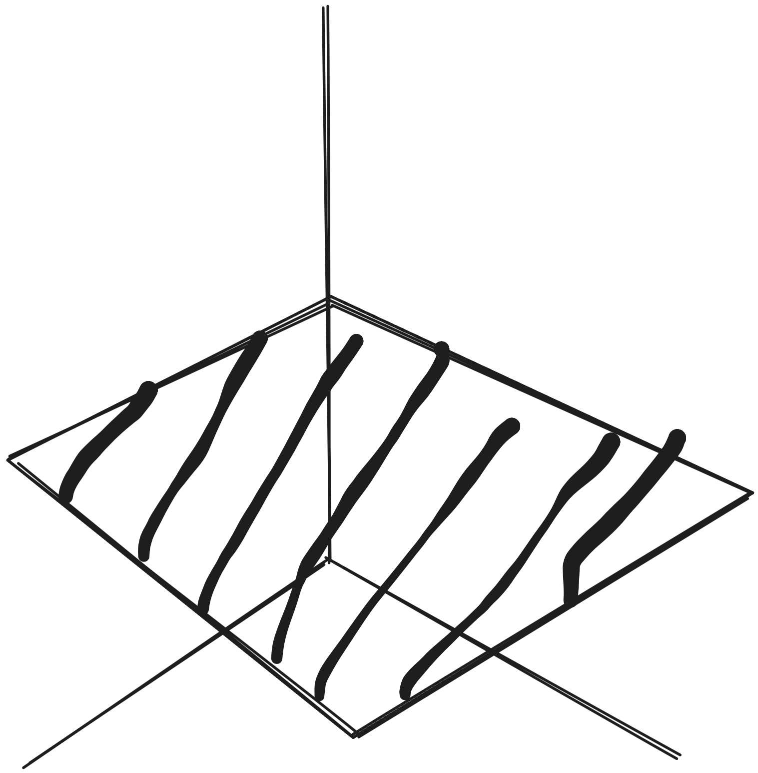


- Graphing $y = -1$



- Hey bucko, we weren't done, we gotta graph something parallel to the xy plane.

- $z = 4$



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- Find the equation of the plane parallel to the yz plane passing through (-5,1,7)
- That whole yz part means it's defined by an $x =$ equation, and uh, $x = -5$. Job done.

dot product!

- they're super nice for finding the angle between two vectors, in which case you'll get some trig involved
- otherwise you're just going to use the algebraic definition
- Given vectors $\vec{u} + \vec{v}$, and θ, (the angle between them), $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$
- If θ is unknown, you can still fanagle definition one, but you'll now need to solve for θ
  - ie, $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$
- You're going to need the actual dot product bit, so $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$ (which works for n many dimensions)
- If $\vec{u} =< 1, 2, 3 >$ and $\vec{v} =< 2, -5, 1 >$, find $\vec{u} \cdot \vec{v}$
  - That right there is $2 - 10 + 3 = -5$. Wowzahs.

couple o' definitions (for the road)

- Dot products will *always always always* shit out a scalar. Never a vector.

- Vector $\vec{u}$ is orthogonal ($\perp$) to $\vec{v}$ if $\vec{u} \cdot \vec{v} = 0$ (not $\vec{0}$, for the record.)
-

MATH213 - 2024-08-23

#notes #math213 #math #calc

more vector review!

- wednesday we chitchatted about the dot product - one if you know θ, in which case you can use $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta), 0 \leq \theta \leq \pi$
- or, the other one, $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$

find the angle

Between $\vec{u} =< \sqrt{3}, 1, 0 >$ and $\vec{v} =< 1, \sqrt{3}, 0 >$
We know that $\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

- I'm also a dot product virst kinda girlie, so we're looking at $\sqrt{3} + \sqrt{3} = 2\sqrt{3}$
  - Magnitude of both of them is 2 ($\sqrt{\sqrt{3}^2 + 1^2}$)
  - $\cos(\theta) = \frac{2\sqrt{3}}{2*2}$
  - $\theta = \cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$

couple important dot product bits

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
   1. That's just the commutative property! Shit's wild.
2. $c(\vec{u} \cdot \vec{v}) = (c\vec{u})\vec{v} + \vec{v}$ but NOT THE C ON BOTH TERMS. THAT WOULD BE SQUARED.
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$
4. Projection of $\vec{u}$ onto $\vec{v}$ = $\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right) \cdot \vec{v}$

fun times over. cross product time.

- Given two non-zero vectors, $\vec{u}$ and $\vec{v}$ in three dimensions, the magnitude of $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta), 0 \leq \theta \leq \pi$
- We have a little bit of a theorem from that

- Given two non-zero vectors $\vec{u}$ and $\vec{v}$
    - $\vec{u}$ and $\vec{v}$ are parallel if and only if the cross product is equal to the 0 vector (ie, $\vec{u} \times \vec{v} = \vec{0}$)
- If $\vec{u}$ and $\vec{v}$ just so happen to coincidentally be sides of a parallelogram, the area of said parallelogram is the cross product's magnitude
    - Not the parallel sides. That would be silly.

Properties

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
    1. This is anti-commutative. Fuck this shit.
2. $\forall a, b \in \mathbb{R}, a\vec{u} \times b\vec{v} = ab(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + (\vec{u} + \vec{w})$ Left distributive property
4. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$ Right distributive
    1. Those are both shenanigans because we're not commutative. See note 1.1 for my opinion

Algebraic Definition of a Cross Product

- So, defintion of $\vec{u} \times \vec{v}$
- Given a 2×2 matrix A, $\det A = ad - bc$
- $$\begin{matrix} a & b \\ c & d \end{matrix} => \det A = ad - bc$$

$$\vec{u} \times \vec{v} = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix} = |\begin{matrix} u_2 & u_3 \\ v_1 & v_3 \end{matrix}|\hat{\imath} - |\begin{matrix} u_1 & u_3 \\ v_1 & v_3 \end{matrix}|\hat{\jmath} + |\begin{matrix} u_1 & u_2 \\ v_1 & v_2 \end{matrix}|\hat{k}$$

- Find a vector $\perp$ to $\vec{u} =< 1, 2, 3 >$ and $\vec{v} =< -2, 4, -1 >$
    - A vector $\perp$ to $\vec{u}$ and $\vec{v}$ is parallel to $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 4 & -1 \end{matrix} = (-2 - 12)\hat{\imath} \ - (-1 + 6)\hat{\jmath} \ + (4 + 4)\hat{k}$$

$$= -14\hat{\imath} - 5\hat{\jmath} + 8\hat{k}$$

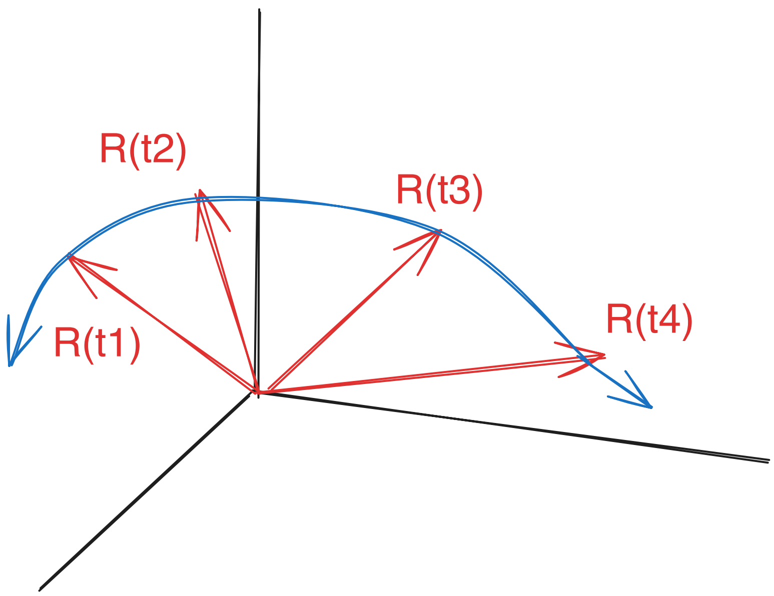
activity time!

MATH213 - 2024-08-26

#notes #math213 #math #calc

more review! (VVF) great fun.

- $\vec{v}(t) =< f(t), g(t), h(t) >$ is a Vector Valued Function
- The input is going to be t, a real number, and the output is a vector



- Each t produces one point, which lives along this path
- A line is given by $\vec{l} =< x_o, y_o, z_o > +t < a, b, c >$
- for $-\infty < t < \infty$
  - The $< x_o, y_o, z_o >$ term is our starting point, $< a, b, c >$ gives us direction
    - (t is magnitude, for vector enthusiasts out there)

Example (that was nice)

Find the line segment as a VVF from $P(1, 2, 3)$ to $Q(-1, 6, 1)$

$$\vec{l} =< 1, 2, 3 > +t < -2, 4, -2 >$$

$$\text{Gotta find your domain, too: } 0 \leq t \leq 1$$

Example (this will not be nice)

- Describe the curve given by the vector valued function $r(t) =< 4\cos t, 4\sin t, t >$
- Ignoring the odd duckling, $x = 4\cos t$, $y = 4\sin t$
  - that looks like a circle!
- however, we do have an odd duckling, so, pondering that
- $z = t$, so as t gets big, so does z, etc
- z works as a free variable, it just gets to go wild
- I'm not sketching this, best to imagine, like, a dough can with the lines tracing up.
- Formal name for this is a *helix*, if you're not a *nerd* though it's a slinky

limit stuff

$$\lim_{t\to a} < f(t), g(t), h(t) > \;=\; < \lim_{t\to\infty} f(t), \lim_{t\to\infty} g(t), \lim_{t\to\infty} h(t) >$$

- Fun one to type

Evaluate $\lim_{t\to 0} < 1 + t^2, 3e^t, \frac{\sin(t)}{t}$
$=< 1, 3, 1 >$, if I'm not tripping

$< 1 + 0^2, 3e^0, \frac{\sin(0)}{0} >$ is mostly fine! except that last part. take that to a hospital.
$< 1, 3, \cos(0) >$

- If one of them doesn't exist, we lump them all in with the bad egg and say the whole thing doesn't exist.

derivative stuff

- If $\vec{r}(t) = \; < f(t), g(t), h(t) >$ and f,g,h are differentiable, then $\vec{r}'(t) = \; < f'(t), g'(t), h'(t) >$
- Another name for $\vec{r}'(t)$ is a tangent vector (which means the exact same thing. do not be scared by the math wizard's tricks)
- This leads us to a unit tangent vector - which has a magnitude of one, and is given by the equation $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

example

- Find the unit tangent vector for
- $\vec{r}(t) =< 2, 4\cos 2t, 4\sin 2t >$, $0 \leq t \leq 2\pi$
- If we ignore our odd duckling again, we, once more, have a circle

- Our odd duckling says that $x = 2$, which is a plane $\|$ to yz-plane
- $\vec{r}'(t) =< 0, -8\sin(2t), 8\cos(2t)$>
  - We, quite unfortunately, need the magntitude
  - $|\vec{r}'(t)| = \sqrt{0^2 + (-8\sin(2t))^2 + (8\cos(2t))^2}$
  - $= \sqrt{64\sin^2(2t) + 64\cos^2(2t)}$
  - $= 8$
    - Pythagorean identity turns all the trig bits into 1, so then you can just... do $\sqrt{64}$, and that'll be that
- Now, going back to what we had
- $\vec{T}(t) = \frac{<0,-8\sin(2t),8\cos(2t)>}{8} =< 0, -\sin(2t), \cos(2t) >$
- Of particular note: We got a nice constant! That was sheer dumb luck. This does not have to be a constant, as t is a real number, so anything in terms of t is also just fine.

More derivative stuff

- Given $\vec{u}$ and $\vec{v}$ are VVF, and f is a differentiable function, all with respect to t, we get a small laundry list of different properties

$$\text{Sum Rule: } \frac{d}{dt}[u(t) \vec{+} \vec{v}(t)] = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$$

$$\text{Product Rule: } \frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

$$\text{Chain Rule: } \frac{d}{dt}\left[\vec{u}(f(t))\right]$$

$$\text{Dot product Rule: } \frac{d}{dt}[\vec{u} \cdot \vec{v}] = (\vec{u}' \cdot \vec{v}) + (\vec{u} + \vec{v}')$$

$$\text{Cross Product Rule: } \frac{d}{dt}[\vec{u} \times \vec{v}] = (\vec{u}' \times \vec{v}) + (\vec{u} \times \vec{v}') \text{ ORDER MATTERS!}$$

If $\vec{v}(t) = \sin(t)\hat{\imath} + 2\cos(t)\hat{\jmath} + \cos(t)\hat{k}$, find $\frac{d}{dt}[t^2\vec{v}(t)]$

- $2t < \sin(t), 2\cos(t), \cos(t) > +t^2 < \cos(t), -2\sin(t), -\sin(t) >$
  - You gotta smush em together
- $$< 2t\sin(t) + t^2\cos(t), 4t\cos(t) - 2t^2\sin(t), 2t\cos(t) - t^2\sin(t) >$$
- There's the tangent vector! Ain't he a cutie.

MATH213 - 2024-08-27

#notes #math213 #math #calc

arc length of a parametrized function

- That's fun to say
- If you have a parametrized curve, r, and $\vec{r}(t) = \ < f(t), g(t)h(t) >$ where f', g', h' are all continuous along the given domain, then the arc length along a given interval $a \leq t \leq b$ is given by the following equation:

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

- That's ugly.

$$L = \int_a^b |\vec{r}(t)|\, dt$$

Find the arc length of the curve

$\vec{r}(t) =< e^{2t}, 2e^{2t} + 5, 2e^{2t} - 20$ from $0 \leq t \leq \ln(2)$

- Alright, welp, we need to find a derivative vector

$$\vec{r}'(t) =< 2e^{2t}, 4e^{2t}, 4e^{2t} >$$

$$\int_0^{\ln(2)} \sqrt{(2e^{2t})^2 + (4e^{2t})^2 + (4e^{2t})^2} dt$$

$$\int_0^{\ln(2)} \sqrt{4e^{4t} + 16e^{4t} + 16e^{4t}} dt$$

$$\int_0^{\ln(2)} \sqrt{36e^{4t}} dt = \int_0^{\ln(2)} 6e^{2t} dt$$

$$= 3e^{2t}\Big|_0^{\ln(2)} = 3[e^{2\ln 2} - e^0]$$

$$= 3[2^2 - 1] = 9$$

  Yippeeeeee!
- In most cases, the radical will simplify
  - Not always though, in those cases you gotta bring out... other integration methods.

example again

- Find the arc length of the curve given by $\vec{r}(t) = \ < 13\sin(2t), 12\cos(2t), 5\cos(2t) >$ from $0 \leq t \leq 2\pi$

$$\vec{r}'(t) = \ < 26\cos(2t), -24\sin(2t), -10\sin(2t) >$$

$$\int_0^{2\pi} \sqrt{(26\cos(2t))^2 + (-24\sin(2t))^2 + (-10\sin(2t))^2 dt}$$

$$\int_0^{2\pi} \sqrt{676\cos^2(2t) + 576\sin^2(2t) + 100\sin^2(2t) dt}$$

$$\int_0^{2\pi} \sqrt{676\cos^2(2t) + 676\sin^2(2t) dt}$$

$$\int_0^{2\pi} \sqrt{676}\, dt = \int_0^{2\pi} 26$$

$$26t\Big|_0^{2\pi} = 52\pi$$

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it keeps going

We try (for about five minutes)
Find the arc length of the curve $\vec{r}(t) = \ < \frac{1}{2}t^2, \frac{1}{3}(2t+1)^{3/2} >$ on $0 \leq t \leq 2$

Okie

$$\vec{r}'(t) =< t, 2 * \frac{3}{2} * \frac{1}{3}(2t+1) >$$

$$\vec{r}'(t) = \ < t, \sqrt{2t+1}) >$$

$$\int_0^2 \sqrt{t^2 + 2t + 1}\, dt$$

$$\int_0^2 (t+1) dt$$

$$\frac{t^2}{2} + t\Big|_0^2 = 4$$

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cursed to live in interesting times

Find the arc length of the curve $\vec{r}(t) = \ < 2t^2+1, 2t^2-1, t^3 >$
from $(1,-1,0)$ to $(9,7,8)$
That'll just be $0 \leq t \leq 2$

Next up! Kid named r'

$$r'(t) =< 4t, 4t, 3t^2 >$$

$$L = \int_0^2 \sqrt{(4t)^2+(4t)^2+(3t^2)^2}$$

$$L = \int_0^2 \sqrt{32t^2+9t^4}$$

$$L = \int_0^2 \sqrt{t^2(32+9t^2)} = \int t\sqrt{32+9t^2}\,dt$$

We're just gonna let $u = 32+9t^2$, so $du = 18tdt$

$$L = \int_0^2 \frac{1}{18}\sqrt{u}\,du$$

$$L = \frac{1}{18} \cdot \frac{2}{3} u^{\frac{3}{2}}$$

$$L = \frac{1}{27}(32+9t^2)^{\frac{3}{2}}\Big|_0^2$$

$$\frac{1}{27}\left((32+9(2)^2)^{\frac{3}{2}} - (32)^{\frac{3}{2}}\right)$$

- And like, you can combine that a bit more

Find the arc length of the curve $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + \ln(\cos(t))\hat{k}$
On the interval $0 \leq t \leq \frac{\pi}{4}$

$$r'(t) =< -\sin(t), \cos(t), \frac{1}{\cos(t)} * -\sin(t)$$

$$r'(t) =< -\sin(t), \cos(t), -\frac{\sin(t)}{\cos(t)} >$$

$$r'(t) =< -\sin(t), \cos(t), -\tan(t) >$$

$$\int_0^{\frac{\pi}{4}} \sqrt{(-\sin(t))^2 + (-\cos(t))^2 - (\tan(t))^2}\,dt$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\sin^2 + \cos^2 - \tan^2(t)}$$

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2(t)}$$

$$\int_0^{\frac{\pi}{4}} \sqrt{\sec^2} = \int_0^{\frac{\pi}{4}} \sec(t)$$

$$\ln(\sec(t)+\tan(t))\Big|_0^{\frac{\pi}{4}}$$

$$\ln|\sqrt{2}+1| - 0$$

- I forgot that darn natural log.

MATH213 - 2024-08-28

#notes #math213 #math #calc

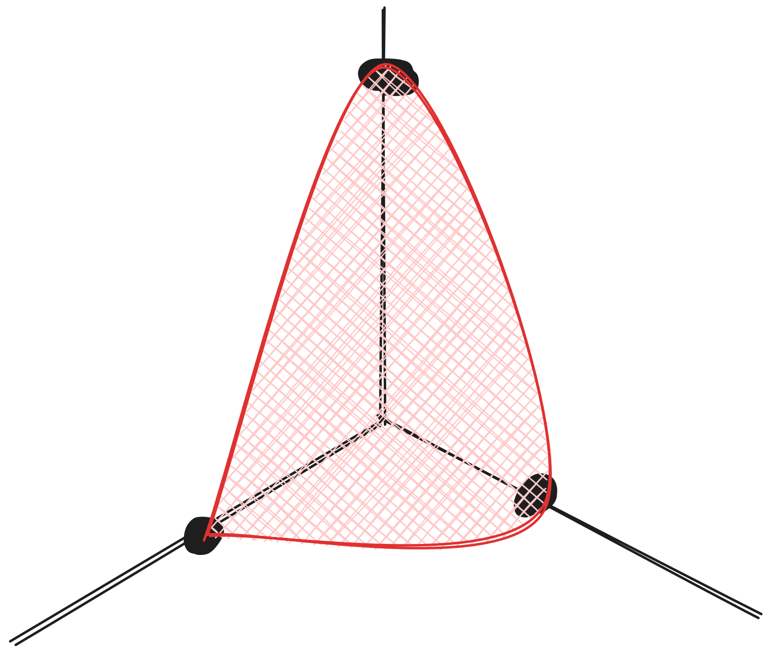
we're finally starting calc 3

hey, uh conics

- You have a circle, $(x-h)^2+(y-k)^2=r^2$
  - where (h,k) is the center and r is the radius
- Ellipse where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
  - if $a>b$ you have a horizontal ellipse (the major axis is the x axis)
  - And vice versa for $b>a$, giving a vertical ellipse with y as the major axis
- Hyperbola 1: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ gives you a right/left hyperbola (open to the far sides, left and right)
- Hyperbola 2: $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is an up/down hyperbola

calc 3 type shit

- A quadric surface is a three dimensional object expressed as a quadratic equation in three variables
- So, general form of a plane ($ax+by+cz=d$) is quirky, it's where all coefficents of quadratic terms are equal to 0
  - Sketch $2x+4y+z=8$
  - $(4,0,0)$ x intercept
  - $(0,2,0)$ y intercept
  - $(0,0,8)$ z intercept

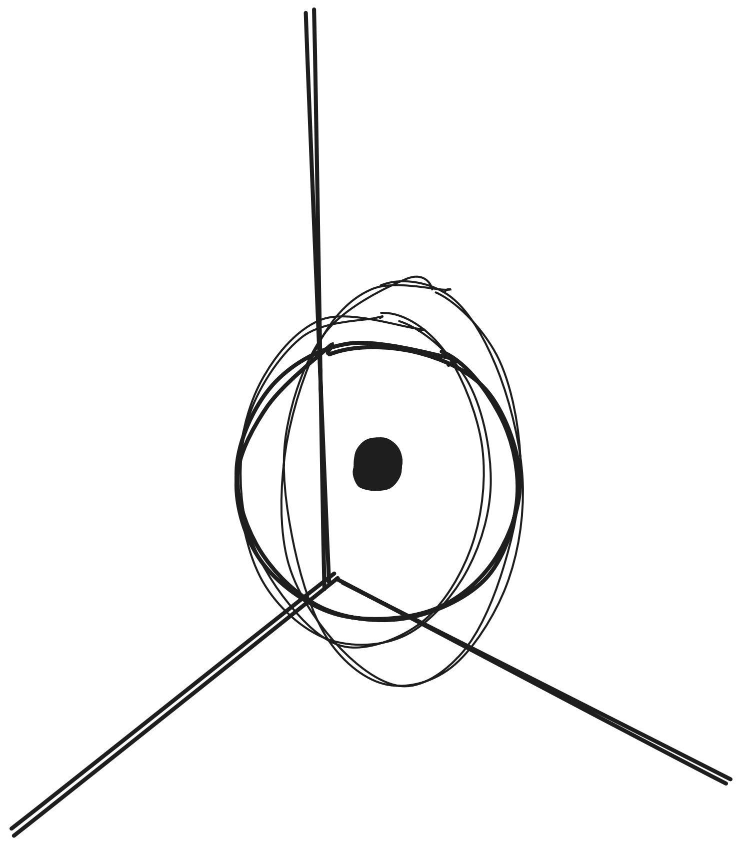


s ph ear

- Spheres are defined by

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

- If we were to sketch the sphere $(x-1)^2+(y-20^2+(z-3)^2=4$
  - Center of this sphere is $(1,2,3)$ with a radius of 2



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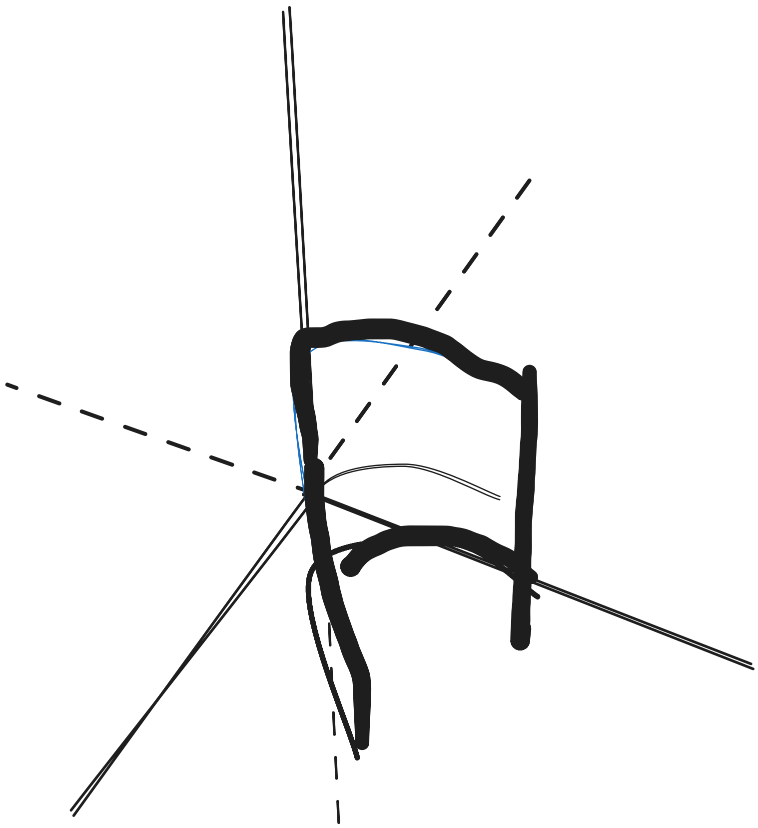
cylinder

- The term cylinder is used to describe any surface parallel to a line (it will not look like a cylinder shape) (always) (because those are also parallel to a line)
- The equation of a cylinder in three dimensions will be parallel to the not wed in to the equation
  - fun term, that
- Sketch $y = x^2$ in $\mathbb{R}^3$
  - In this situation, the z axis is *missing!* Hey! an axis has fallen into the river in lego



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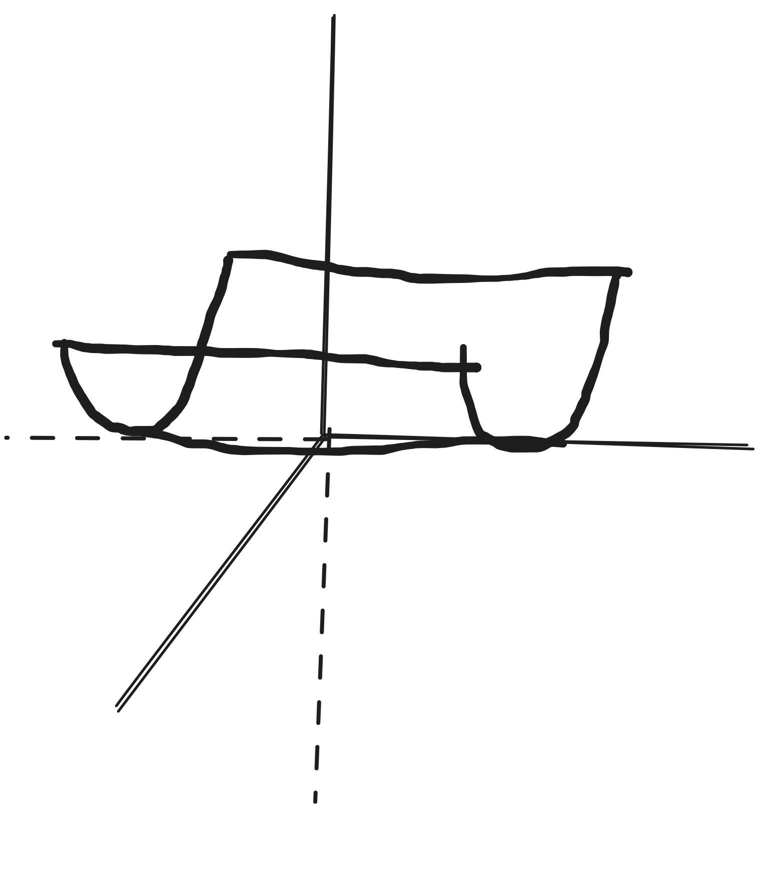
- Ok it's a bad sketch but I drew my angle too harsh. Lesson learned.

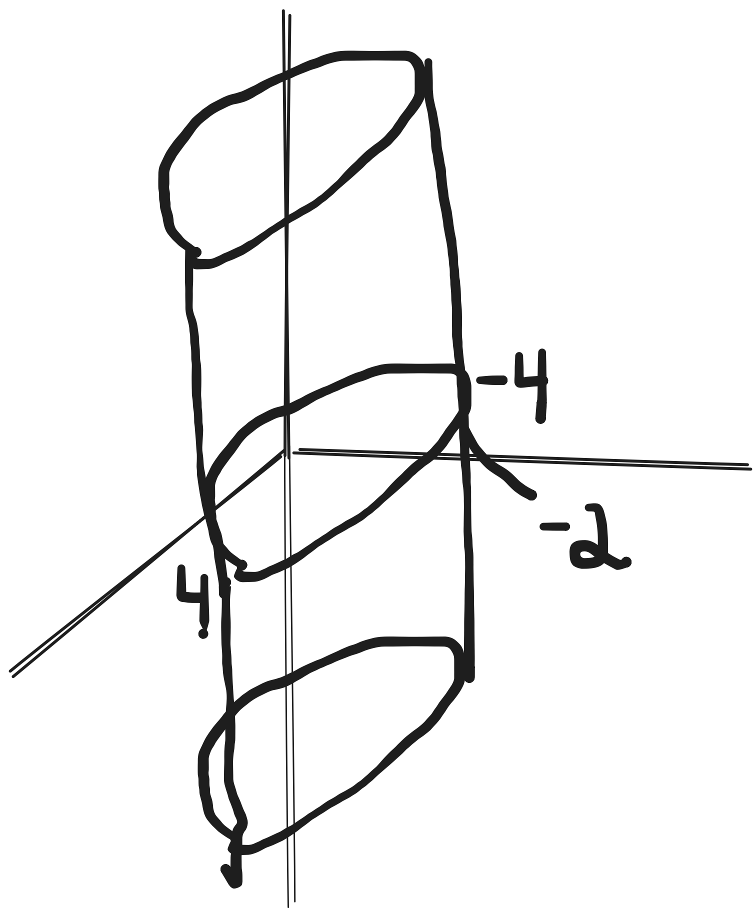
$z = x^2$ in $\mathbb{R}^3$



- This one is noticeably less shit

Trace

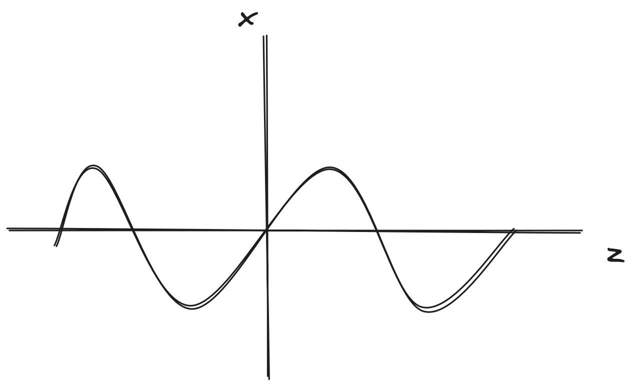
- A trace of a surface is the set of points at which the surface intersects a plane that is $\parallel$ to a coordinate plane.
- Essentially, you do a little slicing.
- Sketch in $\mathbb{R}$ the following cylinders:
    - $x^2 + 4y^2 = 16$
        - $\frac{x^2}{16} + \frac{4y^2}{16} = 1$
        - x is the major axis here - we just care which one is bigger, ignoring the fact that the denominator makes things small

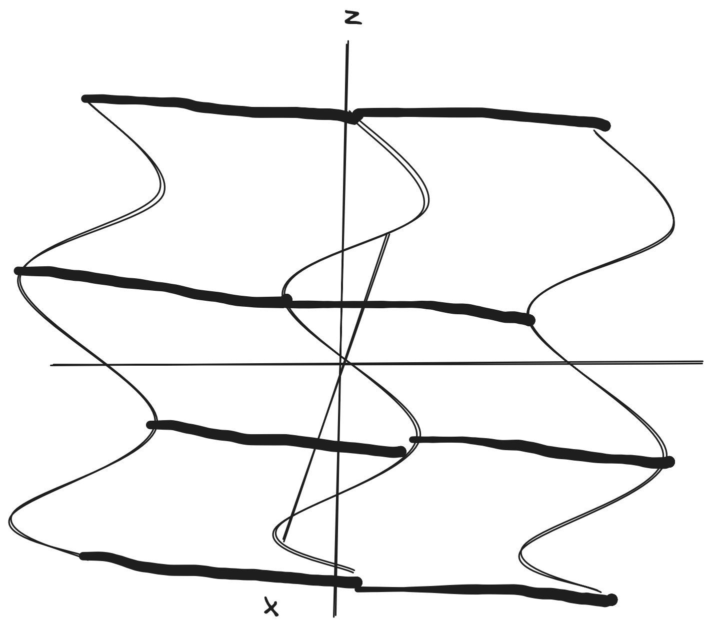


    -

example five (seriously? who counts these things?)

- $x - \sin(z) = 0$ in $\mathbb{R}^3$
- Go isolate a variable here $x = \sin(z)$
    - Hey, we're *pretty good* at $y = \sin(x)$



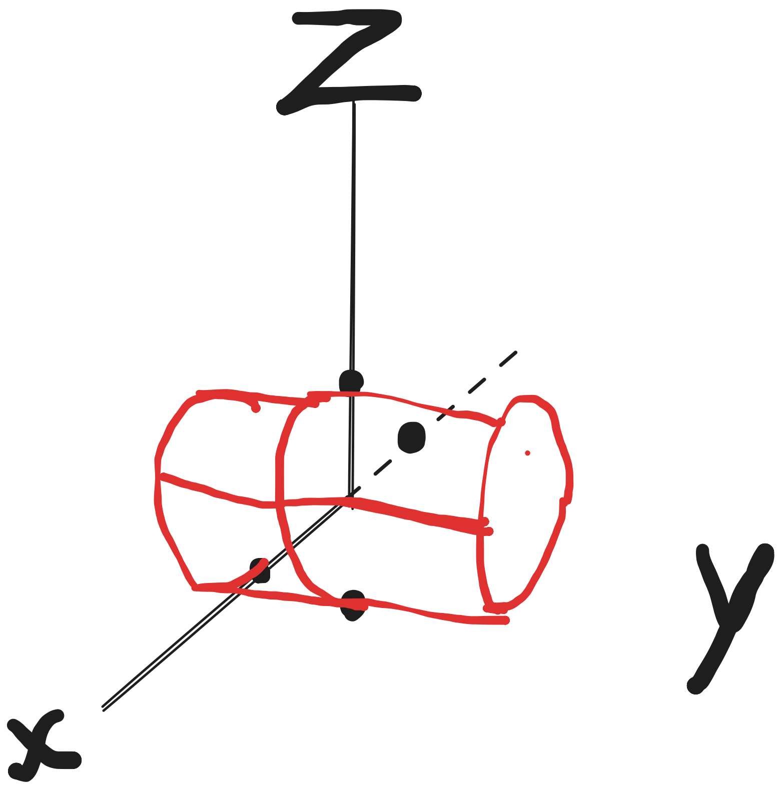


- I mean, turned out pretty good for a woogly sheet

we try!

(sketch the cylinder in $\mathbb{R}^3$)

$$x^2 + z^2 = 4$$



2. $$y - x^3 = 0$$
$$y = x^3$$

MATH213 - 2024-08-30

#notes #math213 #math #calc

ok so this is going to be like, half of quadric surfaces

- A quadric surface is given by

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

- Where, uh ***A THROUGH J*** are all just constants
- Procedure for sketching quadric surfaces:\

1. Find any and all intercepts
    1. Set $x, y, z = 0$ in pairs and solve for the third variable
2. Find all traces
    1. Set $x = 0$ or some $x = x_o$ (this would be the $yz-$ trace
    2. Set $y = 0$ or some $y = y_0$ (xz- trace)

3. Set $z = 0$ or $z = z_o$ (xy-trace)

3. Sketch and graph everything

example

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$$

1. Find all of the intercepts!
   - x intercept means $y, z = 0$, we get left with $\frac{x^2}{9} = 1$ so $x = \pm 3$, $(3, 0, 0)$ and $(-3, 0, 0)$
   - y intercept we set $x, z = 0$, so we get left with $\frac{y^2}{16} = 1$, so $y = \pm 4$, $(0, 4, 0)$ and $(0, -4, 0)$
   - z intercept we set $x, y = 0$, so we get left with $\frac{z^2}{25} = 1$, so $z = \pm 5$, $(0, 0, 5)$ and $(0, 0, -5)$
   - You can only cheat and take the square root of everything when the left side looks the way it does and is all positive and is essentially this exact situation
2. New task: Find all of the traces
   1. $xy-$ trace

      $$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

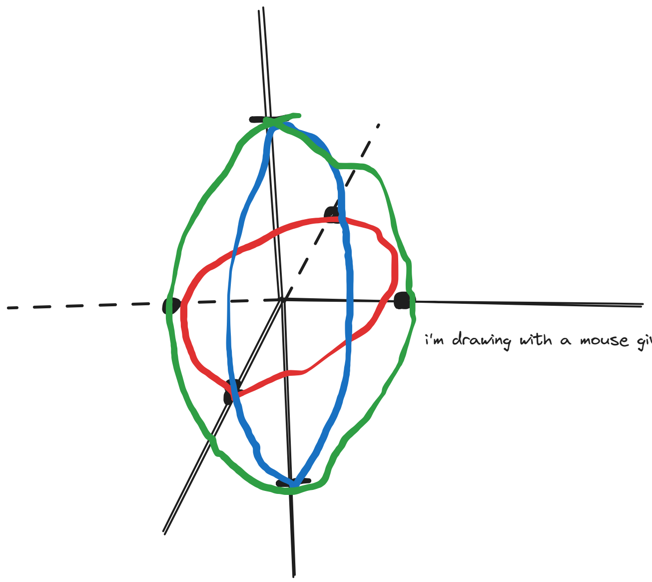
      1. So xy traces are ellipses
   2. xz - trace

      $$\frac{x^2}{9} + \frac{z^2}{25} = 1$$

      1. Ellipse! again.
   3. yz trace

      $$\frac{y^2}{16} + \frac{z^2}{25} = 1$$

      1. Is, believe it or not, also an ellipse.
3. Go forth and sketch



quadric surface but another

$$z = \frac{x^2}{16} + \frac{y^2}{4}$$

1. Intercept!
   1. $$0 = \frac{x^2}{16}$$
      1. $(0, 0, 0)$
   2. y intercept
      1. $$0 = \frac{y^2}{4}$$
      2. $(0, 0, 0)$
   3. z intercept is... also zero.
2. Traces
   1. xy trace! $z = 0$

      $$0 = \frac{x^2}{16} + \frac{y^2}{4}$$

      1. That was a bad choice.

         $$1 = \frac{x^2}{16} + \frac{y^2}{4}$$

      2. That right there? Boy howdy, that's an ellipse.

2. xz trace, try 0, why not

$$z = \frac{x^2}{16}$$

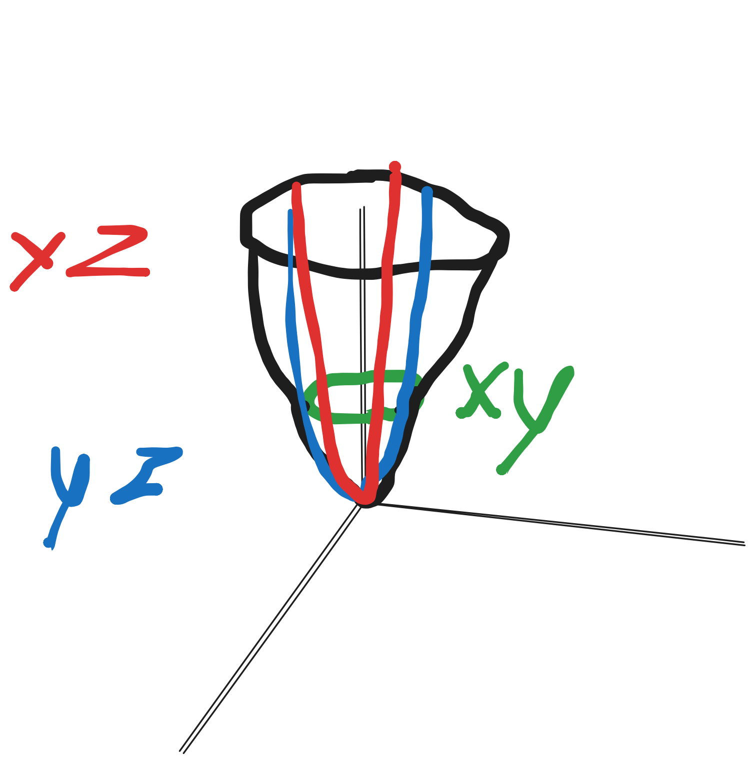
   1. Parabola

3. yz trace, x=0

$$z = \frac{y^2}{4}$$

   1. Parabola!

3. This is going to suck



1.

yet another example

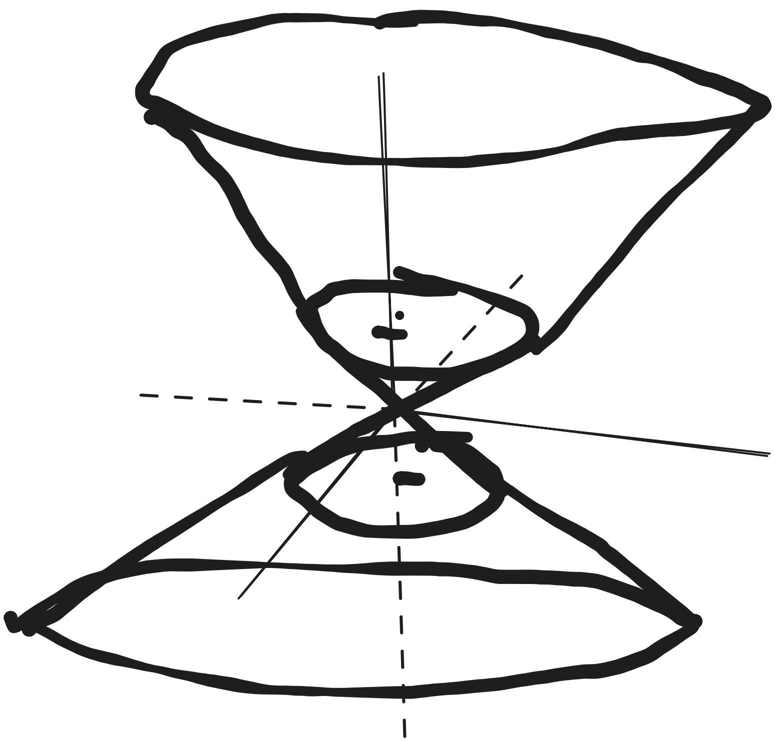
$$z^2 = \frac{x^2}{16} + \frac{y^2}{4}$$

- Intercepts
  - x intercept $0 = \frac{x^2}{16}$
    - $(0, 0, 0)$
  - y and z intercepts are also all zero
- Traces

$$1 = \frac{x^2}{16} + \frac{y^2}{4} \Rightarrow \text{Ellipse}$$

- If z is negative, that's fine, still an ellipse

- $$y = 0, z^2 = \frac{x^2}{16}$$
- $$y = 2, z^2 = \frac{x^2}{16} + \frac{(2)^2}{4} = \frac{z^2 - x^2}{16} = 1$$
- That spits out a hyperbola! Fun.
- $$x = 4, z^2 = \frac{(4)^2}{16} + \frac{y^2}{4}, z^2 - \frac{y^2}{4} = 1$$
    - That spits out another hyperbola, and the negative still works
- So this is actually a cone, for realsises



- $$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

okie intercepts

- $$y, z = 0, x^2 = 1, (1,0,0)(-1,0,0)$$

- $$x, z = 0, \frac{y^2}{4} = 1, y = 2(0, 2, 0)(0, -2, 0)$$

- $$y, x = 0, \frac{z^2}{9} = 1, z = \pm 3, (0, 0, 3)\ (0, 0, -3)$$

  Traces

  $$x = 0, \frac{y^2}{4} + \frac{z^2}{9} = 1$$

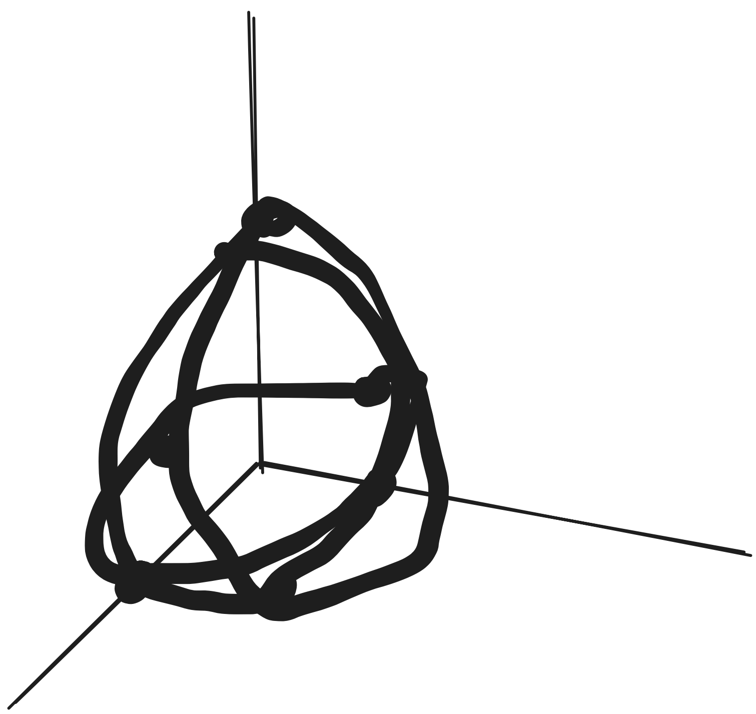
  - Yeah that works as an ellipse with z as the major axis

  $$y = 0, \frac{x^2}{1^2} + \frac{z^2}{9} = 1$$

- Shockingly, also an ellipse

  $$z = 0, \frac{x^2}{1^2} + \frac{y^2}{4} = 1$$

So that's a bunch of ellipses, sketch time
(Should be an ellipsoid)



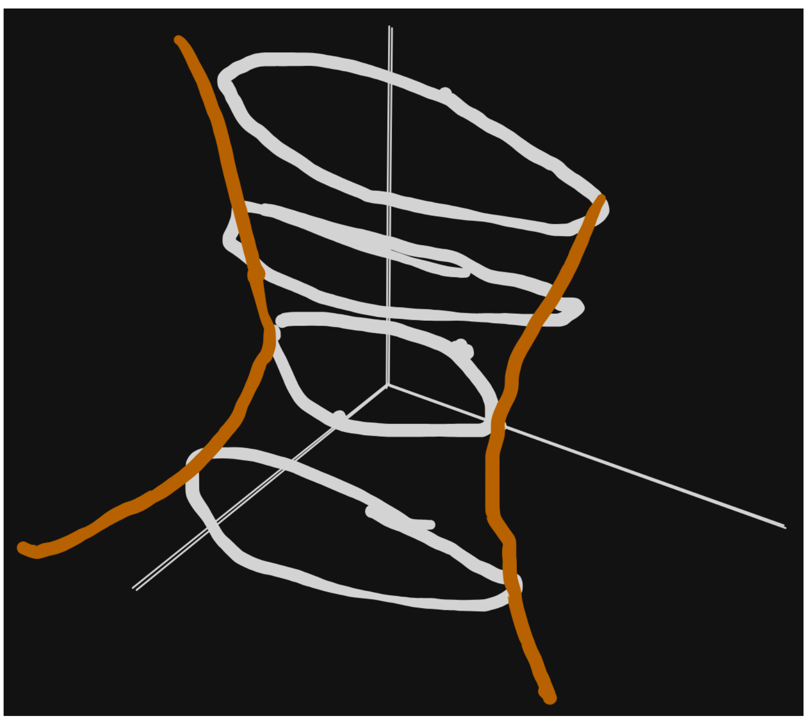
MATH213 - 2024-09-03

#notes #math213 #math #calc

- These shapes are sounding less and less real.

$$\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$$

- Do our same thing of finding intercepts
- x intercept is ± 2
- y intercept is ± 3
- z intercept is just, uh... $-z^2 = 1$
    - That's what we in the business call a problem.
- Moving on to our traces:
  - xy, we can choose 0 to be z, $\frac{x^2}{4} + \frac{y^2}{9} = 1$, oh hey, we get an ellipse
  - If $z^2 = z_0$, we would get some $\frac{x^2}{4} + \frac{y^2}{9} - z_0^2 = 1$
  - Which then pushes the z_0 over, where $\frac{x^2}{4} + \frac{y^2}{9} = 1 + z_0^2$
  - And those will all be ellipses, because z_0 is forced to be positive
  - xz trace, we can call y 0, $\frac{x^2}{4} - z^2 = 1$
  - That there's a hyperbola that opens along the x
  - yz, we can choose 0 to be x, $\frac{y^2}{9} - z^2 = 1$
  - That there's once again a hyperbola, this one opens along the y
  -



- The formal name of this (instead of "tubey thing") is a hyperboloid of one sheet

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Sketch again

$$z^2 - \frac{x^2}{4} - \frac{y^2}{9} = 1$$

Let's.... start with intercepts

- x intercept, $\frac{-x^2}{4} = 1$
  - That's, again, what we call a problem
- y intercept... is also a problem.
- $z^2 = 1$ which is ± 1! Awesome.

Moving on to traces

- xy trace gives us $z_0^2 - \frac{x^2}{4} - \frac{y^2}{9} = 1$
- z_0 can be any real number

$$\frac{-x^2}{4} - \frac{y^2}{9} = 1 - z_0^2$$

$$\frac{x^2}{4} + \frac{y^2}{9} = z_0^2 - 1$$

We know z_0^2 is positive, and is a real number.

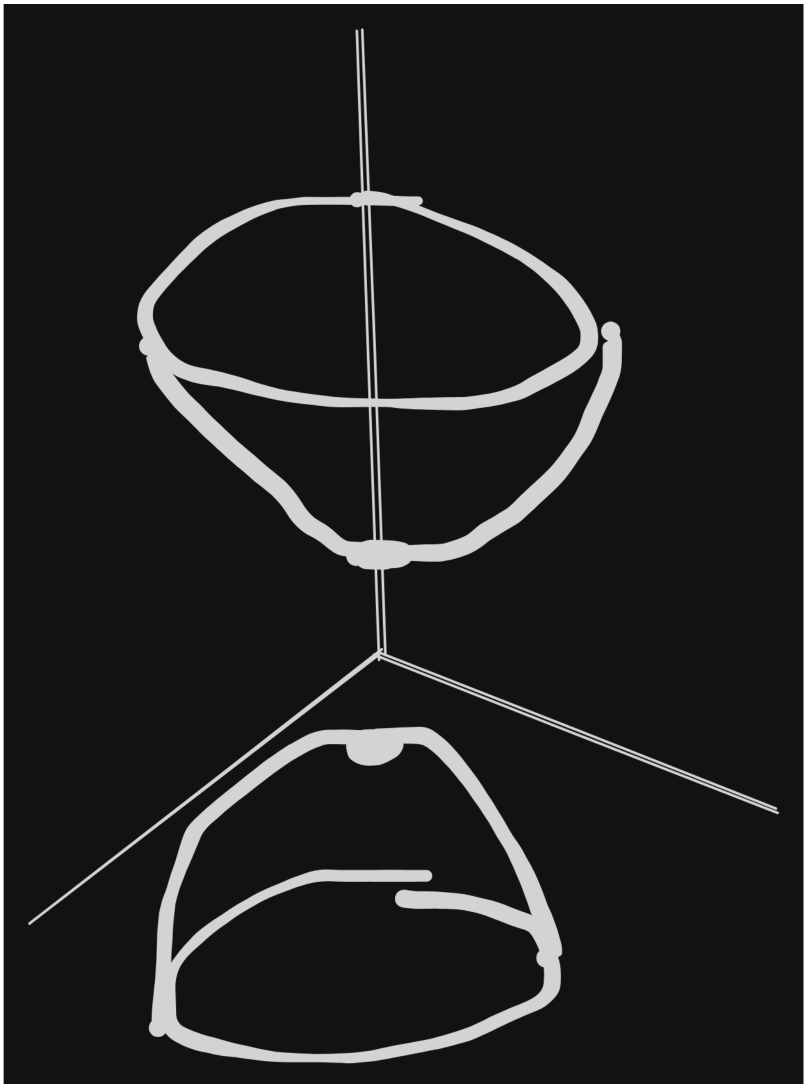
Let $z_0^2 - z = n, n \in \mathbb{R}$

$$\frac{x^2}{4} + \frac{y^2}{9} = n$$

$$\frac{x^2}{4n} + \frac{y^2}{9n} = 1$$

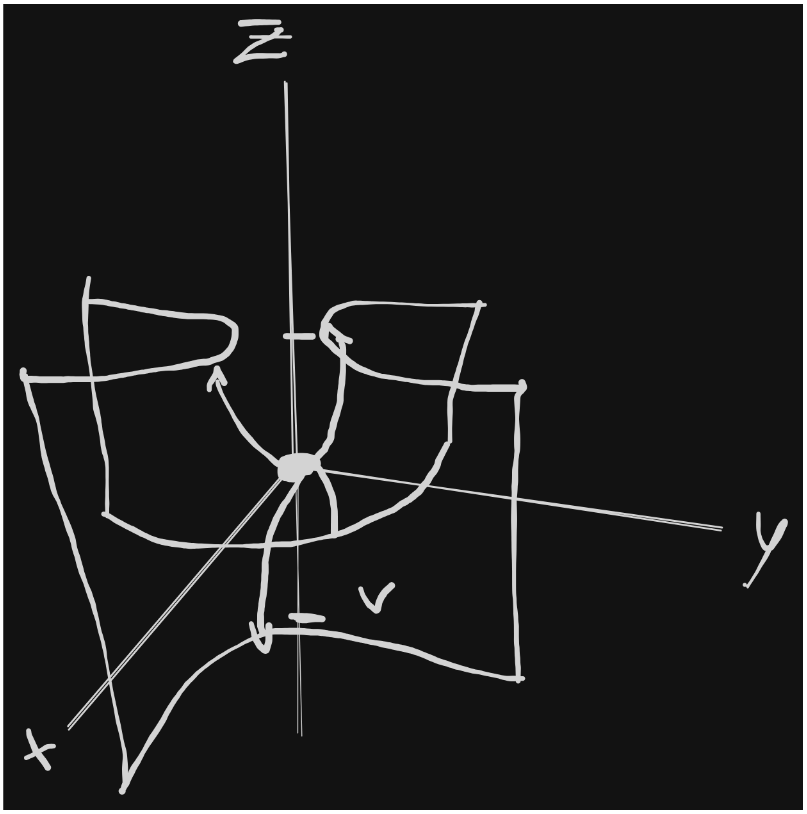
That's an ellipse! I promise.

- This was a nice little baby proof.
- yz trace is just $z^2 - \frac{y^2}{9} = 1$
  - Hyperbola
- xz trace we can also do $x = 0$, giving us $z^2 - \frac{x^2}{4} = 1$
  - One again, a hyperbola

- 
- This is a hyperboloid of two sheets

$$z = y^2 - x^2$$

- You always start with intercepts.
- x intercept, y intercept, and z intercepts are $(0, 0, 0)$
- ok time for traces
  - xy-trace: Let's try $z = 1$, giving us $1 = y^2 - x^2$, which is a cutesy little hyperbola along y
  - Trying $z = -1$, giving us $-1 = y^2 - x^2$ ... kinda works, flip that to be how we like it, giving $1 = x^2 - y^2$, and that works just fine to give a hyperbola along x
  - yz-trace: $x = 0$, giving $z = y^2$, which is honest to god just a parabola (opening to z positive)
  - xz trace, $y = 0$, giving $z = -x^2$, which is yet again just a parabola (opening towards negative z)



- That's a... uh, hyperbolic paraboloid. Yep.
-

Example time

$$-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$$

- Ok I had written

$$-16x^2 - 4y^2 + z^2 + 64x - 80$$
$$-16x^2 + 64x - 4y^2 + z^2 = 80$$
$$-16(x^2 + 4x) - 4y^2 + z^2 = 80$$
$$-16(x^2 - 4x + 4) - 4y^2 + z^2 = 80 - 64$$
$$-16(x-2)^2 - 4y^2 + z^2 = 16$$
$$-(x-2)^2 - \frac{y^2}{4} + \frac{z^2}{16} = 1$$

- That eventually manages to produce a hyperboloid of two sheets.
  ($z = \pm 4$)
  It's the two cups spit out just a little bit from the $(x-2)^2$ having that -2

MATH213 - 2024-09-04

#notes #math213 #math #calc

Graphs and Level Curves

- We're considering several independent variables and one dependent variables
  - Some real life examples:
    - Wind chill depends on at least two (temperature, wind speed)
    - Dosage is number of times administered and patient's body weight

Notation Stuff

$$z = f(x, y)$$

- That's what we call explicit form

$$F(x, y, z) = 0$$

- Is then implicit form, where z has not been dragged out
- We're dealing with functions, and well, functions have domains and ranges, so, we're going to need to find those

Domain

- The domain of a function $z = f(x, y)$ is the set of all points (x, y), in $\mathbb{R}^2$ where f is defined

Range

- The range is the outputs z to the function $f(x, y)$

Notation Aside

- Set Notation to describe domain and range, interval doesn't really make a lot of sense

$$D := \{(x, y) \mid x, y, \in \mathbb{R}\}$$

D is always capitalized, $:=$ means "defined by", and the $|$ is working as a "such that"

Example

Find the domain of

$$g(x,y) = \sqrt{4-x^2-y^2}$$

That square root implies that $4-x^2-y^2 \geq 0$

$$x^2+y^2 \leq 4$$

- Oh hey, a circle, neat

$$D := \{(x,y) \mid x^2+y^2 \leq 4\}$$

- The sketch for this is just a circle of radius two, centered on the origin, with the entire inside shaded for that "less than" part

Level Curve

(or contour curve)

- Curve created by choosing z_0 for $f(x,y) = z_0$
  - you generally pick like, four different points to plug in
- Level curves will essentially be traces
- If the level curves are close together, the "steeper" our 3 dimensional shape is
- Level curves are used for topographical maps to show elevation, which is kinda neat

Example Time

- Find the domain and range of $f(x,y) = \sqrt{9-x^2-y^2}$, then use level curves for $z_0 = 0, 1, 2, 3$ to sketch $f(x,y)$ in 3D

$$D := \{(x,y) \mid x^2+y^2 \leq 9\}$$

- So that's domain, we gotta do some puzzling to get range though.

$$z_0 = 0 => 0 = \sqrt{9-x^2-y^2}, 9 = x^2+y^2$$

Circle. (of radius three)

$$z_0 = 1 => 1 = \sqrt{9-x^2-y^2}, 8 = x^2+y^2$$

Circle. (of radius $\sqrt{8}$)

$$z_0 = 2 => x^2+y^2 = 5$$

Circle. (of radius $\sqrt{5}$)

$$z_0 = 3 => 0 = x^2 + y^2$$

No longer circle. Now has become (0,0)

We get a nice little dollop-y mountain thing. (It's actually the upper half of a sphere)

$$R := \{z \mid 0 \leq z \leq 3\}$$

yet another example

- Find the domain and range of $f(x, y) = e^{-x^2-y^2}$, then sketch f

Domain:

$$D := \{(x, y) \mid x, y \in \mathbb{R}\}$$

Yipeee! Happy happy happy! (That's just from the normal domain of e^x)
Level curves.

0 If $y = e^x$ then $y > 0$
So let's try $z > 0$

In general, let $z = z_0$

$$z_0 = e^{-x^2-y^2}$$

- Quick reminder that level curves are traces in x, y, so we're looking for a function $f(x) = y$

$$\ln(z_0) = \ln(e^{-x^2-y^2})$$

$$\ln(z_0) = -x^2 - y^2$$

$$-\ln(z_0) = x^2 + y^2$$

- Quick tests

$$-\ln(\frac{9}{10}) = \ln(\frac{10}{9}) > 0$$

$$-\ln(\frac{10}{9}) = \ln\left(\frac{9}{10}\right) < 0$$

So our range is

$$R := \{z \mid 0 < z \leq 1\}$$

So these are circles if z is between 0 and 1

That's a circle with a positives radius $r = \sqrt{-\ln(z_0)}$

- What happens at z=1, anyways?
    - That becomes a point! $(0, 0, 1)$
    -

MATH213 - 2024-09-09

#notes #math213 #math #calc

Back in the good old days of Calc 1, we did limits of a single variable, and whatever it spat out was our limit

$$\lim_{(x,y)\to(a,b_)} f(x,y) = L$$

As (x, y) gets close to (a, b), the output of f will get close to L

Continuity of a Limit

The function $f(x, y)$ is continuous at the point (a, b) if three things are true:

- The point must be defined
- Limit has to exist as it approaches the point
- Those things must be the *same*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Then you are continuous

Directional Shenanigans

- Back in calc 1, we had two nice directions we could approach from and that was awesome and wonderful and I loved it.
- Now are in Calc 3. You may approach a point from infinitely many paths.

$$D := \{(x,y) \mid x^2 + y^2 \leq 4\}$$

- That spits out a closed circle, and you can approach any point in that circle from literally wherever the fuck you want.

- We're going to need a theorem, so let $a, b, c \in \mathbb{R}$
-

1. If $f(x, y) = c$, some constant
    1. then $\lim_{(x,y)\to(a,b)} C = c$
2. IF $f(x, y) = x$, then $\lim_{(x,y)\to(a,b)} = \lim_{(x,y)\to(a,b)} x = a$
3. If $f(x, y) = y, \text{ then } \lim_{(x,y)\to(a,b)} = \lim_{(x,y)\to(a,b)} y = b$

All you really need to know is that all limit laws still extend to 3d

$$\lim_{(x,y)\to(e^2,4)} \ln(\sqrt{xy})$$

- First off: PLUG IT IN

$$\ln(\sqrt{4e^2})$$

$$\ln(2e)$$

$$\ln(2) + \ln(e)$$

$$\ln(2) + 1$$

- Job done. We're happy. Yippity ta-tee ta.

$$\lim_{(x,y)\to(2,8)} (3x^2y + \sqrt{xy})$$

- ok we're practicing limit laws here, but just like... $(3(4)(8) + \sqrt{16})$
- $$96 + 4 = 100$$

$$\lim_{(x,y)\to(2,8)} 3x^2y + \lim_{(x,y)\to(2,8)} \sqrt{xy}$$

- What if (a, b) is not in your domain?
    - Then you cry and whine and it eventually works out

$$\lim_{(x,y)\to(6,2)} \frac{x^2 - 3xy}{x - 3y} = \frac{0}{0}$$

- Sobbing. Crying.

$$\lim_{(x,y)\to(6,2)} \frac{(x)(x-3y)}{(x-3y)} = \lim_{(x,y)\to(6,2)} (x) = 6$$

- If plugging in and simplifying both fail, we deploy the "Two-Path Test" (TPT)
- The TPT for nonexistence of limits
    - If we approach from two different paths and those paths don't agree on what you spit out, you have a problem
        - If both of those paths are in the domain and work out fine, then you can conclude that the limit does not exist)
    - Note that this is the TPT for the *non*existence of limits - it does not prove existence! Just that they don't!

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

- if you plug in you get $\frac{0}{0}$, womp womp

$$D := \{(x, y) \mid (x, y) \neq 0\}$$

Algebra?

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x^2 + y^2}$$

- That was WORTHLESS.
- Let's try the tew path test
    - Moving along the x-axis, y is going to be 0

$$\lim_{(x,y)\to(x,0)} \frac{x^2 - 0}{x^2 + 0} = 1$$

- Quickly! Attack from (above) via the y axis!

$$\lim_{(x,y)\to(0,y)} = -\frac{y^2}{y^2} = -1$$

Last I checked, $-1 \neq 1$

$\therefore$, by TPT, this limit quite solidly does not exist

- For the record, I have to type out the word "therefore" to get the $\therefore$ symbol, so it's actually *more* work.

Do your own time

$$\lim_{(x,y)\to(1,0)} \frac{y\ln(x)}{x}$$

- Plug that shit in

$$\frac{0*0}{1} = 0$$

Alright that's fine, Limit is 0

Next!

$$\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x}$$

First of all, plug it in?

$$\frac{(-2)^2+2(1)(-2)}{-2+2}$$

$$\frac{4+-4}{-2+2} = \frac{0}{0}$$

$$\lim_{(x,y)\to(1,-2)} \frac{\cancel{(y+2x)}\,(y)}{\cancel{(y+2x)}} = y = -2$$

$$\lim_{(x,y)\to(-1,1)} = \frac{2+1-3}{0} = \frac{0}{0}$$

$$(x+y)(2x-3y)$$

$$2x^2-3yx+2xy-3y^2$$

$$\frac{(x+y)(2x-3y)}{(x+y)}$$

$$\lim_{(x,y)\to(-1,1)}(2x-3y) = -2-3 = -5$$

One more

$$\lim_{(x,y)\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9}$$

MATH213 - 2024-09-10

#notes #math213 #math #calc

more 15.2 (TPT)

- couple general examples

$$\lim_{(x,y)\to(0,0)} [\frac{xy}{x^2+y^2}]$$

- Plug in will fail you. Algebra will fail you. L'Hopital is dead.
- In this case, let's try the x axis, $y=0$

$$\lim_{(x,y)\to(x,0)} |\frac{xy}{x^2+y^2}|$$

$$= \frac{0}{x^2+0} = 0$$

$$\lim_{(x,y)\to(0,y)} |\frac{xy}{x^2+y^2}|$$

$$\frac{0}{y^2} = 0, \text{ Sadly, } 0 = 0$$

Trying $y=x$

$$\lim_{(x,y)\to(0,0)} \left[\frac{xy}{x^2+y^2}\right]$$

$$\lim_{n\to\infty} \frac{(x)(x)}{x^2+x^2}$$

$$\lim_{n\to\infty} \frac{x^2}{2x^2} = \frac{1}{2}$$

By TPT limit Does Not Exist

- So this is fun, but shooting individual paths is kind of tiring
- So just choose $y = mx$ for literally all the lines going through the origin

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{(x)(mx)}{(x^2)+(mx)^2}$$

$$\lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

$$\lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

---

where lines fail.

$$\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^4}$$

Trying $y = mx$

$$\lim_{(x,y)\to(0,0) \text{ along y=mx}} \frac{3xy^2}{x^2+y^4}$$

$$\lim_{x\to 0} \frac{3x(mx)^2}{x^2+(mx)^4}$$

$$\lim_{x\to 0} \frac{3x^3m^2}{x^2(1+m^4x^2)}$$

$$\lim_{x\to 0} \frac{3xm^2}{1+m^4x^2} = 0$$

- Pro gamer parabola advice:
  - Consider the function that you're working with
  - In this case, let's try $x = y^2$

$$\lim_{(x,y)\to(0,0)\text{x}=y^2} \frac{3xy^2}{x^2+y^4}$$

$$\lim_{x\to 0} \frac{3x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{3x^2}{2x^2} = \frac{3}{2}$$

- Which means our limit does not exist by two path test.

Polar Shenanigans

- only do this if you have to.

$$x = r\cos\theta, y = r\sin\theta, x^2+y^2 = r^2$$

- If we were to do be doing a limit as $(x, y) \to (0, 0)$
  - Radius would be-a shrinking, $r \to 0^+$

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{(r\cos\theta)^3 - (r\sin\theta)^3}{r^2}$$

$$\lim_{r\to 0^+} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

$$\lim_{r\to 0^+} r(\cos^3\theta - \sin^3\theta) = 0$$

Partial Derivatives

- Essentially the way the cookie crumbles is that you look at each rate of change individually for multivariable functions
- We need to consider direction when discussing RoC in $\mathbb{R}^3$
  - f_x is the partial of f with respect to x
  - f_y is the partial of f with respect to y
- The partial derivative of f with respect to x at a point (a, b) is given by

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

- This fine piece of bullshit is the instantaneous rate of change in the x direction

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

- That's the instantaneous rate of change in the y direction. Happy day.

$$f = \lim_{h\to 0} \frac{(x+h)^2 y - x^2 y}{h} = \lim_{h\to 0} \frac{x^2 y + 2xhy + yh^2 - x^2 y}{h}$$

$$\lim_{h\to 0} 2xy + yh$$

$$f_x = 2xy$$

It's just going to be x^2! god damn it! why must we *suffer*

$$f_y = \lim_{h \to 0} \frac{x^2(y+h) - x^2 y}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^2 y} + x^2 \cancel{h} - \cancel{x^2 y}}{\cancel{h}}$$

$$\lim_{h \to 0} x^2 = x^2$$

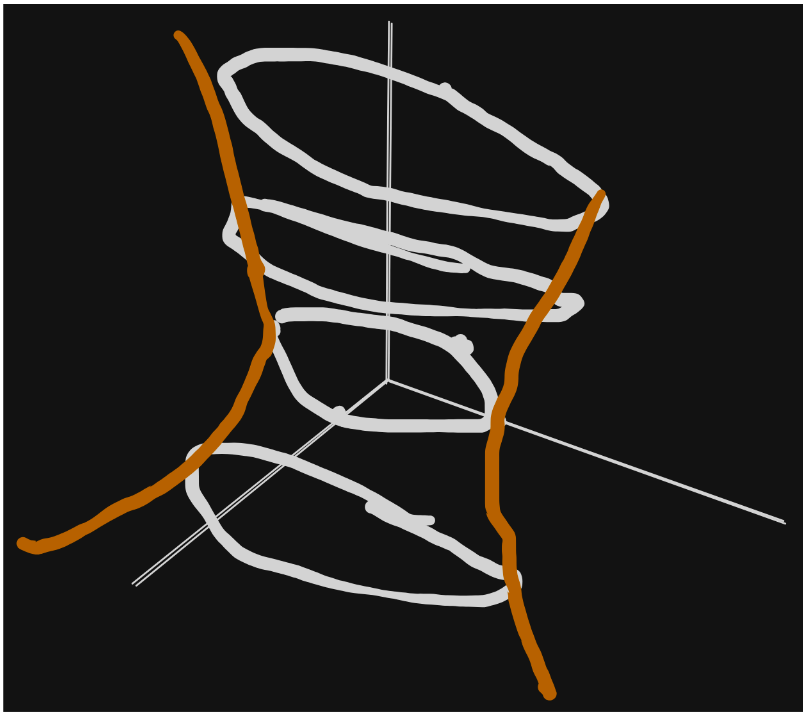
MATH213 - 2024-09-03

#notes #math213 #math #calc

- These shapes are sounding less and less real.

$$\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$$

- Do our same thing of finding intercepts
- x intercept is ± 2
- y intercept is ± 3
- z intercept is just, uh... $-z^2 = 1$
  - That's what we in the business call a problem.
- Moving on to our traces:
  - xy, we can choose 0 to be z, $\frac{x^2}{4} + \frac{y^2}{9} = 1$, oh hey, we get an ellipse
  - If $z^2 = z_0$, we would get some $\frac{x^2}{4} + \frac{y^2}{9} - z_0^2 = 1$
  - Which then pushes the z_0 over, where $\frac{x^2}{4} + \frac{y^2}{9} = 1 + z_0^2$
  - And those will all be ellipses, because z_0 is forced to be positive
  - xz trace, we can call y 0, $\frac{x^2}{4} - z^2 = 1$
  - That there's a hyperbola that opens along the x
  - yz, we can choose 0 to be x, $\frac{y^2}{9} - z^2 = 1$
  - That there's once again a hyperbola, this one opens along the y
  -



- The formal name of this (instead of "tubey thing") is a hyperboloid of one sheet

---

Sketch again

$$z^2 - \frac{x^2}{4} - \frac{y^2}{9} = 1$$

Let's.... start with intercepts

- x intercept, $\frac{-x^2}{4} = 1$
  - That's, again, what we call a problem
- y intercept... is also a problem.
- $z^2 = 1$ which is ± 1! Awesome.

Moving on to traces

- xy trace gives us $z_0^2 - \frac{x^2}{4} - \frac{y^2}{9} = 1$
- z_0 can be any real number

$$\frac{-x^2}{4} - \frac{y^2}{9} = 1 - z_0^2$$

$$\frac{x^2}{4} + \frac{y^2}{9} = z_0^2 - 1$$

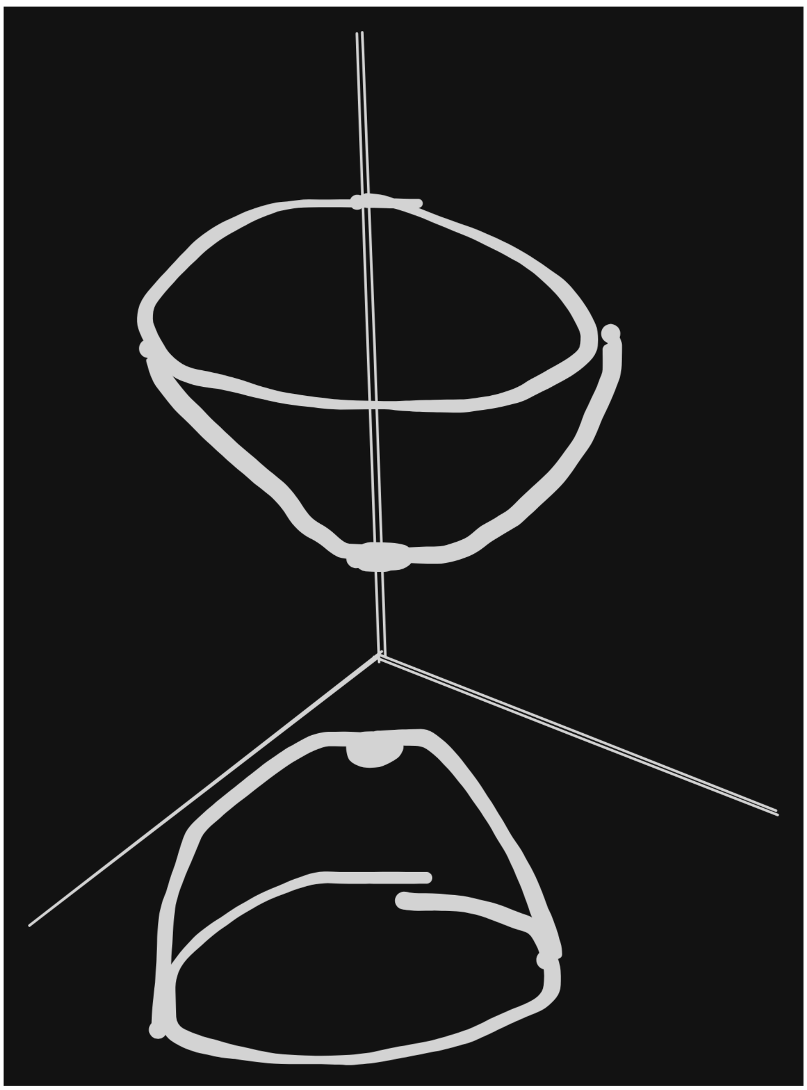
We know z_0^2 is positive, and is a real number.

Let $z_0^2 - z = n, n \in \mathbb{R}$

$$\frac{x^2}{4} + \frac{y^2}{9} = n$$

$$\frac{x^2}{4n} + \frac{y^2}{9n} = 1$$

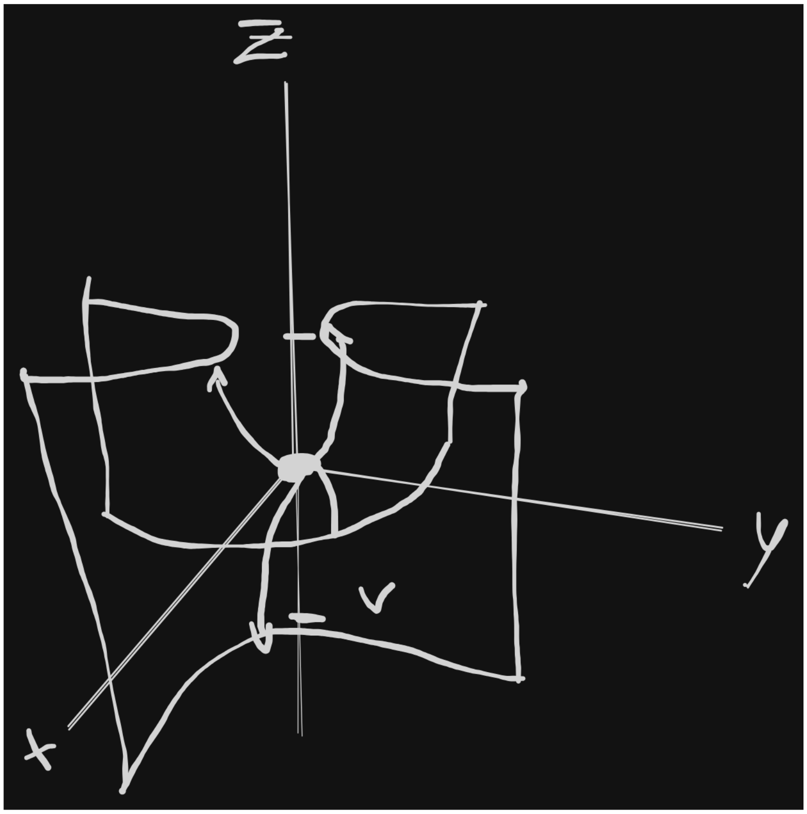
That's an ellipse! I promise.

- This was a nice little baby proof.
- yz trace is just $z^2 - \frac{y^2}{9} = 1$
  - Hyperbola
- xz trace we can also do $x = 0$, giving us $z^2 - \frac{x^2}{4} = 1$
  - One again, a hyperbola
- 
- This is a hyperboloid of two sheets

$$z = y^2 - x^2$$

- You always start with intercepts.

- x intercept, y intercept, and z intercepts are $(0, 0, 0)$
- ok time for traces
  - xy-trace: Let's try $z = 1$, giving us $1 = y^2 - x^2$, which is a cutesy little hyperbola along y
  - Trying $z = -1$, giving us $-1 = y^2 - x^2$ ... kinda works, flip that to be how we like it, giving $1 = x^2 - y^2$, and that works just fine to give a hyperbola along x
  - yz-trace: $x = 0$, giving $z = y^2$, which is honest to god just a parabola (opening to z positive)
  - xz trace, $y = 0$, giving $z = -x^2$, which is yet again just a parabola (opening towards negative z)

- That's a... uh, hyperbolic paraboloid. Yep.
-

Example time

$$-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$$

- Ok I had written

$$-16x^2 - 4y^2 + z^2 + 64x - 80$$

$$-16x^2 + 64x - 4y^2 + z^2 = 80$$

$$-16(x^2 + 4x) - 4y^2 + z^2 = 80$$

$$-16(x^2-4x+4)-4y^2+z^2=80-64$$

$$-16(x-2)^2-4y^2+z^2=16$$

$$-(x-2)^2-\frac{y^2}{4}+\frac{z^2}{16}=1$$

- That eventually manages to produce a hyperboloid of two sheets.
  ($z=\pm 4$)
  It's the two cups spit out just a little bit from the $(x-2)^2$ having that -2

MATH213 - 2024-09-11

#notes #math213 #math #calc

- Last time we were doing the limit definition
  - can we please make this easier

Notation

- Two kinds of notation:
  - Newtonian
    - f_x and f_y is just your function with subscripts
  - Leibniz
    - $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$
  - If defined at (a,b)
    - $f_x(a,b)$, $\frac{\partial f}{\partial x}\Big|_{(a,b)}$

Computing Partials

1. Use the limit definition
   1. (I really don't want to)
2. Or derivative rules! (with a twist)

Example

- Use derivative rules to compute f_x and f_y

$$f(x,y)=\sin(xy)+3x^2+2xy^3-\ln(y)$$

$$f_x = \cos(xy) * y + 6x + 2y^3 - 0$$

$$f_y = \cos(xy)x + 0 + 6xy^2 - \frac{1}{y}$$

again!

$$f(x, y) = (2x - y^3)^4$$

$$f_x = 4(2x - y^3)^3 * 2$$

$$f_y = 4(2x - y^3)^3 * -3y^2$$

Derivative rules will hold for 3 or more variables!

ie, $h(x, y, z, w)$ gives you $h_x, h_y, h_z,$ and h_w

---

beyond the first order (ie, second order)

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

Read from the inside out for ~~leibniz~~ both!

Find all 2nd order partials

$$x^3 + x^2y^2 - 2y^2 + \frac{x}{y}$$

$$f_x = 3x^2 + 2xy^2 + \frac{1}{y}$$

$$f_{xx} = 6x + 2y^2$$

$$f_{xy} = 4xy - \frac{1}{y^2}$$

$$f_y = 2yx^2 - 4y - \frac{x}{y^2}$$

$$f_{yy} = 2x^2 - 4 + \frac{2x}{y^3}$$

$$f_{yx} = 4xy - \frac{1}{y^2}$$

In calc 3, $f_{yx} = f_{xy}$

- This is not true for everything! Just most things you're going to do in calc 3 since we have easy domains

Clairaut's Theorem

- If f is defined on a domain, D in $\mathbb{R}^2$, and both f_{yx} and f_{xy} are continuous on a domain, then they're the sameses

$$f(x,y,z) = e^{xyz} + \sin(\frac{x^2 - \ln(y)}{2y})$$

$$f_y = xz * e^{yxz} + \cos(\frac{x^2 - \ln(y)}{e^{2y}}) * (\frac{-\frac{2y}{y} - 2(x^2 - \ln(y))}{y^2})$$

- Either I missed something or that is just god awful

$$f_{yz} = x * e^{yzx} + x^2 zye^{yzx} + 0$$

$$f_{yzy} = x^2 z * e^{yxz} + x^3$$

I did in fact fucking miss something, you can prove that there's a domain and then you can do z first

$$D := \{x, y, z \in \mathbb{R} \mid y > 0\}$$

We can apply that theorem and do

$$f_{zyy} = yxe^{xyz} + 0$$

$$f_{yy} = x(e^{xyz}) + xy(xze^{xyz})$$

$$f_{yy} = xe^{xyz} + (x^2 yz) * e^{xyz}$$

$$f_{yyy} = x(xze^{xyz}) + (x^2 z)(e^{xyz}) + (x^2 zy)(xze^{xyz})$$

- I was, for the record, doing that right

Activity Stuff

1. Find f_x and f_y for

$$f(x, y) = y^3 \sin(4x)$$

$$f(x, y) = \sqrt{4 + x^2 + y^2}$$

2. Verify $f_{xy} = f_{yx}$ for a domain D for $f(x, y) = 3x^2y^{-1} - 2x^{-1}y^2$

MATH213 - 2024-09-13

#notes #math213 #math #calc

So back in calc 1, we did like

$$f(g(x)) = f'(g(x)) * g'(x)$$

- Which, notably, only really deals with one variable (x, in this case)

Now in this fresh hell

$$z = f(x, y)$$

Chain rule for 1 Independent Variable

- Let z be a differentiable function on its domain where x and y are differentiable with respect to t along an interval I

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left[\frac{dx}{dt}\right] + \frac{\partial z}{\partial y}\left[\frac{dy}{dt}\right]$$

- Note, in this case, t is the only independent variable, even if z is defined as of x and y.
- x and y are known as intermediate variables

Example

Find $\frac{dz}{dt}$ where $z = x \sin y$ and $x = t^2$, $y = 4t^3$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left[\frac{dx}{dt}\right] + \frac{\partial z}{\partial y}\left[\frac{dy}{dt}\right]$$

$$\frac{\partial z}{\partial x} = \sin y$$

$$\frac{\partial z}{\partial y} = x \cos y$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 12t^2$$

$$\frac{dz}{dt} = (\sin y)(2t) + (x \cos y)(12t^2)$$

$$\frac{dz}{dt} = (2t)\sin(4t^3) + 12t^4 \cos(4t^3)$$

Tree Diagrams

- Tree diagrams exist to extend chain rule to more intermediate variables.

$$\frac{dw}{dt} = \sqrt{x^2 + y^2 + z^2}$$

for $x = \sin t, y = \cos t, z = \cos t$

$$\frac{\partial w}{\partial x} = x(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial y} = y(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x} = z(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dt} = -\sin t$$

$$\frac{dz}{dt} = -\sin t$$

$$\frac{dw}{dt} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x \cos t - y \sin t - z \sin t)$$

$$= \frac{1}{\sqrt{\sin^2 t + 2\cos^2 t}}(\cancel{\sin\cos t - \cos t \sin t} - \cos t \sin t)$$

$$= \frac{\cos t \sin t}{\sqrt{\sin^2 t + 2\cos^2 t}}$$

Implicit Differentiation

- Let f be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a function of $x \therefore F(x, y) = F(x, y(x))$
- $$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

example

Find $\frac{dy}{dx}$ for

$$\sin(xy) + \pi y^2 - x = 0$$

$$F_x = y\cos(xy) - 1$$

$$f_y = x\cos(xy) + 2\pi y$$

$$\frac{dy}{dx} = -\frac{y\cos(xy) - 1}{x\cos(xy) + 2\pi y}$$

- as long as they're both 0 this is fine
- Note: implicit differentiation holds when finding partials

example 2: the exampling

If $w = f(x, y, z)$

Let's find $\frac{\partial t}{\partial x}$ and $\frac{\partial x}{\partial y}$
of

$$xy + z = \cos(xz)$$

$$xy + z - \cos(xz) = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{(y + z\sin(xz))}{1 + x\sin(xz)}$$

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = \frac{-x}{y + z\sin(xz)}$$

exampl'nt

$$z = 2x^2 + y^2 + 1$$

and curve in the xy plane given by

$$x = 1 + \cos(t), y = \sin t, 0 \leq t \leq 2\pi$$

Find where z is increasing, aka, find $\frac{dz}{dt} > 0$

$$\frac{dz}{dt} = 4x(-\sin(t)) + (2y)(\cos(t))$$

$$\frac{dz}{dt} = 4(1+\cos(t))(-\sin(t)) + 2(\sin t)(\cos(t))$$

$$-2\sin(t)\left[2(1+\cos t) - \cos t\right]$$

$$= -2\sin t[2+\cos(t)]$$

$$\text{When is sin negative? } \pi < t < 2\pi$$

$$\text{When is cos positive? } 0 < t < 2\pi$$

So our final answer is just $\pi < t < 2\pi$

MATH213 - 2024-09-16

#notes #math213 #math #calc

quick recap

f_x and f_y give the rate of change in the x and y directions

- Minor problem though: three dimensions you can go from *any* direction
    - To do that, it's time to deploy the *directional derivative*
        - Real creative naming here, team.
- Given $z = f(x, y)$ find the RoC of z at x_o, y_o in the direction of an arbitrary unit vector, $\vec{u}$
- There's some shenanigans taking palce, but essentially picking a point on the surface, dropping down to the $x - y$ plane, then using our unit vector for some fun

---

Let f be a differentiable function at (a, b) + let $\vec{u} =< u_1, u_2 >$ be a unit vector in the $x - y$ plane, the directional derivative of f at the point (a, b) in the direction of $\vec{u}$ is

$$D_{\vec{u}}f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

$$D_{\vec{u}}f(a,b) =< f_x(a,b), f_y(a,b) > \cdot \vec{u}$$

Where $\vec{u}$ is a ***UNIT*** vector.

Find $D_{\vec{u}}f(3,2)$. Let $z = \frac{1}{4}(x^2+2y^2+2)$ and $\vec{u} = (\frac{-1}{2}, -\frac{\sqrt{3}}{2})$
Step one.
Check if $\vec{u}$ is a unit vector = yep

Step Two Partials

$$f_x = \frac{1}{4}(2x) = \frac{1}{2}x, f_x(3,2) = \frac{3}{2}$$

$$f_y = \frac{1}{4}(4y) = y, f_y(3,2) = 2$$

$$< f_x, f_y > \cdot \vec{u} =< \frac{3}{2}, 2 > \cdot \vec{u}$$

$$D_{\vec{u}}f(3,2) =< \frac{3}{2}, 2 > \cdot < \frac{1}{2}, \frac{-\sqrt{3}}{2} >$$

$$\frac{3}{2}\left(\frac{1}{2}\right) + 2(\frac{-\sqrt{3}}{2}) = \frac{3}{4} - \sqrt{3} < 0$$

Interpret: Our function is decreasing at $(3,2)$ in the direction of $\vec{u}$ since RoC is negative ($\frac{3}{4} - \sqrt{3}$)

Gradient

- The gradient of f at a point (x,y) is a vector valued function that is used to find rates of change

$$\text{Formal: } \nabla f(x,y) =< f_x(x,y), f_y(x,y) >$$

$$\text{Shorthand: } \nabla f =< f_x, f_y >$$

Compute $\nabla f(x,y) = xe^{2xy}$ for the point $P(1,0)$

$$\nabla f(x,y) = e^{2xy} + x2ye^{2xy}, 2x^2e^{2xy} >$$

$$\nabla f(1,0) = e^0 + 0, 2e^0 =< 1, 2 >$$

Let f be differentiable at a point (a,b) where $\nabla f(a,b) \neq \vec{0}$, Then:

1. f has a maximum rate of increase at (a,b) in the direction of ∇f, given by $|\nabla f(a,b)|$
2. f has a maximum rate of decrease at (a,b) in the direction of $-\nabla f$, $-|\nabla f(a,b)|$
3. f has no change at (a,b) for any orthogonal vector to $\nabla f(a,b)$

$$f(x,y) = x^2 + xy - y^2$$

- Find the RoC of f at P in the direction of $(1,1)$
- Step 1 is finding $\vec{u}$ from $\vec{PQ}$
    - do that, you get $< -1, 2 >$
      - $f_x = 2x + y = 3$
      - $f_y = x - 2y = 4$

$$< 3, 4 > \cdot \frac{< -1, 2 >}{\sqrt{5}} = \frac{1}{\sqrt{5}}(-3+8) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Welp, there it is!

What is the max rate of increase, and in what direction does it occur?

$$|\nabla f(2,-1)| = |< 3, 4 >| = 5$$

in the direction of ∇f
Max rate of decrease is just the negative of that, so negative five.

Find $D_{\vec{u}} f(2,3)$ for $f(x,y) = x^3 - 3xy + 4y^2$ in the direction of $\vec{v} =< 1, 1 >$

- Find $\vec{u}$, current magntitude is $\sqrt{2}$, so entire thing is now $\frac{1}{\sqrt{2}}$ ($\vec{v} =< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$)

Next up is partials!

$$f_x = 3x - 3y, f_x(2,3) = 6 - 9 = -3$$

$$f_y = -3x + 8y, f_y(2,3) = -6 + 24 = 18$$

$$D_u =< -3, 18 > \cdot < \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >= \frac{1}{\sqrt{2}}(-3+18) = \frac{15}{\sqrt{2}}$$

MATH213 - 2024-09-17

#notes #math213 #math #calc

In calc 1, a function described as $y = f(x)$ could be approximated by using tangent lines

In calc 3, a diffferentiable function $z = f(x,y)$ will be approximated using tangent *planes*

- Reminder that planes have a point and a normal vector

Case 1: Implicit

- Let f be differentiable at a point (a,b,c) where $\nabla f(a,b,c) \neq \vec{0}$, then the plane tangent to the surface $f(x,y,z)=0$ is given by:

$$f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)=0$$

$$\nabla f(a,b,c)\cdot <x-a,y-b,z-c>=0$$

- This makes the gradient the normal vector (quirky)
- Also has to end up in standard plane form,

$$Ax+By+Cz=D$$

example

$$2x^2-y^2+z^2=2yz+4$$

$$2x^2-y^2+z^2-2yz-4=0$$

$$F_x=4x=>(1,1,3)=4$$

$$F_y=-2y-2z=>(1,1,3)=-8$$

$$F_z=2z-2y=>(1,1,3)=4$$

$$\nabla F=<4,-8,4>=\text{ Normal vector for this plane.}$$

$$<4,-8,4>\cdot<x-1,y-1,z-3>=0$$

$$4(x-1)-8(y-1)+4(z-3)=0$$

$$4x-4-8y+8+4z-12=0$$

$$4x-8y+4z=-8$$

Case 2: Explicit

$$z=f(x,y)$$

Let f be differentiable at (a,b). The equation of the plane tangent to $z=f(x,y)$ at the point $(a,b,f(a,b))$ is given by

$$z=f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$$

Example

$f(x,y) = \sin(xy) + 2$, Find the plane tangent to the surface at the point $(1,0,2)$

$$f_x = y\cos(xy) => (1,0) = 0$$

$$f_y = x\cos(xy) => (1,0) = 1$$

$$z = 0(x-1) + 1(y-0) + 2$$

$$z = y + 2$$

Linear Approximation

- Back in the days of calc 1,
- For a function $y = f(x)$, we said

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

In calc three, though, we use ✈️s. (Planes, so I don't kill my searchability)

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

---

egg sample

- $f(x,y) = \frac{1}{x+y^2}$, Find $L(x,y)$ of f at $(-3,2,1)$
  - Could rewrite as $z = (x+y^2)^{-1}$

$$f_x = \frac{-1}{(x+y^2)^2} - 1$$

$$f_y = \frac{-1}{(x+y^2)^2} => f_y(3,2) = -4$$

$$L(x,y) = -1(x+3) - 4(y-2) + 1$$

$$L(x,y) = -x - 3 - 4y + 8 + 1$$

$$x + 4y = 6$$

$$L(x,y) = -x - 4y + 6$$

Use $L(x,y)$ to approximate $(-3.1, 2.05)$

$$L(-3.1, 2.05) = -(-3.1) - 4(2.05) + 6$$

$$3.1 - 8.2 + 6 = 0.9$$

MATH213 - 2024-09-18

#notes #math213 #math #calc

The Total Differential

- Something about the total change in z given only x and y? I dunno, formal definition to follow.
- For functions of the form $z = (x, y)$, the linear approximation was given by $L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$
- However, the exact change from a point (a, b) to (x, y) would be given by $\Delta z = f(x, y) - f(a, b)$

$$= \Delta z \approx f_x(a, b)(x - a) + f_y(y - b) + \cancel{f(a, b) - f(a, b)}$$

- So funny story. We accidentally shot the z out of the equation, because $f(a, b)$ was really our only representation. Whoops?
- We kinda need differentials to appear, but conveniently we *already have them*, where

$$x - a => \Delta x \approx dx$$

$$y - b => \Delta y \approx dy$$

$$\Delta z \approx dz = f_x(a, b)dx + f_y(a, b)dy$$

- In general, the differential dz from a point (x, y) to $(x + dx, y + dy)$ is $dz = f_x(x, y)dx + f_y(xy)dy$

Example

$$z = x^2 + 3xy - y^2$$

- Find the total differential dz

$$f_x = 2x + 3y$$

$$f_y = 3x - 2y$$

$$dz = (2x + 3y)dx + (3x - 2y)dy$$

- Use dz to calculate the change from$(3, -1)$ to $(2.96, -0.95)_{\text{what the fuck is that?}}$
    - $dx = -0.04$, $dy = 0.05$
- $$dz = [(2(3) + 3(-1))] + [3(3) - 2(-1)](0.05) = -0.12 + .55 = 0.43$$

Example 2: The wording

- The dimensions of a rectangular box are said to be 4 ± 0.1 cm, 3 ± 0.1 cm, 8 ± 0.2 cm, What is the maximum possible error in computing the volume of the box?
  - $3.9 * 2.9 * 7.8 = 89.218$ cm^3 is the minimum possible volume
  - $4.1 * 3.1 * 8.2 = 104.222$ cm^3 is the largest possible volume
    - So in this case it would be 15.004 cm^3, just from me doing it
    - So I apparently fucked this up
- Formal Method
  - We have a box
  - $v = xyz$
  - $dv = V_x dx + V_y dy + V_z dz$
  - $V_x = yz => V_x(4,3,8) = 24$
  - $V_y = xy => V_y(4,3,8) = 32$
  - $V_z = xy => V_z(4,3,8) = 12$
  -

$$dV = 24(0.1) + 32(0.1) + 12(0.2) = 2.4 + 3.2 + 2.4 = 4.8 + 3.2 = 8 \text{ cm}^3$$

- Sidequest, find the bounding for volume
  - $V(4,3,8) = (4)(3)(8) = 96$
  - $$96 - 8 \leq V \leq 96 + 8$$
  - $$88 \leq V \leq 104$$
    - Oh hey, there's my fuck up. How neat.

MATH213 - 2024-09-20

#notes #math213 #math #calc

Maxes and Mins

- A function $z = f(x,y)$ has a local max at a point (a,b) if there exists a disk, D, such that $f(a,b) \geq f(x,y)$
- The function has a local min at (a,b) if $f(a,b) \leq f(x,y) \; \forall \; (x,y) \; \epsilon \; D$

How do we find Extrema?

- Find Critical Points!
- (a, b) is a critical point for f along a domain D (our disk)
    - If $f_x(a, b) = f_y(a, b) = 0$, that there's a critical a point
        - That means the change in both directions is zero.
    - That's literally the only one you need to know
        - For fun though, if at least one partial, f_x or f_y is nonexistent, that's also a critical point

Example

- Find CPs for $f(x, y) = \frac{1}{3}x^3 + \frac{4}{3}y^3 - x^2 - 3x - 4y - 3$
    - $f_x = x^2 - 2x - 3$
    - $f_y = 4y^2 - 4$
    - Ok those are our partials, where are they zero?
    - $x^2 - 2x - 3 = 0, = (x - 3)(x + 1)$, so $x = 3, -1$
    - $y = -1, 1(4(y^2 - 1))$
    - CPs:
        - $(3, 1), (3, -1), (-1, 1), (-1, 1)$

Two Ways to Classify Extrema

- Note, if $f_x(a, b) = f_y(a, b) = 0$ (if they're a critical point, then this will make sense)
    - If $f_{xx} > 0$ and $f_{yy} > 0$, the concavity is all facing up, so (a, b) is a local minimum
    - If $f_{xx} < 0$ and $f_{yy} < 0$, the concavity is all down, so (a, b) is a local maximum
    - If $f_{xx} < 0$ and $f_{yy} > 0$, the concavity is all fucked up, so (a, b) is a saddle point

Second Derivative Test (but calc 3)

- Suppose the 2nd partials of f are continuous throughout an open disk, D, centered at (a, b) where $f_x(a, b) = f_y(a, b) = 0$.
- Let $D = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2$
    - If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, (a, b) is a local max
    - If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, (a, b) is a local min
    - If $D(a, b) < 0$, we have a saddle point, no need to check anything else.

- If $D(a,b)=0$, inconclusive
- Ok, talking about that D thing we defined earlier
  - The Hessian matrix is a square ($n \times n$) matrix where n is the number of variables that are independent, that (something) for all 2nd order partials.
- $$\begin{matrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{matrix} => \det H = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2$$

Example 2: There's no joke here. It's too early.

- Find and classify CPs using 2nd derivative test for $f(x,y)=x^2+2y^2-4x+4y+6$
  - Partials!
    - $f_x = 2x-4$
    - $f_{xx} = 2$
    - $f_{xy} = 0$
    - $f_y = 4y+4$
    - $f_{yy} = 4$
    - $f_{yx} = 0$
- $f_x = 2x-4=0, x=2$
- $f_y = 4y+4=0, y=-1$
- CP: $(2,-1)$
- $D=2(4)-(0)^2=8$
  - Given that f_{xx} is just two, we have ourselves a local min.

Example 3: The Example Strikes Back

- Yes I know empire is episode five, I don't care, this was supposed to be a brutal one.
- Find and classify all CP for $f(x,y)=x^4+y^4+4xy$
  - $f_x = 4x^3+4y$
  - $f_{xx} = 12x^2$
  - $f_{xy} = 4$
  - $f_y = 4y^3+4x$
  - $f_{yy} = 12y^2$
  - $f_{yx} = 4$
- These things have to be zero at the same time!
  - $4x^3+4y=0$
  - $4y^3+4x=0$

- $4(-x^3)^3 + 4x$
  - $-4x^9 + 4x$
  - $-4x(x^8 - 1) = 0$
- $x = 0, 1, -1$
- For $x = 0 => y = -0^3 => (0, 0)$
- For $x = 1, y = -1^3 = (1, -1)$
- For $x = -1, y = -(-1)^3 = (-1, 1)$
- $D = 12x^2(12y^2) - 4^2 = 144x^2y^2 - 16$

CPs	$D(x,y)$	$f_{xx}(x,y)$	**Classify**
$(0,0)$	$-16 < 0$	Don't care!	Saddle Point
$(1,-1)$	$144 - 16 > 0$	$12(1)^2 > 0$	**Local** Min
$(-1,1)$	$144 - 16 > 0$	$12(-1)^2 > 0$	**Local** Min

MATH213 - 2024-09-23

#notes #math213 #math #calc

- Let $f(x, y)$ be continuous and bounded in the region R, in $\mathbb{R}^3$, containg (a, b), then:
  1. $f(a, b) \geq f(x, y) \forall (x, y) \epsilon R => f(a, b)$ is an absolute max
  2. $f(a, b) \leq f(x, y) \forall (x, y) \epsilon R, => f(a, b)$ is an absolute min
- Procedure
  1. Find all possible critical points for $f_x = f_y = 0$
  2. Test boundary points (your endpoints)
  3. Choose largest and smallest values

This is a long example.

- Find the absolute min/max values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region $R : \{(x, y) \mid 0 \leq x \leq 15, 0 \leq y \leq 15 - x\}$
  - Partials
    - $f_x = y - 8$
    - $f_y = x - 2y + 12$
  - Find Cps
    - $f_x, y = 8$
    - $f_y = x - 2(8) + 12, x = 4$
    - CP: $(4, 8, (4, 8))$

- $f(4,8) = 32 - 32 - 64 + 96 + 160 = 192$
- So we have one CP at $(4, 8, 192)$ that we found from searching around inside of our bounds

- Boundary Testing
  - $R := \{(x,y) | 0 \leq x \leq 15, 0 \leq y \leq 15 - x\}$
  - This shits out our try angel, which has three possible line segments
    - I don't wanna sketch this. Me eepy.
    - Alrighty, so testing our things and finding maxes/mins for segment one (the bottom leg of our triangle)
      - $f(x, 0) = -8x + 160$
      - Hey minor problem, the first derivative test gives us a whopping jack didley squat
      - We still need to test the endpoints though
      - $g(0) = 160, g(15) = 40$
        - Which yields us the points $(0, 0, 160)$ and $(15, 0, 40)$
    - Testing for leg two (upright leg of triangle)
      - $0 \leq y \leq 15, x = 0$
      - $f(0, y) = -y^2 + 12y + 160$
      - We're saying here that $g_2(y) = -y^2 + 12y + 160$
        - $g_2' = -2y + 12$
        - Oh hey, we *do* get a critical point! at $y = 6$
      - $g_2(0) = 160$
      - $g_2(6) = -36 + 72 + 160 = 196$
      - $g_2(15) = -225 + 180 + 160 = 115$
      - Which gives us the points:
        - $(0, 0, 160)$
        - $(0, 6, 196)$
        - $(0, 15, 115)$
    - Testing for segment three (hypotenuse)
      - $0 \leq x \leq 15, y = 15 - x$
      - $f(x, 15 - x) = x(15 - x) - 8x - (15 - x)^2 + 12(15 - y) + 160$
      - $15x - x^2 - 8x - (225 - 30x + x^{2)} + 180 - 12x + 160$
      - $g_3 = -2x^2 + 25x + 115 = g_3(x)$
      - $g_3'(x) = -4x + 25, x = \frac{25}{4}, y = \frac{15-25}{4} = \frac{35}{4}$
      - Endpoint testing
        - $g_3(0) = 115$
        - $g_3(\frac{25}{4}) \approx 193.1$

- $g_3(15) = 40$
- Gives the points
  - $(0, 15, 115)$
  - $(\frac{25}{4}, \frac{35}{4}, 193.1)$
- If we go compare all of our points
  - Our absolute min is $f(15, 0, 40)$
  - And our absolute max is $(0, 6, 196)$

Sweet Suffering (there's another example)

Find absolute mins/maxes along $R := \{(x, y)| -1 \leq x \leq 1, -1 \leq y \leq 1\}$, $z = 4 + 2x^2 + y^2$

- Partials
  - $f_x = 4x$
  - $f_y = 2y$
    - We have ourselves a critical point at $x = 0$, and $y = 0$, and then we gotta plug those back in to our original equation, which gives us a whopping four.
- Boundary Shenanigans
  - We have ourselves a god damn square. That has four sides. I hate it here.
  -

MATH213 - 2024-09-24

#notes #math213 #math #calc

Lagrange Multipliers!

I was late.

- Normal vector to the level curve ∇f is parallel to the normal vector of the function $g(x, y) = k$, ∇g
- Which gives you that $\nabla f = \lambda \nabla g$
- where $\lambda \epsilon \mathbb{R}$

Example of

- Find all absolute min/maxes of $f(x, y) = x^2 + 2y^2$ on $x^2 + y^2 = 1$

    - Oh hey, a paraboloid on the unit circle
- Doing our gradients
    - $\nabla f =< 2x, 4y >$
    - $\nabla g =< 2x, 2y >$
    - Which then yields
        - $< 2x, 4y >= \lambda < 2x, 2y >$
        - $x^2 + y^2 = 1$
    - Strategy: Solve one gradient at a time
        - $2x = \lambda(2x)$
            - $2x - \lambda 2x = 0$
            - $2x(1 - \lambda) = 0$
                - This yields us two cases: either $x = 0$ or $\lambda = 1$
                - For $x = 0$
                    - $x^2 + y^2 = 1$, so $y^2 = 1, y = \pm 1$
                    - This pops out the critical points $(0, 1)$ and $(0, -1)$
                - For $\lambda = 1$
                    - $4y = \lambda(2y)$
                    - $4y = 2y$
                    - $y = 0$ got panicked there for a second
                    - $x^2 = 1, x = \pm 1$
                    - This pops out the critical points $(1, 0)$ and $(-1, 0)$

CP	$f(x, y)$	Extrema
$(0, 1)$	2	Abs. Max
$(0, -1)$	2	Abs. Max
$(1, 0)$	1	Abs. Min
$(-1, 0)$	1	Abs. Min

Application Problem? I sure do have one!

- Find the dimensions of a rectangular box (without a top) that maximizes volume when given 12 units^2 of material
- $v = xyz$
- $g(x, y, z) = xy + 2yz + 2xz = 12$
    - $\nabla V =< yz, xz, xy >$
    - $\nabla g =< y + 2z, x + 2z, 2x + 2y >$

        - $< yz, xz, xy >= \lambda < 2+2z, x+2z, 2x+2y >$
    - $xy + 2xz + 2yz = 12$
- If it's hard to isolate your variables (xyz are fairly intertwined here), try isolating λ
    - $yz = \lambda(y+2z)$
        - Which just gives that λ = $\frac{yz}{y+2z}$
    - $xz = \lambda(x+2z)$
        - $\lambda = \frac{xz}{x+2z}$
    - $xy + \lambda(2x+2y)$
        - $\lambda = \frac{xy}{2x+2y}$
- We have a whole pile of lambdas, so

  -

  $$\frac{yz}{y+2z} = \frac{xz}{x+2z}$$

  $$y\cancel{z}(x+2z) = x\cancel{z}(y+2z)$$

  $xy + 2y = xy + 2xz, y = x$

  $$\frac{xz}{x+2z} = \frac{xy}{2x+2y}$$

  $$y = 2z$$

- Now that we've done all that algebra, we're going to go back to the constraint equation
    - $(2z)(2z) + 2(2z)(z) + 2(2z)z = 12$
    - $4z^2 + 4z^2 + 4z^2 = 12$
    - $12z^2 = 1$
    - $z = 1$
        - Dimensions can't be negative.
    - Given that z is one, that means that
        - $y = 2$, and x is also $= 2$
- Given all of this, the dimensions that maximize the volume are 2 units by 2 units by one unit.
-

MATH213 - 2024-09-25

#notes #math213 #math #calc

Example Ein

- Find the point on the plane $x + y + 2z = 4$ that's closest to the origin
  - Minimize distance
    - That's going to be our function, $d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$
    - Due to some quirky algebra, it actually works out the exact same where $d^2 = x^2 + y^2 + z^2$ is the same.
  - Constraint is going to be the plane
  - So, writing that back out
  - $f(x, y, z) = x^2 + y^2 + z^2$
  - $g(x, y, z) = x + y + 2z = 4$
    - $\nabla f =< 2x, 2y, 2z >= \lambda < 1, 1, 2 >$
      - $2x = \lambda$
      - $2y = \lambda$
      - $2z = \lambda(2) = \lambda = z$
      - $\lambda = 2x = 2y = z$
      - $x = y$
      - $z = 2x$
    - Constraint equation becomes
      - $x + x + 2(2x) = 4$
      - $6x = 4$
      - $x = \frac{4}{6} = \frac{2}{3}$
      - $y = \frac{2}{3}$
      - $z = \frac{4}{3}$
    - Which gives the point $(\frac{2}{3}, \frac{2}{3}, \frac{4}{3})$

Example Du

- Find the min and max values of $z = 4x^2 + 10y^2$ on a disc given by $s := \{(x, y) \mid x^2 + y^2 \leq 4\}$

1. Find critical points on the interior
   1. So we're just going to use partials for this
   2. $f_x = 8x$ which is 0 at 0
   3. $f_y = 20y$ which is also 0 at 0
   4. So our interior CP is $(0, 0, f(x, y))$
2. Finding critical points on the boundary
   1. $f(x, y) = 4x^2 + 10y^2$
   2. $g(x, y) = x^2 + y^2 = 4$

1. $\nabla f = < 8x, 20y >= \lambda < 2x, 2y >$
2. Along the bound $x^2 + y^2 = 4$
   1. $8x = \lambda(2x)$
      1. $2x(4 - \lambda) = 0$
         1. So either x is 0 or lambda is 4.
   2. Plugging back in to constraints, we get that $y = \pm 4$
   3. Yielding the critical points $(0, 2), (0, -2)$
   4. Plugging our lambda back in to the y section
      1. $20y = \lambda(2y)$
      2. $12y = 0$
      3. So that happens at $y = 0$
      4. So now that we have this, plug *back* in to the constraint
      5. $x = \pm 2$
      6. Popping out $(2, 0), (-2, 0)$

CP	$f(x, y)$
$(0, 0)$	0
$(0, 2)$	40
$(0, -2)$	40
$(2, 0)$	16
$(-2, 0)$	16

- Looks like our absolute min is our 0 point on the interior, our 40s are on our edge, and we don't give a single damn about the 16s.

Example Tres

- Find the point on $x^2 + \frac{y^2}{25} + \frac{z^2}{9} = 1$ such that the sum of the point is maximized
  - That's an ellipsoid.
- $x + y + z$ is what we're maximizing
  - $f(x, y, z) = x + y + z$
  - $g(x, y, z) = x^2 + \frac{y^2}{25} + \frac{z^2}{9}$
    - $\nabla f = < 1, 1, 1 >= \lambda < 2x + \frac{2y}{25} + \frac{2z}{9} >$
    - Still restricted on our ellipsoid $x^2 + \frac{y^2}{25} + \frac{z^2}{9}$
  - $1 = \lambda 2x$
    - So $\lambda = \frac{1}{2x}$

    - $1 = \lambda(\frac{2y}{25})$
        - so $\lambda = \frac{25}{2y}$
    - $1 = \lambda(\frac{2z}{9})$
        - $\lambda = \frac{9}{2z}$
- $\frac{1}{2x} = \frac{25}{2y} = \frac{9}{2z}$
    - $2y = 50x$
    - $y = 25x$
- $50z = 18y$
    - $y = \frac{50z}{18}$
        - Or like, $\frac{25z}{9}$, but that's lame.
- $\frac{25z}{9} = 25x = y$
- $z = 9x$, pops out, which is cute.
- $x^2 + \frac{(25x)^2}{25} + \frac{(9x)^2}{9} = 1$
- $x^2 + 25x^2 + 9x^2 = 1$
- $35x^2 = 1$, so $x = \pm\frac{1}{\sqrt{35}}$
- Which pops our point
    - $(\pm\frac{1}{\sqrt{35}}, \pm\frac{25}{\sqrt{35}}, \pm\frac{9}{\sqrt{35}})$
    - That's technically like six options, but the max is just where they're all positive.
    - (so like, $\frac{35}{\sqrt{35}}$)

Example Fou- oh god we've been cut loose

- Use LMM to find points closest to the origin on $xy = 3$, and find what that distance is.
- $d = \sqrt{(x-0)^2 + (y-0)^2}$
- Because we're dealing with distance, we can just square this whole affair, getting that
- $d^2 = x^2 + y^2$
- Giving our equations
    - $f(x, y) = x^2 + y^2$
    - $g(x, y) = xy = 3$
- Aaand then we get to work
    - $\nabla f =< 2x, 2y >= \lambda < y, x >$
    - And then we can solve for some shenanigans
        - $2x = \lambda y$
            - So $\lambda = \frac{2x}{y}$
        - And then in our other equation, $2y = \lambda x$
            - So $\lambda = \frac{2y}{x}$

- So $\frac{2x}{y} = \frac{2y}{x}$
      - $2x^2 = 2y^2$, so $x = y$
  - Plugging back in to our constraint equation
    - $x(x) = 3$
    - $x^2 = 3$
    - So $x = \sqrt{3}$
    - And we know that $x = y$, so $y = \sqrt{3}$
  - Yielding the point $(\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$
  - And the question asks for minimizing distance, which, uh, doesn't matter if it's positive or negative because it's squared, so we just get a distance of $d^2 = 3 + 3$, so $d = \sqrt{6}$

Now number five

- Find absolute extrema of $f(x, y, z) = xyz$ subject $x^2 + 2y^2 + 3z^2 = 6$
- I guess f is just f?
  - $f(x, y, z) = xyz$
  - $g(x, y, z) = x^2 + 2y^2 + 3z^2 = 6$
  - I think we just go here?
    - $\nabla f =< yz, xz, xy >= \lambda < 2x, 4y, 6z >$
    - So now at least vaguely attempting to solve this here
      - $yz = \lambda 2x$
      - $\lambda = \frac{yz}{2x}$
      -

MATH213 - 2024-09-30

#notes #math213 #math #calc

Double Integrals!

- Back in Calc 1, area under a curve was found by... doing the integral. You choose a section, you chop it into sections, yadayadayada
- In Calc 3, we will find the *volume* bounded by a surface over a region R
  - Instead of rectangles when we do our integral chunks, we get rectangular prisms (so l*w*h instead of just l*w)

- Given $f(x,y) \geq 0$ on a rectangular region R in the xy-plane, where $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$, where f is continuous
    - $V = \int\int f(x,y)dA = \int_c^d \int_a^b f(x,y)dx$
        - This is doing x first, you go from the inside out
    - Note: It should be the same solution no matter which way you go, but one of them is *probably* going to be easier.
        - This is because f is continuous, yadayadayada.

How? Why? What must we do?

1. Split the area into n many subrectangles (rectangular prisms, really)
    1. $\Delta A_x = \Delta x_k \Delta y_k$
2. Select points (x_n^*, y_n^*)
    1. The star often means we have a height, so like, evaluate and go crazy
3. Approximate the volume of each subrectangle
    1. $f(x_k^*, y_k^*)\Delta A_k$
4. Add up all rectangles

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

5. Get the exact volume by taking the limit

$$V = \lim_{\Delta A_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k = \iint_R f(x_k^*, y_k^*)\Delta A_k$$

Calc 1 Techniques Still work!

- You can still use basic integral rules, u sub, trig identities, integration by parts, FTC (the ~~federal trade commission?~~ fundamental theorem of calculus. duh.)

Example time

$$\int_0^2 \int_0^1 5xy \, dxdy$$

$$\int_0^2 \left[5y\left(\frac{x^2}{2}\right)\Big|_0^1\right] dy$$

$$\int_0^2 \left[\frac{5y}{2} - 0\right] dy$$

$$\left[\left(\frac{5}{2}\right)\left(\frac{y^2}{2}\right)\right]\Big|_0^2 = \frac{5}{4}(2)^2 - 0 = 5$$

Go on, do it again.

$$\iint_R ye^{xy} dA$$

$$R = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq \ln(2)\}$$

- Now, this wasn't given to us explicitly, so we can choose to do x or y first
  - Let's uh, do x first. I don't want to do integration by parts, largely because I don't remember.

$$\int_0^{\ln(2)} \left[\int_0^1 ye^{xy} dx\right] dy$$

$$\int_0^{\ln(2)} [y(\frac{1}{y})e^{xy)}\big|_0^1]$$

$$\int_0^{\ln(2)} [e^y - e^0] dy$$

$$[e^y - y]\Big|_0^{\ln(2)} = 1 - \ln(2)$$

Example Three: The Third One

$$\iint \frac{x}{(1+xy)^2} dA, R = \{(x, y)|0 \leq x \leq 4, 1 \leq y \leq 2\}$$

- Let's do y first, less complicated

$$\int_0^4 \left[\int_1^2 \frac{x}{(1+xy)^2} dy\right] dx$$

$$u = 1 + xy, du = xdy$$

$$\int_0^4 \left[\int \frac{1}{u^2} du\right] dx$$

$$\int_0^4 \left[\frac{-1}{1+xy}\Big|_1^2\right]$$

$$\int_0^4 [\frac{-1}{1+2x} + \frac{1}{1+x}]dx$$

- You split that. Because like, that's a lot of work.

$$\int_0^4 \frac{-1}{1+2x}dx + \int_0^4 \frac{1}{1+x}$$

$w = 1 + 2x, dw = 2dx$

$$\int \frac{-1}{2u}du + \int_0^4 \frac{1}{1+x}dx$$

$$\frac{-1}{2}\ln(1+2x)\Big|_0^4 + \ln(1+x)\Big|_0^4$$

$$\frac{-1}{2}(\ln(9) - \ln(1)) + (\ln(5) - \cancel{\ln(1)})$$

$$= \frac{-1}{2}\ln(9) + \ln(5)$$

$$= -\ln(3) + \ln(5)$$

$$= \ln(\frac{5}{3})$$

We try!

We try a one...

$$\int_1^2 \int_{-1}^1 (x^2 + xy)dydx$$

$$\int_1^2 (x^2y + \frac{xy^2}{2}\Big|_{-1}^1)dx$$

$$\int_1^2 (-x^2 + \frac{x}{2})dx$$

$$\left(\frac{-x^3}{3} + \frac{x^2}{4}\right)\Big|_1^2$$

$$\left(-\frac{8}{3} + \frac{4}{4}\right) - (\frac{-1}{3} + \frac{1}{4})$$

$$-\frac{8}{3} + \frac{4}{4} + \frac{1}{3} - \frac{1}{4}$$

$$-\frac{7}{3}+\frac{3}{4}=\frac{-28}{12}+\frac{9}{12}=\frac{19}{12}$$

This is apparently $\frac{14}{3}$, somewhere I have gone terribly wrong.

We try a two...

$$\int_0^1\int_1^2 \frac{ye^y}{x}dxdy$$

$$\int_0^1\left(ye^y\int_1^2\frac{dx}{x}\right)dy$$

$$\int_0^1(ye^y\ln(x)\Big|_1^2)dy$$

$$\int_0^1(ye^y\ln(2)-ye^y\ln(1))dy$$

$$\ln(2)\int_0^1 ye^y-\ln(1)\int_0^1 ye^y$$

MATH213 - 2024-10-01

#notes #math213 #math #calc

$$\iint_R y\cos(xy)dA$$

$$R=\{(x,y)\mid 0\le x\le 1, 0\le y\le\frac{\pi}{3}\}$$

$$\int_0^{\frac{\pi}{3}}\int_0^1 y\cos(xy)dxdy$$

$$\int_0^{\frac{\pi}{3}}(\cancel{y}\sin(xy))\Big|_0^1 dy$$

$$\int_0^{\frac{\pi}{3}}\sin(xy)\Big|_0^1 dy$$

$$\int_0^{\frac{\pi}{3}}(\sin(y)-\cancel{\sin(0)})dy$$

$$=-\cos(y)\Big|_0^{\frac{\pi}{3}}=-\cos\left(\frac{\pi}{3}\right)-(-\cos(0))=-\frac{1}{2}+1=\frac{1}{2}$$

- Just as a reading reminder, the volume bounded by this region is $\frac{1}{2}$

Example!

- Find the volume of the solid bounded by $f(x, y) = 9 - xy - y, x = 3, y = 2$ and the coordinate planes (which are $x = 0, y = 0$)
- Hey, uh, what the fuck are our bounds?
  - because like, we're doing $\iint (9 - xy - y)dA$ but like, from where?
- Answer is you do a little shenanigans with the fact that $x = 3$ and $y = 2$ are both planes, and we just go from there
  - With $x = 3$ and $x = 0$, we have $0 \leq x \leq 3$
  - and $0 \leq y \leq 2$

$$\iint_0^3 (9 - xy - y)dxdy$$

$$\int_0^2 (9x - \frac{x^2y}{2} - xy\Big|_0^3)dy$$

$$\int_0^2 (27 - \frac{9}{2}y - 3y)dy$$

$$\int_0^2 (27 - \frac{15}{2}y)dy$$

$$27y - \frac{15}{2}\left(\frac{y^2}{2}\right)\Big|_0^2$$

$$(54 - 15) - 0 = 39$$

Example.. Six? What the hell is counting?

- Find the volume of the solid in the first octant bounded above by $f(x, y) = 9xy\sqrt{1 - x^2}\sqrt{1 - y^2}$ and below by the xy plane
  - We uh, need to find our damn bounds again. They keep running off.
    - xy-plane means that $x, y \geq 0$
    - From the surface shenanigans:
      - $1 - x^2 \geq 0$
        - $x^2 \leq 1$
        - $x \leq \pm 1$, but we're in that whole first octant thing, so $0 \leq x \leq 1$
      - $1 - y^2 \geq 0$
        - It's the exact same thing. $0 \leq y \leq 1$

- $$\int_0^1 \int_0^1 9xy\sqrt{1-x^2}\sqrt{1-y^2}dxdy$$

$$\int_0^1 \left(9y\sqrt{1-y^2}\int_0^1 x\sqrt{1-x^2}dx\right)dy$$

Let $u = 1-x^2, du = -2xdx, \frac{-1}{2}du = xdx$

$$\int_0^1 (9\sqrt{1-y^{2)}} - \frac{1}{2}\sqrt{u}\,du)dy$$

$$\int_0^1 -\frac{9}{2}\sqrt{1-y^2}\left(\frac{2}{3}u^{\frac{3}{2}}\right)$$

$$\int_0^1 -\frac{9}{2}\cdot\frac{2}{3}y\sqrt{1-y^2}((1-x^2)^{\frac{3}{2}})\Big|_0^1$$

$$\int_0^1 -3y\sqrt{1-y^2}(-1)dy$$

$$3\int_0^1 y\sqrt{1-y^2}dy$$

Let w be $1-y^2$, $dw = -2y\,dy$, $\frac{-1}{2}dw = ydy$

$$3\int -\frac{1}{2}dw\sqrt{w}$$

$$\left(-\frac{3}{2}\right)\left(\frac{2}{3}w^{\frac{3}{2}}\right) = -1(1-y^2)^{\frac{3}{2}}\Big|_0^1 = -(0-1) = 1$$

MATH213 - 2024-10-02

#notes #math213 #math #calc

Shape-Matching Warm-Up Thing

A. 4
B. 6
C. 2
D 5
E. 3
F. 1

Maxes and Mins

- Use the 2nd derivative test to classify cps of $f(x,y) = xy(x-2)(y+3)$
  - Now, me personally? I fuck with this all being expanded. So let's do that.

$$f(x,y) = (xyx - 2xy)(y+3)$$

$$f(x,y) = x^2y^2 + 3x^2y - 2xy^2 - 6xy$$

- Not at all ugly. Nope.
  - As a retrospective note, this was not worth it. Don't do that.

$$f_x = 2xy^2 + 6xy - 2y^2 - 6y$$

$$f_y = 2x^2y + 3x^2 - 4xy - 6x$$

$$f_{xx} = 2y^2 + 6y$$

$$f_{yy} = 2x^2 - 4x$$

$$f_{xy} = 4xy + 6x - 4y - 6$$

$$2xy^2 + 6xy - 2y^2 - 6y = 2x^2y + 3x^2 - 4xy - 6x = 0$$

$$-3x^2 + 2xy^2 - 2x^2y + 6xy - 4xy - 6x - 2y^2 - 6y = 0$$

$$x(-3x + 2y^2 - 2xy + 2y - 6) - y(2y - 6) = 0$$

$$x(-3x + 2y^2 - 2xy + 2y - 6) = y(2y - 6)$$

$$x(-3x - 6 + y(2y - 2x + 2)) = y(2y - 6)$$

So, critical-y point at $y = 0$ and $y = 3$, at the very least. I may have gone too deep.
So at $y = 0$,

$$x(-3x - 6) = 0$$

, so x can either be 0 or -2

And at $y = 3$

$$x(-3x - 6 + 3(6 - 2x + 2)) = 0$$

$$x(-3x - 6 + 18 - 6x + 6) = 0$$

$$x(-9x + 18) = 0$$

So x can either be 0 or negative two. Hey howdy.

So it sounds like our critical points are

Point	Value
$(0,0)$	
$(0,3)$	
$(-2,0)$	
$(-2,3)$	

- Ok, os what you actually do, if you have the better partials

  $$f_x = y(y+3)(2x-2)$$

  and

  $$f_y = x(x-2)(2y+3)$$

  So our possible points from f_x are $y=0, y=-3, x=1$

So you do some mapping by plugging in
If, for instance, y=0

$$x(x-2)(3) = 0$$

- This gives us the valid critical points of $(0,0)$ and $(2,0)$
  Now, if y were to be -3

  $$x(x-2)(-3) = 0$$

- this is again, 0 or two. Giving us $(0,-3)$ and $(2,-3)$

Now, if x were to be one

$$1(1-2)(2y+3) = 0$$

$$-2y+3 = 0$$

Which gives $y = -\frac{3}{2}$, and the whopping one critical point of $(1, -\frac{3}{2})$

- How did I lose this point in my original work? The fuck?

D is just slapping our partials together, so

$$D = (2y(y+3))(2x(x-2)) - ((2x-2)(2y+3))$$

Point	Value	Double Partial	Conclusion
$(0,0)$	-36	Don't care!	Saddle point
$(2,0)$	-36	Don't care!	Saddle point

Point	Value	Double Partial	Conclusion
$(0, -3)$	-36	Don't care!	Saddle point
$(2, -3)$	-36	Don't care!	Saddle point
$(1, -\frac{3}{2})$	D>0	Negative	Local max

- Find absolute max and min values using lagrange for $f(x, y) = 2x + y = 10$ subject to the boundary $g(x, y) = 2(x-1)^2 + 4(y-1)^2 = 1$

$$\nabla f =< 2, 1 >= \lambda < 4(x-1), 8(y-1) >$$

On the constraint curve $2(x-1)^2 + 4(y - 10^2 = 1)$

$$2 = \lambda(4x - 4)$$

$$\lambda = \frac{1}{2(x-1)}$$

$$1 = 8\lambda(y-1)$$

$$\lambda = \frac{1}{8(y-1)}$$

$$\frac{1}{2(x-1)} = \frac{1}{8(y-1)}$$

$$8(y-1) = 2(x-1)$$

$$x - 1 = 4y - 4$$

$$x = 4y - 3$$

$$2(4y - 3 - 1)^2 + 4(y-1)^2 = 1$$

$$2(4y-4)^2 + 4(y-1)^2 = 1$$

$$2(4y-4)(4y-4) + 4(y-1)(y-1) = 1$$

$$2(4)(y-1)(4)(y-1) + 4(y-1)(y-1) - 1 = 0$$

$$(4y-4)(8(y-1) + y - 1 - 1) = 0$$

$$(4y-4)(8y - 8 + y - 2) = 0$$

$$(4y-4)(9y-10) = 0$$

$$36y^2 - 40y - 36y + 40 = 0$$

$$36y^2 - 76y + 40 = 0$$

$$4(9y^2 - 19y + 10) = 0$$

MATH213 - 2024-10-04

#notes #math213 #math #calc

Double Integrals over General Regions

- So like, not just rectangular regions

Type 1: Vertical Regions

- Rectangles here are going to be parallel to the y axis, x bounds are going to be from a, b but y is going to vary between your functions, $h(x)$ and $g(x)$
- Type 1 will have one top function and one bottom function for the entire region
- $R := \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$

$$V = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx$$

- This is *y* first, just writing that for posterity

Type 2: Horizontal

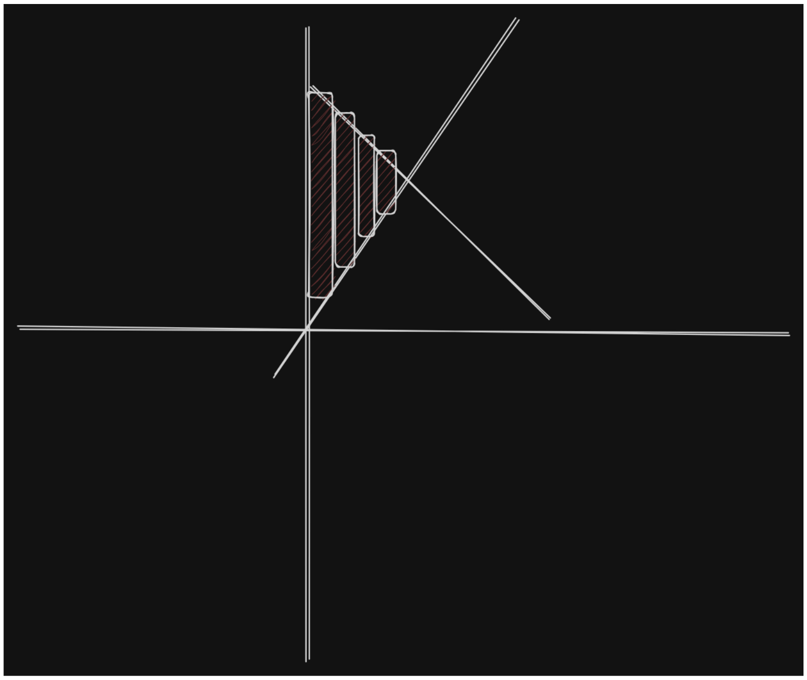
- Rectangles are going to be parallel to the y axis (that shit crazy. who could've seen this coming.
- Guess what? $R := \{(x, y) \mid g(y) \leq x \leq h(y), c \leq y \leq d\}$

$$\int_c^d \int_{g(x)}^{h(x)} f(x, y) dx dy$$

- Fun Note: there will not be functions on the outer bound in calc 3...

Exampling Time

- Let R be the region in the plane between $y = 2x, y = 6x$, and the y axis. Compute $\iint_R (x - 2y) dA$



- Quick bound solving, $6-x=2x$ at $x=2$
- This is going to be a TYPE ONE, with vertical bounds

$$\int_0^2\int_{2x}^{6-x}(x-2y)dydx$$

- And now we integrate

$$\int_0^2(xy-y^2\Big|_{2x}^{6-x})dx$$

$$\int_0^2(x(6-x)-(6-x)^2)-(x(2x)-(2x)^2)dx$$

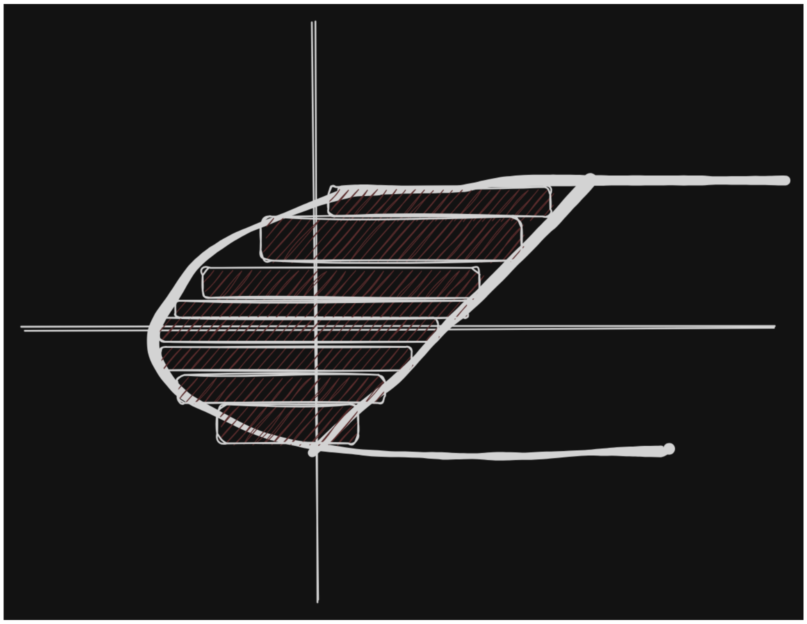
$$\int_0^2(6x-x^2-36+12x-x^2-2x^2+4x^2)dx$$

$$\int_0^2(18x-36)dx$$

$$9x^2-36x\Big|_0^2=36-72=-36$$

Example but again (there was a brief interlude regarding the chopping of fingers before this)

- Compute $\iint_R xy\,dA$ bounded by $y=x-1$ and $y^2=2x+6$
- $x=\frac{1}{2}y^2-3$



$$\int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dxdy$$

$y+1=\frac{1}{2}y^2-3$
$\frac{1}{2}y^2-y-4=0$
$y^2-2y-8=(y+2)(y-4)$

$$\int_{-2}^{4}\frac{x^2y}{2}\Big|_{\frac{1}{2}y^2-3}^{y+1}$$

$$\frac{1}{2}\int_{2}^{4}\left((y+1)^2y-\left(\frac{1}{2}y^2-3\right)^2y\right)dy$$

$$=\frac{1}{2}\int_{-2}^{4}y(y+1)^2dy-\frac{1}{2}\int_{-2}^{4}(\frac{1}{2}y^2-3)^2ydy$$

- Let u = y+1, du = dy, y = u-1

$$\frac{1}{2}\int(u-1)u^2du=\frac{1}{2}(\frac{u^4}{4}-\frac{u^3}{3})$$

$$\frac{1}{2}(\frac{(y+1)^4}{4}-\frac{(y+1)^3}{3}\Big|_{-2}^{4})$$

$$\frac{1}{2}\left(\left(\frac{5^4}{4}-\frac{5^3}{3}\right)-(\frac{1}{4}+\frac{1}{3})\right)$$

$$-\frac{1}{2}\int_{-2}^{4}(\frac{1}{2}y^2-3)^2y\,dy$$

- W = $\frac{1}{2}y^2-3$, $dw=ydy$

$$\frac{-1}{2}\int w^2 dw$$

$$-\frac{1}{2}\frac{w^3}{3}$$

$$\frac{-1}{6}\int\left(\frac{1}{2}y^2-3\right)^3\Big|_{-2}^{4}$$

$$-\frac{1}{6}(5^3-(-1)^3)$$

$$\iint=\frac{1}{2}(\frac{5^4}{4}-\frac{5^3}{3})-\frac{1}{4}-\frac{1}{3}-\frac{1}{6}(5^3+1)$$

$$\iint=\frac{5^4}{8}+\frac{5^3}{6}-\frac{1}{8}-\frac{1}{6}-\frac{5^3}{6}-\frac{1}{6}$$

$$\iint=\frac{5^4-1}{8}-\frac{2(5^3+1)}{6}$$

$$\iint=\frac{5^4-1}{8}-\frac{5^3+1}{3}=78-42=36$$

MATH213 - 2024-10-07

#notes #math213 #math #calc

$$\int_0^9\int_{3x-9}^{2x} xy\ dxdy$$

- This is really nice horizontally, which makes me think I actually did this backwards, hang on.
    - Totally, had in fact, done it backwards.

$$x=\frac{y}{2}$$

$$\frac{y}{3}+3=x$$

Oh, so you just do an arbitrary function

$$\int_0^{18}\int_{\frac{y}{2}}^{\frac{y}{3}+3} f(x,y)dxdy$$

- This one's going to be vertical

$$\int_{-3}^{4}\int_{2x^2}^{2x+24} f(x,y)dydx$$

$$2x+24=2x^2$$

$$2x^2-2x=24$$

$$x^2-x=12$$

$$x=-3,4$$

$$\int_0^4\int_0^{\sqrt{x}}(1)dydx$$

$$\int_0^2\int_{y^2}^4(1)dxdy$$

$$\int_0^4(y)dx$$

$$\int_0^4\sqrt{x}\,dx$$

$$\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^4$$

$$\frac{2x^{\frac{3}{2}}}{3}$$

$$\frac{16}{3}$$

$$\int_0^1\int_y^1 e^{x^2}dxdy$$

- Apparently we need to reverse the order of integration to be able to integrate?
  - I don't know how tf that works, so I'm just going to swap things and pray.

$$\int_y^1\int_0^1 e^{x^2}dydx$$

$$\int_y^1(ye^{x^2}\Big|_0^1)dx$$

$$\int_y^1 e^{x^2}dx$$

$$\frac{e^{x^2}}{2x}\bigg|_y^1$$

$$\frac{e}{2} - \frac{e^{y^2}}{2y}$$

- I have deeply fucked something up.

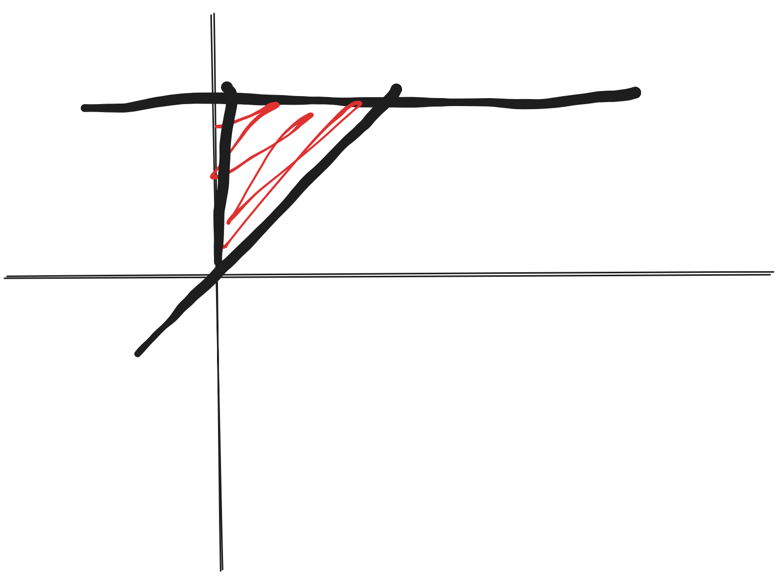
$$\int_0^1 \int_0^x e^{x^2} dy$$

$$\int_0^1 (e^{x^2} y\Big|_0^x)$$

$$\int_0^1 xe^{x^2} dx = \frac{1}{2} \int e^u du$$

$$\frac{1}{2}(e^1 - e^0) = \frac{1}{2}(e - 1)$$

$$\int_0^\pi \int_x^\pi \sin(y^2) dy dx$$



$$\int_0^\pi \int_0^y \sin(y^2) dx dy$$

$$\int_0^\pi (x \sin(y^2)\Big|_0^y) dy$$

$$\int_0^\pi (y \sin(y^2)) dy$$

$$u = y^2, du = 2y, y = \frac{du}{2}$$

$$\frac{1}{2}\int_0^{\pi} du \sin(u)$$

$$-\frac{1}{2}\cos(u) = -\frac{1}{2}\cos(y^2)\Big|_0^{\pi}$$

$$-\frac{1}{2}(\cos(\pi^2) - \cos(0))$$

$$\frac{-1}{2}(\cos(\pi^2) - 1)$$

$$\frac{1}{2}(1 - \cos(\pi^2))$$

MATH213 - 2024-10-08

#notes #math213 #math #calc

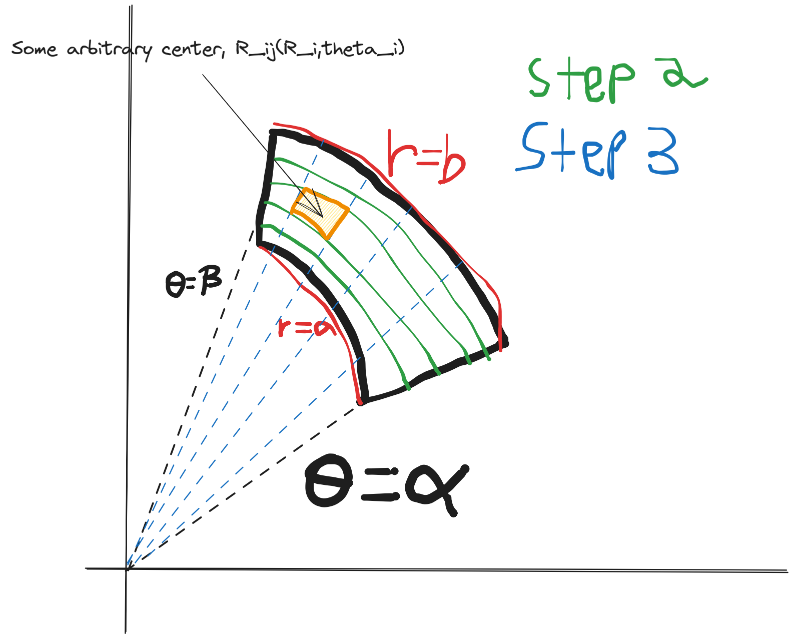
Double integral over *polar* regions

- I don't really like polar.
  - I actually don't hate it that much, but it's the principle of the matter
- Suppose we're given $\iint_R f(x, y)dA$ for some circle, like $R = x^2 + y^2 = 1$
  - Or some donut-y section, like the region between $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$
- This is noticeably easier than Cartesian... for situations that it makes sense in
  - Ie, where $R := \{(r, \theta) \ | \ 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$
    - where, for a unit circle, r is between 0 and 1
    - and θ is between 0 and 2pi
  - Our half donut is just $R := \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

Brief Recap

- $x^2 + y^2 = r^2$
  - $x = r\cos\theta$
  - $y = r\sin\theta$
- Hey wait a second, how do we integrate in polar again?
  1. Divide $[a, b]$ into m-many subintervals $[R_{i-1}, r_i]$ of equal width $\Delta r = \frac{b-a}{m}$
  2. We also have to divide the *other* interval, $[\alpha, \beta]$, into n-many subintervals $[\theta_{j-1}, \theta_j]$ of

equal width $\Delta\theta = \frac{\beta-\alpha}{n}$
3. The "circles," $r = r_i$ and the "rays," $\theta = \theta_j$, will divide R into polar subrectangles.
- That sure is a lot of words.



4. Add up all polar rectangles
    1. We're using the center of each subrectangle, $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$
    2. $\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) * \Delta A_{ij}$
5. As we take the limit and our number of rectangles goes to infinity, the sum is going to become $\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$
    1. Pirates gotta love doing these with how much they get to say *r*

Example!

- Evaluate $\iint_R (3x + 4y^2) dA$ for a region R bounded in the upper half-plane by $x^2 + y^2 = 1$ & $x^2 + y^2 = 4$
    - Look, I'm not drawing this, but it's the top half of a donut, with the unit circle as the inner radius and $r = 2$ as the outer radius.

$$\int_0^\pi \int_1^2 (3r\cos\theta + 4(r\sin\theta)^2) r dr d\theta$$

$$\int_0^\pi \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\; dr d\theta$$

$$\int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta \Big|_1^2\right] d\theta$$

$$\int_0^{\pi} (8\cos\theta + 16\sin^2\theta - \cos\theta - \sin^2\theta) d\theta$$

$$\int_0^{\pi} (7\cos\theta + 15\sin^2\theta) d\theta$$

- Hey, quick recall that $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

$$\int_0^{\pi} 7\cos\theta d\theta + \frac{15}{12}\int_0^{\pi} (1 - \cos(2\theta)) d\theta$$

$$7\sin\theta\Big|_0^{\pi} + \frac{15}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi}$$

$$0 + \frac{15}{2}\theta\Big|_0^{\pi} + 0 = \frac{15\pi}{2}$$

- God damn it the trig identities. They're coming back to haunt me. I hate it here. AAAAARRRRGH

Example but it's less nice and easy (why must we suffer so)

- Find the volume of solid bounded by $z = 9 - x^2 - y^2$ and the xy plane.
  - That shit is a paraboloid.
    - However, we're in just the xy plane! That means we're a circle of radius 3. Awesome.
- $R = \{(r, \theta) \mid 0 \le r \le 3, 0 \le \theta \le 2\pi\}$

$$\int_0^{2\pi}\int_0^{3} (9 - r^2) r dr d\theta$$

Quick Aside

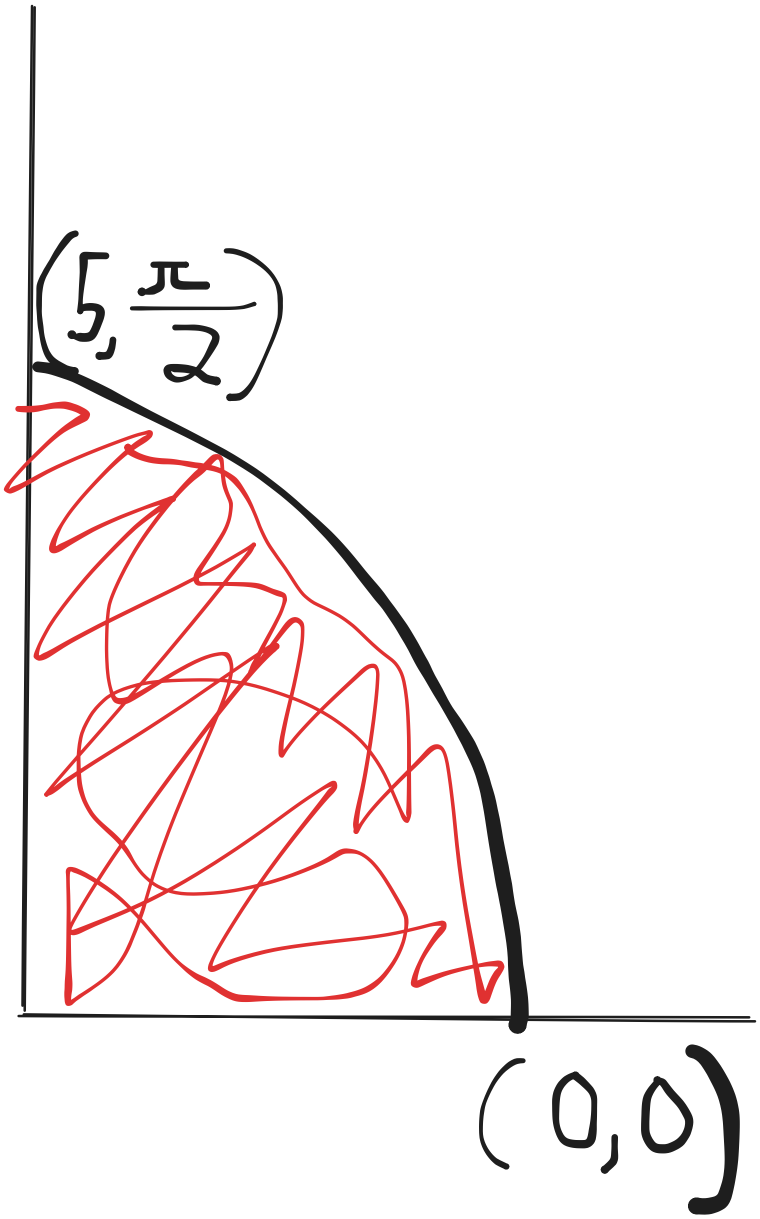
- Motherfucker is named GUIDO FUBINI? AND HE GOT FUBINI'S THEOREM?

Back to the example

$$\int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4}\Big|_0^2\right] d\theta$$

$$= \frac{81}{4}\theta\Big|_0^{2\pi} = \frac{81\pi}{2}$$

1. Sketch $R := \{(r, \theta) \mid 0 \leq r \leq 5, 0 \leq \theta \leq \frac{\pi}{2}\}$



2. Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and region $R := \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

$$\int_0^{2\pi} \int_0^1 (4 - r^2) r dr d\theta$$

MATH213 - 2024-10-09

#notes #math213 #math #calc

more polar things

- A general polar region, but defined by two functions of theta
  - So r is going to go from $f(\theta)$ to $h(\theta)$ (there's not really a top or a bottom, because we're woogling our way around radially, so that just gets kind of weird)

$$R := \{(r, \theta) \mid 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$$

First up: Area of a polar region

- Using our generalized R equation (that I wrote up there)
- $0 < \beta - \alpha \leq 2\pi$
- This is going to be a double integral (shocking)

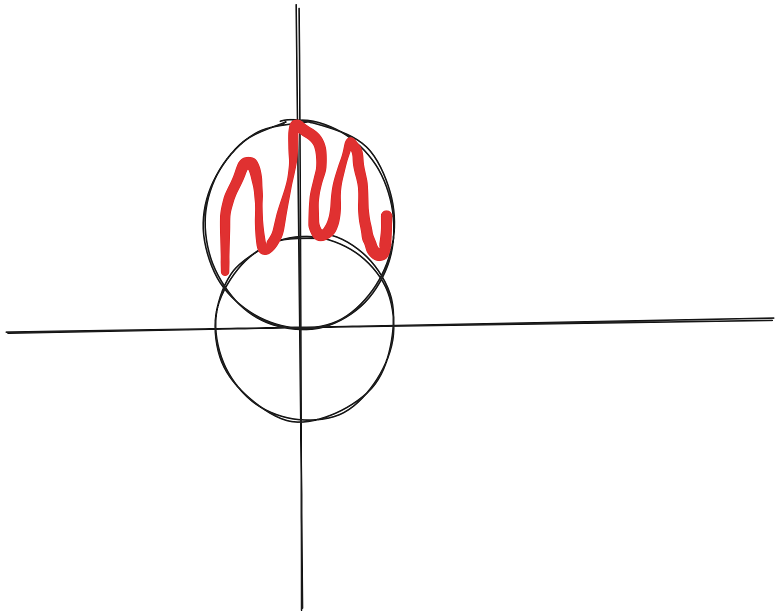
$$\iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} r dr d\theta$$

Recall:

- $r = 2a\cos\theta + 2b\sin\theta$ is a *circle*
  - Note, those 2's are part of the formula, so if you're solving for radius, it's *just* $\sqrt{a^2 + b^2}$
- If $r = 2a\cos\theta$, it's a circle extending along the x axis (sin would be vertical)

Example!

- Find the area of the region outside $r = 1$ and inside $r = 2\sin\theta$
  - so $r = 1$ is just going to be the unit circle
  - and $r = 2\sin\theta$ is going to be a circle opening along y, but you can also make a table if you don't remember that



$$R := \{(r,\theta) \mid 1 \leq r \leq 2\sin\theta\}$$

- We have not a fucking clue what the bounds for theta are. Shucks.
  - Where does $2\sin\theta = 1$ for $\theta \in [0,\pi]$
    - That's just where $\sin\theta = \frac{1}{2}$ for the upper two quadrants, and the answer is $\frac{\pi}{6}$ and $\frac{5\pi}{6}$
- Oh hey, we have everything we need for R, so now we can set up our double integral.

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{1}^{\sin(\theta)} r dr d\theta$$

- That's it! We're set up! Now we just integrate.

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [\frac{r^2}{2}\Big|_{1}^{2\sin\theta}] d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\frac{4\sin^2\theta}{2} - \frac{1}{2}) d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [2(\frac{1-\cos(2\theta)}{2}) - \frac{1}{2}] d\theta$$

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (\frac{1}{2} - \cos(2\theta)) d\theta$$

$$(\frac{1}{2}\theta - \frac{\sin(2\theta)}{2})\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$\left(\frac{1}{2}\left(\frac{5\pi}{6}\right)-\frac{\sin(\frac{5\pi}{3})}{2}\right)-\left(\frac{1}{2}\left(\frac{\pi}{6}\right)-\frac{\sin(\frac{\pi}{3})}{2}\right)$$

$$\frac{5\pi}{12}-\frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right)-\frac{\pi}{12}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

Volume of a general polar region

- Over our same generalized R that we had earlier (I will not, in fact, be writing it out again {even though I could copy paste [do I look like I care]})

$$V=\int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta), rdrd\theta$$

Example (this does not seem fun)

- Find the volume of the solid that lies under $z=x^2+y^2$ and the cylinder $x^2+y^2=2x$ in the xy plane.
  - $z=x^2+y^2$ is a paraboloid (it's the spiral thingy, if you're actually looking at it)
  - $x^2+y^2=2x$ is a cylinder parallel to z
    - Can we uh, learn anything about it?
      - $x^2-2x+y^2=0$
      - Which is then $(x-1)^2+y^2=1)$
        - Oh hey, that's a circle of radius 1 centered at $(1,0)$

$$R:=\{(r,\theta)\mid 0\le r\le 2\cos\theta, 0\le\theta\le\pi\}$$

$$V=\int_0^{\pi}\int_0^{2\cos\theta}(r^2\cos^2\theta+r^2\sin^2\theta)rdrd\theta$$

$$\int_0^{\pi}\int_0^{2\cos\theta} r^3drd\theta$$

$$\int_0^{\pi}(\frac{r^4}{4}\Big|_0^{2\cos\theta})d\theta$$

$$\int_0^{\pi}\left(\frac{16\cos^4\theta}{4}-0\right)d\theta$$

$$\int_0^{\pi}4\left(\frac{1+\cos(2\theta)}{2}\right)^2d\theta$$

$$\int_0^\pi (1 + 2\cos(2\theta) + \cos^2(2\theta))d\theta$$

$$\theta + \sin(2\theta)\Big|_0^\pi + \int_0^\pi \frac{1+\cos(4\theta)}{2}$$

$$\left(\theta + \sin(2\theta) + \frac{\theta}{2} + \frac{1}{8}\sin(4\theta)\right)\Big|_0^\pi$$

$$\frac{3\theta}{2} + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\Big|_0^\pi$$

$$\left(\frac{3\pi}{2} + 0 + 0\right) - (0 - 0 - 0) = \frac{3\pi}{2}$$

---

quick bit of forbidden knowledge?

DO NOT DO THIS

- You will find this if you're googling, or maybe in another professor's office hours

$$\iint_R f(x)g(y)dxdy = \int f(x)dx \; * \int g(y)dy$$

- If you can decouple your functions, you can integrate separately and multiply
- The way that this works is that technically the result of one of the integrals is going to be constant, so you're "pulling out a constant" and that's actually the other integral
- This **ONLY** happens if you can show that over a region, you have decoupled variables

$$\int_1^2 \int_1^e y\ln(x)dxdy = \int_1^e \ln(x)dx \; * \int_1^2 ydy$$

- None of your bounds are in function terms
    > - This is claiming that volume is $L_1W_1 * L_2W_2$ instead of $L * W * H$, which like, doesn't make sense

DO NOT SPLIT INTEGRALS OVER MULTIPLICATION

MATH213 - 2024-10-16

#notes #math213 #math #calc

Triple Integrals

- Consider the function $w = f(x, y, z)$ that is defined on the closed and bounded region D in $\mathbb{R}^3$
- The graph of $f(x, y, z)$ lives in the 4th dimensional space with the point $(x, y, z, f(x, y, z))$ where $(x, y, z) \in D$

How to Build a Triple Integral

- Partition D
  - We now partition with planes, which is kinda quirky.
    - We mostly (entirely?) use the xy, yz, and xz planes
  - So now that we've partitioned a nice chunk of 3d space, we're going to get rectangular prisms, as opposed to rectangles (boxes is occasionally used as a relevant term)
- We're going to add up all of our "boxes," $1 \ldots k$
  - This forms a cute little Riemann sum of the form

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

- If f is continuous, take the limit of the sum as $\Delta \to 0$, where Δ is the change in the length of the diagonal of each box
- Out pops the integral

$$\iiint_D f(x, y, z)dV$$

Two Interpretations

1. If $f(x, y, z) = 1$ then we're finding some volume

$$\iiint_D 1dV = \text{Volume of D}$$

2. If $f(x, y, z)$ is density (per unit volume) of a substance,

$$\iiint_D f(x, y, z)dV = \text{Mass at a point (huh. weird.)}$$

Theorem

- If f is continuous along a region D, where $D := \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$, then $\iiint_D f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\ dxdydz$
  - We end up with six possible combinations from switching all this around
    - Local Italian (Fubini)'s theorem still applies and you can just kinda get jiggy with it so long as they're continuous.

Notation

- If D is a **box**, then you can use $D := [a, b] \times [c, d] \times [e, f]$
  - which is just literally giving the dimensions of a box, blank by blank by blank.

Example

$$\iiint_d xyz^2 dV$$

$$D := \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$$

- Sketch of this sure is a box.

$$\int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dxdydz$$

$$\int_0^3 \int_{-1}^2 (\frac{x^2}{2} yz^2 \Big|_0^1) dydz$$

$$\int_0^3 \int_{-1}^2 (\frac{1}{2} yz^2) dydz$$

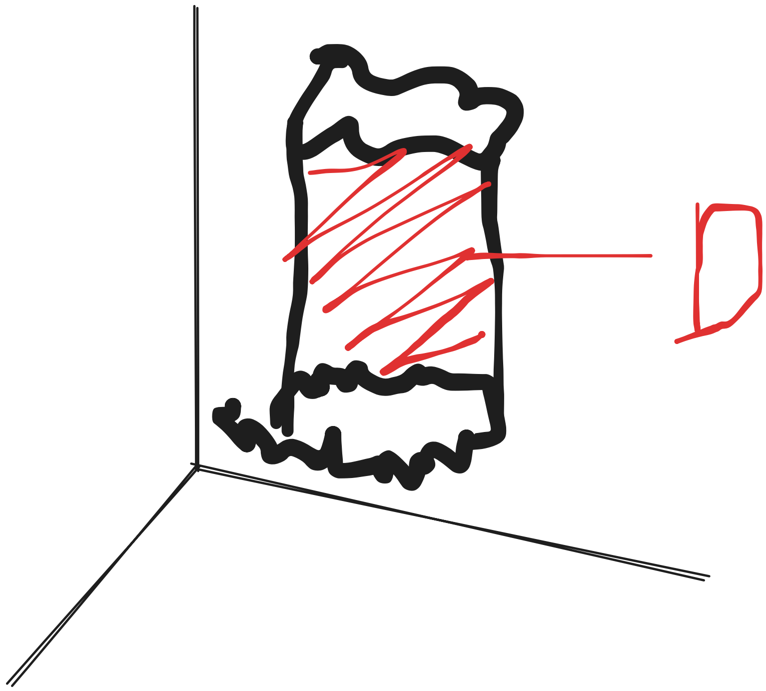
$$\int_0^3 (\frac{y^2}{4} z^2 \Big|_{-1}^2) dz$$

$$\int_0^3 (\frac{3}{4} z^2) dz$$

$$\frac{z^3}{4} \Big|_0^3 = \frac{27}{4}$$

Ok but like what if your region *isn't* a nice box I hate it here.

Top to Bottom

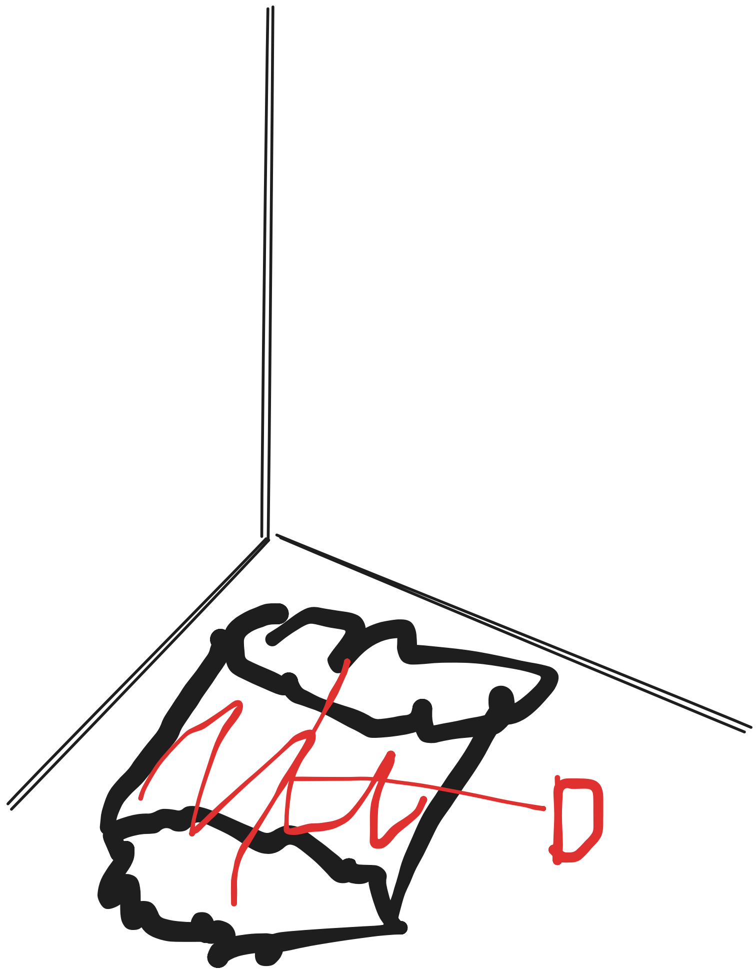


- You got two blobs that you're connecting
- Our rectangles are parallel to the z axis, so you need to do dz first.

$$\iint_R \left[\int (f, x, y, z) dz\right] dA$$

- When you do your projections, you actually just end up projecting down to the xy plane.

Front to Back

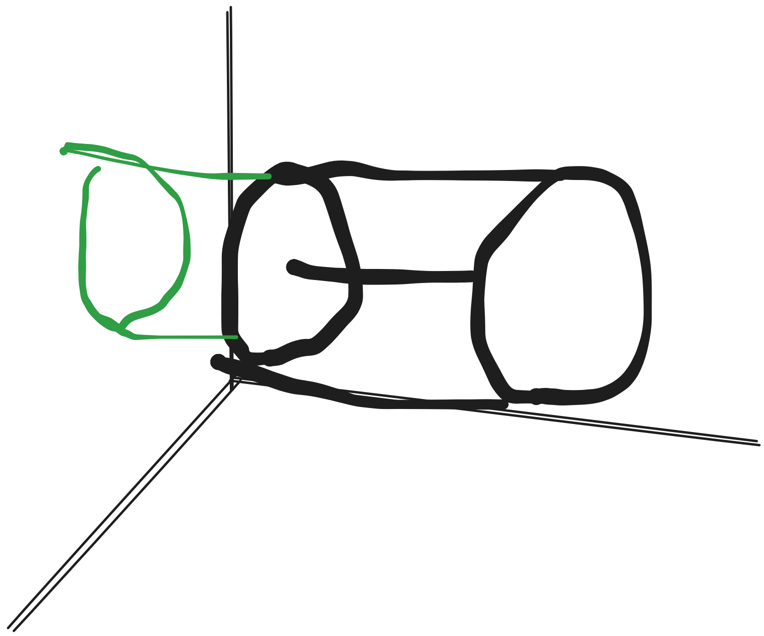


- Here we end up parallel to x (this looks a little squished)

$$\iint_R \left[\int_{G(y,z)}^{H(y,z)} f(x, y, z < dx) \right] dA$$

- Projecting backwards, staying parallel to x, we end up spitting ourselves onto the xy plane

Left to Right



- Here we have

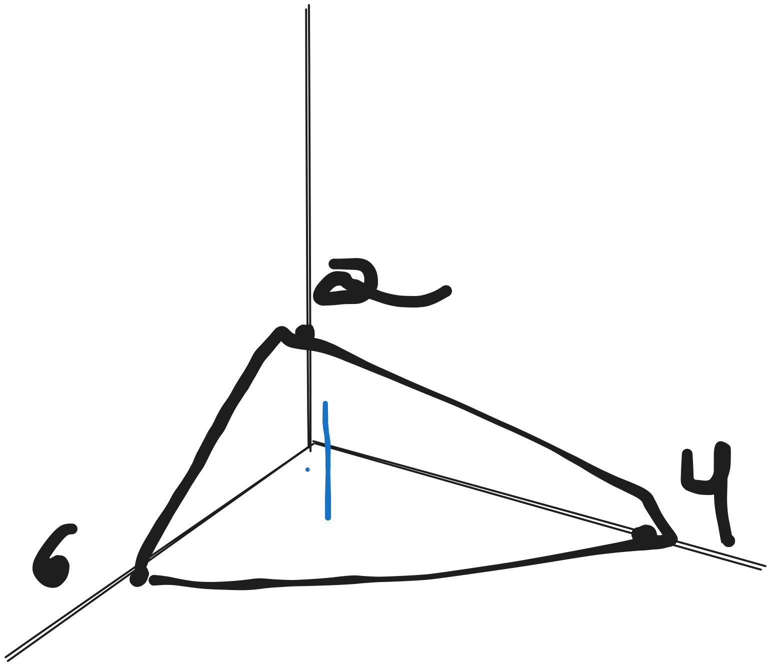
$$\iint_R \left[\int_{G(x,z)}^{H(x,z)} f(x,y,z)dy\right] dA$$

- And we'll be projecting onto the xz plane, we have to figure out what the region we're projecting is, all that fun stuff.

Note: in Each case, R will need to be found, it is not a duplicate of G or H

Example 2 (we're trying here, folks)

- Find the volume of the solid bounded by $2x + 3y + 6z = 12$ and the coordinate planes.



- This is going to be a top to bottom type situation

$$\iint_R [\int_{G(x,y)}^{H(x,y)} 1dz]dA$$

$$H(x,y) : 2x + 3y + 6z = 12$$

$$6z = 12 - 2x - 3y$$

$$z = 2 - \frac{1}{3}x - \frac{1}{2}y$$

$$\iint_R [\int_0^{2-\frac{1}{3}x-\frac{1}{2}y} 1dz]dA$$

MATH213 - 2024-10-18

#notes #math213 #math #calc

Previously On

- Last time we were working with the solid bounded by $2x + 3y + 6z = 12$ and the coordinate planes, which that equation is another plane

$$\iint [\int_0^{2-\frac{1}{3}x-\frac{1}{2}y} 1dz]dA$$

- So if we were to find R, we would be projecting down into the xy plane by letting z be zero.

- That gives us a triangle.
  - (with a vertex at $(0,4)$ and $(6,0)$)
- Let's say we're going vertical, mostly because why not (and it lets us find things in terms of x, which is just about normal)

$$\int_0^6 \int_0^{-\frac{2}{3}x+4} (\int_0^{2-\frac{1}{3}x-\frac{1}{2}y} dz) dydx$$

$$\int_0^6 \int_0^{-\frac{2}{3}x+4} \left(2 - \frac{1}{3}x - \frac{1}{2}y\right) dydx$$

$$\int_0^6 \left(2y - \frac{1}{3}xy - \frac{1}{4}y^2\Big|_0^{-\frac{2}{3}x+4}\right) dx$$

$$\int_0^6 \left(2\left(-\frac{2}{3}x+4\right) - \frac{1}{3}x\left(-\frac{2}{3}x+4\right) - \frac{1}{4}\left(-\frac{2}{3}x+4\right)^2\right) dx$$

$$\int_0^6 \left(-\frac{4}{3}x + 8 + \frac{2}{9}x^2 - \frac{4}{3}x - \frac{1}{4}\left(\frac{4}{9}x^2 - \frac{16}{3}x + 16\right)\right) dx$$

$$\int_0^6 \left(-\frac{8}{3}x + 8 + \frac{2}{9}x^2 - \frac{1}{9}x^2 + \frac{4}{3}x - 4\right) dx$$

$$\int_0^6 \left(-\frac{4}{3}x + \frac{1}{9}x^2 + 4\right) dx$$

$$= (\frac{-2}{3}x^2 + \frac{1}{27}x^3 + 4x)\Big|_0^6$$

$$= 8$$

---

hear me out: we do it again

- Compute $\iiint_D \sqrt{x^2+z^2} dV$ where D is bounded by $y = x^2 + z^2$ and $y = 4$
- This is a nice, friendly paraboloid, that you'd be happy to take home to show your parents, and we're slicing it with the plane $y = 4$ because we are cruel and terrible people.
  - We're going to slice this in line with the y axis so that we're not going to the exact same function every time.
    - So, dy first

$$\int\int \left[\int_{x^2+z^2}^4 \sqrt{x^2+z^2} dy\right] dA$$

In $\mathbb{R}^2$, we are in the $xz-$plane, where $(y = 0)$
In cartesian, $-\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}$
So, $-z \leq x \leq 2$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{4-x^2}\left[\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}dy\right]dzdx$$

$$\int_{-2}^{2}\int_{\sqrt{bs}}^{\sqrt{bs}}(\sqrt{x^2+z^2}(4-(x^2+z^2)))dzdx$$

- Oh hey, once we're a double integral, we can actually treat this like a polar integral
- $x^2+z^2 = r^2$, so $\sqrt{x^2+z^2} = r$

$$\int_{0}^{2\pi}\int_{0}^{2}[r(4-r^2)]rdrd\theta$$

$$\int_{0}^{2\pi}\int_{0}^{2}[4r^2-r^4]drd\theta$$

$$\int_{0}^{2\pi}(\frac{4}{3}r^3-\frac{1}{5}r^5\Big|_0^2)d\theta$$

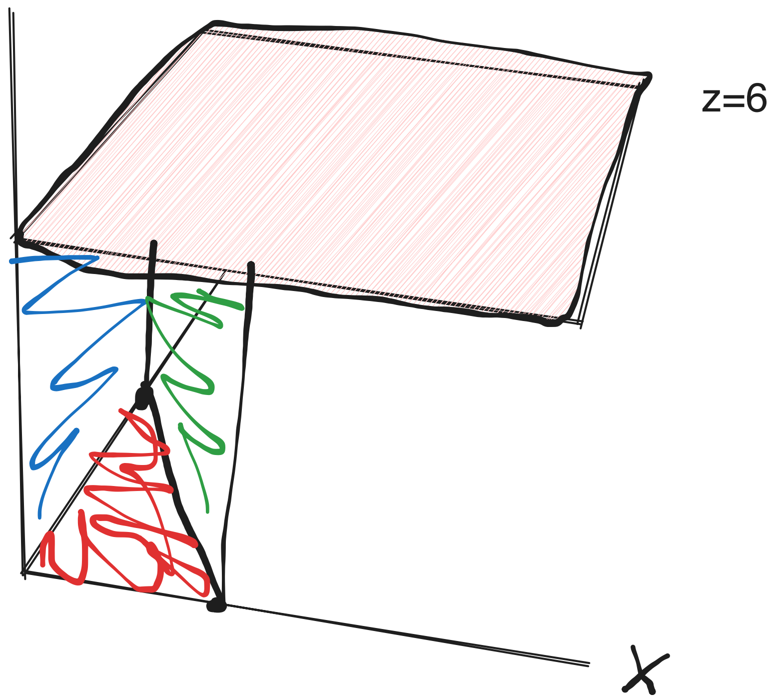
$$\int_{0}^{2\pi}(\frac{32}{3}-\frac{32}{5})d\theta$$

$$32\int_{0}^{2\pi}\left(\frac{1}{3}-\frac{1}{5}\right)d\theta$$

$$= \frac{64}{15}\theta\Big|_0^{2\pi} = \frac{128}{15}\pi$$

I'm scared.

- Find the volume of the prism D in the first octant, bounded by $y = 4-2x$ and $z = 6$



- I think that sketch... kind of works? maybe?

$$\int_0^2 \int_0^{4-2x} \int_0^6 1 dz dy dx$$

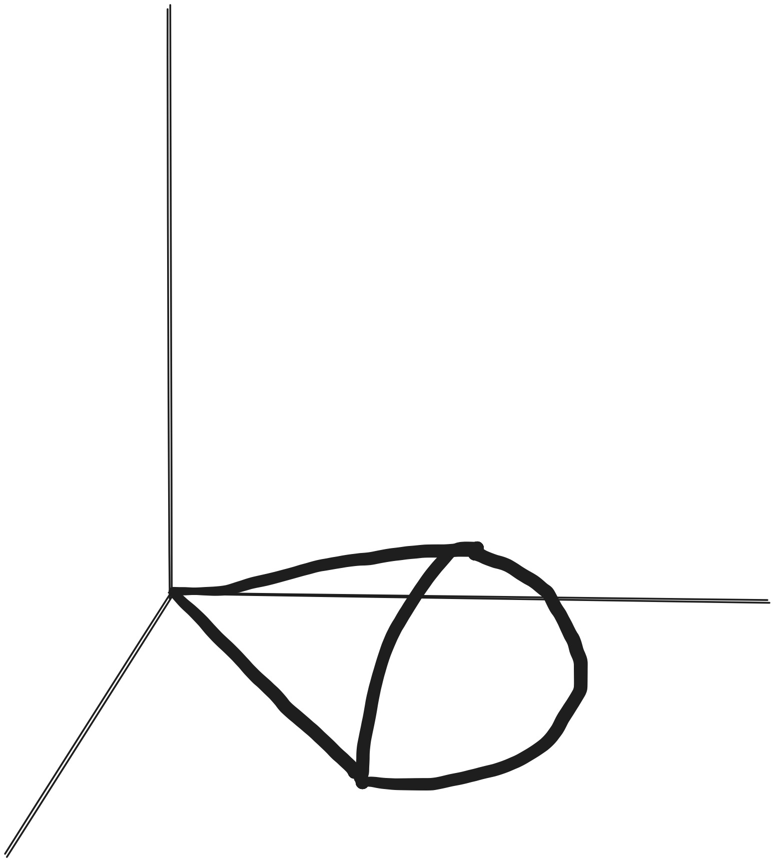
- which is reportedly equal to 24

MATH213 - 2024-10-21

#notes #math213 #math #calc

example - we're still in rectangular

- The solid bounded by the right half of the cone $y^2 = x^2 + z^2$ and the sphere $x^2 + y^2 + z^2 = 8$ is given below



- We want to set up

$$\iiint_D (x^2y + z)dV$$

  This is a snow cone, ice cream cone, whatever, it comes up a lot
- We want to go from a point along the sphere to a point along the cone at every point, so we're going left to right (along the y axis)

$$\iint_R \left[\int_{\sqrt{x^2+z^2}}^{\sqrt{8-x^2-z^2}} (x^2y + z)dy\right] dA$$

- We want to solve our sphere to be explicit in y, so we end up with $y = \pm\sqrt{8 - x^2 - z^2}$, but we actually only care about the positive part, so ditch the $\pm$
- $y^2 = x^2 + z^2$ is a full cone, but we only want the right bit, so we're going to root it and ditch the negative, so $y = \sqrt{x^2 + z^2}$
- Projecting onto the xz plane ($y = 0$)
  - We're going to get a circle, but it's not at the max radius of the sphere, so we gotta solve for that

$$x^2 + (x^2 + z^2) + z^2 = 8$$

$$2x^2 + 2z^2 = 8$$

$$x^2 + z^2 = 4, r = 2\{\text{in } \mathbb{R}^2\}$$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\int_{\sqrt{x^2+z^2}}^{\sqrt{8-x^2-z^2}} (x^2y + z)dy \right] dA$$

- Bounds of $-\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}$, for $-2 \leq x \leq 2$ \
  > - So this sounds like we gotta do z first, because we can't have a function on the outside.
- That's all setup! It, however, would not be remotely enjoyable to solve. Like at all.

on that amundsen type shit (we're doing more polar)

- Polar is for a function in $\mathbb{R}^2$ that assigns values to $r + \theta$, if we need $\mathbb{R}^3$, we will use the Cylindrical Coordinate Systems. Points will be expressed as $P(r, \theta, z)$, where r and θ are our normal polar representation and z is the vertical distance from our point P to the $xy-$plane.

Basic Shapes

$$\{(r, \theta, z) \mid r = \alpha, \alpha > 0\}$$

- This is the set of all points where each point is an α distance away from the z axis.
  - Y'know, that sounds kinda like a circle in two dimensions, which becomes a cylinder out here in three.

$$\{(r, \theta, z) \mid 0 < \alpha \leq r \leq \beta\}$$

- This is, in my own words, a thick cylinder.
  - Alternatively, a tube, a rigatoni, a cylindrical shell, a pipe, a thingamabobber, an extended torus,$_{\text{spitballing here and running out of shapes}}$, a cylindrical chute, a snipped penne

$$\{(r, \theta, z) \mid \theta = \theta_o\}$$

- This here is the set of all points such that theta is constant.
  > - Which ends up being what's known as a "half plane," it's essentially an (infinitesimally thin) slice.

Transformations

- Rectangular $\rightarrow$ cylindrical

- $x^2 + y^2 = r^2$
- $\frac{y}{x} = \tan\theta$
- $z = z$

- Cylindrical $\to$ Rectangular
  - $r\cos\theta = x$
  - $r\sin\theta = y$
  - $z = z$

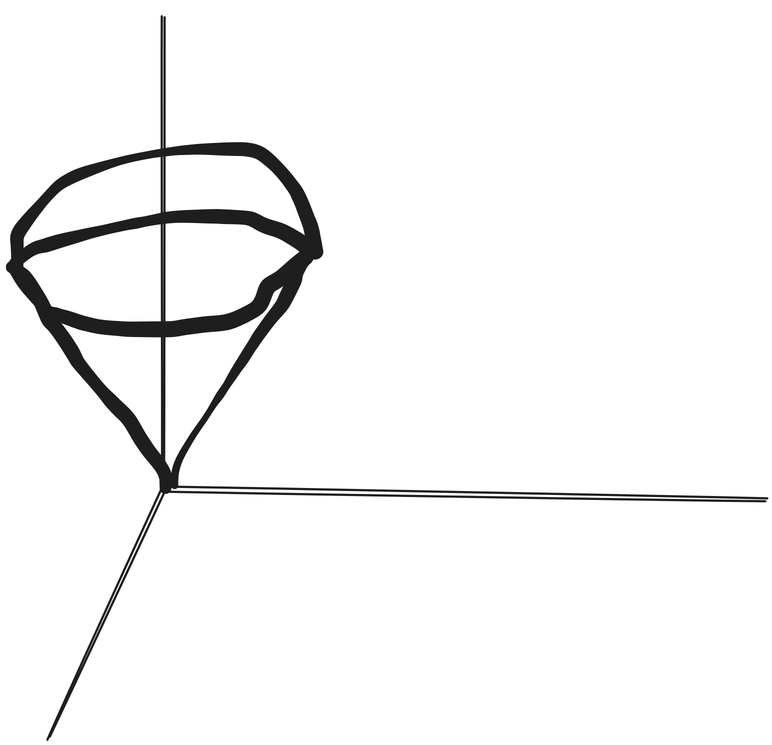
Theorem

- Let f be continuous on D, for
  $D := \{(r, \theta, z) \mid 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq g \leq \beta, G(x, y) \leq z \leq H(x, y)\}$
  - then

$$\iiint_D f(x, y, z)dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

Actual Example

- Find the volume of the solid bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 4$



- We gotta go top to bottom, because we want to be going from one function to the other.

$$\iint_R \left[\int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} 1dz\right] dA$$

- Inside bounds were just slapping our functions on (we had to a little algebra for the top one to get it purely in terms of z, but like, that's doable)

$$z^2 = 3x^2 + 3y^2$$

$$x^2 + y^2 + (3x^2 + 3y^2) = 4$$

$$x^2 + y^2 = 1$$

- Hot damn, we found the unit circle.
  - $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$
  - $-1 \leq x \leq 1$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} 1dz\right] dydx$$

- This is like, objectively awful.
  - So let's convert to cylindrical!
  - $z = \sqrt{3x^2+3y^2} = \sqrt{3(x^2+y^2)} = \sqrt{3}r$
  - $z = \sqrt{4-r^2}$

$$\int_0^1 \int_0^1 \left[\int_{\sqrt{3}r}^{\sqrt{4-r^2}} 1dz\right] rdrd\theta$$

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#notes #math213 #math #calc

- still doing cylindrical integrals, maybe starting spherical?

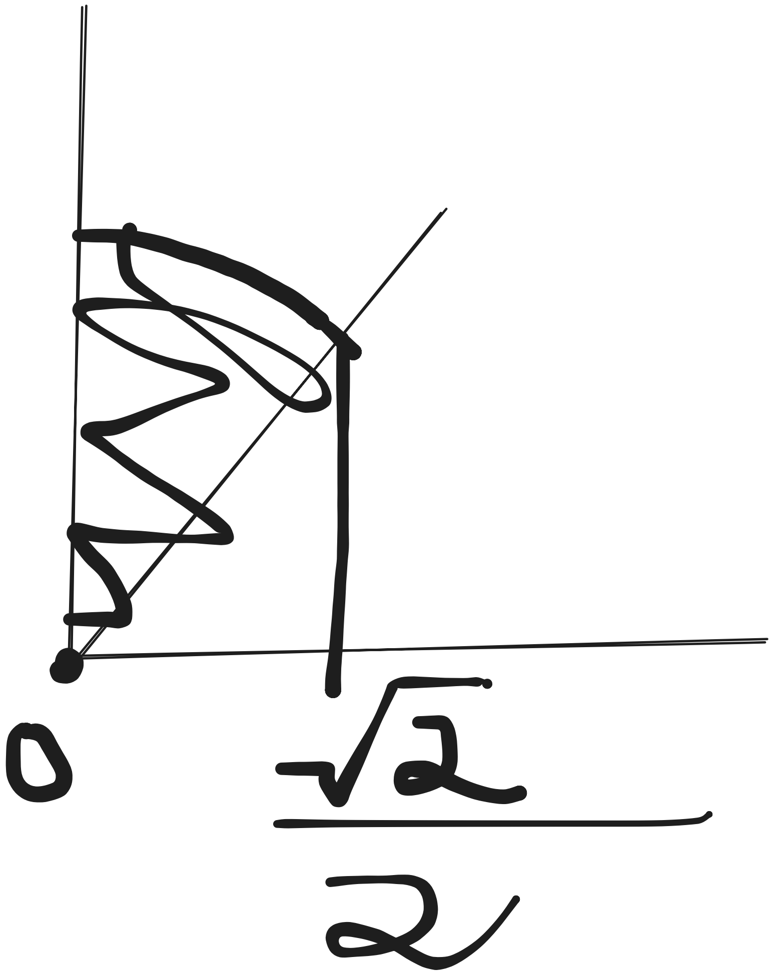
we really don't waste any time

- Compute

$$\int_0^4 \int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2} dydxdz$$

- This is quite definitively not possible in cartesian as we are.
  - we get a funny little bit of information from this problem
  - Inner bound tells us that $x \leq x \leq \sqrt{1-x^2}$

- Middle bound is $0 \leq x \leq \frac{\sqrt{2}}{2}$
- And the outer bound is $0 \leq z \leq 4$

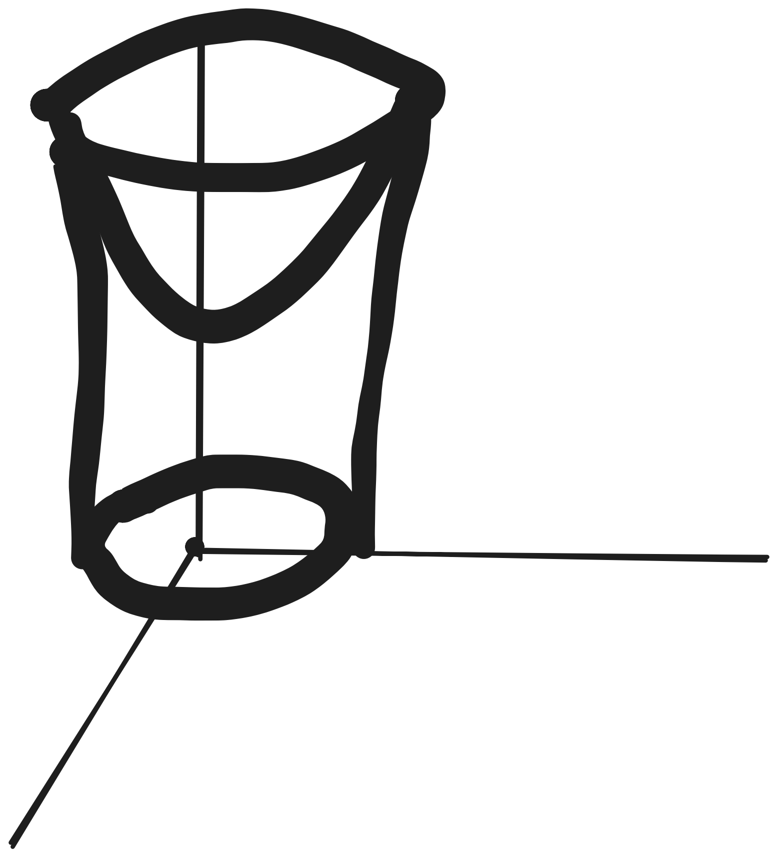


- r is going to be really straightforward, just from $0 \leq r \leq 1$, along the y axis
- θ is going to be a fair smidget more complicated, with $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 [\int_0^4 e^{-r^2} dz] r dr d\theta$$

- This, apparently, works out to $\frac{\pi}{2} - \frac{\pi}{2e}$

Set up the integral $\iiint_D e^z dV$, where D is the solid enclosed by the parabola $z = 1 + x^2 + y^2$ and $x^2 + y^2 = 5$, and the xy p

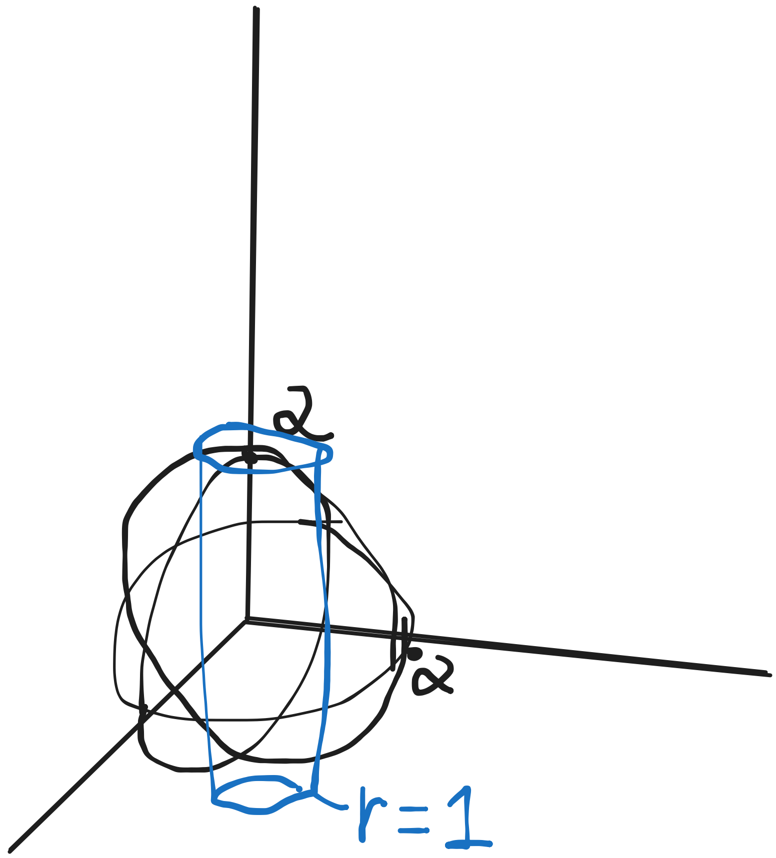


Projecting onto the xy plane we would get a circle of $\sqrt{5}$

- $0 \leq r \leq \sqrt{5}$
- $0 \leq \theta \leq 2\pi$
- Our top function is just $z = 1 + x^2 + y^2 = 1 + r^2$
- Bottom function is just $z = 0$

$$\int_0^{2\pi} \int_0^{\sqrt{5}} \left[\int_0^{1+r^2} e^z dz \right] r dr d\theta$$

- Find a triple integral that computes the volume of the solid contained in $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$



- wow that turned out AWFUL

The cylinder is smaller, so in $\mathbb{R}^2$, we're just going to be a circle of radius 1

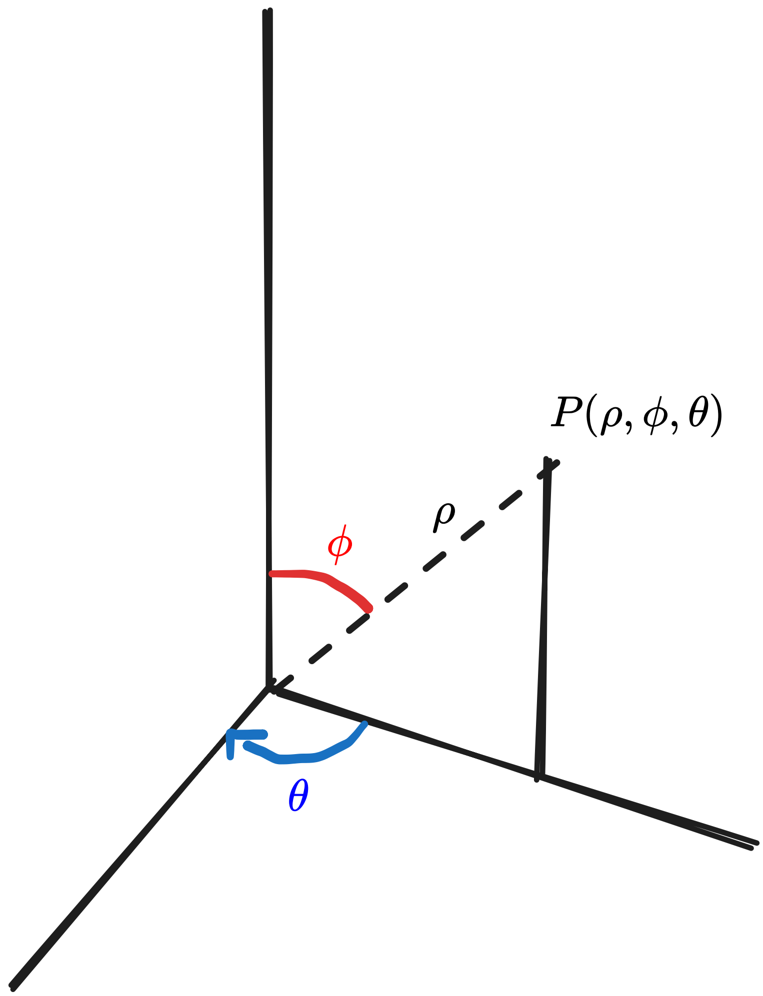
- $0 \leq r \leq 1$
- $0 \leq \theta \leq 2\pi$
- Top function is going to be $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$
- Bottom function is going to be $-\sqrt{4 - r^2}$

$$\int_0^{2\pi} \int_0^1 \left[\int_{\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 dz \right] r dr d\theta$$

and now, for something completely different

Spherical Coordinates

- The spherical coordinate system is composed of points $P(\rho, \phi, \theta)$
    - Where ρ is equal to the distance from the origin to point P
    - ϕ is the angle between the positive z-axis and the line formed from the origin to P
    - θ is the rotation around the z-axis from the positive x-axis
        - θ is the same in all of our coordinate systems! it's so funny like that.



Conversions

$$x^2 + y^2 + z^2 = \rho^2$$
$$r = \rho \sin\phi$$
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Quick Bounds Check

- ϕ is bounded by the z axis, so is going to be $0 \leq \phi \leq \pi$

Example of Suffering

- Convert $(2, \frac{\pi}{3}, \frac{\pi}{4})$

- $\rho = 2$
- $\phi = \frac{\pi}{3}$
- $\theta = \frac{\pi}{4}$

- We actually... know all of our variables, so are literally just out here plugging in
  - $x = 2\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$
  - $y = 2\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) = \sqrt{\frac{3}{2}}$
  - $z = 2\cos(\frac{\pi}{3}) = 1$
  - Which, in rectangular, gives us a final point of $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$
- Convert $(0, 2\sqrt{3}, -2)$ into spherical
- Holy shit I was so fucking wrong I forgot the original equation

$$x^2 + y^2 + z^2 = \rho^2$$

$$\rho^2 = +\sqrt{12+4} = 4$$

$$-2 = \rho\cos\phi$$

$$-2 = 4\cos\phi$$

$$\cos^{-1}(\frac{-1}{2}) = \phi, \phi = \frac{2\pi}{3}$$

$$0 = 4\sin(\frac{2\pi}{3})\cos\theta$$

$$\cos\theta = 0, \theta = \frac{\pi}{2}$$

$$(4, \frac{2\pi}{3}, \frac{\pi}{2})$$

MATH213 - 2024-10-23

#notes #math213 #math #calc

basic shapes in spherical

$$\{(\rho, \phi, \theta) \mid \rho = \alpha, \alpha > 0\}$$

- That's a sphere! Since they're all going to be equidistant

$$\{(\rho, \phi, \theta) \mid \rho = 2\alpha\cos\phi, 0 \leq \phi \leq \frac{\pi}{2}\}$$

- This is going to be a sphere not centered at the origin - instead centered at $(0, 0, \alpha)$ with a radius $r = \alpha$

$$\{(\rho, \phi, \theta), \phi = \phi_0, \phi_0 \neq 0, \frac{\pi}{2}, \pi\}$$

- This is a half cone (where a full cone is the double cone) with its tip on the origin
  - If phi tries to be any of the banned values, your cone stops actually existing. Womp womp.

$$\{(\rho, \phi, \theta) \mid \theta = \theta_0\}$$

- This is a half plane parallel to the z axis, where it's just some slice in line with whatever θ is.
- Two cases for this next one

$$\{(\rho, \phi, \theta) \mid \rho = c \sec \phi, c > 0, 0 \leq \phi < \frac{\pi}{2}\}$$

- This is a positive horizontal half plane

$$\{(\rho, \phi, \theta) \mid \rho = c \sec \phi, c < 0, \frac{\pi}{2} < \phi \leq \pi\}$$

- Is also a horizontal half plane, but the other way.

$$\{\rho, \phi, \theta) \mid \rho = r \csc \phi, 0 \leq \phi \leq \pi\}$$

- That's a cylinder!
- The textbook has a nice like table with equations and pictures.

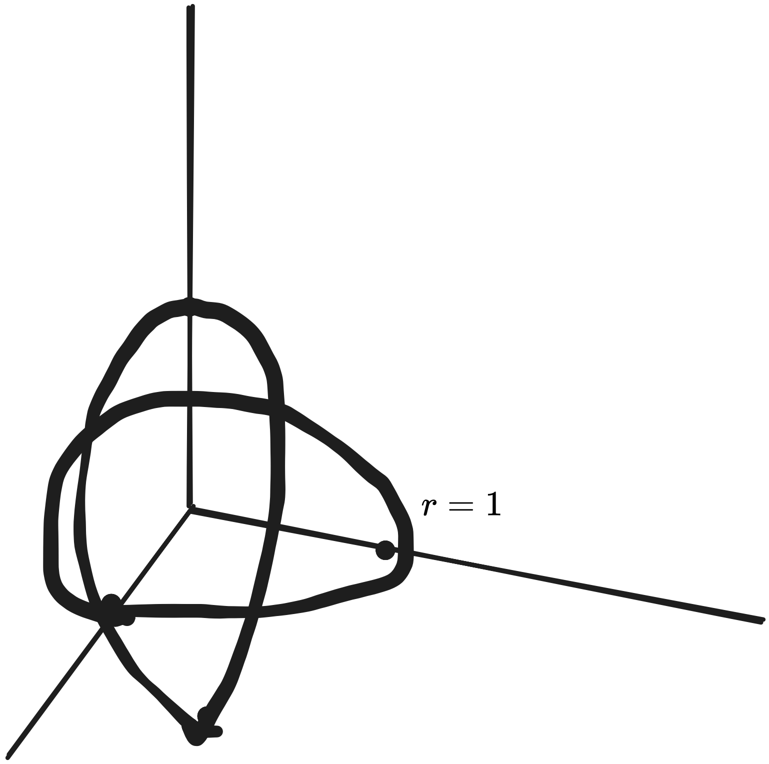
theorem

$$\iiint_D f(x, y, z) dV = \int_\alpha^\beta \int_a^b \int_{g(\phi,\theta)}^{h(\phi,\theta)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi, d\rho\, d\phi\, d\theta$$

example

$$\iiint_D e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV$$

- Where $D := \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$



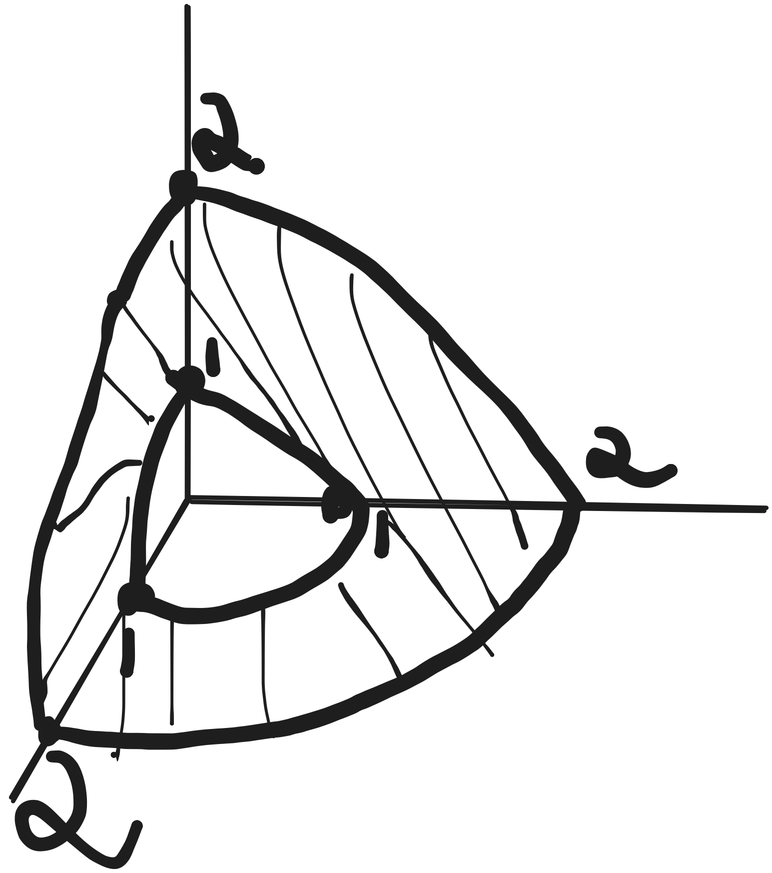
- So ρ is just going to be our radius, so $0 \leq \rho \leq 1$
- Phi's gotta go the whole way around, so $0 \leq \phi \leq \pi$
- Theta's gotta go even more of the whole way around, so $0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 (e^{(\rho^2)^{\frac{3}{2}}})\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi, d\rho \, d\phi \, d\theta$$

egg sample

- Evaluate $\iiint_D z dV$ where the solid lies between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant



$$1 \le \rho \le 2$$

$$0 \le \phi \le \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

---

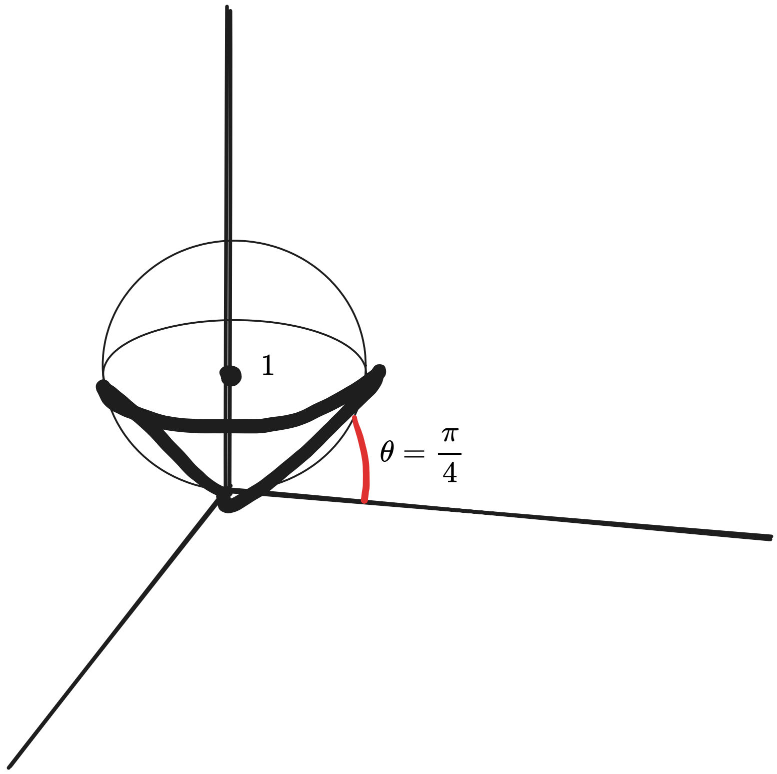
ex ampule

- Use spherical coordinates to find volume of the solid bounded above by $2z = x^2 + y^2 + z^2$ and below by $z = \sqrt{x^2 + y^2}$

$$x^2 + y^2 + z^2 - 2z = 0$$

$$x^2 + y^2 + (z-1)^2 = 1$$

- Boy howdy, that sure looks like a sphere of radius one centered at $(0, 0, 1)$



- Converting our sphere into spherical, we get like

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2 \cos^2\phi - 2\rho\cos\phi = 0$$

$$\rho^2 \sin^2\phi + \rho^2 \cos^2\phi - 2\rho\cos\phi = 0$$

$$\rho^2 - 2\rho\cos\phi = 0$$

$$\rho = 2\cos\phi$$

$$0 \leq \rho \leq 2\cos\phi$$

$$0 \leq \phi \leq \frac{\pi}{4}$$

$$0 \leq \theta \leq 2\pi$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2\cos\phi} 1\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

MATH213 - 2024-10-25

#notes #math213 #math #calc

activity

$$\{(\rho, \phi, \theta) \mid \rho = 4\cos\phi, 0 \le \phi \le \frac{\pi}{2}\}$$

$$\{(r, \theta, z) \mid 2r \le z \le 4\}$$

Integrals for mass calculations

- Set up integrals that compute mass over a domain in 1D, 2D, and 3D

Cases in 1D

1. Discrete (several objects on a line)
    1. If you're asymmetric, it'll pull towards one of the masses
    2. For n objects with masses $m_1, \ldots, m_n$ at locations $x_1, \ldots, x_n$ center of mass is given by $\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$
        1. This is $\frac{\text{Total momement}}{\text{Total mass}}$,
2. Continuous Objects in 1D
    1. Let p be an integrable density function on $[a, b]$, then Center of Mass (CoM) is given by $\bar{x} = \frac{M}{m} \to \frac{\text{Total moment}}{\text{Total mass}}$
        1. where mass is given by $m = \int_a^b p(x)dx$ and $M = \int_a^b xp(x)dx$

Moving on up (to 2D)

- Let $p(x, y)$ be an integrable area density function over a closed and bounded region R in $\mathbb{R}^2$. Then, the coordinates of the center of mass are given by $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R xp(x, y)dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R yp(x, y)dA$$

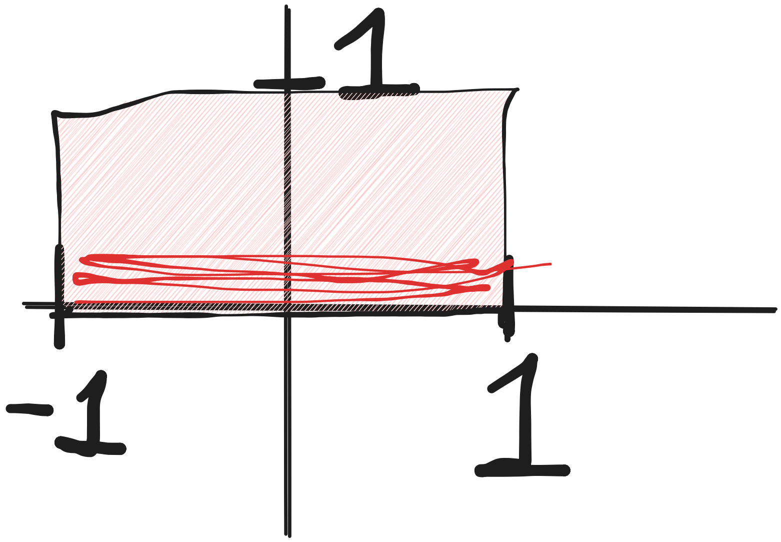
Where $m = \iint p(x, y)dA$ and M_x is the moment with respect to the x-axis, and M_y is the moment with respect to the y axis.

Quick Mathlab Note

- If p is constant, then the CoM is independent of density. When this happens, this is referred to as a centroid (mark that statics jumpscare)

Example

- Find the CoM of a rectangular plate given by $R := \{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1\}$, where the plate is heaviest along the lower edge and lightest at the top edge with density function $p(x, y) = 2 - y$



- So, silly little thing, $p(x, y) = 2 - y$ does not care about x at all
- R is symmetrical about the y axis, which tells us that $\bar{x}$ is going to be zero

$$m = \int\int_R (2-y)dA \to \int_{-1}^{1}\int_{0}^{1}(2-y)dydx$$

$$m = \int_{-1}^{1}\frac{3}{2}dx = 3$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{3}\int_0^1\int_{-1}^{1}x(2-y)dxdy$$

$$\frac{1}{3}\int_0^1\left[(2-y)\frac{x^2}{2}\Big|_{-1}^{1}\right]dy$$

$$\int_0^1 0 = 0 \text{ yippee!}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{3}\int_{-1}^{1}\int_{0}^{1}y(2-y)dydx$$

$$\frac{1}{3}\int_{-1}^{1}\left[\frac{2y^2}{2} - \frac{y^3}{3}\Big|_0^1\right]dx$$

$$\frac{1}{3}\left[\frac{2}{3}x\Big|_{-1}^{1}\right] = \frac{2}{9}(2) = \frac{4}{9}$$

So center of mass pops out to be $(0, \frac{4}{9})$

Movin on Movin on Up (to 3D)

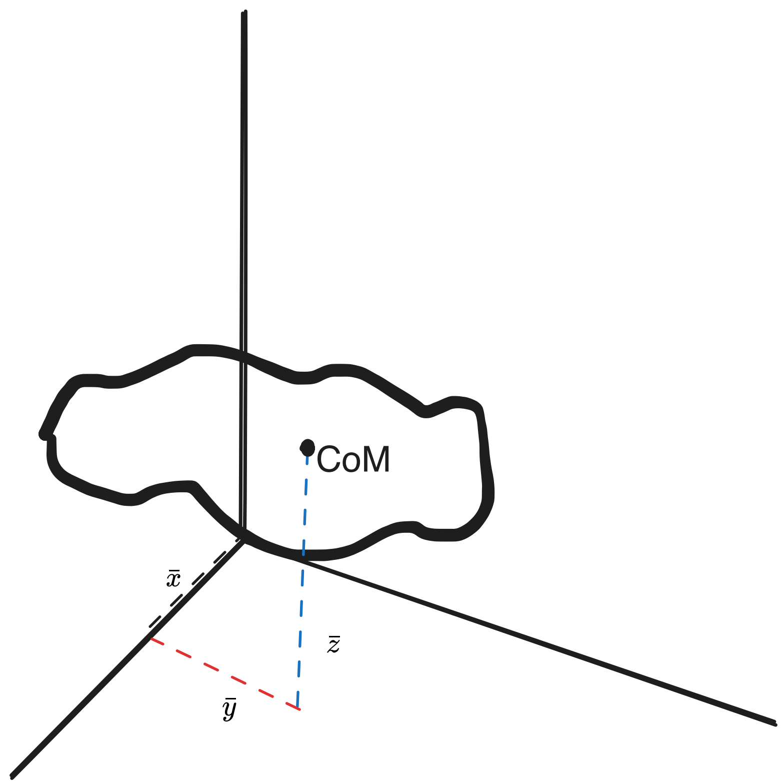
- Let $p(x, y, z)$ be integrable on a closed and bounded region, D, in $\mathbb{R}^3$

$$\bar{x} = \frac{M_{yx}}{m} = \frac{1}{m} \iiint_D xp(x, y, z)dV$$

$$\bar{y} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D yp(x, y, z)dV$$

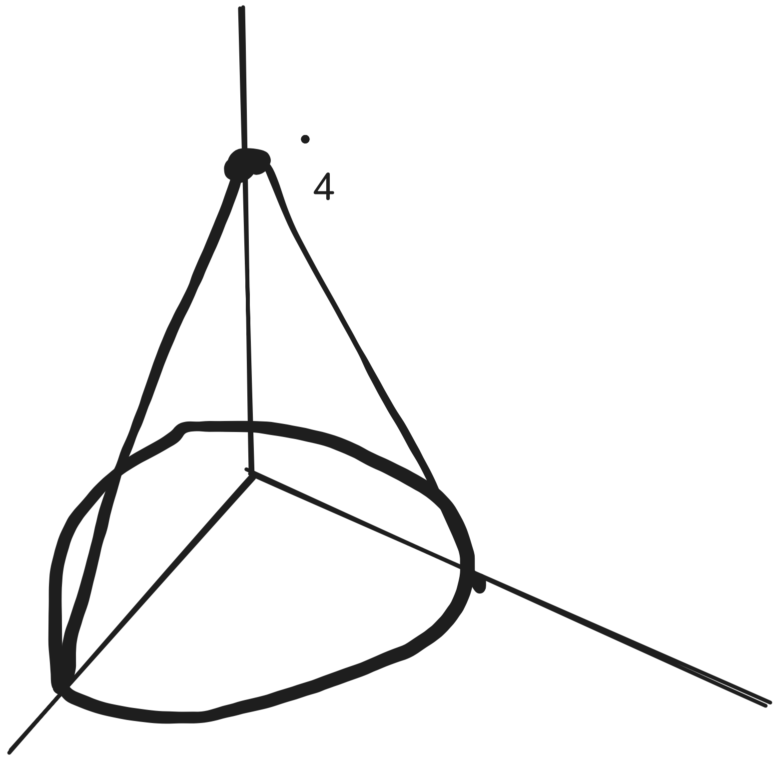
$$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D zp(x, y, z)dV$$

Where $m = \iiint p(x, y, z)dA$ and $M_{xy}, M_{xz},$ and M_{yz} are moments with respect to coordinate planes



Example

- Find CoM of the solid bounded by $z = 4 - \sqrt{x^2 + y^2}$ and $z = 0$ if $p(x, y, z) = 1$



- Oh hey, symmetries exist, so $\bar{x} = 0$ and $\bar{y} = 0$
- $p = 1 \to \text{mass} = \text{ volume of D}$
  - We happen to know the volume of a cone, which is $\frac{1}{3}\pi r^2 h$
  - Finding r
    - $0 = 4 - \sqrt{x^2 + y^2}$
      - $r = 4$
  - Volume is then just $\frac{1}{3}\pi * 16 * 4 = \frac{64\pi}{3}$
- $$\bar{z} = \frac{3}{64\pi} = \iiint z(1)dV$$
- In $\mathbb{R}^2$ we're projecting down onto a circle of radius 2
- $0 \leq z \leq 4 - r$
- Let's hop to cylindrical, by the by

$$\bar{z} = \frac{3}{64\pi} \int_0^{2\pi} \int_0^4 \int_0^{4r} z(1)dz\, rdrd\theta$$

$$\bar{z} = \frac{3}{64\pi} \int_0^{2\pi} \int_0^4 \frac{1}{2}[(4-r)^2]rdrd\theta$$

$$= \frac{3}{128\pi} \int_0^{2\pi} \int_0^4 (16r - 8r^2 + r^3)drd\theta$$

$$\frac{3}{128\pi} \int_0^{2\pi} [8r^2 - \frac{8}{3}r^3 + \frac{r^4}{4}\Big|_0^4]d\theta$$

$$\frac{3}{128\pi}\int_0^{2\pi} 8(4)^2 - \frac{8}{3}(4^3) + 4^3 d\theta$$

$$\frac{3}{8\pi}\int_0^{2\pi} 8 - \frac{32}{3} + 4 d\theta$$

$$\frac{3}{8\pi}\int_0^{2\pi} 12 - \frac{32}{3} d\theta$$

$$\frac{3}{8\pi}[12\theta - \frac{32\theta}{3}\Big|_0^{2\pi}]$$

$$\frac{3}{8\pi}[24\pi - \frac{64\pi}{3}]$$

$$1, CoM = (0,0,1)$$

MATH213 - 2024-10-28

#notes #math213 #math #calc

- Imma be so real I was like five minutes late to this class

Change of Variables

- In single variable calculus, a change in variables is a u sub
  - When you end up with something like $du = 2dx$ and $\frac{1}{2}du = dx$ which says over a small interval, the width in x is equal to half the width of a small interval in u
- In two variable calculus, a change in variables ends up being to polar
  - So, if you take something like $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dx dy$, you can convert using polar identities over into r and θ, so you end up with something with like $\int_0^{\frac{\pi}{2}} \int_0^1 (r^2) r dr d\theta$
- This is a transformation $T: x = r\cos\theta, y = r\sin\theta$, that maps points from cartesian into polar

Transformations

- A transformation, T, from a region S to a region R is one to one if for points $P, Q \in S, T(P) = T(Q)$ for $P = Q$
  - When you have one:one, everything maps to one distinct point
- For us in the oh-so wonderful land of calc 3, all of our transformations should be one to one.

Goal

- We want to take a complicated region and map all points to a distinct point in a simpler region, then integrate.
- Notation, $T : S \to R$ will be mapping from one "field" to another.
- In order to pull this off, we need something called a "Jacobian Determinant Matrix", shorthand is just "Jacobian"
- Given a transformation, T, and $x = g(u, v)$ and $y = h(u, v)$ where $g + h$ are differentiable in the $uv-$plane, the Jacobian is given by

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\right) - \frac{\partial y}{\partial u}\left(\frac{\partial x}{\partial v}\right)$$

- from $T : S \to R$, where $x = g(u, v)$ and $y = h(u, v)$ where T is one-to-one and g+h are continuous in the {something} of $\int$ from

$$\iint_R f(x,y)dA = \iint_S f(g(u,v), h(u,v)) \mid J(u,v) \mid dA$$

- Verify that for a transformation from Rectangular to Polar, $dA = rdrd\theta$
  - $T : x = r\cos\theta, y = r\sin\theta$
- The jacobian is going to be all their partials

$$|\begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix}| = r\cos^2\theta - (-r\sin^2\theta) = r\cos^2\theta + r\sin\theta = r$$

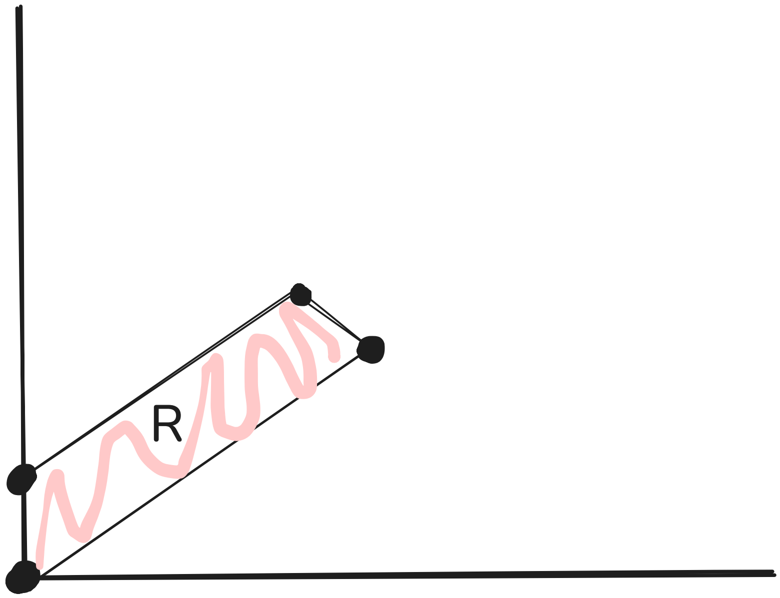
$$|J(r,\theta)| = |r| = r$$

$$\iint_R f(x,y)dA = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta$$

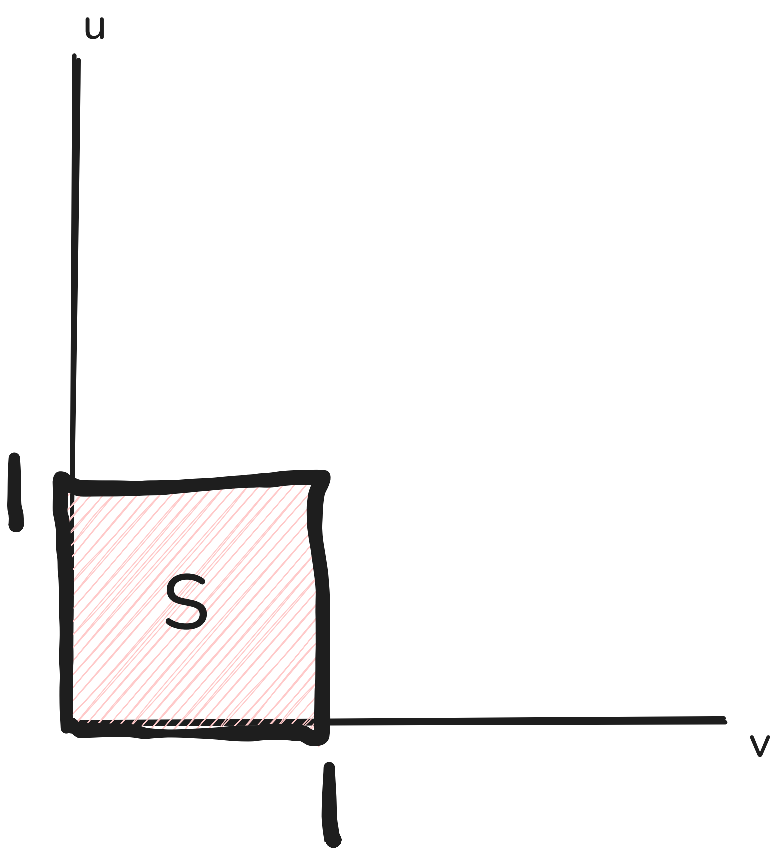
Evaluate some shenanigans

$$\iint_R \sqrt{2x(y-2x)}dA$$

- Where R is a parallelogram with vertices $(0, 0), (0, 1), (2, 4), (2, 5)$
- Use the transformation $x = 2u$ and $y = 4u + v$ to do some mild shenanigans



- Using our given transformation, we get the idea that
  - $u = \frac{1}{2}x$
  - $v = y - 4u = y - 2x$
- Mapping our points around
  - $(0,0) \to^T (0,0)$
  - $(0,1) \to^T (0,1)$
  - $(2,4) \to^T (1,0)$
  - $(2,5) \to^T (1,1)$



- Oh hey, they darn flattened my parallelogram into a square
  - So we'll use $S := \{(u,v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$

$$\int_0^1 \int_0^1 \sqrt{2(2u)(r)} \mid J(u,v) \mid dudv$$

$$J(u,v) = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = |2| = 2$$

$$\int_0^1 \int_0^1 \sqrt{4uv}(2)dudv$$

$$4\int_0^1 \int_0^1 \sqrt{uv}\, dudv = 4\int_0^1 \int_0^1 \sqrt{v}\sqrt{u}\, dudv$$

$$4\int_0^1 \sqrt{v} \left[\frac{2}{3}u^{\frac{3}{2}}\Big|_0^1\right] dV = \frac{8}{3}\left[\frac{2}{3}v^{\frac{3}{2}}\Big|_0^1\right] = \frac{16}{9}$$

MATH213 - 2024-10-29

#notes #math213 #math #calc

moving around to three variable changes of variables

- Where T, our transformation
  - $x = g(u,v,w)$
  - $y = h(u,v,w)$
  - $z = p(u,v,w)$
- maps a bounded region S onto a region D. Assume T is a one-to-one and $g, h, \& \ p$ have continuous partial, then

$$\iiint_D f(x,y,z)dV = \iiint_S f(g,h,p) \mid J(u,v,w) \mid dV$$

  When $J(u,v,w)$ is given by

$$\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{matrix}$$

  - i hate that
- Verify that $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ when transforming from rectangular to spherical
- So our transformation, T, ends up looking something like

> - $x = \rho \sin\phi \cos\theta$
> - $y = \rho \sin\phi \cos\phi$
> - $\theta = \rho \cos\phi$

$$\begin{matrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{matrix}$$

$$= \sin\phi\cos\theta(\rho^2\sin^2\phi\cos\theta) - \rho\cos\phi\sin\theta(\rho\sin\phi\cos\phi\cos\theta)$$

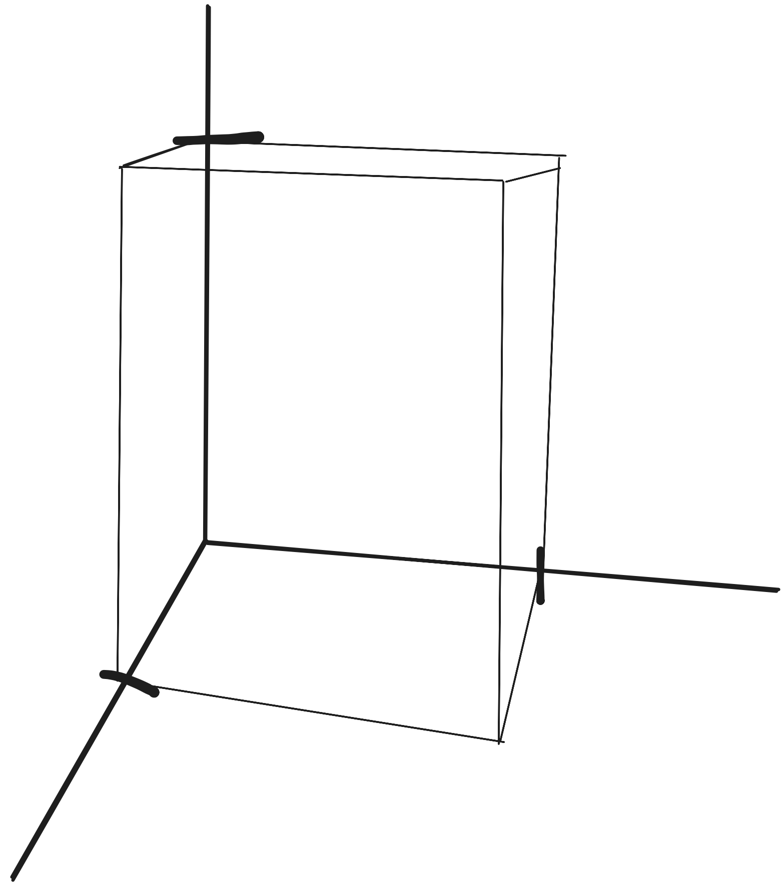
$$-\rho\sin\phi\sin\theta(\sin\phi\sin\theta(-\rho\sin\phi) - \rho\cos\phi\sin\theta\cos\phi)$$

$$\rho^2\sin^3\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta$$

$$\rho^2\sin^3\phi(1) + \rho^2\sin\phi\cos^2\phi(1) = \rho^2\sin\phi(\sin^2\phi + \cos^2\phi) = \rho^2\sin\phi$$

try one!

- Evaluate $\iiint_D xz$, where D is bounded by $y = x, y = x + 2, z = x, z = x + 3, z = 0, z = 4$
  - Oh hey, $y = x$ and $y = x + 2$ are parallel planes (using $y - x$)
  - Oh hey hey, $z = x$ and $z = x + 3$ are parallel (using $z - x$)
  - Oh hey hey hey, $z = 0$ and $z = 4$ are parallel
  - This is what we in the business call a "parallelepiped"
  - We can say that $u = y - x$, which then bounds it between 0 and 2
  - $v = z - x$, which then bounds us between 0 and 3.
  - $w = z$, bounded between 0 and 4



- I've done worse sketches.
- Ok, now to do our actual integral.

$$\iiint_D xzdV = \int_0^4 \int_0^3 \int_0^2 f(u,v,w) \mid J(u,v,w)|dudvdw$$

$$\int_0^4 \int_0^3 \int_0^2 w(w-v) \mid J(u,v,w)|dudvdw$$

$$x = z - v, x = w - v$$

$$y = u + w - v$$

$$z = w$$

$$\begin{array}{ll} & 0 \quad -1 \quad 1 \\ \det(& 1 \quad -1 \quad 1) = (0) - (-1)(1-0) + 1(0) = 1 \\ & 0 \quad\;\; 0 \quad\;\; 1 \end{array}$$

$$\int_0^4 \int_0^3 \int_0^2 (w-v)w(1)dudvdw$$

$$2\int_0^4 \int_0^3 (w^2 - wv)dvdu$$

$$2\int_0^4 (3w^2 - \frac{9}{2}w)dW$$

$$2(w^3 - \frac{9}{4}w^2\Big|_0^4) = 2(64 - 36) = 56$$

Use the transformation

$$u = x + y$$

$$v = 2x - y$$

$$y = 2x - v$$

$$x = u - y$$

$$x = u - 2x + v$$

$$3x = u + v$$

$$x = \frac{u + v}{3}$$

$$y = \frac{2(u + v)}{3} - v$$

$$y = \frac{2}{3}u - \frac{1}{3}v$$

$$\frac{2}{3}u - \frac{1}{3}v = -\frac{1}{3}u - \frac{1}{3}v$$

$$y = -x \to u = 0$$

$$\frac{2}{3}u - \frac{1}{3}v = 2 - \frac{1}{3}u - \frac{1}{3}$$

$$y = 2 - x \to u = 2$$

$$\frac{2}{3}u - \frac{1}{3}v = 2(\frac{1}{3}u + \frac{1}{3}v)$$

$$0 \leq u \leq 2$$

$$-1 \leq v \leq 0$$

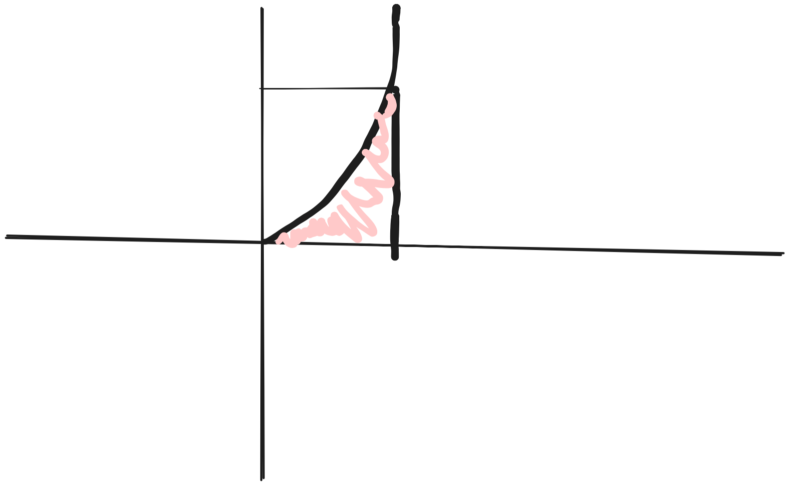
MATH213 - 2024-11-01

#notes #math213 #math #calc

practice problems

- given

$$\int_0^4 \int_{\sqrt{y}}^2 3\sin(x^3)dxdy$$



- Set up an integral by reversing the order (ie, we now become $dydx$)

$$\int_0^2 \int_0^{x^2} 3\sin(x^3)dydx$$

Given

$$\int_0^{\frac{1}{\sqrt{2}}} \int_x^{\sqrt{1-x^2}} x\ dydx$$

- Evaluate the integral by changing to polar
- Alrighty, well,
  > - $x = r\cos\theta$
  > - $y = r\sin\theta$

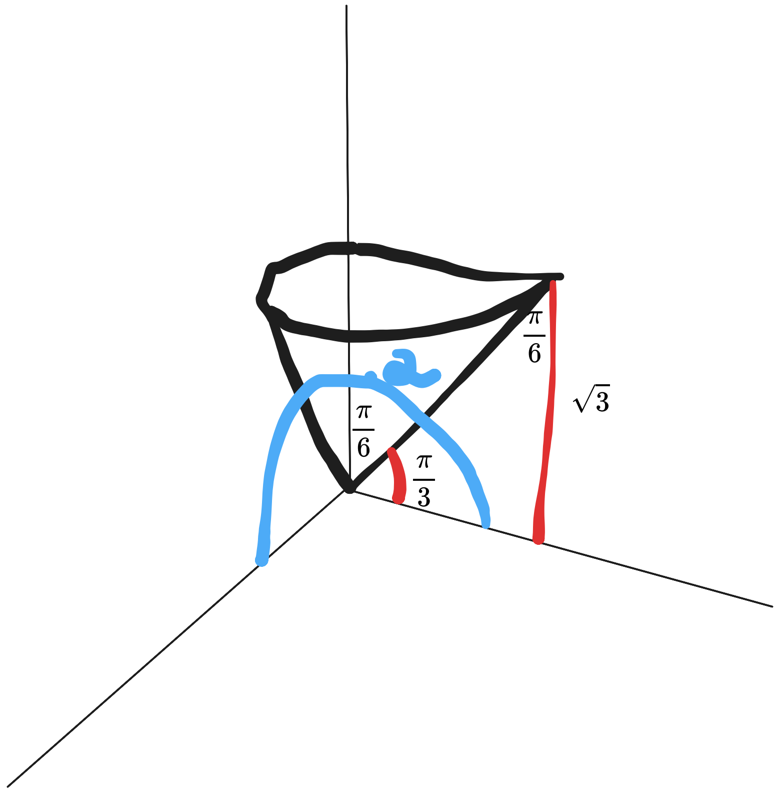
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 r\cos\theta\ rdrd\theta$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{r^3\cos\theta}{3}$$

$$\frac{1}{3}\sin(\frac{\pi}{2}) - \frac{1}{3}\sin(\frac{\pi}{4})$$

$$\frac{1}{3}(1 - \sqrt{\frac{2}{2}}) \to \frac{1}{3} - \frac{\sqrt{2}}{6} \to \frac{2-\sqrt{2}}{6}$$

- Consider the solid bounded by $z = \sqrt{3(x^2+y^2)}$ and $z = \sqrt{4-x^2-y^2}$
  - $\delta_{x,y,z} = \frac{1}{\sqrt{x^2+y^2+z^2}}$



$$z^2 = 4 - x^2 - y^2$$

$$\rho^2 = 4, \rho = 2$$

$$x^2 + y^2 + z^2 = \rho^2, z^2 = \rho^2 - x^2 - y^2$$

$$z^2 = 3(x^2 + y^2)$$

$$3x^2 + 3y^2 = \rho^2 - x^2 - y^2$$

$$\rho^2 = 4x^2 + 4y^2, 1 = x^2 + y^2$$

ϕ is bounded between $0 \leq \phi \leq \frac{\pi}{6}$ due to some mild angle shenanigans

- and we're just projecting the unit circle down, so $0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \frac{1}{\rho} \rho^2 \sin\phi \; d\rho d\phi d\theta$$

- Quick translation into cylindrical does something like this:

$$\int_0^{2\pi} \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} \frac{1}{\sqrt{r^2 + z^2}} r dz dr d\theta$$

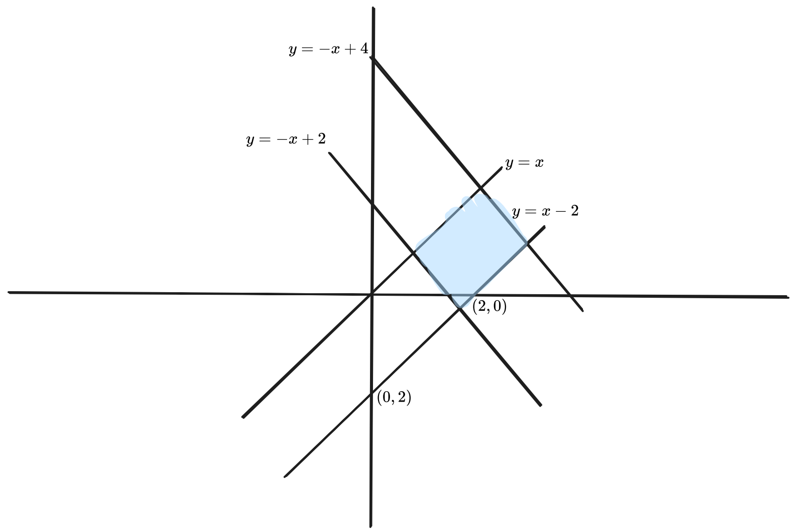
Consider

$$\iint_R \frac{x-y}{x+y}$$

for an R bounded by

$$\begin{array}{ll} x-y=0 & x-y=2 \\ x+y=2 & x+y=4 \end{array}$$

If $u = x+y$ and $v = x-y$



- Finding $x(u,v)$ and $y(u,v)$

$u+v=2x$

$$\frac{1}{2}u+\frac{1}{2}v=x$$

$$y=u-x$$

$$y=u-\frac{1}{2}u-\frac{1}{2}v$$

$$y=\frac{1}{2}u-\frac{1}{2}v$$

- Finding the jacobian, $J(u,v)$

$$\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} = -\frac{1}{4}-\frac{1}{4}=\frac{1}{2}$$

$$\int\int \frac{v}{u}|-\frac{1}{2}|dudv$$

If $u=x+y$, $2 \leq u \leq 4$

If $v=x-y$, $0 \leq v \leq 2$

$$\int_0^2 \int_2^4 \frac{v}{u}| - \frac{1}{2}|dudv$$

MATH213 - 2024-11-06

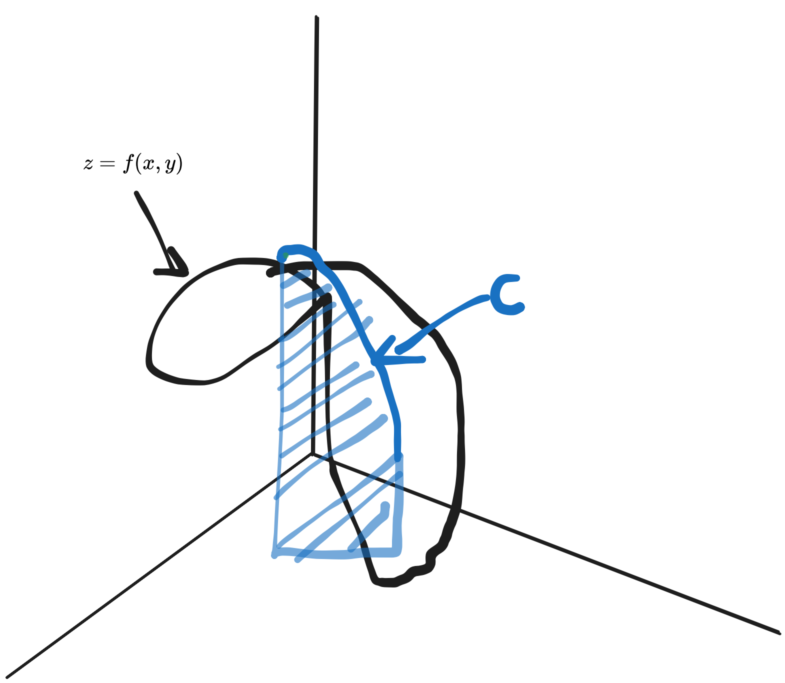
#notes #math213 #math #calc

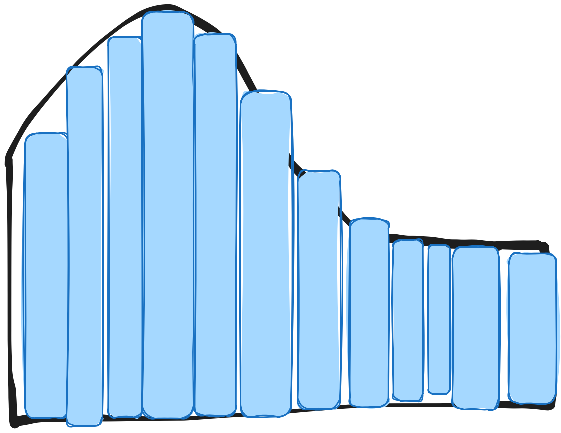
Line integrals!

- What in the world do we even *need* these things for?
  - I mean, we need em for surface area
  - Comes up in work done by a force on an object
- Suppose the scalar valued function f is defined on a region containing a smooth curve C, given by a vector valued function $\vec{r}(t)$, defined as $< x_t, y_t >$ on an integral $a \leq t \leq b$. The line integral is given by

$$\int_C f(x(t), y(t))ds = \lim_{\Delta \to 0} \sum_{k=1}^{\infty} f(r(t_k^*), y(t_k^*))\Delta S_k$$

- Provided t hat the limit exists over all partitions $[a, b]$
  - Note: if the limit exists, t is integrable over C





$$\int_C f(x(t), g(s))ds$$

- Hang on, the hell is ds?
  - Given a path $\vec{r}(t)$ from $a \leq t \leq b$, partition it into cutesy little chunks

$$\Delta s_k = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

  - As Δt goes to zero, we end up with derivatives!

$$= \sqrt{x'(t)^2 + y'(t)^2}dt$$

- This implies that:

$$\int_C f ds = \int_a^b f(x(t), y(t))|\vec{r}'(t)|dt$$

Procedure

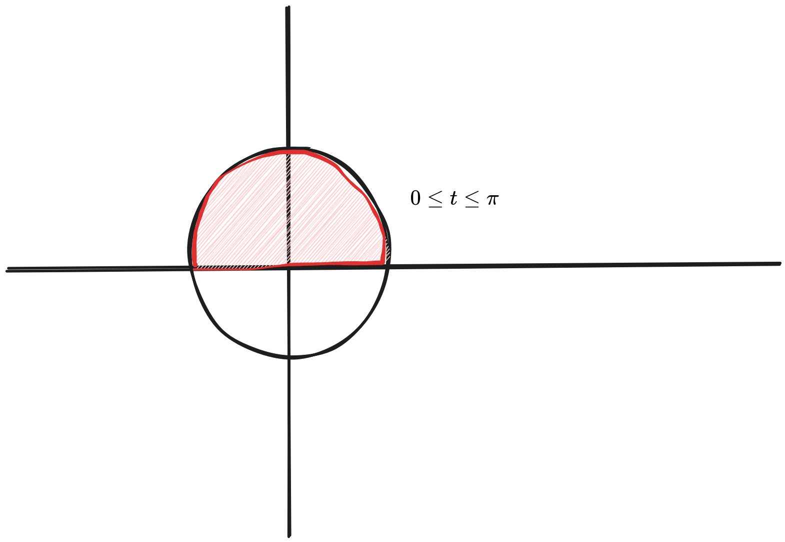
1. Provide a parameterized description of your curves
2. Compute $|\vec{r}'(t)|$
3. Plug in

Quick Application bit

1. $\int_c 1 ds =$ length of the curve
2. $\int_c f ds =$ area of the path of f over c
3. $\int_c \rho(x, y) ds =$ mass of a thin wire or rod (this doesn't really work in higher dimensions

Example

- Compute $\int_c 3x dx$ where c is given by the top half of the unit circle



- Generic circle parameterization is
  - $x = x_0 + r\cos(t)$
  - $y = y_0 + y\sin(t)$
- Which then just gives us $x = \cos(t), y = \sin(t)$ for $\vec{r}(t) =< \cos(t), \sin(t) >$ for $0 \leq t \leq \pi$
- $$\sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$ $$\int_0^\pi 3\cos(t) * (1)dt = 3(0-0) = 0$$

now do it again

- Compute $\int_c (x + y + z)$ where c is given by the line segment from the origin to the point $(1, 1, 1)$
- To generally parameterize a line, we have
  - $x = x_0 + \alpha t$
  - $y = y_0 + \beta t$
  - $z = z_0 \gamma\, t$
    - Where $p_0 = (x_0, y_0, z_0)$
    - And $\vec{PQ} =< \alpha, \beta, \gamma >$
- For this problem, we end up with: $\vec{PQ} =< 1, 1, 1 >$
- $\vec{r}(t) =< t, t, t >$

- for $0 \leq t \leq 1$
- Now, go find the magnitude of $r'(t)$
  - $= \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$$\int_0^1 (t+t+t) * \sqrt{3}\, dt = 3\sqrt{3}\int_0^1 t\, dt = \frac{3\sqrt{3}}{2}$$

wire time

- Find the mass of the wire in the shape of the curve $y = x^2$ from $(0,0)$ to $(1,1)$ followed by $x = 1$ from $(1,1)$ to $(1,2)$ with density function $\rho(x,y) = 2x$
- Parameterize c_1
  - $y = x^2$ from $(0,0) \to (1,1)$
  - Let $x = t$, which then makes $y = t^2$
  - $\vec{r}(t) =< t, t^2 >$ from $0 \leq t \leq 1$
  - $|r'(t)| = \sqrt{1+4t^2}$
- Parameterize c_2
  - $x = 1$ from $(1,1) \to (1,2)$
  - Let $x = 1, y = t$
  - $\vec{r}(t) =< 1, t >$
  - $|r'(t)| = 1$

$$= \int_0^1 2(t)\sqrt{1+4t^2}(dt) + \int_1^2 2(1)(1)dt$$

$$2\int t\sqrt{1+4t^2}dt + 2\int_1^2 dt$$

$$u = 1+4t^2$$

$$\frac{1}{4}\int_0^1 \sqrt{u}du + 2\int_1^2 dt$$

$$\frac{1}{6}(1+4t^2)^{\frac{3}{2}} + 2 = \frac{1}{6}(5^{\frac{3}{2}} - 1) + 2$$

MATH213 - 2024-11-12

#notes #math213 #math #calc

1.

- Set up and evaluate the integral $\int_C (xy+z)ds$ where C is the line segment from $P(0,0,2)$ to $Q(1,2,0)$

$$C = t, 2t, 2-2t$$

$$f(x,y,z) = xy + z \to t = (t(2t) + 2 - 2t) \to 2t^2 - 2t + 2$$

$$2(t^2 - t + 1) = f(\vec{r}(t))$$

$$r'(t) = 1, 2, -2$$

$$||r'(t)|| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\int_0^1 2(t^2 - t + 1)3dt = 6\int_0^1 t^2 - t + 1dt$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t\Big|_0^1\right)$$

$$6\left(\frac{1}{3} - \frac{1}{2} + 1\right) = 6(\frac{2}{6} - \frac{3}{6} + \frac{6}{6}) = 6(\frac{5}{6}) = 5$$

- Trying to write out the parameter bit better, $\vec{PQ} =< 1, 2, -2 >$, where P_0 is $(0,0,2)$. This gives you that $\vec{r}(t) =< 0, 0, 2 > + < 1, 2, -2 > t$, which just pops out what I had gotten of $< t, 2t, 2-2t >$ for $0 \leq t \leq 1$

2.

- Find surface area of the part of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and the plane $z = x + y$ in the first octant using the scalar line integral (parametrize the curve $x^2 + y^2 = 4$)
- Alrighty, time to shoot the moon and hope I don't hit someone I know
  - Parametrizing that curve is actually a good ol circle of radius two, so that's just going to be $r(t) = 2\cos(t), 2\sin(t)$ from $0 \leq t \leq \frac{\pi}{2}$
  - Getting our function
  - $z = x + y = f(t) = 2\cos(t) + 2\sin(t)$
  - Moving on to $r'(t)$, which is $-2\sin(t), 2\cos(t)$
  - So $||r'(t)|| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$
  - So maybe theoretically possibly the integral is

$$2\int_0^{\frac{\pi}{2}} (2\cos t + 2\sin t)dt$$

$$4(\sin t - \cos t\Big|_0^{\frac{\pi}{2}}) = 4(1 - -1) = 8$$

3.

- Find the mass of a thin wire in the shape of the curve $x = 4 - y^2$ in the xy plane, where $x \geq 0$, $y \geq 0$, whose density $\rho(x, y) = y$
  - I mean, I think you can just say $y = t$, so $x = 4 - t^2$, meaning the function is just $f(t) = (4 - t^2), t$ for $0 \leq t \leq 2$
- From there, $\rho(x, y) \to \rho(t) = t$
- Also, finding $r'(t) = -2t, 1$, so the magnitude is $= \sqrt{4t^2 + 1}$
- and from there I think you just kind of plug in to the integral?

$$\int_0^2 (t)(\sqrt{4t^2+1})dt$$

$$u = 4t^2 + 1, du = 8t$$

$$\int \frac{1}{8} du\sqrt{u} = \frac{1}{8}\int u^{\frac{1}{2}} = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{1}{8}\frac{2}{3}u^{\frac{3}{2}} = \frac{2}{24}u^{\frac{3}{2}} = \frac{2}{24}(4t^2+1)^{\frac{3}{2}}\Big|_0^2$$

$$\frac{1}{2}(17^{\frac{3}{2}-1^{\frac{3}{2}})}$$

MATH213 - 2024-11-13

#notes #math213 #math #calc

some shenanigans with special types of curves I dunno man

- Suppose a curve C is given by $\vec{r}(t)$ from $a \leq t \leq b$. C is simple if $\vec{r}(t) \neq \vec{r}(t_2) \forall t_1, t_2 \in [a, b]$
  - Read: the path cannot intersect itself
- C is closed if $\vec{r}(a) = \vec{r}(b)$
  - Read: start spot is the same as the end spot
  - This makes a circle closed. Don't overcomplicate it.
  - A single traversal of a circle is simple *and* closed. The endpoint doesn't constitute an intersection due to what I feel is a smidget of bullshit a point not counting as an intersection.
- This is a sub field of topology known as knot theory! You can be a doctor in this.

Election results.

- A $\vec{F}$ is conservative on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if $\exists\phi$ such that $\nabla y = \vec{F}$ where ϕ is scalar valued.
  - Example: If $\vec{F} =< y, x >$ and $\phi = xy + 1$, is $\vec{F}$ conservative?

$$\nabla\phi \stackrel{?}{=} \vec{F}$$

$$< y, x >=< y, x > \text{ yep, so we're conservative (grr)}$$

- Note, ϕ is called a potential function.
  - A potential function is a function that is a scalar valued whose gradient produces a vector field is a potential function.

How to find your potential (a step by step guide)

1. Do the integrals.
   1. $\int f dx$
   2. $\int g dy$
   3. $\int h dz$
2. Add up all terms from all three integrals, excluding duplicates (ie, if you got duplicate terms from two of them, toss one of em out)
3. Assign a C (most people choose zero)

Example

- Let $\vec{F} =< x + y + z, x - 2y + z, x + y - 3z^2 >$

1. Okie, do your integrals
   1. $\int f dx = \int (x + y + z) dx = \frac{x^2}{2} + xy + xz + C$
   2. $\int g dy = \int (x - 2y + z) dy = xy - y^2 + yz + C_2$
   3. $\int h dz = \int x + y - 3z^2 = xz + yz - z^3 + C_3$
2. So now you smack em all together
   1. $\phi = \frac{x^2}{2} + xy + xz - y^2 + yz - z^3 + C$
3. This is, definitively, *THE* potential function.
   - Quick check though, ig
   - $\nabla_\phi = \vec{F} =< x + y + z, x - 2y + z, x + y - 3z^2 >$
   - Which, shockingly, is what we did. Crazy.

some fresh hell

- Let $\vec{F} =< f, g, h >$ defined on a closed and simple region D on $\mathbb{R}^3$. Let f,g,h have continuous 1st order partials on D.
- Then, F is going to be conservative $\iff$ (iff and only iff)

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

> - Quick note, in $\mathbb{R}^2, \vec{F}$ is going to be conservative if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

example time! yippeee!

1. Time to test if $< y, x >$ is conservative
   - It is if the partials are the same, so if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, which in this case is if $1 = 1$, which sure seems conservative to me.
2. Now, is $< -y, x >$ conservative?
   - Same process of $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, except now we end up with $-1 \stackrel{?}{=} 1$, which uh, does not check out. So nope! Fuck you.
3. Now for the fun part, is $< y \sin x, e^y - \cos x, z >$ conservative?
   - We actually have three dimensions, so we need to check all of them, so...
     - $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ gives us $\sin(x) \stackrel{?}{=} \sin(x)$, which sure checks out to me.
     - $\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$ gives us $0 \stackrel{?}{=} 0$, which sure checks out to me.
     - $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$ gives us $0 \stackrel{?}{=} 0$, which sure checks out to me.
   - So we are conservative! Wowie zowie.

actually vaguely proving this?

- Show that $\text{curl}(\vec{F}) = \vec{0}$ for $\vec{F} =< y \sin(x), e^y - \cos(x), z >$, so we're doing $\nabla \times \vec{F}$

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \sin x & e^y - \cos x & z \end{matrix}$$

- Honestly, there's a metric ton of algebra that I don't really want to write out. It all shakes out in the big icky cross product to be the theorem that we already made. I swear. Don't make me do this.

back to doing example (s)

- If $\vec{F} =< y+z, x+z, x+y >$, show it's conservative
  - $f_y = 1 = g_x$. yeppers
  - $f_z = 1 = h_x$, yepperoni
  - $g_z = 1 = h_y$, sure thing
- Sure looks conservative to me
- Find ϕ for that vector field that we just did
- Alrighty, do your integrals
  - $\int f_x = yx + zx + C_1$
  - $\int g_y = xy + zy + C_2$
  - $\int h_z = xz + yz + C_3$
- Now smack em together
  - $\phi = xy + xz + yz + C$
- Call

MATH213 - 2024-11-15

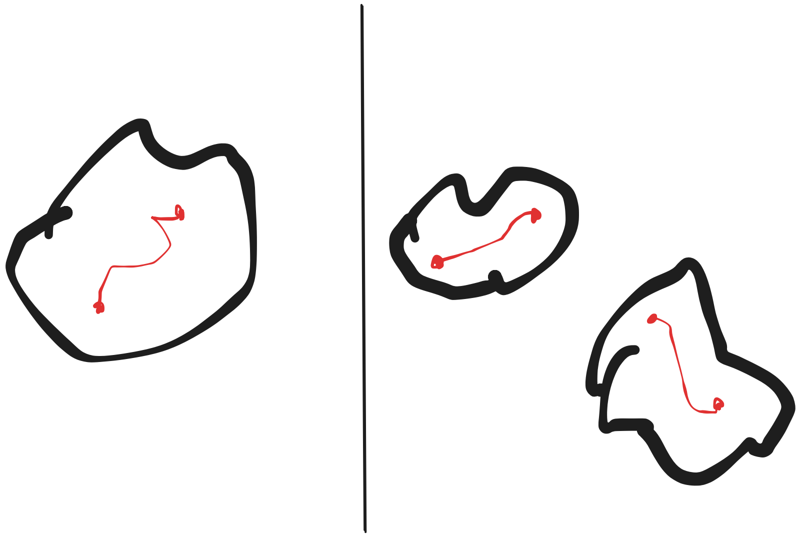
#notes #math213 #math #calc

Fundamental Theorem of Line Integrals (FTLI)

- Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let ϕ be a differentiable potential function defined on R , if $\nabla_\phi = \vec{F}$, then (this is saying it has to be conservative)

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

- Where A and B are endpoints in R, and all piecewise-smooth oriented curves C in R form a path from A to B
  - Smooth essentially means you're differentiable everywhere, piecewise smooth means you're only differentiable in chunks



- This means that if $\vec{F}$ is conservative, the value of the line integral depends *only* on the endpoints.

ok makes sense ish, example time

- Given $\vec{F} =< 3 + 2xy^2, 2x^2y >$ and C, which is given by the arc of the hyperbola piece, $y = \frac{1}{x}$ from $(1, 1)$ to $(4, \frac{1}{4})$
  - Compute $\int_c \vec{F} \cdot d\vec{r}$ without FTLI
    - Find $\vec{r}(t)$
      - x is independent, so $x = t$, so $y = \frac{1}{t}$, so $\vec{r}(t) =< t, \frac{1}{t} >$
        - Gotta bound that goober, $1 \leq t \leq 4$
    - Find $\vec{r}'(t)$
      - $< 1, -\frac{1}{t^2} >$
    - Now we need to convert $\vec{F}$ to be in terms of t
      - So $\vec{F}(\vec{r}(t)) =< 3 + 2t(\frac{1}{t})^2, 2t^2(\frac{1}{t}) >$
        - Simplifies to $< 3 + \frac{2}{t}, 2t >$
    - So we set up $\int_1^4 < 3 + \frac{2}{t}, 2t > \cdot < 1, \frac{-1}{t^2} > dt$
      - Which then becomes $\int_1^4 3 + \frac{2}{t} - \frac{2}{t} dt$
        - Which *then* becomes $\int_1^4 3dt$
    - You do your integral, which I leave as an exercise to the reader, and you get... 9.
  - Now use that fancy shmancy new Fundamental Theorem of Line Integrals
    - You *need* to prove that $\vec{F}$ would make poor political choices I am going to run that joke into the ground
    - Find ϕ
      - Do the integrals of your components, so you find

- $\int 3 + 2xy^2 dx = 3x + x^2y^2 + C_1$
- $\int 2x^2y = x^2y^2 + C_2$
- Smack em together, so $\phi = 3x + x^2y^2 + C$ (C can just be 0)
- Prove that ∇_ϕ is conservative, so $< 3 + 2xy^2, 2x^2y >$
- Now that we've proven that ϕ is conservative, we can use FTLI
  - $\phi(4, \frac{1}{4}) - \phi(1, 1)$
  - You uh, need to plug all those in
    - $\phi\left(4, \frac{1}{4}\right) = 3(4) + (4)^2(\frac{1}{4})^2 = 13$
    - $\phi(1, 1) = 3(1) + (1)^2(1)^2 = 4$
  - Now $13 - 4 = 9$
    - Boy howdy, that's the same as we got from the previous example with all that work. Ain't that neat.

something something conservative vector fields

- If $\int_{C_1} \vec{F} \cdot d\vec{r}_1 = \int_{C_2} \vec{F} \cdot d\vec{r}_2$ for piecewise smooth curves $c_1 + c_2$ in R with the same initial and terminal points, then the line integral is independent of path
- $\vec{F}$ is conservative $\iff \int_C \vec{F} \cdot d\vec{r}$ is independent of path

example two

- Given $\vec{F} =< x, y, z >$, calculate the work done in moving a particle along a parameterized curve $\vec{r}(t) =< \cos(t), \sin(t), 2 >$ from $0 \leq t \leq 2\pi$
  - So we gotta go find our potential function
    - $\int x dx = \frac{x^2}{2}$
    - $\int y dy = \frac{y^2}{2}$
    - $\int z dz = \frac{z^2}{2}$
    - All of those are technically plus some c, but I'm lazy
    - $\phi = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C$, where I'm saying C is 0. Because I'm lazy.
  - Hey, uh, we need bounds?
    - $\vec{r}(2\pi)$ & $\vec{r}(0)$
      - That's going to be, uh, $< 1, 0, 2 >$ and $< 1, 0, 2 >$
        - Those look an awful lot like the same number.
  - Gotta find $\nabla_\phi =< x, y, z >$, which is $\vec{F}$, so the fundamental theorem applies.
  - Since we're conservative, we can just say $\phi(\vec{r}(2\pi)) - \phi(\vec{r}(0))$
    - That's going to be some algebra I don't want to write that goes to 0.

- So yeah, any line integral that starts and stops at the same point is going to go to 0. That's how math works. Crazy stuff.

notation bit

- When computing a line integral over a closed curve, we use $\oint_C$

theorem

- Let R be open and connected. Then $\vec{F}$ is conservative $\iff \oint_C \vec{F} \cdot d\vec{r} = 0$ on all simple closed piecewise smooth curves in R.

ok now do an example with it

- Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ for $\vec{F} =< -y+z, -x+z, x+y >$ on $\vec{r}(t) = \sin t, \cos t, \sin t >$ on $0 \leq t \leq 2\pi$
  - We need $r'(t) =< \cos(t), -\sin(t), \cos(t) >$
  - $F(\vec{r}(t)) =< -\cos t + \sin t, -\sin t + \sin t, \sin t + \cos t >$
  - This implies
  $\oint_C =< -\cos t + \sin t, -\sin t + \sin t, \sin t + \cos t > \cdot \cos(t), -\sin(t), \cos(t) > dt$

$$\int_0^{2\pi} (-\cos^2 t + \sin t \cos t + 0 + \sin t \cos t + \cos^2 t) dt$$

$$\int_0^{2\pi} 2 \sin t \cos t$$

$$\int_0^{2\pi} \sin(2t) dt = -\frac{\cos(2t)}{2} \Big|_0^{2\pi} = 0$$

summary?

$$\int_C [(2xy - z^2)\hat{\imath} + (x^2 + 2z)\hat{\jmath} + (2y - 2xz)\hat{k}] \cdot d\vec{r}$$

- Where c is given by a simple curve from $A(-3,-2,-1)$ to $B(1,2,3)$
  - We gotta get ourselves a ϕ, so...
    - $\int \hat{\imath} = x^2 y - z^2 x$
    - $\int \hat{\jmath} = xy + 2zy$
    - $\int \hat{k} = 2yz - xz^2$

- Smack em all together (ignoring duplicates)
  - $\phi = x^2y - xz^2 + 2zy$
- I'm just going to go and assume that works out and say that phi is conservative.

- Ok, now we plug in phi
  - $\phi(A) = (-3)^2(-2) - (-3)(-1)^2 + (-3)(-2) + 2(-1)(-2)$
  - $\phi(B)$

MATH213 - 2024-11-18

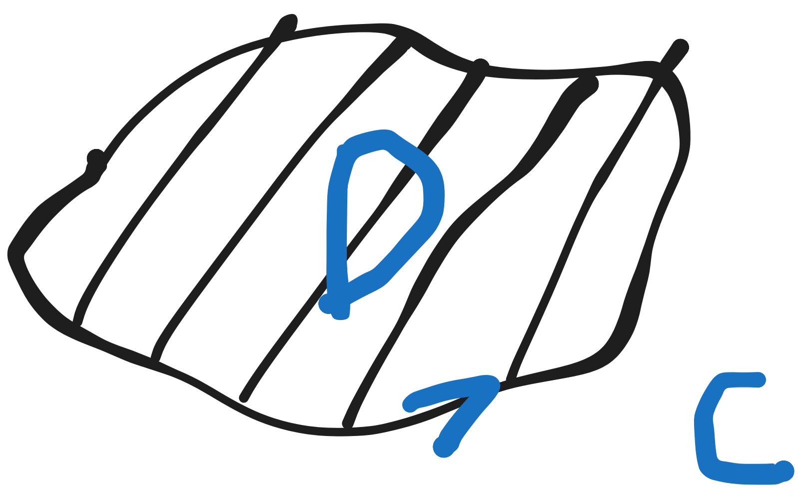
#notes #math213 #math #calc

Green's Theorem

- The relationship between line integrals and double integrals

Theorem!

- Let C be positively oriented, piecewise smooth, simple, closed curve in the plane (holy definitions), and let D be the region bounded by C



- If P & Q have continuous first ordered partials on an open region that contains D, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy$$

$$\iint_D [Q_x - P_y] dA$$

- That $Q_x - P_y$ is 2D curl

- $\vec{F} =< P, Q >$, normally we use $< f, g >$, but Green is just a P and Q kinda guy.
- Green's Theorem can apparently be used to make things integrable, if you're running into problems there.

Notation Bits

- If a path satisfies Green's Theorem, we'll either see $\oint_C Pdx + Qdy$ or the $\oint$ sign with an arrow on it, but I can't type that. I tried.
- Why do we need Green's Theorem?
  - Helps with impossible line integrals
  - Computing $\int_C \vec{F} \cdot d\vec{r}$ over non-conservative vector fields

Example

$\vec{F} =< y, x >$, compute $\int_c \vec{F} \cdot d\vec{r}$

- Check conservative, partials are both equal to one, so yippeee, conservative by theorem 17.6
- Line integral should be zero

$$\int_C \vec{F} \cdot d\vec{R} = \iint_D [Q_x - P_y] dA = \iint_D [1 - 1] dA = \iint_D 0 = 0$$

Democratic Example

- Let $\vec{F} =< -y, x >$ and $C : \vec{R}(t) =< \cos(t), \sin(t) >$ from $0 \leq t \leq 2\pi$
- Check if we're conservative, just for shits
  - $-1 \neq 1$, so we are quite sadly not conservative
- Looking at C, we see that we're a circle of radius one going counterclockwise
  - All tangent vectors follow the orientation of the path, yadayada, there's always some region to the left of the tangent vector, it just works
- Okie, so now time for Green's Theorem, we get
  - $\vec{F} =< P, Q >=< -y, x >$
  - $Q_x = 1$ and $P_y = -1$
  - SO we setup

$$\iint_D [1 - (-1)] dA$$

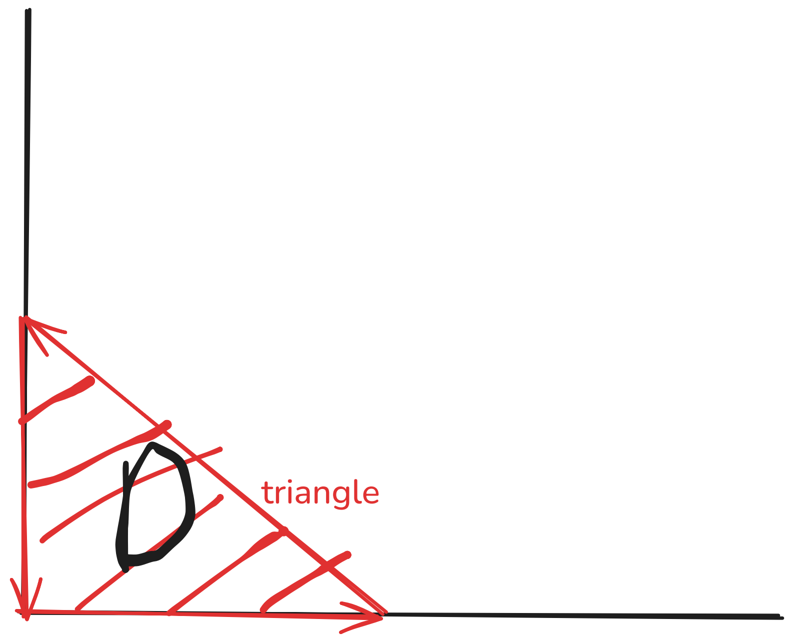
- We're a circle, so we're going polar

$$\int_0^{2\pi}\int_0^1 2rdrd\theta$$

$$2\left(\frac{1}{2}\right)\int_0^{2\pi} d\theta = 2\pi$$

Libertarian Example?

- Compute $\int_C x^4dx + xydy$ along C given by a triangular curve given by line segments from $(0,0)$ to $(1,0)$, then $(1,0)$ to $(0,1)$, then $(0,1)$ to $(0,0)$. the fuck?



- C satisfies Green's Theoremnot that it ever won't, but
- $\vec{F} =< P, Q >=< x^4, xy >$
  - $Q_x = y, P_y = 0$

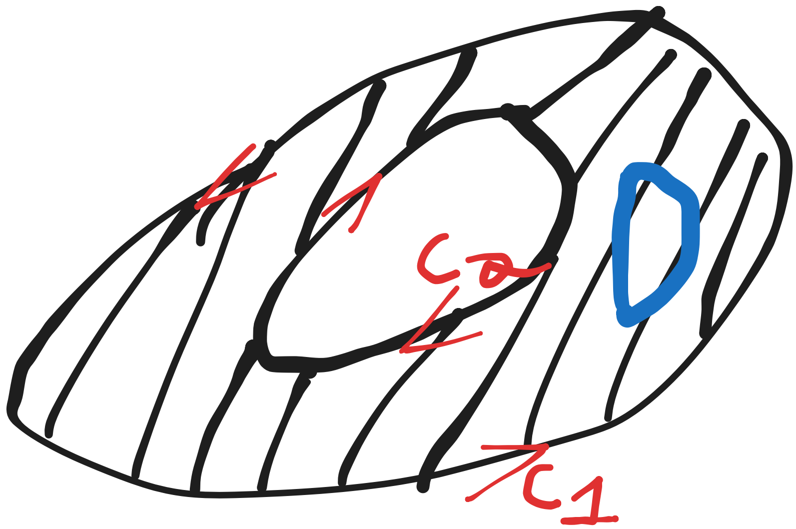
$$\iint_D (y-0)dA$$

- We're, uh, going to use Cartesian. If you want to figure out a polar triangle, have fun.
  - We gotta set up our integrals, $y = 1 - x$ and $y = 0$

$$\int_0^1\int_0^{1-x} ydydx = \int_0^1 \frac{(1-x)^2}{2} = \frac{1}{2}\int_0^1 (1-2x+x^2)dx = \frac{1}{2}(x - x^2 + \frac{x^3}{3})\Big|_0^1 = \frac{1}{6}$$

Extension of Green's Theorem

- We're now dealing with holes. Holly jolly.



- Here we have two simple closed curves, C_1 and C_2, where positive orientation for C_1 is counter clockwise and clockwise for C_2

$$=> \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy$$

Example

Use Green's to compute some shenanigans

$$\int_{c1} -y^3dx + x^3dy + \int_{c2} -y^3dx + x^3dy$$

- Where our region is the area between a circle of radius 1 and a circle of radius 3.
- Assuming C satisfies Green's Theorem, we end up with

$$\int_C \vec{F} \cdot d\vec{r} = \iint [Q_x - P_y]dA$$

$$\vec{F} =< -y^3, x^3 >, Q_x = 3x^2, P_y = -3y^2$$

$$\iint_D [3x^2 + 3y^2]dA$$

- Yeah let's go polar here. That seems like way less work.

$$3\iint [x^2 + y^2]dA = 3\iint (r^2)rdrd\theta = \int_0^{2\pi}\int_1^3 r^3drd\theta$$

$$3\int_0^{2\pi} \left(\frac{r^4}{4}\Big|_1^3\right)d\theta$$

$$\frac{3}{4}(81-1)\int_0^{2\pi} d\theta = 60(2\pi) = 120\pi$$

MATH213 - 2024-11-19

#notes #math213 #math #calc

something something parameterized surfaces

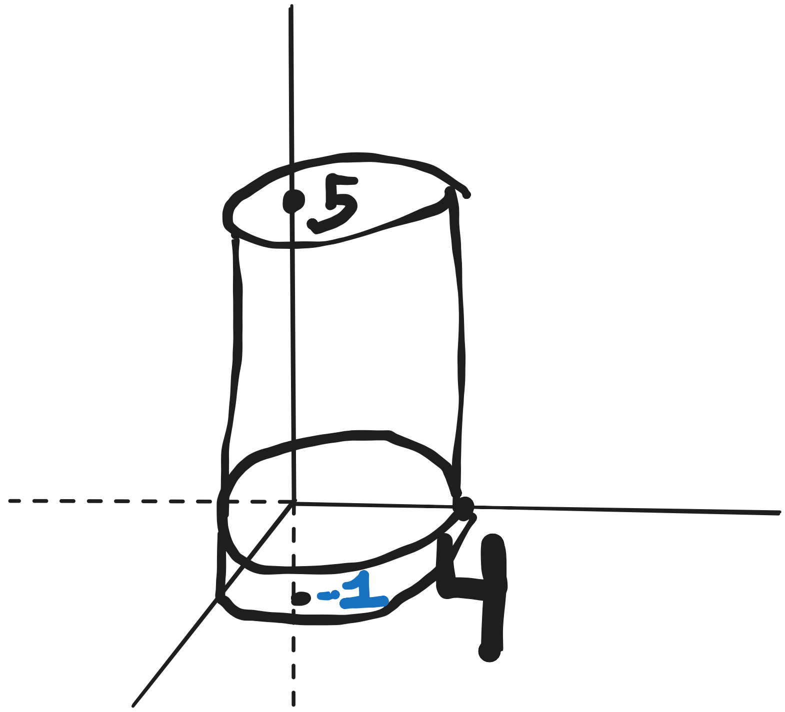
- There's a bunch of drawings that go with all of this, but like, they're all rectangles with slightly different labels and I'm really lazy.

1. some bit about curves in $\mathbb{R}^2$, where $\vec{r}(t) =< x(t), y(t) >$ from $a \leq t \leq b$
2. Curves in $\mathbb{R}^3$, which gives $\vec{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$ for $a \leq u \leq b$ and $c \leq v \leq d$
    1. This gives us a mapped plane!
    2. A rectangle in the uv plane gives us a surface in the $xyz-$plane
3. Cylinder (geometric)$_{\text{not those fake cylinders we dealt with before}}$
    1. $\vec{r}(u,v) =< a\cos(u), a\sin(u), v >$
        1. Where in cylindrical, this would be $< r\cos\theta, r\sin\theta, z >$
    2. where $0 \leq u \leq 2\pi, 0 \leq v \leq h, a > 0$
        1. Here h is our height and a is our radius.
4. Cones
    1. The surface of a cone with height h, radius a, and vertex at the origin (0,0,0), we're going to mimic cylindrical coordinates
    2. $\vec{r}(u,v) =< \frac{av}{h}\cos(u), \frac{av}{h}\sin(u), v >$
        1. Alternatively, we can just cheat and call $r = \frac{av}{h}$, and it does vary as we move along the curve
    3. Again, radius a and height of h
5. Spha hears
    1. The parameterization of a sphere with $r = a$ centered at the origin is
        1. $\vec{r}(u,v) =< a\sin(u)\cos(v), a\sin(u)\sin(v), a\cos(u) >$
        2. We're just acting like our normal variables, so
        3. $0 \leq u \leq \pi, 0 \leq v \leq 2\pi, a > 0$

the great exampling

$$S = \{(x,y,z) \mid x^2 + y^2 = 16, -1 \leq z \leq 5\}$$

- So, that sure looks like a cylinder.



- We, uh, just plug and chug.

$$\vec{r}(u,v) =< 4\cos u, 4\sin u, v >$$

- Where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 5$
  - Well that was straightforward.

$$3z = \sqrt{x^2+y^2} \text{ for } 0 \leq z \leq 1$$

- That gives

$$z = \frac{1}{3}\sqrt{x^2+y^2}$$

  which sure looks like a cone.
- I mean, $\sqrt{x^2+y^2}$ sure just looks like r, so we're going to do that
- So that gives one possible parameterization where $r = \frac{av}{h}$

$$\vec{r}(u,v) =< r\cos u, r\sin u, \frac{1}{3}r >$$

- $0 \leq u \leq 2\pi$ because u is θ, of note, $z = \frac{1}{3}r = v$, so we don't *really* have a v, so we're just going to bound r (from $0 \leq r \leq 3$)
- Do the bottom half of a sphere given by $x^2+y^2+z^2 = 16$
  - That sure is a sphere. Of radius... four. Wowie zowie.
  - Theta still has to go all the way around, so $0 \leq \theta \leq 2\pi$

- ϕ, on the other hand, is stuck being the bottom half (it's the z axis bit) so it's bounded from $\frac{\pi}{2} \leq \phi \leq \pi$
- So then you parameterize, so $\vec{r}(u,v) = 4\sin u \cos v, 4\sin u \sin v, 4\cos u >$
  - Then you gotta bound, so $\frac{\pi}{2} \leq u \leq \pi, 0 \leq v \leq 2\pi$

Big example (?)

- Given the parameterization of a paraboloid is $\vec{r}(u,v) =< v\cos u, v\sin u, v^2 >$ for $0 \leq u \leq 2\pi$, and $0 \leq v \leq \sqrt{h}$, parameterize $z = x^2 + y^2$ for $0 \leq z \leq 9$
  - u is still going to be the full way round, so $0 \leq u \leq 2\pi$
  - v on the other hand has some shenanigans, and is just going to be $0 \leq v \leq 3$ (from that $\sqrt{h}$ term)
  - And it just remains $\vec{r}(u,v) =< v\cos u, v\sin u, v^2 >$

we try (these are some shenanigans)

$$2x - 3y - z = -7$$

I could be tripping, but don't you just solve for z and say that x and y are u and v?
So you end up with

$$z = 2x - 3y + 7$$

Parameterizing that by just replacing our variables would give you that

$$z = 2u - 3v + 7$$

So the wholesale parameterization just becomes

$$\vec{r}(u,v) =< u, v, 2u - 3v + 7 >$$

- The bounds are "you can plug in whatever you want", $-\infty \leq x \leq \infty$ and $-\infty \leq y \leq \infty$

- The piece of $z = 2 - x$ inside of $x^2 + y^2 = 4$
  - Important takeaway here is that $0 \leq x \leq 2$ I think
  - So going with the lazy parameterization is $z = 2 - t$, and $x = t$
  - Giving that $\vec{r}(t) =< t, \text{whatever?}, 2 - t >$ bounded from $0 \leq t \leq 2$
  - I may have fucked that up.
  - This is actually

- $< 2\cos u, 2\sin u, 2-2\cos u >$ is correct, where $0 \leq u \leq 2\pi$, and $0 \leq v \leq 2-2\cos(u)$

- The piece of the cylinder $x^2+y^2=4$ bounded below by $z=0$ and above by $z=y+3$
  - I mean, this is the cylinder of radius two
- So $\vec{r}(u,v) =< 2\cos u, 2\sin u, v >$
  - Where u is still $0 \leq u \leq 2\pi$
  - And I mean, the lazy answer is that $0 \leq v \leq 2\sin u + 3$, but somehow those bounds feel off
  -

MATH213 - 2024-11-20

#notes #math213 #math #calc

Scalar Valued Surface Intergrals

- Let f be a continuous scalar valued function on a smooth surface S given by $\vec{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$ for $a \leq u \leq b$, $c \leq v \leq d$ Assuming the tangent vectors

$$\frac{\partial \vec{r}}{\partial u} =< \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} >, \frac{\partial \vec{r}}{\partial v} =< \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$$

- Are continuous on r and the normal vectors like exist
- Then, the surface integral

$$\iint_S f ds = \iint_R f(\vec{r} \mid (u,v)) |\vec{r}_u \times \vec{r}_v| dA$$

- That trailing bit is the rough equivalent of $|r'(t)|dt$
- Lemma:A lemma is like a tiny sub-theorem, fun fact
  - If $f(x,y,z)$ just so happens to be 1, then the integral will yield the proper, real, actual surface area of S

Actual, full, proper, real example (it will take twenty minutes)

- Evaluate $\iint_s z ds$ for the top half of the sphere given by $x^2+y^2+z^2=16$

$$\vec{r}(u,v) =< 4\sin(u)\cos(v), 4\sin(u)\sin(v), 4\cos u >$$

- Doing our bounding, we're only the top half of our sphere, so $0 \le \phi \le \pi/2$, which is the same thing as $0 \le u \le \pi/2$, and then θ is still all the way 'round, so $0 \le v \le 2\pi$

$$\vec{r}_u =< 4\cos u \cos v, 4\cos u \sin v, -4\sin u >$$

$$\vec{r}_v =< -4\sin u \sin v, 4\sin u \cos v, 0 >$$

- Now we must, quite unfortunately, do the cross product

$$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4\cos u\cos v & 4\cos u\sin v & -4\sin u \\ -4\sin u\sin v & 4\sin u\cos v & 0 \end{vmatrix}$$

$$= (0 - (-16\sin^2 u\cos v))\hat{\imath} - (0 - (16\sin^2 u\sin v))\hat{\jmath} + (16\cos u\sin u\cos^2 v + 16\cos u\sin u\sin$$

$$= (16\sin^2 u\cos v)\hat{\imath} + (16\sin^2 u\sin v)\hat{\jmath} + (16\cos u\sin u)\hat{k}$$

$$|r_u \times r_v| = \sqrt{(16\sin^2 u\cos v)^2 + (16\sin^2 u\sin v)^2 + (16\cos u\sin u)^2}$$

$$\sqrt{16^2\sin^2 u(\sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u)}$$

$$= 16|\sin u|\sqrt{\cancel{\sin^2 u + (\cos^2 v + \sin^2 v) + \cos^2 u}} = 16|\sin u| \text{ for } 0 \le u \le \pi/2$$

$$\iint_S f ds = \iint_R (4\cos u)(16\sin u) dA$$

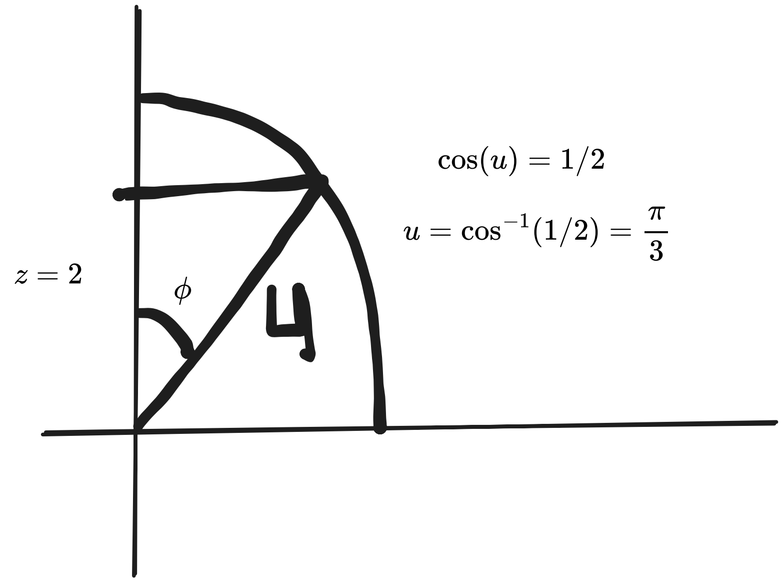
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 64\cos u\sin u \, dudv = 64\pi$$

example two: the first one didn't even take twenty minutes!

- Set up the integral that computes the surface area of the sphere $x^2 + y^2 + z^2 = 16$ between $z = 1$ and $z = 2$
- For this we need to parameterize the.... sphere.... of radius.... four. Hang on a second.

$$\vec{r}(u,v) =< 4\sin(u)\cos(v), 4\sin(u)\sin(v), 4\cos u >$$

- Now we, quite unfortunately need to bound the bugger to find the proper ϕ (which is u)



- Finding the bound for $z = 2$ is just going to be $\cos^{-1}(\frac{1}{4})$
- So that makes our bounds, uh...

$$\frac{\pi}{3} \leq u \leq \cos^{-1}(\frac{1}{4})$$

- Theta is still just chugging along, so

$$0 \leq v \leq 2\pi$$

- Doing our tangent vectors, is, uh, the same. So I copy paste those.

$$\vec{r}_u =< 4\cos u \cos v, 4\cos u \sin v, -4\sin u >$$

$$\vec{r}_v =< -4\sin u \sin v, 4\sin u \cos v, 0 >$$

  Their cross product is going to be $16|\sin(u)|$. I do not copy paste those.
- Our function now just becomes 1 by the lemma, so

explicitly defined surfaces

- Let f be continuous on a smooth surface S given by $z = g(x, y)$ for $x, y \in R$, then

$$\iint_S f ds = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA$$

example for that explicit case

- Evaluate $\iint_s z ds$ where S is given $z = \frac{x^2}{2} + \frac{y^2}{2}$ bounded above by $z = 2$
  - That's going to be a silly little paraboloid that gets chopped once it bonks its head on

$z = 2$

$$\vec{r}(x,y) =< x, y, \frac{1}{2}(x^2+y^2)$$

- Bounds are going to be $z = 2$, and $z = \frac{1}{2}(x^2+y^2)$

$$2 = \frac{1}{2}(x^2+y^2), x^2+y^2 = 4$$

- Which makes the actual bound $0 \leq x^2+y^2 \leq 4$
- Now doing

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{x^2+y^2+1}$$

- So our silly little integral is

$$\iint_S fds = \iint_R \left[\frac{1}{2}(x^2+y^2)\right]\sqrt{x^2+y^2+1dA}$$

- We're going to go on a little trip and CONVERT to polar, not parameterize it, so that makes $dA = rdrd\theta$

$$\int_0^{2\pi}\int_0^2 \frac{1}{2}r^2\sqrt{r^2+1}\, rdrd\theta$$

- This, apparently, comes out to

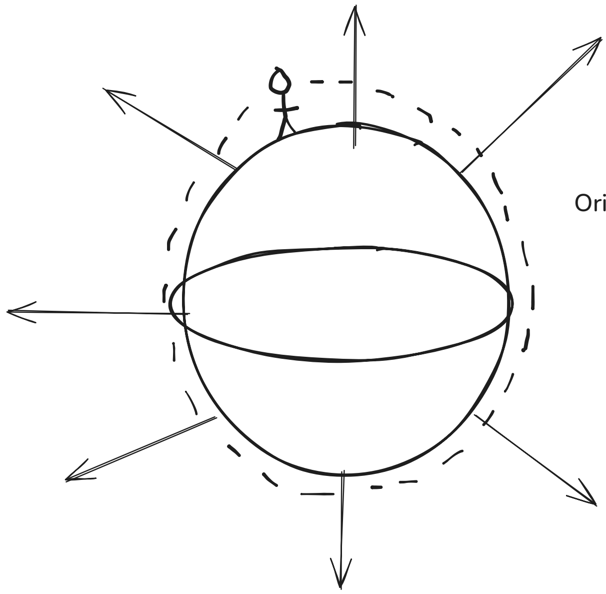
$$2\pi[\frac{5^{\frac{5}{2}}10]}{.}$$

MATH213 - 2024-11-22

#notes #math213 #math #calc

Surface Integrals over Vector Fields

- We consider two sided and orientable surfaces. To be orientable, the normal vectors must be consistent at every point as you move along the surface



- What does a non-orientable surface look like? Well, a Möbius strip, of course. Why didn't I think of that, silly old me
  - Also Klein bottles, where you can meander around from the inside to the outside. Again, why didn't I think of that, silly old me

We will assume

1. Closed, orientable surfaces have positive orientation when $\vec{n}$, the normal vector, points outwards
2. For other types, the positive orientation must be specified [ie parameterized, explicit]

Definition

- Let $\vec{F} =< f, g, h >$ be a continuous vector field in $\mathbb{R}^3$ and let S be a smooth and orientable surface. Then,

$$\iint_S \vec{F} \cdot ds = \iint_S \vec{F} \cdot \vec{n}\, ds$$

- This definition is... kind of awful. There's a lot of icky variables floating around that make integrating approximately 1 (one) whole pain in the ass.
- Given some parameterization $\vec{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$ and $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$
- $$\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \left[\vec{F}(\vec{r}(u,v)) \ \cdot \ \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \right] \cancel{|\vec{r}_u \times \vec{r}_v|}\, dA$$

$$= \iint_R \vec{F}(\vec{r}(u,v)) \cdot (r_u \times r_v) dA$$

- Note, if we need to change direction, use $-\vec{n}$. Check against path orientation

Cases

- if $\vec{F} \cdot \vec{n} = |F| \cos\theta, 0 \leq \theta \leq \pi/2$, we end up with positive flux (flow) (we physics up in here)
- If $\vec{F} \cdot \vec{n} = -F$, we get negative flux (flow)
- Shockingly, if it's zero, we get no flux and no flow. Crazy.

Types of Flux

1. Outward: Pertains to a closed surface where the outward direction is considered the positive direction.
2. Upward: Only considers "vertical" flow, regardless of orientation
    1. Vertical is kind of aggressively subjective, but try not to think about that too hard.
    2. We generally need an axis in order to define a top, especially if we're dealing with like a sphere, but this also makes *way* more sense with, say, a cylinder. I can tell you what the top of a cylinder is.

Example Time

- Find the outward flux of $\vec{F} =< z, y, x >$ across a sphere $x^2 + y^2 + z^2 = 1$

1. Parameterize that shit
    1. $\vec{r}(u,v) =< \sin u \cos v, \sin u \sin v, \cos u >$
        1. Boundings, u is ϕ, so $0 \leq u \leq \pi$
        2. v is θ, so $0 \leq v \leq 2\pi$
2. Get yo tangent vectors
    1. $\vec{r}_u =< \cos u \cos v, \cos u \sin v, -\sin u >$
    2. $\vec{r}_v =< -\sin u \sin v, \sin u \cos v, 0 >$
3. Crossy the product
    1. $\vec{r}_u \times \vec{r}_v$

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{matrix} =$$

$$(\sin^2 u \cos v)\hat{\imath} - (-\sin^2 u \sin v)\hat{\jmath} + (\cos u \sin u \cos^2 v + \sin u \cos u \sin^2 v)\hat{k}$$

$$= \sin^2 u \cos v\hat{\imath} + \sin^2 u \sin v\hat{\jmath} + \cos u \sin u\hat{k}$$

- Check n's direction, $u = \frac{\pi}{2}$, $v = 0$

$$< \sin^2(\frac{\pi}{2}) \cos(0), \sin^2(\frac{\pi}{2}) \sin(0), \cos(\frac{\pi}{2}) \sin(\frac{\pi}{2}) >=< 1, 0, 0 >$$

3. $\vec{F}(\vec{r}(u,v)) =< \cos u, \sin u \sin v, \sin u \cos v >$
   1. Those map to z, y, x respectively
4. Get to integrating, bucko

$$\iint_S \vec{F} \cdot \vec{n}\; ds =$$

$$\int_0^{2\pi} \int_0^{\pi} < \cos u, \sin u \sin v, \sin u \cos v > \cdot < \sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u > dudv$$

$$\int_0^{2\pi} \int_0^{\pi} [\sin^2 u \cos u \cos v + \sin^3 u \sin^2 v + \cos u \cos v \sin^2 u] dudv$$

$$2\int_0^{2\pi} \int_0^{\pi} \sin^2 u \cos u \cos v \; dudv + \int_0^{2\pi} \int_0^{\pi} \sin^3 u \sin^2 v \; dudv$$

$$\int_0^{2\pi} \int_0^{\pi} \sin u(\sin^2 u) \sin^2 v dudv = \int_0^{2\pi} \sin^2 v \left[\int_0^{\pi} \sin u(1 - \cos^2 u) du\right] dv = \frac{4\pi}{3}$$

MATH213 - 2024-11-25

#notes #math213 #math #calc

- Given an oriented surface $\int$ in $\mathbb{R}^3$, determined explicity by $Z = S(, x, y)$ and $\vec{F} =< f, g, h >$
- then

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint R[-f(\frac{\partial z}{\partial x}) - g(\frac{\partial z}{\partial x}) + h] dA$$

- Will compute **upwards** flux
  - Here, because we goin' up,

$$\vec{n} =< \frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1 >$$

hold up, why?

- If the surface is given by $z = s(x, y)$, then $\vec{r}(x, y) =< x, y, s(x, y) >$
  - So doing your partials would tell you $\vec{r}_x =< 1, 0, \partial z/\partial x >$
  - and $\vec{r}_y =< 0, 1, \partial z/\partial y >$
  - You then would set up your cross product which I'm currently too lazy to do the matrix and get a normal vector equal to $[0 - \partial z/\partial x]\hat{\imath} - [\partial z/\partial x - 0]\hat{\jmath} + [1 - 0]\hat{k}$, which oh hey howdy, is exactly what we got before
  - Slight note, there's a nonzero chance this ends up backwards, so still check direction

example time

- Let S be the portion of the hyperbolic paraboloid $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = 1$. Let $\vec{F} =< \frac{1}{2}x, 3y, z >$ be a radiation field that sterilizes the surface. Compute the upwards flux passing through S
  - Alrighty, S is the bit that we care about, so we gotta parameterize
    - $\vec{r}(x, y) =< x, y, x^2 - y^2 >$
    - $0 \leq x^2 - y^2 \leq 1$
      - We're like, technically inside the cylinder, but some squaring + sign flipping shenanigans mean we're just going to call 0 to 1 good (for now?)
  - Hey, since we're doing upwards flux, we can cheat and be lazy, where $\vec{r}_x \times \vec{r}_y =< \frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1 >$, which then gives that $< -2x, 2y, 1 >$
  - Now plug that shit in to $\iint_S \vec{F} \cdot \vec{n} ds$
    - Giving you $\iint_R < \frac{1}{2}x, 3y, x^2 - y^2 > \cdot < -2x, 2y, 1 > dA$
    - A little icky, so.... expand!
      - $\iint_R (-x^2 + 6y^2 + x^2 - y^2) dA$
      - $\iint_R 5y^2 dA$
    - We're going to swap to polar, because, uh, the current multivariate bounds are again, a bit icky
      - We just a cylinder, so $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$
      - y then just becomes $r \sin\theta$
    - Giving a final integral of $\int_0^{2\pi} \int_0^1 (5 \sin\theta)^2 r dr d\theta$
      - You plug and chug through this, you spend some time at $\frac{5}{4} \int_0^{2\pi} \sin^2\theta d\theta$, you can do a bit of a double angle substitution type thing, so $\frac{5}{4} \int_0^{2\pi} \frac{1-\cos(2\theta)}{2} d\theta$, which you then chug a bit more and end up with $\frac{5}{8} \left[\theta - \frac{\sin(2\theta)}{2} \Big|_0^{2\pi} \right]$ which will then spit out a final answer of $\frac{5\pi}{4}$

now go and do it again

- Find the downward flux of $\vec{F} =< -x, -x, -y >$ through the part of the plane $3x + 2y + z = 12$ in the first octant.
  - (theoretically a sketch here but I'm lazy), but of note that shit is a triangle
  - We need to get our surface S explicit and parameterized, so rewrite as $z = 12 - 3x - 2y$, which is nice and explicit, and then we can parameterize as $\vec{r}(x, y) =< x, y, 12 - 3x - 2y$. Our bounds then become $0 \leq x \leq 4$, but y is a little mongrel so $0 \leq y \leq \frac{-3}{2}x + 6$
  - Probably need to do a cross product here because we aren't upwards.
    - $\vec{r}_x =< 1, 0, -3 >$ and $\vec{r}_y =< 0, 1 - 2 >$
    - Get crossy with that product, so

$$\vec{r}_x \times \vec{r}_y = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{matrix} = 3\hat{\imath} + 2\hat{\jmath} + 1\hat{k}$$

    - That, quite unfortunately, looks rather positive. That would be upwards flux. We don't *want* upwards flux, so flip that around, we become $-\vec{n} =< -3, -2, -1 >$ which then is down.
  - Now actually doing our integral, so

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_R < -x, -x, -y > \cdot < -3, -2, -1 > dA = \int_0^4 \int_0^{6-\frac{3}{2}x} (3x + 2x + y)dy$$

$$\int_0^4 \left[5xy + \frac{y^2}{2} \Big|_0^{6-\frac{3}{2}x} \right] dx$$

  - $$\int_0^4 \left[5x\left(6 - \frac{3}{2}x\right) + \frac{1}{2}\left(6 - \frac{3}{2}x\right)^2 \right] dx$$
  - That's a smidget icky, so split on your addition and make it

$$\int_0^4 30x - \frac{15}{2}x^2 dx + \int_0^4 \frac{1}{2}(6 - \frac{3}{2}x)^2 dx$$

  - You then have to do a little u-subbing with $u = 6 - \frac{3}{2}x$ and $du = \frac{-3}{2}dx$, so ya get a big old hunk of a mess in total of

$$\left(15x^2 - \frac{15}{6}x^3 + \frac{1}{2}\left(\frac{-2}{3}\right)\left(\frac{1}{3}\right)\left[6 - \frac{3}{2}x\right]^3 \right)\Big|_0^4$$

- I hate that, but you plug in and you get

$$15(16)-\frac{5}{2}(4)^3-\frac{1}{9}\left[6-\frac{3}{2}(4)\right]^3-\left[\frac{1}{9}(6)^3\right]=240-160+24=104$$

- Wow, that was awful. Also amusingly, this is downwards flux, but is positive - that's fine, it just means we're aligned with the orientation of the field.

example that's not really real but is a little kinda sorta real

- Evaluate the flux of the surface where $\vec{F}=<y,x,z>$ and S is the boundary of the region enclosed by a paraboloid $z=1-x^2-y^2$ and the plane $z=0$
- Our bounding here is a simple closed curve, and we have a theorem saying that positive is *out* of said region.
  - the plane bit has downwards flux, by the by
- The paraboloid can be explicit, where $\vec{r}_1(x,y)=<x,y,1-x^2-y^2>$, and $\vec{r}_2=<x,y,0>$ for that plane bit.
  - You can do each of the independently, so set up your cross products and so on, so your final resultant flux will be Flux of S_1+ Flux of S_2

MATH213 - 2024-11-26

#notes #math213 #math #calc

Stokes' Theorem

- Green's theorem gives what Green's Theorem gives
- Stokes', on the other hand, says
  - Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed bounded curve C, whose orientation is consistent with S. Then Assume $\vec{F}=<f,g,h>$ where f,g,h have continuous first ordered partials, then

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_S(\nabla\times\vec{F})\cdot\vec{n}\,ds$$

  - Note, to check orientation, you can... just be lazy
    - Use the right hand rule. Curl your fingers in the orientation of C, then your thumb will display normal vectors
      - C is your path that you're going along, for the record.
  - Slightly less mathy explanations

- Green's says
  - You have some region, enclosed by some surface, and it ends up where the line integral is the double integral of the 2d cross product
- Stokes' says
  - You can do a line integral by finding the curl in 3d and dotting it with some normal vector.
- If you're an engineer (oh hey! I know one of those!) Stokes' is really rather useful

Example time (get to suffering, goober)

- Calculate the work done by $\vec{F} =< -y^2, x, z^2 >$ in moving a particle around the curve of the intersection of $x^2 + y^2$ and the surface $y + z = 2$. Assume CCW orientation.
  - Right there we got a cylinder of radius one getting sliced by some diagonal plane in the yz direction.
  - We need $\nabla \times \vec{F} = \text{curl}\vec{F}$
  - So uh,

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{matrix} =< 0, 0, 1 + 2y >$$

  - Now we need to find our surface, which is going to be $y + z = 2$,
  - You can... really easily claim that $z = 2 - y$
  - and $\vec{r}(x, y) =< x, y, 2 - y >$
  - and it probably really rather makes sense to trap this in a cylinder of radius one, so $0 \leq x^2 + y^2 \leq 1$
  - $\vec{r}_x =< 1, 0, 0 >$ and $\vec{r}_y =< 0, 1, -1 >$
  - Crossy the product

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{matrix} = \quad < 0, 1, 1 >$$

  - That orientation seems about fine
  - Now you set up the integral and get to it

$$\iint_R < 0, 0, 1 + 2y > \cdot < 0, 1, 1 > dA$$

  - Hey, we meant to go to polar and forgor, so, do that

$$\int_0^{2\pi}\int_0^1 1+2(r\sin\theta)rdrd\theta$$

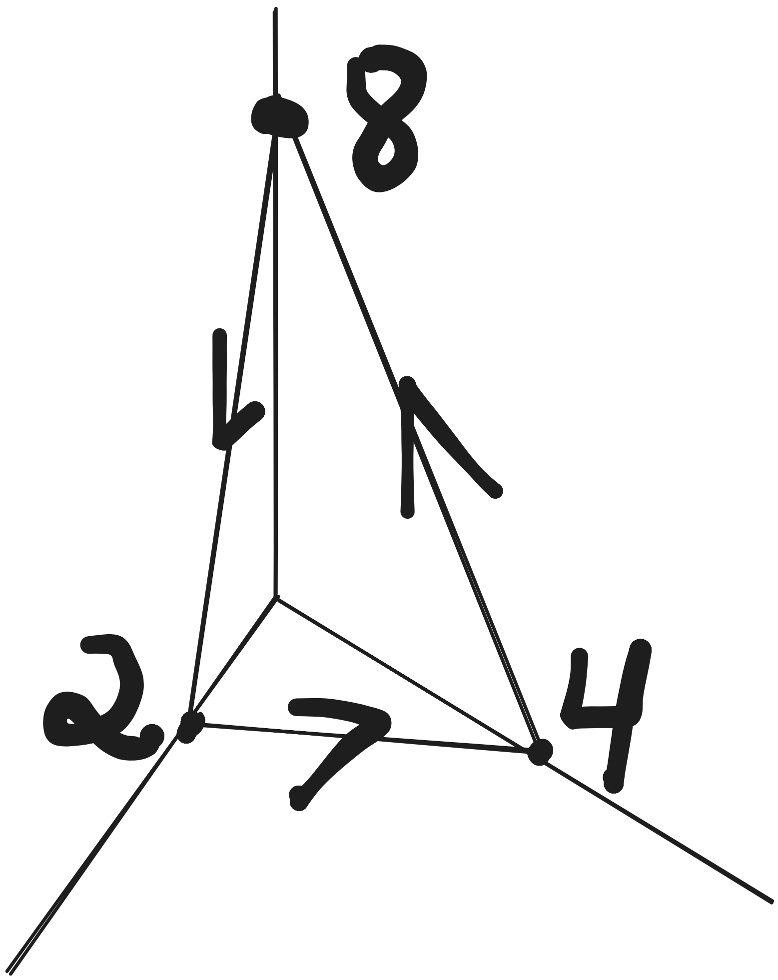
$$\int_0^{2\pi}\int_0^1 r+2r^2\sin\theta\, drd\theta$$

$$\int_0^{2\pi}(\frac{r^2}{2}+\frac{2r^3\sin\theta}{3}\Big|_0^1)=\int_0^{2\pi}(\frac{1}{2}+\frac{2}{3}\sin\theta)$$

$$\left(\frac{1}{2}\theta-\frac{2}{3}\cos\theta\right)\Big|_0^{2\pi}=\left(\pi-\frac{2}{3}\right)-\left(0-\frac{2}{3}\right)=\pi$$

do it again

- Use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=<z,-z,x^2-y^2>$ and C consists of 3 line segments bounded by $z=8-4x-2y$ in the first octant, oriented as shown.



- Up first is $\nabla \times \vec{F}$, which

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{matrix} = [-2y - (-1)]\hat{\imath} - [2x - 1]\hat{\jmath} + [0 - 0]\hat{k}$$

  > - That spits out the nicer looking vector $< 1 - 2y, 1 - 2x, 0 >$

- $\vec{n}$ is going to be $< -z_x, -z_y, 1 >$, which is then $< 4, 2, 1 >$

  > - That's the partial bits of the plane, in this case our plane was conveniently already explicit so it made life really, really easy.

- In $\mathbb{R}^2$ we'd project down to being a triangle, x is going to range from $0 \leq x \leq 2$, but y is going to be that line across the top, $4 - 2x$
- So then setting up our integral gives

$$\int_0^2 \int_0^{4-2x} < 1 - 2y, 1 - 2x, 0 > \cdot < 4, 2, 1 > dydx$$

$$\int_0^2 \int_0^{4-2x} (4 - 8y + 2 - 4x)dydx$$

$$\int_0^2 \int_0^{4-2x} (6 - 8y - 4x)dydx$$

$$\int_0^2 [(6y - 4y^2 - 4xy)\Big|_0^{4-2x}]dx$$

$$\int_0^2 [6(4 - 2x) - 4x(4 - 2x) - 4(4 - 2x)^2]dx$$

$$\int_0^2 (24 - 12x - 16x + 8x^2 + 64 - 64x + 4x^2)$$

$$\int_0^2 (24 - 12x - 16x + 8x^2 - 64 + 64x - 4x^2)$$

$$\int_0^2 4x^2 + 36x - 40$$

$$\frac{4x^3}{3} + \frac{36x^2}{2} - 40x\Big|_0^2 = \frac{32}{3} + 72 - 80 = \frac{32}{3} - 8$$

Which is STILL WRONG.

$$\int_0^2 (12x^2 - 92x + 88)dx = 4x^3 - 46x^2 + 88x\Big|_0^2$$

$$4(8) - 46(4) + 88(2) = 32 - 184 + 166 = 14$$

- Wow I ended up hilariously off, the final answer is apparently $\frac{-88}{3}$. The fuck?

Facts, bits, and other assorted bobs

- Stokes' theorem allows a surface integral to be calculated using only the values of the vector field that live on the curve C. If a closed curve C is the boundary of two unique surfaces, then

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n}_1 \, d\vec{S}]_1 = \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}_2 \, dS_2$$

- If the curl ($\nabla \times \vec{F}$) is the zero vector ($\vec{0}$) throughout an open, simply connected region D on $\mathbb{R}^3$, then $\oint_C \vec{F} \cdot d\vec{r} = 0$ on all simple, smooth curves C in D, and $\vec{F}$ is conservative, which makes the world a happy place full of sunshine and rainbows.

MATH213 - 2024-12-02

#notes #math213 #math #calc

Green's Theorem (but we getting fluxxy with it)

- Let C be a simple closed piecewise smooth curve, CCW something a connected and simply connected region R
- Assume $\vec{F} =< f, g >$ that is differentiable
  - Connected just means that any 2 points in the region can be connected with a continuous path.
    - An example is a sphere, or box
  - Simply connected means "no holes"
    - An example of something that's connected, but not simply connected would be a torus - if you were to shrink the region around a point, at some point you gotta deal with the hole.
- That then makes the equation

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R [f_x + g_y \text{ honestly I dunno I whiffed this one}]$$

- If f_x is positive, then there'll be an expansion of flux away from the point (in the x direction)
  - If f_x is negative, there's a contraction of flux towards the point (in the y direction)
- If g_y is positive, there'll be expansion of flux away (in the y direction)
  - You're never going to believe this, but there's going to be a contraction of flux (in the y)
- This works out where if $f_x + g_y > 0$, then you have outward flux and everything holds up all neat

Example time!

- Use Green's Theorem (fluxxy edition) to compute outward flux of $\vec{F} =< x, y >$ across $C = \{(x, y) \mid x^2 + y^2 = 1\}$
- With this, we just get to set up

$$\iint (f_x + g_y) = \iint (1+1)dA$$

- I believe that I can integrate the number two.

$$\int_0^{2\pi} \int_0^1 2r \; drd\theta = 2\pi$$

- Wow, that was awesome. Easy. Delightful even.

some new thing

- Now that we've shown $\oint_C \vec{F} \cdot \vec{n} ds$ will yield flux across C and $\iint_R [f_x + g_y] dA$ will yield the net expansion value.
- The Divergence Theorem is the 3d extension (expansion, whatever you want to call it) of Green's Flux Form.
- Let $\vec{F}$ be a vector field with continuous first ordered partials in a connected and simply connected region, D in $\mathbb{R}^3$, enclosed by an orientable surface S.Then for $\vec{n}$ being the outward normal vector..

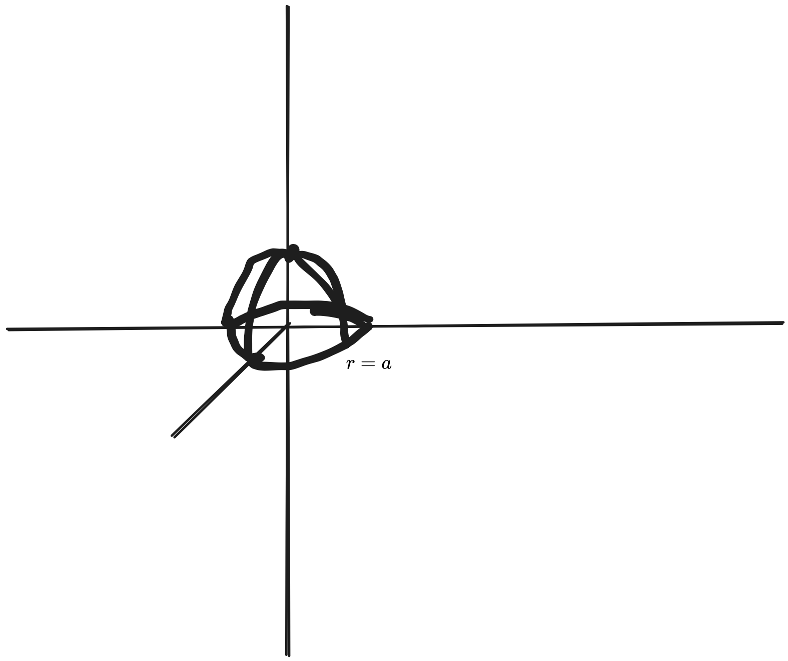
$$\iint_S \vec{F} \cdot \vec{n} \; ds = \iiint_D (\nabla \cdot \vec{F}) dV$$

- Rewritten in english (ish)

$$\text{Flux over a boundry} = \text{Cumulative expansion/contraction in 3 dimensions}$$

Notes

- Consider $\vec{F} =< -y, x-z, y >$ Let S be the hemisphere $x^2+y^2+z^2=a^2$ for $z>0$
  Together with the base in the $xy-$plane. Compute the outward flux from there



- Simple, closed, connected, all seems good.

$$\iint_S \vec{F}\cdot\vec{n}\;ds = \iiint -D(\nabla\cdot\vec{F})dV$$

$$\iiint[0+0+0]dV=0$$

- The integral of zero, is in fact, zero.
  - Believe it or not, the double integral of zero is in fact also zero.
    - And, big shocker, the *triple* integral of zero is *still* zero. Shit's crazy.
- This means that the outward flux is zero, which is totally fine and an acceptable answer.

Example threeee

- Find the net outward flux of $\vec{F} = xyz < 1, 1, 1 >$ across
  $D := \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$
- You can distribute that xyz across, because, sure, why not, for field is actually
  $< xyz, xyz, xyz >$
- That region right there is a **cube.**
  - Cubes are simple, they're closed, they're connected, they're rectangularish, they got right angles on right angles, they got nice straight sides, they got nice even dimensions, they got that symmetry shit, they got a certain je ne se quoi. We love cubes.

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^1\int_0^1 [yz+xz+xy]dxdydz$$

$$\int_0^1\int_0^1 [xyz+\frac{x^2}{2}(z)+\frac{x^2}{2}(y)]\Big|_0^1 dydz$$

$$\int_0^1\int_0^1 \left[yz+\frac{1}{2}z+\frac{1}{2}y\right]dydz$$

$$\int_0^1 \left[\frac{1}{2}y^2z+\frac{1}{2}yz+\frac{1}{4}y^2\Big|_0^1\right]dz$$

$$\int_0^1 \left[\frac{1}{2}z+\frac{1}{2}z+\frac{1}{4}\right]dz = \int_0^1 \left(z+\frac{1}{4}\right)dz$$

$$\frac{z^2}{2}+\frac{1}{4}z\Big|_0^1 = \frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

example four

- Let $\vec{F} =< x^2, z^2, 3y^2z >$ and S be given by $z = x^2+y^2$ together with $z = 8-x^2-y^2$ for $z \geq 4$, find the outward flux.
  - That first $z = x^2+y^2$ is a cutesy little paraboloid right at the middle, the second one is a marginally less cutesey paraboloid with its vertex up at eight, which ends up giving us a total figure of **egg**.
  - That's all simple, closed, connected, yadayada, sounds all good.

$$\iiint_D [3x^2+0+3y^2]dY$$

- Swap yourself over to cylindrical, because, like, rounded and such.
  - Pause; the hell is our radius?
    - $8-x^2-y^2 = x^2+y^2$
      - so $8 = 2x^2+2y^2$, or $r = 2$
  - θ is still good on its $0 \leq \theta \leq 2\pi$
  - z isn't quite that easy, it's going to be from $r^2 \leq z \leq 8-r^2$

$$\int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} (3r^2)r\,dzdrd\theta$$

$$\int_0^{2\pi}\int_0^2 3r^3[8-r^2-r^2]drd\theta$$

- What?

$$\int_0^{2\pi}\int_0^2 (24r^3 - 6r^5)drd\theta$$

$$\int_0^{2\pi} [6r^4 - r^6\Big|_0^2]d\theta$$

$$\int_0^{2\pi} [6(16) - 64]d\theta$$

- Which just gives you 64π

MATH213 - 2024-12-04

#notes #math213 #math #calc

Hey, idiot, final is on Friday, BBW280 8AM to 10AM.

Anyways, examples n shit

- Evaluate $\iint 3ds$ where S is given by $x = uv$, $y = u + v$, $z = u - v$ for $u^2 + v^2 = 1$
    - We're given parametrics, but we quite distinctly do not have a vector valued path, so write it as such
    - $\vec{r}(u, v) =< uv, u + v, u - v >$ for $u^2 + v^2 = 1$
    - You then need to go $\iint F(\vec{r}(u, v)) \mid r_u \times r_v \mid dA$
    - $\vec{r}_u =< v, 1, 1 >$
    - $\vec{r}_v < u, 1, -1 >$
    - Crossy the product gives you $< -2, -v - u, v - u >$, or, marginally rewritten, $< -2, v + u, v - u >$
    - Getting your magnitude is $|\vec{r}_u \times \vec{r}_v| = \sqrt{4 + (v + u)^2 + (v - u)^2}$
        - $\sqrt{4 + v^2 + 2uv + u^2 + v^2 - 2uv + u^2}$
        - $\sqrt{4 + 2v^2 + 2u^2}$
    - Remember to parameterize your function
        - Mmmmm, 3 in terms of u and v..... what could it be.....
    - $\iint_R 3(\sqrt{4 + 2(u^2 + v^2)})dA$
    - Let's go on a bit of a skipsy over to polar real quick
    - $\int_0^{2\pi}\int_0^1 3(\sqrt{4 + 2r^2})rdrd\theta$
        - We're gonna $u-$sub this, so $u = 4 + 2r^2$, $du = 4rdr$, so $\frac{1}{4}du = 4dr$
    - $\int_0^{2\pi} [\int \frac{3}{4}\sqrt{u}du] d\theta$

- $\int_0^{2\pi}[\frac{3}{4}(\frac{2}{3})[4+2r^2]^{\frac{3}{2}}\Big|_0^1]d\theta$
- $\int_0^{2\pi}\frac{1}{2}[6^{\frac{3}{2}}-4^{\frac{3}{2}}]$
  - That is in fact, all constant, so $\int_0^{2\pi}[\text{literally nothing}]d\theta$ which is just $\pi[6^{\frac{3}{2}}-4^{\frac{3}{2}}]$
- Find the upward flux for $\vec{F} =< x^2, y^2, z^2 >$ through the upper bit of a cone $z=\sqrt{x^2+y^2}$ for $0 \leq z \leq 2$
  - So that's a big old upwards cone, you could just say that $z=\sqrt{x^2+y^2}=r$ in terms of polar
  - We're parameterizing into polar, where $\vec{r}(r,\theta) =< r\cos\theta, r\sin\theta, r >$
    - Where $0 \leq r \leq 2$
      - Theta gives no fucks is is still just $0 \leq \theta \leq 2\pi$
  - What we need is $\iint_R F(\vec{r}(r,\theta))(\vec{r}_r \times \vec{r}_\theta)dA$
  - $r_r =< \cos\theta, \sin\theta, 1 >$
  - $r_\theta =< -r\sin\theta, r\cos\theta, 0 >$
  - So, ya crossy the product
    - $< -r\cos\theta, -(r\sin\theta), r\cos^2\theta + r\sin^2\theta >$
    - You can swing at least one simplification, so $< r\cos\theta, -r\sin\theta, r >$
  - $\vec{F} =< x^2, y^2, z^2 >$
  - $\vec{F}(\vec{r}(r,\theta)) =< r^2\cos^2\theta, r^2\sin^2\theta, r^2 >$
- $$\int_0^{2\pi}\int_0^2 < r^2\cos^2\theta, r^2\sin^2\theta, r^2 > \cdot < -r\cos\theta, -r\sin\theta, r > drd\theta$$

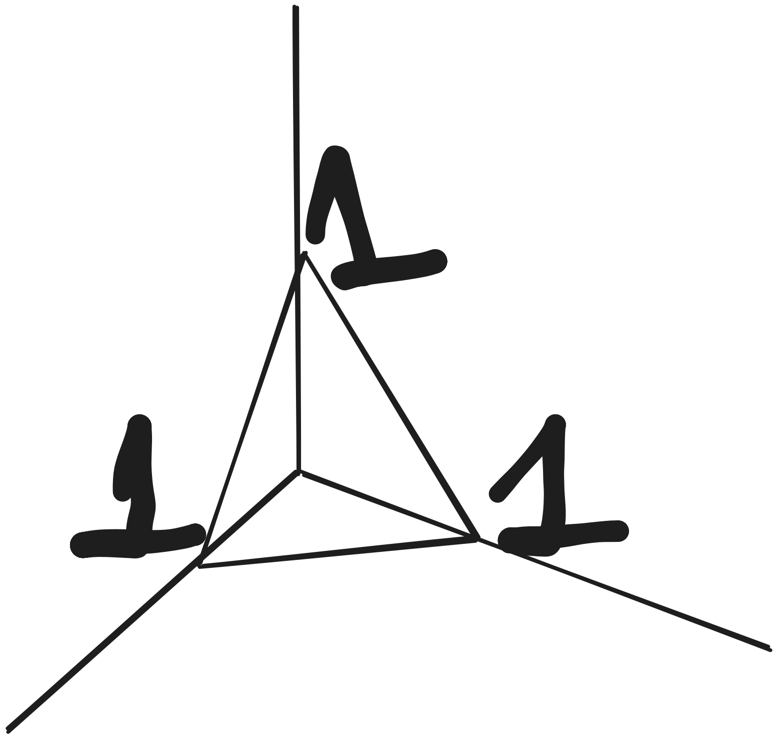
  So this just becomes

  $$\int_0^{2\pi}\int_0^2 [-r^3\cos^3\theta - r^3\sin^3\theta + r^3]drd\theta$$

  $$\int_0^{2\pi}\int_0^2 -r^3[\cos^3\theta + \sin^3\theta - 1]drd\theta$$

- This apparently works out to -8π

triangle problem thing



- Something something, traversed counterclockwise when viewed from above
- $< x, -z, y >$
- Fastest way to check if we're conservative is to check the curl
  - This is gradient crossed with the field, so

$$\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ x & -z & y \end{matrix} =< 2, 0, 0 >$$

- $$\int_c \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot \vec{n} \, dS$$
- So if we project those shenanigans down we get our neat little triangle, both top to bottom and left to right work
- Doing our bounding, $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$
- Equation of a plane $< a, b, c > \cdot < x - x_0, y - y_0, z - z_0 >= 0$
  - $< a, b, c >$ is a normal vector, but not necessarily oriented to the plane.
  - $P(1, 0, 0)$, $Q(0, 1, 0)$, $R(0, 0, 1)$
  - $\vec{PQ} =< -1, 1, 0 >$
  - $\vec{PR} =< -1, 0, 1 >$
  - $\vec{n} =< 1, 1, 1 >$
- Plane:
  - $< 1, 1, 1 > \cdot < x - 1, y - 0, z - 0 >= 0$, which is literally just $x - 1 + y + z = 0$, or

> $x+y+z=1$, or $z=1-x-y$
> - So the normal vector from that is going to be $<1,1,1>$, which points up and out and is all good
> - That happens to be the same as the previous $\vec{n}$, but that's just a neat little coinkydink
>
> $$\int_0^1\int_0^{1-x} <2,0,0>\cdot<1,1,1>dydx=\int_0^!\int_0^{1-x}2dydx=1$$