geometric $\sum_{n \geq k} a_n r^n = \frac{\text{first term}}{1-r}$

Stochastic 1

Continuous MC

Poisson: $\frac{e^{-\lambda}\lambda^k}{k!}$

① Memoryless ② One at a time ③ Avg rate is constant

waiting times $T_1, \dots, T_n$ $\quad T = T_{min}$ $\quad$ rates $\lambda_1, \dots, \lambda_n$

$P(T_1 \geq t) = e^{-\lambda_1 t}$ $\qquad E[T_1] = 1/\lambda_1$

$P(T_1 = T) = \frac{\lambda_1}{\lambda_1 + \dots + \lambda_n}$

$P(T_1 < T_2) = \int_0^\infty \lambda_1 e^{-\lambda_1 t} e^{-\lambda_2 t} dt$

examples

$P(j \text{ calls in 1st hr} \mid k \text{ calls in 4 hrs}) = \binom{k}{j} p^j (1-p)^{k-j}$ $\quad p = 1/4$

$P(4 \text{ customers at store 1} \mid 4 \text{ total customers}) = \binom{4}{4}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^4 \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^0$

$T' :=$ time when 1st customer arrives at store 2

$X :=$ # of customers at store 1 by that time

$P(X_{T'} = k) = \frac{\lambda_2}{\lambda_1+\lambda_2}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k$

rates w/ Finite Spaces

$\alpha(0,1) = 1 \quad \alpha(1,0) = 2 \rightarrow A = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \end{array}$

$P_t = e^{At} = Q e^{tD} Q^{-1}$ ~ has soln $\pi A = 0$ ~ from eigenvalue 0

$b(z) = E[\text{time to get to } z] = [-\tilde{A}]^{-1} \cdot 1$, $\tilde{A} = A$ w/out z row/col

Birth+Death Processes birth λ_n, death μ_n

Transient $\iff \sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty$

→ you can get n w/ derivatives

$\sum_{n=0}^{\infty} x^n \frac{a(a+1)\cdots(a+n-1)}{n!} = (1-x)^{-a}$

$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} = e^{\lambda/\mu}$

$q := \sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$

$\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} q^{-1}$

$E[\text{length}] = \sum_{n=0}^{\infty} n\pi(n)$

explosion $\iff \sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$

Conditionings

① $E[Y|X_1,\dots,X_n]$ ~ function of $x_1,\dots,x_n$

② A - depends on $X_1,\dots,X_n$

$I_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{oth.} \end{cases}$

$E[Y I_A] = E[E[Y|X_1,\dots,X_n] \cdot I_A]$

1. $E[Y] = E[E[Y|F_n]]$ $\quad F_n = (X_1,\dots,X_n)$ ~ info contained in $X_1,\dots,X_n$
2. $E[aY_1 + bY_2 | F_n] = aE[Y_1|F_n] + bE[Y_2|F_n]$
3. Y independent of F_n: $E[Y|F_n] = E[Y]$
4. Y is a function of F_n: $E[Y|F_n] = Y$
5. $E[Y \cdot Z | F_n] = z \cdot E[Y|F_n]$ $\quad Z$ ~ function of F_n
6. $E[E[Y|F_n]|F_m] = E[Y|F_m]$ $\quad m<n$

ex. $E[S_n|F_m] = S_m + (n-m)\mu$

$E[S_m/S_n] = \frac{m}{n} S_m$

$E[Y|X=x] = \frac{\sum_y y\, p(x,y)}{p(x)}$

$E((S_n - S_m)^2) = Var[S_n - S_m]$

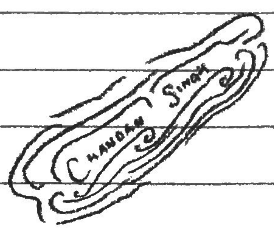
Brownian Motion / Wiener Process - stoch. process s.t.

1. $X_0 = 0$
2. For any $s_1 < t_1 \le s_2 \le t_2 \le \dots \le s_n \le t_n$, $X_{t_i} - X_{s_i}$ are independent
3. For any $s<t$, $X_t - X_s$ has normal distr. w/ mean $=0$ + var $=(t-s)\sigma^2$ $\quad$ Standard: $\sigma^2=1$
4. The paths are continuous: $t \to X_t$ is continuous func. of t

X_t is nowhere differentiable

X_t is Markov

$P(X_t \ge b | X_0 = a) = \int_b^\infty \frac{1}{\sqrt{2\pi t \sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2 t}} dx$



Stochastic 2

Ch 0

Constant coefficient diff eq $\Rightarrow$ try e^{rt}

Solve linear system: find eigenvalues, vectors $P = (v_1\ v_2)$

$P^{-1}\bar{x}(0)$ gives coefficients

$\bar{x}(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + c_2 e^{\lambda_2 t}\bar{v}_2$

difference eqn $\rightarrow$ try α^n

if repeated root, $n\alpha^n$

QDQ^{-1}

Ch 1

(1.4) Find π from $P \Rightarrow$ raise it to a high power

Solve $\pi P = P$

- set $\pi_1 = 1$
- erase the hardest eqn
- renormalize

recurrent class - you can't leave

transient class - you can leave

communicating class - you can travel between the states

irreducible - only 1 communicating class

period - smallest # of steps to return to a state

aperiodic - period = 1

$P = \begin{pmatrix} \tilde{P} & 0 \\ S & Q \end{pmatrix}$ $\quad M = (I-Q)^{-1}$ $\quad \tilde{P}$ is absorbing states

E[# of transient visits to j starting at i] = M_{ij}

E[# of moves before absorptions starting at i] = $\sum_j M_{ij}$

P[starting in i, being absorbed by j] = $(MS)_{ij}$

Mean return time = $1/\pi_i$

$\mathbb{Z}^d$ transient if $d \geq 3$

$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

$\sum_{n=0}^{\infty} r^n \cdot n = \frac{r}{(1-r)^2}$

Ch 2

#returns to a state

return time

Transient $\Leftrightarrow E[R] = \sum_{n=0}^{\infty} p_n(x,x) < \infty \Leftrightarrow P(T<\infty) < 1$

$\alpha(x) = P(X_n \text{ is ever at } 0 | X_0 = x)$ If $\alpha(x) = 1$, recurrent

Recurrent: $P\{X_n = x \text{ for } \infty\ n\} = 1$

Null Recurrent: $E[T] = \infty$ + $\lim_{n\to\infty} p_n(x,y) = 0$ R.W. in $\mathbb{Z}$

Positive Recurrent: $E[T] < \infty$ + $\lim_{n\to\infty} p_n(y,x) = \pi(x) > 0$

Branching Processes $P(k \text{ descendents}) = p_k$

smallest positive root

$a = P[\text{extinction}] = \sum_{k=0}^{\infty} p_k a^k := \phi(a)$

$\mu = \sum_{k=0}^{\infty} p_k k$ $\mu < 1 \Rightarrow a = 1$

$E(X_{n+1} | X_n = k) = k\mu$

$E(X_n) = \mu^n E(X_0)$

$\phi(0) = p_0$ $\phi(1) = 1$

$\phi'(1) = \mu$

$P(E_n | E_{n-1}^C) = \frac{P(E_n) - P(E_{n-1})}{P(E_{n-1}^c)} = \frac{\phi^n(0) - \phi^{n-1}(0)}{1 - \phi^{n-1}(0)}$

E_n extinct at gen. n

Ch 7

Reversible MC

$\pi(x) p(x \to y) = \pi(y) p(y \to x)$ $\forall x, y$ reversible $\Rightarrow$ invariant

Random Walk: $\pi(x) = \frac{d_x}{2E}$

Symmetric, Reversible $\Rightarrow$ Uniform π

Nonuniform π: $\pi(w) = \frac{f(w)}{\sum_{v \in V} f(v)}$

$p(u,w) = \frac{1}{K} \min\left(1, \frac{f(w)}{f(u)}\right)$ ~K large

Continuous MC + Poisson

$E[\exp(\lambda)] = 1/\lambda$

$P(j \text{ calls in 1st hr} / K \text{ calls in 4hrs}) = \binom{K}{j} p^j (1-p)^{K-j}$ $p = 1/4$

Poisson: (1) Memoryless (2) Customers arrive one at a time (3) avg. rate is constant

$= \frac{e^{-\lambda} \lambda^k}{k!}$ waiting times $T_1, \ldots$ are independent, exponential(λ): $P(T_1 \geq t) = e^{-b_1 t}$

Finite state space: $T = \min\{T_1, \ldots, T_n\}$ $P(T \geq t) = e^{-(b_1 + \cdots + b_n)t}$

A: $\alpha(x,y)$ $x \neq y$ $-\alpha(x)$ $x = y$ $P(T_1 = T) = \frac{b_1}{b_1 + \cdots + b_n}$

ex. $\alpha(0,1) = 1$ $\alpha(1,0) = 2$ $\rightarrow \begin{array}{c} 0 \\ 1 \end{array}\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$

$P_t = e^{At} = Q e^{D} Q^{-1}$, has soln $\pi A = 0$ (from eigenvalue 0, all others are negative)

$b(z)$ = expected time to get to $z = [-\tilde{A}]^{-1} \mathbb{1}$, $\tilde{A} = A$ w/out z row + z col