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Review

Volterra-series-based nonlinear system modeling and its engineering applications: A state-of-the-art review

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A B S T R A C T

Nonlinear problems have drawn great interest and extensive attention from engineers, physicists and mathematicians and many other scientists because most real systems are inherently nonlinear in nature. To model and analyze nonlinear systems, many mathematical theories and methods have been developed, including Volterra series. In this paper, the basic definition of the Volterra series is recapitulated, together with some frequency domain concepts which are derived from the Volterra series, including the general frequency response function (GFRF), the nonlinear output frequency response function (NOFRF), output frequency response function (OFRF) and associated frequency response function (AFRF). The relationship between the Volterra series and other nonlinear system models and nonlinear problem solving methods are discussed, including the Taylor series, Wiener series, NARMAX model, Hammerstein model, Wiener model, Wiener-Hammerstein model, harmonic balance method, perturbation method and Adomian decomposition. The challenging problems and their state of arts in the series convergence study and the kernel identification study are comprehensively introduced. In addition, a detailed review is then given on the applications of Volterra series in mechanical engineering, aeroelasticity problem, control engineering, electronic and electrical engineering.

1. Introduction

Nonlinear problems are very common, and have been researched by engineers, physicists, mathematicians and many other scientists. To model and analyze nonlinear systems and solve related problems, people have carried out extensive studies, and developed a variety of mathematical theories and methods, among which the Volterra series is one of the most widely used and well-established methods. It can be traced back to the work of the Italian mathematician Vito Volterra about theory of analytic functional in 1887 [1]. Then, Norbert Wiener applied his theory of Brownian motion to investigate the integration of Volterra analytic functional and firstly used it for system analysis in 1942 [2,3]. As a general method for the design and analysis of nonlinear systems, it came into use after about 1957. Using Volterra series, many nonlinear phenomena could be explained, but it was very complicated, and could only be applied to the analysis of some relatively simple nonlinear systems. This problem restricted its application in practical engineering and the progress of the application research was very slow. This situation continued until the 1990s, then, because of the development and popularization of computer technology, the application of Volterra series has been widely ranged from aeroelastic systems, biomedical engineering, fluid dynamics, electrical engineering, to mechanical engineering, etc. Especially during the last ten years, the global scholars have published nearly one thousand SCI papers about the theory and application of Volterra series, and the number of citations is over ten thousand, which shows the powerful vigor and broad application prospects in

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Nomenclature

FRF	frequency response function
GFRF	general frequency response function
NOFRF	nonlinear output frequency response function
OFRF	output frequency response function
AFRF	associated frequency response function
ALE	associated linear equation
NARMAX	nonlinear autoregressive moving average model with exogenous inputs
HBM	harmonic balance method
$h_n(\tau_1,...,\tau_n)$	the nth Volterra kernel function
$H_n(\omega_1, \ldots, \omega_n)$	the nth orderGFRF
$G_n(\omega)$	the nth order NOFRF
$U_n(\omega)$	the Fourier transform of the system input $u(t)$ raised to nth power
$G_n[x(t)]$	the nth Wiener G functional
$\lambda_1^{j_1}\ldots\lambda_{s_n}^{js_n}$	monomials in OFRF
$\Phi^{(n:j_1\ldots j_{s_n})}_{\lambda_1\ldots\lambda_{s_n}}(\omega_1, \ldots, \omega_n)$	coefficients of monomials $\lambda_1^{j_1}\ldots\lambda_{s_n}^{js_n}$ in OFRF
K_n	the nth nonlinear gain constant in AFRF

this research field. Unfortunately, although the number of research papers is very large, hitherto the related research books are only two or three copies [4–6], and there is still no one review paper which summarizes the related research achievements and status of Volterra series. These problems restrict the further promotion of Volterra series. In order to let the beginners master Volterra series faster and better, this paper tries to concisely and comprehensively introduce it, summarize the related research achievements, and discuss its application prospects.

2. Volterra series

Volterra series is one of the earliest approaches to achieve a systematic characterization of a nonlinear system. It is a powerful mathematical tool for nonlinear system analysis. Essentially, it is an extension of the standard convolution description of linear systems to nonlinear systems. Therefore, in order to help people understand the theory better, the convolution integral and its related concepts in linear systems are taken as references.

2.1. The definition in time domain

If a system is linear and time-invariant, then the linear input-output relation of the system can be represented by the convolution integral, which is shown as follows,

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)u(\tau)d\tau \tag{1}$$

where, $u(t)$ is the input, $y(t)$ is the output, Eq. (1) can be interpreted as Duhamel integral, and the system is determined uniquely by the impulse response function $h(t)$.

Implementing the Fourier transform at both the left and right sides of Eq. (1), the linear frequency domain relational expression between the system input and output can be obtained,

$$Y(\omega) = H(\omega)U(\omega) \tag{2}$$

where $U(\omega)$, $Y(\omega)$, $H(\omega)$ are the Fourier transform of $u(t)$, $y(t)$, $h(t)$, respectively, $H(\omega)$ is also known as the frequency response function (FRF). For a linear system, $H(\omega)$ or $h(t)$ includes all the information in the system.

In contrast, for nonlinear continuous time-invariant systems with fading memory, under zero initial conditions, if the energy of input signal $u(t)$ is limited, the system response can be represented by Volterra series [5,7–9]. It is an extension of Eq. (1) for linear systems to nonlinear systems, which can be represented as,

$$\begin{cases} y(t) = y_0 + \sum_{n=1}^{\infty} y_n(t) \\ y_n(t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)d\tau_1 \cdots d\tau_n \end{cases} \tag{3}$$

where, $h_1(\tau)$, $h_2(\tau_1,\tau_2)$, ..., $h_n(\tau_1,...,\tau_n)$ are each order Volterra kernel functions, which are the extensions of the impulse response function for the linear system to the nonlinear system. In addition, generally, the equilibrium position of the system is set to be zero. This means that the DC term y_0 equals zero [10] Eq. (3) reveals that, if all the Volterra kernel functions except the first order are zero, the system degenerates into a linear system.

For the discrete nonlinear time invariant system, using Volterra series, it can be represented as [11–14],

$$y(k) = y_0 + \sum_{n=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} h_n(m_1, \cdots m_n)u(k-m_1)\cdots u(k-m_n) \tag{4}$$

where, $u(k)$, $y(k)\in R$, are the system input and output, respectively, $h_n(m_1,...,m_n)$ is the nth discrete Volterra kernel function.

A significant characteristic of the Volterra kernel function is the symmetry, which can be represented as,

$$h_n(\tau_1, \tau_2\ldots, \tau_n) = h_n(\tau_2, \tau_1\ldots, \tau_n) = \cdots = h_n(\tau_{i_1}, \tau_{i_2}\ldots, \tau_{i_n}) \qquad i_j \neq i_k$$
$$i_1, i_2, \ldots, i_n \in (1, 2, \ldots, n), j, k \in (1, 2, \ldots, n) \tag{5}$$

At present, Volterra series is widely applied to the analysis and design of the nonlinear system, including polynomial nonlinear system [15,16], piecewise linear system [17,18], bilinear system [19,20], saturation system [21,22], spatio-temporal nonlinear system [23–25], hysteresis nonlinear system [26,27], fractional order nonlinear system [28,29]. In addition, although the traditional Volterra series can only describe nonlinear time invariant systems, by replacing the time invariant kernel function with the time-varying kernel function, the Volterra series is expanded to describe nonlinear time-varying systems in recent years [30–35]. It can be represented as,

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n(t, \tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_1 \cdots d\tau_n \tag{6}$$

where, $h_n(t, \tau_1, \ldots, \tau_n)$ is the nth time-varying Volterra kernel function.

2.2. The frequency domain

It is well known that the relationship between the input and output of the linear system can be represented by the frequency response function (FRF) in frequency domain, which has greatly facilitated the analysis and design of the linear system. Unfortunately, FRF of the linear system cannot be used to analyze the nonlinear system in frequency domain. In order to overcome this problem, based on Volterra series, some concepts have been developed to carry out the analysis of nonlinear systems in frequency domain, including generalized frequency response function (GFRF), nonlinear output frequency response function (NOFRF), output frequency response function (OFRF) and associated frequency response function (AFRF).

2.2.1. Generalized frequency response function (GFRF)

The first concept is the generalized frequency response function (GFRF) proposed by George [36] from the MIT in 1959, named as nonlinear frequency response function at that time. Afterwards, Bussgang [37] extended this concept for the study of the nonlinear system subjected to multiple inputs; in mathematics, Victor and Knight [38] put forward a more rigorous formulation to the Volterra kernel function in frequency domain. GFRF is defined as the multi-dimensional Fourier transform of Volterra kernel function $h_n(\tau_1,\ldots,\tau_n)$, which can be represented as [39],

$$H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) e^{-j(\omega_1\tau_1 +, \ldots, +\omega_n\tau_n)} d\tau_1 \ldots d\tau_n \tag{7}$$

The multi-dimensional inverse Fourier transform of GFRF is the Volterra kernel function, which can be represented as,

$$h_n(\tau_1, \ldots, \tau_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H_n(\omega_1, \ldots, \omega_n) e^{+j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)} d\omega_1 \cdots d\omega_n \tag{8}$$

The Volterra kernel function is symmetrical, so its multi-dimensional Fourier transform also is symmetrical, and the GFRF is symmetrical, for example, $H_2(\omega_1, \omega_2)= H_2(\omega_2, \omega_1)$.

Using GFRF, the output frequency response of the Volterra nonlinear system to a general input can be expressed as the following form [40,41],

$$\begin{cases} Y(\omega) = \sum_{n=1}^{\infty} Y_n(\omega) \\ Y_n(\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1 + \cdots + \omega_n = \omega} H_n(\omega_1, \ldots, \omega_n) \prod_{i=1}^{n} U(\omega_i) d\sigma_{n\omega} \end{cases} \tag{9}$$

where, $Y(\omega)$ is the spectra of the system output, $U(\omega)$ is the spectra of the system input, $Y_n(\omega)$ represents the nth order output frequency response of the system, $\sigma_{n\omega}$ represents the whole integral field satisfying the constraint $\omega_1+\ldots+\omega_n=\omega$ and $d\sigma_{n\omega}$ denotes an infinitely small element within this integral field.

For linear systems, using Laplace transform, the frequency response function can be directly deduced from the differential dynamic equation. In order to derive the GFRF of nonlinear systems, Bedrosian and Rice [42] proposed the harmonic probing method, but this method is only suitable for continuous nonlinear systems subjected to a single input. Worden [10] extended this method for the analysis of nonlinear systems with multiple inputs and multiple outputs, and the GFRF can be used to describe the coupling between frequencies of multiple inputs. Billings and Tsang [43,44] extended this method for the analysis of nonlinear discrete systems. Peyton Jones and Billings [45] further developed the harmonic probing method and put forward an effective method which could recursively determine the GFRFs from low to high orders. Later, Billings and his colleagues [46] extended this recursive method to the case of multiple inputs. The algorithm can be easily implemented by a computer using the symbolic operation method. To a certain extent, this algorithm can facilitate the application of the GFRFs. Moreover, Billings and his colleagues developed special software packages [47].

2.2.2. Nonlinear output frequency response function (NOFRF)

A significant property of the GFRF is that it is multidimensional. The dimension of each order GFRF equals its order. GFRF is much more complicated than the linear FRF and is often difficult to measure, display, and interpret in practice. In order to overcome this problem, Lang and Billings [41,48] proposed another concept named as nonlinear output frequency response function (NOFRF). The concept can be considered to be an alternative extension of the classical FRF for the linear system to the nonlinear case. NOFRF is a one-dimensional function of frequency, which allows the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems, and provides a great insight into the mechanisms which dominate many nonlinear behaviors. NOFRF is defined as,

$$G_n(\omega) = \frac{\int_{\omega_1+\cdots+\omega_n=\omega} H_n(\omega_1, \ldots, \omega_n) \prod_{i=1}^{n} U(\omega_i) d\sigma_{n\omega}}{\int_{\omega_1+\cdots+\omega_n=\omega} \prod_{i=1}^{n} U(\omega_i) d\sigma_{n\omega}} \tag{10}$$

under the condition

$$U_n(\omega) = \int_{\omega_1+\cdots+\omega_n=\omega} \prod_{i=1}^{n} U(\omega_i) d\sigma_{n\omega} \neq 0 \tag{11}$$

by introducing the NOFRF concept, the output frequency response of nonlinear systems can be written as,

$$Y(\omega) = \sum_{n=1}^{N} Y_n(\omega) = \sum_{n=1}^{N} G_n(\omega) U_n(\omega) \tag{12}$$

where, $U_n(\omega)$ is defined as Eq. (11). Eq. (12) is similar to the description of the output frequency response for the linear system. Figs. 1 and 2 illustrate how the NOFRF concept can be used to describe the output spectra of the linear and nonlinear systems, respectively, which clearly show the advantage of the NOFRF' one-dimensional nature in describing the system output frequency responses.

According to the definition of NOFRF, it can be seen that the NOFRF is not only related to the nonlinear characteristics of the system, but also related to the system input. It reflects a combined contribution of the system and the input to the system output frequency response behavior. The most important property of the NOFRF is that it is a one-dimensional function, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Peng and Lang [16] derived the analytical expression between NOFRF and GFRF of the nonlinear system subjected to the harmonic input, and further analyzed and explained the appearance of super-harmonics and sub-resonance for a wide class of nonlinear systems, and the effects of linear damping on the resonances and resonant frequencies are analyzed; they further extended this concept to the nonlinear system with multiple inputs and multiple outputs [49]. In addition, in order to estimate each order NOFRF, Lang and his colleagues [41] proposed a multilevel excitation method required to excite the system under study many times by inputs with the same form but different intensities. By using this method, the estimation of the NOFRF can be implemented directly using the system input-output data.

2.2.3. Output frequency response function (OFRF)

Although GFRF and NOFRF can be used to analyze the nonlinear system in frequency domain, neither the GFRF nor the NOFRF can provide a clear explicit relationship for the system output spectrum and the nonlinear characteristics parameters. If the explicit relationship between them can be established, it is no doubt that it will facilitate the analysis of the effect of nonlinear parameters on the system output frequency response. To overcome this problem, Lang and his colleagues [50] recently had developed another new concept- output frequency response function (OFRF). Using OFRF, the system output spectrum and GFRF can be expressed as a polynomial function of the nonlinear characteristics parameters for polynomial nonlinear systems.

Denote C as the set of the nonlinear parameters of the nonlinear system, Lang and his colleagues [50] found that any order GFRF can be expressed as a polynomial function of the elements in the set C. For example, the nth GFRF can be represented as,

$$H_n(\omega_1, \ldots, \omega_n) = \sum_{(j_1,\ldots,j_{s_n})\in J_n} \Theta_{\lambda_1\ldots\lambda_{s_n}}^{(n:j_1\ldots j_{s_n})}(\omega_1, \ldots, \omega_n)\lambda_1^{j_1}\ldots\lambda_{s_n}^{j s_n} \tag{13}$$

where, $\lambda_1, \ldots, \lambda_{s_n}$ are the elements in C, J_n is a set of s_n dimensional nonnegative integer vectors which contains the exponents of those monomials $\lambda_1^{j_1}\cdots\lambda_{s_n}^{j_{s_n}}$, as presented in the polynomial representation(13). $\Theta_{\lambda_1\ldots\lambda_{s_n}}^{(n:j_1\ldots j_{s_n})}(\omega_1, \ldots, \omega_n)$ are the coefficients of those monomials $\lambda_1^{j_1}\cdots\lambda_{s_n}^{j_{s_n}}$, among which n represents the order of GFRF, and the superscripts and subscriptis are corresponding to those monomials. $\Theta_{\lambda_1\ldots\lambda_{s_n}}^{(n:j_1\ldots j_{s_n})}(\omega_1, \ldots, \omega_n)$ are only related to the linear characteristics parameters and the frequency $\omega_1,\ldots,\omega_n$, and not related to the nonlinear characteristics parameters. Eq.(13) gives the explicit relationship between GFRF and the nonlinear characteristics parameters, which provides an important basis for the analytical study and design of a wide class of nonlinear systems in the frequency domain.

Moreover, substituting Eq. (13) into Eq. (9), output frequency response $Y(\omega)$ can be represented as,

$$Y(\omega) = H_1(\omega)U(\omega) + \sum_{n=2}^{N} \sum_{(j_1,\ldots,j_{s_n})\in J_n} \Phi_{\lambda_1\ldots\lambda_{s_n}}^{(n:j_1\ldots j_{s_n})}(\omega_1, \ldots, \omega_n)\lambda_1^{j_1}\ldots\lambda_{s_n}^{j s_n} \tag{14}$$

where,

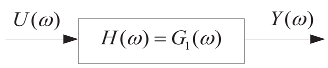

Fig. 1. The NOFRF based representation for the output frequency response of linear systems.

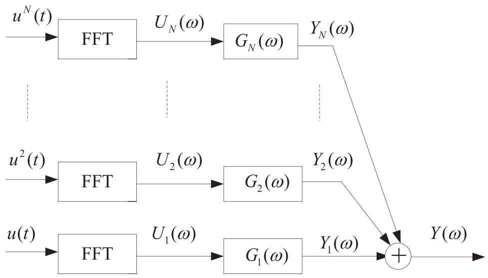

Fig. 2. The NOFRF based representation for the output frequency response of nonlinear systems.

$$\Phi^{(n:j_1\ldots j_{s_n})}_{\lambda_1\ldots\lambda_{s_n}}(\omega_1,\,\ldots,\,\omega_n)=\frac{1/\sqrt{n}}{(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\Theta^{(n:j_1\ldots j_{s_n})}_{\lambda_1\ldots\lambda_{s_n}}(\omega_1,\,\ldots,\,\omega_n)\prod_{i=1}^{n}U(\omega_i)d\sigma_{n\omega} \tag{15}$$

Eq. (14) reveals that, the output frequency response can be expressed as a polynomial function of nonlinear characteristics parameters, among which the coefficient $\Phi^{(n:j_1\ldots j_{s_n})}_{\lambda_1\ldots\lambda_{s_n}}(\omega_1,\,\ldots,\,\omega_n)$ is only related to the linear characteristics parameters and the frequency ω_1,..., ω_n, and not related to the nonlinear characteristics parameters. Lang named it as output frequency response function (OFRF).

By introducing OFRF, the relationship between system output frequency response and nonlinear characteristics parameters can be explicitly expressed. But to clearly know how the nonlinear characteristics parameters affect system output frequency response using Eq. (14), the priori information of the monomials $\lambda_1^{j_1}\ldots\lambda_{s_n}^{j_{s_n}}$and their coefficients $\Phi^{(n:j_1\ldots j_{s_n})}_{\lambda_1\ldots\lambda_{s_n}}(\omega_1,\,\ldots,\,\omega_n)$ are required, thus they have to be firstly determined. To solve this problem and ensure that the new expression of the output frequency response can be used practically to conduct the nonlinear system analysis and design, Jing [15] proposed a monomial extraction operator which can extract the concerned parameters involved in a separable parameterized function series; Peng and Lang [51] also proposed a recursive algorithm to determine the form and number of monomials. The algorithm can be easily implemented by a computer using the simple symbolic operation. Assuming that all the form and number of monomials are available in the polynomial form OFRF(14), Lang [50] proposed an algorithm to evaluate the values of OFRF by the least square method.

2.2.4. Associated frequency response function (AFRF)

The multidimensional property of GFRF makes it difficult to measure, display, and interpret in practice. In order to overcome this problem, besides NOFRF proposed by Lang, Feijoo and Worden [52,53] proposed the concept-associated frequency response function (AFRF) which is derived from the associated linear equation (ALE) of the nonlinear system. For systems with a nonlinearity in the output (the generalized Duffing oscillator) and/ or systems with a nonlinearity in the input (Hammerstein systems), their research results revealed that the nth Volterra output could be derived by taking the combination function of the lower order Volterra outputs as the inputs of the associated linear system. This relationship can be represented as,

$$Ly_n=a_nf(y_1,\,\ldots,\,y_{n-1})(n=1,\,\ldots,\,\infty) \tag{16}$$

Eq. (16) defines the associated linear equation(ALE), and the response of each order ALE can be presented as,

$$y_n(t)=a_n\int_{-\infty}^{\infty}h_1(t-\tau)f(y_1(\tau),\,\ldots,\,y_{n-1}(\tau))d\tau \tag{17}$$

where, L is a linear differential operator, and it is independent of n, h_1 is the Green's function of operator L, and a_n is the coefficient of the nth ALE.

For the second order differential operator,

$$L=m\frac{d^2}{dt^2}+c\frac{d}{dt}+k \tag{18}$$

based on each order ALE, their associated AFRF can be obtained,

$$H_{1n}(\Omega_n)=\frac{a_n}{-m\Omega_n^2+ic\Omega_n+k}\quad(n=1,\,\ldots,\,\infty) \tag{19}$$

From Eq. (19), it can be seen that each order AFRF own the same functional form, but the amplitudes of each order AFRF are associated with the coefficient a_n of each order ALE. Therefore, Feijoo and Worden [52,53] defined a nonlinear gain constant K_n,

$$K_n=\frac{a_n}{a_1} \tag{20}$$

According to Eqs. (19) and (20), any order AFRF can be determined by the first order AFRF and the nth nonlinear gain constant K_n, and the response also can be derived through them. Comparing with GFRF, each order AFRF are linear frequency response functions, and based on AFRF, nonlinear systems can be analyzed in a convenient manner similar to the analysis of linear systems. The property of AFRF greatly facilitates the analysis and design of nonlinear systems in frequency domain. In addition, the AFRF contains the same information as the GFRF, and through the AFRF, the corresponding GFRF of the original system can be obtained straightforwardly. Based on AFRF, Feijoo and Worden [54] proposed a novel method to identify the linear and nonlinear coefficients of a single-degree-of-freedom nonlinear system. Furthermore, Feijoo and Worden [55] extended this method for the analysis, identification and control of the multi- degree-of-freedom nonlinear system.

2.3. The relationships between Volterra series and other nonlinear models

To describe the nonlinear system, a variety of nonlinear models have been put forward, including Taylor series [56], Wiener series [57], Volterra series [5], NARMAX model [58], Wiener model [59,60], Hammerstein model [61] and Wiener-Hammerstein model [62–64], etc. Research results revealed that there are close relationships between Volterra series and many other nonlinear models.

2.3.1. Taylor series

Taylor series [56] is the most typical model which is used to approximately describe the nonlinear relationship between two variables. Assume that the function relationship between variable y and u can be described as function $y= f(u)$, if the function f at point u_0 is infinitely differentiable, then in the vicinity of u_0, variable y can be expressed as the power series of the variable u. Specifically, when u_0=0, the series is also called a Maclaurin series which can be represented as,

$$\begin{aligned} y &= f(0) + f'(0)u + \frac{f''(0)}{2!}u^2 + \cdots + \frac{f^{(n)}(0)}{n!}u^n + \cdots \\ &= \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}u^i \end{aligned} \tag{21}$$

According to the definition of Volterra series, it can be seen that if the Volterra kernel function is multidimensional Dirac function which can be expressed as,

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) = a_n\delta(\tau_1)\delta(\tau_2)\cdots\delta(\tau_n) \tag{22}$$

where, $\delta(.)$ is delta function, then Eq. (3) can be written as,

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} a_n\delta(\tau_1)\cdots\delta(\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_1\cdots d\tau_n \\ &= \sum_{n=1}^{\infty} a_n u^n(t) \end{aligned} \tag{23}$$

In this case, the Volterra series degenerates into the ordinary series which is similar to Taylor series. The delay τ_i among Volterra kernel function describe the effect of the past and current input on the current response. If Volterra kernel function is chosen as Eq. (22), it means that only when all the time delay τ_i equal zero, Volterra kernel function $h_i(.)$ may not be equal to zero; otherwise, all Volterra kernel functions are equal to zero. It implies that the past input can't affect the current response. Therefore, Volterra series is the extension of Taylor series, which can be regarded accordingly as a Taylor series with memory [7]. It takes the dynamic characteristic of the system into account, thus it can describe the nonlinear dynamic system, whereas Taylor series can only represent the nonlinear static system that instantaneously map the input to the output, so the application field of Volterra series is wider than Taylor series.

2.3.2. Wiener series

Although Volterra series are widely used in the analysis and design of nonlinear systems, in some cases it also has some disadvantages. For example, the convergence region of Volterra series is very strict, and the identification of Volterra kernel function is difficult. It is well-known that, to avoid the convergence problem of power series, the orthogonal functions are used to describe nonlinear systems; similarly, in order to avoid the convergence problem of Volterra series, Wiener [57] proposed Wiener series or Wiener G functional in 1958. In mathematics, Wiener series essentially is the orthogonal expansion of functional series for nonlinear time-invariant systems, and it has a close relationship with Volterra series. Both of them are the functional series expansions for nonlinear systems, and the truncation forms of them can be represented by each other. Given that the Wiener series representation is known, then each order Volterra series can be derived by adding each order Wiener series; otherwise, given that the Volterra series representation is known, then each order Wiener series can be obtained by applying a Gram-Schmid orthogonalization procedure. In addition, if the system input is Gaussian white noise with mean zero, then the Volterra system can be represented by the equivalent inhomogeneous Wiener series [65–67],

$$\begin{cases} y(t) = G_0 + \sum_{n=1}^{\infty} G_n[x(t)] \\ G_n[x(t)] = G_{0,n} + G_{1,n}[x(t)] + \cdots + G_{n,n}[x(t)] \end{cases} \tag{24}$$

where, $x(t)$ is the input, G_0 is the zero order G functional, $G_n[x(t)]$ is the nth G functional which is constituted by the nth order inhomogeneous Volterra series, and $G_{r,n}(\tau_1,\ldots, \tau_r)(r \le n)$ can be represented as,

$$G_{r,n}[x(t)] = \int_0^{\infty}\cdots\int_0^{\infty} G_{r,n}(\tau_1, \ldots, \tau_r)x(t-\tau_1)\cdots x(t-\tau_r)d\tau_1\cdots d\tau_r \tag{25}$$

where, $G_{r,n}(\tau_1,\ldots, \tau_r)(r \le n)$ is the Wiener kernel function.

Each order G functional $G_n[x(t)]$ are mutually orthogonal, which can be expressed as,

$$E(G_n[x(t)]G_m[x(t)]) = 0 \quad n \neq m \tag{26}$$

In addition, the research results of Rugh [5] indicated that the nth order G functional can be expressed by the first n order Volterra series,

$$G_n[x(t)] = \sum_{i=0}^{[n/2]} \frac{(-1)^i n!\sigma_x^{2i}}{2^i (n-2i)!i!} \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty G_{n,n}(\tau_1, \ldots, \tau_{n-2i}, \xi_1, \xi_1, \ldots, \xi_i, \xi_i)$$
$$\times x(t-\tau_1)\cdots x(t-\tau_{n-2i})d\tau_1\cdots d\tau_{n-2i}d\xi_1\cdots d\xi_i \tag{27}$$

The key issue involved in modeling nonlinear systems using the Wiener series is the identification of its kernel functions. To address this problem, based on the cross-correlation method, Lee and Schetzen [68] proposed a method which can be used to directly estimate Wiener kernel functions. The expression can be written as,

$$G_{n,n}(\tau_1, \ldots, \tau_n) = \frac{1}{n!\sigma_x^{2n}} \overline{\left(y(t) - \sum_{i=1}^{n-1} G_i[x(t)]\right) x(t-\tau_1)\cdots x(t-\tau_n)} \tag{28}$$

where, the bar indicates the average over time, σ_x^2 is the variance of input. Moreover, Palm and Pöpel [69] derived the mathematical validity of the method of Lee and Schetzen for identifying a nonlinear system. Later, they reviewed the nonlinear system identification method based on Volterra series and Wiener series, and presented the limitations of identification based on Wiener series [70]. Franz and Schölkopf [65] reviewed the relationship between Volterra series and Wiener series, and identified the Wiener kernel function of nonlinear systems based on the theory of reproducing kernel Hilbert space. Ogunfunmi and Chang [71] proposed a nonlinear LMS adaptive filter by using the nonlinear discrete Wiener series.

2.3.3. NARMAX model

Nonlinear autoregressive moving average model with exogenous inputs (NARMAX) was firstly proposed by Leontaritis and Billings [58] in 1985. There are a wide range of nonlinear systems that can be described by this model, which is defined as,

$$y(k) = F(y(k-1), \ldots, y(k-n_y), u(k-1), \ldots, u(k-n_u), e(k-1), \ldots, e(k-n_e)) + e(k) \tag{29}$$

where $u(k)$, $y(k)$ and $e(k)$ are the sampled input, output and prediction error sequences respectively; n_y, n_u and n_e are the maximal time delay of output, input and prediction error at sampled point k; $F(.)$ is an arbitrary nonlinear function, and generally is assumed to be a polynomial function.

Comparing with the discrete Volterra series defined in Eq.(4), it can be seen that Volterra series can be represented by NARMAX model when $y(k\text{-}i)$ (i=1,..., n_y) and $e(k\text{-}i)$ (i=0,..., n_e) equal zero. Therefore, NARMAX model can be regarded as an extension of the discrete Volterra series, which introduces the past output into the discrete Volterra series. By introducing the past output, NARMAX representation of nonlinear systems is more concise than Volterra series representation, namely, the number of terms among NARMAX model is less than Volterra series, that is similar to the difference between infinite impulse response filter (IIR) and finite impulse response filter (FIR).

The NARX model, which is obtained by removing all the terms associated with the noise $e(t)$ in an identified NARMAX model, is normally used for the system analysis. According to the research results of Leontaritis and Billings [72], when the system input is not equal zero and the state-space model of NARX system takes a special form that actually corresponds to the structure of a cascade of linear multivariable systems with polynomial interconnections, the NARX system can be exactly represented by a Volterra series. In this case, Billings [43,44] proposed a recursive algorithm to determine the GFRF of NARX model from the low to high order. Then, through implementing multi-dimensional inverse Fourier transform at each order GFRF, the temporal Volterra kernel function can be obtained. Therefore, the NARX model and the discrete Volterra series can be mutually represented. There is a variety of nonlinear system identification methods, among which an effective method is to firstly identify the NARX model of nonlinear systems under identification through the input/output signal, and then calculate the GFRF of nonlinear systems based on the identified NARX model.

In addition, researchers have developed a lot of methods for the identification of NARMAX model. For example, Chen and Billings [73] developed a practical algorithm for identifying NARMAX models based on radial basis functions from noise-corrupted data. Kukreja [74] developed a bootstrap structure detection (BSD) algorithm to determine the structure of NARMAX model. Pawlus [75] proposed a method to reproduce car kinematics during a collision using NARX model, among which parameters are estimated by the use of a feedforward neural network. Based on reversible jump Markov chain Monte Carlo (RJMCMC), Baldacchino [76] proposed a computational bayesian approach to estimate the parameters of polynomial NARMAX models. This new method can numerically obtain posterior distributions for both model structures and parameters via sampling methods. Worden [77] proposed a novel identification method for the NARX model based on Gaussian Process regression. Pillonetto [78] proposed a new kernel-based approach to identify the NARX model. Li [79] proposed a new approach to identify a new class of NARX models based on kernel machine and space projection.

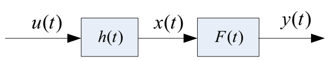

Fig. 3. Wiener model.

2.3.4. Wiener model

The Wiener model has a linear dynamic subsystem followed by a static nonlinear one, as shown in Fig. 3. The Wiener model [80,81] can be modeled by,

$$x(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \tag{30}$$

$$y(t) = F(x(t)) \tag{31}$$

where, $u(t)$ is the input to the system, and $x(t)$ is the unmeasured output of the linear dynamic subsystem, and $y(t)$ is the measured output of the static nonlinear subsystem. $h(t)$ is the impulse response function of the linear system. $F(t)$ is the static nonlinear function. According to the Weierstrass approximation theorem [82], any continuous function on a closed and bounded interval can be uniformly approximated on that interval by a polynomial function to any degree of accuracy, so the nonlinear function $F(t)$ usually is assumed to be a polynomial nonlinear function, which can be modeled by,

$$F(u(t)) = \sum_{n=1}^{N} a_n[u(t)]^n \tag{32}$$

where, a_n (n=1,...,N) are the coefficients of polynomial nonlinear function $F(t)$, and N is the maximal truncation order of polynomial nonlinear function.

For the Wiener model, according to Eqs. (30)–(32), the relationship between the input $u(t)$ and output $y(t)$ can be expressed as,

$$\begin{aligned} y(t) &= F(x(t)) = \sum_{n=1}^{N} a_n[x(t)]^n = \sum_{n=1}^{N} a_n\left[\int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau\right]^n \\ &= \sum_{n=1}^{N} y_n(t) \end{aligned} \tag{33}$$

where,

$$y_n(t) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} a_n h(\tau_1)\cdots h(\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_1\cdots d\tau_n \tag{34}$$

According to the definition of Volterra series, it can be seen that the Wiener model can be equalized to a truncated Volterra series whose kernel functions are,

$$h_n(\tau_1,\ \ldots,\ \tau_n) = a_n h(\tau_1)\cdots h(\tau_n) \tag{35}$$

Therefore, Wiener model can be represented as Volterra series. Le Caillec [83] blindly identified the second order Volterra model by using Wiener model. Based on the theory of reproducing kernel Hilbert space, Chen [84] identified the Volterra kernel function of Wiener model. Kibangou and Favier [85] presented the relationship between a parallel-cascade Wiener system and its associated Volterra model, then the coefficients of the linear and nonlinear subsystems can be obtained using third-order Volterra kernel slices.

2.3.5. Hammerstein model

The Hammerstein model has a static nonlinear subsystem followed by a linear dynamic one, which is represented in Fig. 4. The Hammerstein model [86,87] can be modeled by,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{36}$$

$$x(t) = F(u(t)) \tag{37}$$

where, the nonlinear function $F(t)$ is usually be chosen as a polynomial nonlinear function. For Hammerstein model, according to Eqs. (36), (37) and Eq. (32), the relationship between the input $u(t)$ and output $y(t)$ can be derived as,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\sum_{n=1}^{N} a_n[u(t-\tau)]^n d\tau = \sum_{n=1}^{N}\int_{-\infty}^{\infty} a_n h(\tau)[u(t-\tau)]^n d\tau \\ &= \sum_{n=1}^{N} y_n(t) \end{aligned} \tag{38}$$

where,

$$y_n(t) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} a_n h(\tau_1)\delta(\tau_1-\tau_2)\cdots\delta(\tau_1-\tau_n)/[\Delta\tau]^{n-1}u(t-\tau_1)\cdots u(t-\tau_n)d\tau_1\cdots d\tau_n \tag{39}$$

According to the definition of Volterra series, it can be seen that the Hammerstein model can be equalized to a truncated Volterra series whose kernel functions are,

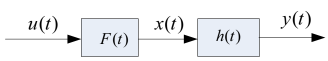

Fig. 4. Hammerstein model.

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) = a_n h(\tau_1)\delta(\tau_1 - \tau_2)\cdots\delta(\tau_1 - \tau_n)/[\Delta\tau]^{n-1}$$
$$= a_n h(\tau)/[\Delta\tau]^{n-1} \tag{40}$$

or

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) = \begin{cases} 0 & \tau_1 \neq \tau_2 \neq \cdots \neq \tau_n \\ a_n h(\tau)/[\Delta\tau]^{n-1} & \tau_1 = \tau_2 = \cdots = \tau_n \end{cases} \tag{41}$$

Therefore, Hammerstein model can also be represented as Volterra series, and only the diagonal elements of the kernel function don't equal zero.

Westwick and Kearney [88] presented the relationship between Hammerstein model and Volterra series for nonlinear time-invariant systems. Ralston and Zoubir [89] derived the relationship between them for nonlinear time-variant systems. Kibangou and Favier [90] derived the Volterra series representation of Hammerstein model, and identified the coefficients of Hammerstein model using Volterra series.

2.3.6. Wiener-Hammerstein model

Wiener-Hammerstein model [91–93] has the structure shown in Fig. 5. In such a system, a static nonlinear subsystem is preceded and followed by a linear dynamic one. Wiener-Hammerstein model can be used to describe a wider range of nonlinear systems than Wiener and Hammerstein models. Both of Wiener and Hammerstein models are the specific cases of Wiener-Hammerstein model.

The Wiener-Hammerstein can be modeled by,

$$z(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \tag{42}$$

$$x(t) = F(z(t)) \tag{43}$$

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau \tag{44}$$

According to Eqs. (32), (42)–(44), the relationship between the input $u(t)$ and output $y(t)$ can be derived as,

$$y(t) = \int_{-\infty}^{\infty} g(\tau) \sum_{n=1}^{N} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} a_n h(\tau_1)\cdots h(\tau_n) \prod_{i=1}^{n} u(t-\tau-\tau_i)d\tau_1\cdots d\tau_n d\tau \tag{45}$$

Define $\tau+\tau_i=\sigma_i$, then Eq. (45) can be rewritten as,

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left[a_n \int_{-\infty}^{\infty} g(\tau)h(\sigma_1-\tau)\cdots h(\sigma_n-\tau)d\tau\right] \prod_{i=1}^{n} u(t-\sigma_i)d\sigma_1\cdots d\sigma_n \tag{46}$$

According to the definition of Volterra series, it can be seen that the Wiener-Hammerstein model can also be equalized to a truncated Volterra series whose kernel functions are,

$$h_n(\tau_1, \ldots, \tau_n) = a_n \int_{-\infty}^{\infty} g(\tau)h(\tau_1-\tau)\cdots h(\tau_n-\tau)d\tau \tag{47}$$

Kibangou and Favier [91,92] presented the explicit relationship between Wiener- Hammerstein model and Volterra series, and showed that the estimation of the diagonal coefficients of the Volterra kernels associated with Wiener-Hammerstein model is sufficient to recover the overall model. Tan and Godfrey [94] proposed a new method to identify the linear subsystems of a Wiener-Hammerstein model through the measurement of the second-order Volterra kernel function; based on this method, Tan [95] further estimated the linear subsystems of Wiener-Hammerstein model for the bilinear system. Westwick and Kearney [96] discussed the link between some block-oriented models and their associated Volterra kernels. Westwick and Schoukens [97] developed a novel method for the initial estimates of the linear subsystems of Wiener-Hammerstein model. They proposed a new scanning technique that can efficiently evaluate each of the proposed initializations using estimates of some carefully constructed nonlinear characteristics of the system. This method resulted in a much smaller number, often only one, of potential starting points for the optimization. Paduart, Pintelon and Schoukens [98] presented a new identification method for Wiener-Hammerstein model using the polynomial nonlinear state space approach.

2.4. The relationships between Volterra series and other nonlinear solving methods

Except for some simple nonlinear equations, people have not yet developed a mature nonlinear solving algorithm to precisely solve general nonlinear differential equations as the solving algorithms for linear systems. In order to circumvent this problem,

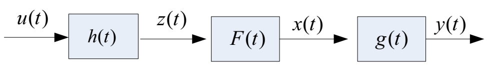

Fig. 5. Wiener-Hammerstein model.

researchers have developed a variety of nonlinear approximate solution methods, including the average method [99], KBM method [100], perturbation method [101], multiscale method [100], harmonic balance method [100] and the Adomian decomposition method [102], etc.

On the other hand, theoretically given that the dynamic differential equation of the nonlinear system is known, using Volterra series or its frequency concepts, such as GFRF and NOFRF, the system output response can be obtained. In recent years, the relationships between Volterra series and some approximate solution methods are studied.

Peng and Lang [103] investigated the relationship between NOFRF and harmonic balance method (HBM), the results showed that the harmonic components in the nonlinear system response to a sinusoidal input calculated using the nonlinear output frequency response functions (NOFRFs) are one of the solutions obtained using the harmonic balance method. In addition, studies had shown that both of these two methods had advantages and disadvantages. For example, the HBM can capture the well-known jump phenomenon, but is restricted by computational limits for some strongly nonlinear systems, and can fail to provide accurate predictions for some harmonic components. Although the NOFRFs can't capture the jump phenomenon, the method has few computational restrictions, and can accurately calculate the value of the harmonic components. Fig. 6(A) and (B) show the forced response result of the second order harmonic for a nonlinear damping oscillator using HBM and NOFRF, respectively. From Fig. 6, it can be seen that the HBM method has successfully captured the jump phenomenon, but the NOFRF expansion fails to capture the jump phenomenon. From Fig. 6(A), it can be seen that the estimated amplitudes of frequencies over $0.8\omega_0$ using HB3 method cannot match the result by the Runge-Kutta method well, and the difference between the results is mainly introduced by ignoring the higher-order harmonics. Unfortunately, when more harmonics are taken into account in the application of the HBM, the software often fails to find a solution representing the harmonic components of the responses at this frequency range. Clearly, from Fig. 6(B), the result by the NOFRF expansions matches the result by the Runge-Kutta method very well for the second order harmonic except for a few points around the maximal peak, and the reason may be that the representation of Volterra series is divergent at this frequency range.

The perturbation method is one of the classical methods for solving weak nonlinear equations, and the Volterra series is a classical mathematical tool for modeling systems with weak nonlinearity. Following different lines, both the perturbation method and Volterra series can find an approximate solution for a wide class of weakly nonlinear systems, among which the former belongs to numerical computation methods and the latter belongs to functional expansion methods. Is there any connection or similarity between the two methods? This problem has already been partly answered by a few researchers. In the study of the nonlinear Schrödinger equation, Vannucci, Serena and Bononi [104] had shown that the solutions obtained by the perturbation method and the Volterra series respectively coincide with each other. The study about the nonlinear distortion in analog integrated circuit by Buonomo and Schiavo [105] had also shown that the perturbation solution coincides with the Volterra series solution. Peng and his colleagues [106] investigated the connection between the Volterra series and the regular perturbation method in polynomial nonlinear systems analysis. It is revealed for the first time that, for a forced polynomial nonlinear system, if its associated linear system is a damped dissipative system, the steady response obtained through the regular perturbation method is exactly identical to the response given by the Volterra series. On the other hand, if the associated linear system is an undamped conservative system, then the Volterra series is incapable of modeling the forced polynomial nonlinear system. The results indicate that the Volterra series is only valid for modeling the polynomial nonlinear systems whose associated linear systems are damped dissipative systems, and is incapable of modeling nonlinear systems whose associated linear systems are undamped conservative systems. Fig. 7(A) and (B) show the steady part of the responses obtained by using the perturbation method, the Volterra series and the Runge-Kutta method for the damped and undamped associated linear system, respectively. Fig. 7(C) and (D) show the errors of responses obtained by using the perturbation method and the Runge-Kutta method, the Volterra series and the Runge-Kutta method for the damped and undamped associated linear system, respectively. It can be seen that the responses calculated by the three different methods are almost identical and no visible difference can be observed from the time waveforms of the responses for the damped dissipative

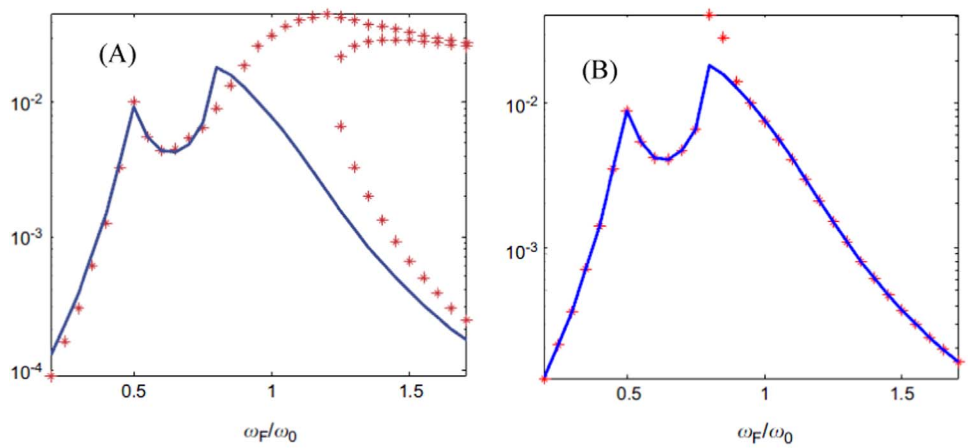

Fig. 6. Comparison between the HBM(A), NOFRF(B) and the Runge-Kutta method (star-HBM or NOFRF, solid-Runge-Kutta): the second harmonic.

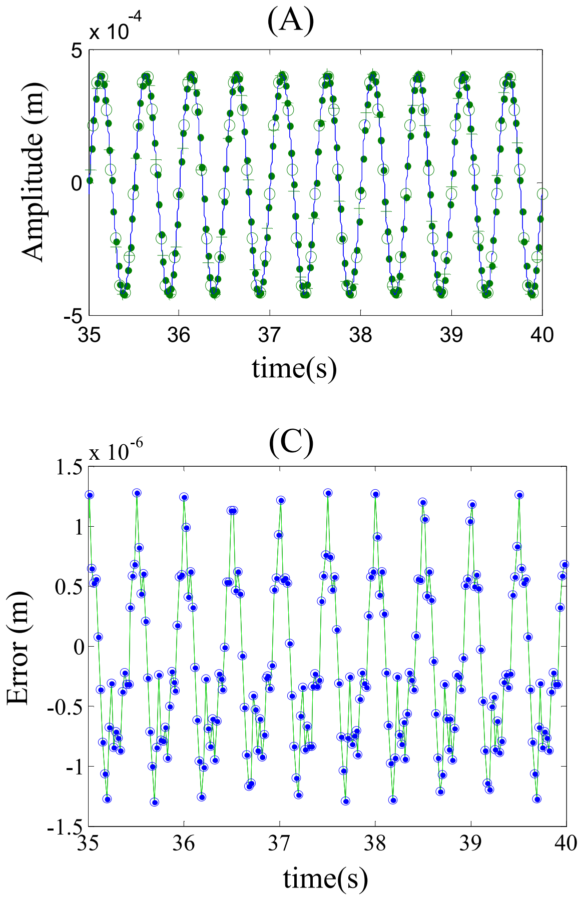

Fig. 7. (A) the steady response for the damped dissipative associated linear system; (B) the steady response for the undamped conservative associated linear system; (C) the error for the damped dissipative associated linear system; (D) the error for undamped conservative associated linear system [• – perturbation method; ○ – Volterra Series; + – Runge-Kutta method].

associated linear system, and for the undamped conservative associated linear system, both of Volterra series and perturbation method can't calculate the accurate response of the nonlinear system.

Adomian decomposition [107] is a powerful tool for solving nonlinear equations, including algebraic, differential, and integral equations, and provides a bridge between the analytical and numerical solutions. The Adomian decomposition method can be used to solve/approximate a wide class of nonlinear problems with only a few limitations, and provides an infinite series approximation in an analytical form. Guo and his colleagues [108] investigated the relationship between the Adomian decomposition and Volterra series, and it is shown that the Volterra series can be considered as a specialization of the Adomian decomposition. Based on the relationship, the Volterra series can be calculated using the Adomian decomposition method whenever a convergent Volterra series representation exists. The simulated output and the predicted one using the obtained Volterra series for a Duffing oscillator are

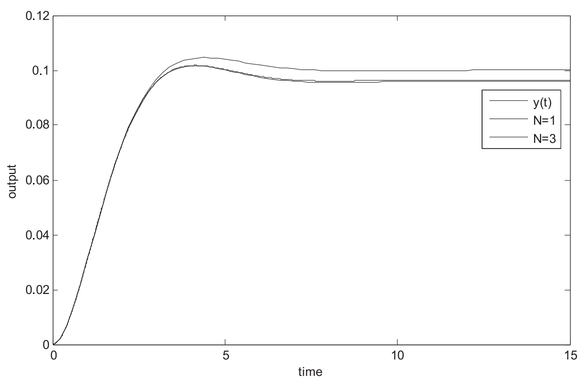

Fig. 8. The simulated output and the predicted output using the obtained Volterra series.

shown in Fig. 8. It can be seen that the predicted output using the truncated Volterra series matches the simulated output very well, even only using the first few order Volterra series.

2.5. The convergence of Volterra series

The Volterra system representation is an infinite series, and there must be associated convergence conditions to guarantee that the representation is meaningful, which is similar as Taylor series. The convergence research of Volterra series representation is important, which mainly includes two aspects, the first one is whether the Volterra series is convergence, the other is that if the Volterra series representation is convergence, how is the rule of convergence? The solution of the first problem can be used to determine whether a given nonlinear system can be expressed as Volterra series. Depending on the selected convergence criterion, a wider or a smaller class of systems can be approximated by the Volterra series [109]. If the mean square convergence of the system and model output is selected, the most general class of systems is retrieved. In this case, the discontinuities and saturation nonlinearity can be modeled by Volterra series. If choose the uniform convergence, saturation nonlinearity can be modeled, but the discontinuities nonlinearity cannot be modeled. The systems having a Volterra series expression that converge uniformly around a given working point are mainly considered in this paper. The solution of the second problem can be used to analyze the approximation accuracy. Based on the rule of convergence, the truncation order of Volterra series can be chosen in accordance with the requirement of approximation accuracy. The convergence of Volterra series is a challenging problem, and many researchers have carried out a lot of researches, but so far there is no a general method which can be used to determine the interval of convergence of Volterra series representation. The existing criterions are quite restrictive, which only can roughly determine the convergence region. The early research for convergence problem of Volterra series was done by American scholars Boyd, Chua [7,8] and Sandberg [110]. Their basic idea is to seek a convergence region of power series which is more restrictive than the convergence region of Volterra series. Suppose that for all time,

$$|u(t)| < K \tag{48}$$

and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |h_n(\tau_1, \ldots, \tau_n)| d\tau_1 \cdots d\tau_n \le a_n \tag{49}$$

then, according to the definition of Volterra series, it can be seen that the system output meets the following equation,

$$|y(t)| \le \sum_{n=1}^{\infty} a_n K^n \tag{50}$$

Obviously, if the power series on the right side of Eq. (50) is convergent, then the Volterra series is convergent. The convergence condition is quite restrictive. It is the unnecessary and sufficient condition for the convergence of Volterra series representation. This method needs to know the upper bound of each order kernel functions, and the criterion is conservative. Drawing lessons from the method, people further studied the convergence of Volterra series. For example, Sandberg [111] derived a Volterra series representation theorem which provides a bound on the error in approximating the series with its first p+1 terms, and the theorem provides a good foundation for the study of convergence of Volterra series. Czarniak and Kudrewicz [112] derived the sufficient conditions for the existence of the convergent Volterra series representation of nonlinear time-invariant networks, and proposed the calculation expression for the radius of convergence of Volterra series representation. Billings and Lang [113] derived a bound expression for the magnitude frequency domain characteristics associated with the outputs for a wide class of nonlinear systems which is a relatively simple function of the generalized frequency response functions and the system inputs, and analyzed the effect of GFRF and input on the output magnitude bound. The research results provided a good foundation to determine the bound of system input magnitude. Thouverez [114] determined the upper bound of sinusoidal input amplitude by using the harmonic balance method and Volterra series. For the Duffing's oscillator subjected to harmonic inputs, Tomlinson [115], Chatterjee [116] and Peng [117] investigated the uniform convergence issue of the Volterra series representations respectively, and proposed different criterions to determine the upper bounds of the magnitude of harmonic inputs. Tomlinson [115] and Chatterjee [116] used the concept of GFRF, and Peng [117] used the concept of NOFRF, and Peng considered the sub-resonance of each order NOFRFs, so the new criterion proposed by Peng can provide a more accurate prediction about the convergence of the Volterra series representation for the Duffing's oscillator. Li and Billings [118] investigated the uniform convergence analysis of the Volterra series model for a quadratic nonlinear system driven by harmonic inputs, and a simple criterion was proposed to provide an estimation of the upper limit of the magnitude of the harmonic inputs to ensure convergence; moreover, to accommodate the analysis of nonlinear oscillators subjected to a harmonic excitation [119], they extended the approach proposed by Barrett [120] who derived the convergence condition using the infinite norm boundary method in time domain to the frequency domain. Hélie and his colleagues [121] proposed a novel approach to compute the bound of the convergence radius, and provided a guaranteed error bound for the truncated series. Moreover, they extended the algorithm to the multi-input nonlinear system [122]. Glass and Franchek [123] presented an algorithm for computing the convergence of Volterra series representation of the sinusoidal input describing function, and the results revealed that the upper bound of the harmonic input is a function of frequency and the coefficients of the differential equation. For the NARX model, Jing [124–127] studied the bound characteristics of both GFRF and the output frequency response, and the results showed that the magnitudes of the GFRF and the system output spectrum can all be bounded by a polynomial function of the magnitude bound of the first order GFRF, and the parameters of the NARX model. These new bound characteristics of the NARX model provide an important insight into the relationship between the model parameters and the magnitudes of the system frequency response functions, revealing the effect of the model parameters on the stability of the NARX model to a certain extent, and provide a useful technique for the magnitude based analysis for nonlinear systems in the frequency domain. Later, Xiao and Jing [128,129] presented the analytical representation of the relationship between the upper bound of nonlinear output spectrum and characteristic parameters including model parameters, magnitude bound of the first order GFRF (related to linear model parameters), input magnitude, and frequency variables, and estimated the parametric bound of convergence for the NARX model by using this analytical representation.

2.6. The identification of Volterra kernel function

The key issue involved in modeling nonlinear systems using Volterra series is the identification of its kernel functions in time domain and frequency domain. The kernel functions in time domain are often called Volterra kernel functions, and the kernel functions in frequency domain are often called GFRF. For the identification of these two kinds of kernel functions, people have developed different identification methods.

2.6.1. The identification of Volterra kernel function in time domain

The identification of Volterra kernel functions is difficult as the number of terms for modeling the kernels is usually quite large. For example, we consider a system with a memory of N samples. A simple discrete Volterra model for such a system requires N^p coefficients to represent the pth order kernel function. Therefore, to represent the nonlinear system, a Volterra model requires a large number of coefficients even only the first few kernels are included. To address this problem, Schetzen [130] developed a variation of the Volterra series that is orthogonal provided that the input signal is a Gaussian white noise. A number of statistical approaches, such as the cross correlation technique, have been developed to identify Wiener kernels. A problem within the Wiener kernels identification is that it is difficult to generate a Gaussian white noise input in experimental systems, if not impossible. Glentis and Koukoulas [131] identified the Volterra kernels via input-output statistics or directly in terms of the input-output data. It is shown that the normal equations for a finite support Volterra system excited by the Gaussian input with zero mean have a unique solution if, and only if, the power spectral of the input signal is nonzero at least at m distinct frequencies, where m is the memory of the system. According to this result, they designed a suitable excitation signal, and proposed an efficient algorithm for the identification of Volterra kernel functions by introducing a multichannel embedding approach. Reed and Hawksford [132] firstly identified the Wiener series by using a modified binary maximum sequence, then obtained the Volterra kernels from the Wiener model by a change of basis. Nowak and Van Veen [133] studied input signals for the identification of nonlinear discrete-time systems modeled via a truncated Volterra series representation. It is shown that i.i.d. sequences and deterministic pseudorandom multilevel sequences (PRMLS) are the persistence of excitation conditions for a truncated Volterra series with nonlinearities of polynomial degree N, if and only if the sequences take on N+1 or more distinct levels. However, short data records are used in which case it is demonstrated that Volterra series identification is much more accurate using PRMLS inputs than Gaussian white noise inputs. Based on some characteristics of i.i.d. circularly-symmetric zero-mean complex-valued Gaussian random variables, Cheng [134] proposed a fifth-order Volterra kernel estimation algorithm which is optimal in the least mean square error sense. The proposed algorithm can be used to identify a nonlinear system under the uniformly i.i.d. rectangular M-QAM input and under the uniformly i.i.d. M-PSK input with modest modification. Mathews [135] presented a novel method for orthogonalizing correlated Gaussian input signals for the identification of truncated Volterra systems of arbitrary order of nonlinearity P and memory length N. Based on the constrained optimization, Stathaki and Scohyers [136] proposed a novel approach to estimate the parameters of Volterra model. The equations required for the determination of the Volterra kernels are formed entirely from the second and higher order statistical properties of the output signal to be modeled, and can therefore be classed as blind in nature. Chang [137] investigated the identification problem of nonlinear discrete-time systems using Volterra filter series model using improved particle swarm optimization. Toker and Emara-Shabaik [138] studied the generation of pseudo-random multilevel sequences (PRMLS) which are suitable for the identification of nonlinear systems. A closed form solution is provided for the optimal signal level selection, which is formulated as a nonlinear optimization problem to maximize the similarity to a white Gaussian noise. Cho [139] proposed a digital spectral method for evaluating the second-order distortion of a nonlinear system, which can be represented by Volterra kernels up to second order and which is subjected to a random noise input. Wray and Green [140] showed how a certain class of artificial neural networks are equivalent to Volterra series and gave the equation for the nth order Volterra kernel in terms of the internal parameters of the network. The technique was then illustrated using a specific nonlinear system. Based on an LMS variant or an errors-in-variables method, Sigrist [141] estimated the second order Volterra series from the input and the output disturbed by additive white Gaussian noise. Recently, Birpoutsoukis and Schoukens [142] extended the Bayesian- regularization based approach for the impulse response identification of a linear time invariant (LTI) system to the identification of Volterra kernels. Numerical example illustrated the enormous benefit for the identification of Volterra kernels due to the regularization.

To reduce the required number of parameters to be identified, another common approach for kernel identification is to expand the kernels in terms of a set of basis functions. In this way, the kernel function identification problem can be translated into the estimation problem of a few expansion coefficients. For example, Marmarelis [143] estimated the kernels for biological systems in terms of discrete Laguerre functions. Moodi et al. [144] used Laguerre functions and wavelet packets as orthogonal basis functions to represent Volterra kernel functions. da Rosa et al. [145] derived an analytical solution to expand the Volterra kernels with a set of Kautz functions. Asyali and Juusola [146] expanded the Volterra kernels with a set of discrete time Meixner basis functions, and identified the Volterra series representation of nonlinear systems with a slow initial onset or delay. Campello and Favier [147,148] extended the identification method of the linear system proposed by Fu and Dumont [149] who identified the linear system via the optimal expansion of the impulse response function using Laguerre functions to the Volterra system. By expanding Volterra kernel functions with optimal Laguerre functions, this identification method can minimize the number of Laguerre functions associated with a given series truncation error, thus it reduced the complexity of the resulting finite-dimensional representation, and improved the identification accuracy. Hacioğlu and Williamson [150] proposed a new identification method for Volterra series using a fixed pole expansion technique. This identification method reduced the required number of estimated parameters, thus improved the identification efficiency. Essentially, these methods seek to represent the kernels in terms of a relatively compact set of globally or locally supported basis functions. The advantage of these methods is that they only require the input with sufficient bandwidth for accurate kernels identification, but they do not require any specific excitations.

There are a variety of basis functions which can be used for the expansion of Volterra kernel functions, among which wavelet basis is one of the most widely used and well-established functions because of its several desirable properties such as orthogonality, multiscale and relatively compact support. Nikolaou [151] used biorthogonal wavelets to express the first and second order Volterra kernels, and Chou [152] derived sparse representations of Green's function operators with the orthogonal wavelet basis. Coca and Billings [153] identified NARMAX model from the data contaminated by noise based on semi-orthogonal wavelet multiresolution approximation. When wavelet basis is used to represent the Volterra kernels, the coefficients are non-varying. Therefore, the Volterra kernel functions are linear equations with the non-varying coefficients. Based on the advantages, Kurdila [154] considered the reduced order Volterra kernel representations in terms of biorthogonal wavelets. This approach utilized a family of biorthogonal wavelets proposed by Cohen et al. [155]. These wavelets possess several desirable properties such as biorthogonality, symmetry or antisymmetry, and relatively compact support. In addition, this wavelet family can be extended to include functions of arbitrary approximation order. However, these biorthogonal wavelets do not exist in closed form but are rather defined by a limiting procedure. This is a disadvantage in that the functions can be more difficult to work with, particularly when computing quadrature. In order to overcome this problem, Prazenica [156] constructed wavelets over the domain of the support of the triangular form of the second order Volterra kernel. These triangular wavelets are orthogonal, compactly supported, and symmetric or antisymmetric. However, these functions are piecewise constant, and consequently do not yield very smooth kernel estimation. In addition, the triangular wavelet construction is specific to the second order kernel. It is not straightforward to extend this approach to higher order kernels. Recently, to overcome the problem, Prazenica [157] further presented an approach in which the multiwavelets constructed from the classical finite element basis functions using the technique of intertwining are used to obtain low order estimations of the first, second, and third order Volterra kernels. These results indicate the potential of the multiwavelet based algorithm in obtaining the reduced order models for a large class of weakly nonlinear systems. As pointed by Prazenica [157], there was one problem in applying the algorithm that the first, second and third order Volterra kernel functions must be identified simultaneously. This problem reduces the identification accuracy of Volterra kernel functions. To overcome this problem, Cheng and Peng [158] proposed a wavelet basis expansion based Volterra kernel function identification through multilevel excitations. Moreover, recently, they extend this method to identify the spatio-temporal Volterra kernel functions for nonlinear distributed parameter systems [159].

2.6.2. The identification of Volterra kernel function in frequency domain

The multi-dimensional Fourier transform of Volterra kernel function is named as Volterra frequency kernel function, or generalized frequency response function (GFRF), which can be used to analyze nonlinear systems in frequency domain. The key issue involved in analyzing nonlinear systems using Volterra series in the frequency domain is the identification of its frequency kernel function. In order to identify the frequency kernel function, researchers have done extensive researches. For example, Bedrosian and Rice [42] studied how to identify GFRF of nonlinear systems under the harmonic excitation and Gaussian noise excitation. Worden [10] extended this concept for the analysis of nonlinear systems with multiple inputs and multiple outputs. Peyton Jones and Billings [45] further developed the harmonic probing method and put forward an effective method which could recursively determine the GFRFs from low to high order if a differential or difference equation is available for the nonlinear system. Later, Billings and his colleagues [46] extended this recursive method to the case of multiple inputs. The algorithm can be easily implemented by a computer using the symbolic operation method. To a certain extent, this algorithm can facilitate the application of the GFRFs. Unfortunately, the differential or difference equation of the nonlinear system must be firstly derived. In practical engineering, especially for the complex nonlinear system, it is often difficult to firstly derive the nonlinear dynamic equation of systems. In order to circumvent the difficulty in the identification method of GFRF, the data based direct identification approach for GFRF of nonlinear systems had been employed. The identification method is based on the principle of black box model, and doesn't have to understand the internal mechanisms and the physical properties of systems, only identifies the system from the input/output data, thus it is more practical. For example, Evans and Rees [160] presented a review of the identification method of Volterra frequency kernels for a nonlinear system using periodic multisine signals, after which a range of new periodic signals is defined by minimizing the signal crest factors. By introducing periodic multisine signals, Boyd and Tang [9] identified the second Volterra frequency kernel of an electro-acoustic transducer; later, Chua and Liao [161] extended this method for the identification of higher order Volterra frequency kernels for weakly nonlinear systems by designing suitable periodic multisine signals; furthermore, they [162] presented a practical algorithm for determining the highest significant order of nonlinear systems. Powers and his colleagues [163,164] identified each order Volterra frequency kernels using high order spectrum. Bicken et al. [165] identified second order Volterra frequency kernel of nonlinear systems using random multi-tone (harmonic) excitation. Pavlenko et al. [166] identified Volterra frequency kernel functions of nonlinear dynamical systems using the interpolation of excitation amplitude. Based on the assumption that frequency-domain kernels are locally smooth, Németh and Schoukens [167] presented a new method for the identification of frequency domain Volterra kernels. The kernel surface can be approximated by the interpolation technique, thus reduces the complexity of the model. Li et al. [168] proposed a new method for estimating the Volterra frequency kernels, which used the time domain measurements directly to estimate the frequency response functions. However, this method can only identify the values of GFRFs in the frequencies or the combination frequencies of excitations. Tseng [169] presented a new mixed-domain method for identifying the second order Volterra frequency kernel functions of quadratic nonlinear systems.

In addition, people also have tried to identify the nonlinear dynamic equation for continuous time nonlinear systems using GFRF. For example, Billings and Li [170,171] presented a new algorithm for identifying the nonlinear differential equation of continuous time nonlinear systems based on kernel invariance method [172,173]. The algorithm avoided the computation of derivatives by using the generalized frequency response functions to reconstruct the model. It was shown that the model could be constructed sequentially by building the linear terms firstly, then the quadratic terms and so on, and provided unbiased estimates in the presence of noise. Because the algorithm detects model structure from the lowest order to high order items, it can effectively reduce the complexity of the model, and can derive parsimonious system description. Moreover, Li et al. [171] presented a new instrumental- variable- based identification procedure to estimate linear and nonlinear continuous-time models using a shifted Chebyshev basis in the presence of noise. In addition, Guo and Billings [23,174,175] proposed a novel approach to identify the nonlinear partial differential equation or CML model (Coupled Map Lattice) for continuous spatio-temporal dynamical systems from discrete observations. The proposed approach is a combination of the implicit Adams integration and an orthogonal least-square algorithm, in which the operators are expanded using polynomials as basis functions, and the spatial derivatives are estimated by the finite difference method.

3. The applications in engineering

Volterra series is a powerful mathematical tool for nonlinear system analysis. It has been successfully used in areas such as mechanical engineering, aeroelasticityproblem, control engineering, electronic and electrical engineering etc.

3.1. Mechanical engineering

There are a lot of nonlinear factors in the practical engineering, such as the geometric nonlinearity caused by large deformation, nonlinear inertial force, the nonlinear constitutive relation in material, all kinds of complex boundary conditions and nonlinear resistances, etc. Especially in the complex electromechanical system, the nonlinear dynamic problems are outstanding, which usually are caused by the internal nonlinearity in subsystems and the electro-mechanical, rigid-elastic, fluid-solid coupling between subsystems or the nonlinear control. The application of Volterra series in mechanical engineering can be generally divided into system modeling and identification, system analysis and design, fault diagnosis etc.

3.1.1. Mechanical system modeling

Based on Volterra series theory and field measurements, Liu et al. [176] proposed a hybrid method for modeling the time history of structural vibrations resulting from the impact loading in the vicinity of a structure, and used it to predict nonlinear structural vibrations induced by ground impact loading. Mirri and Luculano [177] described different solutions for the identification of nonlinear dynamic systems, all based on a modified Volterra series, which are characterized by a reduced number of operators with respect to the classical Volterra approach. The operator reduction does not affect the accuracy of the obtained models if a mild assumption on the system memory time duration is satisfied. Based on Volterra series, Carassale and Wu [178] presented a frequency domain approach for nonlinear bridge aerodynamics and aeroelasticity. Moreover, Wu and Kareem [179] used a truncated Volterra series to model the vortex-induced vibration (VIV) of bridge decks. The results showed that the relative contribution of nonlinear effects in VIV is around 50% of the total response for a range of bridge cross sections. Roze and Hélie [180] introduced a Green-Volterra series formalism which extended the Green's function and the Volterra series. This series allows the representation of the space-time solution of weakly nonlinear boundary problems excited by an input which depends on space and time. Burke and Baumann [181] used parametric Volterra series to model dynamic diesel engine emissions; Nichols and Olson [182] derived the expressions for the bispectrum and bicoherence functions for multi-degree-of-freedom spring-mass systems with quadratic nonlinearities subject to inputs described by a wide class of random processes, and this expression was then used to determine the optimal probability distribution of the input and the optimal bispectrum to compute for the goal of maximizing the probability of detecting the nonlinearity; Furthermore, they [183] derived an expression for the trispectrum of a multi-degree-of-freedom system subject to a Gaussian excitation applied at an arbitrary location, and the analytical result was compared to those obtained via estimation using the direct method. Badji and Fenaux [184] presented a new approach based on the Volterra series theory to analyze a nonlinear single track model, which was considered to describe the vehicle dynamics behavior in the nonlinear domain. Wan and Dodd [185] presented a new data-based modeling approach named the generalized Fock space nonlinear autoregressive with exogeneous inputs (GFSNARX) model. This model allows Volterra and polynomial NARX models of arbitrary degree to be estimated using finite data. The results of modeling friction dynamics using the GFSNARX model are better than the DNLR Maxwell slip and AVDNN models. Naess and Gaidai [186] presented a detailed study of the structure and asymptotic behavior of a second-order stochastic Volterra series model of the slow drift response of large volume compliant offshore structures subjected to random seas. Moan and Zheng [187] investigated the effects of second-order nonlinear random waves on the structural response of slender fixed offshore platforms based on frequency-domain Volterra-series approach and correlation function/ FFT-based cumulant spectral method. Han [188] analyzed the orthotropic bearing nonlinearity in rotor system using GFRF, and used it to detect rotor failure. The results of simulation showed that this damage detection method is theoretically feasible as that of non-rotating structures. Liaw and

Zheng [189] presented the Volterra series representation of the nonlinear wave forces for the fixed offshore structural system. Based on GFRF, Petkovska and Do [190] analyzed and compared four different kinetic mechanisms of nonlinear adsorption systems. It was shown that, contrary to the linear frequency response characteristic functions, the higher-order FRFs corresponding to different mechanisms differ in shape. This result offers a great potential for the identification of adsorption-diffusion mechanism governing the process. Billings and Swain [191,192] identified the continuous time nonlinear differential equation models of wave forces acting on fixed cylinders and responding cylinders. Using Volterra series, Worden and Manson [193] studied the random vibration of Duffing oscillator under a random excitation, moreover, they [194] extended the analysis to multiple-degree-of-freedom nonlinear systems subjected to the random excitation.

3.1.2. Mechanical system identification

The identification of stiffness and damping are two important fields for the application of Volterra series. Various methods have been developed to estimate the stiffness and damping parameters. Most of these are based on modal analysis techniques [195,196], which were essentially derived from the frequency response function(FRF). For example, Arruda and Santos [195] estimated the mechanical parameters via curve fitting for measured FRFs using a nonlinear least squares method. Also based on the FRF, Hwang [196] put forward an identification method for stiffness and damping parameters of connections using test data for a structure attached to another structure via connection. Unfortunately, modal analysis method can only be used for the analysis of linear systems, and in practical engineering, most of the structures are nonlinear systems, and the linear model is just a simple approximation under some suppositions, so the linear FRF are no longer suitable to investigate the nonlinear system dynamics. In order to identify the parameters in the nonlinear system, some researchers have put forward nonlinear system parameter estimation methods for nonlinear systems using the GFRF. For example, Lee [197] proposed a straightforward method to estimate the nonlinear system parameters using the GFRFs. Khan and Vyas [198] employed the relationships between higher-order GFRF and first-order one to estimate the nonlinear parameters, and they further used this identification method to estimate nonlinear bearing stiffness parameter in flexible rotor-bearing systems [199]. Later, Chatterjee and Vyas [200,201] further developed this method by using a recursive iteration method; moreover, they used the recursive iteration method identified the nonlinear parameters in the rotor-bearing system [202] and bilinear oscillator [19]; also based on this recursive method, they identified the linear and nonlinear parameters in Duffing oscillator through multi-tone excitations [203]. Recently, Peng et al. [204,205] derived a series of important properties about the NOFRF of multiple-degree-of-freedom nonlinear system, and based on these properties they [206] presented a novel algorithm to estimate the linear stiffness and damping parameters of multi-degree-of-freedom (MDOF) nonlinear systems; moreover, they [207] extended this identification method to estimate the nonlinear stiffness and damping parameters for multi-degree-of-freedom (MDOF) nonlinear systems. Thouverez and Jezequel [208] identified the Volterra kernel functions for structures with local nonlinearity.

3.1.3. Design of mechanical systems

Volterra series is also applied to guide the design of mechanical systems, among which its application in the design of nonlinear vibration isolators has been widely concerned. For example, based on OFRF, Lang [209] investigated the effects of nonlinear viscous damping on vibration isolation of single degree of freedom systems. The theoretical analysis revealed that the cubic nonlinear viscous damping can produce an ideal vibration isolation such that only the resonant region is modified by the damping and the non-resonant regions remain unaffected, regardless of the levels of damping applied to the system. Peng [210] theoretically investigated the force transmissibility of single degree of freedom (SDOF) passive vibration isolators with a nonlinear antisymmetric damping characteristic. The results also revealed that a nonlinear antisymmetric damping characteristic has almost no effect on the transmissibility of SDOF vibration isolators over the range of frequencies, which are much lower or higher than the isolator's resonance frequency. On the other hand, the introduction of a nonlinear antisymmetric damping can significantly reduce the transmissibility of the vibration isolator over the resonance frequency region. The results indicated that nonlinear vibration isolators with an antisymmetric damping characteristic have great potential to overcome the dilemma encountered in the design of passive linear vibration isolators; furthermore, they [211] studied the effects of cubic non-linear viscous damping and the strong linear damping on the force transmissibility of MDOF structures, and the results also indicated that a strong linear damping may shift the system resonances and compromise the beneficial effects of cubic non-linear viscous damping on the force transmissibility of MDOF structures. Therefore, Peng et al. suggested that a less significant linear damping together with a strong cubic non-linear damping can be used in MDOF structures to achieve a desired vibration isolation performance. Later, Peng and Lang [212] investigated the force transmissibility of multiple degrees of freedom (MDOF) structures with nonlinear anti-symmetric viscous damping, and derived some conclusions which are similar as single degree of freedom systems with nonlinear anti-symmetric viscous damping. In addition, using OFRF, they [51] investigated the effects of system nonlinear parameters on the output frequency response of a passive engine mount, and presented an effective algorithm to determine the monomials in the OFRF-based representation of the output frequency response of nonlinear systems. Xiao and Jing [213] studied the force or displacement transmissibility of vibration isolators with cubic nonlinear damping under both force and base excitations, and derived the explicit relationships between the force or displacement transmissibility and the nonlinear damping coefficient in the frequency domain for the isolator systems subjected to both force and base excitations. The advantages of nonlinear damping provide a new idea for the analysis and design of nonlinear vibration isolator. Ho and Lang [214] theoretically investigated the effects of the nonlinear stiffness and nonlinear damping on the output spectra as well as the output energy spectra over different frequency ranges, and the results reveal that the nonlinear damping can be used in conjunction with the nonlinear stiffness to achieve a better vibration isolation; moreover, they [215] proposed a novel approach for the design of a new type of single degree of freedom nonlinear vibration isolation systems that can deal with harmonic excitations and take advantage of both spring and damping nonlinearities, and a detailed step-by-step procedure is developed to systematically determine the nonlinear parameters from a small set of simulation or experimental data. Based on the advantages of nonlinear damping in nonlinear vibration isolator performance, Ho and Lang [216] realized the ideal nonlinear damping characteristic using a feedback-controlled MR damper; Laalej and Lang [217] presented an experimental verification of previous theoretical finding and the selection of the cubic damping characteristic parameter required to achieve a desired performance for a single degree of freedom vibration isolation system. These results provide an important basis for the design and practical application of nonlinear damped vibration isolation systems in engineering practice.

3.1.4. Damage detection for mechanical systems

The mechanical or structural damage usually introduces nonlinearity into the mechanical or structural characteristics, as a result, the relationship between the input and output of the structural systems become nonlinear, and the extent of resultant nonlinear effects is often very sensitive to the damage. Therefore, the nonlinear characteristics of systems can be used for the structural condition monitoring and fault diagnosis, among which many researchers have studied how to use Volterra series analysis method for the damage detection and fault diagnosis. For example, Surace [218] used generalized frequency response function (GFRF) or higher order frequency response function to detect cracks in beam-like structures. Chatterjee [219] used higher order frequency response functions to assess the structural damage in a cantilever beam with a breathing crack, and investigated the effect of crack severity on the response harmonic amplitudes. Using the prediction error between the values of an unknown structural condition and the values of the reference structure in healthy condition, Shiki [220] proposed some new indexes, and these indexes are used to show the importance of considering the nonlinear behavior of the structure and estimate whether there is any damage in structure or not. The results of simulation revealed that these indexes are extremely sensitive to the appearance of damage, and therefore can be used for damage detection. Bazargan-Sabet [221] used Volterra kernel functions and GFRF to detect the cracking threshold of the geo-materials under loading. The occurrence of micro-crack was characterized by the changes in the linear and nonlinear parts of the measured signal energy. Based on Volterra series, Tang [222] presented a novel identification method of Volterra kernel function and a fault diagnosis method. This method first identified the Volterra kernels using the genetic algorithm (GA), then implementing multidimensional Fourier transformation at the Volterra kernels, derived the GFRF, finally, compared the GFRF with the results obtained for a damage-free structure to conduct the fault diagnosis. The effectiveness of this fault diagnosis method was verified by the computer simulations and engineering experiments. Cao [223] presented a novel fault detection and diagnosis approach for nonlinear complex systems by combining nonlinear output frequency response function and evidence theory. Rébillat [224] used nonlinear convolution method [225,226] to estimate the linear and each-order nonlinear parts of the outputs; then the ratio of the energy contained in the nonlinear part of an output versus the energy contained in its linear part can be calculated to determine whether there is damage in the structure or not. Peng [49] applied NOFRF to detect cracks in a beam using the frequency domain information. However, this crack detection method needs to excite the structural system under inspection many times. In order to overcome the disadvantage, Peng et al. [227] further proposed one novel damage detection approach which is based on NARMAX modeling and NOFRF deduced from the Volterra series. This new method only needed to excite the structural system one time. Cheng [228] proposed a Volterra kernel functions-based index, and the crack detection was conducted by comparing the values of the Volterra kernel functions-based indexes of the inspected beam with the values of the indexes for a uncrack beam. The numerical simulation results revealed that the indexes can be used as crack detection indicators to indicate the existence and the size of cracks. In addition, Peng [229,230] derived a series of important properties about the NOFRFs of 1D periodic structures with local nonlinearity. Furthermore, based on these important properties of the NOFRFs, they [229,231] developed a novel method to detect the position of the nonlinear component in a 1D nonlinear periodic structure. Recently, Cheng et al. [232] presented a series of important properties about the NOFRFs of 2D periodic structures with local nonlinearity, and based on these important properties, a novel method was developed to detect the position of the nonlinear components in a 2D periodic structures with local nonlinearity.

3.2. Aeroelasticity problem

Aeroelasticity problem is a very complicated nonlinear problem, and it is almost impossible to derive the accurate solution. In order to overcome this difficulty, researchers have put forward various approximate methods for solution and simplified analysis models, among which Volterra series is one of the most widely used and well-established methods. Based on Volterra series, Marzocca [233] investigated the subcritical aeroelastic response to an arbitrary time-dependent external excitation, and the flutter instability of open/closed-loop two-dimensional nonlinear airfoils. Moreover, they [234] proposed an analysis method for the supersonic flutter and aeroelastic response to a blast loading of flat panels in a supersonic flowfield via a combined Galerkin-Volterra series approach. In addition, they [235] addressed the problem of the determination of the subcritical aeroelastic response and flutter instability of nonlinear two-dimensional lifting surfaces in an incompressible flow-field via Volterra series approach. Balajewicz and Dowell [236] demonstrated that sparse Volterra reduced-order models are capable of efficiently modeling the aerodynamically induced limit-cycle oscillations. Balajewicz and Nitzsche [237] presented a reduced-order-modeling approach for nonlinear aerodynamic systems utilizing a pruned Volterra series. Wu [238] conducted linear and nonlinear aeroelastic analysis frameworks for cable-supported bridges using the artificial neural network(ANN) and Volterra series. Kurdila [239] identified the reduced-order Volterra models for aeroelastic systems with multiple inputs and multiple outputs by utilizing multiwavelet basis expansion method. Silva [240] presented a new method for the aeroservoelastic analysis and design of nonlinear aeroelasticity systems, and this method relied on the identification of nonlinear kernels that can be used to predict the response of a nonlinear system due to an arbitrary input. Brenner and Prazenica [241] proposed reasonable data processing procedures to reduce the uncertainty of the F/A-18 active aeroelastic wing aircraft data, and improve the estimation stability and identification accuracy of nonlinear systems. Sincethe traditional approaches are able to identify the optimal models of the aeroelastic dynamics based on flight data, but are not able to predict the responses at all airspeeds, in order to overcome this problem, Lind and Prazenica [242] introduced parameter-varying Volterra series model, and based on the identified model, the onset of flutter can be accurately predicted by analyzing data obtained at lower air speeds. Prazenica and Reisenthel [243] investigated the modeling of parameter-varying nonlinear aeroelastic systems using Volterra theory. Omran and Newman [244] presented a global piecewise Volterra kernel approach, which uses submodel Volterra kernels to build global kernels, and a piecewise interpolation is employed to switch between the submodels. In 2005, Silva [245] had written a review paper about the identification of nonlinear aeroelasticity systems by using Volterra series.

3.3. Control engineering

Because the controlled object, the observer or the controller is nonlinear, in order to obtain an ideal control performance, it is necessary to take nonlinear factors into consideration. Volterra series is one of the main methods for the nonlinear system modeling, which is widely used for control problems within chemical process, communications electronics, machinery, and other fields. Based on it, researchers have developed a variety of effective control methods for nonlinear systems. For example, using Volterra series, Aliyev and Gatzke [246] proposed a nonsquare nonlinear model predictive control formulation that used prioritized soft output constraints and hard input constraints, and the simulation results showed an improvement when compared with traditional linear control methods. Gruber and Guzmán [247] investigated the design of a nonlinear model predictive control strategy for greenhouse temperature control using natural ventilation based on a second-order Volterra series model identified from experimental input/output data of a greenhouse. Gruber and Bordons [248] presented the design of a nonlinear model predictive control (NMPC) strategy for the airflow in a PEM fuel cell, which manipulated the air flow rate in order to maintain the oxygen excess ratio in a desired value, both for safety and performance reasons. Yamanaka and Ohmori [249] investigated the design method of exact model matching controls for finite Volterra series systems, and clarified the condition under which the control system is stable for the reference input magnitude within a certain range. Yoon and Sun [250] presented a nonlinear system identification method and a robust tracking control method of a camless engine valve actuator based on a Volterra series representation. Maner and Doyle III [251] presented two formulations of a nonlinear model predictive control scheme based on the second order Volterra series model, and the simulation results showed that the predictive ability of the second order Volterra model controller is superior to the traditional linear controller. Kumar and Budman [252] proposed a robust nonlinear model predictive controller (NMPC) based on Volterra series and polynomial chaos expansions (PCEs), and PCEs were used to represent the uncertainty in the Volterra series coefficients. Gruber and Ramirez [253] presented a computationally efficient nonlinear Min-Max model predictive controller using Volterra series model. Vazquez and Krstic [254,255] presented stabilizing boundary control designs for a broad class of nonlinear parabolic PDEs in 1-D using spatial Volterra series. Li and Qi [256] presented a spatio-temporal Volterra model with a series of spatio-temporal kernels for modeling unknown nonlinear distributed parameter systems (DPS), and used this model for the prediction and control of nonlinear DPS. Based on the nonlinear Laguerre-Volterra observer, Zhang and Tischenko [257] proposed a novel adaptive control algorithm, and by expanding each kernel using an orthonormal Laguerre series, the number of parameters to-be-identified can be effectively reduced. Mastromauro and Liserre [258] investigated the effects of inductor nonlinear behavior on the performance of current controllers for single-phase PV grid converters. Glass and Franchek [259] investigated the H_∞ performance of nonlinear feedback systems modeled by a truncated Volterra series. Tan and Jiang [260] presented a Volterra filtered-X least mean square (LMS) algorithm for the feed-forward active noise control, and the numerical simulation results showed that the developed algorithm achieved performance improvement over the standard filtered-X LMS algorithm when the reference noise is a nonlinear noise process, and at the same time, the secondary path estimate is of non-minimum phase or when the primary path exhibits the nonlinear behavior.

3.4. Electronic and electrical engineering

The application of Volterra series in electronic and electrical engineering is also widespread, which is ranged from nonlinear distortion compensation, echo cancellation in communication channel, channel equalization, speech modeling to image processing, etc. For example, in the field of distortion analysis, Ishikawa and Kimura [261] compensated the nonlinear distortion caused by the self-heating effect in an InGaP/GaAs heterojunction bipolar transistor (HBT). In addition, a compensation condition for the distortion caused by the thermal influence was also successfully derived based on distortion analysis. Wambacq and Gielen [262] presented a method for the nonlinear high-frequency distortion analysis of analog integrated circuits using Volterra series. Huang [263] analyzed QPSK-based digital modulation signal impairments in microwave MESFET amplifiers using the Volterra series method under weakly nonlinear conditions, and evaluated the signal quality expressions, including the error vector magnitude(EVM), the adjacent channel power ratio (ACPR), and the eye diagram for a π/4-DQPSK system. Cabral and Pedro [264] theoretically investigated the effects of nonlinear memory on the AM/AM and AM/PM conversions and swept frequency-separation two-tone intermodulation distortion (IMD) tests made on microwave power amplifiers circuits; Huang and Pai [265,266] conducted the analysis of microwave MESFET power amplifiers for digital wireless communications by using the hybrid Volterra series method. In the field of echo cancellation in communication channel, Abd-Alhameed and Excell [267] designed a laser diode predistorter over a frequency range, and used it to conduct the nonlinear equalization for the intermodulation distortion of mobile communication systems; Based on an inverse modified Volterra series transfer function, Guiomar and Reis [268] proposed a noniterative digital backward propagation technique to postcompensate transmission linear and nonlinear impairments in the presence of optical noise. In order to reduce the complexity in the parameter identification of Volterra series, Liu and Li [269] proposed a new electronic nonlinearity compensation scheme for communication channel based on inverse Volterra series transfer function. Based on a discrete frequency-domain, third-order Volterra filter with the multidimensional overlap-save filtering technique, Sungbin [270] presented a new structure for an adaptive nonlinear equalizer for the digital transmission over a nonlinear satellite channel with PSK modulation. In the field of speech modeling, Patil [271] modeled the speech signal using Volterra series, and predicted the speech signal using the identified Volterra model; Despotovic and Goertz [272] presented a long-term nonlinear prediction based on second-order Volterra filters, and the results revealed that the presented predictor can outperform conventional linear prediction techniques in terms of prediction gain and "whiter" residuals; Hari and Raj [273] presented a quadratic predictor based differential encoding and decoding of speech signals using Volterra series; Alipoor and Savoji [274] examined the adaptive differential pulse code modulation(ADPCM) coding technique with nonlinear prediction based on quadratic Volterra filters using backward prediction schemes based on LMS and RLS algorithms; furthermore, they [275] wrote a review paper about the exploitation of nonlinear Volterra filters in the context of the ADPCM-based speech coding technique. In the field of image processing, Kumar and Banerjee [276] presented a novel image classification system, and used it for face recognition, and the results revealed that the Volterra kernel classifiers consistently outperformed various state-of-the-art methods in the same category; Aysal and Barner [277] proposed quadratic weighted median (QWM) filters by combining quadratic Volterra filters and weighted median filters, and used it to enhance the edges of the image with noise, and compared with the quadratic Volterra sharpener, the QWM filters exhibits superior qualitative and quantitative performance in noisy image sharpening; Hari and Jagathy [278] summarized the design and implementation of a quadratic edge detection filter for enhancing calcifications in mammograms using Volterra series; Le Caillec and Garello [279] investigated how to model the synthetic aperture radar (SAR) process of the ocean surface mapping using a decomposition based on a Volterra model.

4. Conclusion

This paper summarizes the research progress about Volterra series theory and its application in mechanical engineering, aeroelasticity problem, control engineering, electronic and electrical engineering. In this paper, the basic definition of the Volterra series is recapitulated, together with some frequency domain concepts that are derived from the Volterra series. The connection between the Volterra series and other nonlinear system description models and nonlinear problem solving methods are discussed, including the Taylor series, the Wiener series, the NARMAX model, the Hammerstein model, the Wiener model, the Wiener-Hammerstein model, the harmonic balance method, the perturbation method and the Adomian decomposition. The challenging problems and their state of arts in the series convergence study and the kernel identification study are comprehensively introduced. In addition, a detailed review is then given on the applications of Volterra series in mechanical engineering, aeroelasticity problem, control engineering, electronic and electrical engineering.

Although people have done extensive researches about Volterra series and made some progress in the past decades, there are still many challenges. For example, the convergence of Volterra series is still a challenging problem. Hitherto, there isn’t an uniform method that can determine the interval of convergence of Volterra series representation. The convergence criterions for some relatively simple systems can only be derived using the existing methods, and the criterions are more conservative, which can only determine a rough convergence domain. The number of parameters for Volterra kernel functions representation is very large. Although people have put forward some effective methods to reduce the number of parameters to be identified, the number of Volterra kernel function identification is still large. The high number of parameters to be identified reduces the efficiency of Volterra kernel function identification, and brings difficulties to the online identification. In addition, at present, Volterra series is mainly used for the analysis and modeling of nonlinear systems in the simulation or laboratory stage, so if Volterra series is used for the analysis and design of practical systems, more practical factors need to be considered.

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