Extending Defeasible Reasoning beyond Rational Closure

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ABSTRACT

Defeasible reasoning is a vital component of many intelligent systems, as it provides a way to reason about uncertain or exceptional information. Rational closure has been at the forefront of research in defeasible reasoning, serving as a basis for a class of rational defeasible entailment relations. Lexicographic closure is one such entailment relation which has been thoroughly researched, though other members of this class do exist. This literature review aims to explore the various extensions proposed in the literature for defeasible entailment beyond rational closure. The findings of this review can provide insights into the characteristics, advantages, and limitations of these forms of entailment, and how they compare to rational closure and lexicographic closure.

CCS CONCEPTS

• **Theory of computation → Automated reasoning**; • **Computing methodologies → Nonmonotonic, default reasoning and belief revision**.

KEYWORDS

artificial intelligence, knowledge representation and reasoning, nonmonotonic reasoning, defeasible reasoning, rational closure

1 INTRODUCTION

A vital component of any intelligent system is the ability to contain, in a suitable format and language, knowledge about the world. *Knowledge representation* is the process of representing knowledge in a way that can be processed by an intelligent system. It involves selecting a suitable representation for encoding the knowledge, and then translating the knowledge into that representation [13]. This enables the system to reason about the world and make informed decisions based on that knowledge. For the purposes of this paper, we will use *propositional logic* as our knowledge representation scheme of choice, but we acknowledge that there are alternative schemes that could be used. Propositional logic is favourable because of its simplicity and wide use in the field of artificial intelligence.

Propositional logic is a branch of mathematical logic that studies the logical relationships between propositions, which are expressions that can be either true or false, but not both, such as $2 + 2 = 4$. Propositional logic provides a set of rules and symbols for representing, combining, and manipulating propositions to derive new conclusions based on the truth values of the original propositions [1]. Classical propositional logic, however, does not allow for the notion of exceptions. Take, for instance, the statements "Birds can fly" and "Penguins are birds". It is perfectly reasonable, from the rules of inference in propositional logic, to conclude that "Penguins can fly", though we know this is not true. *Defeasible logic and reasoning* allows us to handle exceptions to statements, such as penguins being an exception to the statement "Birds can fly".

Defeasible reasoning is a type of reasoning that allows for the handling of incomplete and uncertain information. It involves making assumptions about the truth of certain propositions and revising these assumptions as new information becomes available. It extends classical propositional logic by allowing us to reason *non-monotonically* [14]. In classical propositional logic, the addition of new information does not affect the truth of previously established conclusions. However, in many real-world situations, new information can invalidate or modify previously accepted conclusions, as we saw before. Defeasible reasoning allows us to retract or revise conclusions in the presence of *defeaters* such as "Penguins cannot fly".

The most prominent approach to formalising defeasible reasoning is the KLM framework, developed by Kraus, Lehmann, and Magidor [10]. It is based on a *preferential* approach to defeasible reasoning, whereby statements are ranked according to their relative strength or priority. This then allows one to, given a set of statements, make conclusions based on those statements, some of which may be defeasible.

The method of forming these conclusions is called *entailment*, the defeasible form of which is called *defeasible entailment*. The KLM framework was extended to include a form of defeasible entailment, known as *rational closure* [12]. Rational closure is the most conservative form of defeasible entailment, and thus forms the core of defeasible entailment relations [9]. This conservative nature lends itself to being easily extended, as in the case of *lexicographic closure* [11].

This literature review aims to explore the various extensions proposed in the literature for defeasible reasoning beyond rational closure. In particular, it will focus on lexicographic closure. The findings of this literature review can guide the design and implementation of algorithms for a well-defined class of defeasible reasoning approaches that extend rational closure.

2 PROPOSITIONAL LOGIC

Propositional logic is the building block of many forms of logic [1]. It allows us to assign true or false values to propositions, and subsequently use these *atomic propositions* to form more complex statements, or *formulas*. These formulas are produced from atomic propositions by using logical connectives called *Boolean operators*. For instance, "2 + 2 = 4" and "The sky is blue" are both atomic propositions, also referred to as *atoms*. They can be joined to form a more complex statement, such as "2 + 2 = 4 *and* the sky is blue", whose truth can be deduced from the truth of the atoms themselves as well as the semantics of the Boolean operator between them. This sections aims to introduce the syntax and semantics of propositional logic, and its applicability to defeasible reasoning.

2.1 Syntax

Logical formulas constitute two components: atoms and Boolean operators.

2.1.1 Atoms. Atoms are understood to be single propositions, and are denoted by lowercase letters. That is, the set of all propositional atoms, $\mathcal{P}$, consists of the elements $a, b, c, \ldots$ so that $\mathcal{P} = \{a, b, c, \ldots\}$.

2.1.2 Boolean operators. Boolean operators are the logical connectives between atoms and they are given by the set $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$. Note that they are all binary operators, save $\neg$, which is a unary operator. The names of these connectives are, respectively: negation, disjunction, conjunction, implication and equivalence (or bi-implication).

2.1.3 Formulas. These components are useless alone, however. To form more complex statements, which is our end goal in knowledge representation, they need to be used in tandem to produce formulas. The grammar we can use to produce formulas is given below [1]:

$$
\begin{aligned}
fml &::= p && \text{for some } p \in \mathcal{P} \\
fml &::= \neg fml \\
fml &::= fml \; op \; fml \\
op &::= \vee \mid \wedge \mid \rightarrow \mid \leftrightarrow
\end{aligned}
$$

From this we see that atoms themselves are formulas, and we can create ever larger formulas by joining two formulas with a Boolean operator or negating a single formula with $\neg$. We denote the set of all possible formulas by $\mathcal{L}$[9].

2.2 Object-Level Semantics

2.2.1 Valuations. A valuation $u : \mathcal{P} \rightarrow \{T, F\}$ is a function which maps an atom $p \in \mathcal{P}$ to either T or F, understood to be true or false, respectively [9]. We denote the set of all valuations by $\mathcal{U}$.

For instance, given the atoms $\mathcal{P} = \{p, q, r\}$ and a random valuation $u \in \mathcal{U}$, we could have $u(p) = T, u(q) = F$ and $u(r) = F$. It is useful to note that a shorthand notation for this valuation is $\mathrm{p}\overline{\mathrm{q}}\overline{\mathrm{r}}$, where a bar above an atom indicates it is false, and the absence of a bar indicates it is true.

2.2.2 Atoms. Atoms are the smallest possible unit of a formula. While a single atom p has no meaning in and of itself, we can use it to encode information or knowledge that we are considering. For instance, if we wanted to consider the statements "penguins fly" and "birds fly", we could use the atoms p and b to represent these statements, respectively. We could equivalently have used these atoms to represent the statements "penguins exist" and "birds exist". However, in the first case p is false while in the second case p is true. This example aims to show that an atom has no intrinsic meaning - its usefulness is in our understanding of what the atom represents.

2.2.3 Boolean operators. Boolean operators, in contrast, do have meaning. They join two formulas whose truth values are already known to form a new formula. The truth value of the new formula is determined by the Boolean operators and the truth values of the original formulas.

Given two atoms, $p, q \in \mathcal{P}$, we can enumerate the possible valuations $u \in \mathcal{U}$ by using a *truth table*, where each row represents a different valuation. Furthermore, we can use the truth table to define the semantics of the binary Boolean operators. The truth table for the aforementioned Boolean operators is given below [1]:

p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

Figure 1: The truth table for binary Boolean operators

Finally, recalling that $\neg$ is a unary operator, we have that if p is T, then $\neg p$ is F. Conversely, if p is F, then $\neg p$ is T.

It is important to note that the semantics of the Boolean operators remain consistent when operating on two formulas, as they do when operating on two atoms. For instance, consider the formula $A := (p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ and the valuation $u := \mathrm{p}\overline{\mathrm{q}}$. Using the truth table above, we see that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are both F. Therefore, we have that A is T under u.

2.3 Meta-Level Semantics

2.3.1 Satisfiability. If an atom p is assigned T by a valuation u, it is said that the valuation *satisfies* the atom, written $u \Vdash p$. Conversely, if it is assigned F, we write $u \nVdash p$ [1].

We can extend this definition to include satisfaction of formulas. We say that a valuation $u \in \mathcal{U}$ satisfies a formula $A \in \mathcal{L}$ if and only if, according to the truth values assigned to the atoms of A by u, A has a truth value of T [1]. In our example from earlier, then, we can say that u satisfies A, written $u \Vdash A$, since the valuation $u := \mathrm{p}\overline{\mathrm{q}}$ causes A to evaluate to T.

We can extend this definition even further by considering sets of formulas, known as *knowledge bases.* We say that a valuation $u \in \mathcal{U}$ satisfies a knowledge base $\mathcal{K}$, written $u \Vdash \mathcal{K}$, if and only if it satisfies every formula $A \in \mathcal{K}$ [1]. We then call u a *model* of $\mathcal{K}$ and write $u \in Mod(\mathcal{K})$ where $Mod(\mathcal{K})$ is the set of all models of the knowledge base $\mathcal{K}$ [1]. For instance, given the valuation u and formula A from earlier, we can define $\mathcal{K} = \{A\}$. In this case, $u \Vdash \mathcal{K}$, so $u \in Mod(\mathcal{K})$. However, if $\mathcal{K} = \{A, p \wedge q\}$, then $u \nVdash \mathcal{K}$ since $u \nVdash p \wedge q$.

If there is at least one valuation which satisfies $\mathcal{K}$, we say that $\mathcal{K}$ is *satisfiable.* More formally, $\mathcal{K}$ is satisfiable if and only if $Mod(\mathcal{K}) \neq \emptyset$ [1].

2.3.2 Entailment. Our goal in using propositional logic for defeasible reasoning is to systematically reason about statements, or formulas. That is, given a knowledge base $\mathcal{K}$, we want to consider which formulas are a consequence of $\mathcal{K}$. Equivalently in terminology, we want to consider which formulas are *entailed* by $\mathcal{K}$.

We say that a formula A is entailed by a knowledge base $\mathcal{K}$, written $\mathcal{K} \models A$, if and only if $Mod(\mathcal{K}) \subseteq Mod(A)$ [1]. That is to say, whenever a valuation u satisfies $\mathcal{K}$, it satisfies A as well.

2.4 Object vs Meta Levels

It is worth noting, to avoid any confusion, the distinction between object-level concepts and meta-level concepts.

The object level consists of the formulas themselves, and the logical implications that follow from them. For example, given the formulas $p \wedge q$ and $q \rightarrow r$, we can derive the object-level consequence $p \wedge r$ using the logical connectives.

The meta level, on the other hand, is a higher level of abstraction that is concerned with the logic itself, rather than with particular formulas. At the meta level, we can reason about logical concepts such as entailment, satisfaction, and validity.

For example, the entailment symbol ($\models$) is a meta-level symbol that we can use to express the relationship between two formulas. The satisfaction symbol ($\Vdash$) is also a meta-level symbol that we can use to express the relationship between a formula and a valuation.

Therefore, the difference between object-level and meta-level concepts lies in the level at which we reason with the formulas. Object-level consequences are derived using the logical connectives and propositional variables of the object language, while meta-level consequences are expressed using meta-level symbols and involve reasoning about the language itself.

3 DEFEASIBLE REASONING

It is clear that we still require some form of reasoning which allows us to deal with exceptions. For the sake of exhibiting classical propositional logic's inability to deal with exceptions, we will consider the example from earlier. We consider a knowledge base $\mathcal{K} = \{p \rightarrow b, b \rightarrow f, p \rightarrow \neg f\}$ which contains the knowledge that penguins are birds, birds can fly and penguins cannot fly, respectively. Since $p \rightarrow b$ and $b \rightarrow f$, we conclude, according to basic logical implication, that $p \rightarrow f$. However, $p \rightarrow \neg f$ according to our knowledge base $\mathcal{K}$. Thus we must conclude that p is not true, so penguins do not exist. We know, however, that this is not the case; rather, penguins are merely exceptions to the statement $b \rightarrow f$. Defeasible reasoning allows us to reason appropriately in these cases.

3.1 The KLM Framework

In [10], KLM introduced the notion of *typicality* by defining a meta-level consequence relation, denoted by $\mid\!\sim$. For formulas $\alpha, \beta \in \mathcal{L}$, we can then state the *defeasible implication* $\alpha \mid\!\sim \beta$, read "typically, if α, then β" [3]. They proposed a set of six postulates for *preferential consequence relations*, which attempted to introduce defeasibility [9]. Furthermore, they showed that the semantics of such preferential consequence relations can be defined using *preferential interpretations* [16]. In [12], a seventh postulate was added to define a new, more refined set of consequence relations called *rational consequence relations*. Similarly to preferential consequence relations, their semantics can be defined using *ranked interpretations*, a subclass of preferential interpretations. These ranked interpretations are the concern of this paper, as they allow us to investigate only those consequence relations which are rational. [12]

3.1.1 Ranked Interpretations. A ranked interpretation [3] is a function $\mathcal{R} : \mathcal{U} \rightarrow \mathcal{N} \cup \infty$ which satisfies the property that for every $i \in \mathcal{N}$, if there exists an $u \in \mathcal{U}$ s.t. $\mathcal{R}(u) = i$ then there exists a $v \in \mathcal{U}$ s.t. $0 \leq \mathcal{R}(v) < i$. That is to say, $\mathcal{R}$ maps all the valuations of a set of atoms to a natural number or ∞; it also has a *convexity* property which has the consequence of not permitting empty ranks. The natural number that $\mathcal{R}$ assigns to a valuation is indicative of that valuations typicality - a lower number indicates that the valuation is more typical, while a higher number indicates that the valuation is less typical. A valuation which is assigned ∞ is understood to be impossible.

For example, given the atoms $\mathcal{P} := \{p, q, r\}$, one possible ranked interpretation is given below:

∞	pqr
2	$\mathrm{pq\bar{r}}$ $\mathrm{p\bar{q}r}$ $\mathrm{\bar{p}qr}$
1	$\overline{\mathrm{pqr}}$ $\mathrm{p}\overline{\mathrm{qr}}$
0	$\mathrm{\bar{p}q\bar{r}}$ $\overline{\mathrm{pq}}\mathrm{r}$

Figure 2: A possible ranked interpretation of $\mathcal{P}$

This ranked interpretation would imply that the valuation $u := \overline{\mathrm{pq}}\mathrm{r}$ is more likely than the valuation $v := \mathrm{pq\bar{r}}$, since $\mathcal{R}(u) < \mathcal{R}(v)$. We can also infer that the valuation pqr is impossible.

3.1.2 Defeasible Implication. We want to consider meta-level concepts such as satisfiability and entailment in a defeasible setting. For this, it is necessary to redefine $\mid\!\sim$ on the object level [9, 12]. We have been working with $\mid\!\sim$ as a meta-level consequence relation, which is focused on what can be inferred from the language, rather than the language itself. We redefine $\mid\!\sim$ as a logical connective in propositional logic, and consider it to be the defeasible counterpart to $\rightarrow$. A statement $\alpha \mid\!\sim \beta$ is then read "α *typically* implies β", and is called a *defeasible implication*. A set of defeasible implications is called a *defeasible knowledge base*, and is denoted (as before) by $\mathcal{K}$ [9].

3.1.3 Satisfiability. Intuitively, we would like to say that a ranked interpretation satisfies a defeasible implication if it is satisfied in the most typical valuations (also called *worlds*). That is, it is satisfied in the worlds with the lowest rank, also called the *minimal* worlds. This leads us to the following definition [9]: given a ranked interpretation $\mathcal{R}$ and any formula $\alpha \in \mathcal{L}$, $u \in [\![\alpha]\!]^{\mathcal{R}}$ is minimal if and only if there is no $v \in [\![\alpha]\!]^{\mathcal{R}}$ such that $\mathcal{R}(v) < \mathcal{R}(u)$, where $[\![\alpha]\!]^{\mathcal{R}}$ denotes the models of α in $\mathcal{R}$. For instance, given the ranked interpretation from earlier, and the formula $\alpha := p$ we have that $[\![\alpha]\!]^{\mathcal{R}} = \{\mathrm{p}\overline{\mathrm{qr}}, \mathrm{pq\bar{r}}, \mathrm{p\bar{q}r}\}$, since these valuations all evaluate p as true. The minimal world in $\mathcal{R}$ which satisfies the formula α is then $\mathrm{p}\overline{\mathrm{qr}}$, since it has the lowest rank (it's rank is 1, while the rank of the other two is 2).

Given this definition of minimal worlds, we have that, for a given ranked interpretation $\mathcal{R}$ and defeasible implication $\alpha \mid\!\sim \beta$, $\mathcal{R}$ satisfies $\alpha \mid\!\sim \beta$, written $\mathcal{R} \Vdash \alpha \mid\!\sim \beta$, if and only if for every u minimal in $[\![\alpha]\!]^{\mathcal{R}}$, $u \Vdash \beta$. In this case, $\mathcal{R}$ is called a model of $\alpha \mid\!\sim \beta$ [9]. In other words, for $\mathcal{R}$ to satisfy $\alpha \mid\!\sim \beta$, all the minimal worlds which satisfy α (also called α worlds) need to satisfy β as well. Drawing on our example from before (where $\alpha := p$), we have that $\mathcal{R} \Vdash p \mid\!\sim \neg q$, since the minimal α world, $\mathrm{p}\overline{\mathrm{qr}}$, satisfies $\neg q$.

We can extend this definition to knowledge bases. A defeasible knowledge base $\mathcal{K}$ is satisfied by a ranked interpretation $\mathcal{R}$, written

$\mathcal{R} \Vdash \mathcal{K}$, if and only if every formula in $\mathcal{K}$ is satisfied by $\mathcal{R}$. In this case, $\mathcal{R}$ is called a model of $\mathcal{K}$ [9].

A defeasible knowledge base may contain classical propositional formulas as well as defeasible implications. Since we already have a definition for satisfaction of defeasible implications, we aim to represent these classical formulas as defeasible implications. Clearly, for a ranked interpretation to satisfy a classical formula, the formula ought to be satisfied in all finitely-ranked worlds, since classical formulas do not allow for typicality [9]. This intuition allows us to represent classical formulas as defeasible implications (thus extending propositional logic [2]) in the following way: Given a classical formula $\alpha \in \mathcal{L}$, α can be expressed as a defeasible implication $\neg\alpha \mid\sim \bot$ ($\bot$ is a statement that is always false). Thus, for any $\alpha \in \mathcal{L}$, $\mathcal{R} \Vdash \alpha$ if and only if $\mathcal{R} \Vdash \neg\alpha \mid\sim \bot$. For $\mathcal{R}$ to satisfy $\neg\alpha \mid\sim \bot$, we require (by definition) that in the minimal worlds where $\neg\alpha$ is satisfied by $\mathcal{R}$, $\bot$ must also be satisfied. However, $\bot$ is always false, so it is never satisfied. That is to say, there are no minimal worlds where α is false, so α must be true in $\mathcal{R}$.

3.1.4 Entailment. It was mentioned earlier that KLM [10] proposed six postulates (later increased to seven by Lehmann and Magidor [12]) which any reasonable meta-level consequence relation, $\mid\sim$, should conform to. These postulates were later reformulated [3] to characterise *defeasible entailment*, denoted by $\approx$, on the meta level:

(LLE) $\dfrac{\mathcal{K} \models\!\approx \alpha \leftrightarrow \beta, \mathcal{K} \models\!\approx \alpha \mid\sim \gamma}{\mathcal{K} \models\!\approx \beta \mid\sim \gamma}$ **(Or)** $\dfrac{\mathcal{K} \models\!\approx \alpha \mid\sim \gamma, \mathcal{K} \models\!\approx \beta \mid\sim \gamma}{\mathcal{K} \models\!\approx \alpha \lor \beta \mid\sim \gamma}$

(RW) $\dfrac{\mathcal{K} \models\!\approx \alpha \to \beta, \mathcal{K} \models\!\approx \gamma \mid\sim \alpha}{\mathcal{K} \models\!\approx \gamma \mid\sim \beta}$ **(CM)** $\dfrac{\mathcal{K} \models\!\approx \alpha \mid\sim \gamma, \mathcal{K} \models\!\approx \alpha \mid\sim \beta}{\mathcal{K} \models\!\approx \alpha \land \beta \mid\sim \gamma}$

(Ref) $\mathcal{K} \models\!\approx \alpha \mid\sim \alpha$ **(RM)** $\dfrac{\mathcal{K} \models\!\approx \alpha \mid\sim \gamma, \mathcal{K} \not\models\!\approx \alpha \not\mid\sim \beta}{\mathcal{K} \models\!\approx \alpha \land \beta \mid\sim \gamma}$

(And $\dfrac{\mathcal{K} \models\!\approx \alpha \mid\sim \beta, \mathcal{K} \models\!\approx \alpha \mid\sim \gamma}{\mathcal{K} \models\!\approx \alpha \mid\sim \beta \land \gamma}$

If it is the case that a defeasible entailment relation $\models\!\approx$ satisfies these postulates, we refer to $\models\!\approx$ as being *LM-rational*.

KLM [10] defined *preferential entailment* to be an entailment relation conforming to all seven postulates, save RM. They showed that preferential entailment can be defined using preferential interpretations, while Lehmann and Magidor [12] showed that *ranked entailment* can be defined using ranked interpretations. These two entailment relations were shown to be equivalent [12] and not LM-rational [3], since they do not conform to RM.

More importantly, perhaps, is that they are monotonic. While this is unfortunate, ranked entailment can nevertheless be seen as the monotonic core [2] of what is to follow - *rational defeasible entailment.*

3.2 Extending the KLM Framework

3.2.1 Ranked Entailment. In order to extend defeasible entailment within the KLM framework, it is necessary to define ranked entailment [3]: A defeasible implication $\alpha \mid\sim \beta$ is said to be rank entailed by a knowledge base $\mathcal{K}$, denoted as $\mathcal{K} \models\!\approx_R \alpha \mid\sim \beta$, if every *ranked model* of $\mathcal{K}$ (a ranked interpretation which satisfies $\mathcal{K}$) satisfies $\alpha \mid\sim \beta$. As mentioned earlier, however, this form of defeasible entailment is not LM-rational [10].

3.2.2 Basic Defeasible Entailment. The KLM framework was extended in [2] to define a new class of defeasible entailment relations, called *basic defeasible entailment relations*. This class of entailment relations not only satisfies LM-rationality, but the following two properties:

(Inclusion) $\mathcal{K} \models\!\approx \alpha \mid\sim \beta$ for every $\alpha \mid\sim \beta \in \mathcal{K}$

(Classic Preservation) $\mathcal{K} \models\!\approx \alpha \mid\sim \bot$ if and only if $\mathcal{K} \models\!\approx_R \alpha \mid\sim \bot$

Inclusion states that all defeasible implications already in $\mathcal{K}$ should be defeasible entailed by $\mathcal{K}$. Classic Preservation states that all classical defeasible implications (those of the form $\alpha \to \beta$) which are defeasibly entailed by $\mathcal{K}$ should correspond to the classical defeasible implications rank entailed by $\mathcal{K}$. This demonstrates that rank entailment is indeed the monotonic core of other forms of defeasible entailment.

3.2.3 Rational Defeasible Entailment. It was claimed in [2] that basic defeasible entailment is too permissive. That is to say, it does not always entail the same defeasible implications as *rational closure* (discussed later). However, just as ranked entailment is seen as the *monotonic* core of defeasible entailment, rational closure is seen as the *nonmonotonic* core [3]. As we will see, rational closure is defined by the *minimal ranked model* of a knowledge base $\mathcal{K}$, and thus is the baseline of what should be entailed; other reasonable forms of defeasible entailment should entail *at least* what rational closure entails [9].

This leads us to the next class of defeasible entailment relations, called *rational defeasible entailment relations* [2]. This class of entailment relations are basic defeasible entailment relations which satisfy yet another property [2]:

(Rational Closure Extension) If $\mathcal{K} \models\!\approx_{RC} \alpha \mid\sim \beta$, then $\mathcal{K} \models\!\approx \alpha \mid\sim \beta$

Rational Closure Extension states that any rational entailment relation should entail, at least, all the defeasible implications entailed by rational closure (denoted $\models\!\approx_{RC}$). That is, they should extend rational closure. Two such rational defeasible entailment relations have been thoroughly researched in the literature [3–5, 11, 12]: *rational closure* and *lexicographic closure*. We will cover the core concepts introduced in the former, and focus on the specifics of the latter.

3.3 Rational Closure

Rational closure (RC) can be defined either semantically via a *minimal ranked model* or syntactically via *base ranks* [9]. As we will see, the former is concerned with typicality *between* ranked models of a knowledge base $\mathcal{K}$, while the latter is concerned with typicality *within* ranked models of $\mathcal{K}$ [2].

3.3.1 Minimal Ranked Model. We begin by defining a partial order $\preceq_{\mathcal{K}}$ over all the ranked models of a knowledge base $\mathcal{K}$ as follows [2]: given $\mathcal{R}_1, \mathcal{R}_2 \in Mod(\mathcal{K}), \mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_2$ if and only if for every $u \in \mathcal{U}$, $\mathcal{R}_1(u) \le \mathcal{R}_2(u)$. That is to say, a ranked model of $\mathcal{K}$ will be lower down in the order $\preceq_{\mathcal{K}}$ if it assigns lower rankings to all worlds (valuations). This implies that more typical ranked models will be lower down the order [2].

Interestingly, it was shown that such an ordering has a minimal element, denoted $\mathcal{R}^{RC}_{\mathcal{K}}$ [8]. This element is used in defining the rational closure of $\mathcal{K}$ as follows [9]: the minimal ranked model, $\mathcal{R}^{RC}_{\mathcal{K}}$, defines an entailment relation, $\models\!\approx_{RC}$, such that for any defeasible implication $\alpha \mid\sim \beta$, $\mathcal{K} \models\!\approx_{RC} \alpha \mid\sim \beta$ if and only if $\mathcal{R}^{RC}_{\mathcal{K}} \Vdash \alpha \mid\sim \beta$. That is to say, a defeasible implication is in the rational closure of $\mathcal{K}$ if it is satisfied by the minimal ranked model $\mathcal{R}^{RC}_{\mathcal{K}}$.

In [12], it was shown that a defeasible entailment relation is LM-rational if it is generated from a ranked interpretation. Therefore, rational closure is indeed LM-rational, since it is solely defined by $\mathcal{R}^{RC}_{\mathcal{K}}$.

As an example, let $\mathcal{K} = \{p \rightarrow b, b \mid\sim f, b \mid\sim w, p \mid\sim \neg f\}$ be a knowledge base, whose statements are that penguins are birds, birds typically fly, birds typically have wings and penguins do not typically fly. Consider the minimal ranked model $\mathcal{R}^{RC}_{\mathcal{K}}$ of $\mathcal{K}$, given below [3]:

∞	$\overline{b}fp\overline{w}$ $\overline{b}\overline{f}pw$ $\overline{b}\overline{f}p\overline{w}$ $\overline{b}fpw$
2	$bfpw$ $bfp\overline{w}$
1	$b\overline{f}\overline{p}w$ $b\overline{f}\overline{p}\overline{w}$ $b\overline{f}pw$ $b\overline{f}p\overline{w}$
0	$bf\overline{p}w$ $\overline{b}f\overline{p}w$ $\overline{b}f\overline{p}\overline{w}$ $\overline{b}\overline{f}\overline{p}w$ $\overline{b}\overline{f}\overline{p}\overline{w}$ $bf\overline{p}\overline{w}$

Figure 3: The minimal ranked model $\mathcal{R}^{RC}_{\mathcal{K}}$ of $\mathcal{K}$

Now, given the query $p \mid\sim w$ (whether penguins typically have wings), we check whether this statement is in the rational closure of $\mathcal{K}$. As defined earlier, we must check whether $\mathcal{R}^{RC}_{\mathcal{K}} \Vdash p \mid\sim w$. That is, we must check whether all the minimal p worlds in $\mathcal{R}^{RC}_{\mathcal{K}}$ (circled in the figure above) satisfy w. It is clear that this is not the case, since $b\overline{f}p\overline{w} \nVdash w$. Thus $\mathcal{K} \not\approx_{RC} p \mid\sim w$, and so rational closure does not conclude that penguins have wings.

3.3.2 *Base Ranks.*

We begin by introducing terminology necessary for understanding *base ranks*. Recalling that $\approx_R$ denotes rank entailment, we say that a formula $\alpha \in \mathcal{L}$ is *exceptional* with regards to a knowledge base $\mathcal{K}$ if $\mathcal{K} \approx_R \top \mid\sim \neg\alpha$. That is, α is exceptional if it is false in the most typical valuations in every ranked model of $\mathcal{K}$ [3], by definition of rank entailment. Equivalently, we could say that $\alpha \in \mathcal{L}$ is exceptional with regards to $\mathcal{K}$ if $\overrightarrow{\mathcal{K}} \models \neg\alpha$ where $\overrightarrow{\mathcal{K}}$ is the *materialisation* of $\mathcal{K}$ and is defined as $\overrightarrow{\mathcal{K}} := \{\alpha \rightarrow \beta \mid \alpha \mid\sim \beta \in \mathcal{K}\}$ [9].

Now let $\varepsilon(\mathcal{K}) := \{\alpha \mid\sim \beta \mid \overrightarrow{\mathcal{K}} \models \neg\alpha\}$. We can then define a non-increasing sequence of knowledge bases $\mathcal{E}^{\mathcal{K}}_0, \ldots, \mathcal{E}^{\mathcal{K}}_\infty$ as follows [3]:

$$\mathcal{E}^{\mathcal{K}}_0 := \mathcal{K}$$
$$\mathcal{E}^{\mathcal{K}}_i := \varepsilon(\mathcal{E}^{\mathcal{K}}_{i-1}) \text{ for } 0 < i < n$$
$$\mathcal{E}^{\mathcal{K}}_\infty := \mathcal{E}^{\mathcal{K}}_n$$

where n is the smallest i such that $\mathcal{E}^{\mathcal{K}}_i = \mathcal{E}^{\mathcal{K}}_{i+1}$ (note that since $\mathcal{K}$ is finite, n must exist [3]).

It is important to note that if $\alpha \in \mathcal{K}$ and α is a classical formula, then α will always be in $\mathcal{E}^{\mathcal{K}}_\infty$. Recall that a classical formula $\alpha \in \mathcal{L}$ can be expressed as the defeasible implication $\neg\alpha \mid\sim \bot$. Then, to check if α is exceptional, we have to check (by definition) if the negation of the antecedent of $\neg\alpha \mid\sim \bot$ is entailed by $\overrightarrow{\mathcal{K}}$. That is, we have to check if $\overrightarrow{\mathcal{K}} \models \alpha$, which is trivially true.

Finally, the base rank of a formula α with regards to a knowledge base $\mathcal{K}$, denoted $br_{\mathcal{K}}(\alpha)$, is defined to be the smallest integer r for which α is not exceptional with regards to the knowledge base $\mathcal{E}^{\mathcal{K}}_r$ [3]. That is to say, $br_{\mathcal{K}}(\alpha) := min\{r \mid \overrightarrow{\mathcal{E}^{\mathcal{K}}_r} \not\models \neg\alpha\}$. It is important to note that the base rank of a defeasible implication is defined to be equal to the base rank of its *antecedent*, so $br_{\mathcal{K}}(\alpha \mid\sim \beta) := br_{\mathcal{K}}(\alpha)$ [9].

We can now define rational closure in terms of base ranks [8]: $\mathcal{K} \approx_{RC} \alpha \mid\sim \beta$ if and only if $br_{\mathcal{K}}(\alpha) < br_{\mathcal{K}}(\alpha \wedge \neg\beta)$ or $br_{\mathcal{K}}(\alpha) = \infty$. Furthermore, there is a correlation between the base rank of a formula α in $\mathcal{K}$ and the rank of its models in $\mathcal{R}^{RC}_{\mathcal{K}}$ [8]: $br_{\mathcal{K}}(\alpha) = min\{i \mid \exists v \in Mod(v) \ s.t. \ \mathcal{R}^{RC}_{\mathcal{K}}(v) = i\}$. In other words, the base rank of α is equal to the minimum rank of all the models of α with regards to the minimal ranked model $\mathcal{R}^{RC}_{\mathcal{K}}$ of $\mathcal{K}$.

We can explore this in the following example. Consider our knowledge base $\mathcal{K}$ from earlier. We can compute the sequence of knowledge bases $\mathcal{E}^{\mathcal{K}}_0, \ldots, \mathcal{E}^{\mathcal{K}}_\infty$ as follows:

$$\mathcal{E}^{\mathcal{K}}_0 = \{p \rightarrow b, b \mid\sim f, b \mid\sim w, p \mid\sim \neg f\} \quad \text{(by definition)}$$
$$\mathcal{E}^{\mathcal{K}}_1 = \{p \rightarrow b, p \mid\sim \neg f\} \quad (\text{since } \overrightarrow{\mathcal{E}^{\mathcal{K}}_0} \models \neg p, p \rightarrow b \text{ but } \overrightarrow{\mathcal{E}^{\mathcal{K}}_0} \not\models \neg b)$$
$$\mathcal{E}^{\mathcal{K}}_2 = \{p \rightarrow b\} \quad (\text{since } \overrightarrow{\mathcal{E}^{\mathcal{K}}_1} \models p \rightarrow b \text{ but } \overrightarrow{\mathcal{E}^{\mathcal{K}}_1} \not\models \neg p)$$
$$\mathcal{E}^{\mathcal{K}}_\infty = \mathcal{E}^{\mathcal{K}}_2 \quad (\text{since } \overrightarrow{\mathcal{E}^{\mathcal{K}}_2} \models p \rightarrow b \text{ again})$$

Given the query $p \mid\sim w$ from earlier, we would like to compute $br_{\mathcal{K}}(p \mid\sim w)$. Since $br_{\mathcal{K}}(p \mid\sim w) = br_{\mathcal{K}}(p)$, it remains to determine the smallest integer i such that p is not exceptional in $\mathcal{E}^{\mathcal{K}}_i$. Since $\overrightarrow{\mathcal{E}^{\mathcal{K}}_0} \models \neg p$ but $\overrightarrow{\mathcal{E}^{\mathcal{K}}_1} \not\models \neg p$, $br_{\mathcal{K}}(p) = 1$. One can confirm that this is indeed the rank of the minimal model of $p \mid\sim w$ in $\mathcal{R}^{RC}_{\mathcal{K}}$ (this world is $b\overline{f}pw$, whose rank is 1 in $\mathcal{R}^{RC}_{\mathcal{K}}$).

3.3.3 *Algorithm for RC Entailment Queries.*

With the base rank of a formula defined and linked to rational closure, we can define an algorithm for computing the rational closure of a knowledge base $\mathcal{K}$. This algorithm is split into two stages: `BaseRank` and `RationalClosure`. `BaseRank` partitions a knowledge base's materialisation $\overrightarrow{\mathcal{K}}$ according to the base rank of its statements in the following way [3]: $R_i := \{\alpha \rightarrow \beta \mid \alpha \mid\sim \beta \in \mathcal{K}, br_{\mathcal{K}}(\alpha) = i\}$. These partitions are then used in `RationalClosure` which, given a query $\alpha \mid\sim \beta$, checks if it is in the rational closure of $\mathcal{K}$. The algorithm achieves this by verifying whether $\alpha \mid\sim \beta$ is exceptional in $R_0 \cup \cdots \cup R_\infty$, and subsequently removing $R_0, \ldots, R_{i-1}$ until $\alpha \mid\sim \beta$ is no longer exceptional in $R_i \cup \cdots \cup R_\infty$ or only R_∞ remains. It then checks if the materialisation of the query, $\alpha \rightarrow \beta$, is entailed by the remaining partitions $R_i, \ldots, R_\infty$, returning true if it is, and false otherwise [3]. It was shown in [7] that `RationalClosure` returns true if and only if $\mathcal{K} \approx_{RC} \alpha \mid\sim \beta$.

In our example from earlier, our partitions $R_0, \ldots, R_\infty$ would be:

R_∞	$p \rightarrow b$
R_1	$p \rightarrow \neg f$
R_0	$b \rightarrow f$ $b \rightarrow w$

Figure 4: The partitions of $\overrightarrow{\mathcal{K}}$ produced by `BaseRank`

Take the query $p \mid\sim w$ from earlier, and let us check if it is in the rational closure of $\mathcal{K}$ using the algorithm described. First,

$R_0 \cup R_1 \cup R_\infty \Vdash \neg p$, so $p \mid\sim w$ is exceptional in $R_0 \cup R_1 \cup R_\infty$. Thus, we remove R_0 and continue. Now, $R_1 \cup R_\infty \nVdash \neg p$, so $p \mid\sim w$ is not exceptional in $R_1 \cup R_\infty$. We proceed to the next step and check if $R_1 \cup R_\infty$ entails $p \mid\sim w$. However, it does not, so we conclude that it is not in the rational closure of $\mathcal{K}$. Note that this is the same conclusion we arrived at earlier, using the semantic definition of rational closure.

The conservative nature of rational closure is evident in the fact that it does not attribute any properties to atypical birds that would normally be associated with typical birds. For example, since penguins are flightless, rational closure would classify them as atypical birds and therefore would not ascribe them any characteristics that typical birds possess, such as having wings.

This example highlights an important distinction: *prototypical* vs *presumptive* reasoning. Rational closure is considered prototypical in its reasoning, whereas lexicographic closure is viewed as more presumptive in its approach [9]. Specifically, when presented with $\alpha \mid\sim \beta$, rational closure assumes that from α it is typically possible to conclude β, while lexicographic closure assumes that from α, we conclude β unless it is explicitly stated otherwise. This will become apparent when we answer the query $p \mid\sim w$ using lexicographic closure.

3.4 Lexicographic Closure

Lexicographic closure was first described in [11], and is based on the pattern of default reasoning [15]. Like rational closure, lexicographic closure can be defined via a ranked model, called the *lexicographic ranked model* [3]. We will see that two different types of lexicographic closure exist, which are generated from distinct definitions of a lexicographic ordering [6]. The first type is described in [11], while the second is described in [3]. Furthermore, we will see that lexicographic closure can also be defined in terms of a lexicographic rank function, similar to base ranks in rational closure [3].

3.4.1 Lexicographic Ranked Model in [11]. We begin by defining the *order* of a defeasible knowledge base $\mathcal{K}$ to be the maximum base rank of any formula $\alpha \in \mathcal{K}$, excluding ∞ [9]. Then, for a given knowledge base $\mathcal{K}$ of order k, we can assign tuples of size $k+1$ to each subset of $\mathcal{K}$ as follows: given $D \subseteq \mathcal{K}$, we assign D the tuple $n_D = \langle n_0, n_1, \ldots, n_k \rangle$ where $n_0 = |\{\alpha \mid\sim \beta \in D \mid br_{\mathcal{K}}(\alpha) = \infty\}|$ and $n_i = |\{\alpha \mid\sim \beta \in D \mid br_{\mathcal{K}}(\alpha) = k - i\}|$ for $i = 1, \ldots, k$. That is to say, n_0 is the number of formulas in $\mathcal{K}$ with a base rank of ∞, and n_i is the number of formulas in $\mathcal{K}$ with a base rank of $k - i$ [9].

Using these tuples, we can impose an order $\prec_S$ on the subsets of $\mathcal{K}$ by ordering them lexicographically according to their associated tuples. More specifically, for $D_1, D_2 \subseteq \mathcal{K}$, $D_1 \prec_S D_2$ if and only if $n_{D_1} < n_{D_2}$, where $<$ is the natural lexicographic order on tuples of natural numbers, e.g. $\langle 1,1,0,2 \rangle < \langle 1,1,2,1 \rangle$ [9]. Furthermore, we can impose an order $\preceq_{LC}$ on the valuations $\mathcal{U}$. Given valuations $u, v \in \mathcal{U}$ and a knowledge base $\mathcal{K}$, we say that $u \preceq_{LC} v$ if and only if $V(u) \preceq_S V(v)$ where $V(x) \subseteq \mathcal{K}$ denotes the set of defeasible implications violated (not satisfied) by the valuation x [9]. This ordering can be used to generate a ranked interpretation, denoted $\mathcal{R}^{LC}_{\mathcal{K}}$.

Finally, $\mathcal{R}^{LC}_{\mathcal{K}}$ defines an entailment relation, $\approx_{LC}$, such that for any defeasible implication $\alpha \mid\sim \beta$, $\mathcal{K} \approx_{LC} \alpha \mid\sim \beta$ if and only if $\mathcal{R}^{LC}_{\mathcal{K}} \Vdash \alpha \mid\sim \beta$. That is to say, a defeasible implication is in the lexicographic closure of $\mathcal{K}$ if it is satisfied by the lexicographic ranked model $\mathcal{R}^{LC}_{\mathcal{K}}$ [9].

3.4.2 Lexicographic Ranked Model in [3]. Given a knowledge base $\mathcal{K}$ we define a function $C^{\mathcal{K}} : \mathcal{U} \to \mathbb{N}$ such that $C^{\mathcal{K}}(v) = |\{\alpha \mid\sim \beta \in \mathcal{K} \mid v \Vdash \alpha \to \beta\}|$ [3]. That is to say, $C^{\mathcal{K}}$ maps a valuation to the number of defeasible implications whose materialisations are satisfied by that valuation [3]. We can then impose an order $\preceq_{LC}$ on the valuations $\mathcal{U}$. Given valuations $u, v \in \mathcal{U}$ and a knowledge base $\mathcal{K}$, we say that $u \preceq_{LC} v$ if and only if [3]:

- $\mathcal{R}^{RC}_{\mathcal{K}}(v) = \infty$, or
- $\mathcal{R}^{RC}_{\mathcal{K}}(u) < \mathcal{R}^{RC}_{\mathcal{K}}(v)$, or
- $\mathcal{R}^{RC}_{\mathcal{K}}(u) = \mathcal{R}^{RC}_{\mathcal{K}}(v)$ and $C^{\mathcal{K}}(u) \geq C^{\mathcal{K}}(v)$

This ordering can be used to generate a ranked interpretation, denoted $\mathcal{R}^{LC}_{\mathcal{K}}$.

Finally, $\mathcal{R}^{LC}_{\mathcal{K}}$ defines an entailment relation, $\approx_{LC}$, such that for any defeasible implication $\alpha \mid\sim \beta$, $\mathcal{K} \approx_{LC} \alpha \mid\sim \beta$ if and only if $\mathcal{R}^{LC}_{\mathcal{K}} \Vdash \alpha \mid\sim \beta$. That is to say, a defeasible implication is in the lexicographic closure of $\mathcal{K}$ if it is satisfied by the lexicographic ranked model $\mathcal{R}^{LC}_{\mathcal{K}}$ [3].

It is evident that the ordering of valuations proposed by Lehmann in [11] and Casini et al. in [3] share similarities.

Lehmann orders the valuations by considering the set of defeasible implications that each valuation violates, and determining how *serious* each set of violations is. This is achieved by considering the tuples assigned to the subsets of $\mathcal{K}$, where violations with higher base ranks are considered more serious than violations with lower base ranks. Thus a set of violations (and its corresponding valuation) will be preferred if it has a lower tuple [9].

Casini et al. refine the order imposed on the valuations by $\mathcal{R}^{RC}_{\mathcal{K}}$. The valuations within a single rank of $\mathcal{R}^{RC}_{\mathcal{K}}$ are ordered by how many defeasible implications they satisfy. A valuation with more satisfactions will be lower down in the order, and will thus be regarded as more typical [3].

Both orders rank the valuations within a rank defined by $\mathcal{R}^{RC}_{\mathcal{K}}$, thus extending rational closure [9]; however, they are not equivalent. We note that a more detailed explanation of the differences between the two definitions can be found in [6].

3.4.3 Lexicographic Rank. It was shown in [3] that since lexicographic closure can be defined by a ranked model which preserves the rankings from $\mathcal{R}^{RC}_{\mathcal{K}}$, we can find a rank function which can generate lexicographic closure. This rank function, denoted $r^{LC}_{\mathcal{K}}$, is defined as $r^{LC}_{\mathcal{K}}(\alpha) := min\{\mathcal{R}^{LC}_{\mathcal{K}}(v) \mid v \in [\![a]\!]\}$ and is called the *lexicographic rank* with regards to a knowledge base $\mathcal{K}$ [3]. We can now define lexicographic closure in terms of this rank function [3]: for any defeasible implication $\alpha \mid\sim \beta$, $\mathcal{K} \approx_{LC} \alpha \mid\sim \beta$ if and only if $r^{LC}_{\mathcal{K}}(\alpha) < r^{LC}_{\mathcal{K}}(\alpha \wedge \neg\beta)$ or $r^{LC}_{\mathcal{K}}(\alpha) = \infty$.

3.4.4 Algorithm for LC Entailment Queries. As mentioned earlier, the KLM framework was extended by Casini et al. in [2] to include two new classes of entailment relations, basic defeasible entailment and rational defeasible entailment. While defining these classes, Casini et al. define an algorithm called `DefeasibleEntailment` that takes a knowledge base $\mathcal{K}$ and a rank function r satisfying

$\mathcal{K}$*-faithfulness* [2], and computes the entailment relation generated by r [3]. That is, given $\mathcal{K}, r$ and $\alpha \mid\sim \beta$ as input, it returns true if $\mathcal{K} \mid\approx_r \alpha \mid\sim \beta$ and false otherwise, where $\mid\approx_r$ is the entailment relation generated by r [3]. For instance, given the rank function $br_{\mathcal{K}}$ (base rank) and a knowledge base $\mathcal{K}$, `DefeasibleEntailment` would compute $\mid\approx_{RC}$.

In fact, `DefeasibleEntailment` is a modified version of `RationalClosure`, where instead of calling `BaseRank` to rank the defeasible implications of $\mathcal{K}$ (more specifically, the statements of $\overrightarrow{\mathcal{K}}$), it calls a more generalised algorithm `Rank`. `Rank` accepts r and $\mathcal{K}$ as input and outputs classical formulas $R_0, \ldots, R_\infty$ [3]. These formulas are similar to the partitions that are output by `BaseRank`, but rather than R_i being a set of classical implications, R_i will be a single classical formula [3]. While the formulas $R_0, \ldots, R_\infty$ might not look like formulas in $\overrightarrow{\mathcal{K}}$, they are in fact logically equivalent. That is to say, if $R_0, \ldots, R_{n-1}, R_\infty$ are the formulas output by `Rank`, then $\{R_\infty\} \cup \bigcup_{0 \leq i < n}\{R_i\} \equiv \overrightarrow{\mathcal{K}}$ [3].

We can therefore algorithmically check lexicographic entailment by calling `DefeasibleEntailment`, passing $\mathcal{K}, r_{\mathcal{K}}^{LC}$ and $\alpha \mid\sim \beta$ as input. That is to say, `DefeasibleEntailment` will return true if and only if $\mathcal{K} \mid\approx_{LC} \alpha \mid\sim \beta$ [3].

In an attempt to compare rational closure and lexicographic closure, consider our example from earlier, with $\mathcal{K} = \{p \rightarrow b, b \mid\sim f, b \mid\sim w, p \mid\sim \neg f\}$ as our knowledge base. Then the lexicographic ranked model $\mathcal{R}_{\mathcal{K}}^{LC}$ (by the definition in [3]) is given below [3]:

∞	$\overline{b}fp\overline{w}$ $\overline{b}\overline{f}pw$ $\overline{b}\overline{f}p\overline{w}$ $\overline{b}fpw$
5	$bfp\overline{w}$
4	$bfpw$
3	$b\overline{f}p\overline{w}$ $b\overline{f}\overline{p}\overline{w}$
2	$b\overline{f}pw$ $b\overline{f}\overline{p}w$
1	$bf\overline{p}\overline{w}$
0	$bf\overline{p}w$ $\overline{b}f\overline{p}w$ $\overline{b}f\overline{p}\overline{w}$ $\overline{b}\overline{f}\overline{p}w$ $\overline{b}\overline{f}\overline{p}\overline{w}$

Figure 5: The lexicographic ranked model $\mathcal{R}_{\mathcal{K}}^{LC}$ of $\mathcal{K}$

Now, given the query $p \mid\sim w$ from earlier (whether penguins typically have wings), we check whether this statement is in the lexicographic closure of $\mathcal{K}$. As defined earlier, we must check whether $\mathcal{R}_{\mathcal{K}}^{LC} \Vdash p \mid\sim w$. That is, we must check whether all the minimal p worlds in $\mathcal{R}_{\mathcal{K}}^{LC}$ (circled in the figure above) satisfy w. It is clear that this is the case, since $b\overline{f}pw \Vdash p \mid\sim w$. Thus $\mathcal{K} \mid\approx_{LC} p \mid\sim w$, and so lexicographic closure concludes that penguins have wings, in contrast to rational closure.

It is important to note that, despite lexicographic closure answering our query "more correctly" than rational closure in the example above, there is no one correct form of defeasible entailment. Suppose, for instance, that we replaced penguins for kiwis (a flightless, wingless bird) in the example above. In that case, rational closure would answer our query "more correctly" than lexicographic closure. Rather than searching for a "correct" form of defeasible entailment, we would like to explore various types of defeasible entailment and attempt to classify them as being more or less conservative, prototypical or presumptive, etc.

4 CONCLUSIONS

We began by exploring propositional logic as a means of representing and reasoning about knowledge, but soon realized that classical propositional logic lacks the ability to handle typicality or defeasibility. Consequently, we explored the KLM framework for defeasible reasoning [10], where the KLM postulates were introduced to establish what constitutes a "reasonable" form of defeasible entailment. [10]

This framework was extended by Casini et al. [3] to include two new classes of defeasible entailment: basic defeasible entailment and rational defeasible entailment. Rational defeasible entailment is of particular interest as it has rational closure as its nonmonotonic core (rational closure is particularly important as it is the most conservative LM-rational entailment relation) [3].

This class was shown [3] to include two thoroughly researched forms of entailment, namely rational closure and lexicographic closure [3–5, 11, 12]. We saw that rational closure is prototypical and less willing to jump to conclusions, while lexicographic closure is presumptive and more willing to jump to conclusions. However, it has been shown that there are forms of rational defeasible entailment which are even more "adventurous" than lexicographic closure [3].

No other forms of rational defeasible entailment have been explored in the literature, besides rational closure and lexicographic closure. We hope to close this gap by examining various forms of rational defeasible entailment which extend rational closure. Our analysis will aim to provide insights into the characteristics, advantages, and limitations of these forms of entailment, and how they compare to rational closure and lexicographic closure.

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