Diff Eq 1

All 1st Order

$\frac{dy}{dx} = f(x,y)$

Unique soln provided

1. $f(x,y)$ bounded in neighborhood of (x_0, y_0)
2. $f(x,y)$ continuous
3. Lipschitz → no vertical lines

Phase-plane portrait

$0 = M\,dx + N\,dy$

exact: $M_y = N_x$
- integrate

$IF: \frac{\partial \mu}{\partial y}M - \frac{\partial \mu}{\partial x}N = \mu\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$

$\mu = \mu(u) \quad \frac{\partial \mu}{\partial y} = \frac{d\mu}{du}\cdot\frac{\partial u}{\partial y}$

homogenous

$f(tx, ty) = f(x,y)$

$u = y/x$

$\frac{d}{dx}(x\cdot u) = u + x\frac{du}{dx}$

FOLDE

divide by coefficient of y'

$IF = e^{\int \text{coeff } y}$

$\int_{x_0}^{x}\frac{d}{dx}(IF\cdot y) = \int_{x_0}^{x} IF\cdot f$

All 2nd Order

Euler: $ax^2y'' + bxy' + cy = 0$

use $y = x^r$

$ar(r-1) + br + c = 0$

$y = c_1x^{r_1} + c_2x^{r_2}$

same: $y = c_1x^{r_1} + c_2\ln x\cdot x^{r_1}$

complex: $r = \lambda \pm i\mu$

$y = x^{\lambda}(c_1\cos(\mu\ln x) + c_2\sin(\mu\ln x))$

SOLDE: $a_2y'' + a_1y' + a_0y = f(x)$

Uniqueness

f, a_2, a_1, a_0 all continuous at x_0

$W(x_0) \neq 0$

Wronskian

$W = e^{-\int_{x_0}^{x}\frac{a_1}{a_2}} = y_2'y_1 - y_2y_1'$

Finds 2nd soln given 1st

* Find homogenous first

$(r+1)^2 = -4 \rightarrow e^{-x}\cos(2x), e^{-x}\sin(2x)$

Variation of Parameters

$$y_p = y_1\int_{x_0}^{x}\frac{-y_2}{y_1y_2' - y_2y_1'}\cdot\frac{f}{a_2} + y_2\int_{x_0}^{x}\frac{y_1}{y_1y_2' - y_2y_1'}\cdot\frac{f}{a_2}$$

Judicious Guessing

Bump up if in homogenous + include all

- Plug in + solve

Laplace

$$Y(s) = \frac{sy_0 + ay_0 + \dot{y}_0}{s^2 + as + b} + \frac{F(s)}{s^2 + as + b}$$

Systems

① Find $P^{-1}\bar{x}(0)$

Multiply by exponentials + eigenvectors

② $\bar{x}(t) = e^{At}\bar{x}(0) + e^{At}\int_0^t e^{-A\xi}\bar{F}(\xi)d\xi$

③ $\bar{x}(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\bar{x}(0) + (sI-A)^{-1}\mathcal{L}\{\bar{f}\}\}$

Degenerate

answer has "t" in it

pick generalized eigenvector

Find $(A - \lambda I)\bar{u}$

Find $\bar{x}(0)$ in terms of $\bar{u} + \bar{v}$

pull out $e^{\lambda t}$, expand $(1 + t(A - \lambda I))$

express in terms of $\bar{u}, \bar{v}, t\bar{v}$

Nonlinear

Find fixed points

$f(x,y) \approx f_x(x_0,y_0)x + f_y(x_0,y_0)y$

make matrix: $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$

Integrals

$\sin^{-1}\left(\frac{u}{a}\right) = \int\frac{du}{\sqrt{a^2-u^2}}$

$\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) = \int\frac{du}{a^2+u^2}$

$\ln|\sec| = \int\tan$

$\ln|\sin| = \int\cot$

$\ln|\sec+\tan| = \int\sec$

$\ln xy = \ln x + \ln y$

Laplace

t^n	$\frac{n!}{s^{n+1}}$
e^{ct}	$\frac{1}{s-c}$
$e^{ct}f$	$F(s-c)$
t^nf	$(-1)^nF^{(n)}$
$\sin ct$	$\frac{c}{s^2+c^2}$
$f(t+T)$	$\frac{\int_0^T f e^{-st}}{1-e^{-sT}}$
$\cos ct$	$\frac{s}{s^2+c^2}$
$f*g$ = $\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-sc}
$\Theta(t-c)f(t-c)$	$e^{-sc}F(s)$

ex. $2\Theta(t-t_0) \rightarrow \frac{e^{-st_0}}{s}$

Solns w/ δ won't be differentiable

Solns w/ Θ won't be twice differentiable

Diff Eq 2

+

Partial Fractions (use powers)

$\frac{A}{ax+b} + \frac{Bx+C}{ax^2+bx+c}$ ~ cover up works for the highest powers

Heaviside

use $h(t) = \ldots$

Dirac Delta

Bromwich

$f(t) = \frac{1}{2\pi i}\int_\Gamma F(s)e^{st}ds$

Residue is power of s^{-1}

$\frac{1}{1-x} \cong 1+x+x^2+x^3+\ldots$

$(\frac{1}{1-x})^2 \cong 1+2x+3x^2+\ldots$

$e^x \cong 1+x+\frac{x^2}{2}+\ldots$

Convolution

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau$$

Systems

- you can substitute

① Find $P^{-1}\bar{x}(0)$
Multiply by exponentials + eigenvectors

② $\bar{x}(t) = e^{At}\bar{x}(0)$

③ Laplace
$\bar{x}(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\bar{x}(0)\}$

Degenerate

- answer has "t" in it

① Pick generalized eigenvector
find $(A-\lambda I)\bar{u}$, find $\bar{x}(0)$ in terms of $\bar{u}+\bar{v}$
pull out $e^{\lambda t}$ + do expansion
express in terms of $\bar{u}, \bar{v}, t\bar{v}$

② Laplace
take Laplace of inhomogeneity

Nonlinear

You can substitute $u = \dot{x}$
Find fixed points: both equal 0
$f(x,y) \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
Make into a matrix

Fixed points

Sink - both negative
Source - both positive
Saddle - one each
vortex depends on real part

Inhomogeneity $= \bar{f}$

① $\bar{x}(t) = e^{At}\bar{x}(0) + e^{At}\int_0^t e^{-A\xi}\bar{f}(\xi)d\xi$

② $\bar{x}(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\bar{x}(0) + (sI-A)^{-1}\mathcal{L}\{\bar{f}\}\}$

Integrals

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\sin^{-1}(\frac{u}{a}) = \int\frac{du}{\sqrt{a^2-u^2}}$

$\frac{1}{a}\sec^{-1}(\frac{u}{a}) = \int\frac{du}{u\sqrt{u^2-a^2}}$

$\frac{1}{a}\tan^{-1}(\frac{u}{a}) = \int\frac{du}{u^2+a^2}$

$\ln|\sec| = \int\tan$

$\ln|\sin| = \int\cot$

$\ln|\sec+\tan| = \int\sec$

$\ln xy = \ln x + \ln y$