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Highlights:

- An enhanced indexation model is proposed with an explicit objective to track and outperform the naive diversification (1/N) strategy.
- A ratio-based objective is used to trade off between the excess return and a generalized downside risk measure.
- Simulation demonstrates that the enhanced indexation model can perform better than the 1/N strategy with a practical number of historical samples as input.
- Promising empirical results are presented using 21 representative US and non-US datasets.

Improving the Naive Diversification: An Enhanced Indexation Approach

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Abstract

This paper employs enhanced indexation to derive an optimization model with an explicit objective to track and outperform the naive diversification (1/N) strategy. The proposed model is data-driven and can start from any number of historical return samples. Simulation shows that the number of samples needed for the new model to outperform the 1/N benchmark is much smaller than the number documented in existing literature for other models. Our out-of-sample tests show that the proposed enhanced indexation model with the 1/N strategy as benchmark can achieve higher expected returns and significantly higher Sharpe ratios in most of the test cases.

Keywords: Enhanced indexation; Enhanced index tracking; Naive diversification; Portfolio management; Benchmark portfolio

JEL classification: G10, G11, C60

1. Introduction

Enhanced indexation (or enhanced index-tracking) models are portfolio selection models that try to efficiently track a benchmark index while also capturing higher potential excess returns. The benchmark index is usually a market index, as market indices are standard benchmarks for evaluating the performance of fund managers or

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actively-managed investment funds. Enhanced indexation funds, which are investment funds with an explicit objective to outperform a certain market index, have undergone rapid developments in recent decades (Filippi et al., 2016). Although using a real market index as benchmark is practical and intuitive in its own right, this paper constructs enhanced indexation models using the naive diversification (1/N) strategy as benchmark. We want to explore whether the success of enhanced indexation models can be extended to outperforming a specific strategy, and in that regard, whether enhanced indexation has the potential of serving as a general-purpose portfolio selection framework.

Many enhanced indexation models have been proposed in the literature. These models differ with respect to the way the tracking error is measured, and the mechanism of trading-off between the expected excess return and the tracking error. With respect to the tracking error measure, early studies are inspired by the Markowitz mean-variance framework and use the variance of the excess return as a tracking error measure (Hodges, 1976; Roll, 1992). This measure remains popular nowadays (e.g., Lejeune and Samatlı-Paç (2012); Huang et al. (2018); Kim and Kim (2020)). The second type of tracking error is defined using the price series dissimilarity instead of the return series dissimilarity. One widely used measure of such type is the mean-absolute deviation (MAD) between the historical values of the tracking portfolio and the index (e.g., Konno and Wijayanayake (2001); Guastaroba and Speranza (2012); Filippi et al. (2016)). The third type of tracking error is defined using only downside scenarios, i.e., the scenario when the tracking portfolio performs worse than the benchmark index. Examples of such type are the downside variance of the excess return (Meade and Beasley, 2011) the expected downside absolute deviation of the excess return (Guastaroba et al., 2016), and the maximum under-performance measure (Bruni et al., 2015). With respect to the mechanism of trading-off between the expected excess return and the tracking error, many models use a weighted linear combination of the expected excess return and the tracking error as the objective function (e.g., Beasley et al. (2003)). This linear trade-off approach is similar to tracking error minimization with a constraint that prescribes a minimum expected excess return (e.g., Gnägi and Strub (2020)), or with the maximization of the expected excess return as a second competing objective (e.g., Li et al. (2011); Filippi et al. (2016)). The second approach uses certain ratios as the objective function. Meade and Beasley (2011) maximize the modified Sortino ratio, which is the expected excess return divided by the downside variance of the excess return, while Guastaroba et al. (2016) maximize the Omega ratio, which is the expected excess return divided by the expected downside absolute deviation of the excess return. All these models have been applied on real datasets to track and outperform a certain market index and the average results have been successful. However, there does not exist a dominating tracking error measure or a dominating trade-off mechanism, as the relative empirical performances of different enhanced indexation models vary when additional real-life constraints and risk-controlling constraints are present.

Literature on enhanced indexation models has been focusing on its native application: tracking and outperforming a target index. It is unknown whether it is possible to expand the the applicability of enhanced indexation and use it for general-purpose portfolio selection. In this paper, we explore this possibility by constructing an enhanced indexation model with an explicit objective to outperform the 1/N strategy. The 1/N strategy has been serving as an important benchmark when evaluating the performance of modern portfolio selection models and it has been proved to be approximately optimal under high model ambiguity (Pflug et al., 2012) or under a well-securitized market with abundant assets (Platen and Rendek, 2012). Benartzi and Thaler (2001) document that some investors allocate their wealth across funds using the 1/N rule. DeMiguel et al. (2009b) demonstrate that the 1/N strategy has superior out-of-sample performance over many modern portfolio selection models. They show that the sample number needed for the classical sample-based mean-variance strategy and its extensions to outperform the 1/N benchmark is around 3000 for a portfolio with 25 assets and about 6000 for a portfolio with 50 assets, and when the sample number is small, the gain from optimal diversification is more than offset by estimation error. Given this background, if a representative enhanced indexation model can perform better than the 1/N strategy with a practical number of historical samples as input, then we can confirm that enhanced indexation has the potential of serving as a general-purpose portfolio selection framework and hopefully stimulate further application and investigation of enhanced indexation models in the future.

The contribution of this paper is two-folded. Firstly, we propose an enhanced indexation model using a ratio-based objective to trade off between the excess return of a tracking portfolio over the 1/N strategy, and a generalized downside risk measure with respect to the 1/N benchmark. Our model incorporates representative aspects in the enhanced indexation literature and it includes several existing models as its special cases. In addition, the proposed model is data-driven and can start from any number of historical return samples. Secondly, we demonstrate the advantage of our enhanced indexation model using two different simulation schemes. In the first schemes, the returns of the risky assets are assumed to follow a multivariate normal distribution with mean and covariance matrix estimated from real world portfolios. In the second scheme, the returns are assumed to be generated from a single-factor market model as in DeMiguel et al. (2009b). Our simulation shows that the sample number needed for the enhanced indexation model to outperform the 1/N benchmark is around dozens for a portfolio with 48 assets in the first simulation scheme and around 100 to 200 for a portfolio with 50 assets in the second simulation scheme. These sample numbers are much smaller than the number documented in existing literature for other models and are within practical ranges, making enhanced indexation models feasible in outperforming the 1/N benchmark in real world situations. To verify our simulation findings, we conduct extensive empirical tests using 21 representative datasets, covering both US and non-US markets, accounting for both fundamental and anomalistic asset characteristics. Our out-of-sample tests show that our model has higher expect returns and significantly higher Sharpe ratios than the 1/N strategy for most of the situations.

The remaining paper is organized as follows. In Section 2 we introduce our enhanced indexation model with the 1/N strategy as the benchmark. In Section 3 we demonstrate the advantage of our enhanced indexation model using two different simulation schemes. In Section 4 we conduct out-of-sample tests to compare the performance of the 1/N strategy with our model. We conclude in Section 5.

2. Enhanced indexation model

Consider the situation where an investor wants to invest on a portfolio of n securities with the objective of outperforming a specific benchmark for a coming period. The random return of security $i, i \in \{1, 2, \ldots, n\}$, is denoted by R_i. The random return of the benchmark is denoted by I. In this paper, we are interested in the case where the benchmark I is the return generated by the naive diversification (1/N) strategy, where an equal proportion of the total wealth in invested on each security, i.e., $I = \sum_{i=1}^{n} R_i/n$. We propose the following ratio-based enhanced indexation model:

$$\max_{x} \quad \frac{\mathbf{E}\left[\sum_{i=1}^{n} x_i R_i - I\right]}{\mathbf{E}\left[\max\left(I - \sum_{i=1}^{n} x_i R_i,\ 0\right)^{\alpha}\right]^{\frac{1}{\alpha}} + \epsilon}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1,\ x_i \in \mathbb{R}_+,\ i \in \{1, 2, \ldots, n\}.$$

The decision variable x_i represents the proportion of total wealth invested on the i-th asset. Short-selling is not allowed in the model. The numerator of the objective function is the expected excess return of the tracking portfolio over the 1/N portfolio. The expectation in the denominator is the expected downside risk (Beasley et al., 2003) for this excess return. The ϵ is a small positive constant used to prevent the denominator from falling to zero. We set $\epsilon = 10^{-6}$ in our model through out this paper. The investor would want to maximize the expected excess return while minimizing the expected downside risk and these two objectives are traded-off via a ratio in our model. The parameter α determines the way we define the downside risk. Our model includes three special cases: a) If $\alpha = 1$, the objective function becomes the Omega ratio employed in Guastaroba et al. (2016). b) If $\alpha = 2$, the objective function becomes the modified Sortino ratio proposed in Meade and Beasley (2011). c) If $\alpha = \infty$ and the security returns follow bounded distributions, the expectation in the denominator becomes the maximum under-performance tracking error measure proposed in Bruni et al. (2015), which is defined as the maximal possible value of $(I - \sum_{i=1}^{n} x_i R_i)$.

When enhanced indexation models are applied to real world problems, the security returns are usually assumed to follow a joint discrete probability distribution. This discrete distribution often comes from historical return scenarios within a given time window, where each scenario is assumed to have an equal probability weight. If the historical returns in the past T periods are used as the possible scenarios, denoted by $R_{it},\ t \in \{1, 2, \ldots, T\},\ i \in \{1, 2 \ldots, n\}$, then the proposed model can be rewritten as

$$(\mathrm{P}(\alpha)) \quad \max_{x} \quad \frac{\frac{1}{T}\sum\limits_{t=1}^{T}\left(\sum\limits_{i=1}^{n} x_i R_{it} - I_t\right)}{\left(\frac{1}{T}\sum\limits_{t=1}^{T}\max\left(I_t - \sum\limits_{i=1}^{n} x_i R_{it},\ 0\right)^{\alpha}\right)^{\frac{1}{\alpha}} + \epsilon}$$
$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = 1,\ x_i \in \mathbb{R}_+,\ i \in \{1, 2, \ldots, n\}.$$

When the benchmark is the return generated by the naive 1/N strategy, we have $I_t = \sum_{i=1}^{n} R_{it}/n$.

When $\alpha \geq 1$, the denominator in the objective of the model $(\mathrm{P}(\alpha))$ is nonnegative and convex in x, thus the overall problem is a pseudo-concave maximization problem and any local optimal solution is globally optimal (Stancu-Minasian, 2012). In this case, the model is solvable using existing nonlinear optimizers. When $\alpha = 1$, the problem $(\mathrm{P}(\alpha))$ can also be transformed into a linear programming problem (Guastaroba et al., 2016).

3. Results from simulated data

In this section, we focus on the sample periods T in problem $(\mathrm{P}(\alpha))$. A similar simulation method as in DeMiguel et al. (2009b) is used to find out how large T is required in order for the proposed enhanced indexation model to beat the 1/N strategy.

We assume the returns of the n securities follow a multivariate normal distribution[1]. Then T samples are drawn from this distribution, based on which problem $(\mathrm{P}(\alpha))$ is solved. The performance of the resulted optimal solution is evaluated using one more

[1]As one referee pointed out that multivariate normal distribution might be inaccurate in modeling real-world distributions for security returns, we have also conducted simulation by re-sampling directly from historical stock return scenarios. The results are similar to those displayed in Figure 1.

random sample from the same multivariate normal distribution. In this way we generate one simulated out-of-sample return $\hat{r}$ for our enhanced indexation model. This process is simulated for K times, generating out-of-sample returns $\hat{r}_1, \ldots, \hat{r}_K$. Then we can calculate the average return, standard deviation and Sharpe ratio of the enhanced indexation model $(\mathrm{P}(\alpha))$. We assume an annual risk-free rate of 2% and set the simulation iteration number $K = 100,000$.

Before running the simulation, we need to specify the mean and covariance matrix of the multivariate normal distribution. We adopt two different schemes to obtain these parameters. The first scheme uses the sample mean and covariance of monthly returns from the 48 equal-weight US industry portfolios provided by Kenneth R. French's data website[2]. The second scheme uses the same settings as in DeMiguel et al. (2009b), where the returns of the n securities are assumed to be generated by the single-factor model $r_i = B_i r_b + \epsilon_i, i = 1, \ldots, n$. The common factor r_b follows a normal distribution with an annual average of 8% and standard deviation of 16%. The noise $\epsilon_i, i = 1, \ldots, n$, follow a multivariate normal distribution with zero mean and a diagonal covariance matrix with elements drawn from a uniform distribution with support $[0.10, 0.30]$. The factor loading $B_i, i = 1, \ldots, n$, are evenly spread between 0.5 and 1.5. We set the asset number $n = 50$ in the second simulation scheme.

Figure 1 presents comparison of the 1/N strategy and our enhanced indexation model with different α in the simulation scheme using an empirical covariance matrix. As the number of samples T changes, the dynamics of average return, standard deviation, and Sharpe ratio of different strategies are plotted. For the 1/N strategy, because its allocation does not changes with T, the average return, standard deviation and Sharpe ratio are constants. For enhanced indexation models, the average returns first decrease as T increases, and then stay at a stable level higher than the 1/N benchmark level after $T > 20$. As T increases, the standard deviations decrease in an exponential speed to the 1/N benchmark level. Because the average returns of the enhanced indexation model stay at a level higher than the 1/N benchmark level while the standard

[2]The portfolio data is available at `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. The data in the time period from July 1963 to April 2019 is used to estimate the mean and covariance matrix.

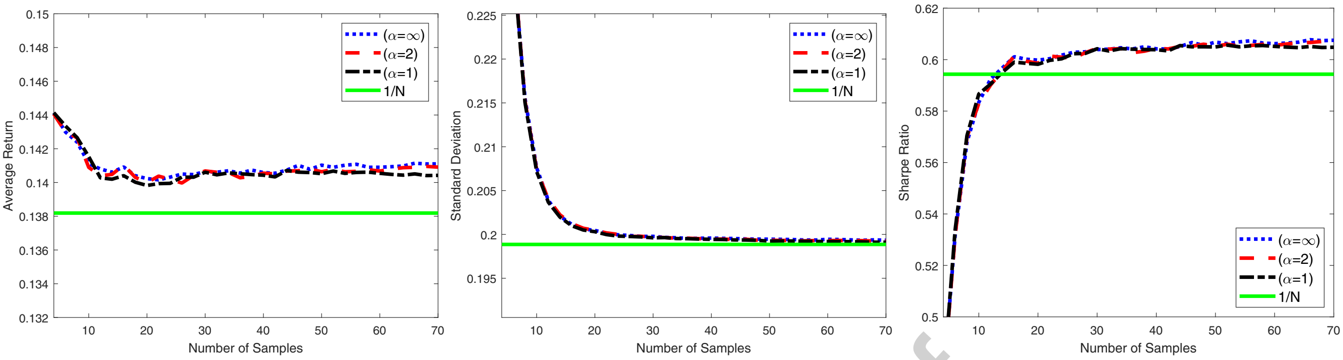

Figure 1: Comparison on average return (left), standard deviation (middle), and Sharpe ratio (right) of the 1/N strategy and the enhanced indexation model with different α in the simulation scheme using an empirical covariance matrix.

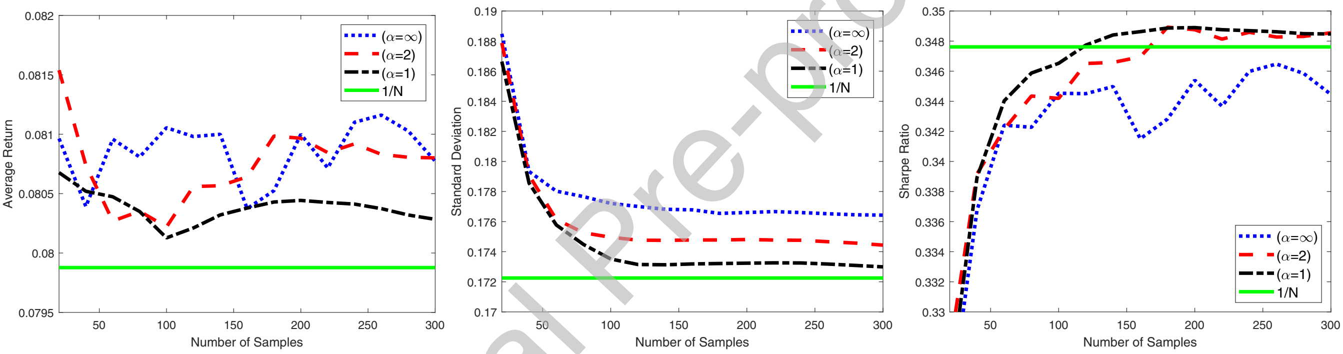

Figure 2: Comparison on average return (left), standard deviation (middle), and Sharpe ratio (right) of the 1/N strategy and the enhanced indexation model with different α in the simulation scheme using a single-factor covariance matrix.

deviations drop to the 1/N benchmark level, the Sharpe ratios of the enhanced indexation models become higher than the 1/N benchmark level as T increases. The dynamics of different enhanced indexation models with $\alpha = 1, 2$, and ∞ are almost the same, and the required sample number for the enhanced indexation model to outperform the 1/N strategy (in terms of Sharpe ratio) is around 14.

Figure 2 presents the results in the simulation scheme using a single-factor covariance matrix. The general trends of the dynamics are similar to those in Figures 1, but there are several differences. Firstly, the average return dynamics becomes volatile when $\alpha = 2$ or ∞. Secondly, as the number of samples T increase, the model with

$\alpha = 1$ has lower average return and standard deviation than the model with $\alpha = 2$ or ∞. In terms of Sharpe ratio, the model with $\alpha = 1$ performs the best and model with $\alpha = \infty$ performs the worst. Thirdly, only the model with $\alpha = 1$ or 2 performs better than the 1/N benchmark (in terms of Sharpe ratio) within a sample size of 300. In those cases, the required sample number for our enhanced indexation model to outperform the 1/N strategy is around 100 to 200.

In summary, our simulation shows that the number of historical samples needed for the enhanced indexation model with $\alpha = 1$ or 2 to outperform the 1/N benchmark is 14 in the first simulation scheme and 100 to 200 in the second simulation scheme, much smaller than the number (6000 for $n = 50$) documented in DeMiguel et al. (2009b) for other models and is within a practical time range.

4. Results from empirical datasets

In this section, we carry out out-of-sample tests on US and non-US market data to compare the performance of the 1/N strategy with our model ($\mathrm{P}(\alpha)$) under three cases of α as discussed in Section 2: $\alpha = 1$, $\alpha = 2$ and $\alpha = \infty$.

In our out-of-sample tests, we choose 21 datasets as summarized in Table 1. The datasets in the first 10 rows are from Kenneth R. French's data website. For each of the first 10 rows, there is an equal-weight version and a value-weight version. In the equal-weight version, the individual asset return of that dataset is constructed using the simple average return of a corresponding asset sub-group; in the value-weight version, the individual asset return of that dataset is constructed using the market-capitalization-weighted average return of the corresponding asset sub-group. Datasets No.1-3 are constructed by grouping US common stocks based on size and anomalistic characteristics of short-term reversal, momentum and variance; Datasets No.4-10 are constructed by grouping common stocks based on size and fundamental characteristics and among them, No.7-9 are constructed using assets from non-US developed countries. The dataset in the 11-th row is used in the empirical tests in Guastaroba et al. (2016), which contains weekly returns of the S&P500 index constituents.

Table 1: List of datasets considered.

This table lists the various datasets of asset returns used in the empirical tests, the number of risky assets n in each dataset, the time period spanned by the dataset, and the abbreviation used to refer to each dataset. Monthly datasets in the first 10 rows are from Kenneth R. French's data website `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. For each of the 10 rows, there is an equal-weight version and a value-weight version. In the equal-weight version, the individual asset return of that dataset is constructed using the simple average return of a corresponding asset sub-group; In the value-weight version, the individual asset return of that dataset is constructed using the market-capitalization-weighted average return of the corresponding asset sub-group. The weekly dataset in the 11-th row is from `http://people.brunel.ac.uk/~mastjjb/jeb/orlib/indtrackinfo.html` and is used in the empirical tests in Guastaroba et al. (2016).

No.	Dataset	n	Time Period	Abbreviation
1	25 Monthly US Portfolios Formed on Size and Short-Term Reversal	25	07/1963-04/2019	25SR
2	25 Monthly US Portfolios Formed on Size and Momentum	25	07/1963-04/2019	25SM
3	25 Monthly US Portfolios Formed on Size and Variance	25	07/1963-04/2019	25SV
4	32 Monthly US Portfolios Formed on Size, Book-to-Market, and Investment	32	07/1963-04/2019	32SBI
5	32 Monthly US Portfolios Formed on Size, Book-to-Market, and Operating Profitability	32	07/1963-04/2019	32SBO
6	32 Monthly US Portfolios Formed on Size, Operating Profitability, and Investment	32	07/1963-04/2019	32SOI
7	32 Monthly Developed ex US Portfolios Formed on Size, Book-to-Market, and Investment	32	07/1990-04/2019	32SBI (exUS)
8	32 Monthly Developed ex US Portfolios Formed on Size, Book-to-Market, and Operating Profitability	32	07/1990-04/2019	32SBO (exUS)
9	32 Monthly Developed ex US Portfolios Formed on Size, Operating Profitability, and Investment	32	07/1990-04/2019	32SOI (exUS)
10	48 Monthly US Industry Portfolios	48	07/1963-04/2019	48IND
11	Weekly Returns of S&P500 Index Constituents	457	156 weeks	SP500

We use a "rolling-horizon" approach with monthly re-balance for the out-of-sample tests. Before each monthly re-balance, we adopt a cross-validation procedure similar to the approach in DeMiguel et al. (2009a) to select a better window length T. Ranging from 24 months to 120 months for the monthly datasets and from 52 weeks to 104 weeks for the weekly dataset. we choose T to maximize portfolio return in the latest period in history.

Table 2 presents the out-of-sample results using the datasets listed in Table 1. For each dataset, the 1/N strategy is compared with our model $(\mathrm{P}(\alpha))$ under the cases $\alpha = 1, 2$, or ∞. The mean, standard deviation and Sharpe ratio of each strategy are reported. Our enhanced indexation model performs better in terms of mean return and Sharpe ratio in all the test cases. For each enhanced indexation strategy, we test the hypothesis of its Sharpe ratio being equal to that of the corresponding 1/N strategy. Among the 63 Sharpe ratio equality tests, 49 tests show that our model has a significantly higher Sharpe ratio than that of the 1/N strategy. For the value of α, it does not seem to have a material effect on the performance of the enhanced indexation model. This fact matches the pattern in the simulation using the first scheme in the previous section, where the covariance matrix is estimated from real world portfolios.

In summary, using representative datasets from US and non-US markets with different firm characteristics, our out-of-sample results show that the enhanced indexation model with the 1/N strategy as the benchmark has a high potential in out-performing the 1/N strategy with higher Sharpe ratios and expected returns.

5. Conclusion and future research

We have built an enhanced indexation model with an explicit objective of tracking and outperforming the naive 1/N diversification strategy. Our simulation has demonstrated that the enhanced indexation model can perform better than the 1/N strategy with a practical number of historical samples as input, which is also supported by empirical out-of-sample tests using 21 representative datasets. Our results provide support for the effectiveness of the enhanced indexation framework and show that it has the potential of serving as a general-purpose portfolio selection framework. This paper does not consider transaction cost and other practical real-life constraints. It would be interesting to explore the effects of those constraints in future research.

Table 2: Out-of-sample results of the 1/N strategy versus enhanced indexation strategies.

The datasets are described in Table 1. The enhanced indexation model ($\mathrm{P}(\alpha)$) is implemented with different values of α. "1/N" denotes the naive diversification strategy. "SR" denotes the Sharpe ratio. The notations (***), (**) and (*) indicate that the Sharpe ratio is significantly different from the Sharpe ratio of the corresponding 1/N strategy at the $1\%, 5\%$ and 10% statistical levels, respectively. The bootstrapping methodology in Ledoit and Wolf (2008) is used to test the equality hypothesis of two Sharpe ratios.

		Equal-Weight				Value-Weight			
		1/N	$\alpha=1$	$\alpha=2$	$\alpha=\infty$	1/N	$\alpha=1$	$\alpha=2$	$\alpha=\infty$
25SR	Mean	0.141	0.150	0.149	0.150	0.136	0.141	0.143	0.142
	Std	0.185	0.186	0.186	0.188	0.184	0.187	0.185	0.185
	SR	0.515	0.562(***)	0.556(***)	0.556(***)	0.492	0.510	0.524(***)	0.524(***)
25SM	Mean	0.144	0.149	0.147	0.150	0.138	0.144	0.145	0.145
	Std	0.183	0.185	0.185	0.182	0.183	0.182	0.182	0.182
	SR	0.539	0.556	0.549	0.575(***)	0.504	0.539(***)	0.543(***)	0.549(***)
25SV	Mean	0.148	0.152	0.154	0.154	0.140	0.146	0.147	0.146
	Std	0.179	0.179	0.179	0.178	0.181	0.181	0.181	0.182
	SR	0.574	0.598(***)	0.606(***)	0.610(***)	0.523	0.557(***)	0.560(***)	0.553(**)
32SBI	Mean	0.157	0.161	0.162	0.164	0.143	0.144	0.143	0.144
	Std	0.182	0.182	0.182	0.182	0.169	0.169	0.169	0.169
	SR	0.613	0.637(***)	0.641(***)	0.651(***)	0.573	0.581	0.580	0.580
32SBO	Mean	0.157	0.162	0.163	0.162	0.141	0.146	0.144	0.145
	Std	0.186	0.185	0.185	0.185	0.174	0.174	0.174	0.174
	SR	0.601	0.629(**)	0.632(**)	0.628(**)	0.547	0.576(***)	0.569(**)	0.572(***)
32SOI	Mean	0.157	0.160	0.162	0.164	0.138	0.142	0.142	0.145
	Std	0.180	0.180	0.180	0.180	0.171	0.171	0.170	0.171
	SR	0.618	0.639(***)	0.647(***)	0.657(***)	0.541	0.564(**)	0.567(***)	0.579(***)
32SBI(exUS)	Mean	0.102	0.108	0.110	0.110	0.076	0.078	0.079	0.080
	Std	0.174	0.174	0.174	0.173	0.164	0.164	0.165	0.164
	SR	0.497	0.537(***)	0.54(***)	0.547(***)	0.374	0.385	0.389	0.397(**)
32SBO(exUS)	Mean	0.099	0.114	0.114	0.111	0.070	0.081	0.082	0.081
	Std	0.177	0.176	0.176	0.177	0.166	0.165	0.164	0.166
	SR	0.473	0.562(***)	0.562(***)	0.545(***)	0.331	0.397(***)	0.411(***)	0.397(**)
32SIO(exUS)	Mean	0.100	0.104	0.105	0.107	0.066	0.069	0.068	0.072
	Std	0.175	0.174	0.173	0.173	0.165	0.164	0.165	0.165
	SR	0.486	0.510(**)	0.521(***)	0.530(***)	0.311	0.331(**)	0.323	0.343(***)
48IND	Mean	0.148	0.160	0.160	0.158	0.128	0.131	0.132	0.132
	Std	0.196	0.195	0.195	0.197	0.168	0.169	0.170	0.169
	SR	0.524	0.587(***)	0.587(***)	0.568(**)	0.491	0.504	0.506	0.512
SP500	Mean	0.194	0.210	0.227	0.253				
	Std	0.157	0.156	0.157	0.161				
	SR	1.238	1.345	1.439	1.575(*)				

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