North American Journal of Economics and Finance 59 (2022) 101582

Contents lists available at ScienceDirect

North American Journal of Economics and Finance

journal homepage: www.elsevier.com/locate/najef

# Lessons from naïve diversification about the risk-reward trade-off

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ARTICLE INFO

*JEL codes:*
G11

*Keywords:*
Naïve diversification
Risk-return tradeoff
Risk analysis
Idiosyncratic risk
Sharpe ratio
Modigliani risk-adjusted return
Stationary block bootstrap

ABSTRACT

Studies of naïve diversification show that average total portfolio risk declines asymptotically as number of stocks increases. Recent work shows that a significant amount of idiosyncratic risk remains, even for portfolios with large numbers of stocks. The corresponding shocks are non-trivial. For example, more than half of all equal-weighted portfolios with 100 stocks have better than a 16 percent chance of an annual shock at least as large as about half of the annualized mean excess return on the U.S. total stock market index over July 1963–June 2018. I perform a simulation analysis of portfolio reward-to-risk as well as the components of total portfolio risk. On average, investors do not appear to be rewarded for exposure to non-systematic risk. The cross-sectional distribution of the true Sharpe ratio rises and its dispersion shrinks significantly as the number of stocks in the portfolio increases, whereas the cross-sectional distribution of the true non-systematic risk falls and its dispersion shrinks significantly as the number of stocks in the portfolio increases. This pattern appears regardless of the true asset pricing model for generating security returns, the portfolio weighting method, or specification of security alphas.

## 1. Introduction

Many finance textbooks, especially those about investing, present a classic chart that shows average total portfolio risk declining asymptotically to some positive value as the number of randomly drawn stocks increases. The intuitive explanation is that total portfolio risk converges to market risk as idiosyncratic risk is diversified away. Because average total risk appears to level out at about 10 to 30 stocks, authors often conclude that most of the benefit of diversification has been obtained for a relatively small number of stocks.

This analysis is incomplete. When we look only at portfolio risk, we leave out something that is just as important to investors: the portfolio return. A complete analysis should consider the risk-reward trade-off and not just the risk. In addition, averages can be misleading. The analysis should include the cross-sectional dispersion of risk and risk-reward measures at each portfolio size in order to determine the likelihood of selecting a portfolio with a poor risk-reward trade-off.

I perform a simulation analysis of the cross-sectional distributions of total risk, its components, and reward-to-risk under naïve diversification. The distributions are conditional on $N$, the portfolio size (*i.e.*, number of securities in the portfolio). This analysis has an advantage over earlier studies of naïve diversification that draw conclusions based only on estimates (since true values are unknown in the real world). The simulation is structured so that—within the context of the simulated market—we know the true process that generates returns and thus the true total, systematic, and non-systematic risks as well as the true Sharpe ratio for every simulated portfolio. Thus, we can draw conclusions about actual risk and reward-to-risk regardless of whether our (simulated) investor knows

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https://doi.org/10.1016/j.najef.2021.101582
Received 18 November 2020; Received in revised form 4 October 2021; Accepted 21 October 2021
Available online 28 October 2021
1062-9408/

these true values.

The simulator also generates cross-sectional distributions of estimators of each type of risk as well as the Sharpe ratio. Most analysis under naïve diversification is based on large-sample statistical behavior of the *estimators*. By performing simulations in which the true asset return generating process and its parameters are known, we can compare the performance of the estimators directly with the true values that they are intended to measure. The relevance for investors is that, in practice, they do *not* know the true returns generating process and must rely on estimates for knowledge about risk and return.

I start by confirming an important point first raised by Bennett and Sias (2011). They show that idiosyncratic risk exposes the investor to a nontrivial chance of significant shocks, even in large portfolios with 100 stocks or more. In principle, statistical estimates of risk could be biased and inaccurate; even if they are not, a given analysis is subject to statistical error. By analyzing the shocks in a simulation with returns determined by a known asset pricing model with known parameters, I show that the *true* idiosyncratic risk leads to significant shocks even in large portfolios. Thus, Bennett and Sias's result is fundamental and not conditional on their particular statistics (*e.g.*, choice of securities, pricing models, or historical periods).

The presence of significant idiosyncratic risk in large portfolios motivates three questions addressed in this paper. First, is there a relationship between the level of idiosyncratic risk and the level of reward-to-risk? Second, are investors compensated for this risk? Finally, at what portfolio size does an investor have the best odds of maximizing reward-to-risk? (Alternatively, does an investor need to hold a market portfolio in order to have the best shot at this goal?).

To address these questions, I examine the cross-sectional distributions of the true Sharpe ratio and Modigliani risk-adjusted performance measure as well as estimates of these values and compare them to the cross-sectional distributions of non-systematic risk. This paper contributes to the literature on naïve diversification by showing that the cross-sectional distribution of the Sharpe ratio shifts upward as $N$ increases. Hence, on average, the levels of the true Sharpe ratio and true idiosyncratic risk are inversely related, because the latter declines as $N$ increases. I also show that dispersion of the true cross-sectional distribution of the Sharpe ratio shrinks significantly as $N$ increases, indicating that for large values of $N$, most portfolios have nearly the same reward-to-risk even as the cross-sectional distribution of idiosyncratic risk continues to shift downward. These two results suggest that idiosyncratic risk is not compensated; otherwise, the Sharpe ratio would fall as $N$ increases.

This paper also contributes to the literature by demonstrating that least squares-based estimators of the Sharpe ratio perform poorly: they are biased upward and exhibit poor accuracy. These results are consistent with studies showing that naïve diversification generally is not dominated by portfolio optimization techniques, including mean–variance optimization.

To get a better handle on compensation for idiosyncratic risk, I compare two hypothetical markets. In the first market, all true alphas are zero. Hence, by construction, the market is efficient, and investors gain no benefit from exposure to diversifiable risk. The second market is not efficient, because I set the simulated true alphas equal to the nonzero least squares estimates for the real-world counterparts of the simulated stocks. Nonetheless, we observe the same pattern in both types of markets: the cross-sectional distribution Sharpe ratio shifts upward while the cross-sectional distribution of the non-market risk shifts downward as $N$ increases.

Finally, I construct a total market portfolio that is value-weighted and includes the universe of stocks in this study. Thus, it represents the extreme in diversification. I show that the Modigliani risk-adjusted performance of this total market portfolio is better than the majority of portfolios at each portfolio size in the inefficient market. Moreover, the proportion of portfolios that underperform the total market portfolio increases as $N$ ***decreases***. This result is contrary to what we would expect if investors in poorly diversified portfolios (with their relatively high levels of idiosyncratic risk) were compensated for risk that that they could have reduced by further diversification.

All results that I report in this paper are robust to the choice of the asset pricing model for security returns. The general conclusions that I report in Sections 4 and 5 hold regardless of whether we assume that returns are governed by the single index market model, the Fama-French three-factor model, or a two-stage industry effects model. I report results under all three models.

I organize the remainder of this paper as follows. In section 2, I review the literature on the sampling methods applied in the simulation (specifically, naïve diversification and the block bootstrap) and on the key topics in this paper (specifically, the pricing of idiosyncratic risk and the reward-to-risk tradeoff). In section 3, I describe the methods of analysis. In Section 4, I summarize my evidence about risk under naïve diversification. In Section 5, I report results concerning the risk-reward trade-off as the number of stocks increases under naïve diversification. In the final section, I discuss the implications for investors.

## 2. Review of the literature

### 2.1. Literature on naïve diversification

The seminal papers on naïve diversification show that adding stocks to an equal-weighted portfolio decreases total risk, on average. Moreover, the marginal reduction declines rapidly. Based on these results, the authors typically conclude that a portfolio is well diversified for a relatively small number of stocks. For example, Evans and Archer (1968) conclude that portfolios are sufficiently diversified at about 10 stocks; Latané & Young (1969), eight to 16 stocks; Fielitz (1974), eight stocks; Johnson and Shannon (1974), nine stocks; Bird and Tippett (1986), 22 stocks; Beck, Perfect, and Peterson (1996), 14 to 19 stocks.

The common methodological approach in most of these papers is an empirical analysis of the average total portfolio risk as a function of portfolio size under naïve diversification without reference to any asset pricing model. Often, the authors also examine linear regression of total portfolio risk on some function of portfolio size. For example, Latané & Young (1969) analyze regression on $1/N$ or $1/\sqrt{N}$, where $N$ is the number of stocks in the portfolio; the estimated coefficient is assumed to be the average diversifiable risk of individual stocks. With rare exceptions (*e.g.*, Johnson and Shannon (1974), who analyze portfolio risk using the single index market model but do not list their assumptions), the early papers do not frame the analysis in terms of an asset pricing model. Instead, market risk (equivalently, systematic risk) either is estimated from a market index or is assumed to be the asymptote for total portfolio risk as $N$ increases.

Others examine the cross-sectional dispersion of total portfolio risk and provide evidence that conclusions based only on the averages are misleading. For example, Upson, Jessup, and Matsumoto (1975) conclude that many more than 16 stocks are necessary to significantly reduce uncertainty of achieving the market return. Elton and Gruber (1977, p. 426) state, "The gains in decreased risk from adding stocks beyond 15 would appear to be significant." Later papers present evidence that the range of cross-sectional total risk is not negligible even for portfolios substantially larger than 30 stocks. For example, Newbould and Poon (1996) show that to have 95 percent confidence of being within five percent of the average portfolio risk requires more than 100 stocks. Surz and Price (2000, p. 94) conclude, "Even 60-stock portfolios achieve less than 90% of full diversification." Bennett and Sias (2011) demonstrate that significant levels of non-systematic risk are still present even for portfolios as large as 1,000 stocks.

The methodological approach in these papers varies. Some papers, *e.g.*, Upson, Jessup, and Matsumoto (1975) and Newbould and Poon (1996), simply carry out empirical analyses of the cross-sectional dispersion of total portfolio risk, conditional on number of stocks in the portfolio. Other papers apply asset pricing models to directly examine the decomposition of total portfolio risk. For example, Elton and Gruber (1977) apply the single index market model with the standard least squares assumptions about the error terms, while Bennett and Sias (2011) examine risk decomposition under various k-factor index models. They assume that the idiosyncratic error terms are stationary with zero expected values, finite variances, and zero covariances with the factors.

### 2.2. Pricing of idiosyncratic risk

If it is not easy to diversify away non-systematic risk, then perhaps it is priced. A number of papers have presented evidence that idiosyncratic risk is significantly related to expected return on individual securities. For example, Levy (1978) finds that the residual variance has a significant impact on the risk-return relationship when added to the basic capital asset pricing model. On the other hand, Ang, Hodrick, Xing, and Zhang (2006) document a *negative* cross-sectional relationship between idiosyncratic volatility and return in a two-factor model with a market factor and aggregate volatility factor. Bhootra and Hur (2015, p. 295) argue that prospect theory provides an explanation: "Consistent with risk-seeking investors' preference for high-volatility stocks in the loss domain, we find that the negative relationship between idiosyncratic volatility and stock returns is concentrated in stocks with unrealized capital losses, but is nonexistent in stocks with unrealized capital gains."

These results are puzzling and not simply because they contradict each other. They are contrary to finance theory that idiosyncratic risk can be diversified away and thus should have no effect (positive or negative) on returns. Indeed, others argue that results showing a relation between return and idiosyncratic volatility are artifacts of the methods applied. For example, Bali and Cakici (2008, p. 29) examine the cross-sectional relation between idiosyncratic volatility and expected stock returns and conclude, "No robustly significant relation exists between idiosyncratic volatility and expected returns." Significance of the relation depends on factors such as the frequency of the data used to estimate volatility and method for weighting security returns in portfolios. Huang, Liu, Rhee, and Zhang (2010) propose that the idiosyncratic volatility puzzle is due to the short-term reversals in returns documented in the literature. They find that the coefficient on idiosyncratic volatility is not statistically significant when cross-sectional regressions control for the previous month's return.

### 2.3. Relevant literature on the reward-to-risk trade-off

Relatively little attention has been given to the risk-reward trade-off under naïve diversification. Statman (1987) implicitly addresses the risk-reward relation in terms of expected return on a levered portfolio. Newbould and Poon (1996) determine the minimum number of stocks needed to be within a given percent of average risk and within a given percent of average return. De Wit (1998) develops a model for estimating the required return on an undiversified (equal-weighted) portfolio in excess of the market risk premium. However, as best as I can determine, no one has applied naïve diversification to study the cross-sectional distributions of reward-to-risk ratios such as the Sharpe ratio.

A number of papers in the literature examine the performance of portfolio optimization methods and use equal-weighted portfolios as benchmarks. See, *e.g.*, Bloomfield, Leftwich, and Long (1977); DeMiguel, Garlappi, and Uppal (2009); Allen, McAleer, Powell, and Singh (2016); and Bessler, Opfer, and Wolff (2017). These papers refer to their benchmark portfolios as $1/N$ naïve diversification. However, their analyses differ from the mainstream work on naïve diversification that focuses on behavior of risk as portfolio size, $N$, increases. A more accurate label for their benchmark strategy would be "naïve equal-weighting," because $N$ generally is fixed and small. For example, Allen, McAleer, Powell, and Singh compare optimization methods applied to portfolios constructed from ten European stock indices; their benchmark is an equal-weighted portfolio of these indices. That said, this branch of the literature is relevant, because these authors demonstrate that equal-weighted portfolios constructed from all available assets generally are not dominated by portfolios constructed with mean–variance optimization or optimization techniques that account for departures from the assumptions that underlie mean–variance optimization. The estimation error problem that I report in the context of conventional naïve diversification is consistent with the conclusions in this thread of the literature.

### 2.4. Block bootstrap methods

Efron (1979a, 1979b) generally is credited with the initial development of the bootstrap. These methods provide nonparametric estimates based on resampling from the original sample. They are appealing when the probability distribution of the statistic of interest is unknown or whose properties are unwieldy to calculate mathematically.

Consider a sample, $\{x_i, i = 1, 2, \cdots, n\}$, where $x_i$ is the $i$-th observation and might be univariate or multivariate. Perform simple random sampling with replacement of size $n$ from this original sample; label the new sample as $\{x_i^{*(1)}, i = 1, 2, \cdots, n\}$ Repeat this process $N$ times, where $\left\{x_i^{*(j)}, i = 1, 2, \cdots, n\right\}$ is the $j$-th sample generated in this manner. Let $s$ be the statistic of interest calculated from the original sample. For each new sample $j$, calculate this statistic and label it $s^{*(j)}$. Construct the empirical distribution for $\{s^{*(j)}, j = 1, 2, \cdots, N\}$. From this distribution, calculate the characteristics of interest for $s$, *e.g.*, the 5-percentile and 95-percentile confidence bounds. Generally, the resampling is carried out a large number of times, *e.g.*, $N = 10,000$. Hence, the method is computationally intensive.

Suppose that the original sample is a time series whose observations are not independent and identically distributed. Then the simple bootstrap will not work, because the simple random sampling destroys any serial dependence. The **block bootstrap** overcomes this problem by dividing the original time series $\{x_i, i = 1, 2, \cdots, n\}$ into blocks of $k$ consecutive observations. Blocks are sampled independently and with replacement to produce a new sample of length $n$ (or approximately so).

The block bootstrap has many variants. Lahiri (1999) reports that four of the most common are the nonoverlapping block bootstrap, the moving block bootstrap, the circular block bootstrap, and the stationary block bootstrap.

- Carlstein (1986) is credited with the original work on the **nonoverlapping block bootstrap** (NBB). In this variant, the original time series is divided up into *nonoverlapping* blocks of length $k$. The first block is$\{x_i, i = 1, 2, \cdots, k\}$; the second block is $\{x_i, i = k+1, k+2, \cdots, 2k\}$; the third block is $\{x_i, i = 2k+1, 2k+2, \cdots, 3k\}$; and so forth. Then draw $m = [n/k]$ blocks with replacement to construct a new sample. Repeat this process to create $N$ new time series. As with the simple bootstrap, calculate the statistic of interest for each new sample, construct the empirical distribution of this statistic, and estimate its properties based on the empirical distribution.
- Künsch (1989) and Liu and Singh (1992) generally are credited with the development of the **moving block bootstrap** (MBB). In this variant, the original time series is divided up into *overlapping* blocks of length $k$. The first block is$\{x_i, i = 1, 2, \cdots, k\}$; the second block is $\{x_i, i = 2, 3, \cdots, k+1\}$; the third block is $\{x_i, i = 3, 4, \cdots, k+2\}$; and so forth up through the block $\{x_i, i = n-k+1, n-k+2, \cdots, n\}$. Then draw $m = [n/k]$ blocks with replacement to construct a new sample. Repeat this process to create $N$ new time series and proceed as before for the nonoverlapping block bootstrap.
- Politis and Romano (1992) proposed the **circular block bootstrap** (CBB). The original series is divided up into *overlapping* blocks of length $k$ in the following manner. The block construction starts in the same way as the MBB and continues through the block consisting of the last $k$ observations in the original time series. In the CBB, the next block "circles back" to the start of the original series: $\{x_i, i = n-k+2, n-k+3, \cdots, n, 1\}$. The construction continues in this manner, *e.g.*, the next block is $\{x_i, i = n-k+3, n-k+4, \cdots, n, 1, 2\}$, and the final block is $\{x_i, i = n, 1, 2, \cdots, k-1\}$. The CBB has two advantages over the NBB. The CBB uses the original data slightly more efficiently if $n/k$ is not an integer (unless the last block for the NBB is allowed to have fewer than $k$ observations), and the CBB captures serial dependence that crosses consecutive nonoverlapping blocks in the NBB. The CBB has one important advantage over the MBB. By construction, the MBB underweights the representation of observations at the beginning and end of the original time series, *e.g.*, $x_1$ is only in the first block and $x_n$ is only in the last block, $x_2$ is only in the first two blocks and $x_{n-1}$ is only in the last two blocks, and so forth.
- Politis and Romano (1994) are credited with proposing the **stationary block bootstrap** (SBB). In this procedure, the resampled blocks have random length, where the block length $k$ has a geometric distribution with the parameter $p, 0 < p < 1$, and $E\left\{k\right\} = p^{-1}$. Under the SBB, resampled blocks are drawn to form the new series until the length of the new series equals or exceeds $n$.

The SBB has two important advantages over the other methods. First, Politis and Romano show that the SBB generates a bootstrap sample that is stationary, whereas the NBB and MBB methods do not. This feature is important in my simulation analysis, because I construct the simulated stock returns under the assumption that the asset pricing model parameters are stationary.

Second, the other three methods generate resampled series from blocks with fixed length $k$. But if the true underlying process has serial dependence of higher order than $k$, then the resampled series generated by these methods will not reflect this dependence. Estimates in the literature for serial dependence of stock returns range from one to 12 months (*e.g.*, see Jegadeesh and Titman, 1993, for evidence of positive serial correlation of returns on equity portfolios) to three to five years or longer (*e.g.*, see De Bondt and Thaler, 1989, for evidence of negative serial correlation in individual stock returns). This situation can be addressed in fixed-length block bootstrap methods by choosing $k$ to be very large (*e.g.*, greater than 60 when working with monthly returns). Alternatively, we can apply the SBB with random length blocks and set the parameter $p$ sufficiently large so that $E\left\{k\right\}$ is in the ballpark of the likely order of dependence. I take this approach in the simulation.

Block bootstrap methods have been used widely in the finance literature over the last two decades. Fixed length block bootstrap methods have been applied to investigate portfolio optimization (*e.g.*, Hansson and Persson, 2000); monetary policy (*e.g.*, Bae, Kakkar, and Ogaki, 2006; and Bekaert, Cho, and Moreno, 2010); mutual fund performance (*e.g.*, Barras, Scaillet, and Wermers, 2010); properties of equity returns (*e.g.*, Chen, Da, and Zhao, 2013; and Colacito, Ghysels, Meng, and Siwasarit, 2016); and market efficiency (*e.g.*, Da, Engelberg, and Gao, 2011; and Daniel and Titman, 1999). The stationary block bootstrap has been applied to study portfolio rebalancing (*e.g.*, Beber, Brandt, and Kavajecz, 2011), information and trading volatility (*e.g.*, Fleming, Kirby, and Ostdiek, 2006), and hedge funds (*e.g.*, Patton, 2009).

## 3. Data and methods

### 3.1. Overview

In each simulation run, I specify an asset pricing model that governs security returns. In Section 3.2, I elaborate on the asset pricing models used in this paper and the decomposition of total risk determined by each model. I also define the corresponding reward-to-risk measures reported in this paper.

Under each asset pricing model, ***I set each simulated stock's true parameters equal to the least squares estimates for its real world counterpart based on data from 2007 to 2016***. In Section 3.3, I describe the sources of market index data and individual stock data on which the parameters in the models are based and justify use of this period. ***These parameters are global in the sense that they hold for all simulation runs, given the asset pricing model.***

To help the reader understand the steps in the simulation, I outline the procedures in Table 1 with reference to the sections that provide the details. Briefly, each simulation run proceeds in two stages.

In the first stage (described in Section 3.4), the simulator draws 10,000 portfolios independently for each portfolio size $N$. Stocks are selected for a portfolio by applying simple random sampling without replacement (*i.e.*, naïve diversification). Once the stocks have been selected, the simulator calculates the ***true*** risk characteristics and ***true*** reward-to-risk measures for the portfolio from the ***true*** parameters for the simulated stocks.

In the second stage (described in Section 3.5), the simulator generates a 60-month series of simulated returns for each stock. For each portfolio, the simulator applies a stationary block bootstrap to construct 60-month series of the market factor returns and risk-free rate. Then, given the asset pricing model for the simulation run, the simulator generates the simulated returns for stocks in the portfolio and, in turn, the portfolio returns. From these returns, the simulator generates cross-sectional distributions of the ***estimates*** of total risk and its components as well as ***estimates*** of the reward-to-risk measures, conditional on $N$.

### 3.2. Asset pricing models

In this section, I summarize key information about the asset pricing models that I use. The baseline scenarios assume that the single index market model (SIMM) governs the stock returns. A long-running discussion in the literature concerns whether or not single index models are satisfactory for describing the relation between return and risk. Model misspecification is one potential problem, *e.g.*, see

**Table 1**
Flow Chart of Simulation Procedures.

| Steps | Description of step | Section of paper where discussed |
|---|---|---|
| | Given the asset pricing model, type of weights (equal-weighted or value-weighted), and portfolio size $N$ | |
| ***Stage 1: construction of cross-sectional distributions of true risks and true reward-to-risk, conditional on portfolio size. Carry out steps 1 and 2 for each of 10,000 portfolios.*** | | |
| 1 | Draw $N$ stocks using simple random sampling without replacement. | 3.4 |
| 2 | Calculate *true* total risk, components of total risk, and reward-to-risk measures for the portfolio, given the true parameters for each simulated stock in the portfolio. | 3.3 |
| 3 | For each *true* portfolio characteristic, construct the corresponding cross-sectional distribution at portfolio size $N$ based on results for the 10,000 portfolios. | 3.3 |
| ***Stage 2: construction of cross-sectional distributions of estimated risks and reward-to-risk, conditional on portfolio size. Carry out steps 4 through 10 for each of the 10,000 portfolios selected in Stage 1.*** | | |
| 4 | Apply a stationary block bootstrap procedure to draw a 60-month series of historical excess monthly market returns and monthly risk-free rates (and, if the asset pricing model requires them, other market factors). | 3.5.1 |
| 5 | For each stock in the portfolio, generate a 60-month series of error term returns using a random number generator under the assumption that the error terms follow a stationary lognormal distribution. (When the industry effects model applies, also generate a 60-month series of industry error term returns using a random number generator under the assumption that error terms follow a stationary lognormal distribution.) | 3.5.2 |
| 6 | Apply the asset pricing model governing simulated security returns to generate a 60-month series of excess returns for each stock in the portfolio. (When the industry effects model governs security returns, first construct 60-month series of industry excess returns needed to generate the simulated security returns for stocks in each industry.) | 3.5.2 |
| 7 | From the excess returns on the stocks in the portfolio and given the portfolio weighting method, generate a 60-month series of excess portfolio returns. | 3.5.3 |
| 8 | For each portfolio, estimate total portfolio risk as the sample standard deviation of portfolio excess returns. Also, estimate systematic and diversifiable components of portfolio risk based on least squares methods applied to the asset pricing model. | 3.5.4 |
| 9 | Estimate expected excess portfolio return by applying the expected return version of the asset pricing model that governs the security returns. | 3.5.4 |
| 10 | Calculate *estimates* of the Sharpe ratio and the Modigliani risk-adjusted measure for each portfolio. | 3.5.4 |
| 11 | Generate cross-sectional distributions of the *estimates* of total risk and its components as well as the reward-to-risk measures, conditional on number of stocks in the portfolio. | 3.5.4 |

Cheng and Lee (1986) and Johnson and Sprinkle (1993). To examine the robustness of the results based on the SIMM, I replicate each scenario with the Fama-French three-factor model (FF3M) and a two-stage industry effects model (IEM). Please see Appendices A.2 and A.3, respectively, for details on the specification and risk decomposition for the latter two models.

### 3.2.1. Single index market model

In the single index market model (SIMM), first proposed by Sharpe (1963), the total monthly return on stock *i* in period *t*, in excess of the risk-free rate, can be written as

$$r_{it} = \alpha_i + \beta_{mi} r_{mt} + e_{it}, \tag{1}$$

where $r_{it}$ has finite standard deviation $\sigma_i$; $r_{mt}$ is the total monthly return on the market index *m* in period *t*, in excess of the risk-free rate, with finite standard deviation $\sigma_m$ and expected value $\mu_m$ assumed constant over time; and $e_{it}$ is the error term for stock stock *i* in period *t*, with mean zero and finite standard deviation $\sigma_{ei} > 0$. I adopt the standard assumptions about the error terms:

- error terms are independent: $covar\left\{e_{it}, e_{jt}\right\} = 0$ for all stocks, $i \neq j$;
- error terms are not autocorrelated: $covar\left\{e_{is}, e_{it}\right\} = 0$ for all stocks *i* and all periods $s \neq t$;
- error terms and the market index are independent: $covar\left\{e_{it}, r_{mt}\right\} = 0$ for all stocks *i* and all periods *t*.

The total monthly return on a portfolio of *N* stocks, in excess of the risk-free rate, is

$$r_{pt} = \sum_{i=1}^{N} w_{it} r_{it} = \sum_{i=1}^{N} w_{it}\left(\alpha_i + \beta_{mi} r_{mt} + e_{it}\right) = \alpha_{pt} + \beta_{mpt} r_{mt} + e_{pt}, \tag{2}$$

where $w_{it}$ is the weight on stock *i* at the start of period *t*, and the portfolio parameters, $\alpha_{pt}$ and $\beta_{mpt}$ are defined as weighted sums of the corresponding parameters for the stocks in the portfolio. In particular, portfolio beta is

$$\beta_{mpt} = \sum_{i=1}^{N} w_{it} \beta_{mi}. \tag{3}$$

The portfolio error term, $e_{pt}$, is a weighted sum of the error terms for the stocks, and the portfolio residual variance,

$$\sigma^2_{ept} = Var\left\{e_{pt}\right\} = \sum_{i=1}^{N} w^2_{it} \sigma^2_{ei}, \tag{4}$$

is a weighted sum of the residual variances for the stocks in the portfolio. Under the above assumptions for the SIMM, the partition of total risk for portfolio *p* is

$$\sigma^2_{pt} = Var\left\{r_{pt}\right\} = \beta^2_{mpt}\sigma^2_m + \sigma^2_{ept}. \tag{5}$$

If the portfolio weights are time varying, then the portfolio parameters also are time-varying, even if the SIMM parameters are stationary for individual stocks. Because the residuals are uncorrelated with each other and the market, they represent company-unique risk in the portfolio. The random nature of company-unique events explains why they offset each other in a portfolio.

In order to properly interpret shocks due to the diversifiable risk, Bennett and Sias (2011) recommend working with risk in terms of rate of return rather than squared rates of return as with variances. In this form, $\sigma_{pt}$, the true total portfolio risk, is *not* the sum of its components, because the square root of a sum is not equal to the sum of the square roots of the elements of that sum. The advantage of working on the scale of rate of return outweighs the loss of additivity of the components of total risk, because risk is scaled in a manner that not only is easier to interpret but also is less likely to lead to incorrect conclusions. When the SIMM governs the simulated returns, I define **systematic risk** of portfolio *p* to be $\beta_{mpt}\sigma_m$. I define **idiosyncratic risk** to be $\sigma_{ept}$, the square root of the portfolio residual variance. For details on the estimators for the systematic and idiosyncratic risks, please see Appendix A.1.

In this paper, I use the terms "non-systematic risk" and "diversifiable risk" interchangeably as is common in the literature. However, idiosyncratic risk is not always equivalent to non-systematic risk. In the two-stage industry effects model, non-market risk can be partitioned into industry-unique and idiosyncratic (*i.e.*, company-unique) risk.

### 3.2.2. Reward-to-risk measures

The Sharpe ratio for a portfolio, first mentioned in Sharpe (1966), is defined as the excess return (*i.e.*, the expected return minus the risk-free rate) divided by the standard deviation of the excess return. In the notation of Section 3.2.1, the Sharpe ratio for portfolio *p* is

$$SR_{pt} = \frac{E\left\{r_{pt}\right\}}{\sigma_{pt}}, \tag{6}$$

where $r_{pt}$ is the excess return on portfolio $p$ in period $t$, and the denominator is total portfolio risk in terms of standard deviation of excess return.

The literature addresses the strengths and weaknesses of the Sharpe ratio in depth; see, *e.g.*, Sharpe (1994) and Lo (2002). Hence, I simply summarize the points most relevant to my analysis. The most important strength is that the Sharpe ratio is consistent with the Markowitz mean–variance analysis. In this framework, the risky portfolio with the highest Sharpe ratio is optimal because it offers the highest reward-to-risk trade-off. However, if portfolio returns follow a non-normal distribution, then the Sharpe ratio will not take higher moments into account, because the denominator is solely in terms of the variance. Nonetheless, Vidal-García and Vidal (2016) find that the Sharpe ratio and alternatives such as the Treynor ratio and Modigliani risk-adjusted performance measure yield the same performance ranking of mutual funds. Moreover, the Sharpe ratio provides comparable results across countries.

Within the framework of the simulation, we know the true underlying asset pricing model and its parameters in each simulation run. Hence, the simulator determines the true Sharpe ratio for each portfolio in the simulated market. For example, when the SIMM is the model, the true Sharpe ratio for portfolio $p$ is

$$SR_{pt} = \frac{E\left\{r_{pt}\right\}}{\sigma_{pt}} = \frac{\alpha_{pt} + \beta_{mpt}E\left\{r_{mt}\right\}}{\sqrt{\beta_{mpt}^2\sigma_m^2 + \sigma_{ept}^2}}, \tag{7}$$

where the true expected excess return on the market, $E\left\{r_{mt}\right\}$, and the true variance on the market excess return, $\sigma_m^2$, are set equal to values estimated from the monthly time series of the excess market returns over July 1963–June 2018. The simulator also calculates the *estimated* Sharpe ratio for each portfolio based on least squares estimates of the portfolio parameters over the portfolio's simulated 60-month estimation period.

The Modigliani risk-adjusted performance measure (M2), first proposed by Graham and Harvey (1997) and Modigliani and Modigliani (1997), is defined for portfolio $p$ as

$$M2_{pt} = SR_{pt}\sigma_B + r_{ft}, \tag{8}$$

where $SR_{pt}$ is the Sharpe ratio for portfolio $p$; $\sigma_B$ is the standard deviation of return on some benchmark portfolio, $B$; and $r_{ft}$ is the risk-free rate for the period in question. We also can rearrange this formula as

$$M2_{pt} = E\left\{\upsilon_{pt}R_{pt} + (1 - \upsilon_{pt})r_{ft}\right\}, \tag{9}$$

where $R_{pt}$ is the total return on portfolio $p$, and $\upsilon_{pt} = \sigma_B/\sigma_{pt}$. Thus, M2 is the expected return on a portfolio with a proportion $\upsilon_{pt}$ invested in portfolio $p$ and $(1 - \upsilon_{pt})$ invested in the risk-free asset, *i.e.*, portfolio $p$ is levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the benchmark (the market portfolio in the simulation).

An important advantage of M2 is that it is measured in terms of rate of return. Thus, its scale is easier to interpret than a dimensionless ratio, and the difference between M2 for any two portfolios is the difference in risk-adjusted return. When our objective is simply to rank portfolios based on the risk-reward trade-off, M2 and the Sharpe ratio yield the same ordering. However, when we wish to interpret differences between two portfolios, M2 is more useful.

### 3.3. *The simulated stock market*

Each of the stocks in the simulation has a real-world counterpart. The universe of securities in the simulation consists of 1,465 U.S. common stocks whose real-world counterparts have complete monthly returns for the 10-year period, 2007–2016. For details on how I identified the real-world counterparts and constructed monthly total returns for each, please see Appendix B.

The returns for the stocks in the simulation are designed to emulate the returns of their real-world counterparts. For each of the three asset pricing models mentioned in Section 3.2, I define the true parameters for each simulated stock by setting them equal to the values estimated by applying least squares methods to the monthly returns for their real-world counterparts over 2007–2016 (except for alphas in scenarios where I set all alphas equal to zero).

I chose the historical period 2007–2016 for determining the asset pricing model parameters for four reasons (aside from the fact that I began collecting data for this project in spring 2017). First, the period is 120 months, which is sufficient for reasonably good quality least squares estimates with monthly data. Second, the period includes the bear market of 2008–2009 and the bull market that followed and thus reflects the types of volatility experienced in the stock market. Third, the variance of the excess monthly market returns is approximately the same for 2007–2016 as it is for the longer historical period that the simulator samples in the stationary block bootstrap described in Section 3.5.1. Finally, inflation was low over this period, averaging about 1.8% per year. Hence, returns used to estimate the asset pricing model parameters are not significantly distorted by the effects of high levels of inflation on the economy and security markets.

Within the framework of the simulation, we know the true model parameters conditional on the assumed asset pricing model. In the first stage of each simulation run, once the stocks are selected for a portfolio, the simulator carries out the calculations of *true* total risk, the *true* components of total risk, the *true* expected excess return, and the *true* reward-to-risk measures for the portfolio based on the true model parameters for the individual stocks. Then, for each of these true portfolio characteristics, the simulator constructs the corresponding cross-sectional distributions at each portfolio size *N* based on results for the 10,000 portfolios.

## 3.4. How the simulator constructs portfolios

### 3.4.1. Portfolio selection

Studies of naïve diversification in the literature examine how risk changes as the number of stocks in the portfolio, *N*, increases. In this paper, I also look at how the reward-to-risk relation changes as *N* increases. I examine portfolios of sizes 10, 20, 30, 50, 100, 200, 300, 400, and 500 stocks. In the first stage of each simulation run, the simulator constructs 10,000 independently drawn portfolios for each portfolio size, where stocks are selected for each portfolio by simple random sampling without replacement. Once the stocks for a given portfolio have been selected, the simulator calculates the ***true*** risk characteristics and ***true*** reward-to-risk measures for the portfolio from the ***true*** parameters for the simulated stocks as explained in Sections 3.2 and 3.3.

### 3.4.2. Portfolio weights

My baseline analysis is for equal-weighted portfolios with monthly rebalancing. Specifically, $w_{it} = 1/N$ for all stocks $i$ and all periods $t$. This approach is attractive for several reasons. First, the formulas for true and estimated portfolio risk are less complicated, and the theoretical convergence of the diversifiable components of risk to zero is easier to describe and analyze mathematically. Second, by assuming monthly rebalancing, the analysis is not tethered to a specific holding period. Hence, length of the estimation period can be selected based on statistical objectives such as reducing bias and increasing efficiency of the estimators. Third, for each asset pricing model, the partitions of total portfolio risk are stationary, because I treat other parameters as stationary in the simulation. Finally, it facilitates comparisons with earlier results in the literature on naïve diversification, where use of equal weighting with implicit periodic rebalancing is common; see, *e.g.*, Evans and Archer (1968), Latané and Young (1969), Fielitz (1974), Elton and Gruber (1977), Tole (1982), Bird and Tippett (1986), Statman (1987), Beck, Perfect, and Peterson (1996), and Chance, Shynkevich, and Yang (2011).

However, the analysis with equal-weighted portfolios has important drawbacks. In particular, it overstates portfolio returns if transactions costs are not taken into account. With monthly rebalancing, these costs are likely to pile up as the holding period increases. In addition, Goetzmann and Kumar (2008) show that individual investors and professional active managers construct portfolios whose weights differ significantly from equal weighting.

Despite these drawbacks, I ignore transactions costs in the analysis of equal-weighted portfolios. Doing so facilitates comparison with results in the literature on naïve diversification where analysis has been conducted primarily with equal-weighted portfolios while also ignoring transactions costs. A more subtle reason is that including transactions costs would make the results sensitive to choice of length of the estimation period due to the effect of compounding.

Weights based on market capitalization have two important advantages over equal weighting. First, portfolio weights update with changes in market capitalization. Hence, the portfolio automatically rebalances with no need for costly trading. Second, value weighting corresponds to that for an investor with an index fund, provided that components of the index do not change (as is the case in the simulation). In recent decades, index investing has become increasingly important, so results for value-weighted portfolios are relevant to many investors.

Still, value weighting has two important drawbacks. First, when *N* is relatively small, some portfolios will be highly undiversified. If one of the stocks in a small portfolio is a mega-cap stock, then the portfolio essentially behaves as if it were a one-stock portfolio. The second important drawback is that we do not have an unambiguous definition of true portfolio risk, as is the case when portfolios are equal weighted with monthly rebalancing. The components of total portfolio risk for all three asset pricing models used in the simulation are dependent on the portfolio weights, $w_{it}$. If portfolio weights are not rebalanced back to the initial weights each month—as is the case for equal weighting in the simulation—then the formulas for true systematic risk, industry risk, and idiosyncratic risk are functions of time.

## 3.5. Simulation of procedures for estimating risk and reward-to-risk

In the second stage of each simulation run, the simulator generates a 60-month series of simulated excess returns for each stock in each of the 10,000 portfolios drawn for the simulation run, conditional on *N*. The simulator then calculates the corresponding 60-month series of excess portfolio returns. Finally, the simulator applies least squares methods to these returns to construct simulated estimates of the portfolio's total risk, its risk decomposition, and its reward-to-risk measures.

### 3.5.1. Generating simulated factor returns

The market data consists of excess monthly returns on the market and monthly risk-free rates from the period July 1963 through June 2018 from the Fama/French 5 Factors (2x3) series available at the Kenneth R. French Data Library (2019). (I use monthly data through the month of June to avoid any seasonality bias that might arise from including a partial 12-month period in the total history.) When the asset pricing model is the Fama-French three-factor model, the simulator also draws the Fama and French SMB and HML factors.

For each portfolio, the simulator generates a 60-month series of excess total market returns and, if the security returns are governed by the FF3M, the 60-month series of the SMB and HML index returns. The simulator uses the same factor returns for all stocks in a given portfolio but generates new series of factor returns for each portfolio.

To generate these returns, the simulator applies a stationary block bootstrap (SBB) procedure to construct a 60-month time series of market factor returns and the risk-free rate from July 1963–June 2018. At the initial time step in each simulated estimation period, the simulator draws a historical month at random with equal probability. At the next time step, with probability $1/\gamma$, the simulator draws the next month at random; otherwise, it draws the next consecutive historical month. (If the previous historical month is June 2018, then it draws July 1963. Hence, the approach adds the "circling back" feature of the circular block bootstrap to the SBB.) The parameter gamma, $\gamma$, is the expected length of a sequence of consecutive historical months arising from the bootstrap. The simulation results are for $\gamma = 24$. Test runs for gammas of 12 and 36 show that the simulation results are robust to gamma in this range.

The primary purpose in applying the SBB is to capture serial dependence in the market factors. My application of the SBB differs slightly from that described in Section 2.4. In the basic SBB, the resampled series is constructed so that its length equals or exceeds the length of the original time series. In my analysis, blocks are resampled until the length is 60 months rather than the length of the period July 1963 through June 2018 (*i.e.*, 660 months). In addition, the last "block" sampled is truncated so that the new series is exactly 60 months. However, in principle, the simulator could continue to construct each resampled series until the length equaled or exceeded 660 months. Thus, each resampled series in my analysis can be treated as the first 60 observations of a full SBB resampled series.

When determining the values of the asset pricing model parameters for the real-world counterparts of the simulated stocks and industries, I used the monthly variance on excess market returns (0.002056) for January 2007–December 2016. This variance is only modestly higher than the monthly variance of $\sigma_m^2 = 0.001916$ for July 1963–June 2018, the sampling period for the SBB described in Section 3.5.1. Thus, the simulated true stock and industry parameters are consistent with the market volatility for the longer historical period sampled by the SBB.

#### *3.5.2. Generating simulated stock returns*

For each stock in the portfolio, the simulator generates a 60-month series of error term returns using a random number generator under the assumption that the error terms follow a stationary lognormal distribution with a true variance equal to the error term variance assigned to the stock. When the IEM applies, the simulator also generates a 60-month series of industry error term returns using a random number generator under the assumption that error terms follow a stationary lognormal distribution with a true variance equal to the error term variance assigned to the industry.

Next, the simulator applies the appropriate asset pricing model governing the simulated security returns to generate a 60-month series of excess returns for each stock in the portfolio. The simulator uses the simulated monthly market factor returns (Section 3.5.1) and the simulated error term returns for each stock in the asset pricing model, where the model parameters are those assigned to the simulated stock (Section 3.3). When the IEM governs security returns, the simulator first constructs 60-month series of industry excess returns needed to generate the simulated security returns for stocks in each industry.

#### *3.5.3. Generating simulated portfolio returns*

When portfolios are weighted by market cap and/or the IEM applies, the market cap of each stock evolves over the 60-month period. The market cap for a stock at the start of each month equals the previous month's initial market cap times one plus the simulated rate of total return for the previous month. Because the simulator generates excess returns for each stock, to back out the simulated total return requires that the simulator add the risk-free rate to the excess return. These risk-free rates are drawn in tandem with the market index in the block bootstrap.

When the analysis requires market capitalization weights for stocks in a portfolio or industry, the simulator assigns an initial market cap to each simulated stock at the start of the simulated estimation period. The simulator initializes the market cap for a stock as the float-adjusted market capitalization of the corresponding real-world stock at the end of 2016. The market cap for each stock then evolves over the simulated 60-month estimation period based on the simulated total return each month for the stock. Initially in each simulated estimation period, large cap stocks are those with a market cap greater than $7,521 million (70th percentile); mid-cap stocks are those with a market cap between $1,512 million and $7,521 million; and small cap stocks are those with market cap less than $1,512 million (30th percentile). The largest initial market cap in the simulation is $598,511 million, while the smallest is $40 million. Total initial market cap for the simulated universe of stocks is $19,870,749 million.

#### *3.5.4. Constructing cross-sectional distributions for estimators*

Once the simulator has generated the 60-month series of total excess portfolio returns, it estimates the characteristics for the portfolio. The simulator estimates the total portfolio risk by calculating the sample standard deviation of total excess return. It also estimates systematic and diversifiable components of portfolio risk by applying least squares methods as described in Section 3.2.1 for the SIMM and as described in Appendices A.2 and A.3, respectively, for the FF3M and IEM. The simulator also estimates expected total excess portfolio return by applying the expected return version of the asset pricing model that is governing security returns. Finally, the simulator calculates estimates of the Sharpe ratio and the Modigliani risk-adjusted measure for each portfolio.

Once these calculations have been completed for all 10,000 portfolios in the current simulation run, the simulator generates cross-sectional distributions of the estimates of total risk and its components as well as the reward-to-risk measures, conditional on number of stocks in the portfolio. Because we know the true values (constructed as described in Section 3.3), we are able to evaluate how well the corresponding estimators perform in terms of accuracy and precision.

### 3.6. Evaluation of assumptions about serial correlation

In the simulation, I apply the SIMM (described in Section 3.2.1). To evaluate robustness to choice of asset pricing model, I also run the simulation with the FF3M (described in Appendix A.2) and the IEM (described in Appendix A.3). All three models include two general assumptions about serial correlation. They assume that the error terms for individual securities (as well as industry error terms in the IEM) are not serially correlated. The models also assume that the market factors (the market index in all three models, and the SMB and HML factors in the FF3M) do not exhibit serial correlation. Because the market index is the only factor in the SIMM and the IEM and is the dominant factor in the FF3M, the effects of serial correlation of error terms for individual stocks and of market factors are likely to be similar under all three models. Hence, I use the SIMM to illustrate the effects of serial correlation.

In this section, I evaluate whether the error terms for the real-world stocks emulated in the simulation exhibit serial correlation. I also assess whether the market factors exhibit serial correlation. Then I evaluate the likely effects on the true decomposition of total portfolio risk and on the cross-sectional distributions of the true components of total risk. Finally, I examine the effects on least squares estimators of the parameters in the SIMM and the implications for the estimators of the components of total portfolio risk.

#### 3.6.1. Evidence of serial correlation for error terms for individual stocks

In each simulation run, I treat a given asset pricing model (SIMM, FF3M, or IEM) as governing the returns of individual stocks. The parameters of each simulated stock are defined in the simulation to equal the least squares estimates under the given model for their real world counterparts based on monthly data from 2007 to 2016.

Do the error terms of the real world counterparts exhibit serial correlation? Evidence in the literature suggests that the answer should be yes. For example, Jegadeesh (1990) finds that monthly returns on individual U.S. stocks exhibit large and highly significant negative first order serial correlations as well as positive and statistically significant serial correlations at other lags.

For the emulated stocks, I estimated the $k$th lag autocorrelation, $\rho_k$, using the estimator proposed by Box and Jenkins (1976, p. 32):

$$r_k = c_k / c_0, \tag{10}$$

where

$$c_k = \frac{1}{n}\sum_{t=1}^{n-k}(z_t - \bar{z})(z_{t+k} - \bar{z}), \tag{11}$$

for the time series $z_t$ of $n$ observations, where $\bar{z}$ is the mean. Box and Jenkins (p. 34) provide an approximate variance of $r_k$; for the first order lag,

$$\widehat{var}\{r_1\} \cong \frac{1}{n} \tag{12}$$

Table 2 summarizes estimates of first order autocorrelation for the real-world stocks that are emulated in the simulation. The monthly residuals are calculated assuming the SIMM under standard assumptions. Although the residuals are approximate values of the true error term, autocorrelation of the residuals would indicate autocorrelation of the underlying error terms.

The results in Table 2 indicate that the first order autocorrelations for monthly excess returns are significantly positive more often than we would expect from random chance, while the first order autocorrelations for monthly residuals are significantly negative more often than we would expect from random chance. The latter is consistent with evidence in Jegadeesh (1990), but the former is not. In Section 3.6.2, I report that the market index exhibits *positive* first order autocorrelation for January 2007–December 2016 (the time period for the estimates reported in Table 2). Thus, the positive first order autocorrelation of the market index dominates the negative first order autocorrelations in the error terms and results in a greater tendency for monthly excess returns on individual stocks to have positive first order autocorrelation.

The tendency for significant negative first order autocorrelation in the error terms under the SIMM for the real-world counterparts of the simulated stocks suggests that the decomposition of total portfolio risk and the cross-sectional distributions of the decomposition might be affected. Nonetheless, the autocorrelation of the error terms is weak (since only about 14 percent of the stocks have significant positive or negative first order autocorrelation) and is not consistently positive or negative. If the autocorrelation function

**Table 2**
Statistics on First Order Autocorrelation for Real-world Stocks Emulated in the Simulation Assuming the Single Index Market Model.

| | Monthly excess returns | Monthly residuals |
|---|---|---|
| Percent of stocks with $r_1 > 0$ | 53.45% | 32.35% |
| Percent of stocks with $r_1 > 0$ and $\|r_1\| > 2\sqrt{\widehat{var}\{r_1\}}$ | 7.51% | 1.71% |
| Percent of stocks with $r_1 < 0$ | 46.55% | 67.65% |
| Percent of stocks with $r_1 < 0$ and $\|r_1\| > 2\sqrt{\widehat{var}\{r_1\}}$ | 3.00% | 12.29% |

*Notes*. Data is for the real-world counterparts of the 1,465 simulated stocks. Monthly residuals are calculated assuming the SIMM under standard assumptions. Monthly returns are from January 2007–December 2016. In the second column, the statistics on the first order autocorrelation are for the monthly excess returns. In the third column, the statistics on the first order autocorrelation are for the monthly residuals.

damps out exponentially, as suggested by the results reported in Jegadeesh (1990), then the primary source of autocorrelation for the error terms will the first order lags. Consistent with this expectation, Table 3 shows that the 12th order autocorrelation of the error terms is weak (since only about five percent of the stocks have significant positive or negative 12th order autocorrelation), and this autocorrelation is not consistently positive or negative.

### 3.6.2. Evidence of serial correlation for market factors

In each simulation run, I apply a stationary block bootstrap procedure that samples monthly returns on the market factors from July 1963–June 2018. The estimated first order autocorrelation for the monthly excess total market return is positive for the entire period and for non-overlapping 10-year subperiods, including January 2007–December 2016. Please see Table 4. However, the first order autocorrelation is not significantly different from zero in any period.

Similarly, the first order autocorrelation for the Fama and French SMB (small minus big) factor is not significantly different from zero in any period. Moreover, the autocorrelation is not consistently positive or negative over the non-overlapping 10-year subperiods. On the other hand, the first order autocorrelation for the Fama and French HML (high minus low) factor is significantly positive in three of the five non-overlapping 10-year subperiods.

### 3.6.3. Effects of first order serial correlation on risk decomposition

Given the results in Sections 3.6.1 and 3.6.2, I simplify the current analysis by assuming that only the first order autocorrelation for error terms for individual stocks is significantly different from zero. Similarly, I assume that only the first order autocorrelation for the excess total market return index is nonzero.

I address three questions. First, if the error terms for individual stocks exhibit first order autocorrelation, then how is the level of the ***true*** diversifiable risk affected? Second, if the market index exhibits first order autocorrelation, then how is the level of the ***true*** systematic risk affected? Third, if both types of first autocorrelation are present, how is the ***decomposition*** of the ***true*** total portfolio risk affected?

Recall from Section 3.2 that the excess monthly return on stock $i$ in month $t$ under the SIMM is

$$r_{it} = \alpha_i + \beta_{mi} r_{mt} + e_{it}. \tag{13}$$

Now assume that the excess market return and the individual stock error terms are first-order autoregressive (AR1) processes:

$$r_{mt} = \lambda r_{m,t-1} + \xi_{mt}, \tag{14}$$

$$e_{it} = \phi_i e_{i,t-1} + \eta_{it}, \tag{15}$$

where

- $covar\left\{\xi_{ms}, \xi_{mt}\right\} = \sigma^2_{m\xi}$ if s = t and zero otherwise, and $|\lambda| < 1$;
- $covar\left\{\eta_{is}, \eta_{jt}\right\} = \sigma^2_{i\eta}$ if s = t and i = j and zero otherwise, and $|\phi_i| < 1$;
- $covar\left\{\eta_{is}, \xi_{mt}\right\} = 0$ for all stocks i and all periods s and t.

The variances of these AR1 processes are

$$var\left\{r_{mt}\right\} = \frac{\sigma^2_{m\xi}}{1-\lambda^2}, \tag{16}$$

$$var\left\{e_{it}\right\} = \frac{\sigma^2_{i\eta}}{1-\phi_i^2}; \tag{17}$$

see Box and Jenkins (1976, p. 58).

**Table 3**
Statistics on 12th Order Autocorrelation for Real-world Stocks Emulated in the Simulation Assuming the Single Index Market Model.

| | Monthly excess returns | Monthly residuals |
|---|---|---|
| Percent of stocks with $r_{12} > 0$ | 70.92% | 45.46% |
| Percent of stocks with $r_{12} > 0$ and $\|r_{12}\| > 2\sqrt{\widehat{var}\{r_{12}\}}$ | 9.22% | 2.05% |
| Percent of stocks with $r_{12} < 0$ | 29.08% | 54.54% |
| Percent of stocks with $r_{12} < 0$ and $\|r_{12}\| > 2\sqrt{\widehat{var}\{r_{12}\}}$ | 0.55% | 3.41% |

*Notes*. Data is for the real-world counterparts of the 1,465 simulated stocks. Monthly residuals are calculated assuming the SIMM under standard assumptions. Monthly returns are from January 2007–December 2016. In the second column, the statistics on the 12th order autocorrelation are for the monthly excess returns. In the third column, the statistics on the 12th order autocorrelation are for the monthly residuals.

**Table 4**
Statistics on First Order Autocorrelation for Monthly Returns on Market Factors.

| Time period | Excess total market index $r_1$ | $2\bullet$st.dev.$\{r_1\}$ | SMB factor $r_1$ | $2\bullet$st.dev.$\{r_1\}$ | HML factor $r_1$ | $2\bullet$st.dev.$\{r_1\}$ |
|---|---|---|---|---|---|---|
| 7/1963–6/2018 | 0.0722 | 0.0776 | 0.0576 | 0.0777 | 0.1714* | 0.0767 |
| 10-yr. subperiods | | | | | | |
| 1/1967–12/1976 | 0.0870 | 0.1819 | 0.1722 | 0.1798 | 0.1575 | 0.1803 |
| 1/1977–12/1986 | 0.0000 | 0.1826 | 0.1497 | 0.1805 | 0.2297* | 0.1777 |
| 1/1987–12/1996 | 0.1182 | 0.1813 | 0.1669 | 0.1800 | 0.1835* | 0.1795 |
| 1/1997–12/2006 | 0.0410 | 0.1824 | −0.0651 | 0.1822 | 0.0819 | 0.1820 |
| 1/2007–12/2016 | 0.1522 | 0.1804 | −0.1688 | 0.1800 | 0.2802* | 0.1753 |

*Notes*. The autocorrelations are for monthly returns on the excess total market index, the Fama and French SMB (small minus big) factor, and HML (high minus low) factor from the Fama/French 5 Factors (2x3) series available at the Kenneth R. French Data Library (2019). These series begin with returns for July 1963. The estimated standard deviation of the estimate of the first order autocorrelation ($r_1$) is defined in Equation (12). *: estimated autocorrelation is significantly different from zero.

Recall from Section 3.2.1 that the excess return on the portfolio with weight $w_{it}$ on stock $i$ at the start of period $t$ is

$$r_{pt} = \sum_{i=1}^{N} w_{it} r_{it} = \alpha_{pt} + \beta_{mpt} r_{mt} + \sum_{i=1}^{N} w_{it} e_{it}, \tag{18}$$

where the portfolio parameters, $\alpha_{pt}$ and $\beta_{mpt}$ are weighted sums of the corresponding parameters for the stocks in the portfolio. Then, given the above assumptions about the autoregressive processes,

$$\sigma_{pt}^2 = Var\left\{r_{pt}\right\} = \beta_{mpt}^2 \frac{\sigma_{m\xi}^2}{1-\lambda^2} + \sum_{i=1}^{N} w_{it}^2 \frac{\sigma_{i\eta}^2}{1-\phi_i^2}. \tag{19}$$

When the parameters $\lambda$ and $\phi_i$ are zero for all $i$, Equation (19) reduces to the decomposition of total portfolio risk described in Equations (5)-(6) for the scenario with no autocorrelation. When the decomposition is expressed on the scale of standard deviation of return, the portfolio systematic risk $= \beta_{mpt}\sigma_{m\xi}\sqrt{1/(1-\lambda^2)}$ . Suppose that all stocks have the same autoregressive parameter $\phi$. Then portfolio idiosyncratic risk $= \sigma_{\eta pt}\sqrt{1/(1-\phi^2)}$, where

$$\sigma_{\eta pt} = \sqrt{\sum_{i=1}^{N} w_{it}^2 \sigma_{i\eta}^2}. \tag{20}$$

Table 5 illustrates the effect on the true portfolio idiosyncratic risk for various values of $\phi$. Even if this parameter were equal to the extreme values of the estimated first-order autocorrelation for the real-world stocks emulated in the simulation, the effect on the level of the true portfolio idiosyncratic risk would not be large (about 12 percent). Since the autocorrelation parameter for most stocks ranges between -0.1952 and 0.0919, the effect on idiosyncratic risk for most portfolios would be negligible. Similarly, Table 6 illustrates the effect on the true portfolio systematic risk for various values of $\lambda$ that are consistent with the estimated first-order autocorrelations of the excess monthly market returns in the simulation. The effect on the level of the true portfolio systematic risk is, at most, about one percent.

The results in Tables 5 and 6 show that first order autocorrelations consistent with the data have a negligible effect on the decomposition and level of the components of true total portfolio risk. Hence, the results reported in this paper under the assumption of no autocorrelations would change little if returns on the simulated stocks were modeled with the autoregressive processes in Equations (15) and (16).

**Table 5**
Effect on Portfolio Idiosyncratic Risk of First Order Autocorrelation Assuming a Single Index Market Model Where the SIMM Error Terms Are First-order Autoregressive Processes.

| Population statistic for $r_1$ | Autocorrelation parameter $\phi$ | $\sqrt{1/(1-\phi^2)}$ |
|---|---|---|
| Maximum | 0.4573 | 1.1244 |
| 90 percentile | 0.0919 | 1.0043 |
| Median | −0.0525 | 1.0014 |
| 10 percentile | −0.1952 | 1.0196 |
| Minimum | −0.3912 | 1.0866 |

*Notes*. The table assumes that all stocks have the same autoregressive parameter $\phi$. The values of $\phi$ in this table are set equal to statistics for the estimated first order autocorrelation, $r_1$, of the residuals for the real-world stocks emulated in the simulation. The excess stock returns are monthly.

**Table 6**
Effect on Portfolio Systematic Risk of First Order Autocorrelation Assuming a Single Index Market Model Where the Market Index Is a First-order Autoregressive Process.

| Historical period | Autocorrelation parameter$\lambda$ | $\sqrt{1/(1-\lambda^2)}$ |
|---|---|---|
| Jan 1967-Dec 1976 | 0.086990 | 1.0038 |
| Jan 1977-Dec 1986 | 0.000019 | 1.0000 |
| Jan 1987-Dec 1996 | 0.118202 | 1.0071 |
| Jan 1997-Dec 2006 | 0.040969 | 1.0008 |
| Jan 2007-Dec 2016 | 0.152233 | 1.0118 |
| Jul 1963-Jun 2018 | 0.072213 | 1.0026 |

*Notes*. The values of $\lambda$ in this table are set equal to statistics for the estimated first order autocorrelation, $r_1$, of the monthly excess total market return.

### 3.6.4. *Effects of first order serial correlation on estimators*

How would serial correlation of the error terms for individual stocks and/or of the market index affect the ***estimators*** of the components of total portfolio risk? I continue to assume that the asset pricing model is the SIMM.

Tables 7, 8, and 9 show the effects on ordinary least squares (OLS) estimates when various levels of first-order autocorrelation are present. The results are for values of the autocorrelation parameters $\phi_i$ and $\lambda$ that are consistent with the estimates (based on data from 2007 to 2016) of first-order autocorrelation for the real-world stocks emulated in the simulation and for the monthly excess total market returns over July 1963–June 2018. For these ranges of the two autocorrelation parameters, we observe the following for the individual stocks emulated in the simulation: the asymptotic efficiency of OLS estimates of beta is within five percent of one (Table 7); the asymptotic proportional bias of OLS estimates of the variance of beta is small (Table 8); and the bias of OLS estimates of the error term variance is negligible (Table 9). Hence, the magnitude and cross-sectional distributions of the estimates of the components of total portfolio risk reported in this paper would exhibit relatively little change if the first-order autocorrelation in the market index and in the individual stocks' error terms were explicitly taken into account.

## 4. Naïve diversification and risk

### 4.1. *Overview*

In this section, I report and discuss results concerning the cross-sectional distributions of total portfolio risk and its components. I focus on non-systematic portfolio risk because its presence motivates my analysis of reward-to-risk, *i.e.*, is this risk compensated? In the SIMM and FF3M, this risk is idiosyncratic risk due to company-unique events. In the IEM, non-systematic risk has two independent components: industry-unique risk and idiosyncratic risk.

In the framework of the simulation where returns are governed by known asset pricing models with known parameters, we can easily determine true values of total risk and its components for each portfolio. (For details, please see Section 3.2 for the SIMM, Appendix A.2 for the FF3M, and Appendix A.3 for the IEM.) Thus, estimation error does not cloud our assessment of the *true* cross-sectional distributions of total, systematic, and non-systematic risks. The 10-percentile, median, and 90-percentile values for each risk measure, conditional on $N$, are sufficient to provide a good sense of the true location and true dispersion. Although I calculate these statistics based on a sample from the universe of all possible equal-weighted portfolios at each size $N$ (and a second sample from the universe of all value-weighted portfolios at each size $N$), the size of the sample—10,000 portfolios—is sufficiently large so that the sampling distribution of the true measures is a good approximation of the actual distribution.

**Table 7**
Asymptotic Efficiency of OLS Estimates of Beta for an Individual Stock, Given the Presence of First-order Autocorrelation In the Market Index Returns and the Stock Error Terms.

| | Parameter$\phi$ | | | | | | |
|---|---|---|---|---|---|---|---|
| Parameter$\lambda$ | −0.15 | −0.10 | −0.05 | 0.00 | 0.05 | 0.10 | 0.15 |
| 0.000 | 0.9560 | 0.9802 | 0.9950 | 1.0000 | 0.9950 | 0.9802 | 0.9560 |
| 0.025 | 0.9562 | 0.9803 | 0.9950 | 1.0000 | 0.9950 | 0.9802 | 0.9559 |
| 0.050 | 0.9564 | 0.9803 | 0.9950 | 1.0000 | 0.9950 | 0.9801 | 0.9558 |
| 0.075 | 0.9567 | 0.9805 | 0.9951 | 1.0000 | 0.9950 | 0.9802 | 0.9558 |
| 0.100 | 0.9570 | 0.9806 | 0.9951 | 1.0000 | 0.9950 | 0.9802 | 0.9558 |
| 0.125 | 0.9574 | 0.9807 | 0.9951 | 1.0000 | 0.9951 | 0.9803 | 0.9559 |
| 0.150 | 0.9578 | 0.9809 | 0.9952 | 1.0000 | 0.9951 | 0.9803 | 0.9560 |

*Notes*. The first-order autocorrelation parameter $\phi$ is for individual stock error terms in the SIMM; *i.e.*, the error terms follow the autoregressive process in Equation (16). The first-order autocorrelation parameter $\lambda$ is for the market index; *i.e.*, the error terms follow the autoregressive process in Equation (15). Values are consistent with the estimates over 2007–2016 of first-order autocorrelation for the real-world stocks emulated in the simulation and for the monthly excess total market returns over July 1963–June 2018. The formula for asymptotic efficiency is Equation (8–58) in Johnston (1984, p. 311).

**Table 8**
Asymptotic Proportional Bias of OLS Estimates of Variance of Beta for an Individual Stock, Given the Presence of First-order Autocorrelation In the Market Index Returns and the Stock Error Terms.

| | Parameter$\phi$ | | | | | | |
|---|---|---|---|---|---|---|---|
| Parameter$\lambda$ | −0.15 | −0.10 | −0.05 | 0.00 | 0.05 | 0.10 | 0.15 |
| 0.000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.025 | 0.0075 | 0.0050 | 0.0025 | 0.0000 | −0.0025 | −0.0050 | −0.0075 |
| 0.050 | 0.0151 | 0.0101 | 0.0050 | 0.0000 | −0.0050 | −0.0100 | −0.0149 |
| 0.075 | 0.0228 | 0.0151 | 0.0075 | 0.0000 | −0.0075 | −0.0149 | −0.0222 |
| 0.100 | 0.0305 | 0.0202 | 0.0101 | 0.0000 | −0.0100 | −0.0198 | −0.0296 |
| 0.125 | 0.0382 | 0.0253 | 0.0126 | 0.0000 | −0.0124 | −0.0247 | −0.0368 |
| 0.150 | 0.0460 | 0.0305 | 0.0151 | 0.0000 | −0.0149 | −0.0296 | −0.0440 |

*Notes*. The first-order autocorrelation parameter $\phi$ is for individual stock error terms in the SIMM; *i.e.*, the error terms follow the autoregressive process in Equation (16). The first-order autocorrelation parameter $\lambda$ is for the market index; *i.e.*, the error terms follow the autoregressive process in Equation (15). Values are consistent with the estimates over 2007–2016 of first-order autocorrelation for the real-world stocks emulated in the simulation and for the monthly excess total market returns over July 1963–June 2018. The formula for asymptotic proportional bias is Equation (8–59) in Johnston (1984, p. 312).

**Table 9**
Bias of OLS Estimates of the Error Term Variance for an Individual Stock, Given the Presence of First-order Autocorrelation In the Market Index Returns and the Stock Error Terms.

| | Parameter$\phi$ | | | | | | |
|---|---|---|---|---|---|---|---|
| Parameter$\lambda$ | −0.15 | −0.10 | −0.05 | 0.00 | 0.05 | 0.10 | 0.15 |
| 0.000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.025 | 1.0001 | 1.0001 | 1.0000 | 1.0000 | 1.0000 | 0.9999 | 0.9999 |
| 0.050 | 1.0003 | 1.0002 | 1.0001 | 1.0000 | 0.9999 | 0.9998 | 0.9997 |
| 0.075 | 1.0004 | 1.0003 | 1.0001 | 1.0000 | 0.9999 | 0.9997 | 0.9996 |
| 0.100 | 1.0005 | 1.0003 | 1.0002 | 1.0000 | 0.9998 | 0.9997 | 0.9995 |
| 0.125 | 1.0006 | 1.0004 | 1.0002 | 1.0000 | 0.9998 | 0.9996 | 0.9994 |
| 0.150 | 1.0007 | 1.0005 | 1.0003 | 1.0000 | 0.9997 | 0.9995 | 0.9992 |

*Notes*. The first-order autocorrelation parameter $\phi$ is for individual stock error terms in the SIMM; *i.e.*, the error terms follow the autoregressive process in Equation (16). The first-order autocorrelation parameter $\lambda$ is for the market index; *i.e.*, the error terms follow the autoregressive process in Equation (15). Values are consistent with the estimates over 2007–2016 of first-order autocorrelation for the real-world stocks emulated in the simulation and for the monthly excess total market returns over July 1963–June 2018. The bias in error term variance is calculated using the formula presented by Johnston (1984, p. 313). The above values assume 60 observations as in the simulation.

In Sections 4.2 and 4.3, I report results under all three asset pricing models (although, in some cases, the results under the FF3M or IEM are so similar to those under the SIMM that I make note of this fact and only report details for the SIMM). All results concerning *estimators* of risk and reward-to-risk are based on the application of the stationary block bootstrap to generate resampled 60-month multivariate time series of the factor returns (the market factor for all three models and the SMB and HML factors under the FF3M) and the risk-free rate.

### 4.2. Key results

The true portfolio non-systematic risk behaves much differently than the true portfolio systematic risk. In theory, non-systematic risk shrinks to zero as $N$ increases, whereas the median true systematic risk should be about the same for all $N$. For example, for equal-weighted portfolios when stock returns follow the SIMM or FF3M, the portfolio idiosyncratic risk is

$$\sigma_{ep} = \left(\frac{1}{N}\right)\sqrt{\sum_{i=1}^{N}\sigma_{ei}^{2}}. \tag{21}$$

Analogously, for the IEM, we can show that, for equal-weighted portfolios, the portfolio idiosyncratic risk is

$$\sigma_{\varepsilon p} = \left(\frac{1}{N}\right)\sqrt{\sum_{i=1}^{k}\sum_{j(i)=1}^{n_i}\sigma_{\eta j(i)}^{2}}, \tag{22}$$

(See Equation (A21) in Appendix A.3 for the general formula.) In both cases, the idiosyncratic risk declines at a rate roughly equal to $-1/N^2$ (*i.e.*, the first derivative of $\sigma_{ep}$ with respect to $N$) provided that the variances of residuals are bounded.

We observe this decline in the true idiosyncratic risk for both equal-weighted and value-weighted portfolios; see Figs. 1 and 2, respectively, and Table 10, Panels A and B, respectively, for results when the SIMM governs security returns. By comparison, the true median portfolio systematic risk is approximately the same at all portfolio sizes, conditional on the method for portfolio weights.

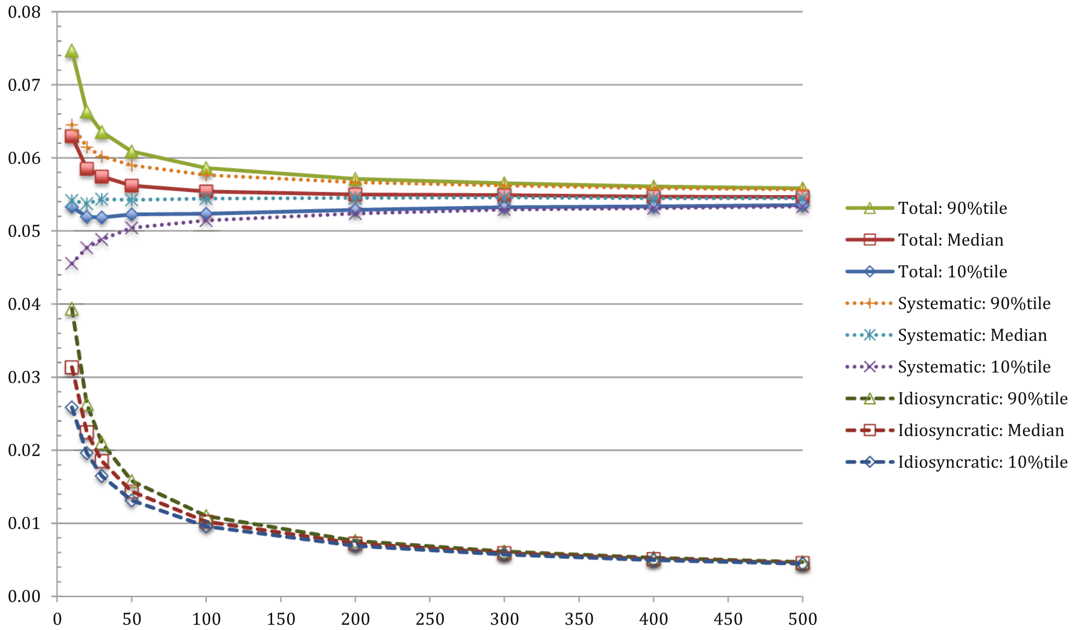


**Fig. 1.** Cross-sectional Distributions of True Values of Total Risk and Its Components When Portfolios are Equal-weighted and the Single Index Market Model Governs Returns.

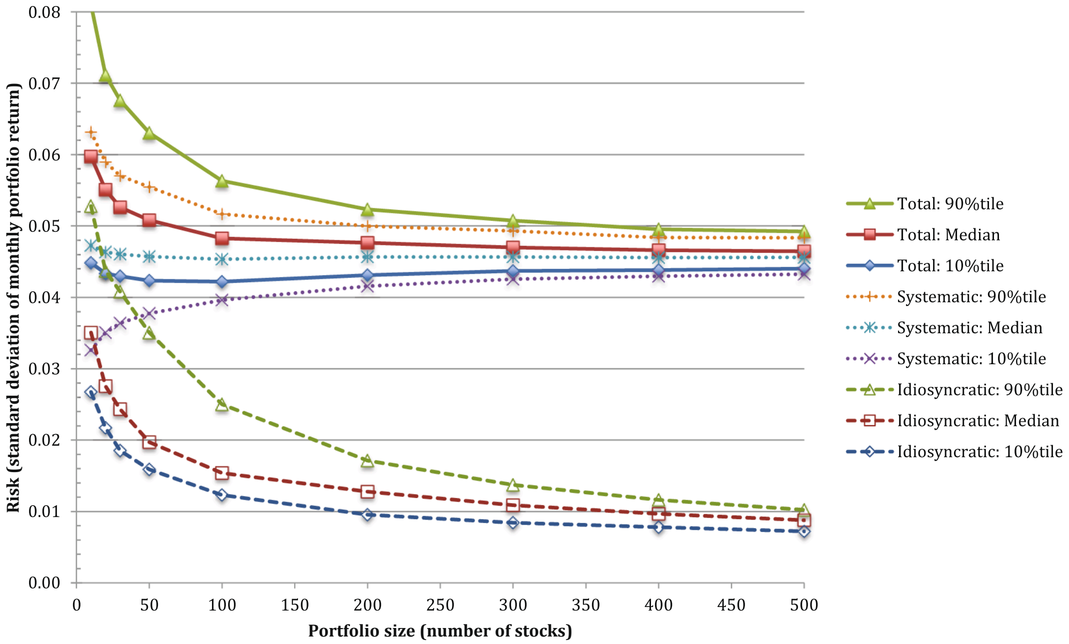


**Fig. 2.** Cross-sectional Distributions of True Values of Total Risk and Its Components When Portfolios Are Value-weighted and the Single Index Market Model Governs Returns.

A significant amount of true idiosyncratic risk remains even for portfolios as large as 500 stocks. For equal-weighted 500-stock portfolios when returns follow the SIMM, the median is 0.46% per month (approximately 1.59% annualized), and 80 percent of these portfolios have true idiosyncratic risk within 0.01% per month of this median. For value-weighted 500-stock portfolios, the median is 0.88% per month (approximately 3.05% annualized), and 80 percent of these portfolio have true idiosyncratic risk within 0.16% per month of the median. More importantly, few of the 500-stock portfolios have idiosyncratic risk close to zero. The 10 percentile of the true idiosyncratic risk is well above zero: 0.45% per month for equal-weighted portfolios (approximately 1.56%

**Table 10**
Cross-sectional Distributions of True Risk as a Function of Portfolio Size In Terms of Monthly Returns (%) Where Returns Are Generated with the Single Index Market Model.

| Panel A. Equal-weighted Portfolios Rebalanced Monthly | | | | | | |
|---|---|---|---|---|---|---|
| | Systematic Risk | | | Idiosyncratic Risk | | |
| Portfolio size | 10 percentile | Median | 90 percentile | 10 percentile | Median | 90 percentile |
| 10 | 4.56 | 5.42 | 6.45 | 2.59 | 3.14 | 3.94 |
| 20 | 4.77 | 5.37 | 6.15 | 1.96 | 2.25 | 2.63 |
| 30 | 4.88 | 5.43 | 6.02 | 1.65 | 1.85 | 2.10 |
| 50 | 5.04 | 5.43 | 5.90 | 1.31 | 1.44 | 1.58 |
| 100 | 5.14 | 5.44 | 5.76 | 0.96 | 1.02 | 1.10 |
| 200 | 5.24 | 5.45 | 5.67 | 0.69 | 0.72 | 0.76 |
| 300 | 5.29 | 5.46 | 5.62 | 0.57 | 0.59 | 0.62 |
| 400 | 5.31 | 5.44 | 5.59 | 0.50 | 0.51 | 0.53 |
| 500 | 5.33 | 5.45 | 5.57 | 0.45 | 0.46 | 0.47 |
| Panel B. Value-weighted Portfolios | | | | | | |
| | Systematic Risk | | | Idiosyncratic Risk | | |
| Portfolio size | 10 percentile | Median | 90 percentile | 10 percentile | Median | 90 percentile |
| 10 | 3.26 | 4.73 | 6.32 | 2.67 | 3.51 | 5.28 |
| 20 | 3.51 | 4.64 | 5.89 | 2.17 | 2.76 | 4.37 |
| 30 | 3.64 | 4.60 | 5.70 | 1.85 | 2.43 | 4.08 |
| 50 | 3.77 | 4.57 | 5.55 | 1.59 | 1.97 | 3.51 |
| 100 | 3.96 | 4.53 | 5.17 | 1.23 | 1.54 | 2.50 |
| 200 | 4.16 | 4.56 | 5.00 | 0.96 | 1.28 | 1.71 |
| 300 | 4.25 | 4.57 | 4.93 | 0.84 | 1.09 | 1.37 |
| 400 | 4.30 | 4.56 | 4.84 | 0.78 | 0.97 | 1.16 |
| 500 | 4.32 | 4.56 | 4.83 | 0.72 | 0.88 | 1.02 |

*Notes*. Portfolio size is the number of stocks in the portfolio. Results when the underlying asset pricing model is the Fama-French three-factor model are similar to those when the underlying model is the single index market model. The components of risk when portfolios are value-weighted are conditional on the market-value weights, which change over the simulated 60-month estimation period. The results reported in Panel B are the components calculated with the market-value weights at the start of each estimation period. Market capitalization for each simulated stock is initialized at the same value for every 60-month estimation period. Hence, these initial weights are global in the sense that they are not determined by the evolution of stock returns over estimation periods. Thus, the measures of *true* risk reported in this table, given a specific set of stocks in a portfolio, do not change due to simulated returns histories used to calculate *estimators*.

annualized), and 0.72% per month for value-weighted portfolios (approximately 2.49% annualized).

When the IEM generates the stock returns, the cross-sectional distributions of true total risk and its components are similar to the corresponding cross-sectional distributions when the SIMM holds. In particular, the medians for both types of non-systematic risk (industry-unique and idiosyncratic) decrease. Please see Table 11. For the true cross-sectional distributions for each component of total portfolio risk, the dispersion shrinks as $N$ increases. The dispersion decreases more rapidly when portfolios are equal-weighted than when portfolios are value-weighted. Compare the respective components of total risk in Panels A and B in Table 11.

In the real world, of course, investors and portfolio managers do not know the true asset pricing model or its parameters. They must choose models and estimate model parameters. For both equal-weighted and value-weighed portfolios, the cross-sectional distributions for estimated systematic risk have much greater dispersion than the true distributions. Compare Fig. 3 (estimates) to Fig. 1 (true values) for equal-weighted portfolios under the SIMM, and Fig. 4 (estimates) to Fig. 2 (true values) for value-weighted portfolios under the SIMM. These charts indicate that, for a given portfolio, the accuracy of the estimator for systematic risk is poor. This accuracy gets worse as $N$ increases, because the dispersion of *true* systematic risk shrinks, but the dispersion of the *estimated* systematic risk is fairly constant for $N \geq 100$ when portfolios are equal-weighted and $N \geq 200$ when portfolios are value-weighted.

In a pairwise comparison of the cross-sectional distributions of estimates of systematic risk to the cross-sectional distributions of true systematic risk, the Kolmogorov test statistic for a difference between the estimate and its true value is highly significant. See the first and third sections of Panel A of Table 12 for scenarios where the SIMM holds and the first and fourth sections of Panel B for scenarios where the IEM applies. (Results for the FF3M are similar to those when the SIMM holds and thus are not presented here.)

On the other hand, estimators of idiosyncratic risk appear to be much more accurate. For equal-weighted portfolios, the **10/90 range** (the range from the 10 percentile to the 90 percentile) displayed in Fig. 3 (where the SIMM holds) converges to the median estimate for $N \geq 200$. The p-values for the Kolmogorov statistic for pairwise comparisons between the estimated and true idiosyncratic risk are not significant at the one percent level for $N \leq 200$ and are only marginally significant for larger sizes. Please see the second row concerning idiosyncratic risk for equal-weighted portfolios in Panel A of Table 12. When the IEM holds, we observe similar results; please see the third row concerning idiosyncratic risk for equal-weighted portfolios in Panel B of Table 12. (Results under the FF3M are similar to those when the SIMM governs returns and thus are not reported here.)

For value-weighted portfolios, the 10/90 range for the estimator of idiosyncratic risk displayed in Fig. 4 (where the SIMM holds) are similar to those for the true idiosyncratic risk (Fig. 2). The p-values for the Kolmogorov statistic for pairwise comparisons between the estimated and true idiosyncratic risk are significant at the one percent level for the majority of values of $N$. Please see the fourth row concerning idiosyncratic risk of value weighted portfolios in Panel A in Table 12. Nonetheless, as a comparison of Figures 2 and 4 shows, the magnitude of this difference does not appear to be important. When the IEM holds, the estimator of idiosyncratic risk is not

**Table 11**
Cross-sectional Distributions of True Risk as a Function of Portfolio Size In Terms of Monthly Returns (%) When Returns Are Generated with the Two-stage Industry Effects Model.

| Panel A. Equal-weighted Portfolios Rebalanced Monthly | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Systematic Risk | | | Industry-unique Risk | | | Idiosyncratic Risk | | |
| Portfolio size | 10%-tile | Median | 90%-tile | 10%-tile | Median | 90%-tile | 10%-tile | Median | 90%-tile |
| 10 | 3.91 | 4.77 | 5.75 | 1.21 | 1.54 | 2.01 | 2.30 | 2.84 | 3.69 |
| 20 | 4.20 | 4.82 | 5.46 | 0.98 | 1.18 | 1.41 | 1.75 | 2.05 | 2.46 |
| 30 | 4.30 | 4.80 | 5.38 | 0.87 | 1.02 | 1.22 | 1.47 | 1.70 | 1.95 |
| 50 | 4.39 | 4.80 | 5.19 | 0.76 | 0.86 | 1.00 | 1.18 | 1.33 | 1.49 |
| 100 | 4.53 | 4.81 | 5.09 | 0.65 | 0.73 | 0.82 | 0.87 | 0.94 | 1.01 |
| 200 | 4.62 | 4.81 | 5.00 | 0.59 | 0.64 | 0.71 | 0.63 | 0.67 | 0.71 |
| 300 | 4.66 | 4.81 | 4.96 | 0.57 | 0.61 | 0.66 | 0.52 | 0.54 | 0.57 |
| 400 | 4.69 | 4.81 | 4.94 | 0.56 | 0.60 | 0.64 | 0.46 | 0.47 | 0.49 |
| 500 | 4.70 | 4.80 | 4.91 | 0.56 | 0.59 | 0.62 | 0.41 | 0.42 | 0.43 |
| Panel B. Value-weighted Portfolios | | | | | | | | | |
| | Systematic Risk | | | Industry-unique Risk | | | Idiosyncratic Risk | | |
| Portfolio size | 10%-tile | Median | 90%-tile | 10%-tile | Median | 90%-tile | 10%-tile | Median | 90%-tile |
| 10 | 3.21 | 4.61 | 6.14 | 1.48 | 2.18 | 3.70 | 1.81 | 2.69 | 4.16 |
| 20 | 3.34 | 4.52 | 5.80 | 1.29 | 1.80 | 3.01 | 1.43 | 2.03 | 2.88 |
| 30 | 3.56 | 4.56 | 5.88 | 1.18 | 1.62 | 2.94 | 1.23 | 1.71 | 2.44 |
| 50 | 3.79 | 4.60 | 5.58 | 1.04 | 1.42 | 2.47 | 1.00 | 1.38 | 1.96 |
| 100 | 3.95 | 4.58 | 5.39 | 0.90 | 1.16 | 1.90 | 0.79 | 1.02 | 1.37 |
| 200 | 4.12 | 4.62 | 5.17 | 0.78 | 0.99 | 1.48 | 0.62 | 0.74 | 0.92 |
| 300 | 4.23 | 4.64 | 5.10 | 0.72 | 0.90 | 1.24 | 0.53 | 0.61 | 0.73 |
| 400 | 4.30 | 4.65 | 4.99 | 0.69 | 0.83 | 1.06 | 0.48 | 0.54 | 0.62 |
| 500 | 4.36 | 4.65 | 4.93 | 0.67 | 0.80 | 0.95 | 0.43 | 0.48 | 0.54 |

*Notes*. Portfolio size is the number of stocks in the portfolio. In the two-stage industry effects model, diversifiable risk arises from two sources: industry-unique risk, and company-unique risk (*a.k.a.* idiosyncratic risk). The components of risk when portfolios are value-weighted are conditional on the market-value weights, which change over the simulated 60-month estimation period. The results in Panel B are the components calculated with the market-value weights at the start of each estimation period. Market capitalization for each simulated stock is initialized at the same value for every 60-month estimation period. Hence, these initial weights are global in the sense that they are not determined by the evolution of stock returns over estimation periods. Thus, the measures of *true* risk reported in this table, given a specific set of stocks in a portfolio, do not change due to simulated returns histories used to calculate *estimators*.

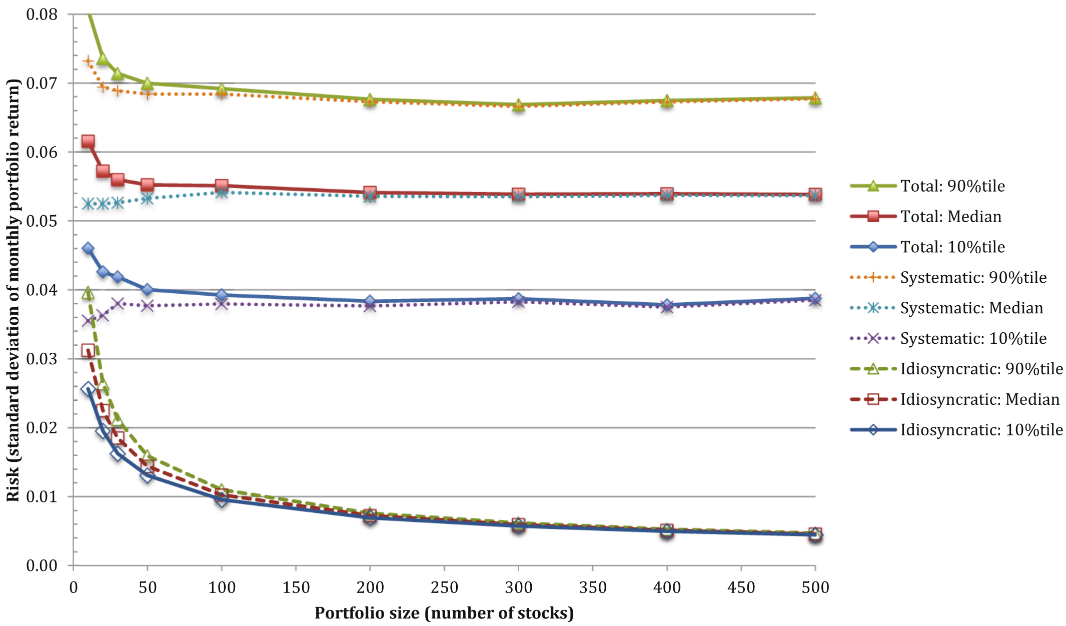


**Fig. 3.** Cross-sectional Distributions of Estimators of Total Risk and Its Components When Portfolios Are Equal-weighted and the Single Index Market Model Governs Returns

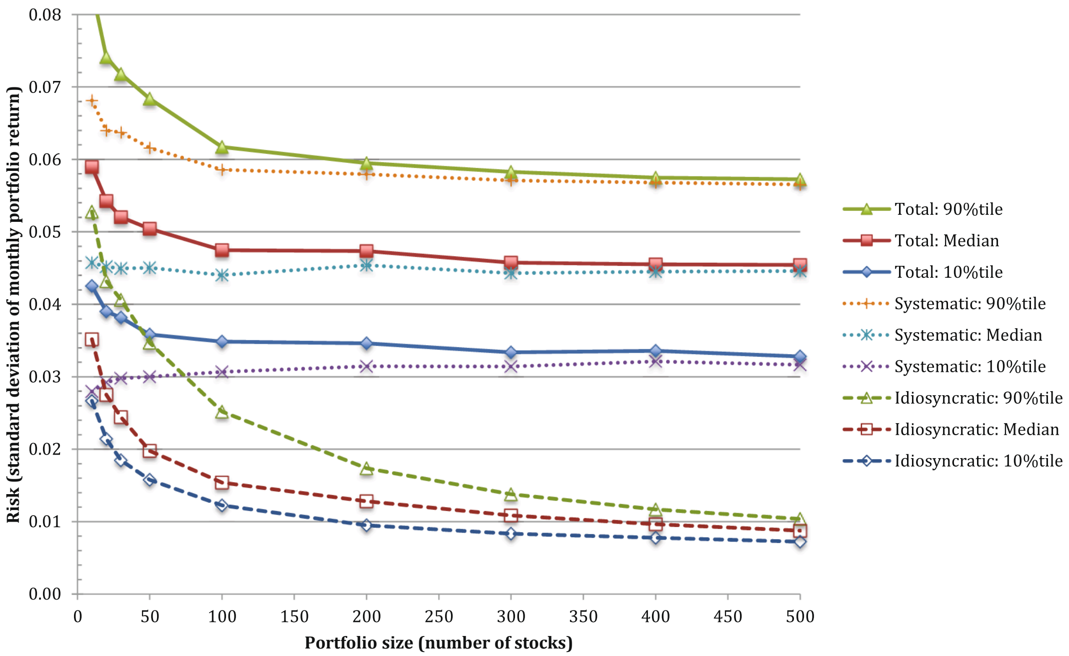


**Fig. 4.** Cross-sectional Distributions of Estimators of Total Risk and Its Components When Portfolios Are Value-weighted and the Single Index Market Model Governs Returns.

**Table 12**

Comparison of Cross-sectional Distribution of Risk Estimates with Cross-sectional Distribution of True Risk: Goodness-of-fit Tests (Kolmogorov test statistic; p-value in parentheses, except that * represents p less than 0.01).

Panel A. Underlying Asset Pricing Model Is the Single Index Market Model

| Type of risk | Portfolio weighting | Portfolio size | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| Systematic | Equal | 0.1939 * | 0.2409 * | 0.2815 * | 0.3130 * | 0.3618 * | 0.4008 * | 0.4319 * | 0.4343 * | 0.4600 * |
| Idiosyncratic | Equal | 0.0138 (0.03) | 0.0152 (0.02) | 0.0151 (0.02) | 0.0147 (0.03) | 0.0147 (0.03) | 0.0154 (0.02) | 0.0169 * | 0.0168 * | 0.0220 * |
| Systematic | Value | 0.1125 * | 0.1457 * | 0.1573 * | 0.1802 * | 0.2334 * | 0.2914 * | 0.3323 * | 0.3470 * | 0.3691 * |
| Idiosyncratic | Value | 0.0183 * | 0.0185 * | 0.0175 * | 0.0164 (0.01) | 0.0178 * | 0.0133 (0.06) | 0.0161 (0.01) | 0.0154 (0.02) | 0.0230 * |

Panel B. Underlying Asset Pricing Model Is the Two-stage Industry Effects Model

| Type of risk | Portfolio weighting | Portfolio size | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| Systematic | Equal | 0.1820 * | 0.2358 * | 0.2698 * | 0.3050 * | 0.3429 * | 0.3859 * | 0.4198 * | 0.4306 * | 0.4390 * |
| Industry | Equal | 0.0341 * | 0.0332 * | 0.0344 * | 0.0404 * | 0.0307 * | 0.0303 * | 0.0370 * | 0.0400 * | 0.0527 * |
| Idiosyncratic | Equal | 0.0135 (0.05) | 0.0104 (0.2+) | 0.0130 (0.07) | 0.0149 (0.03) | 0.0134 (0.06) | 0.0196 * | 0.0143 (0.04) | 0.0197 * | 0.0199 * |
| Systematic | Value | 0.1044 * | 0.1241 * | 0.1376 * | 0.1644 * | 0.2156 * | 0.2624 * | 0.3025 * | 0.3275 * | 0.3564 * |
| Industry | Value | 0.0200 * | 0.0167 * | 0.0169 * | 0.0109 (0.19) | 0.0155 (0.02) | 0.0119 (0.12) | 0.0135 (0.05) | 0.0177 * | 0.0238 * |
| Idiosyncratic | Value | 0.0118 (0.13) | 0.0106 (0.20) | 0.0152 (0.02) | 0.0106 (0.20) | 0.0157 (0.02) | 0.0094 (0.2+) | 0.0130 (0.07) | 0.0139 (0.04) | 0.0120 (0.11) |

*Notes.* Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risk for each portfolio and the corresponding estimate. Stock and market returns are monthly in excess of the one-month Treasury bill rates. Market returns are generated from stationary block bootstrap samples of historical returns. Excess stock returns are calculated from the appropriate asset pricing model with known, stationary values of alpha, beta, and error term variance for each stock (and industry) and assuming a stationary lognormal distribution for each stock's error term (and each industry's error term). Results when using the Fama-French three-factor model are similar to those when using the single index market model and thus are not reported here. The table lists the Kolmogorov goodness-of-fit statistic for the difference between the cross-sectional distribution of the risk estimate and the corresponding cross-sectional distribution of true risk. The p-values for two-sided tests are listed and are determined from Table A14 in Conover (1980). "0.2+" reads "p-value is greater than 0.20." Statistical tests are independent across portfolio sizes but not across risk measures for a given portfolio size and weighting method.

significantly different from its true value. Please see the last row concerning idiosyncratic risk of value-weighted portfolios in Panel B in Table 12.

In summary, least squares based estimators of portfolio diversifiable risk (idiosyncratic risk in all three models as well as industry-unique risk in the IEM) are far more accurate than the least squares based estimators of portfolio systematic risk. Moreover, this accuracy improves as $N$ increases. Thus, for questions concerning diversifiable risk, least squares based estimators are relatively accurate and precise. However, least squares based estimators for portfolio systematic risk exhibit such poor accuracy that their use provides little information about true values. This problem carries over to estimators of portfolio total risk, because—in its variance form, such as in Equation (5) for the SIMM—it equals the sum of the systematic and non-systematic risks.

### 4.3. Shocks due to idiosyncratic risk

I start by explaining how I quantify shocks due to the true idiosyncratic risk. (The explanation for shocks due to industry-unique risk in the IEM is analogous.) The cross-sectional distributions of true idiosyncratic risk are positively skewed, but the normal approximation is reasonably good for $N \geq 50$ and improves as $N$ increases. Recall from Equations (2) and (4) in Section 3.2.1 that the portfolio residual variance in the SIMM is $\sigma^2_{ept} = Var\left\{e_{pt}\right\}$, where the portfolio error term, $e_{pt}$, is the true idiosyncratic risk in the portfolio. (For the FF3M, the same notation applies; see Equations (A6) and (A8) in Appendix A.2. For the IEM, I present the analogous notation for the idiosyncratic risk in Equations (A14) and (A21) in Appendix A.3.) The expected value of the idiosyncratic risk is zero, *i.e.*, $E\left\{e_{pt}\right\} = 0$. Therefore, under the normal approximation, the probability that the idiosyncratic risk falls one standard deviation or more below zero is

$$Pr\left\{e_{pt} < -\sigma_{ept}\right\} \cong 0.16. \tag{23}$$

**Table 13**

Shocks Due to True Diversifiable Risk at Each Portfolio Size (Approximate Annualized Rates in %)

| Panel A. Equal-weighted Portfolios Rebalanced Monthly | | Portfolio size | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| Shocks due to idiosyncratic risk when SIMM holds | 90%-tile | 13.65 | 9.12 | 7.29 | 5.48 | 3.81 | 2.63 | 2.13 | 1.83 | 1.63 |
| | Median | 10.86 | 7.78 | 6.41 | 4.97 | 3.54 | 2.51 | 2.06 | 1.77 | 1.59 |
| | 10%-tile | 8.96 | 6.80 | 5.71 | 4.55 | 3.32 | 2.40 | 1.99 | 1.72 | 1.55 |
| Shocks due to idiosyncratic risk when FF3M holds | 90%-tile | 13.39 | 8.85 | 7.02 | 5.37 | 3.69 | 2.56 | 2.07 | 1.78 | 1.59 |
| | Median | 10.51 | 7.53 | 6.24 | 4.85 | 3.45 | 2.45 | 2.00 | 1.73 | 1.55 |
| | 10%-tile | 8.70 | 6.52 | 5.50 | 4.39 | 3.21 | 2.33 | 1.92 | 1.68 | 1.51 |
| Shocks due to idiosyncratic risk when IEM holds | 90%-tile | 12.77 | 8.52 | 6.77 | 5.18 | 3.51 | 2.45 | 1.97 | 1.69 | 1.50 |
| | Median | 9.83 | 7.10 | 5.88 | 4.60 | 3.25 | 2.31 | 1.89 | 1.64 | 1.46 |
| | 10%-tile | 7.97 | 6.07 | 5.08 | 4.10 | 3.00 | 2.20 | 1.81 | 1.58 | 1.42 |
| Shocks due to industry risk when IEM holds | 90%-tile | 6.98 | 4.90 | 4.22 | 3.47 | 2.84 | 2.46 | 2.30 | 2.22 | 2.16 |
| | Median | 5.34 | 4.08 | 3.52 | 2.99 | 2.52 | 2.23 | 2.12 | 2.07 | 2.04 |
| | 10%-tile | 4.18 | 3.39 | 3.01 | 2.62 | 2.25 | 2.04 | 1.98 | 1.95 | 1.93 |
| **Panel B. Value-weighted Portfolios** | | Portfolio size | | | | | | | | |
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| Shocks due to idiosyncratic risk when SIMM holds | 90%-tile | 18.29 | 15.14 | 14.13 | 12.15 | 8.66 | 5.93 | 4.76 | 4.03 | 3.54 |
| | Median | 12.15 | 9.55 | 8.43 | 6.83 | 5.33 | 4.43 | 3.77 | 3.35 | 3.04 |
| | 10%-tile | 9.26 | 7.53 | 6.42 | 5.51 | 4.26 | 3.31 | 2.92 | 2.70 | 2.50 |
| Shocks due to idiosyncratic risk when FF3M holds | 90%-tile | 17.83 | 14.37 | 13.05 | 10.94 | 8.00 | 5.67 | 4.44 | 3.73 | 3.26 |
| | Median | 12.05 | 9.44 | 8.16 | 6.83 | 5.14 | 4.21 | 3.55 | 3.12 | 2.80 |
| | 10%-tile | 9.11 | 7.38 | 6.32 | 5.43 | 4.17 | 3.24 | 2.82 | 2.48 | 2.31 |
| Shocks due to idiosyncratic risk when IEM holds | 90%-tile | 14.40 | 9.98 | 8.46 | 6.80 | 4.73 | 3.17 | 2.52 | 2.15 | 1.88 |
| | Median | 9.32 | 7.03 | 5.92 | 4.79 | 3.53 | 2.57 | 2.12 | 1.86 | 1.67 |
| | 10%-tile | 6.28 | 4.97 | 4.26 | 3.45 | 2.74 | 2.13 | 1.83 | 1.65 | 1.49 |
| Shocks due to industry risk when IEM holds | 90%-tile | 12.83 | 10.43 | 10.17 | 8.56 | 6.57 | 5.13 | 4.29 | 3.68 | 3.29 |
| | Median | 7.56 | 6.25 | 5.61 | 4.92 | 4.02 | 3.42 | 3.11 | 2.87 | 2.78 |
| | 10%-tile | 5.14 | 4.48 | 4.09 | 3.59 | 3.13 | 2.69 | 2.49 | 2.40 | 2.32 |

*Notes.* Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions for the true components of total portfolio risk are constructed by drawing 10,000 portfolios at random, then calculating the true idiosyncratic risk for each portfolio (and, in the case of the IEM, the true industry risk for each portfolio). Models: SIMM (single index market model), FF3M (Fama-French three-factor model), IEM (two-stage industry effects model). The industry effects model has two components of diversifiable risk: industry-unique risk due to the residuals in the industry equation, and idiosyncratic risk. Annualizing from the monthly rates assumes that monthly residuals are approximately independent.

That is, given the true idiosyncratic risk for portfolio $p$, the probability of a shock of magnitude equal to *or greater than* the true idiosyncratic risk is about 16 percent. Consider, for example, a portfolio with true monthly idiosyncratic risk of $\sigma_{ep} = 1\%$; annualized, the idiosyncratic risk is approximately $\sqrt{12}\sigma_{ep} = 3.46\%$. Thus, the probability of a shock of this magnitude ***or worse*** due to idiosyncratic risk in the portfolio is about 16 percent.

A subtle point when interpreting the magnitude of a given shock is that the 16 percent probability is associated with the event that the decline in the portfolio return is this magnitude ***or worse***.

Table 13 lists the magnitudes of the potential shocks due to the non-systematic risk in portfolios at the 10-percentile, median, and 90-percentile levels of the risk, given the portfolio size $N$. To put these numbers in perspective, the approximate annualized mean excess return on the U.S. total stock market index was 6.37% over July 1963–June 2018, and the approximate annualized standard deviation of the market excess return was 15.16% for the same period.

I illustrate how to read Table 13 with two examples. First, consider a 100-stock, equal-weighted portfolio with the median true idiosyncratic risk. Please see the $N = 100$ column in Panel A of Table 13. This portfolio has a 16 percent chance of a potential shock due to idiosyncratic risk of 3.54% per year under the SIMM, 3.45% per year under the FF3M, and 3.25% per year under the IEM. Moreover, because the portfolio has the ***median*** true idiosyncratic risk, the probability of this magnitude of a shock ***is greater than 16 percent for half of all portfolios*** (specificially, all portfolios with higher levels of idiosyncratic risk than the median). For these portfolios, in ***at least*** one out of every six years, the idiosyncratic risk could cause a shock equivalent to about half of the long-term average annual excess return on the market.

Next consider a 500-stock, value-weighted portfolio with the 10-percentile level of idiosyncratic risk. Please see the $N = 500$ column in Panel B of Table 13. This portfolio has a 16 percent chance of a potential shock due to idiosyncratic risk of 2.50% per year under the SIMM, 2.31% per year under the FF3M, and 1.49% per year under the IEM. (In addition, if the IEM determines returns, the portfolio has a second *independent* potential shock of 2.32% due to industry risk.) Moreover, because the portfolio has the 10-percentile true idiosyncratic risk, the probability of a shock of this magnitude ***is greater than 16 percent for 90 percent of all portfolios*** (specifically, all portfolios with higher levels of idiosyncratic risk than the 10-percentile level).

## 5. The Risk-Reward Trade-off

### 5.1. Overview

I examine the simulation results for the cross-sectional distributions of the Sharpe ratio, conditional on portfolio size. This ratio is an ordinal measure (*i.e.*, observations can be arranged from largest to smallest). In order to interpret differences between portfolios, we turn to M2, which converts the Sharpe ratio into an interval measure (*i.e.*, the numerical value of observations and differences have meaning).

In Section 5, I report results under all three asset pricing models (although, in some cases, the results under the FF3M or IEM are so similar to those under the SIMM that I make note of this fact and only report on the latter). All results concerning *estimators* of risk and reward-to-risk are based on the application of the stationary block bootstrap to generate resampled 60-month multivariate time series of the factor returns (the market factor for all three models and the SMB and HML factors under the FF3M) and the risk-free rate.

### 5.2. Market with all true alphas equal to zero

In the first pass, I set all true alphas equal to zero. This market is efficient in the sense that the expected return due to non-systematic risk is zero. This assumption does not affect the total portfolio risk or its decomposition under each asset pricing model. However, it does affect the risk-reward measures. When all alphas are zero, the true Sharpe ratio for portfolio $p$ under the SIMM becomes

$$SR_{pt} = \frac{\beta_{mpt}E\left\{r_{mt}\right\}}{\sqrt{\beta_{mpt}^2\sigma_m^2 + \sigma_{ept}^2}}. \tag{24}$$

This formula follows from Equation (7). All true betas for stocks in the simulation are positive. Thus, for any portfolio $p$ other than the market index portfolio, $m$,

$$SR_{pt} = \frac{E\left\{r_{mt}\right\}}{\sqrt{\sigma_m^2 + \sigma_{ept}^2 \Big/ \beta_{mpt}^2}} < \frac{E\left\{r_{mt}\right\}}{\sigma_m}. \tag{25}$$

(This transformation assumes that $\beta_{mpt} > 0$. Since market beta for most individual stocks is positive, the market beta for portfolios with 10 or more stocks—as in my analysis—is highly likely to satisfy this assumption.) This relation is consistent with the conclusion that the market portfolio is the optimal risky portfolio in mean–variance space. When all alphas are zero, the Sharpe ratio for portfolio $p$ under the IEM is

$$SR_{pt} = \frac{\beta_{pt} E\{r_{mt}\}}{\sqrt{\beta_{pt}^2 \sigma_m^2 + \sigma_{\varepsilon pt}^2 + \sigma_{\eta pt}^2}}. \tag{26}$$

(Please see Appendix A.3 for the derivations of the excess portfolio return and the total portfolio risk under the IEM.) For any portfolio $p$ other than $m$,

$$SR_{pt} = \frac{E\{r_{mt}\}}{\sqrt{\sigma_m^2 + \left(\sigma_{\varepsilon pt}^2 + \sigma_{\eta pt}^2\right)/\beta_{pt}^2}} < \frac{E\{r_{mt}\}}{\sigma_m}. \tag{27}$$

(This transformation assumes that $\beta_{mpt} > 0$. As for the SIMM, this assumption is highly likely to hold.)

The properties of the cross-sectional distributions of the true Sharpe ratio, conditional on $N$, are similar under the SIMM and IEM. For equal-weighted portfolios, please see Fig. 5 and the first and second sections (SIMM and IEM, respectively) of Panel A in Table 14. For value-weighted portfolios, please see Fig. 6 and the first and second sections (SIMM and IEM, respectively) of Panel B in Table 14. Whether portfolios are equal-weighted or value-weighted, the cross-sectional distributions of the true Sharpe ratios exhibit the following characteristics under the SIMM and IEM.

- The median true Sharpe ratio increases asymptotically as $N$ increases, where the asymptote is the true Sharpe ratio for the market index portfolio; $E\{r_{mt}\}/\sigma_m = 0.1212$ in the simulation based on monthly returns for July 1963–June 2018. (We anticipate this behavior based in Equation (25) for the SIMM and Equation (27) for the IEM.) Convergence is somewhat quicker when portfolios are equal-weighted.
- The 10-percentile and 90-percentile statistics for the true Sharpe ratio converge quickly to the median, *i.e.*, the dispersion shrinks quickly.
- Consistent with the theoretical results in Equations (25) and (27), the sample maximum of the true Sharpe ratio among all 10,000 portfolios (not listed in Table 14) at each $N$ is less than the true Sharpe ratio for the market portfolio.
- The 90-percentile statistic for the true Sharpe ratio for relatively undiversified portfolios (*i.e.*, small values of $N$) is less than the 10-percentile statistic for relatively diversified portfolios (*i.e.*, large values of $N$). That is, most of the relatively undiversified portfolios have lower true reward-to-risk ratios than most of the relatively diversified portfolios. For example, under either the SIMM or IEM, 90 percent of equal-weighted portfolios with $N \geq 100$ have a higher true Sharpe ratio than 90 percent of equal-weighted portfolios with 50 or fewer stocks. 90 percent of value-weighted portfolios with $N \geq 300$ have a higher true Sharpe ratio than 90 percent of equal-weighted portfolios with 50 or fewer stocks.

Estimators of the Sharpe ratio under either the SIMM or IEM perform poorly. The estimators are positively biased: the median

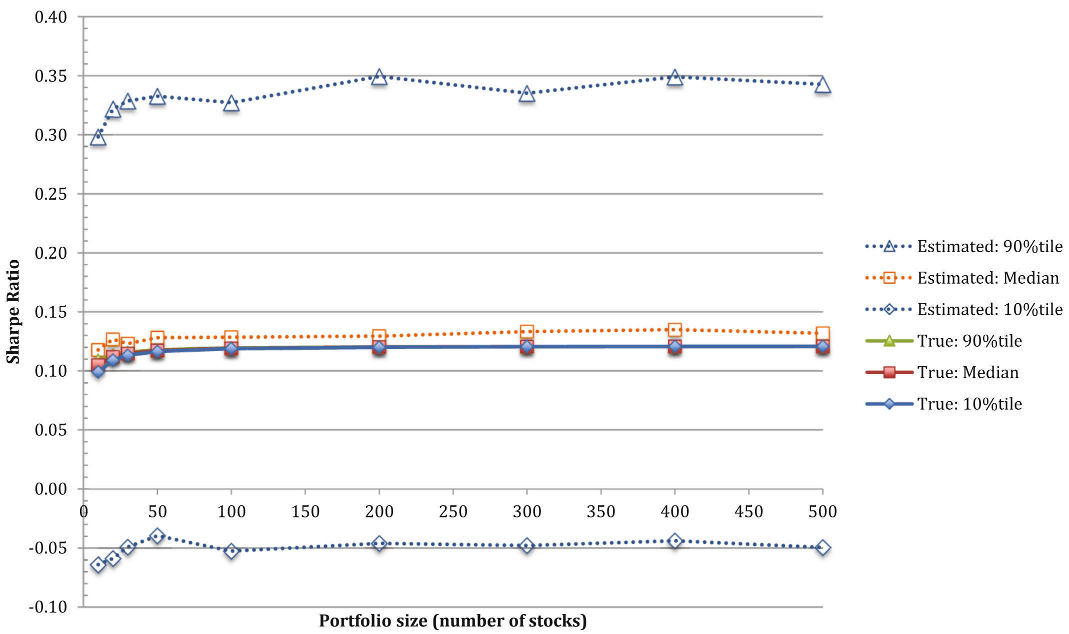


**Fig. 5.** Cross-sectional Distributions of Sharpe Ratio for Equal-weighted Portfolios When Returns Follow the Single Index Market Model and All True Alphas Are Zero.

**Table 14**
Cross-sectional Distribution of True and Estimated Sharpe Ratios When Monthly Stock Returns Follow a Specified Asset Pricing Model and All Alphas Equal Zero.

Panel A. Equal-weighted Portfolios Rebalanced Monthly

| | | Portfolio size | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True ratio when SIMM holds | 90%-tile | 0.1090 | 0.1137 | 0.1159 | 0.1178 | 0.1194 | 0.1203 | 0.1206 | 0.1207 | 0.1208 |
| | Median | 0.1049 | 0.1119 | 0.1147 | 0.1172 | 0.1191 | 0.1202 | 0.1205 | 0.1207 | 0.1208 |
| | 10%-tile | 0.0990 | 0.1091 | 0.1133 | 0.1165 | 0.1189 | 0.1201 | 0.1205 | 0.1207 | 0.1208 |
| Estimated ratio when SIMM holds | 90%-tile | 0.2982 | 0.3218 | 0.3286 | 0.3327 | 0.3272 | 0.3494 | 0.3352 | 0.3490 | 0.3426 |
| | Median | 0.1179 | 0.1267 | 0.1230 | 0.1282 | 0.1286 | 0.1294 | 0.1333 | 0.1350 | 0.1318 |
| | 10%-tile | -0.0640 | -0.0588 | -0.0491 | -0.0395 | -0.0526 | -0.0460 | -0.0479 | -0.0439 | -0.0495 |
| True ratio when IEM holds | 90%-tile | 0.1054 | 0.1114 | 0.1138 | 0.1161 | 0.1181 | 0.1192 | 0.1196 | 0.1198 | 0.1200 |
| | Median | 0.0999 | 0.1086 | 0.1120 | 0.1151 | 0.1177 | 0.1190 | 0.1195 | 0.1197 | 0.1198 |
| | 10%-tile | 0.0918 | 0.1046 | 0.1098 | 0.1138 | 0.1172 | 0.1188 | 0.1193 | 0.1196 | 0.1198 |
| Estimated ratio when IEM holds | 90%-tile | 0.2788 | 0.3113 | 0.3064 | 0.3243 | 0.3440 | 0.3332 | 0.3448 | 0.3431 | 0.3426 |
| | Median | 0.1028 | 0.1187 | 0.1188 | 0.1338 | 0.1263 | 0.1410 | 0.1356 | 0.1365 | 0.1410 |
| | 10%-tile | -0.0530 | -0.0583 | -0.0539 | -0.0543 | -0.0530 | -0.0351 | -0.0386 | -0.0355 | -0.0401 |
| True ratio when FF3M holds | 90%-tile | 0.1404 | 0.1463 | 0.1469 | 0.1483 | 0.1481 | 0.1475 | 0.1473 | 0.1468 | 0.1466 |
| | Median | 0.1250 | 0.1343 | 0.1373 | 0.1403 | 0.1425 | 0.1437 | 0.1441 | 0.1443 | 0.1444 |
| | 10%-tile | 0.1093 | 0.1203 | 0.1263 | 0.1318 | 0.1368 | 0.1399 | 0.1410 | 0.1417 | 0.1422 |
| Estimated ratio when FF3M holds | 90%-tile | 0.3114 | 0.3146 | 0.3370 | 0.3300 | 0.3382 | 0.3433 | 0.3419 | 0.3527 | 0.3420 |
| | Median | . 1336 | 0.1486 | 0.1562 | 0.1565 | 0.1599 | 0.1593 | 0.1530 | 0.1625 | 0.1572 |
| | 10%-tile | -0.0442 | -0.0153 | -0.0255 | -0.0244 | -0.0150 | -0.0118 | -0.0122 | -0.0148 | -0.0265 |

Panel B. Value-weighted Portfolios

| | | Portfolio size | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True ratio when SIMM holds | 90%-tile | 0.1055 | 0.1102 | 0.1125 | 0.1147 | 0.1169 | 0.1185 | 0.1191 | 0.1194 | 0.1197 |
| | Median | 0.0963 | 0.1032 | 0.1066 | 0.1103 | 0.1142 | 0.1167 | 0.1180 | 0.1186 | 0.1191 |
| | 10%-tile | 0.0787 | 0.0879 | 0.0923 | 0.0996 | 0.1070 | 0.1138 | 0.1163 | 0.1175 | 0.1184 |
| Estimated ratio when SIMM holds | 90%-tile | 0.2889 | 0.3003 | 0.3052 | 0.2926 | 0.3246 | 0.3260 | 0.3437 | 0.3247 | 0.3497 |
| | Median | 0.1070 | 0.1098 | 0.1172 | 0.1159 | 0.1250 | 0.1301 | 0.1283 | 0.1328 | 0.1337 |
| | 10%-tile | -0.0696 | -0.0584 | -0.0690 | -0.0599 | -0.0531 | -0.0586 | -0.0437 | -0.0461 | -0.0511 |
| True ratio when IEM holds | 90%-tile | 0.1035 | 0.1089 | 0.1114 | 0.1139 | 0.1163 | 0.1180 | 0.1187 | 0.1190 | 0.1193 |
| | Median | 0.0950 | 0.1023 | 0.1064 | 0.1103 | 0.1145 | 0.1170 | 0.1179 | 0.1185 | 0.1188 |
| | 10%-tile | 0.0780 | 0.0899 | 0.0964 | 0.1035 | 0.1102 | 0.1147 | 0.1166 | 0.1176 | 0.1183 |
| Estimated ratio when IEM holds | 90%-tile | 0.2724 | 0.2944 | 0.3116 | 0.3143 | 0.3300 | 0.3336 | 0.3331 | 0.3371 | 0.3348 |
| | Median | 0.1036 | 0.1021 | 0.1157 | 0.1233 | 0.1217 | 0.1368 | 0.1303 | 0.1392 | 0.1374 |
| | 10%-tile | -0.0621 | -0.0582 | -0.0548 | -0.0441 | -0.0497 | -0.0405 | -0.0464 | -0.0461 | -0.0377 |
| True ratio when FF3M holds | 90%-tile | 0.1341 | 0.1382 | 0.1404 | 0.1447 | 0.1480 | 0.1459 | 0.1427 | 0.1420 | 0.1408 |
| | Median | 0.1045 | 0.1107 | 0.1147 | 0.1210 | 0.1238 | 0.1270 | 0.1274 | 0.1282 | 0.1290 |
| | 10%-tile | 0.0700 | 0.0786 | 0.0851 | 0.0903 | 0.0939 | 0.1042 | 0.1103 | 0.1140 | 0.1166 |
| Estimated ratio when FF3M holds | 90%-tile | 0.2749 | 0.2758 | 0.2959 | 0.3097 | 0.3198 | 0.3412 | 0.3349 | 0.3495 | 0.3685 |
| | Median | 0.1091 | 0.1063 | 0.1212 | 0.1321 | 0.1343 | 0.1418 | 0.1442 | 0.1422 | 0.1564 |
| | 10%-tile | -0.0628 | -0.0624 | -0.0562 | -0.0391 | -0.0547 | -0.0368 | -0.0324 | -0.0244 | -0.0279 |

*Notes*. Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risk for each portfolio and the corresponding estimate. Stock and market returns are monthly in excess of the one-month Treasury bill rates. Market returns are generated from block bootstrap samples of historical returns. Excess stock returns are calculated from a specified model (SIMM: single index market model; IEM: two-stage industry effects model; and FF3M: Fama-French three-factor model) with known, stationary values of alpha, beta, and error term variance for each stock and industry, assuming a stationary lognormal distribution for each stock and industry error term. In the simulation, the true Sharpe ratio for the market index portfolio is approximately 0.1212 under both the SIMM and IEM and approximately 0.1640 under the FF3M.

values of the cross-sectional distributions of the estimated Sharpe ratio are consistently above the median of the true Sharpe ratio. The estimators also exhibit poor accuracy: the 10/90 ranges for the estimates are substantially wider than the corresponding ranges for the true values. Please see Figs. 5 and 6. In Table 14, the estimated Sharpe ratios under each asset pricing model are listed immediately below the true Sharpe ratios for the corresponding model, with results for equal-weighted portfolios in Panel A and value-weighted portfolios in Panel B. I carried out a goodness of fit test using the Kolmogorov test statistic for each scenario. The difference between the cross-sectional distributions of the true and estimated Sharpe ratio is significant at the one percent level for all portfolio sizes and regardless of asset pricing model (SIMM or IEM) or portfolio weighting.

The general behavior of the cross-sectional distributions of the true Sharpe ratio under the FF3M is similar to that when the SIMM is the underlying asset pricing model. For equal-weighted portfolios, please see the third section of Panel A in Table 14; for value-weighted portfolios, please see the third section of Panel B in Table 14. Whether portfolios are equal-weighted or value-weighted, the cross-sectional distributions of the true Sharpe ratio exhibit the following characteristics under the FF3M.

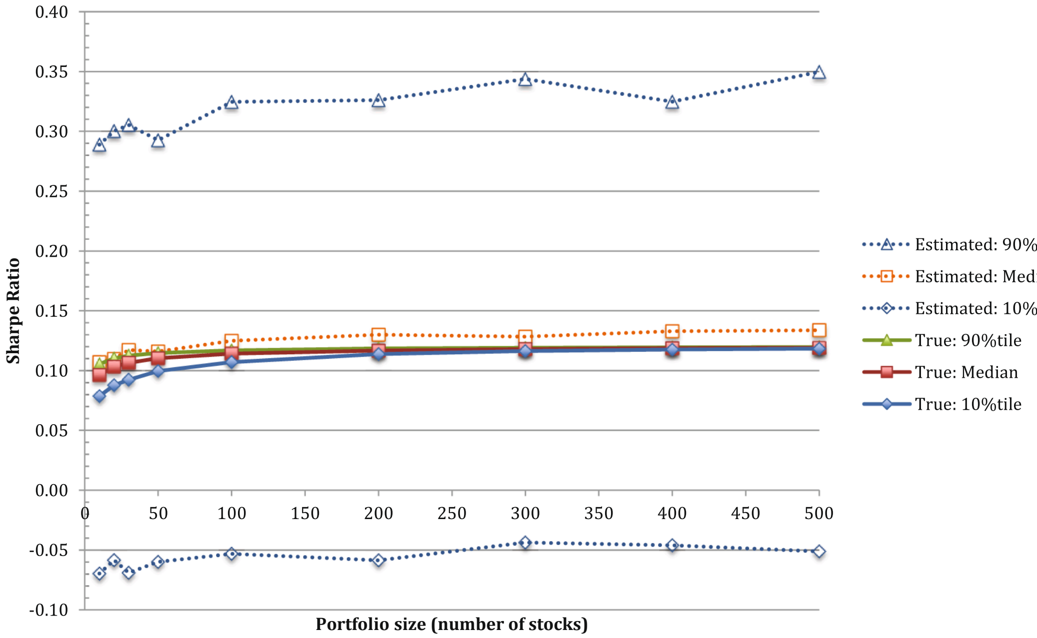


**Fig. 6.** Cross-sectional Distributions of Sharpe Ratio for Value-weighted Portfolios When Stock Returns Follow the Single Index Market Model and All True Alphas Are Zero.

- The median true Sharpe ratio increases asymptotically as $N$ increases. However, unlike when the SIMM or IEM govern returns, the asymptote for the median does not appear to be the true Sharpe ratio for the market index portfolio. Under the FF3M, $E\left\{r_{mt}\right\}/\sigma_m =$ 0.1640 in the simulation based on monthly returns for July 1963–June 2018.
- The 10-percentile and 90-percentile statistics for the true Sharpe ratio converge to the median. The dispersion shrinks quickly when portfolios are equal-weighted. However, when portfolios are value-weighted, the dispersion still is significant for $N = 500$ stocks.
- As in scenarios where the SIMM or IEM govern returns, the sample maximum of the true Sharpe ratio among all 10,000 portfolios (not listed in Table 14) at each $N$ is close to the true Sharpe ratio for the market index portfolio (and, in fact, slightly exceeds it for portfolio sizes from 30 to 200 stocks). An important difference, is that, unlike the SIMM and IEM scenarios, the 90-percentile for the true Sharpe ratio is not close to the true Sharpe ratio for the market and, in fact, decreases for $N$ greater than 400 stocks when portfolios are equal-weighted and for $N$ greater than 200 stocks when portfolios are value-weighted.
- Unlike the scenarios when the SIMM or IEM govern returns, the 10/90 ranges for the true Sharpe ratios for relatively undiversified portfolios (*i.e.*, small values of $N$) overlap with the 10/90 ranges for relatively diversified portfolios (*i.e.*, large values of $N$).

**Table 15**
Correlations Between True and Estimated Sharpe Ratios When Monthly Stock Returns Follow a Specified Asset Pricing Model and All Alphas Equal Zero.

| Panel A. Equal-weighted Portfolios | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| SIMM | 0.0682* | 0.0350* | 0.0015 | 0.0262* | 0.0113 | -0.0245* | 0.0295* | -0.0517* | -0.0229* |
| IEM | 0.0495* | 0.0651* | 0.0131 | -0.0149 | 0.0378* | -0.0017 | -0.0260* | -0.0355* | -0.0098 |
| FF3M | 0.1414* | 0.1512* | 0.0936* | 0.0595* | 0.0935* | 0.0219* | 0.0274* | 0.0808* | 0.0646* |
| Panel B. Value-weighted Portfolios | | | | | | | | | |
| Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| SIMM | 0.1262* | 0.1757* | 0.0610* | 0.0680* | 0.0015 | 0.0212* | -0.0128 | -0.0082 | -0.0329* |
| IEM | 0.0330* | 0.1199* | 0.0172 | 0.0283* | 0.0094 | 0.0265* | -0.0253* | 0.0828* | -0.0264* |
| FF3M | 0.1666* | 0.1956* | 0.1990* | 0.1620* | 0.1199* | 0.1003* | 0.1172* | 0.0590* | 0.0417* |

*Notes*. Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risk for each portfolio and the corresponding estimate. Stock and market returns are monthly in excess of the one-month Treasury bill rates. Market returns are generated from block bootstrap samples of historical returns. Excess stock returns are calculated from the specified asset pricing model with known, stationary values of alpha, beta, and error term variance for each stock (and for each industry) and assuming a stationary lognormal distribution for each stock's error term (and for each industry's error term). Models: SIMM (single index market model), IEM (two-stage industry effects model), and FF3M (Fama-French three-factor model). The Pearson approximation to the standard deviation of the correlation, given 10,000 observations, is approximately 0.01. Sample correlations more than two standard deviations from zero are flagged with *.

Nonetheless, the 10-percentile statistic rises substantially as $N$ increases, especially for equal-weighted portfolios. Thus, the risk that a randomly selected portfolio will have a Sharpe ratio well below that of the market index portfolio decreases as $N$ increases.

- Estimators of the Sharpe ratio under the FF3M perform poorly. As is the case under the SIMM and IEM, these estimators are positively biased and exhibit poor accuracy.

In order to evaluate whether the *estimated* Sharpe ratios preserve the ranking of portfolios based on the true Sharpe ratios, I examined the correlations between estimates and true values at each portfolio size. Please see Table 15. While most of the correlations are significantly different from zero in a statistical sense, the magnitude of the correlation is small: less than 0.20 in all cases. When the SIMM and IEM govern the stock returns, the correlations are not consistently positive or negative. Overall, these results indicate that the *estimated* Sharpe ratios do not consistently preserve the true ranking of portfolios in terms of the reward-to-risk ratio. When the underlying model is the FF3M, the correlations are positive for both equal- and value-weighted portfolios and all values of $N$. However, the low magnitude of the correlations indicates that the *estimated* Sharpe ratios do not strongly preserve the true portfolio ranking at each portfolio size.

### 5.3. True Sharpe ratios interpreted as risk-adjusted rates of return

M2 converts the Sharpe ratio into a more easily interpreted risk-adjusted rate of return. Table 16 displays the true values for M2 under the assumption that all alphas are zero. The statistics correspond to the true Sharpe ratios in Table 14. Diversification (in the sense of increasing $N$) improves the odds that a portfolio selected at random will have a risk-adjusted return close to that of the market.

For example, consider markets where returns are determined by the SIMM or the IEM.

- At least 90 percent of equal-weighted portfolios with 200 stocks or more have M2 that differs from that of the optimal portfolio (*i.e.*, the market index portfolio) by 0.2% or less in return. Please see the first two sections of Panel A in Table 16.
- By comparison, at least 90 percent of equal-weighted portfolios with 30 stocks or less have M2 that differs from the optimal value by approximately 0.3% or more in annual returns.
- At least 50 percent or more of value-weighted portfolios with 300 stocks or more have M2 that differs from that of the optimal portfolio by 0.2% or less in return. Please see the first two sections of Panel B in Table 16.
- By comparison, at least 90 percent of value-weighted portfolios with 100 stocks or fewer have M2 that differs from the optimal M2 by more than 0.2% in annual returns.

**Table 16**

Modigliani Risk-adjusted Performance Measure (M2) as a Function of Portfolio Size In Terms of Annualized Returns (%) When True Alphas Equal Zero.

| Panel A. Equal-weighted Portfolios Rebalanced Monthly | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 when SIMM holds | 90%-tile | 10.34 | 10.58 | 10.69 | 10.80# | 10.88# | 10.93# | 10.94* | 10.95* | 10.95* |
| | Median | 10.12 | 10.49 | 10.64 | 10.77 | 10.87# | 10.92# | 10.94* | 10.95* | 10.95* |
| | 10%-tile | 9.81 | 10.34 | 10.56 | 10.73 | 10.85# | 10.92# | 10.94* | 10.95* | 10.95* |
| True M2 when IEM holds | 90%-tile | 10.15 | 10.46 | 10.58 | 10.71 | 10.81# | 10.87# | 10.89* | 10.90* | 10.91* |
| | Median | 9.86 | 10.31 | 10.49 | 10.65 | 10.79# | 10.86# | 10.89* | 10.90* | 10.91* |
| | 10%-tile | 9.43 | 10.10 | 10.37 | 10.58 | 10.76 | 10.85# | 10.88* | 10.89* | 10.90* |
| True M2 when FF3M holds | Maximum | 12.90 | 13.17* | 13.01* | 13.17* | 12.96 | 12.76 | 12.59 | 12.51 | 12.53 |
| | 90%-tile | 11.98 | 12.30 | 12.33 | 12.40 | 12.39 | 12.36 | 12.34 | 12.32 | 12.31 |
| | Median | 11.18 | 11.66 | 11.82 | 11.98 | 12.09 | 12.16 | 12.18 | 12.19 | 12.19 |
| | 10%-tile | 10.35 | 10.93 | 11.25 | 11.53 | 11.79 | 11.96 | 12.01 | 12.05 | 12.08 |
| Panel B. Value-weighted Portfolios | | | | | | | | | | |
| | Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 when SIMM holds | 90%-tile | 10.15 | 10.40 | 10.52 | 10.64 | 10.75 | 10.83# | 10.86# | 10.88* | 10.90* |
| | Median | 9.67 | 10.03 | 10.21 | 10.40 | 10.61 | 10.74 | 10.80# | 10.84# | 10.86# |
| | 10%-tile | 8.74 | 9.22 | 9.46 | 9.84 | 10.23 | 10.58 | 10.72 | 10.78# | 10.83# |
| True M2 when IEM holds | 90%-tile | 10.05 | 10.33 | 10.46 | 10.59 | 10.72 | 10.81# | 10.84# | 10.86# | 10.87# |
| | Median | 9.60 | 9.98 | 10.20 | 10.40 | 10.62 | 10.75 | 10.80# | 10.83# | 10.85# |
| | 10%-tile | 8.71 | 9.33 | 9.67 | 10.04 | 10.40 | 10.64 | 10.73 | 10.79# | 10.82# |
| True M2 when FF3M holds | Maximum | 13.18* | 13.22* | 13.26* | 13.32* | 13.51* | 13.47* | 13.12* | 12.77 | 12.90 |
| | 90%-tile | 11.65 | 11.87 | 11.99 | 12.21 | 12.39 | 12.27 | 12.10 | 12.07 | 12.01 |
| | Median | 10.10 | 10.42 | 10.64 | 10.96 | 11.11 | 11.28 | 11.30 | 11.34 | 11.39 |
| | 10%-tile | 8.29 | 8.73 | 9.08 | 9.35 | 9.54 | 10.08 | 10.40 | 10.60 | 10.73 |

*Notes*. Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risk for each portfolio. Models: SIMM (single index market model), FF3M (Fama-French three-factor model), IEM (two-stage industry effects model). M2 is the expected return on a portfolio that is levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the market portfolio. Statistics reported in this table are for the cross-sectional distributions of the true values of M2 in the simulation. The maximum at each portfolio size is the sample maximum for the 10,000 portfolios drawn. The true maximum is M2 for the market index portfolio, $m$. In the simulation, when the SIMM or IEM govern stock returns, $M2_m = 10.98\%$; when the FF3M governs stock returns, $M2_m = 13.22\%$. * = M2 differs from this optimum by 0.10% or less in annualized return (or, in the case of the FF3M, slightly exceeds it). # = M2 differs from this optimum by between 0.10% and 0.20% in annualized return.

Diversification also improves the odds that the investor will avoid a portfolio with relatively low risk-adjusted return. Under all three asset pricing models considered in this paper, the 10-percentile statistic for the true M2 rises substantially as *N* increases. Please see the last row in each section of Panel A and Panel B of Table 16. For example, when the SIMM holds and portfolios are value-weighted, the floor on the true annualized M2 for 90 percent of 10-stock portfolios is 8.74%, and this floor rises to 10.83% for 90 percent of 500-stock portfolios.

Differences in risk-adjusted rates of return in Table 16 do not appear to be large. However, small differences in annual rates translate into large differences in ending wealth over long holding periods. For example, for value-weighted portfolios under the SIMM, consider the difference between the 90 percentile for 30-stock portfolios (10.52%) versus the 10 percentile for 500-stock portfolios (10.83%). A representative dollar invested for 30 years at 10.52% is worth \$20.10, while the same dollar invested for 30 years at 10.83% is worth \$21.86. Thus, ending wealth on most (specifically, 90 percent) of the 500-stock portfolios is ***at least*** 8.8% greater than ending wealth for most (specifically, 90 percent) of the 30-stock portfolios.

### *5.4. Market with non-zero alphas*

The least squares estimates of alphas for the real-world counterparts of the simulated stocks (and industry alphas in the case of the IEM) are not zero. Please see Table 17. For example, if the SIMM is the asset pricing model, then the real-world estimates of monthly alphas range from −2.555% to 3.805%, and 69.8 percent of the stocks have a positive alpha. However, only 3.75 percent of these alphas are more than two standard errors away from zero. The alpha estimates for individual stocks when the FF3M or IEM govern returns follow a similar pattern. In addition, under all three models, the correlation between estimates for the alphas of individual stocks and the corresponding residual mean squares are positive, and the difference from zero is statistically significant at the five percent level.

These results are contradictory. On the one hand, the estimated real-world alphas are not statistically different from zero. This result is consistent with the hypothesis that the true alphas of the real-world counterparts are zero. On the other hand, the positive correlation between the estimates of the real-world alphas and the real-world idiosyncratic risk is consistent with the market rewarding investors for taking on this risk.

In an effort to resolve this contradiction, I perform a simulation analysis to see if the inverse relation between the true Sharpe ratio and true idiosyncratic risk as *N* increases changes when we shift to a simulated market in which more than two-thirds of the stock alphas are positive. I repeat the previous simulation runs but now set the true alphas equal to the estimates for their real-world counterparts. For each scenario (*i.e.*, the choice of asset pricing model and portfolio weighting method) the simulator draws exactly the same portfolios as before. Thus, we can make pairwise comparisons of risk-reward measures for the same portfolios when true alphas are zero versus when true alphas are non-zero.

The inverse relation between the true Sharpe ratio and true idiosyncratic risk still is present when we use the non-zero values of the alphas. The cross-sectional distributions of the true Sharpe ratio shift upward and dispersion decreases as *N* increases, while the cross-sectional distributions of the true idiosyncratic risk shift downward and dispersion decreases as *N* increases. Please see Table 18. Panels A and B list results for equal-weighted and value-weighted portfolios, respectively, when returns are generated using the SIMM. The pattern under the FF3M and IEM is similar; hence, those results are not reported heare. In short, even if alphas are non-zero (and consistent with real-world estimates), the evidence is not consistent with a market in which non-systematic risk is rewarded.

If investors are compensated for non-systematic risk, then the true M2 should be higher when this risk is greater, all else equal. I rank each sample of 10,000 portfolios by systematic risk, partition the ranked portfolios into deciles, and calculate the Spearman's Rho statistic to test the null hypothesis that M2 and non-systematic risk are mutually independent versus the alternative that they are positively correlated. Under the SIMM and FF3M, the non-systematic risk is the idiosyncratic risk; under the IEM, it consists of

**Table 17**
Estimated Alphas for Real-world Stocks and Industries Emulated in the Simulation (Estimated from Excess Monthly Total Returns Over 2007–2016; values in %).

| Asset pricing model | Min | 25%-tile | Median | 75%-tile | Max | Correlation between alpha and RMS | % with alpha greater than zero | % with alpha estimate greater than 2 standard errors from zero |
|---|---|---|---|---|---|---|---|---|
| SIMM: stock alphas | −2.555 | −0.100 | 0.296 | 0.692 | 3.805 | 0.1587 | 69.8 | 3.75 |
| FF3M: stock alphas | −2.243 | −0.046 | 0.339 | 0.704 | 3.981 | 0.1942 | 72.3 | 3.96 |
| IEM: industry alphas | −1.785 | −0.218 | 0.065 | 0.305 | 1.398 | −0.4230 | 57.0 | 4.76 |
| IEM: stock alphas | −2.775 | −0.032 | 0.313 | 0.715 | 1.195 | 0.2540 | 72.9 | 4.10 |

*Notes*. Models: SIMM (single index market model), FF3M (Fama-French three-factor model), IEM (two-stage industry effects model). Alpha estimates are for the real-world stocks and industries emulated in the simulation. In the case of the SIMM and FF3M, alphas and their standard deviations are estimated by applying linear regression methods for their respective linear models. In the case of the IEM, the industry and stock alphas are estimated by applying linear regression methods for the respective equation in the two-stage industry effects model. All four correlations between estimated alphas and the residual mean square (RMS) are significantly different from zero at the 5 percent level. The monthly excess total returns for stocks and industries are for the 120 months in 2007–2016. The database consists of 1,465 common stocks listed on U.S. stock exchanges with complete monthly returns over this period.

**Table 18**
Cross-sectional Distributions of True Sharpe Ratio and True Idiosyncratic Risk as a Function of Portfolio Size In Terms of Monthly Returns (%) Where Returns Are Generated with the Single Index Market Model and True Alphas Equal the Estimated Values of the Real-world Counterparts for Simulated Stocks and Industries.

| Panel A. Equal-weighted Portfolios Rebalanced Monthly | | | | | | |
|---|---|---|---|---|---|---|
| | Sharpe Ratio | | | Idiosyncratic Risk | | |
| Portfolio size | 10 percentile | Median | 90 percentile | 10 percentile | Median | 90 percentile |
| 10 | 0.1133 | 0.1540 | 0.1980 | 2.59 | 3.14 | 3.94 |
| 20 | 0.1313 | 0.1643 | 0.2019 | 1.96 | 2.25 | 2.63 |
| 30 | 0.1402 | 0.1694 | 0.1992 | 1.65 | 1.85 | 2.10 |
| 50 | 0.1488 | 0.1734 | 0.1973 | 1.31 | 1.44 | 1.58 |
| 100 | 0.1584 | 0.1757 | 0.1932 | 0.96 | 1.02 | 1.10 |
| 200 | 0.1656 | 0.1768 | 0.1895 | 0.69 | 0.72 | 0.76 |
| 300 | 0.1672 | 0.1775 | 0.1868 | 0.57 | 0.59 | 0.62 |
| 400 | 0.1706 | 0.1777 | 0.1854 | 0.50 | 0.51 | 0.53 |
| 500 | 0.1712 | 0.1779 | 0.1848 | 0.45 | 0.46 | 0.47 |
| Panel B. Value-weighted Portfolios | | | | | | |
| | Sharpe Ratio | | | Idiosyncratic Risk | | |
| Portfolio size | 10 percentile | Median | 90 percentile | 10 percentile | Median | 90 percentile |
| 10 | 0.1063 | 0.1604 | 0.2321 | 2.67 | 3.51 | 5.28 |
| 20 | 0.1235 | 0.1750 | 0.2473 | 2.17 | 2.76 | 4.37 |
| 30 | 0.1307 | 0.1820 | 0.2486 | 1.85 | 2.43 | 4.08 |
| 50 | 0.1330 | 0.1905 | 0.2489 | 1.59 | 1.97 | 3.51 |
| 100 | 0.1549 | 0.2002 | 0.2540 | 1.23 | 1.54 | 2.50 |
| 200 | 0.1704 | 0.2069 | 0.2459 | 0.96 | 1.28 | 1.71 |
| 300 | 0.1769 | 0.2080 | 0.2426 | 0.84 | 1.09 | 1.37 |
| 400 | 0.1843 | 0.2112 | 0.2408 | 0.78 | 0.97 | 1.16 |
| 500 | 0.1912 | 0.2119 | 0.2366 | 0.72 | 0.88 | 1.02 |

*Notes*. Portfolio size is the number of stocks in the portfolio. Results when the underlying asset pricing model is the Fama-French three-factor model (FF3M) or the two-stage industry effects model are similar to those when the underlying model is the single index market model (SIMM) and hence are not reported here.

industry-unique and idiosyncratic risk.

I report results for cases when the SIMM governs the stock returns. (Correlations when either the FF3M or the IEM govern returns exhibit the same general pattern and hence are not reported here.) When portfolios are equal-weighted, the correlation is positive and statistically significant for portfolios with 20 or more stocks. Please see Table 19. Within each decile for systematic risk, the correlation tends to improve as $N$ increases.

When portfolios are value-weighted, the correlation increases dramatically as $N$ increases. Please see Table 20. For portfolios with 50 or fewer stocks, the correlation either is negative or, when positive, is not statistically different from zero. For portfolios of 200 stocks or more, the positive correlations are highly significant across all deciles of systematic risk.

These results suggest a puzzling outcome: the market appears to compensate investors for the diversifiable risk, ***provided that the investor holds a sufficiently diversified portfolio***. Moreover, the closer that the portfolio is to the market portfolio in terms of size ($N$) and portfolio weighting (*i.e.*, value weighting rather than equal weighting), the higher the correlation between the true risk-adjusted return and the true diversifiable risk. These results are not consistent with Levy (1978) and Merton (1987), who argue that idiosyncratic risk should affect asset prices if investors hold *undiversified* portfolios. But if that were the case, then M2 and idiosyncratic risk should be at least as strongly correlated for small portfolios as large ones. The evidence reported in Tables 19 and 20 is not consistent with this hypothesis.

Given these results, I look to see how the characteristics of the market portfolio in the simulated *inefficient* market (non-zero alphas) differ from characteristics of the market portfolio in an efficient market (all alphas zero). I treat the value-weighted portfolio of the 1,465 stocks in this study as a proxy for the market portfolio and label it as the Total Market Portfolio (TMP). This list of stocks is roughly the same as the S&P 1500. S&P Dow Jones Indices (2020) reports that the S&P 1500 constitutes about 90% of market capitalization in the U.S. stock markets. Thus, the TMP is a plausible proxy for the market.

I use the total stock market index from the Kenneth R. French Data Library (2019) for the monthly excess market returns. For July 1963–June 2018, the Sharpe ratio for this market index is 0.1212, and the annualized M2 is 10.98%. By comparison, under the SIMM, the TMP has a Sharpe ratio of 0.1204 and annualized M2 of 10.94% when the simulation is run with all alphas set equal to zero. These values are slightly below the corresponding statistics for the market. In other words, when true alphas are all zero, the TMP behaves like the true total stock market index. Please see Table 21 for details about the properties of the TMP under each asset pricing model.

When I set the true alphas of the simulated stocks equal to the estimated non-zero values of their real-world counterparts, the TMP has a Sharpe ratio of 0.2160 and annualized M2 of 15.95% under the SIMM. Moreover, the TMP has annualized idiosyncratic risk of approximately 1.805%. In the simulation, the true alphas do not change (as they might in the real world if non-zero alphas were arbitraged away). When non-zero alphas are persistent, M2 for the TMP is higher than its M2 when the simulated market is efficient; please see Table 21. However, we must be cautious about concluding that this difference represents compensation for idiosyncratic risk, given that the cross-sectional distributions of the true Sharpe ratio and the true idiosyncratic risk shift in opposite directions as $N$ increases.

**Table 19**
Correlation Between True Modigliani Risk-adjusted Performance Measure (M2) and True Idiosyncratic Risk: Equal-weighted Portfolios and Returns Generated by the Single Index Market Model (Correlation Measured by Spearman's Rho).

| Systematic risk decile | Portfolio size (number of stocks in the portfolio) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| 10 | Rho | 0.223 | 0.276 | 0.347 | 0.305 | 0.345 | 0.354 | 0.275 | 0.274 | 0.289 |
| | p-val. | (0.013) | (0.003) | (0.000) | (0.001) | (0.000) | (0.000) | (0.003) | (0.003) | (0.002) |
| | *Beta* | *1.48–2.08* | *1.40–1.83* | *1.38–1.62* | *1.34–1.60* | *1.31–1.49* | *1.29–1.43* | *1.28–1.35* | *1.28–1.34* | *1.27–1.32* |
| 9 | Rho | 0.251 | 0.304 | 0.302 | 0.302 | 0.360 | 0.342 | 0.378 | 0.379 | 0.370 |
| | p-val. | (0.006) | (0.001) | (0.001) | (0.001) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.39–1.48* | *1.35–1.40* | *1.33–1.38* | *1.31–1.34* | *1.29–1.31* | *1.28–1.29* | *1.27–1.28* | *1.27–1.28* | *1.26–1.27* |
| 8 | Rho | 0.253 | 0.268 | 0.344 | 0.351 | 0.376 | 0.391 | 0.354 | 0.383 | 0.358 |
| | p-val. | (0.006) | (0.004) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.33–1.39* | *1.31–1.35* | *1.30–1.33* | *1.28–1.31* | *1.27–1.29* | *1.27–1.28* | *1.26–1.27* | *1.26–1.27* | *1.26–1.26* |
| 7 | Rho | 0.220 | 0.277 | 0.303 | 0.358 | 0.354 | 0.341 | 0.360 | 0.362 | 0.339 |
| | p-val. | (0.014) | (0.003) | (0.001) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.28–1.33* | *1.27–1.31* | *1.27–1.30* | *1.26–1.28* | *1.26–1.27* | *1.25–1.27* | *1.25–1.26* | *1.25–1.26* | *1.25–1.26* |
| 6 | Rho | 0.224 | 0.312 | 0.266 | 0.331 | 0.372 | 0.349 | 0.357 | 0.349 | 0.367 |
| | p-val. | (0.013) | (0.001) | (0.004) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.24–1.28* | *1.24–1.27* | *1.24–1.27* | *1.24–1.26* | *1.24–1.26* | *1.24–1.25* | *1.25–1.25* | *1.25–1.25* | *1.25–1.25* |
| 5 | Rho | 0.146 | 0.260 | 0.269 | 0.326 | 0.356 | 0.337 | 0.341 | 0.379 | 0.379 |
| | p-val. | (0.073) | (0.005) | (0.004) | (0.001) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.19–1.24* | *1.21–1.24* | *1.22–1.24* | *1.22–1.24* | *1.23–1.24* | *1.24–1.24* | *1.24–1.25* | *1.24–1.25* | *1.24–1.25* |
| 4 | Rho | 0.142 | 0.213 | 0.312 | 0.354 | 0.380 | 0.360 | 0.376 | 0.359 | 0.428 |
| | p-val. | (0.078) | (0.017) | (0.001) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.15–1.19* | *1.18–1.21* | *1.19–1.22* | *1.20–1.22* | *1.22–1.23* | *1.23–1.24* | *1.23–1.24* | *1.23–1.24* | *1.23–1.24* |
| 3 | Rho | 0.132 | 0.198 | 0.272 | 0.319 | 0.333 | 0.343 | 0.359 | 0.336 | 0.352 |
| | p-val. | (0.094) | (0.025) | (0.003) | (0.001) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.10–1.15* | *1.14–1.18* | *1.16–1.19* | *1.18–1.20* | *1.20–1.22* | *1.21–1.23* | *1.22–1.23* | *1.22–1.23* | *1.23–1.23* |
| 2 | Rho | 0.157 | 0.209 | 0.249 | 0.294 | 0.342 | 0.326 | 0.359 | 0.339 | 0.333 |
| | p-val. | (0.059) | (0.019) | (0.007) | (0.002) | (0.000) | (0.001) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.03–1.10* | *1.09–1.14* | *1.12–1.16* | *1.15–1.18* | *1.18–1.20* | *1.20–1.21* | *1.21–1.22* | *1.21–1.22* | *1.22–1.23* |
| 1 | Rho | 0.070 | 0.166 | 0.238 | 0.188 | 0.283 | 0.257 | 0.294 | 0.311 | 0.289 |
| | p-val. | (0.242) | (0.050) | (0.009) | (0.030) | (0.002) | (0.005) | (0.002) | (0.001) | (0.002) |
| | *Beta* | *0.66–1.03* | *0.80–1.09* | *0.88–1.12* | *0.94–1.15* | *1.06–1.18* | *1.11–1.20* | *1.15–1.21* | *1.14–1.21* | *1.17–1.22* |

*Notes*. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risks and expected return for each portfolio. M2 is the expected return on a portfolio that is levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the benchmark portfolio. In the simulation, I treat the market portfolio as the benchmark. P-values for Spearman's Rho are listed in parentheses; the hypothesis test is a one-tailed test for positive correlation between the true M2 and the true idiosyncratic risk for portfolios of *N* stocks within each systematic risk decile.

**Table 20**

Correlation Between True Modigliani Risk-adjusted Performance Measure (M2) and True Idiosyncratic Risk: Value-weighted Portfolios and Returns Generated by the Single Index Market Model (Correlation Measured by Spearman's Rho).

| Systematic risk decile | Portfolio size (number of stocks in the portfolio) | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | Rho | −0.489 | −0.599 | −0.633 | −0.499 | −0.097 | 0.314 | 0.499 | 0.587 | 0.660 |
| | p-val. | (1.000) | (1.000) | (1.000) | (1.000) | (0.833) | (0.001) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.44–2.50* | *1.34–2.31* | *1.30–2.08* | *1.25–1.79* | *1.19–1.53* | *1.14–1.39* | *1.12–1.27* | *1.11–1.24* | *1.10–1.20* |
| 9 | Rho | −0.122 | −0.179 | −0.178 | −0.167 | 0.080 | 0.393 | 0.612 | 0.591 | 0.652 |
| | p-val. | (0.888) | (0.962) | (0.962) | (0.951) | (0.213) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.30–1.44* | *1.23–1.34* | *1.20–1.30* | *1.17–1.25* | *1.14–1.19* | *1.11–1.14* | *1.09–1.12* | *1.08–1.11* | *1.08–1.10* |
| 8 | Rho | −0.219 | −0.051 | −0.018 | 0.104 | 0.247 | 0.472 | 0.546 | 0.639 | 0.666 |
| | p-val. | (0.985) | (0.693) | (0.569) | (0.151) | (0.007) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.22–1.30* | *1.17–1.23* | *1.15–1.20* | *1.12–1.17* | *1.10–1.14* | *1.08–1.11* | *1.07–1.09* | *1.07–1.08* | *1.06–1.08* |
| 7 | Rho | −0.100 | 0.093 | 0.132 | 0.150 | 0.261 | 0.475 | 0.555 | 0.631 | 0.670 |
| | p-val. | (0.840) | (0.178) | (0.094) | (0.068) | (0.005) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.15–1.22* | *1.11–1.17* | *1.10–1.15* | *1.08–1.12* | *1.07–1.10* | *1.06–1.08* | *1.06–1.07* | *1.05–1.07* | *1.05–1.06* |
| 6 | Rho | −0.022 | 0.050 | 0.109 | 0.096 | 0.267 | 0.456 | 0.563 | 0.623 | 0.671 |
| | p-val. | (0.585) | (0.311) | (0.138) | (0.170) | (0.004) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.09–1.15* | *1.06–1.11* | *1.05–1.10* | *1.05–1.08* | *1.04–1.07* | *1.04–1.06* | *1.04–1.06* | *1.04–1.05* | *1.04–1.05* |
| 5 | Rho | −0.056 | −0.007 | 0.069 | 0.086 | 0.235 | 0.430 | 0.567 | 0.602 | 0.681 |
| | p-val. | (0.711) | (0.530) | (0.246) | (0.196) | (0.010) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *1.02–1.09* | *1.01–1.06* | *1.01–1.05* | *1.01–1.05* | *1.01–1.04* | *1.02–1.04* | *1.03–1.04* | *1.03–1.04* | *1.03–1.04* |
| 4 | Rho | −0.054 | −0.037 | 0.014 | 0.098 | 0.262 | 0.460 | 0.574 | 0.662 | 0.624 |
| | p-val. | (0.705) | (0.645) | (0.444) | (0.164) | (0.005) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *0.95–1.02* | *0.96–1.01* | *0.96–1.01* | *0.97–1.01* | *0.99–1.01* | *1.00–1.02* | *1.01–1.03* | *1.02–1.03* | *1.02–1.03* |
| 3 | Rho | −0.040 | −0.074 | −0.025 | 0.038 | 0.092 | 0.423 | 0.527 | 0.576 | 0.641 |
| | p-val. | (0.655) | (0.769) | (0.599) | (0.351) | (0.179) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *0.87–0.95* | *0.89–0.96* | *0.91–0.96* | *0.93–0.97* | *0.95–0.99* | *0.98–1.00* | *0.99–1.01* | *1.00–1.02* | *1.01–1.02* |
| 2 | Rho | −0.174 | −0.161 | −0.162 | 0.009 | 0.073 | 0.352 | 0.491 | 0.552 | 0.607 |
| | p-val. | (0.958) | (0.945) | (0.946) | (0.464) | (0.233) | (0.000) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *0.75–0.87* | *0.80–0.89* | *0.83–0.91* | *0.86–0.93* | *0.91–0.95* | *0.95–0.98* | *0.97–0.99* | *0.98–1.00* | *0.99–1.01* |
| 1 | Rho | −0.337 | −0.400 | −0.334 | −0.247 | −0.078 | 0.230 | 0.413 | 0.484 | 0.530 |
| | p-val. | (1.000) | (1.000) | (1.000) | (0.993) | (0.781) | (0.011) | (0.000) | (0.000) | (0.000) |
| | *Beta* | *0.35–0.75* | *0.50–0.80* | *0.54–0.83* | *0.65–0.86* | *0.68–0.91* | *0.80–0.95* | *0.84–0.97* | *0.87–0.98* | *0.88–0.99* |

*Notes*. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risks and expected return for each portfolio. M2 is the expected return on a portfolio that is levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the benchmark portfolio. In the simulation, I treat the market portfolio as the benchmark. P-values for Spearman's Rho are listed in parentheses; the hypothesis test is a one-tailed test for positive correlation between M2 and idiosyncratic risk for portfolios of $N$ stocks within each systematic risk decile.

**Table 21**
Properties of a Value-weighted Portfolio of the 1,465 Stocks in the Simulation.

| | | | | |
|---|---|---|---|---|
| Panel A. Properties Under the SIMM | | | | |
| | Stock alphas set equal to zero | | Stock alphas set equal to real-world estimates | |
| | Monthly (%) | Annualized (%) | Monthly (%) | Annualized (%) |
| Portfolio alpha | 0.0000 | 0.000 | 0.4384 | 5.261 |
| Portfolio market beta | 1.0416 | | 1.0416 | |
| Portfolio idiosyncratic risk | 0.5209 | 1.805 | 0.5209 | 1.805 |
| Expected excess return | 0.5528 | 6.633 | 0.9912 | 11.894 |
| Expected total return | 0.9368 | 11.241 | 1.3752 | 16.502 |
| Sharpe ratio | 0.1204 | | 0.2160 | |
| M2 | 0.9113 | 10.935 | 1.3295 | 15.954 |
| Panel B. Properties Under the IEM | | | | |
| | Stock alphas set equal to zero | | Stock alphas set equal to real-world estimates | |
| | Monthly (%) | Annualized (%) | Monthly (%) | Annualized (%) |
| Portfolio alpha | 0.0000 | 0.000 | 0.4112 | 4.934 |
| Portfolio market beta | 1.0625 | | 1.0625 | |
| Portfolio industry risk | 0.6386 | 2.212 | 0.6386 | 2.212 |
| Portfolio idiosyncratic risk | 0.2842 | 0.985 | 0.2842 | 0.985 |
| Portfolio diversifiable risk | 0.6990 | 2.421 | 0.6990 | 2.421 |
| Expected excess return | 0.5638 | 6.766 | 0.9750 | 11.700 |
| Expected total return | 0.9478 | 11.374 | 1.3590 | 16.308 |
| Sharpe ratio | 0.1199 | | 0.2073 | |
| M2 | 0.9088 | 10.906 | 1.2915 | 15.498 |
| Panel C. Properties Under the FF3M | | | | |
| | Stock alphas set equal to zero | | Stock alphas set equal to real-world estimates | |
| | Monthly (%) | Annualized (%) | Monthly (%) | Annualized (%) |
| Portfolio alpha | 0.0000 | 0.000 | 0.4603 | 5.523 |
| Portfolio market beta | 1.0225 | | 1.0225 | |
| Portfolio SMB beta | −0.0100 | | −0.0100 | |
| Portfolio HMB beta | 0.1047 | | 0.1047 | |
| Portfolio idiosyncratic risk | 0.4858 | 1.683 | 0.4858 | 1.683 |
| Expected excess return | 0.5748 | 6.897 | 1.0350 | 12.421 |
| Expected total return | 0.9588 | 11.505 | 1.4190 | 17.029 |
| Sharpe ratio | 0.1298 | | 0.2337 | |
| M2 | 0.9522 | 11.427 | 1.4073 | 16.887 |

*Notes*. Models: SIMM (single index market model), IEM (two-stage industry effects model), FF3M (Fama-French three-factor model). M2 is the expected return on a portfolio that is levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the benchmark portfolio. Stock and industry parameters are estimated from monthly historical returns over 2007–2016 using least squares regression. These parameters are then treated as the true parameter values for the simulated stocks and industries. Market value weights are calculated from the float-adjusted market capitalization of the corresponding real-world stocks at the end of 2016.

Let's examine the true M2 in the scenarios where the simulated true alphas are non-zero. True M2 for the TMP serves as a performance benchmark in the sense that the investor could simply invest in the TMP as a proxy for the market. When portfolios are equal-weighted, the vast majority of portfolios have lower risk-adjusted total return than the TMP. Please see the fifth row in each panel in Table 22. Moreover, if stock returns are governed by either the SIMM or FF3M, then more than 90 percent of portfolios at each size underperform the TMP on a risk-adjusted basis. When portfolios are value-weighted, the great majority of relatively undiversified portfolios (those with 50 stocks or less) have lower risk-adjusted total return than the TMP. Please see the fifth row in each panel of Table 23. As $N$ increases, the proportion of portfolios at each size $N$ that outperform the TMP on a risk-adjusted basis increases, and the median value of the true M2 appears to converge to the true M2 of the TMP.

Table 23 appears to suggest that if an investor could correctly identify the value-weighted portfolio with the highest true M2 (equivalently, the highest true Sharpe ratio), then diversification would be unnecessary. For example, if the SIMM governs stock returns, then the globally optimal portfolio appears to have only about 20 stocks, because the highest sample maximum for the true M2 occurs when $N = 20$.

Unfortunately, estimation error swamps any attempt to identify the optimal portfolio. Figs. 7 and 8 illustrate this problem for the Sharpe ratio when returns follow the SIMM, true alphas are non-zero as described above, and portfolios are equal-weighted or value-weighted, respectively. Since M2 is simply a linear transformation of the Sharpe ratio, its estimators exhibit the same relative estimation error as the Sharpe ratio. The median estimated Sharpe ratio is biased, and the dispersion of the cross-sectional distribution for the estimates of the Sharpe ratio at each size $N$ is at least four times that of dispersion of the cross-sectional distribution for the true values.

## 6. Conclusions

The majority of earlier papers in the naïve diversification literature implicitly or explicitly assume that convergence of total to systematic portfolio risk is evidence that non-systematic risk is reduced to zero for relatively small numbers of stocks. I follow up on work by Bennett and Sias (2011) and demonstrate that this conclusion is incorrect. I show that even for portfolios as large as 500

**Table 22**
True Modigliani Risk-adjusted Performance Measure (M2) When True Alphas Are Set to Estimated Values of the Real-world Counterparts for Simulated Stocks and Industries: Equal-weighted Portfolios (M2 in Terms of Annualized Returns Expressed in %).

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Panel A. SIMM is underlying asset pricing model | | | | | | | | | | |
| | Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 | Maximum | 18.42 | 18.61 | 18.82 | 17.31 | 15.96 | 15.61 | 15.28 | 14.76 | 14.86 |
| | 90%-tile | 15.01 | 15.21 | 15.07 | 14.97 | 14.76 | 14.56 | 14.42 | 14.35 | 14.32 |
| | Median | 12.70 | 13.24 | 13.51 | 13.72 | 13.84 | 13.90 | 13.93 | 13.94 | 13.95 |
| | 10%-tile | 10.56 | 11.51 | 11.97 | 12.42 | 12.93 | 13.31 | 13.39 | 13.57 | 13.60 |
| % of portfolios below TMP M2 | | *94.6%* | *95.8%* | *97.0%* | *98.8%* | *99.7%* | *100%* | *100%* | *100%* | *100%* |
| Panel B. IEM is underlying asset pricing model | | | | | | | | | | |
| | Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 | Maximum | 21.56 | 19.98 | 21.28 | 18.80 | 18.11 | 17.04 | 16.72 | 16.58 | 16.17 |
| | 90%-tile | 16.52 | 16.30 | 16.51 | 16.44 | 16.18 | 15.99 | 15.84 | 15.73 | 15.71 |
| | Median | 13.47 | 14.17 | 14.57 | 14.87 | 15.07 | 15.18 | 15.26 | 15.27 | 15.30 |
| | 10%-tile | 10.88 | 12.24 | 12.75 | 13.45 | 13.99 | 14.50 | 14.71 | 14.79 | 14.91 |
| % of portfolios below TMP M2 | | *82.4%* | *77.9%* | *73.6%* | *69.9%* | *68.9%* | *69.6%* | *70.1%* | *71.9%* | *74.3%* |
| Panel C. FF3M is underlying asset pricing model | | | | | | | | | | |
| | Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 | Maximum | 20.09 | 19.28 | 19.00 | 18.70 | 18.31 | 16.85 | 16.64 | 16.48 | 16.44 |
| | 90%-tile | 16.61 | 16.65 | 16.73 | 16.40 | 16.24 | 16.05 | 15.97 | 15.95 | 15.87 |
| | Median | 14.19 | 14.78 | 15.11 | 15.19 | 15.38 | 15.49 | 15.50 | 15.52 | 15.52 |
| | 10%-tile | 11.92 | 13.00 | 13.57 | 13.96 | 14.46 | 14.90 | 15.00 | 15.13 | 15.20 |
| % of portfolios below TMP M2 | | *92.4%* | *92.6%* | *92.8%* | *95.5%* | *98.3%* | *99.8%* | *99.9%* | *100%* | *100%* |

*Notes*. Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risk for each portfolio and expected return. Asset pricing models: SIMM (single index market model), FF3M (Fama-French three-factor model), IEM (two-stage industry effects model). M2 is the expected return on a portfolio levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the market portfolio. Statistics in the top part of each section are for cross-sectional distributions of true values of M2 in the simulation. The maximum at each portfolio size is the sample maximum among the 10,000 portfolios. "% of portfolios below TMP M2" is the percent of portfolios at each size *N* whose true M2 is less than the true M2 of the "Total Market Fund (TMP)," a value-weighted portfolio of all 1,465 stocks in this simulation and thus serves as a proxy for the market portfolio.

**Table 23**
True Modigliani Risk-adjusted Performance Measure (M2) When True Alphas Are Set to Estimated Values of the Real-world Counterparts for Simulated Stocks and Industries: Value-weighted Portfolios (M2 in Terms of Annualized Returns Expressed in %).

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Panel A. SIMM is underlying asset pricing model | | | | | | | | | | |
| Portfolio size | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 | Maximum | 22.27 | 23.71 | 22.93 | 22.08 | 22.40 | 19.98 | 19.36 | 18.86 | 19.10 |
| | 90%-tile | 16.80 | 17.60 | 17.67 | 17.69 | 17.95 | 17.53 | 17.35 | 17.26 | 17.04 |
| | Median | 13.03 | 13.80 | 14.17 | 14.62 | 15.12 | 15.48 | 15.53 | 15.70 | 15.74 |
| | 10%-tile | 10.19 | 11.10 | 11.48 | 11.59 | 12.75 | 13.56 | 13.90 | 14.29 | 14.65 |
| *% of portfolios below TMP M2* | | *84.4%* | *79.5%* | *76.5%* | *71.9%* | *67.0%* | *61.1%* | *60.3%* | *56.8%* | *55.8%* |
| Panel B. IEM is underlying asset pricing model | | | | | | | | | | |
| Portfolio size | | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 | Maximum | 22.41 | 23.30 | 23.78 | 22.42 | 21.04 | 21.72 | 19.08 | 18.53 | 18.30 |
| | 90%-tile | 17.18 | 17.79 | 17.96 | 18.07 | 17.79 | 17.41 | 17.07 | 16.88 | 16.81 |
| | Median | 13.22 | 14.13 | 14.49 | 14.89 | 15.20 | 15.30 | 15.37 | 15.37 | 15.42 |
| | 10%-tile | 10.23 | 10.72 | 10.75 | 11.39 | 12.13 | 13.15 | 13.50 | 13.87 | 14.11 |
| *% of portfoliosbelow TMP M2* | | *76.3%* | *70.1%* | *66.8%* | *61.2%* | *56.9%* | *54.5%* | *53.0%* | *54.1%* | *53.3%* |
| Panel C. FF3M is underlying asset pricing model | | | | | | | | | | |
| | Portfolio size | 10 | 20 | 30 | 50 | 100 | 200 | 300 | 400 | 500 |
| True M2 | Maximum | 24.22 | 21.83 | 21.72 | 21.68 | 20.80 | 19.46 | 20.03 | 19.69 | 18.84 |
| | 90%-tile | 17.30 | 17.53 | 17.90 | 18.03 | 17.85 | 17.72 | 17.67 | 17.62 | 17.53 |
| | Median | 13.83 | 14.66 | 15.03 | 15.58 | 16.01 | 16.34 | 16.59 | 16.65 | 16.75 |
| | 10%-tile | 10.89 | 11.92 | 12.06 | 13.18 | 14.20 | 15.09 | 15.49 | 15.74 | 15.88 |
| *% of portfoliosbelow TMP M2* | | *87.9%* | *83.9%* | *81.0%* | *76.8%* | *71.8%* | *68.2%* | *64.0%* | *61.9%* | *59.7%* |

*Notes*. Portfolio size is the number of stocks in the portfolio. Cross-sectional distributions are constructed by drawing 10,000 portfolios at random, then calculating the true risk for each portfolio and expected return. Asset pricing models: SIMM (single index market model), FF3M (Fama-French three-factor model), IEM (two-stage industry effects model). M2 is the expected return on a portfolio levered up or down using the risk-free asset so that risk of the adjusted portfolio matches that of the market portfolio. Statistics in the top part of each section are for cross-sectional distributions of true values of M2 in the simulation. The maximum at each portfolio size is the sample maximum among the 10,000 portfolios. "% of portfolios below TMP M2" is the percent of portfolios at each size *N* whose true M2 is less than the true M2 of the "Total Market Fund (TMP)," a value-weighted portfolio of all 1,465 stocks in this simulation (which serves as a proxy for the market portfolio).

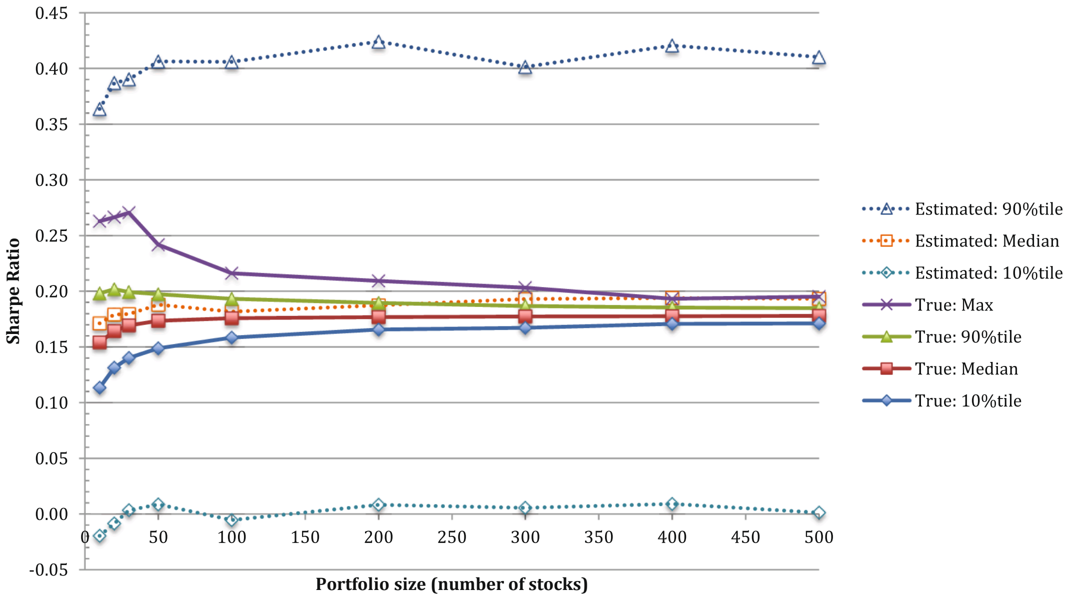


**Fig. 7.** Cross-sectional Distributions of Sharpe Ratio. Equal-weighted Portfolios. Returns Follow Single Index Market Model; Stocks Have Nonzero Alphas.

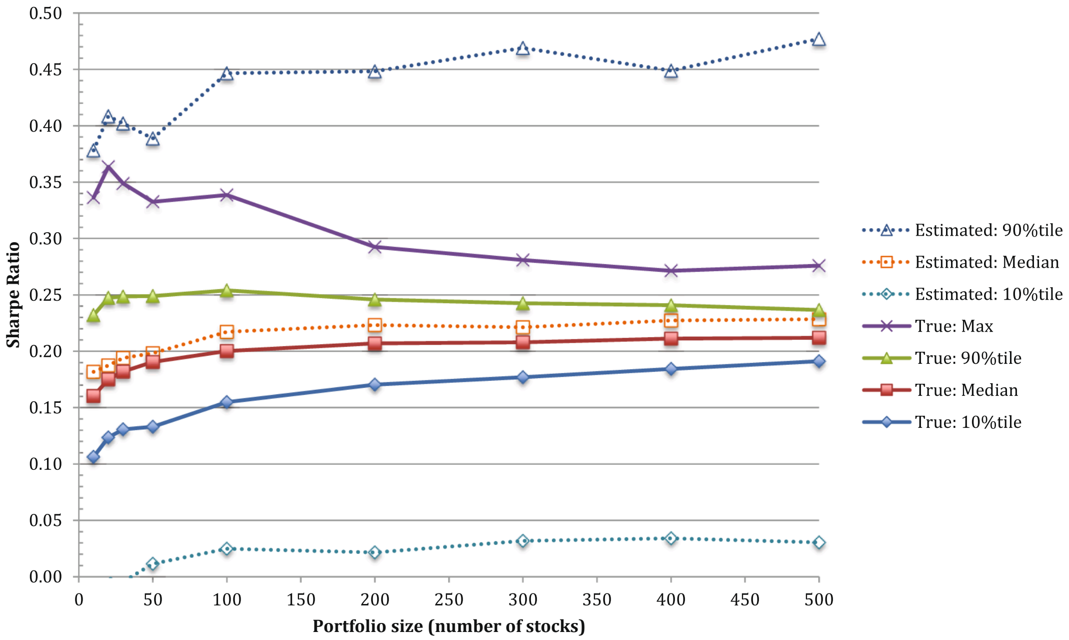


**Fig. 8.** Cross-sectional Distributions of Sharpe Ratio. Value-weighted Portfolios. Returns Follow Single Index Market Model; Stocks Have Nonzero Alphas

stocks, the presence of theoretically diversifiable risk exposes the investor to a nontrivial chance of significant shocks that are large relative to the excess market return. Specifically, for relatively diversified portfolios (*i.e.*, large values of *N*), the idiosyncratic risk could cause a shock equivalent to about half of the long-term average excess return on the market in about one out of every six years. The shocks are even larger for relatively undiversified portfolios.

The simulation shows that, on average, investors are not rewarded for exposure to this non-systematic risk. The cross-sectional distribution of the true Sharpe ratio *rises* and its dispersion shrinks significantly as the number of stocks in the portfolio increases, whereas the cross-sectional distribution of the true non-systematic risk *falls* and its dispersion shrinks significantly as the number of stocks in the portfolio increases. This pattern appears regardless of the true asset pricing model for generating security returns (in this study, the single index market model, the Fama-French three-factor model, or a two-stage industry effects model), the portfolio weighting method (equal-weighted and rebalanced monthly, or value-weighted), or specification of security alphas (all true alphas set to zero, or the great majority of true alphas set to some positive value).

What lessons does the simulation hold for investors? Under some circumstances, the total market portfolio has the highest true Sharpe ratio and thus is optimal. Given the availability of low cost, no-load total market index funds today, this option is available to just about any investor. But even under circumstances where the total market portfolio is not the true optimal portfolio, it still is a good choice. Uncertainty about the true underlying asset pricing model and poor accuracy of estimators for the Sharpe ratio (even if we "know" the true model as in the simulation) mean that correctly identifying the optimal portfolio is difficult. The simulation results show that the total market portfolio gives an uninformed investor the best odds of holding a portfolio with a Sharpe ratio that is better than that for the majority of portfolios.

The main limitation of this simulation study is that the simulated market consists of 1,465 stocks whose simulated parameters are based on estimates for their real-world counterparts over 2007–2016. This set of stocks corresponds approximately to the S&P 1500 and thus accounts for roughly 90 percent of the float-adjusted capitalization of the U.S. stock market. Thus, size of the simulated population is not the limitation. Rather, the results are conditional on the stocks that traded during 2007–2016. Hence, a natural extension of this study would be to look at parameterization of the asset pricing models based on other historical time periods, because using market returns from 1963 to 2018 in the block bootstrap to generate data for the 60-month estimation periods implicitly assumes that these stocks would have behaved similarly over the longer historical period. It also assumes that they or similar stocks had been trading.

A second important limitation is the explicit assumption that the parameters in the asset pricing models are stationary. Evidence in the literature, *e.g.*, French, Schwert, and Stambaugh (1987), shows that estimated volatility of returns is significantly different in different historical periods. The stationary block bootstrap used in the simulation captures this variability to some extent in the series of market returns drawn for each 60-month simulated estimation period. The simulation assumes that the asset pricing models parameters are stationary, but these should change over time if the true market volatility changes. The results in the current paper could serve as a baseline for a later simulation study in which parameters are not stationary.

## 7. Author statement

This manuscript is under review only at *The North American Journal of Economics and Finance*. There are no "competing interests." No special funding supported this research, which was completed in the normal course of employment as faculty member at The University of Texas of the Permian Basin.

## Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Appendix A

A.1. Single index market model

This section includes details about the estimators for the components of total portfolio risk in the single index market model (SIMM) described in Section 3.1. When the SIMM holds, we can estimate systematic risk in the general case as $\widehat{\beta}_{mpt}\widehat{\sigma}_m$, where $\widehat{\beta}_{mpt}$ is the least-squares estimator of portfolio beta, given portfolio weights at time $t$, and

$$\widehat{\sigma}_m = \sqrt{\widehat{\sigma}_m^2} \tag{A1}$$

where $\widehat{\sigma}_m^2$ is the estimated variance of the excess market return. When portfolio returns are available, we can estimate portfolio beta directly from these returns.

Due to the linearity of portfolio beta,

$$\widehat{\beta}_{mpt} = \sum_{i=1}^{N} w_{it}\widehat{\beta}_{mi}. \tag{A2}$$

Thus, an equivalent approach is to estimate the betas for the individual stocks and then apply Equation (A2). A problem arises, however, when we estimate the standard deviation of the excess market return, $\widehat{\sigma}_m$, as in Equation (A1). Taking the square root of $\widehat{\sigma}_m^2$ introduces a downward bias due to Jensen's Inequality for concave functions; that is,

$$E\left\{\sqrt{\widehat{\sigma}_m^2}\right\} < \sqrt{E\left\{\widehat{\sigma}_m^2\right\}}. \tag{A3}$$

Despite this theoretical problem, Haensly (2020) shows that the approximation in Equation (A1) is a reasonably good estimate of $\sigma_m$ even if the market returns are not normally distributed.

When the SIMM holds, we can estimate idiosyncratic risk as the square root of the residual mean square (RMS) for the portfolio,

$$\widehat{\sigma}^2_{ept} = \sum_{i=1}^{N} w_{it}^2 \widehat{\sigma}^2_{ei}, \tag{A4}$$

in the general case, and

$$\widehat{\sigma}^2_{ep} = \left(\frac{1}{N}\right)^2 \sum_{i=1}^{N} \widehat{\sigma}^2_{ei}, \tag{A5}$$

when portfolios are equal weighted. Taking the square root of the portfolio RMS introduces a downward bias due to Jensen's Inequality for concave functions. Thus, when interpreting the simulation results, it is important to keep in mind that the estimates of portfolio idiosyncratic risk are conservative.

A.2. Fama-French three-factor model

In the Fama-French three-factor model (FF3M) (Fama and French, 1992; and Fama and French, 1993), the total monthly return on stock *i* in period *t*, in excess of the risk-free rate, can be written as

$$r_{it} = \alpha_i + \beta_{mi} r_{mt} + \beta_{SMB,i} r_{SMB,t} + \beta_{HML,i} r_{HML,t} + e_{it}, \tag{A6}$$

where $r_{mt}$ is the total return on a market index *m* in period *t*, in excess of the risk-free rate, with standard deviation $\sigma_m$; $r_{SMB,t}$ is the return on the Fama and French SMB (small minus big) portfolio, with standard deviation $\sigma_{SMB}$; $r_{HML,t}$ is the return on the Fama and French HML (high minus low) portfolio, with standard deviation $\sigma_{HML}$; and $e_{it}$ is the error term for security *i* in period *t*, with mean zero and finite standard deviation $\sigma_{ei} > 0$. I assume that the error terms have zero covariance with each of the three factors and each other. Because the model in Equation (A6) is linear, the portfolio return, $r_{pt}$, can be defined analogously. The portfolio market beta, $\beta_{mpt}$, is defined as

$$\beta_{mpt} = \sum_{i=1}^{N} w_{it} \beta_{mi}, \tag{A7}$$

and the SMB and HML portfolio betas are defined analogously. The portfolio residual variance, $\sigma^2_{ept}$, takes the same form as in the SIMM:

$$\sigma^2_{ept} = Var\left\{e_{pt}\right\} = \sum_{i=1}^{N} w_{it}^2 \sigma^2_{ei} \tag{A8}$$

For convenience, I write the variance decomposition as

$$\sigma^2_{pt} = Var\left\{r_{pt}\right\} = SRV_t + SRVC_t + \sigma^2_{ept}, \tag{A9}$$

where

$$SRV_t = \beta^2_{mpt}\sigma^2_m + \beta^2_{SMB,pt}\sigma^2_{SMB} + \beta^2_{HML,pt}\sigma^2_{HML}, \tag{A10}$$

and

$$SRVC_t = 2\left(\beta_{mpt}\beta_{SMB,pt}\sigma_{m,SMB} + \beta_{mpt}\beta_{HML,pt}\sigma_{m,HML} + \beta_{SMB,pt}\beta_{HML,pt}\sigma_{SMB,HML}\right), \tag{A11}$$

where $SRV_t$ includes the factor variances while $SRVC_t$ includes the factor covariances. Factor models often are constructed so that the covariances are zero; in that case, the variance decomposition simplifies, because $SRVC_t$ is zero. However, this condition is not necessary for the simulation, because I only wish to distinguish systematic from idiosyncratic risk.

The sum of the first two terms on the right-hand side of Equation (A9) is the systematic risk of portfolio *p*, conditional on portfolio weights at time *t*, and $\sigma^2_{ept}$ is the portfolio's idiosyncratic risk. As with the SIMM, I work with the square roots of these values. In the analysis, I define the systematic risk of portfolio *p* to be $\sqrt{SRV_t + SRVC_t}$ and the idiosyncratic risk as $\sigma_{ept}$.

Under the FF3M, the true expected excess portfolio return is

$$E\left\{r_{pt}\right\} = \alpha_{pt} + \beta_{mpt}E\left\{r_{mt}\right\} + \beta_{SMB,pt}E\left\{r_{SMB,t}\right\} + \beta_{HML,pt}E\left\{r_{HML,t}\right\}, \tag{A12}$$

where portfolio alpha is a weighted sum of the alphas of the stocks in the portfolio.

Analogous to the SIMM, we can calculate multivariate least squares estimates for the parameters of the individual stocks in the portfolio and then estimate each portfolio beta as a weighted sum of the corresponding security betas. Similarly, we can estimate the portfolio idiosyncratic risk as the square root of the RMS for the portfolio.

A.3. Two-stage industry effects model

The two-stage industry effects model (IEM) is similar to that used by Campbell, Lettau, Malkiel, and Xu (2001). The monthly return on industry $i$ in month $t$, is

$$r_{it} = \alpha_i + \beta_{mi} r_{mt} + \varepsilon_{it}, \tag{13}$$

where $r_{mt}$ is the return on the market index in month $t$, returns are in excess of the risk-free rate, and the industry residual variance, $\sigma^2_{\varepsilon i} = Var\left\{\varepsilon_{it}\right\} > 0$, is finite. The excess return on security $j(i)$ in industry $i$ in month $t$ is

$$r_{j(i)t} = \alpha_{j(i)} + \beta_{j(i)} r_{it} + \eta_{j(i)t}, \tag{14}$$

where the company residual variance, $\sigma^2_{\eta j(i)} = Var\left\{\eta_{j(i)t}\right\}>0$, is finite. In the simulation, I assume the following.

- The market return has zero covariance with both types of error terms: $Covar\left\{r_{mt}, \varepsilon_{it}\right\} = 0$ and $Covar\left\{r_{mt}, \eta_{j(i)t}\right\} = 0$, for all securities $j(i)$ and all industries $i$.
- The industry and individual firm error terms have zero covariance: $Covar\left\{\varepsilon_{it}, \eta_{j(i)t}\right\} = 0$, for all securities $j(i)$ and all industries $i$.
- To assure that the industry effect is not otherwise systematic, the industry error terms have zero covariances with each other: $Covar\left\{\varepsilon_{it}, \varepsilon_{kt}\right\} = 0$, for all industries $i \neq k$.
- To assure that the company-unique risk truly is idiosyncratic, the individual firm error terms have zero covariances with each other: $Covar\left\{\eta_{j(i)t}, \eta_{l(k)t}\right\} = 0$, for all firms $j(i) \neq l(k)$, and all industries $i$ and $k$.

From the first two assumptions, we can show that the industry return has zero covariance with the individual firm error term: $Covar\left\{r_{it}, \eta_{j(i)t}\right\} = 0$, for all securities $j(i)$ in industry $i$ and all industries $i$. Thus, the two-stage model can be analyzed with least squares methods.

Write the return on portfolio $p$, in excess of the risk-free rate, as

$$r_{pt} = \sum_{i=1}^{k} \sum_{j(i)=1}^{n_i} w_{j(i)t} r_{j(i)t}, \tag{15}$$

where portfolio $p$ has $n_i$ securities from industry $i$, $i = 1, 2, \ldots, k$;

$$\sum_{i=1}^{k} n_i = N; \tag{16}$$

and market value weights total to one:

$$\sum_{i=1}^{k} \sum_{j(i)=1}^{n_i} w_{j(i)t} = 1. \tag{17}$$

Under the assumptions for this model, the variance decomposition for portfolio returns is

$$Var\left\{r_{pt}\right\} = \beta^2_{pt}\sigma^2_m + \sigma^2_{\varepsilon pt} + \sigma^2_{\eta pt}, \tag{18}$$

where portfolio beta is

$\beta_{pt} = \sum_{i=1}^{k} \sum_{j(i)=1}^{n_i} w_{j(i)t}\beta_{j(i)}\beta_{mi}$.(A19)

We can show that the component of non-systematic risk resulting from industry-unique risk is

$$\sigma^2_{\varepsilon pt} = \sum_{i=1}^{k} \left( \sum_{j(i)=1}^{n_i} w_{j(i)t}\beta_{j(i)} \right)^2 \sigma^2_{\varepsilon i}, \tag{20}$$

and the idiosyncratic volatility due to company-unique risk is

$$\sigma^2_{\eta pt} = \sum_{i=1}^{k} \sum_{j(i)=1}^{n_i} w^2_{j(i)t}\sigma^2_{\eta j(i)}. \tag{21}$$

Provided that the portfolio weights $w_{j(i)t}$ are properly bounded and number of securities in the market is sufficiently large, both components of diversifiable risk, $\sigma^2_{\varepsilon pt}$ and $\sigma^2_{\eta pt}$, converge to zero as $N$ increases. Analogous to my treatment of risk in the SIMM and FF3M, I define systematic risk as $\beta_{pt}\sigma_m$. In the IEM, we have two sources of non-systematic risk: the portfolio industry-unique risk, $\sigma_{\varepsilon pt}$, and the portfolio idiosyncratic risk, $\sigma_{\eta pt}$.

Under the assumptions above, we first calculate least squares estimates for the industry parameters in Equation (A13) with the market excess return as the independent variable. Then we calculate least squares estimates for the individual security parameters in Equation (A14) with the appropriate industry excess return as the independent variable. Finally, estimate portfolio beta by substituting the industry and security beta estimates into Equation (A19). Estimate the portfolio industry volatility by substituting security beta estimates and industry RMS statistics into Equation (A20), and estimate the portfolio idiosyncratic volatility by substituting the security RMS statistics into Equation (A21).

When working with the Sharpe ratio, we also need the expected excess portfolio return. Under the IEM, the expected excess portfolio return is

$$E\left\{r_{pt}\right\} = \sum_{i=1}^{k}\sum_{j(i)=1}^{n_i} w_{j(i)t}E\left\{r_{j(i)t}\right\}, \tag{22)}$$

where

$$E\left\{r_{j(i)t}\right\} = \alpha_{j(i)} + \beta_{j(i)}E\left\{r_{it}\right\} = \left(\alpha_{j(i)} + \beta_{j(i)}\alpha_i\right) + \beta_{j(i)}\beta_{mi}E\left\{r_{mt}\right\} \tag{23)}$$

When all true alphas are zero,

$$E\left\{r_{j(i)t}\right\} = \beta_{j(i)}\beta_{mi}E\left\{r_{mt}\right\}, \tag{24)}$$

and thus

$$E\left\{r_{pt}\right\} = \beta_{pt}E\left\{r_{mt}\right\}. \tag{25)}$$

## Appendix B

Each of the stocks in the simulation has a real-world counterpart. I assembled a list of potential candidates from the portfolio composition files (PCFs) available on May 2017 for 12 Vanguard index exchange traded funds (ETFs): nine equity index funds designed to track the performance of the CRSP U.S. market indexes defined by market capitalization and value-growth characteristics, *e.g.*, the Vanguard Value ETF that tracks the CRSP U.S. Large Cap Value Index; and three equity index funds designed to track the performance of components of the S&P 1500 index, *e.g.*, the Vanguard S&P Small-Cap 600 ETF. For these stocks, I downloaded daily-adjusted closing prices from the Yahoo! Finance web site (https://finance.yahoo.com/) when available. These prices are adjusted to reflect splits and cash dividends. From these daily closing prices, I calculated monthly total returns for each stock. Of these stocks, 1,465 had complete monthly returns for the 10-year period, 2007–2016.

The IEM requires that each stock be categorized by industry. To determine an industry for a given stock, I recorded its sector and industry as listed on the Profile page at Yahoo! Finance. I then attempted to match this description with a Global Industry Classification Standard (GICS) category; see S&P Global Market Intelligence (2018). The GICS classification is a four-tiered, hierarchical system based on a company's primary business activity. Sectors are subdivided into industry groups, which are subdivided by industries, and then into sub-industries. Occasionally, a collection of the stocks in an industry could be subdivided into distinct sub-industries with more than three stocks each; in these cases, I use the GICS sub-industry as the simulated industry. On rare occasion, some industries of a given GICS industry had fewer than three stocks, so I either combined industries or used the GICS industry group as a simulated industry. The result is that the stocks in the simulation are divided into 105 non-overlapping simulated industries. The size of simulated industries ranges from three to 98 stocks with a median size of 10 stocks.

Yahoo! Finance reportedly uses the GICS system (*e.g.*, see Benzinga (2016)), but their industry categories do not always match up with the GICS categories. I sometimes made a judgment call about how to assign a company or group of companies to an industry. Nonetheless, once stocks are assigned to a simulated industry, the IEM imposes a strict partition between industries, because the error terms in the industry equation are uncorrelated with each other, the market index, and error terms in the security returns equation. The model in Equation (A14) determines the comovement of the returns of individual firms within each industry.

To determine the true parameters for the IEM in the simulation, I first construct an industry returns series for the real-world counterparts. To construct the market cap weights of stocks within each industry, I use the float-adjusted shares outstanding reported by Yahoo! Finance as of May 2017 and the adjusted closing prices at the start of each month over 2007–2016 to estimate time series of the float-adjusted market capitalization for each real-world counterpart. Then I construct the value weighted excess returns series for each industry. I calculate the parameter values to use for the true industry equations by applying least squares to the industry returns model in Equation (A13). Finally, I calculate the values to use for the true stock equation parameters by applying least squares to the stock returns model in Equation (A14), where the independent variable is the excess industry return for each stock.

I set the market cap for each simulated stock equal to the same value at the start of the estimation period each time that the simulator draws a new portfolio. Specifically, I set it equal to the float adjusted market cap of its real-world counterpart at the end of 2016. I also define the true values of the components of total risk in terms of these initial market cap weights. This approach standardizes the measurement of true values of the components of total risk.

Unfortunately, this approach also has the undesirable effect of optimistically biasing the dispersion downward for the cross-sectional distributions of the true values of the components, because market caps evolve over time conditional on their initial magnitude. For example, if the initial market cap is $100 million, and the stock realizes a capital gain of 10% over a given time period, then its market cap increases by $10 million. By comparison, if the initial market cap of a second stock is $200 million and it realizes the same percentage capital gain over the same time period as the first stock, then its market cap grows by $20 million. Hence, over time, a portfolio with $N$ stocks tends to become more concentrated, and the cross-sectional distributions of the components of risk for a given size $N$ tend to exhibit dispersion more characteristic of a portfolio with fewer stocks. This effect contributes to the greater dispersion of the cross-sectional distributions that I report when portfolios are value weighted. At the other extreme, market capitalization of stocks at the end of each portfolio's 60-month estimation period overstates the dispersion of the cross-sectional distributions of true risk.

Nonetheless, results presented in this paper are relatively robust to choice of the market value weights for initializing the simulated estimation periods (*i.e.*, from which month during 2007–2016 to draw the market caps for the real world counterparts of the simulated stocks). Specifically, the cross-sectional dispersions of *estimates* of risk and risk-reward measures always are much wider than the cross-sectional dispersion of the corresponding *true* values. Hence, the estimation problem discussed in Section 5 arises regardless of the precise specification of the initializing market caps. This problem generally does not arise when working solely with historical data and an unknown returns generating process, because the estimators typically give equal weight to returns from each discrete unit of time in the estimation period.

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