Journal of Financial Econometrics, 2019, 1–27
doi: 10.1093/jjfinec/nbz016
Article

Investing with Cryptocurrencies—a Liquidity Constrained Investment Approach*

Simon Trimborn[1], Mingyang Li[2], and Wolfgang Karl Härdle[3]

[1]National University of Singapore and Humboldt-Universität zu Berlin, [2]Xiamen University, and [3]Humboldt-Universität zu Berlin

Address correspondence to Mingyang Li, Wang Yanan Institute for Studies in Economics, Xiamen University, No. 422, Siming South Road, Xiamen, China, 361005 or e-mail: limingyang@stu.xmu.edu.cn

Received July 19, 2017; revised April 5, 2019; editorial decision April 7, 2019; accepted April 26, 2019

Abstract

Cryptocurrencies have left the dark side of the finance universe and become an object of study for asset and portfolio management. Since they have low liquidity compared to traditional assets, one needs to take into account liquidity issues when adding them to a portfolio. We propose a Liquidity Bounded Risk-return Optimization (LIBRO) approach, which is a combination of risk-return portfolio optimization under liquidity constraints. Cryptocurrencies are included in portfolios formed with stocks of the S&P 100, US Bonds, and commodities. We illustrate the importance of the liquidity constraints in an in-sample and out-of-sample study. LIBRO improves the weight optimization in the sense that it only adds cryptocurrencies in tradable amounts depending on the intended investment amount. The returns greatly increase compared to portfolios consisting only of traditional assets. We show that including cryptocurrencies in a portfolio can indeed improve its risk–return trade-off.

Key words: asset classes, blockchain, crypto-currency, CRIX, portfolio investment

JEL classification: C01, C58, G11

1 Introduction

With the emergence of cryptocurrencies, not only has a new kind of currency and transaction network arisen, but also a new kind of investment product. The cryptocurrency (CC) market has shown strong gains over the past years, which can be inferred from the CRIX, developed by Trimborn and Härdle (2018) and visualized at thecrix.de. The CRIX index indicates a gain of the market of 500% over the past 2 years, which makes it attractive for investors. Simultaneously the market bears high risk in terms of price variations and operational risk. In the past years, users and exchanges were vulnerable in many ways, for

* The authors would like to thank the editor and two anonymous referees for their valuable comments to this article.

example, the traders on the exchange Mt. Gox experienced fraud and exchanges like Bitfinex got hacked. Also single users were subject to larceny. The situation has already improved a lot but there still remains a problem of trust, since the market is not fully developed. It has often been pointed out that a procedure called "cold-storage" should be used to secure one's CCs. This refers to storing the access codes for the coins in such a way that they are disconnected from any device under threat of a hostile attack. While this source of risk can be managed comparably easily, the market risk is difficult to handle.

A natural question is why an investor should engage in such a risky market given these volatility effects. Advantages beside the opportunity for strong gains need to be present to make an investment worth the risk. An important perk is that CCs have a low linear dependency with each other and also the top 10 CCs (by market capitalization) have low correlations with traditional assets, Elendner et al. (2018). Since CCs have low correlations with each other and are uncorrelated with traditional assets, they are indeed interesting for investors due to the diversification effect. Making use of this advantage, Brière, Oosterlinck and Szafarz (2015) and Eisl, Gasser, and Weinmayer (2015) added Bitcoin to a portfolio of traditional assets and found an enhanced portfolio in terms of risk-return. Since alternative CCs (alt-coins, other than Bitcoin), have favorable properties too, we aim to construct portfolios consisting of traditional assets and several cryptocurrencies. Lee Kuo Chuen, Guo, and Wang (2017) worked in a related direction by investigating a portfolio mimicking the CRIX, the CRyptocurrency IndeX. Treating CRIX as a financial asset, Chen et al. (2017, 2018) investigated options pricing based on CRIX.

When investing in CCs, one is confronted with a higher volatility than for traditional assets, see Figure 1. Markowitz (1952) developed a method which optimizes the portfolio allocation in terms of a minimum variance portfolio according to a target return. The approach was applied in a broad variety of applications and showed its usefulness especially in the case of Gaussian distributed data. But CCs are known to behave differently from the normal distribution, Elendner et al. (2018). In particular, the stronger tails come with higher risk, arising from higher moments, Scaillet, Treccani, and Trevisan (2018). Tail risk optimized portfolios might be worth considering in this market, for example, taking into account Conditional Value-at-Risk (CVaR), Rockafellar and Uryasev (2000). But there is another issue that cannot be handled by these risk optimization methods, namely the low liquidity of the CC market. Figure 1 (right plot) presents a comparison of the liquidity measured by median daily trading amount of CCs and S&P 100 component stocks. It is obvious that the median daily trading amounts of CCs are all lower than the 25% quantile of S&P 100 stocks.

If we want to include CCs and stocks in the same portfolio, we need to avoid giving CCs too big a weight since this will induce a severe liquidity problem on adjusting the position when reallocating the portfolio. For example, if one holds a long position on an asset that is equal to twice its average daily trading amount, then one expects to take about two days to clear this position, following the same pace of the market. However, this may result in missing a trading opportunity. A proper way to deal with such a liquidity issue is the introduction of liquidity constraints on the weights. Krokhmal, Palmquist, and Uryasev (2002) utilized liquidity constraints in the sense of restricting the change in a position. Darolles, Gourieroux, and Jay (2012) choose a related approach by incorporating a penalty term into the optimization function, balancing the risk and change of positions in the portfolio. However we intend to be able to clear all positions at once, which is assumed to be in the interest of an investor engaging in a risky market like the CC market. Instead our definition

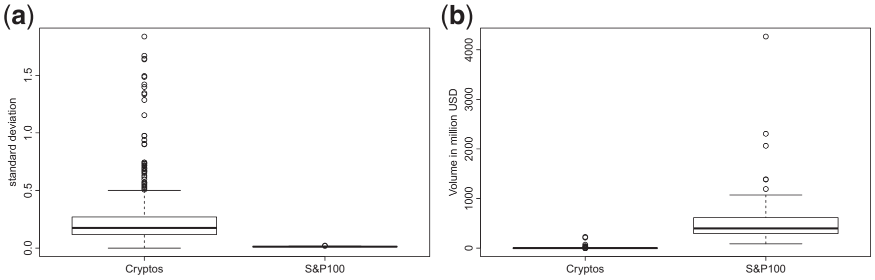

Figure 1 Boxplots of standard deviation and median trading volume (measured in US dollars) of all CCs and S&P 100 components, from April 22, 2014 to October 30, 2017. Obviously CCs have much lower daily trading volumes and higher volatilities than the stocks, highlighting the importance of volatility and liquidity risk management when investing in them.

(a) Comparison of standard deviation of CCs and S&P100 Equity Index components. (b) Comparison of median trading volume of CCs and S&P100 Equity Index components.

of liquidity constraints is concentrated on the entire weight given to a CC, rather than the allowed change in a position. Additionally, such an approach has the advantage of tackling a drawback of Markowitz portfolios. Minimum Variance optimized portfolios often suffer from extreme positive and negative weights, Härdle et al. (2018). This may result from a single dominant factor in the covariance matrix, Green and Hollifield (1992). In an empirical study, Jagannathan and Ma (2003) find nonnegativity constraints on the weights to have an equal efficacy at removing the effect of a single dominant eigenvalue from the covariance matrix. Fan, Zhang, and K. Yu.June (2012) provide theoretical insights into their findings and find that constraining the weights from taking extreme positions is more effective than nonnegativity constraints. Thus introducing weight constraints gives us the opportunity to "kill two birds with one stone."

Due to these challenges and the advantage from investing in cryptocurrencies, we aim at a portfolio optimization method which takes into account volatility or tail risk and low liquidity. We call it LIBRO—which is a combination of a risk optimization portfolio formation method and an additional restriction, which prevents big weights on assets with low liquidity. The portfolios are formed with Mean-Variance (Markowitz) and Conditional Value-at-Risk as risk measures. Due to the huge dimensionality of the asset universe and limited data availability, the sample covariance matrix may not be a well-conditioned estimator of its theoretical counterpart (well-conditioned in the sense that inverting the covariance matrix does not amplify the estimation error, Ledoit and Wolf (2004)). A well-conditioned and more accurate estimator was introduced by Ledoit and Wolf (2004), which we apply to the estimation of the Markowitz portfolios. Reduced factor model approaches were, for example, investigated by Kozak, Nagel, and Santosh (2017) and sparse estimation by, for example, Friedman, Hastie, and Tibshirani (2008). To investigate the robustness of the results, the reallocation dates in the out-of-sample study are set to be monthly and weekly. In order to overcome estimation difficulties driven by too short time series, we work under an extending window approach. Two datasets are compared in the application. The first one is a portfolio formed with S&P 100 components and CCs. The excess returns from the portfolio with CCs over the pure stock one range from 13.5 to 88% (gained over

3.5 years) in the in-sample analysis, and from 13.7 to 60% (gained over 2.75 years) in the out-of-sample analysis. When using stocks, bonds and commodities as the traditional assets, the results still range from 6 to 20.43% in-sample (3.5 years) and 6.7–24.38% out-of-sample (2.75 years). Summary statistics of the return series indicate that including CCs can increase the Sharpe Ratio, thus we show that including CCs can indeed improve the risk-return trade-off of the portfolio. Furthermore, the present article illustrates the importance of the liquidity constraints and their effects.

This article is organized as follows. Section 2 introduces the data. Section 3 presents the portfolio optimization methods. Section 4 introduces the liquidity constraints and Section 5 gives an in-sample and out-of-sample application with S&P 100 component stocks, Barclays Capital US Aggregate Index (US Bonds Index), S&P GSCI (Commodities Index), and CCs. The portfolios based on stocks are indicated by S, while the Stock, Bonds, and Commodities ones are indicated by SBC. The results are summarized in Section 6. The codes used to obtain the results in this article are available at www.quantlet.de, Borke and Härdle (2018) and Borke and Härdle (2017).

2 Data Description

In this article, 42 CCs are used to form portfolios together with traditional financial assets, with a sample period from April 22, 2014 to October 30, 2017. The daily prices (in USD) and trading volumes were downloaded from the CRIX cryptocurrencies database (thecrix.de), kindly provided by CoinGecko. The CCs were selected so that the average market cap during the sample period is no less than 10,000 US dollar. This criterion was applied since we target portfolios consisting of a reasonably high investment size, thus the CCs should have enough market capitalization to be added to a portfolio.

For the traditional financial assets, we chose the components of the S&P 100, Barclays Capital US Aggregate Index (US-Bonds Index), S&P GSCI (Commodities Index). The daily closing price (in USD) and trading volume, dated from April 22, 2014 to October 30, 2017, are downloaded from Datastream. To get the daily trading volume of stocks measured in US dollars, we multiplied the daily trading volume by the daily closing price. Three stocks were omitted: DowDuPont Inc., The Kraft Heinz Company, and PayPal Holdings, Inc., since they have a shorter sample period due to company mergers or spin-offs. For the indices we did not chose a particular Exchange Traded Fund, instead we assume perfect liquidity for both of them.

We show the summary statistics for the top 10 CCs over time in Table 1. For the full list, see Table A1 in the Appendix. In both tables, the CC statistics are arranged in decreasing order of their mean daily trading volume. For comparison, we list the summary statistics of stocks, a bond index, and a commodity index too. The summary statistics for the stocks are the average values for all stocks. The first 5 columns focus on the return series, while the remaining two list the mean trading volume and market capitalization. We will focus on Table 1 to analyze the summary statistics of the CCs. Compared to the average annualized mean returns for stocks, which is 8%, and for the bond index, 5%, those for CCs can be quite shocking: except for PPC and BLK, all the other eight have returns that exceed 10%. What's more, five of them exceed 20%, three of them exceed 50%, and there is even one of them, DASH, that has an annualized average return that exceeds 100%. One can observe that three of the alt-coins, viz., Ripple (XRP), Dashcoin (DASH), and DigiByte

Table 1 Summary statistics of top ten CCs by trading volume. Ann.Ret and Ann.STD indicates annualized mean and standard deviation of the return of each CC, which are calculated by multiplying their daily counterparts by 250 and $\sqrt{250}$ respectively. For the purpose of comparison, we list the summary statistics of traditional financial assets at the bottom part of the table as well. "Bond" and "Commodity" indicate the summary statistics of the daily return of the bond index and commodity index, while "Stocks" indicates the average level of the summary statistics for the daily return of each individual stock

	Ann.Ret	Ann.STD	skewness	kurtosis	ρ	mean volume	market cap
BTC	0.49	0.55	−0.61	10.87	−0.01	4.71e + 08	1.07e + 10
LTC	0.29	0.87	0.34	23.70	0.02	9.04e + 07	3.22e + 08
XRP	0.68	1.09	2.72	37.76	0.01	4.88e + 06	8.35e + 08
DASH	1.16	1.37	0.62	50.60	−0.14	1.61e + 06	1.52e + 08
DOGE	0.11	0.96	0.97	15.54	0.01	3.87e + 05	3.11e + 07
NXT	0.17	1.08	0.80	8.42	−0.03	2.28e + 05	2.08e + 07
DGB	0.73	1.70	2.93	29.89	−0.03	1.63e + 05	7.24e + 06
PPC	−0.17	1.00	0.62	12.51	−0.05	1.05e + 05	1.54e + 07
BLK	−0.05	1.27	1.82	17.72	−0.08	7.15e + 04	4.21e + 06
VTC	0.33	1.76	1.84	17.83	−0.01	5.93e + 04	2.42e + 06
Stocks	0.08	0.20	−0.26	10.29	0.00	5.40e + 08	
Bond	0.05	0.05	−1.91	12.38	−0.04		
Commodity	−0.20	0.20	0.05	4.84	−0.07		

(DGB), have a higher return than BTC, the dominant CC in the market, indicating that it is time to take these alt-coins into account for portfolio formation as well. However, the outstanding returns come at a price. Judging from Table 1, CCs also have much higher volatility and tail risk. All CCs have an annualized volatility that exceeds 50%, with 7 of them even exceeding 100%. In contrast, S&P 100 stocks have an average annualized volatility of 20%, as does the commodity index, and the bond index has only 5%. Thus there is a trade-off in terms of high returns yet high standard deviations for the CCs. This finding is consistent with the size effect reported by Elendner et al. (2018). A similar picture is observed for the kurtosis: all the listed CCs, except for BTC and NXT, have a higher kurtosis than the stocks (10.29), bond index (12.38), and commodity index (4.84). Among these CCs, BTC has the lowest volatility of the 10, which is not surprising since it has the largest market capitalization, largest trading volume, and longest trading history, which makes it the most mature CC. Besides, BTC is the only CC that has a negative skewness, akin to stocks and bonds, while the other CCs all have positive skewness. This may imply that the other CCs are in a different developmental phase than BTC. For the auto-correlation, most of the top 10 CCs have negative or slightly positive ρ, like stocks and bonds.

The influence of liquidity is a major focus of this study. The evolution of the log of the trading volume of the top 10 CCs by trading volume is shown in Figure 2, which, for each of the CCs, shows the daily changes. Table 2 provides descriptive statistics of the log returns of the turnover value. For all of the 10 CCs one can see that they vary around a mean value of roughly 0. The median is slightly negative for all CCs, suggesting that there are more frequent decreases in the liquidity than increases. However the first and third quantiles show more or less opposing values, which hints that the variation around the

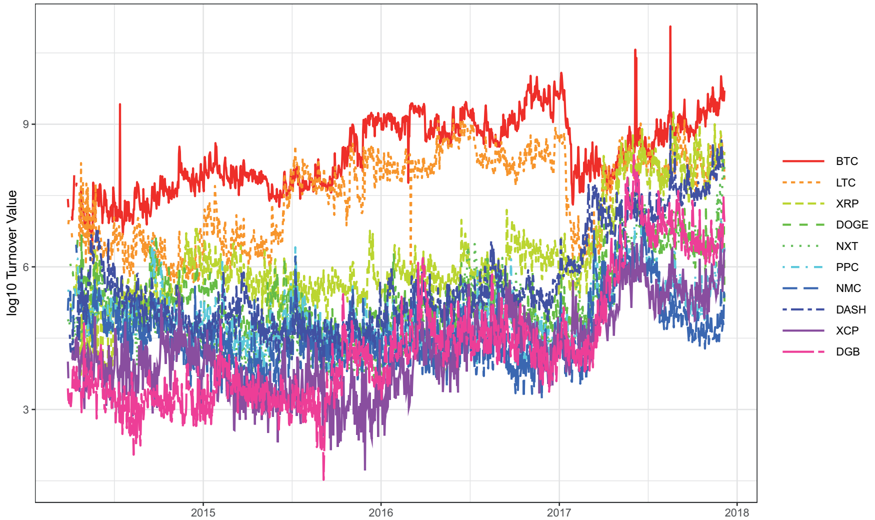

Figure 2 Common Logarithm of turnover value of top 10 CCs by their daily trading volume, BTC, LTC, XRP, DOGE, NXT, PPC, NMC, DASH, XCP, DGB.

LIBROliquidity

Table 2 Summary statistics of the trading volume of top 10 CCs

	BTC	LTC	XRP	DOGE	NXT	PPC	NMC	DASH	XCP	BLK
min	−5.50	−4.85	−3.65	−4.00	−2.06	−3.00	−2.95	−3.07	−2.64	−4.15
1st Quantile	−0.26	−0.36	−0.39	−0.37	−0.38	−0.50	−0.57	−0.35	−0.56	−0.50
mean	0.01	0.00	0.01	−0.00	0.01	0.00	−0.00	0.01	0.00	0.00
median	−0.03	−0.06	−0.02	−0.01	−0.02	−0.05	−0.04	−0.01	−0.04	−0.03
3rd Quantile	0.23	0.29	0.36	0.33	0.36	0.46	0.48	0.32	0.53	0.47
max	5.44	4.47	4.86	3.39	2.53	4.50	4.90	2.89	4.40	4.28
variance	0.26	0.41	0.48	0.41	0.38	0.63	0.84	0.33	0.83	0.67

median value is symmetric. In extreme cases, min and max values, this does not hold. The variance and extreme values suggest that fixing liquidity weights based on the mean turnover value could result in boundaries that are too high. Thus, an approach based on a robust measure, the median, will be employed. In the next section, we will introduce the portfolio optimization methods that we will use.

3 Constrained Portfolio Optimization

Markowitz (1952) introduced the theory of optimizing weights so that the variance of the portfolio would be minimized according to a certain target return. When the variance serves as a risk measure, this translates into risk minimization. Consider now N assets with T returns given by an $(N \times T)$ matrix X and let $\hat{\Sigma}$ be the estimated covariance matrix of the respective assets. Then the Markowitz portfolio is defined as, Härdle and Simar (2015):

$$\begin{aligned} &\min_{w} \quad w^{\top}\hat{\Sigma}w \\ &\text{s.t. } \mathbf{1}_{\mathrm{N}}^{\top}\mathrm{w} = 1, \ \mu \leq \mathrm{x}^{\top}\mathrm{w} \end{aligned} \tag{1}$$

where $w = (w_1, w_2, \ldots, w_N)^{\top}$ are the weights on the assets, x is the $(N \times 1)$ vector of expected returns of the assets, $\mathbf{1}_N$ is an $(N \times 1)$ matrix (vector) with all elements equal to 1, and μ is the target return. The optimization problem is extended by a bound for each weight. The vector of constraints $a = (a_1, \ldots, a_N)^{\top}$ with $a_i \in [0, \infty)$ for all $i = \{1, \ldots, N\}$ is an $(N \times 1)$ vector and can be given (or estimated) upfront. Furthermore, an upper bound for the sum over the absolute values of the weights is introduced. Then, a constrained Markowitz portfolio is defined as

$$\min \quad w^{\top}\hat{\Sigma}w \tag{2}$$

$$\begin{aligned} &\text{s.t. } \mathbf{1}_{\mathrm{N}}^{\top}\mathrm{w} = 1, \ \mu \leq \mathrm{x}^{\top}\mathrm{w}, \\ &\quad ||w||_1 \leq c, \ |w_i| \leq a_i \, \forall i. \end{aligned} \tag{3}$$

The parameter c controls the amount of shortselling, $c \in [1, \infty)$. Fan et al. (2012) showed how the risk of the estimated portfolio is influenced by the choice of c while $a_i = \infty$ for all a_i. The estimation of $\hat{\Sigma}$ is crucial for the method, yet the huge dimensionality of the asset universe and limited data availability make the estimation of $\hat{\Sigma}$ difficult. Thus we employ the covariance estimator of Ledoit and Wolf (2004). It has been shown to be invertible, well-conditioned, and asymptotically more accurate than the sample covariance matrix. The estimator is a weighted average of the identity matrix and the sample covariance matrix. The identity matrix is a well-conditioned matrix and due to its being combined with the sample covariance matrix under a quadratic loss function, the resulting estimator has respective property and is more accurate than the sample covariance matrix, Ledoit and Wolf (2004). For more details, we refer to Section A.1 in the Appendix.

However, Markowitz portfolio optimization neglects the effect of higher moments when minimizing the risk. Due to the often occurring strong decreases in the CC market, portfolios optimized for Conditional Value-at-Risk (CVaR) will be employed to compare their performance with the Markowitz portfolio.

Define $y(w) = w^{\top}X$ as the returns of the portfolio with weights w and α as the probability level such that $0 < \alpha < 1$, the Value-at-Risk (VaR) is defined by

$$\mathrm{VaR}_{\alpha}(w) = -\inf \{y | F(y|w) \geq \alpha\} \tag{4}$$

with $F(y|w)$ being the distribution function of the portfolio returns with weights w. $\mathrm{VaR}_{\alpha}(w)$ is the corresponding α-quantile of the cdf, defining the loss to be expected $(\alpha \cdot 100)\%$ of the time. A negative sign is added to turn the negative return into a loss, which is defined on the positive domain. Considering the VaR only captures one quantile rather than the whole shape of the tails of the return distribution, it is silent on losses beyond that critical point. Overcoming the limitations of the VaR measure, the CVaR was introduced which measures the expected loss larger or equal to the $\mathrm{VaR}_{\alpha}(w)$. Since CCs are at times subject to huge losses, the CVaR will give a more accurate impression of the investment at risk.

The Conditional Value-at-Risk is defined as, Rockafellar and Uryasev (2000),

$$\mathrm{CVaR}_\alpha(w) = -\frac{1}{1-\alpha} \int_{y(w) \leq -\mathrm{VaR}_\alpha(w)} y f(y|w) dy, \tag{5}$$

with $\frac{\partial}{\partial y} F(y|w) = f(y|w)$ the probability density function for the portfolio returns y with weights vector w. Thus $\mathrm{CVaR}_\alpha(w)$ includes the expected value over the tail of the pdf left of the $\mathrm{VaR}_\alpha(w)$.

The optimization problem is, then,

$$\min \quad \mathrm{CVaR}_\alpha(w) \tag{6}$$

$$\begin{aligned} &\text{s.t. } \mathbf{1}_N^\top \mathrm{w} = 1, \ \mu \leq \mathrm{x}^\top \mathrm{w}, \\ &||w||_1 \leq c, \ |w_i| \leq a_i \, \forall i. \end{aligned} \tag{7}$$

So far we have taken into consideration the parameter c. However, short-selling is still rare in the CC market. Thus, the value of c is an issue. First exchanges started to offer the possibility of short selling for larger CCs, and the launch of Bitcoin Futures allows it too. But it is still not possible for most CCs to be shorted. Due to the inability to sell short in the CC market, the exposure is set to $c = 1$, which produces a no short-sell constraint combined with $\mathbf{1}_N^\top w = 1$. Surely it is only a matter of time until short selling is common in the CC market too. In that case, our optimization problem could be amended to allow for it, which would enable one to use hedging effects to decrease the historical risk. However, this approach can cause extreme weights on single assets, so that the position is not tradable in a real market situation.

4 LIBRO

So far, the actual measure for liquidity for the constraints was not further explained. Yet this is a central point of this study, because CCs have far lower daily trading amounts than traditional financial assets, causing a liquidity problem for any portfolio construction. To address this issue, one tries to avoid holding too many illiquid assets by employing weight constraints $|w_i| \leq a_i$ for all $i = 1, \ldots, N$.

Many different liquidity measures have been proposed in the literature, tackling either one aspect of liquidity or several aspects at the same time, Wyss (2004). In the context of this paper, we are interested in

1. being able to trade the assets on the reallocation date,
2. being able to sell or buy between two reallocation dates, if necessary.

Naturally, a more liquid asset should be allowed to have a higher weight in the portfolio.

Our data set consists of daily price and turnover value observations, which enables us to use the turnover value as a proxy for liquidity. An even better measure would be Limit Order Book based measures since they allow a deeper look into the behaviour of the markets. As we do not possess of a sufficient history of these data to run an analysis, for the moment the liquidity measures using such information are not applicable. Since the time period of interest for a trading action is one day, daily closing data for the standard assets and CCs are going to be used. The trading volume of asset i at date t, measured in USD, is defined to be[1]

$$TV_{it} = p_{it} \cdot q_{it} \tag{8}$$

where p_{it} is the closing price of asset i at date t, and q_{it} is the number of shares of asset i traded at date t. The liquidity of asset i in a sample with time length T can be measured by, e.g. the mean or median of the daily trading volume. The mean has the disadvantage that days with outliers in the trading volume will distort the liquidity measure. The median is a robust measure and therefore better suited for this situation. We define the sample median of the trading volume to be

$$TV_{i,m} = \frac{1}{2}(TV_{i,u} + TV_{i,l}) \tag{9}$$

where $TV_{i,u} = TV_{i,\lceil \frac{T+1}{2} \rceil}$, $TV_{i,l} = TV_{i,\lfloor \frac{T+1}{2} \rfloor}$ and $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ indicate the ceiling and floor operators respectively.

Next we construct the liquidity bound. Taking into account that investors enter the market with different sizes of portfolios, this has to be reflected in the liquidity bound. For example, an investor with a portfolio of 100 USD will not be as strongly affected by low liquidity as an investor with $10,000,000$ USD. Denote by M the total amount we are going to invest and recall that $w_i,\ i = 1, \dots, N$ is the weight for asset i, so Mw_i is the market value of the position in asset i. Hence the constraint on w_i concerning the liquidity of asset i is

$$Mw_i \leq TV_{i,m} \cdot f_i, \tag{10}$$

where f_i is a factor controlling the maximum ratio of the position in asset i to its median trading volume, i.e. liquidity. The larger the f_i, the more bullish the investor is on asset i, and the more likely the position in asset i will suffer from a low liquidity problem when clearing or rebalancing. For example, setting $f_i = 0.1$ corresponds to a position in asset i not larger than 10% of the median trading amount of asset i. Dividing both sides of Equation (10) by M yields the bound for w_i:

$$w_i \leq \frac{TV_{i,m} \cdot f_i}{M} = \hat{a}_i.$$

Hence, the Markowitz portfolio optimization framework we will use in this article is

$$\begin{aligned} &\min \quad w^\top \hat{\Sigma} w \\ &\text{s.t. } \mathbf{1}_{\mathrm{N}}^\top \mathrm{w} = 1,\ \mu \leq \mathrm{x}^\top \mathrm{w},\ ||\mathrm{w}||_1 = 1, \\ &w_i \leq \frac{1}{M} \cdot \widehat{Liq}_i = \hat{a}_i\ \forall i, \end{aligned} \tag{11}$$

4 Note this is an approximation of the actual trading volume. The trading volume of asset i on date t, measured in USD, is calculated as $TV_{it} = \sum_{j=1}^{N_t} p_{it,j} \cdot q_{it,j}$, where N_t is the number of times that asset i is traded on date t, and $p_{it,j}$ and $q_{it,j}$ are the trading price and number of traded shares for the jth trade, $j = 1, \cdots, N_t$. However, due to the lack of sufficient historical data on the per-trade data of the CCs, the theoretical definition is not available for a sufficiently large sample size.

where $\widehat{Liq} = (TV_{1,m} \cdot f_1, \cdots, TV_{N,m} \cdot f_N)^\top$. The CVaR optimization problem can thus be expressed as

$$\begin{aligned} &\min \quad \mathrm{CVaR}_\alpha(w) \\ &\text{s.t. } \mathbf{1}_N^\top \mathrm{w} = 1,\ \mu \leq \mathrm{x}^\top \mathrm{w},\ \|\mathrm{w}\|_1 = 1, \\ &w_i \leq \frac{1}{M} \cdot \widehat{Liq}_i = \hat{a}_i\, \forall i. \end{aligned} \tag{12}$$

5 Application

For the application we treat three kinds of settings. First we perform an in-sample analysis without liquidity constraints to investigate whether including CCs in traditional financial portfolios increases the risk–return trade-off. Furthermore, we intend to determine whether including alt-coins (i.e. CCs other than Bitcoin) is profitable and confirm whether the introduction of liquidity constraints is necessary. Second, when liquidity constraints are included in the in-sample analysis, we set $f_i = 0.01$ for all $i = 1, \ldots, N$, that is, we assume that our position in a certain asset cannot exceed 1% of its daily trading volume. This is a quite conservative setting, and investors who want to be more aggressive can enlarge this factor. We choose to be conservative because the CC market, especially the alt-coin market, exhibits swings in its daily trading volume. Thus we are securing our portfolio choice against these swings. Furthermore, trading on the entire daily trading volume would be a rather strict anticipation. In both cases, the portfolio and weights are formed over the entire time period. For the out-of-sample portfolio formation, we choose an extending window approach. The initial portfolio weights are derived for the time period from April 22, 2014 until December 31, 2014. Due to the limited data availability in the CC market, the April 2014 data are not omitted, so as to enhance the estimation. Two kinds of portfolio rebalancing frequencies are employed: weekly and monthly, while the underlying data period is extended. Thus for the monthly case, on the next re-evaluation date, which is February 1, 2015, the derivation period is extended to the period from April 22, 2014 until January 1, 2015. For the portfolio formation under CVaR, the quantile level in all cases is chosen to be $\alpha = 0.05$. For the liquidity constraint, we maintain the same setting for the whole of the analysis, which is either unbounded (without liquidity constraint), or bounded with investment amount equal to one of either 1.0×10^5, 1.0×10^6, or 1.0×10^7 US dollars, see Equations (11) and (12). For selecting the target return μ, the Sharpe Ratio is maximized for the Markowitz portfolio, and the Return-to-CVaR Ratio for the CVaR portfolio. The median over the trading volume, necessary for the constraints, is chosen in-sample over the entire sample and out-of-sample over the extending window. We compare two different data sets consisting of traditional assets by adding CCs to them: one based solely on stocks from the S&P 100 (S), and one on stocks plus US Bonds and Commodities (SBC). We denote the S and SBC plus CCs portfolios as S-CC and SBC-CC, respectively.

5.1 In-sample portfolio formation

5.1.1 Without liquidity bounds

The cumulative returns of the portfolios formed by the S&P 100 component stocks and SBC with or without CCs for both definitions of risk are shown in Figure 3. One can see

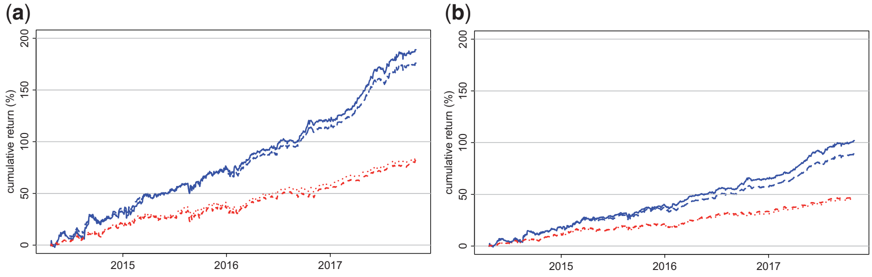

Figure 3 The panels 3a and 3 b indicate the cumulative return performance of S/S-CC and SBC/SBC-CC portfolios. The dashed (- - -) and dotted (· · · ·) lines indicate the S/SBC portfolios formed with CVaR and Markowitz method respectively, while the solid (—) and long-dashed (– –) lines correspond to the S-CC/SBC-CC portfolios formed with CVaR and Markowitz method. The coloured version is available online.

(a) A comparison of the cumulative returns of the S-CC portfolios. (b) A comparison of the cumulative returns of the SBC-CC portfolios.

that the improvement in the returns is remarkable and consistent throughout the sample. Starting from the very beginning, the portfolio with CCs (S-CC/SBC-CC) outperforms the one without, with the difference becoming larger and larger as time goes on. At the end of the sample, the S-CC Markowitz portfolio gives a cumulative return of 173.3%, while the S portfolio ends at 85.3%, only one-half of the former. However, comparing the S-CC and the SBC-CC, the former outperforms the latter, with double the cumulative return. This is because the latter reaches the maximum Sharpe ratio at a lower target return. In fact, the optimal Sharpe ratio of the latter portfolio is always higher than those of the former portfolio, see Tables 3 and 4. The case is similar for the CVaR portfolio, with the difference being even larger. It is interesting to observe that the portfolio returns for the CVaR portfolio are higher than those for the Markowitz, since this suggests that the returns increase due to holding CCs rises more significantly than does the risk induced from the tails. The summary statistics of the Markowitz portfolio returns are given in Tables 3 and 4, where the S column is the one for the S portfolio and the SBC column is that for the SBC portfolio, and the "unbounded" column is for the S-CC and SBC-CC without implementing the liquidity constraints. As an example of the results of both strategies (Markowitz and CVaR) with S and SBC, we will now look in more detail at the S Markowitz portfolio results, Table 3. Coinciding with the previous finding, the annualized average returns of the S-CC (48%) is twice that of the S portfolio (23%). Though the volatility is a bit higher, the whole risk–return trade-off has improved after adding CCs, since the Sharpe Ratio increases from 0.12 to 0.18. The higher moments of portfolio returns are also improved: after adding CCs, the skewness changes from –0.31 to 0.14, and the kurtosis decreases from 5.50 to 4.55. The Maximum drawdown, which measures the downside risk of the portfolio, stays the same. Similar results can be observed when forming a CVaR portfolio, see Tables 3 and 4. When using the SBC portfolio, the average returns roughly halve, while the standard deviation shrinks by more than one-half, resulting in an improvement in the Sharpe ratio.

Even though these are striking results, a check on the weights suggests that a liquidity constraint aiming at lowering the weight on illiquid assets is needed. The weights different

Table 3 Summary statistics of in-sample S/S-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter

			S-CC			
		S	unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.23	0.48	0.33	0.27	0.27
	Ann.STD	0.13	0.17	0.14	0.13	0.13
	Sharpe-ratio	0.12	0.18	0.15	0.14	0.13
	Skewness	–0.31	0.14	–0.24	–0.54	–0.55
	Kurtosis	5.50	4.55	5.20	5.70	5.68
	Max drawdown	0.10	0.10	0.11	0.11	0.11
	Auto correlation	–0.02	–0.04	–0.01	0.00	0.01
CVaR	Ann.Ret	0.22	0.52	0.32	0.26	0.26
	Ann.STD	0.12	0.18	0.14	0.12	0.12
	Sharpe-ratio	0.12	0.18	0.15	0.14	0.13
	Skewness	–0.27	0.28	–0.00	–0.47	–0.48
	Kurtosis	5.50	4.81	4.89	5.60	5.54
	Max drawdown	0.10	0.09	0.10	0.10	0.10
	Auto correlation	–0.03	–0.05	–0.02	–0.02	–0.02

Table 4 Summary statistics of in-sample SBC/SBC-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter

			SBC-CC			
		SBC	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.13	0.25	0.18	0.15	0.15
	Ann.STD	0.06	0.08	0.07	0.06	0.06
	Sharpe-ratio	0.14	0.19	0.17	0.16	0.15
	Skewness	–0.34	0.01	–0.27	–0.46	–0.48
	Kurtosis	4.95	4.33	4.81	5.04	5.20
	Max drawdown	0.04	0.05	0.05	0.05	0.05
	Auto correlation	0.03	–0.01	0.01	0.03	0.04
CVaR	Ann.Ret	0.13	0.28	0.19	0.16	0.15
	Ann.STD	0.06	0.09	0.07	0.06	0.06
	Sharpe-ratio	0.13	0.19	0.16	0.16	0.15
	Skewness	–0.26	0.29	0.03	–0.39	–0.38
	Kurtosis	4.77	4.59	4.69	4.90	4.86
	Max drawdown	0.05	0.05	0.05	0.05	0.05
	Auto correlation	0.01	–0.02	–0.02	0.00	0.01

from 0 given to CCs in the unbounded case for the Markowitz and CVaR portfolios are shown in Tables 5–8.

The key information conveyed from the tables is that the more liquid (traded) CCs are not generally given a larger weight. The biggest weight on a CC is given to NLG (Gulden),

Table 5 Weights (in %) given to CCs in in-sample S-CC Markowitz portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts *M*. Underline/bold indicates that the weight equals to its liquidity upper bound

	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	5.37	8.28	9.01
XRP	3.72	4.53	**0.80**	**0.08**
DASH	3.17	**2.13**	**0.21**	**0.02**
DGB	0.95	**0.13**	**0.01**	**0.00**
VTC	0.18	**0.10**	**0.01**	**0.00**
NLG	5.30	**0.02**	**0.00**	**0.00**
FLO	1.53	**0.01**	**0.00**	**0.00**
RBY	0.84	**0.03**	**0.00**	**0.00**
NOTE	0.18	**0.01**	**0.00**	**0.00**
CBX	0.12	**0.00**	**0.00**	**0.00**
total	16.00	12.32	9.32	9.11

Table 6 Weights (in %) given to CCs in in-sample SBC-CC Markowitz portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts *M*. Underline/bold indicates that the weight equals to its liquidity upper bound

	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	2.11	3.14	3.68
XRP	1.47	1.77	**0.80**	**0.08**
DASH	1.34	1.41	**0.21**	**0.02**
DGB	0.43	**0.13**	**0.01**	**0.00**
VTC	0.13	**0.10**	**0.01**	**0.00**
NLG	2.26	**0.02**	**0.00**	**0.00**
FLO	0.64	**0.01**	**0.00**	**0.00**
RBY	0.37	**0.03**	**0.00**	**0.00**
total	6.65	5.57	4.18	3.79

which has a medium level of liquidity. However, Bitcoin (BTC), the CC that has the largest trading volume, is given a zero weight. Considering the CCs in the unbounded S-CC Markowitz portfolio, their weights account for 16.3% of the whole portfolio. The top three CCs by weight are NLG (Gulden), XRP (Ripple), and DASH (Dash coin), at 5.3%, 3.7% and 3.2% respectively. This shows that alt-coins are more appealing than BTC in terms of variance minimization, at least during the period covered by this paper. Furthermore, the inclusion of liquidity constraints appears necessary, since the highly weighted CCs have, partly, low liquidity compared to BTC. For instance, with an investment amount of $M = 1.0 \times 10^6$ US dollars, and considering S-CC, one needs to hold a position of 5, 300 US

Table 7 Weights (in %) given to CCs in in-sample S-CC CVaR portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts *M*. Underline/bold indicates that the weight equals to its liquidity upper bound

	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	3.82	8.71	9.61
XRP	3.30	5.57	**0.80**	**0.08**
DASH	3.72	**2.15**	**0.22**	**0.02**
DGB	0.00	**0.12**	**0.01**	**0.00**
NLG	6.22	**0.02**	**0.00**	**0.00**
FLO	2.79	**0.01**	**0.00**	**0.00**
RBY	1.11	**0.03**	**0.00**	**0.00**
MAX	0.50	**0.01**	**0.00**	**0.00**
CBX	0.28	0.00	**0.00**	**0.00**
ZEIT	0.12	**0.00**	**0.00**	**0.00**
total	18.03	11.73	9.74	9.71

Table 8 Weights (in %) given to CCs in in-sample SBC-CC CVaR portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The 'unbounded' refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts *M*. Underline/bold indicates that the weight equals to its liquidity upper bound

	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	1.27	4.38	4.73
XRP	1.32	2.33	**0.80**	**0.08**
DASH	1.80	**2.15**	**0.22**	**0.02**
DGB	0.00	**0.12**	**0.01**	**0.00**
NLG	3.18	**0.02**	**0.00**	**0.00**
FLO	1.35	**0.01**	**0.00**	**0.00**
RBY	0.53	**0.03**	**0.00**	**0.00**
MAX	0.26	**0.01**	**0.00**	**0.00**
total	8.44	5.94	5.40	4.84

dollars in NLG, which is equal to 47.3% of its average daily trading volume. Taking into consideration the price impact, this position is neither easy to obtain nor to clear. Table 7 shows the weights for the CVaR p1.0×10^6ortfolio, providing similar results. In fact, the influence of the CCs and NLG in particular is even higher. However, when relying on the SBC portfolio, the weights given to CCs shrink markedly, Table 8. It seems the inclusion of bonds and commodities shifts the Mean–Variance and Mean–CVaR Frontier so strongly that the resulting portfolio favors having fewer CCs. However one can see that this harms the returns achieved from the portfolio.

5.1.2 Including liquidity constraints

The cumulative returns of portfolios with liquidity constraints included are shown in Figure 4, with the summary statistics of these returns in Table 3. When liquidity constraints are imposed, the cumulative returns shift downward compared to those without liquidity constraints. The larger the amount to invest, the lower the cumulative returns. This is not surprising, since adding a liquidity constraint makes the global optimal Sharpe ratio point and Return-to-CVaR point unreachable, and the larger the investment amount, the tighter the liquidity constraint. Hence, the constrained optimal Sharpe ratio point is further from the unconstrained one. When the investment amount is set to 1.0×10^5 or US dollars, the cumulative returns of the Markowitz portfolios still outperform the one formed by only traditional assets throughout the sample, and by the end of the sample period, the cumulative returns are 128.5% and 117.0% respectively, still 43.2% and 31.7% higher than the S portfolio, by considering this setting as an example. When the investment amount is increased to 1.0×10^7 US dollar, the portfolio does not outperform the one containing only stocks until 2017, however it ends at 98.8%, still 13.5% higher. For the CVaR portfolios, the constrained portfolios still outperform the S and SBC portfolios, however only after 2017 and with excess returns of 12.2% and 5.8%, respectively.

Summary statistics also favour the portfolios with CCs, see the last three columns of Tables 3 and 4. When the investment amounts are set to 1.0×10^5, 1.0×10^6 or 1.0×10^7 US dollars, the liquidity constrained portfolios have higher average return and higher Sharpe-ratio than the S/SBC portfolios, with, however, a slightly higher standard deviation and almost the same downside risk. Kurtosis and Skewness show a mixed picture. Mostly the absolute value of either increases, making it less of a Gaussian distribution. Yet this observation is not surprising, taking into account the strong deviations of CC return series from normality. Overall, the summary statistics provide support for adding CCs to portfolios consisting of the chosen traditional assets.

To see how the liquidity constraint affects the optimization procedure, we turn to an analysis of the weights. For the comparison, we focus on the weights given to the CCs, since the impact of CCs on the portfolio is of greater interest and no liquidity bound on the traditional assets is binding in any situation. The weights are shown in Tables 5–8, where we show the weights for the CCs included in the portfolio in any considered situation. The first column shows the weights when no liquidity constraint is implemented, while the remaining three columns show those when liquidity constraints are included with the three different investment amounts. The weights coloured red indicate that its liquidity upper bound is binding, i.e. the weight given to this CC just equals its liquidity upper bound. Before implementing the liquidity upper bound, the CCs account for 16% of the total position for the S-CC Markowitz portfolio, with the largest weight 5.3% given to NLG, and zero weight to Bitcoin. After including a liquidity constraint, the total weights on the CCs decrease to 12.3%, 9.3% and 9.1% as the investment amount increases from 1.0×10^5 to 1.0×10^7 US dollars. This is not surprising, since the liquidity upper bounds limit the weight given to the CCs. When the investment amount equals 1.0×10^5 US dollars, the lower 8 CCs are binding, including NLG, which only has a weight of 0.02%. When the investment increases to 1.0×10^6 and 1.0×10^7 US dollars, only the bound on Bitcoin is not binding.

As the liquidity constraints tighten, the weight on Bitcoin increases from 5.4% to 9.0%, which shows the great investment potential of the Bitcoin market, since it can account for about 9% of the portfolio when formed together with S&P 100 stocks, while not being

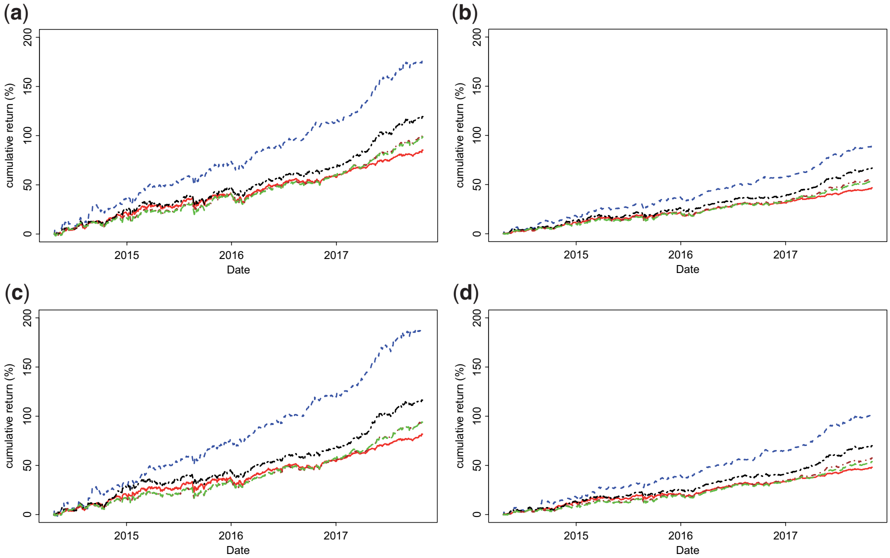

Figure 4 The solid and dashed lines indicate the cumulative return performance of Markowitz and CVaR portfolios. The solid line (—) and dashed line (- - -) stand for S/SBC and S-CC/SBC-CC without liquidity constraints respectively. The remaining 3 portfolios are S-CC/SBC-CC ones containing the bounds $M = 1 \times 10^5$ USD (▬ ▪ ▬), $M = 1 \times 10^6$ USD (–·–), $M = 1 \times 10^7$ USD (– –). The coloured version is available online.

LIBROinsample

(a) In-sample cumulative returns of S-CC Markowitz portfolios (b) In-sample cumulative returns of SBC-CC Markowitz portfolios (c) In-sample cumulative returns of S-CC CVaR portfolios (d) In-sample cumulative returns of SBC-CC CVaR portfolios

constrained. When considering the CVaR portfolio, the cumulative weight on the CCs in the unbound case is 18%, thus even larger, however in the constrained cases this shrinks to 9.71%. For the SBC portfolios, with either definition of risk (Markowitz or CVaR), the weight given to CCs is considerably lower, so the portfolios favour bonds and commodities. Still no weight is given to BTC in the unbounded case. However, when optimizing with liquidity constraints, BTC receives a weight and the constraints become active on various CCs. This shows that the constraints are necessary for achieving the goal set in this study, namely constructing portfolios with CCs in which the positions in the CCs can be easily cleared. After illustrating the potential effect of CCs on portfolio performance and the effect of liquidity constraints on the in-sample analysis, we turn next to an out-of-sample study to investigate the performance of the portfolios under pseudo-real conditions.

5.2 Out-of-sample portfolio formation

After having analysed the potential of CCs for enhancing the performance of a combined portfolio, an out-of-sample analysis remains to justify their applicability in real-world investments. The S portfolio and the SBC portfolio will be constructed with monthly rebalanced weights, calculated using all the sample data before the rebalancing day. First, the portfolio is formed on January 1, 2015 and held to January 31, 2015, with weights calculated using the sample from April 22, 2014 to December 31, 2014. Then, data from April 22, 2014 to January 31, 2015 is used to calculate the new weights, and the portfolio is rebalanced accordingly at February 1, 2015. For subsequent periods, the portfolio will be rebalanced at the first day of each month, with the weights calculated using all the sample data before that day. We choose this extending window approach to calculate the rebalancing weights due to the limited amount of data in the sample.

The cumulative returns of the Markowitz portfolios are illustrated in Figure 5, where panel 5a shows S/S-CC portfolios, and panel 5c shows S/SBC-CC portfolios. For each panel, it is obvious that the S-CC/SBC-CC portfolio, no matter whether liquidity bounded or not, outperforms its counterpart without CCs at the end of the sample. For the S-CC portfolio, when no liquidity constraint is applied, the cumulative returns exceed the S portfolio at the very beginning of 2015, with the difference continuing to increase until the end of the sample. In the whole out-of-sample period, the cumulative returns of the S-CC portfolio without liquidity constraints is over 80%, while that of the S portfolio is 15.5%, a quite substantial improvement. The SBC-CC portfolio without liquidity constraints outperforms the SBC portfolio from March 2016 onwards, and the difference keeps enlarging in the remaining periods. At the end of the sample, the cumulative returns of the SBC-CC portfolio reach 37.4%, which is substantially higher than the 12.1% of the SBC portfolio. The big improvement of the S-CC/SBC-CC portfolio over the S/SBC one indicates the huge potential investment gain that can be obtained by including CCs in a portfolio. However, as stated in the in-sample cases, only when these improvements persist when liquidity constraints are included, can one infer that the profits are feasible in practice.

Now comes the situation when liquidity constraints are included. In Figure 5, we label the cumulative returns calculated with liquidity constraints with black, brown and green, indicating an investment amount equal to 1×10^5, 1×10^6, and 1×10^7 US dollars, respectively. For all three investment amounts, the S-CC portfolios exceed the S one starting from March of 2016, and the difference does not become large until 2017. At the end of the sample, the liquidity bounded portfolios with investment amounts 1×10^5, 1×10^6, 1×10^7 US dollars end with a cumulative return equal to 40.0%, 28.6% and 29.2%, which is 24.5%, 13.1% and 13.7% higher than the pure stock portfolio. When bond and commodity indexes are included, the liquidity bounded cumulative return under an investment amount of 1×10^5, 1×10^6, 1×10^7 US dollars reaches 21.2%, 17.7% and 16.4%, all of which outperform the SBC portfolio. All in all, adding CCs into portfolios is profitable even after controlling for low liquidity by imposing constraints.

Summary statistics also favour the portfolios with CCs added, see Tables 9 and 10. In either the S-CC or SBC-CC case, the portfolios with CCs always dominate the one without, regarding returns and Sharpe ratio. When no liquidity bound is applied or when the investment amount is 1×10^5 US dollars, the portfolio show less negative skewness and less heavy tails. Although the skewness and kurtosis may get worse under a tighter liquidity constraint when the investment amount gets larger, the maximum drawdown improves after CCs are added, which is somehow surprising, since the CC market is considered highly risky. Interestingly, the mean returns on SBC/SBC-CC are lower than for S/S-CC, yet in combination with a lower volatility as well. The SBC-CC portfolio outperforms S-CC in two ways: first, it only has about one-half the max drawdown, which is a substantial

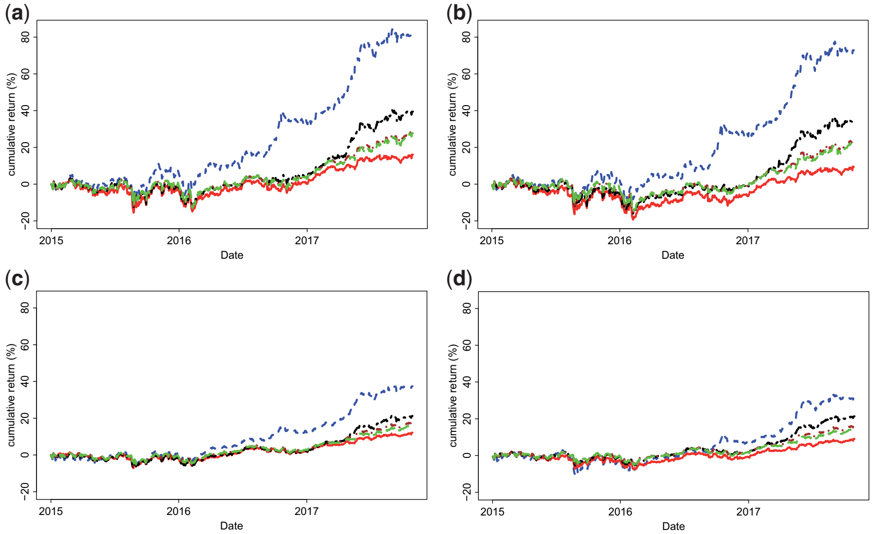

Figure 5 Out-of-sample cumulative returns with monthly and weekly adjusted Markowitz portfolios. The solid line (—) and dashed line (- - -) stand for S/SBC and S-CC/SBC-CC without liquidity constraints respectively. The remaining 3 portfolios are S-CC/SBC-CC ones containing the bounds $M = 1 \times 10^5$ USD (▬ ▪ ▬), $M = 1 \times 10^6$ USD (–·–), $M = 1 \times 10^7$ USD (– –). The coloured version is available online.

(a) Out-of-sample cumulative returns for monthly adjusted S-CC Markowitz portfolios (b) Out-of-sample cumulative returns for weekly adjusted S-CC Markowitz portfolios (c) Out-of-sample cumulative returns for monthly adjusted SBC-CC Markowitz portfolios (d) Out-of-sample cumulative returns for weekly adjusted SBC-CC Markowitz portfolios

Table 9 Summary statistics of out-of-sample monthly rebalanced S/S-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter

			S-CC			
		S	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.05	0.28	0.13	0.10	0.09
	Ann.STD	0.13	0.19	0.15	0.14	0.14
	Sharpe-ratio	0.03	0.09	0.06	0.04	0.04
	Skewness	–0.38	0.09	–0.29	–0.44	–0.44
	Kurtosis	5.98	5.23	5.95	6.20	6.15
	Max drawdown	0.17	0.17	0.16	0.16	0.16
	Auto correlation	0.03	–0.02	0.03	0.04	0.04
CVaR	Ann.Ret	0.07	0.28	0.12	0.08	0.16
	Ann.STD	0.14	0.21	0.16	0.17	0.18
	Sharpe-ratio	0.03	0.08	0.05	0.03	0.06
	Skewness	–0.21	0.12	–0.48	–0.31	–0.11
	Kurtosis	5.83	5.13	6.39	9.36	9.44
	Max drawdown	0.16	0.19	0.18	0.25	0.23
	Auto correlation	–0.00	–0.05	0.04	0.10	0.12

Table 10 Summary statistics of out-of-sample monthly rebalanced SBC/SBC-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter

			SBC-CC			
		SBC	Unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.04	0.13	0.07	0.06	0.06
	Ann.STD	0.07	0.10	0.08	0.07	0.07
	Sharpe-ratio	0.04	0.08	0.06	0.06	0.05
	Skewness	−0.33	−0.10	−0.34	−0.37	−0.38
	Kurtosis	5.98	5.63	6.33	6.31	6.40
	Max drawdown	0.08	0.10	0.08	0.07	0.07
	Auto correlation	0.08	0.03	0.07	0.08	0.08
CVaR	Ann.Ret	0.03	0.17	0.07	0.05	0.11
	Ann.STD	0.07	0.11	0.11	0.14	0.14
	Sharpe-ratio	0.03	0.10	0.04	0.02	0.05
	Skewness	−0.36	0.30	−0.94	−0.92	−0.51
	Kurtosis	5.47	5.87	14.16	14.93	15.48
	Max drawdown	0.11	0.06	0.14	0.22	0.23
	Auto correlation	0.05	−0.02	0.09	0.19	0.22

decrease in the downside risk; second, at larger investment amounts, it has higher Sharpe ratios.

Looking at the weights of the CCs gives an answer for how CCs influence the performance of the portfolios. For the monthly rebalanced weights, see Figures 7 and 8. Obviously the CCs taking strong positions in the portfolios in the unbounded case, for example, NLG, which even reaches 8% in the fourth quarter of 2016, get ruled out when liquidity constraints are added. Furthermore, by the red rectangles we indicate that the weight reaches its respective upper bound on that reallocation date. The constraints are mostly in place, giving support for their introduction into the method. For the S-CC Markowitz portfolios, when the investment amount equals 1.0×10^5 US dollars, 6 CCs are included over time. When the investment amount increases to 1.0×10^6 and 1.0×10^7 US dollars, only 3 and 1 CCs are included over time, respectively, and BTC becomes the only one that is not affected by the liquidity bound. A further observation is the absence of Bitcoin, the largest and most liquid CC, in the unbounded portfolio. Yet with liquidity constraints, it is always part of the portfolio and does not reach its upper bound. Additionally, it becomes apparent that Bitcoin and also Ripple receive higher weights in 2017, due to their better Sharpe ratios, thus adding value to the portfolios due to the strong gains in the CC market in this period.

Shifting our analysis to the CVaR portfolios provides partly different observations. Still the unbounded portfolios with CCs clearly outperform those without CCs, and in the constrained portfolios, the cumulative returns perform better at the end of the sample, Figure 6. However, for the constrained case with investment amounts $M = 1.0 \times 10^5$ and $M = 1.0 \times 10^6$ US dollars, the improvements are not consistent throughout the sample: the pure S portfolio still outperforms until March of 2017 and August of 2017. It is interesting to observe that in this case the portfolio having the highest investment amount, and therefore the strongest constraints, performs the best among the constrained ones, with a stable

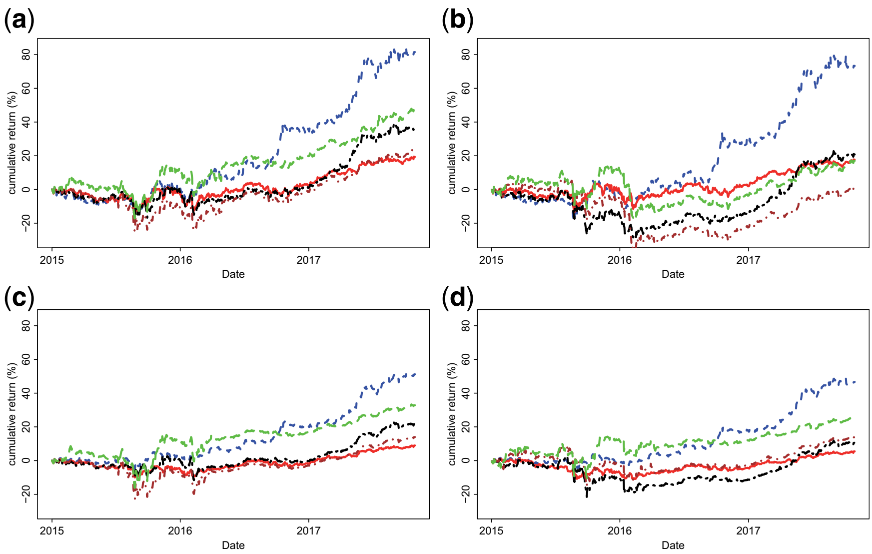

Figure 6 Out-of-sample cumulative returns with monthly and weekly adjusted CVaR portfolios. The solid line (—) and dashed line (- - -) stand for S/SBC and S-CC/SBC-CC without liquidity constraints, respectively. The remaining 3 portfolios are S-CC/SBC-CC ones containing the bounds $M = 1 \times 10^5$ USD (▬ ▪ ▬), $M = 1 \times 10^6$ USD (–·–), $M = 1 \times 10^7$ USD (– –). The coloured version is available online.

(a) Out-of-sample cumulative returns for monthly adjusted S-CC CVaR portfolios (b) Out-of-sample cumulative returns for weekly adjusted S-CC CVaR portfolios (c) Out-of-sample cumulative returns for monthly adjusted SBC-CC CVaR portfolios (d) Out-of-sample cumulative returns for weekly adjusted SBC-CC CVaR portfolios

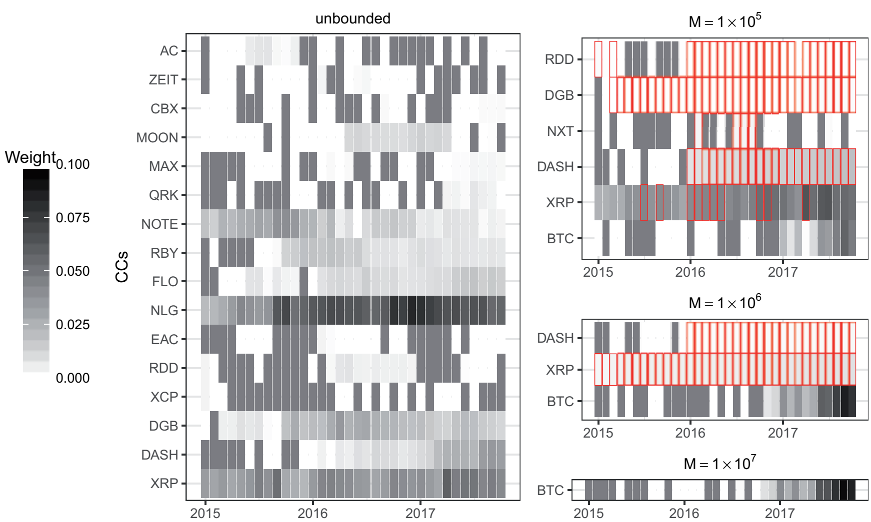

Figure 7 Weights given to CCs for S-CC Markowitz portfolios at each monthly rebalancing date under the three different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing date are given. The darker the color, the larger the weight. Weights in ☐ are bounded by their upper bounds.

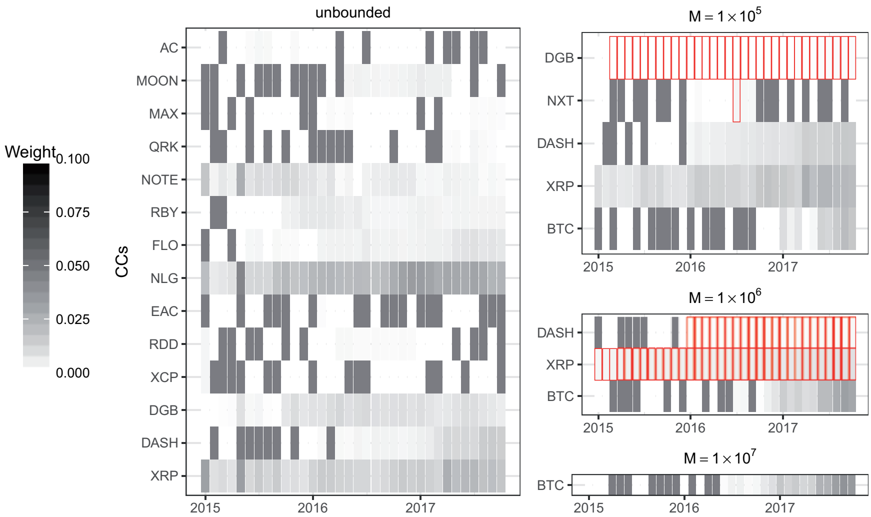

Figure 8 Weights given to CCs for SBC-CC Markowitz portfolio at each monthly rebalancing date under the three different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing date are given. The darker the color, the larger the weight. Weights in $\boxed{}$ are bounded by their upper bounds.

improvement compared to the pure S portfolio, and the highest cumulative returns at the sample end. A similar observation can be made for the SBC-CC portfolio. Looking at the weights for the monthly reallocation, Figures 9 and 10, one observes only Bitcoin being included in the most strongly restricted portfolio. Since CVaR gives less weight to assets having high tail risk, it can be inferred that for larger investment amounts, only the tail risk of Bitcoin is sufficiently low to be appropriate for the portfolio. Interestingly, this causes the portfolio to outperform the other constrained ones, although the unbounded one still outperforms.

Looking at the summary statistics, Tables 9 and 10, the SBC-CC unbounded portfolio performs better, with an annualized return of 0.17, than the corresponding Markowitz portfolio, with an annualized return 0.13. Also the Sharpe Ratio is enhanced. It is quite remarkable that the annualized return is high for $M = 1.0 \times 10^7$ US dollars, yet the Sharpe Ratio (0.05) is the same as for the corresponding Markowitz portfolio. For the S-CC portfolios, again they perform better than their SBC-CC CVaR counterparts, both in terms of annualized return and Sharpe Ratio. Again the $M = 1.0 \times 10^7$ US dollars portfolio shows a remarkably high annualized return. Comparing the weights, Figures 7–10, one observes that only BTC was included, yet for the CVaR portfolios with higher values, only from mid-2016 onwards. For the Markowitz portfolios, BTC was also included in 2015.

5.3 Robustness: Weekly versus monthly rebalancing

Since the CCs' market has a great deal of variation, it is an interesting question to ask how will the portfolio perform if we readjust the portfolio more frequently, for example weekly

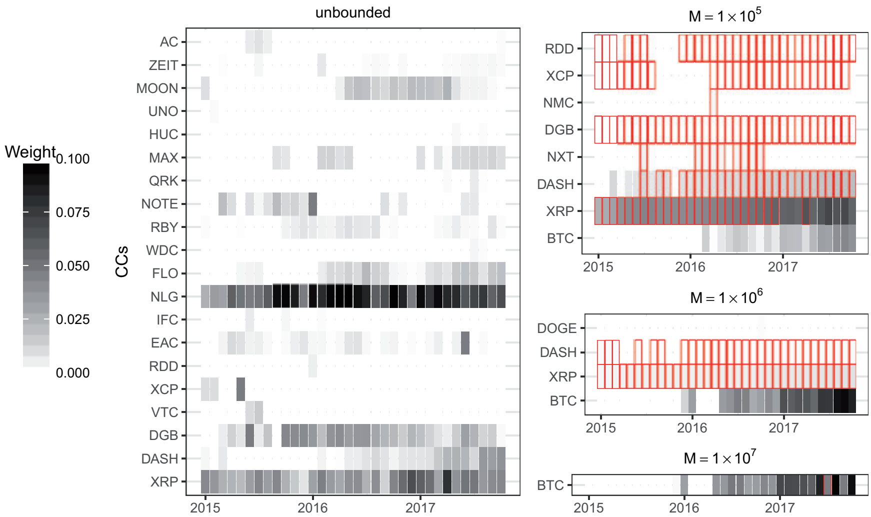

Figure 9 Weights given to CCs for S-CC CVaR portfolio at each monthly rebalancing date under the 3 different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing date are given. The darker the color, the larger the weight. Weights in □ are bounded by their upper bounds.

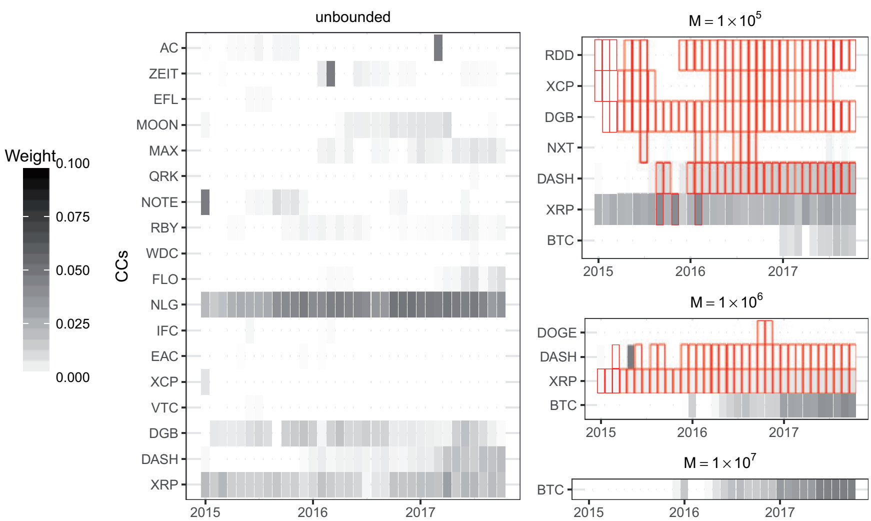

Figure 10 Weights given to CCs for SBC-CC CVaR portfolio at each monthly rebalancing date under the three different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing date are given. The darker the color, the larger the weight. Weights in □ are bounded by their upper bounds.

rather than monthly. In this section, a weekly rebalancing portfolio is constructed, with weights updated every Wednesday. Again, the weights are calculated using an extending window approach: all the data before the readjustment date are used for the calculation, and the first portfolio is formed on January 1, 2015.

The cumulative return plots for the Markowitz and CVaR portfolios are shown in panels 5 b, 5d, 6 b and 6d. An overview of the results leads to the conclusion that the cumulative returns of the weekly rebalanced portfolios show almost the same pattern as those of the monthly ones; however, in most cases, they perform worse. For the Markowitz method, the cumulative returns at the end of the sample of the S and S-CC portfolio are 9.4% and 72.6% respectively, which are 6.7% and 8.5% smaller than those of their monthly readjusted counterparts. When liquidity constrained, the cumulative returns are 4.9%, 4.7% and 5.3% lower than for the monthly readjusted case at investment amounts 1×10^5, 1×10^6, 1×10^7 US dollars. The same deterioration happens when bond and commodity indices are included. For the case with CVaR portfolios, the situation is similar: in both the S-CC and SBC-CC cases, all the portfolios have a smaller cumulative return than their monthly counterpart.

Furthermore, rebalancing the portfolio weekly instead of monthly harms the performance. Apart from this issue, the return curves appear almost similar, suggesting robust results regarding the reallocation frequency. However the better performance with monthly rebalancing gives support for the interpretation that at times, swings in the return series of CCs have to be endured to ensure a better performance at the end of the day.

6 Conclusion

In this article, we have explored the potential gain of including CCs into risk optimized portfolios, taking into consideration the low liquidity of the CC market. On the one hand, the rapid rise in CCs make them promising investment assets, while on the other hand, they are more volatile, have heavy tails, and relatively low liquidity, so investing in them is somewhat challenging. To control the risk as well as the liquidity problem, we have proposed LIBRO method, which extends the framework employed in Fan et al. (2012) to contain an additional liquidity constraint, depending on the intended investment amount. Applying this method to monthly and weekly re-allocated Markowitz and CVaR portfolios consisting of S&P 100 component stocks, Barclays Capital US Aggregate Index (US Bonds Index), S&P GSCI (Commodities Index), and adding CCs to them, the results show a strong improvement in terms of volatility/quantile risk to returns. However, it is worth noting that Bitcoin (BTC), the earliest and most dominant CC, is given a zero weight when no liquidity constraint is included, and this under both definitions of risk: volatility and quantile risk. Two key conclusions can be inferred from this result: first, although the one most discussed in the literature, BTC is not the most appealing CC in terms of risk–return optimization, at least during the period covered by the present paper, which highlights our contribution to include CCs other than BTC for portfolio formation. Second, the inclusion of liquidity constraints appears necessary, since some high-weight CCs are less liquid than BTC. In this situation, one can no longer assume that positions in these CCs would not distort the market or be tradable in the necessary amounts. To improve the applicability of the portfolio formation strategy, including an upper bound becomes necessary. It should be correlated with the daily trading volume of these assets.

The results of the in-sample analysis are already remarkable, while the out-of-sample analysis provides impressive results too. In the Markowitz portfolio the increase in cumulative returns reaches up to 80%. When including the liquidity upper bounds, the S-CC and SBC-CC portfolios still outperform the ones without constraints. For the Markowitz portfolios with monthly and weekly reallocation, the cumulative excess returns range from 10 to 22% with investment amounts equal to 1×10^5, 1×10^6, 1×10^7 US dollars. Over an investment period of roughly three years, this is a substantial gain. For the largest investment amount, which is, by construction, the most strongly restricted portfolio, one observes that the only CC included is BTC. Since CVaR gives less weight to assets having high tail risks, it can be inferred that for larger investment amounts, only the tail risk of BTC is sufficiently low to be appropriate for the portfolio. Furthermore, the monthly reallocated portfolios clearly outperformed the weekly adjusted ones. The better performance with monthly rebalancing gives support for the interpretation that, at times, swings in the return series of CCs have to be endured to ensure a better performance at the end of the day.

The main implications of this article are that including CCs into a portfolio can bring huge gains for the investor, even under the situation with the largest investment amount, which incurs the tightest liquidity constraint. In addition investing in alt-coins can provide much higher gains in the returns than just including BTC, but is more likely to encounter a liquidity problem; thus we have proposed LIBRO to tackle the low liquidity issue of certain CCs.

Supplementary Data

Supplementary data are available at *Journal of Financial Econometrics* online.

Funding

This work was supported by the Deutsche Forschungsgemeinschaft via IRTG 1792 “High Dimensional Non Stationary Time Series”, Humboldt-Universität zu Berlin, and the China Scholarship Council. We gratefully acknowledged their support.

Appendix

A.1 A Well-Conditioned Estimator for Large-Dimensional Covariance Matrices

For an $(N \times T)$ de-meaned matrix X with T iid observations, define S to be the sample covariance matrix $S = XX^\top/T$, I_N to be the N-dimensional identity matrix, and $\mathrm{tr}(\cdot)$ to indicate the trace of a matrix. Define $||\cdot||$ to be the Frobenius norm normalized by the dimension N, that is, $||X|| = \sqrt{\mathrm{tr}(XX^\top)/N}$. Further define $\zeta = \mathrm{tr}(\Sigma I_N)/N$, $\gamma^2 =$

Table A1 Summary statistics of CCs. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, which are calculated by multiplying 250 and $\sqrt{250}$ by their daily counterparts. Mean volume and market cap are measured in US dollars.

	Ann.Ret	Ann.STD	Skewness	kurtosis	ρ	mean volume	market cap
BTC	0.49	0.55	−0.61	10.87	−0.01	4.71e + 08	1.07e + 10
LTC	0.29	0.87	0.34	23.70	0.02	9.04e + 07	3.22e + 08
XRP	0.68	1.09	2.72	37.76	0.01	4.88e + 06	8.35e + 08
DASH	1.16	1.37	0.62	50.60	−0.14	1.61e + 06	1.52e + 08
DOGE	0.11	0.96	0.97	15.54	0.01	3.87e + 05	3.11e + 07
NXT	0.17	1.08	0.80	8.42	−0.03	2.28e + 05	2.08e + 07
DGB	0.73	1.70	2.93	29.89	−0.03	1.63e + 05	7.24e + 06
PPC	−0.17	1.00	0.62	12.51	−0.05	1.05e + 05	1.54e + 07
BLK	−0.05	1.27	1.82	17.72	−0.08	7.15e + 04	4.21e + 06
VTC	0.33	1.76	1.84	17.83	−0.01	5.93e + 04	2.42e + 06
NMC	−0.17	1.07	1.30	25.81	−0.09	5.26e + 04	8.60e + 06
POT	0.44	1.57	0.79	19.40	−0.05	3.95e + 04	2.30e + 06
XCP	0.36	1.46	1.16	9.52	−0.10	3.69e + 04	7.51e + 06
RDD	0.90	2.38	0.36	14.59	−0.25	2.55e + 04	2.83e + 06
XPM	−0.39	1.29	0.85	16.66	−0.07	2.05e + 04	1.54e + 06
NVC	−0.07	1.09	2.39	24.07	−0.01	1.73e + 04	1.58e + 06
EMC2	0.56	1.93	1.63	14.47	−0.02	1.57e + 04	7.16e + 05
EAC	−0.04	2.70	1.08	25.71	−0.27	1.50e + 04	7.92e + 05
IFC	−0.09	2.15	1.73	19.11	−0.13	1.26e + 04	8.37e + 05
NLG	1.00	1.46	0.90	12.35	−0.08	1.12e + 04	4.28e + 06
FTC	−0.00	1.76	0.59	13.66	−0.04	1.02e + 04	1.84e + 06
FLO	0.77	1.83	1.35	10.87	−0.07	8.35e + 03	7.74e + 05
ZET	−0.35	2.03	0.96	19.83	−0.16	5.80e + 03	7.54e + 05
WDC	−0.33	1.86	1.45	29.43	−0.12	5.65e + 03	7.94e + 05
RBY	1.16	2.18	0.98	28.41	−0.26	5.16e + 03	2.83e + 06
NOTE	0.79	1.60	1.59	15.01	−0.10	4.76e + 03	1.11e + 06
QRK	−0.27	2.91	−0.05	36.12	−0.35	4.38e + 03	1.29e + 06
MAX	−0.48	4.37	0.54	96.94	−0.38	3.69e + 03	4.13e + 05
HUC	−0.04	2.31	0.42	8.39	−0.19	3.25e + 03	2.31e + 05
SLR	0.72	1.92	0.44	9.59	−0.21	3.01e + 03	2.39e + 06
AUR	−0.03	1.51	1.08	19.12	−0.08	2.55e + 03	1.17e + 06
UNO	0.57	1.53	−0.24	15.64	−0.21	2.10e + 03	9.14e + 05
DMD	0.66	1.47	0.85	12.79	−0.20	1.57e + 03	6.89e + 05
GRS	0.99	2.86	1.52	15.80	−0.22	1.45e + 03	4.72e + 05
MINT	−0.01	3.63	0.27	7.76	−0.32	1.04e + 03	9.71e + 05
DGC	−0.43	1.92	0.71	27.86	−0.15	8.42e + 02	2.95e + 05
MOON	0.66	2.76	0.24	18.84	−0.16	6.83e + 02	1.09e + 06
EFL	0.58	1.87	−0.56	18.73	−0.16	6.19e + 02	2.24e + 05
NET	1.76	5.84	7.84	201.18	−0.22	5.45e + 02	2.84e + 05
CBX	0.17	3.33	0.81	28.53	−0.33	2.45e + 02	1.80e + 05
ZEIT	0.15	4.47	0.24	63.85	−0.25	2.10e + 02	3.78e + 05
AC	−0.30	3.05	1.17	26.52	−0.20	7.66e + 01	4.13e + 05
S&P stocks average	0.08	0.20	−0.26	10.29	0.00	5.40e + 08	
Bond index	0.05	0.05	−1.91	12.38	−0.04		
Commodity index	−0.20	0.20	0.05	4.84	−0.07		

$||\Sigma - \zeta I_N||^2$, $\beta^2 = \mathrm{E}(||S - \Sigma||^2)$ and $\delta^2 = \mathrm{E}(S - \zeta I_N)$. Assuming $\mathrm{E}(X^4) < \infty$, the optimization problem considered is

$$\min_{\rho_1 \rho_2} = \mathrm{E}(||\hat{\Sigma} - \Sigma||^2)$$
$$\text{s.t. } \hat{\Sigma} = \rho_1 \mathrm{I_N} + \rho_2 \mathrm{S}$$

which solves to $\rho_1 = \frac{\beta^2}{\delta^2}\zeta$ and $\rho_2 = \frac{\gamma^2}{\delta^2}$, thus the estimator is

$$\hat{\Sigma} = \frac{\beta^2}{\delta^2}\zeta I_N + \frac{\gamma^2}{\delta^2} S.$$

Since the estimator depends on the true covariance matrix Σ, a consistent estimator $\hat{\Sigma}^*$ has been introduced. Define $x_{\cdot k}$ as the $(N \times 1)$ column k of X and rewrite the sample covariance matrix S as $S = \frac{1}{T}\sum_{k=1}^{T} x_{\cdot k} x_{\cdot k}^\top$. Since the matrices $x_{\cdot k} x_{\cdot k}^\top$ are iid across k, β^2 can be estimated by $\overline{b}^2 = 1/N^2 \sum_{i=1}^{N} ||x_{\cdot i} x_{\cdot i}^\top - S||^2$, Ledoit and Wolf (2004). Further define $m = \mathrm{tr}(SI_N)/N$, $d^2 = ||S - mI_N||^2$, $\overline{b}^2 = \min(\overline{b}^2, d^2)$ and $k^2 = d^2 - b^2$. A consistent estimator is then

$$\hat{\Sigma}^* = \frac{b^2}{d^2} mI + \frac{k^2}{d^2} S.$$

References

Borke, L., and W. K. Härdle. 2017. "GitHub API based QuantNet Mining infrastructure in R". SFB 649 Discussion Paper.

Borke, L., and W. K. Härdle. 2018. "Q3-D3-LSA". *Handbook of Big Data Analytics*. Springer Verlag.

Brière, M., K. Oosterlinck, and A. Szafarz. 2015. Virtual Currency, Tangible Return: Portfolio Diversification with Bitcoin. *Journal of Asset Management* 16: 365–373.

Chen, C. Y., W. K. Härdle, A. J. Hou, and W. Wang. 2018. Pricing Cryptocurrency Options: The Case of CRIX and Bitcoin. *Journal of Financial Econometrics (Submitted)*.

Chen, S., C. Y.-H. Chen, W. K. Härdle, T. M. Lee, and B. Ong. 2017. "CRIX Index: Analysis of a Cryptocurrency Index for Portfolio Investment." In D. Lee Kuo Chuen and R. Deng. (eds), *Handbook of Digital Finance and Financial Inclusion: Cryptocurrency, FinTech, InsurTech, and Regulation*, Vol. 1. London, UK: Elsevier.

Darolles, S., C. Gourieroux, and E. Jay. 2012. "Robust Portfolio Allocation with Systematic Risk Contribution Restrictions". SSRN Scholarly Paper ID 2192399. Rochester, NY: Social Science Research Network.

Eisl, A., S. M. Gasser, and K. Weinmayer. 2015. "Caveat Emptor: Does Bitcoin Improve Portfolio Diversification?". SSRN Scholarly Paper ID 2408997. Rochester, NY: Social Science Research Network.

Elendner, H., S. Trimborn, B. Ong, and T. M. Lee. 2018. "The Cross-Section of Crypto-Currencies as Financial Assets". In D. Lee Kuo Chuen and R. Deng. (eds), *Handbook of Digital Finance and Financial Inclusion: Cryptocurrency, FinTech, InsurTech, and Regulation*. Vol. 1. London, UK: Elsevier.

Fan, J., J. Zhang, and K. Yu.June. 2012. Vast Portfolio Selection with Gross-Exposure Constraints. *Journal of the American Statistical Association* 107: 592–606.

Friedman, J., T. Hastie, and R. Tibshirani. 2008. Sparse Inverse Covariance Estimation with the Graphical Lasso. *Biostatistics* 9: 432–441.

Green, R. C. and B. Hollifield. 1992. When Will Mean-Variance Efficient Portfolios Be Well Diversified? *The Journal of Finance* 47: 1785–1809.

Härdle, W. K., and L. Simar. 2015. "Applications in Finance". *Applied Multivariate Statistical Analysis*. 4th ed. Heidelberg, Germany: Springer, pp. 487–499.

Härdle, W. K., D. K. C. Lee, S. Nasekin, and A. Petukhina. 2018. Tail Event Driven ASset Allocation: Evidence from Equity and Mutual Funds' Markets. *Journal of Asset Management* 19: 49–63.

Jagannathan, R., and A. T. Ma. 2003. Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. . *The Journal of Finance* 58: 1651–1683.

Kozak, S., S. Nagel, and S. Santosh. 2017. Interpreting Factor Models. *Journal of Finance (Forthcoming)*.

Krokhmal, P., J. Palmquist, and S. Uryasev. 2002. Portfolio Optimization with Conditional Value-at-Risk Objective and Constraints. *The Journal of Risk* 4: 43–68.

Ledoit, O. and M. Wolf. 2004. A Well-Conditioned Estimator for Large-Dimensional Covariance Matrices. *Journal of Multivariate Analysis* 88: 365–411.

Lee Kuo Chuen, D., L. Guo, and Y. Wang. 2017. Cryptocurrency: A New Investment Opportunity? *The Journal of Alternative Investments* 20: 16–40.

Markowitz, H. 1952. Portfolio Selection. *The Journal of Finance* 7: 77–91.

Rockafellar, R. T., and S. Uryasev. 2000. Optimization of Conditional Value-at-Risk. *The Journal of Risk* 2: 21–42.

Scaillet, O., A. Treccani, and C. Trevisan. 2018. High-Frequency Jump Analysis of the Bitcoin Market. *Journal of Financial Econometrics (Forthcoming)*.

Trimborn, S., and W. K. Härdle. 2018. CRIX an Index for Cryptocurrencies. *Journal of Empirical Finance* 49: 107–122.

Wyss, R. v. 2004. "Measuring and Predicting Liquidity in the StockMarket". Doctoral thesis.