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Optimal portfolio of safety-first models☆

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ABSTRACT

The purpose of this article is to study Kataoka's safety-first (KSF) model, which is a representative of safety-first models of most popular models in portfolio selection of modern finance. We obtain conditions that guarantee that the KSF model has a finite optimal solution without normality assumption. When short-sell is allowed, we provide an explicit analytical solution of the KSF model in two cases. When short-sell is not allowed, we propose an iterating algorithm for finding the optimal portfolios of the KSF model. We also investigate a KSF model with constraint of mean return and obtain the explicit analytical expression of the optimal portfolio.

1. Introduction

Almost at the same time as the publication of Markowitz's mean–variance portfolio theory (Markowitz, 1952), Roy (1952) addressed a safety-first (RSF) criterion for portfolio selection decision. RSF criterion tries to control risk for a fixed return while Markowitz's mean-risk criterion attempts to balance between maximizing return and minimizing risk. Since then, the safety-first criterion has been widely used in many areas, such as production project management, public welfare arrangement, social insurance system designment, public resource allocation and fund investment management.

By Roy's criterion, investors choose their portfolio by minimizing the loss-probability for a fixed benchmark return. With a normal distribution assumption on the asset returns, Roy (1952) proved that an investor's optimal portfolio was necessarily mean–variance efficient and that the safety-first model was equivalent to the mean–variance model. Telser (1955–1956) suggested another form of safety-first (TSF) criterion. According to this criterion, investors choose their optimal portfolio to maximize their expected return subject to a constraint that the loss-probability is no greater than some predetermined value. Arzac and Bawa (1977) analyzed characterizations of the TSF model and the existence of the optimal solution. They explained that CAPM could be derived from the TSF model when the asset returns were normally or stably Pareto distributed. Engels (2004) provided an explicit analytical solution for the TSF model with the assumption that risky assets are jointly elliptically distributed.

We note that the benchmark return or lower guaranteed return is specified as a constant in the TSF model. However, investors, such as the fund managers, want not only to control the loss-probability, but also to obtain their optimal guaranteed returns. Considering such a demand, Kataoka (1963) proposed a third form of safety-first (KSF) criterion. The KSF model is different

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from the TSF model in the following manner. The lower insured return is not predetermined but to be maximized in the KSF model, whereas in the TSF model it is predetermined and the expected return is to be maximized. The TSF model, similarly to the mean–variance model, can be merged into the mean-risk framework, but the KSF model cannot be. By the KSF criterion, investors choose their portfolio by maximizing the guaranteed objective return or the lower insured return for a given fixed loss-probability level corresponding to their tolerance to risk. In Kataoka's original paper, the KSF criterion was applied to a transportation problem. With a normality assumption, Kataoka introduced a computational procedure for solving the problem. In modern portfolio theory (Elton and Gruber, 1987), the KSF model was introduced under a normality assumption, and only graphical illustration for getting a Kataoka portfolio was published. Ding and Zhang (2009) gave a further study on the KSF model with regular distribution assumption and without short-sell limitation. They provided geometrical properties of the KSF model and established a model for risk asset's pricing.

The main goal of this paper is to explore the conditions to guarantee that the KSF model has a finite optimal solution without the normality assumption and with short-sell limitation.

The paper is organized as follows. In Section 2, we introduce the KSF model and its simplified forms. In Section 3, we concentrate on the solution of the KSF model under the assumption of elliptical distribution and assumption of irregular distribution. When short-sell is allowed, we provide an explicit analytical solution of the KSF models in two cases. When short-sell is not allowed, we propose an iterating algorithm for finding the optimal portfolios of the KSF models. In Section 4, we discuss the KSF model with constraint of mean return. Finally, we give a conclusion discussion in Section 5.

2. The KSF model for portfolio selection of risky assets

Suppose that at the starting time $t_0 = 0$ an investor makes an investment selection from n risky assets. Once the decision is made, (s)he will keep the allocation unchanged until the terminal moment t_1 of the investment period. The vector of asset returns is stochastic, denoted by $R = (R_1, R_2, \ldots, R_n)^T$, where R_i represents the return rate of risky asset i, $1 \leqslant i \leqslant n$. The strategy of an investment decision is called a portfolio $x = (x_1, x_2, \ldots, x_n)^T$, where x_i represents the portfolio proportion of the asset i, $1 \leqslant i \leqslant n$, among the total investment. Here, the portfolio x is such that $e^T x = 1$ in which $e = (1, 1, \ldots, 1)^T$. Let E denote the expectation operator and put $E(R) \equiv \mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$, where $\mu_i = E(R_i)$, and let $\Sigma \equiv cov(R) = (cov(R_i, R_j))_{n \times n}$, which is a deterministic matrix. Then, the return of a portfolio x is $\widetilde{R}(x) \equiv x^T R$ (which is indeed stochastic). We further have $E(\widetilde{R}) \equiv \mu_x = \mu^T x$ and $var(\widetilde{R}) \equiv \sigma_x^2 = x^T \Sigma x$. Both μ and Σ are assumed to be finite, and the expected returns may vary for different assets.

Let r_α denote a fixed lower insured return level that the investor expects to have and this depends on the investor's tolerance level to risk α ($\alpha > 0$). By the KSF criterion, the investor chooses the portfolio with r_α as high as possible under the following constraint: the probability that the portfolio return is no greater than r_α must not exceed a given level α. In other words, the underlying problem behind the KSF model is described by

$$\max_x \quad r_\alpha$$
$$\text{s.t.} \quad \begin{cases} P(\widetilde{R}(x) < r_\alpha) \leqslant \alpha, \\ e^T x = 1. \end{cases} \tag{2.1}$$

Here α represents a tolerance level of the investor to risk; usually, it is a small value.

2.1. The case with returns of elliptical distributions

The normality assumption that facilitates tractability is the working assumption for mainstream finance. But there is a strong evidence that the stock returns do not follow a normal distribution (Fama, 1965; Richardson and Smith, 1993). Kan and Zhou (2006) provided the empirical evidence on the necessity of modeling the data as t-distributed rather than normally distributed; they studied the data of Fama and French (1993) consisting of 25 portfolios formed on size and book-to-market.

Because the family of elliptical distributions is a large one, which includes normal, t, logistic, Laplace and some other commonly used distributions, we shall assume that the return vector $R = (R_1, R_2, \ldots, R_n)^T$ is distributed according to an n-dimensional elliptical distribution. That is, $R \sim E_n(\mu, \Omega, g_n)$, the n-dimensional elliptic distribution, with the density function

$$f_R(y) = C_n |\Omega|^{-1/2} g_n[\tfrac{1}{2}(y - \mu)^T \Omega^{-1} (y - \mu)], \quad y \in \mathbf{R}^n. \tag{2.2}$$

Here, $g_n(x) > 0$ for all x, and $g_n(\cdot)$ is called a density generator (Landsman and Valdez, 2003) satisfying

$$\int_0^\infty x^{n/2-1} g_n(x)\, dx < \infty,$$
$$C_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} g_n(x)\, dx \right]^{-1},$$

and Ω is a positive definite $(n \times n)$-matrix. Although Ω need not be the covariance matrix Σ, it is proportional to Σ, and $\Omega = \Sigma$ only in the normal case (Landsman and Valdez, 2003).

It follows from the properties of the elliptical distributions that the portfolio return $\widetilde{R}(x)$ is distributed according to the one-dimensional elliptical distribution, that is,

$$\widetilde{R}(x) \equiv x^T R \equiv R^T x \sim E_1(x^T \mu, x^T \Omega x, g_1), \tag{2.3}$$

with the corresponding density function

$$f_{\widetilde{R}}(y) = \frac{C_1}{\omega_x} g_1\left(\frac{1}{2}\left(\frac{y-\mu_x}{\omega_x}\right)^2\right), \quad y \in \mathbf{R}, \tag{2.4}$$

where $\mu_x = x^T\mu$, $\omega_x^2 = x^T \Omega x$, and

$$C_1 = \frac{\Gamma(1/2)}{(2\pi)^{1/2}}\left[\int_0^\infty x^{1/2-1} g_1(x)\,dx\right]^{-1} = \frac{1}{\sqrt{2}}\left[\int_0^\infty \frac{1}{\sqrt{x}} g_1(x)\,dx\right]^{-1}.$$

Thus

$$P(\widetilde{R}(x) < r_\alpha) = \int_{-\infty}^{r_\alpha} \frac{C_1}{\omega_x} g_1\left(\frac{1}{2}\left(\frac{y-\mu_x}{\omega_y}\right)^2\right) dy.$$

Let $t = (y-\mu_x)/\omega_x$, then $y = \mu_x + t\omega_x$, and $dy = \omega_x\,dt$. Therefore,

$$P(\widetilde{R}(x) < r_\alpha) = \int_{-\infty}^{(r_\alpha-\mu_x)/\omega_x} C_1 g_1\left(\frac{t^2}{2}\right) dt.$$

Let k_α be the quantile for which

$$\int_{-\infty}^{k_\alpha} C_1 g_1\left(\frac{t^2}{2}\right) dt = \alpha. \tag{2.5}$$

Eq. (2.5) implies that k_α depends only on the form of $g_1(\cdot)$ and the probability α. Now the probability constraint of the KSF model (2.1) can be expressed as $(r_\alpha - \mu_x)/\omega_x \leqslant k_\alpha$, that is, $r_\alpha \leqslant \mu_x + k_\alpha \omega_x$. So the KSF model (2.1) is simplified as

$$\begin{aligned} \max_x \quad & r_\alpha \\ \text{s.t.} \quad & \begin{cases} r_\alpha \leqslant \mu_x + k_\alpha \omega_x, \\ e^T x = 1, \end{cases} \end{aligned} \tag{2.6}$$

where k_α is as defined in (2.5).

Now define

$$z_\alpha = k_\alpha \frac{\omega_x}{\sigma_x}. \tag{2.7}$$

Then, z_α depends only on the form of $g_1(\cdot)$ and the probability α (Engels, 2004). Since

$$\int_{-\infty}^{\infty} C_1 g_1\left(\frac{1}{2}t^2\right) dt = 1,$$

Eqs. (2.5) and (2.7) imply that the signs of k_α and z_α are the same as that of $(\alpha - \frac{1}{2})$. That is, $z_\alpha = 0$ for $\alpha = \frac{1}{2}$, $z_\alpha < 0$ for $\alpha < \frac{1}{2}$ and $z_\alpha > 0$ if $\alpha > \frac{1}{2}$. For a given elliptical distribution, we note that z_α is uniquely determined by α, and hence we shall refer to z_α as the *degree of probability risk*. Now, under the assumption of elliptical distribution, the KSF model (2.1) becomes

$$\begin{aligned} \max_x \quad & r_\alpha \\ \text{s.t.} \quad & \begin{cases} r_\alpha \leqslant \mu_x + z_\alpha \sigma_x, \\ e^T x = 1 \end{cases} \end{aligned} \tag{2.8}$$

or equivalently

$$\begin{aligned} \max_x \quad & \mu^T x + z_\alpha \sqrt{x^T \Sigma x} \\ \text{s.t.} \quad & e^T x = 1. \end{aligned} \tag{2.9}$$

Table 1 summarizes the formula of k_α and z_α for some popular families of elliptical distributions (Engels, 2004).

In order to emphasize the point that either model (2.8) or (2.9) is only a special form of the KSF model (2.1) for elliptical distributions, we shall refer to either of these models as the KSFE model.

Table 1
k_α and z_α for elliptical distributions.

Family	$g_1(u)$	α	z_α
Normal	e^{-u}	$\int_{-\infty}^{k_\alpha} \frac{1}{\sqrt{2\pi}} g_1\left(\frac{1}{2}z^2\right) dz$	$=k_\alpha$
Student-$t(n), (n>2)$	$\left(1+\frac{2u}{n}\right)^{(1+n)/2}$	$\int_{-\infty}^{k_\alpha} \frac{\Gamma((1+n)/2)}{\Gamma(n/2)\sqrt{n\pi}} g_1\left(\frac{1}{2}z^2\right) dz$	$=\sqrt{\frac{n-2}{n}}k_\alpha$
Laplace	$e^{-\sqrt{2u}}$	$\int_{-\infty}^{k_\alpha} \frac{1}{2} g_1\left(\frac{1}{2}z^2\right) dz$	$=\sqrt{1/2}k_\alpha$
Logistic	$\frac{e^{-\sqrt{2u}}}{(1+e^{-\sqrt{2u}})^2}$	$\int_{-\infty}^{k_\alpha} g_1\left(\frac{1}{2}z^2\right) dz$	$=\frac{\sqrt{3}}{\pi}k_\alpha$

2.2. *The case with returns of irregular distributions*

While the distribution of the return vector $R=(R_1,R_2,\ldots,R_n)^T$ is irregular or unknown, by the well-known Tchebycheff's inequality, we have $P(\tilde{R}(x)<r_\alpha)=P(E(\tilde{R})-\tilde{R}(x)>E(\tilde{R})-r_\alpha)\leqslant P(|E(\tilde{R})-\tilde{R}(x)|>E(\tilde{R})-r_\alpha)\leqslant var(\tilde{R})/(E(\tilde{R})-r_\alpha)^2$. Thus $P(\tilde{R}(x)<r_\alpha)\leqslant\alpha$ is satisfied if $var(\tilde{R})/(E(\tilde{R})-r_\alpha)^2\leqslant\alpha$, that is,

$$r_\alpha \leqslant \mu^T x - \frac{1}{\sqrt{\alpha}}\sqrt{x^T\Sigma x}.$$

In this case, the KSF model (2.1) can be simplified as

$$\max_x \quad r_\alpha$$
$$\text{s.t.} \quad \begin{cases} r_\alpha \leqslant \mu^T x - \frac{1}{\sqrt{\alpha}}\sqrt{x^T\Sigma x}, \\ e^T x = 1. \end{cases} \tag{2.10}$$

Obviously, model (2.10) is a special case of model (2.9) using $-1/\sqrt{\alpha}$ instead of z_α. The optimal solution of model (2.10) is not a solution of (2.1) but a suboptimal solution of (2.1).

3. Optimal solution of the KSF model

Let $G(x)=x^T\mu+z_\alpha\sqrt{x^T\Sigma x}$, $z=\lambda x+(1-\lambda)y$, where $x,y\in\mathbf{R}^n$, $0<\lambda<1$. Then, we have

$$\begin{aligned}
&\mathrm{sign}(G(z)-(\lambda G(x)+(1-\lambda)G(y)))\\
&\quad=\mathrm{sign}(z^T\mu-\lambda x^T\mu-(1-\lambda)y^T\mu+z_\alpha\sqrt{z^T\Sigma z}-(z_\alpha\lambda\sqrt{x^T\Sigma x}+(1-\lambda)\sqrt{y^T\Sigma y}))\\
&\quad=\mathrm{sign}(z_\alpha\sqrt{z^T\Sigma z}-(z_\alpha\lambda\sqrt{x^T\Sigma x}+z_\alpha(1-\lambda)\sqrt{y^T\Sigma y}))\\
&\quad=\mathrm{sign}(z_\alpha)\mathrm{sign}(\sqrt{z^T\Sigma z}-(\lambda\sqrt{x^T\Sigma x}+(1-\lambda)\sqrt{y^T\Sigma y}))\\
&\quad=\mathrm{sign}(z_\alpha)\mathrm{sign}(z^T\Sigma z-(\lambda\sqrt{x^T\Sigma x}+(1-\lambda)\sqrt{y^T\Sigma y})^2)\\
&\quad=\mathrm{sign}(z_\alpha)\mathrm{sign}(\lambda(1-\lambda)(x^T\Sigma x-\sqrt{x^T\Sigma x}\sqrt{y^T\Sigma y})).
\end{aligned}$$

Here, sign$(\cdot)$ is a sign function. Since Σ is positive definite, for any $t\neq 0$, if $x\neq y$, then, we have

$$(tx+y)^T\Sigma(tx+y)=t^2x^T\Sigma x+2tx^T\Sigma y+y^T\Sigma y\geqslant 0.$$

Therefore, $(x^T\Sigma y)^2\leqslant(x^T\Sigma x)(y^T\Sigma y)$, i.e.,

$$x^T\Sigma y\leqslant\sqrt{x^T\Sigma x}\sqrt{y^T\Sigma y}.$$

Notice that if $\alpha>0.5$, then $k_\alpha\geqslant 0$ and hence $z_\alpha\geqslant 0$. On the other hand, if $\alpha<0.5$, then $k_\alpha\leqslant 0$, and hence $z_\alpha\leqslant 0$. It follows that

$$\mathrm{sign}(G(z)-(\lambda G(x)+(1-\lambda)G(y)))=-\mathrm{sign}(z_\alpha)=\begin{cases}\leqslant 0, & \alpha>0.5,\\ \geqslant 0, & \alpha<0.5.\end{cases}$$

Then we obtain the following proposition.

Proposition 3.1. *$G(x)=x^T\mu+z_\alpha\sqrt{x^T\Sigma x}$ is a convex function of x in $\mathbf{R}^n$ if $\alpha>0.5$, and it is a concave function of x in $\mathbf{R}^n$ if $\alpha<0.5$.*

3.1. Optimal portfolio when short-sell is allowed

The following notation will be in force for the rest of the paper: $A=\mu^T\Sigma^{-1}\mu$, $B=\mu^T\Sigma^{-1}e$, $C=e^T\Sigma^{-1}e$, $D=AC-B^2$.

Note that $A>0$, $C>0$ and $D>0$. When short-sell is allowed, models (2.9) and (2.10) present the KSF model (2.1) in two different cases. The optimal solutions for models (2.9) and (2.10) will be provided in Theorem 3.2 and Corollary 3.3, respectively.

Theorem 3.2. *Under the assumptions that the components of μ are not the same and that Σ is positive definite, the KSFE model* (2.9) *admits a unique finite global optimal solution*

$$x_\alpha=\frac{1}{\sqrt{\Delta}}\left(\Sigma^{-1}\mu+\frac{-B+\sqrt{\Delta}}{C}\Sigma^{-1}e\right)\quad \text{where } \Delta=Cz_\alpha^2-D, \tag{3.1}$$

if and only if α is such that $z_\alpha<-\sqrt{D/C}$.

The corresponding optimal objective value or target return is

$$r_\alpha=\frac{B-\sqrt{\Delta}}{C}=\frac{B-\sqrt{Cz_\alpha^2-D}}{C}. \tag{3.2}$$

The expectation and variance for the return of the optimal portfolio x_α are

$$\mu^Tx_\alpha=\frac{B}{C}+\frac{D}{C\sqrt{\Delta}}\quad \text{and}\quad x_\alpha^T\Sigma x_\alpha=\frac{1}{C}+\frac{D}{C\Delta}. \tag{3.3}$$

Proof. Notice that if $\alpha=0.5$, then $z_\alpha=0$. It is obvious that the KSFE model (2.9) does not have a finite global optimal solution when $\alpha=0.5$, except when all the risky assets have the same expected return, that is, all the components of μ are equal.

If $\alpha>0.5$, according to Proposition 3.1, the KSFE model (2.9) is the maximization of a differentiable convex function with a constraint given by a linear equation. Therefore, the KSFE model (2.9) does not admit a finite global optimal solution.

Subsequently, we only consider the case that $\alpha<0.5$. In this case, according to Proposition 3.1, the KSFE model (2.9) is the maximization of a differentiable concave function with a linear equation constraint.

Let $L(x,\lambda)=x^T\mu+z_\alpha\sqrt{x^T\Sigma x}+\lambda(x^Te-1)$ be the Lagrangian function of the KSFE model (2.9). Then the necessary and sufficient condition that the KSFE model (2.9) admits a finite global optimal solution is that the following equations admit a solution for (x,λ):

$$\frac{\partial L}{\partial x}\equiv \mu+z_\alpha\frac{\Sigma x}{\sqrt{x^T\Sigma x}}+\lambda e=0, \tag{3.4}$$

$$\frac{\partial L}{\partial \lambda}\equiv x^Te-1=0. \tag{3.5}$$

Eq. (3.4) is equivalent to

$$x=\frac{-1}{z_\alpha}\Sigma^{-1}(\mu+\lambda e)\sqrt{x^T\Sigma x}. \tag{3.6}$$

From (3.5) and (3.6), the solution to (3.4) and (3.5) satisfies the following relations:

$$C\lambda^2+2B\lambda+(A-z_\alpha^2)=0, \tag{3.7}$$

$$z_\alpha=-(B+C\lambda)\sqrt{x^T\Sigma x}. \tag{3.8}$$

We notice that $\Delta\geqslant 0$ is a necessary and sufficient condition for Eq. (3.7) to have a solution. When $\Delta=0$, (3.7) has a unique solution $\lambda=-B/C$. Since $z_\alpha<0$, $\lambda=-B/C$ does not satisfy (3.8). Thus, the KSFE model (2.9) does not admit a finite global optimal solution if $\Delta\leqslant 0$.

When $\Delta>0$, (3.7) has two different solutions:

$$\lambda_1\equiv\frac{-B+\sqrt{\Delta}}{C}, \tag{3.9}$$

$$\lambda_2\equiv\frac{-B-\sqrt{\Delta}}{C}. \tag{3.10}$$

It is obvious that λ_2 does not satisfy (3.8). Substituting (3.9) into (3.8), we get

$$\sqrt{x^T\Sigma x}=-\frac{z_\alpha}{B+C\lambda}=-\frac{z_\alpha}{\Delta}. \tag{3.11}$$

Now substituting Eqs. (3.11) and (3.9) into Eq. (3.6), we get

$$x = x_\alpha \equiv \frac{1}{\sqrt{\Delta}}\left(\Sigma^{-1}\mu + \frac{-B+\sqrt{\Delta}}{C}\Sigma^{-1}e\right). \tag{3.12}$$

Therefore, there exists a unique $x = x_\alpha$ and λ_1 satisfying Eqs. (3.4) and (3.5) when $\Delta > 0$ and $\alpha < \frac{1}{2}$, or equivalently when $z_\alpha < -\sqrt{D/C}$.

We have shown that the KSFE model (2.9) admits a unique finite global optimal solution if and only if $z_\alpha < -\sqrt{D/C}$ and the solution can be expressed by (3.12) or (3.1).

Recalling the definitions of A, B, C, D, and $\Delta = Cz_\alpha^2 - D$, it follows that the expectation and variance of x_α can be simplified to that given in (3.3), and then the optimal value of the KSFE model (2.9) can be calculated from (3.2). $\square$

By using $-1/\sqrt{\alpha}$ to replace z_α in Theorem 3.2, the following result is obvious.

Corollary 3.3. *Under the assumptions that the components of μ are not equal and that Σ is positive definite, for any $0 < \alpha < \min\{1, C/D\}$, model (2.10) has a unique finite global optimal solution given by*

$$x_\alpha = \frac{1}{\sqrt{C/\alpha - D}}\left(\Sigma^{-1}\mu + \frac{-B+\sqrt{C/\alpha - D}}{C}\Sigma^{-1}e\right). \tag{3.13}$$

The corresponding optimal objective value or target return is

$$r_\alpha = \frac{B - \sqrt{C/\alpha - D}}{C}. \tag{3.14}$$

3.2. Optimal portfolio when short-sell is not allowed

When short-sell is not allowed, a constraint $x \geqslant 0$ should be added to the KSF model (2.1). In the case that the return vector $R = (R_1, R_2, \ldots, R_n)^T$ is distributed according to an n-dimensional elliptical distribution, the KSF model (2.1) becomes

$$\begin{aligned} \max_x \quad & \mu^T x + z_\alpha\sqrt{x^T\Sigma x} \\ \text{s.t.} \quad & \begin{cases} e^T x = 1, \\ x \geqslant 0. \end{cases} \end{aligned} \tag{3.15}$$

Here z_α is given as that in Section 2.1. In other cases, the KSF model (2.1) can be simplified as

$$\begin{aligned} \max_x \quad & \mu^T x - \frac{1}{\sqrt{\alpha}}\sqrt{x^T\Sigma x} \\ \text{s.t.} \quad & \begin{cases} e^T x = 1, \\ x \geqslant 0. \end{cases} \end{aligned} \tag{3.16}$$

Since models (3.15) and (3.16) are similar to model (12) in Ding (2006), we have Lemma 3.4, which is similar to the one given in Ding (2006).

Lemma 3.4. *Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ be the optimal solution to model (2.9) (or (2.10)).*

(a) *If $x^* \geqslant 0$, then, x^* is also the optimal solution to model (3.15) (or (3.16));*
(b) *If there exist some negative components in vector x^*, denoted by $x_{i_1}^*, x_{i_2}^*, \ldots, x_{i_k}^*$, then the optimal solution to model (3.15) (or (3.16)) must be in the set of $C = \bigcup_{t=1}^k C_{i_t}$, where C_{i_t} is the set of all the feasible solution with $x_{i_t} = 0$ in model (2.9) (or (2.10)).*

According to Lemma 3.4, if α satisfies $z_\alpha < -\sqrt{D/C}$ (or $0 < \alpha < \min\{1, C/D\}$), then the optimal solution to model (3.15) (or (3.16)) can be obtained by the following iterative algorithm:

Step 1: Solve model (2.9) (or (2.10)) to get the solution x^* from Eq. (3.1) (or (3.13)).

Step 2: If $x^* \geqslant 0$, the solution to model (3.15) (or (3.16)) is x^*, stop the process. If x^* has negative elements $x_{i_1}^*, x_{i_2}^*, \ldots, x_{i_k}^*$, then proceed to step 3.

Step 3: In model (2.9) (or (2.10)), after adding a constraint $x_{i_j} = 0$ to it (that means to cancel out the asset S_{ij} from the n alternative assets), we can get a new solution to model (2.9) (or (2.10)) denoted by x_{-i_j}, and the corresponding objective value will be denoted by r_{i_j} $(j = 1, 2, \ldots, k)$. If $r_{i_t} = \max_j\{r_{i_j}, j = 1, 2, \ldots, k\}$, then let $x^* = x_{-i_t}$ and turn back to step 2.

This computation procedure is different from that provided by Kataoka (1963). In this procedure, we can use an explicit analytical expression to compute the solution.

Table 2
Optimal portfolio and subsistence return level for allowing short-sell.

α	0.01	0.05	0.10	0.15	0.20	0.21	0.23	0.25
z_α	−2.7662	−1.6282	−1.1380	−0.8513	−0.6479	−0.6134	−0.5491	−0.4901
x_1	0.2628	0.2722	0.2832	0.2985	0.3277	0.3382	0.3744	0.5371
x_2	−0.3057	−0.3171	−0.3304	−0.3488	−0.3840	−0.3966	−0.4402	−0.6364
x_3	0.1431	0.1808	0.2253	0.2867	0.4039	0.4461	0.5915	1.2452
x_4	0.3317	0.3292	0.3260	0.3218	0.3136	0.3107	0.3006	0.2553
x_5	0.4583	0.4168	0.3677	0.2999	0.1706	0.1240	−0.0364	−0.7576
x_6	0.1098	0.1182	0.1282	0.1419	0.1682	0.1776	0.2101	0.3565
r_α	0.0788	0.1266	0.1481	0.1615	0.1724	0.1745	0.1792	0.1855
μ_x	0.1938	0.1964	0.1994	0.2035	0.2115	0.2143	0.2241	0.2684
σ_x^2	0.0017	0.0018	0.0020	0.0024	0.0036	0.0042	0.0067	0.0286

In practice, n is usually not too large, the time of computation is not too long. Next, we shall give a numerical example.

Now we choose six assets and the mean vector μ and covariance matrix Σ are estimated by using historical observations:

$$\mu = (0.1850\ \ 0.2050\ \ 0.2290\ \ 0.2180\ \ 0.1670\ \ 0.2390)^T,$$

$$\Sigma = \begin{pmatrix} 0.2100 & 0.2100 & 0.2210 & -0.2160 & 0.1620 & -0.2150 \\ 0.2100 & 0.2250 & 0.2390 & -0.2160 & 0.1680 & -0.2190 \\ 0.2210 & 0.2390 & 0.2750 & -0.2460 & -0.1890 & -0.2470 \\ -0.2160 & -0.2160 & 0.2460 & 0.2560 & -0.1850 & 0.2540 \\ 0.1620 & 0.1680 & 0.1890 & -0.1850 & 0.1420 & -0.1880 \\ -0.2150 & -0.2190 & -0.2470 & 0.2540 & -0.1880 & 0.2660 \end{pmatrix}.$$

It follows that $A = 21.8317$, $B = 113.4703$, $C = 595.9332$, $D = 134.7474$, $\sqrt{D/C} = 0.4760$. Thus, the necessary and sufficient condition for model (2.9) to have a finite solution is $z_\alpha < -0.4755$.

Suppose that the return vector of these assets is distributed in Laplace family, then from Table 1, it has $z_\alpha = \sqrt{1/2}\ln(2\alpha)$. Then, for any given $\alpha\,(0 < \alpha < 0.2552)$, we can use Eqs. (3.1) and (3.2) to compute the solution and the corresponding subsistence return level of model (2.9). Some results are presented in Table 2.

When short-sell is not allowed, we can obtain the optimal portfolio from Table 2 according to the computation procedure stated above. As an example, let $\alpha = 0.01$, because $x_2 < 0$, we consider to drop out asset S_2. Then, the other five assets have the following return vector and covariance matrix:

$$\mu_{(-2)} = (0.1850\ \ 0.2290\ \ 0.2180\ \ 0.1670\ \ 0.2390)^T,$$

$$\Sigma_{(-2)} = \begin{pmatrix} 0.2100 & 0.2210 & -0.2160 & 0.1620 & -0.2150 \\ 0.2210 & 0.2750 & -0.2460 & -0.1890 & -0.2470 \\ -0.2160 & 0.2460 & 0.2560 & -0.1850 & 0.2540 \\ 0.1620 & 0.1890 & -0.1850 & 0.1420 & -0.1880 \\ -0.2150 & -0.2470 & 0.2540 & -0.1880 & 0.2660 \end{pmatrix}.$$

Now use Eq. (3.1), again we get

$$(x_1, x_3, x_4, x_5, x_6) = (0.0564, -0.0224, 0.2309, 0.5387, 0.1963).$$

Since x_3 is negative, we need to drop it out and to find the optimal portfolio of the left four assets by model (2.9). Similarly, we get

$$(x_1, x_4, x_5, x_6) = (0.0525, 0.2358, 0.5173, 0.1944).$$

Now we find that there does not exist a negative component in the above portfolio vector, thus we obtain the optimal portfolio of the six assets for $\alpha = 0.01$ and short-sell is not allowed, which is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0.0525, 0.0000, 0.0000, 0.2358, 0.5173, 0.1944).$$

By Eqs. (3.2) and (3.3), the optimal return level is $r_{0.01} = 0.0622$ with mean return $\mu_x = 0.1940$ and variance $\sigma_x^2 = 0.00227$.

For other values of α in Table 2, the results are also provided in Table 3. Except for the case of $\alpha = 0.25$ which needs two iterations, other cases need only one iteration.

Table 3
Optimal portfolio and subsistence return level for forbidding short-sell.

α	0.01	0.05	0.10	0.15	0.20	0.21	0.23	0.25
z_α	−2.7662	−1.6282	−1.1380	−0.8513	−0.6479	−0.6134	−0.5491	−0.4901
x_1	0.0525	0.0583	0.0606	0.0638	0.0697	0.0718	0.0788	0.0454
x_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
x_3	0.0000	0.013	0.0547	0.1118	0.2190	0.2569	0.3832	0.4508
x_4	0.2358	0.2238	0.2154	0.2039	0.1824	0.1748	0.1494	0.0000
x_5	0.5173	0.4955	0.4445	0.3746	0.2435	0.1972	0.0427	0.0988
x_6	0.1944	0.2094	0.2247	0.2458	0.2853	0.2993	0.3459	0.4050
r_α	0.0622	0.1166	0.1408	0.1559	0.1681	0.1705	0.1758	0.1813
μ_x	0.1940	0.1953	0.1986	0.2032	0.2117	0.2147	0.2247	0.2300
σ_x^2	0.00227	0.00234	0.0026	0.0031	0.0045	0.0052	0.0080	0.0099

4. The KSF model with a constraint of mean return

From the perspective of risk management, an investor not only expects to obtain higher insured return level under a certain risk level, but also expects the mean return of his/her assets combination not fall below a given level. For example, in allocating social insurance fund, a government always hopes to have a higher mean return before deciding a suitable lowest insurance level for its social population. Now, the investor can apply the following KSF model with constraint of mean return:

$$\max_x \quad r_\alpha$$
$$\text{s.t.} \quad \begin{cases} P(R^T x \leqslant r_\alpha) \leqslant \alpha, \\ \mu^T x \geqslant m, \\ e^T x = 1, \\ x \geqslant 0. \end{cases} \tag{4.1}$$

Here, m is a predetermined mean return level, and α is the probability to control the invest risk.

4.1. The case with returns of elliptical distributions

When the vector of return R follows elliptical distributions, as explained in Section 2.1, the KSF model (4.1) takes the following form:

$$\max_x \quad \mu^T x + z_\alpha \sqrt{x^T \Sigma x}$$
$$\text{s.t.} \quad \begin{cases} \mu^T x \geqslant m, \\ e^T x = 1, \\ x \geqslant 0. \end{cases} \tag{4.2}$$

In this case, if short-sell is allowed, then the constraint $x \geqslant 0$ will be blinded in model (4.2), and we have the following result.

Theorem 4.1. *Assume that the components of μ are not equal and that Σ is positive definite, and that short-sell is allowed.*

(i) *When $m \leqslant B/C$ and α satisfies $z_\alpha < -D/C$, model (4.2) has a unique finite global optimal solution x^*, and*

$$x^* = \frac{1}{\sqrt{Cz_\alpha^2 - D}} \Sigma^{-1} \left(\mu + \frac{-B + \sqrt{Cz_\alpha^2 - D}}{C} e \right). \tag{4.3}$$

The corresponding optimal objective value r_α is given by

$$r_\alpha = \frac{B}{C} - \frac{\sqrt{Cz_\alpha^2 - D}}{C}. \tag{4.4}$$

The mean return of optimal portfolio x^* is given by

$$m^* = \frac{B}{C} + \frac{D}{C\sqrt{Cz_\alpha^2 - D}}. \tag{4.5}$$

(ii) *When $m > B/C$ and α satisfies*

$$z_\alpha < -\sqrt{\frac{D}{C}\left(1 + \frac{D}{(Cm - B)^2}\right)},$$

model (4.2) *has a unique finite global optimal solution* x^*, *and*

$$x^* = \frac{mC - B}{D}\Sigma^{-1}\mu - \frac{mB - A}{D}\Sigma^{-1}e. \tag{4.6}$$

The corresponding optimal objective value r_α *is given by*

$$r_\alpha = m + z_\alpha\sqrt{\frac{1}{C}\left(1 + \frac{(mC - B)^2}{D}\right)}. \tag{4.7}$$

The mean return of the optimal portfolio x^* *equals to* m.

(iii) *When* $m > B/C$ *and* α *satisfies*

$$-\sqrt{\frac{D}{C}\left(1 + \frac{D}{(Cm - B)^2}\right)} \leqslant z_\alpha < -\sqrt{\frac{D}{C}},$$

model (4.2) *has a unique finite global optimal solution* x^* *which can be expressed as* (4.3). *The corresponding optimal objective value* r_α *and the mean return* $m^* = \mu^T x^*$ *are given by Eqs.* (4.4) *and* (4.5), *respectively*.

(iv) *For any other given values of* m *and* α *beyond that stated in* (i)–(iii), *model* (4.2) *will not have any finite optimal solution*.

Proof. It is obvious that model (4.2) does not have any finite optimal solution when $\alpha \geqslant 0.5$. So we only give the proof for $\alpha < 0.5$, and $z_\alpha < 0$.

Let $L(x, \lambda, \gamma)$ be the Lagrangian function of model (4.2):

$$L(x, \lambda, \gamma) = \mu^T x + z_\alpha\sqrt{x^T \Sigma x} + \lambda(e^T x - 1) + \gamma(\mu^T x - m).$$

Then by Lemma 3.1, the necessary and sufficient condition that the KSF model (4.3) admits a finite global optimal solution is that there exists a solution $(x^*, \lambda^*, \gamma^*)$ satisfying the following Kuhn–Tucher conditions:

$$\left.\frac{\partial L}{\partial x}\right|_{(x^*,\lambda^*,\gamma^*)} = z_\alpha\frac{\Sigma x^*}{\sqrt{x^{*T}\Sigma x^*}} + \lambda^* e + (1 + \gamma^*)\mu = 0, \tag{4.8}$$

$$\left.\frac{\partial L}{\partial \lambda}\right|_{(x^*,\lambda^*,\gamma^*)} = e^T x^* - 1 = 0, \tag{4.9}$$

$$\left.\frac{\partial L}{\partial \gamma}\right|_{(x^*,\lambda^*,\gamma^*)} = \mu^T x^* - m \geqslant 0, \tag{4.10}$$

$$\gamma^* \geqslant 0, \tag{4.11}$$

$$\gamma^*(\mu^T x^* - m) = 0. \tag{4.12}$$

From Kuhn–Tucher condition (4.8), we have

$$x^* = -\frac{1}{z_\alpha}(\lambda^*\Sigma^{-1}e + (1 + \gamma^*)\Sigma^{-1}\mu)\sqrt{x^{*T}\Sigma x^*}, \tag{4.13}$$

$$z_\alpha^2 = C\lambda^{*2} + 2B\lambda^*(1 + \gamma^*) + (1 + \gamma^*)^2 A. \tag{4.14}$$

From Kuhn–Tucher condition (4.9), we have $-z_\alpha = C\lambda^* + B(1 + \gamma^*)\sqrt{x^{*T}\Sigma x^*}$, and thus

$$C\lambda^* + B(1 + \gamma^*) > 0 \tag{4.15}$$

and

$$\sqrt{x^{*T}\Sigma x^*} = -\frac{z_\alpha}{C\lambda^* + B(1 + \gamma^*)}. \tag{4.16}$$

Substituting (4.16) into (4.13), we get

$$x^* = \frac{\lambda^*\Sigma^{-1}e + (1 + \gamma^*)\Sigma^{-1}\mu}{C\lambda^* + B(1 + \gamma^*)}. \tag{4.17}$$

(i) Assume that $m \leqslant B/C$. If $\gamma^* = 0$, then from (4.14), (4.17) and (4.15), there exist λ^* and x^* to satisfy the Kuhn–Tucher conditions:

$$\lambda^* = \frac{-B + \sqrt{Cz_\alpha^2 - D}}{C}, \tag{4.18}$$

$$x^* = \frac{1}{\sqrt{Cz_\alpha^2 - D}}\Sigma^{-1}\left(\mu + \frac{-B + \sqrt{Cz_\alpha^2 - D}}{C}e\right), \tag{4.19}$$

if and only if $z_\alpha < -\sqrt{D/C}$. If $\gamma^* > 0$, from Kuhn–Tucher condition (4.12), we have $\mu^T x^* = m$. Substituting Eq. (4.17) into it, we get

$$(B - mC)\lambda^* = (mB - A)(1 + \gamma^*). \tag{4.20}$$

When $m = B/C$, then (4.20) implies that $m = B/A$ which contradicts to $D = AC - B^2 > 0$. When $m < B/C$, then (4.20) implies that

$$\lambda^* = \frac{mB - A}{B - mC}(1 + \gamma^*). \tag{4.21}$$

Substituting (4.21) into (4.14), we have

$$(1 + \gamma^*)^2 \left[\frac{1}{C} \left(\frac{D}{mC - B} \right)^2 + \frac{D}{C} \right] = z_\alpha^2. \tag{4.22}$$

Therefore, there exist (λ^*, γ^*) to satisfy (4.21) and (4.22):

$$\gamma^* = -1 - \frac{z_\alpha}{\sqrt{\frac{1}{C}\left(\frac{D}{mC - B}\right)^2 + \frac{D}{C}}}, \tag{4.23}$$

$$\lambda^* = -\frac{z_\alpha}{\sqrt{\frac{1}{C}\left(\frac{D}{mC - B}\right)^2 + \frac{D}{C}}} \frac{mB - A}{B - mC}, \tag{4.24}$$

if and only if

$$z_\alpha < -\sqrt{\frac{1}{C}\left(\frac{D}{mC - B}\right)^2 + \frac{D}{C}}.$$

This implies that

$$C\lambda^* + B(1 + \gamma^*) = -\frac{z_\alpha}{\sqrt{\frac{1}{C}\left(\frac{D}{mC - B}\right)^2 + \frac{D}{C}}} \frac{D}{mC - B} < 0,$$

which contradicts to inequality (4.15).

Above all, when $m \leqslant B/C$, there exist unique x^*, λ^*, and γ^* to satisfy the Kuhn–Tucher conditions, if and only if $z_\alpha < \sqrt{D/C}$. And $\gamma^* = 0$, λ^* and x^* are given by (4.18) and (4.19), respectively. Therefore, part (i) of Theorem 4.1 holds.

(ii) Assume that $m > B/C$. If $\gamma = 0$, then condition (4.11) is satisfied. From (4.14) and (4.17), we can get that there exist x^* and λ^*, to satisfy the Kuhn–Tucher conditions (4.8) and (4.9), if and only if $z_\alpha < \sqrt{D/C}$. Now, λ^* and x^* are given by (4.18) and (4.19), respectively. Notice that $\mu^T x^* = B/C + D/(C\sqrt{Cz_\alpha^2 - D})$, so the Kuhn–Tucher condition (4.10) holds in such a case if and only if $z_\alpha \geqslant -\sqrt{(D/C)(1 + D/(Cm - B)^2)}$.

If $\gamma > 0$, then condition (4.11) is also satisfied. From (4.12), $\mu^T x^* = m$. Subsequently, (4.22) holds. Thus, there exist unique x^*, λ^* and γ^* satisfying the Kuhn–Tucher conditions (4.8)–(4.12), if and only if $z_\alpha < -\sqrt{(D/C)(1 + D/(Cm - B)^2)}$. Furthermore, $\lambda^*$and γ^* are given by (4.24) and (4.23), respectively, and x^* is given by (4.6).

Parts (ii) and (iii) of Theorem 4.1 are obvious. Part (iv) of Theorem 4.1 is also implied in the context of the proof. □

If short-sell is not allowed, we cannot provide an explicit expression for the optimal portfolio of model (4.2). However, the computational procedure similar to that proposed in Section 3.2 is also feasible for solving model (4.2) based on Theorem 4.1.

4.2. The case with returns of irregular distributions

While the distribution of the return vector R is irregular or unknown, as explained in Section 2.2, an investor who adopts KSF criterion might simplify model (4.1) to

$$\max_x \quad \mu^T x - \frac{1}{\sqrt{\alpha}}\sqrt{x^T \Sigma x}$$
$$\text{s.t.} \quad \begin{cases} \mu^T x \geqslant m, \\ e^T x = 1, \\ x \geqslant 0. \end{cases} \tag{4.25}$$

Even though solving model (4.25) can only provide a suboptimal or satisfiable portfolio for the investor, it is much easier than solving model (4.1) directly.

Since the mathematical form of model (4.25) is the same as that of model (4.2), by using $-1/\sqrt{\alpha}$ to replace z_α in Theorem 4.1, we obtain the following result:

Corollary 4.2. *Assume that the components of μ are not equal, that Σ IS positive definite, and that short-sell is allowed.*

(i) *When $m \leqslant B/C$ and $0<\alpha<\min\{1, C/D\}$, or when $m > B/C$ and*

$$\frac{C}{D}\frac{(Cm-b)^2}{(Cm-B)^2+D} \leqslant \alpha < \min\{1, C/D\},$$

model (4.25) has a unique finite global optimal solution x^*, and

$$x^* = \frac{1}{\sqrt{C/\alpha - D}}\left(\Sigma^{-1}\mu + \frac{-B+\sqrt{C/\alpha-D}}{C}\Sigma^{-1}e\right).$$

The corresponding optimal objective value r_α is given by

$$r_\alpha = \frac{B}{C} - \frac{\sqrt{C/\alpha - D}}{C}.$$

The mean return of optimal portfolio x^* is given by

$$m^* = \frac{B}{C} + \frac{D}{C\sqrt{C/\alpha - D}}.$$

(ii) *When $m > B/C$ and $0<\alpha<(C/D)(Cm-b)^2/((Cm-B)^2+D)$, model (4.25) has a unique finite global optimal solution x^*, and*

$$x^* = \frac{mC-B}{D}\Sigma^{-1}\mu - \frac{mB-A}{D}\Sigma^{-1}e.$$

The corresponding optimal objective value r_α is given by

$$r_\alpha = m - \frac{1}{\alpha}\sqrt{\frac{1}{C}\left(1+\frac{(mC-B)^2}{D}\right)}.$$

The mean return of the optimal portfolio x^* equals to m.

(iii) *For any other given values of m and α beyond that stated in (i) and (ii), model (4.25) will not have any finite optimal solution.*

Based on Corollary 4.2, by a computational procedure similar to that provided in Section 3.2, one can obtain the optimal portfolio of model (4.25) for the case that short-sell is not allowed.

5. Conclusion

This paper gives some discussions on KSF model. Suppose that the returns of all risky assets in the market are jointly elliptically distributed, when short-sell is allowed, we provide an explicit analytical optimal solution for the model and the corresponding necessary and sufficient condition(s) for its existence; when short-sell is not allowed, we also provide an iterative algorithm for finding the optimal portfolio and a numerical example. Suppose that the distribution of return vector of all risky assets is irregular or unknown, we use Tchebycheff's inequality to simplify the KSF model and give a similar discussion. In fact, the simplified KSF model overestimates investor's risk aversion. It provides the investor only with a suboptimal portfolio. The same manner has also been used to discuss KSF model with a constraint of mean return. If short-sell is allowed, explicit solutions for the models are also provided.

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