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A differential evolution copula-based approach for a multi-period cryptocurrency portfolio optimization

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Abstract

Recent years have seen a growing interest among investors in the new technology of blockchain and cryptocurrencies and some early investors in this new type of digital assets have made significant gains. The heuristic algorithm, differential evolution, has been advocated as a powerful tool in portfolio optimization. We propose in this study two new approaches derived from the traditional differential evolution (DE) method: the GARCH-differential evolution (GARCH-DE) and the GARCH-differential evolution t-copula (GARCH-DE-t-copula). We then contrast these two models with DE (benchmark) in single and multi-period optimizations on a portfolio consisting of five cryptoassets under the coherent risk measure CVaR constraint. Our analysis shows that the GARCH-DE-t-copula outperforms the DE and GARCH-DE approaches in both single- and multi-period frameworks. For these notoriously volatile assets, the GARCH-DE-t-copula has shown risk-control ability, hereby confirming the ability of t-copula to capture the dependence structure in the fat tail.

Keywords Cryptocurrencies · GARCH · Differential evolution · t-copula · CVaR · Portfolio optimization

JEL Classification C02 · G11 · G17

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1 Introduction

The increasing adoption of cryptocurrencies has contributed to the global dependence between cryptoassets. Due to both contagion effects and volatility spillovers, modeling this dependence is important for asset allocation and management. Cryptocurrency, which is still in its infancy of development and adoption, is known to be highly volatile and susceptible to important changes driven by speculations and institutional regulations. These extreme dynamics can result in dependence shifts and portfolio losses (see, e.g., Brunnermeier 2009; Moshirian 2011; Florackis et al. 2014; Bekiros et al. 2015). It is therefore important for cryptocurrency portfolio optimization, to find technical tools that are able to deal with such underlying interactions. For cryptocurrency portfolios, several models have been developed following the traditional approach based on the bilateral correlation coefficient algorithm. However, these models appear to be restrictive. An example of these models is given by the class of multivariate GARCH models developed to alleviate the normality assumption sustaining the pioneered mean–variance approach. Deterministic by nature, these models rely on parametric multivariate distribution likely to be erroneously specified when the distribution of all the variables is not the same. This is likely the case for financial assets in general and for cryptoasset returns in particular, exhibiting various underlying distribution properties such as non-normality, asymmetric correlations, volatility clustering, heavy tail behavior (see, e.g., Osterrieder and Lorenz 2017; Phillip et al. 2018; Alvarez-Ramirez et al. 2018). A positive response to these deterministic and restrictive shortcomings of the traditional models can be obtained through the copula approach, particularly the t-copula (see, e.g., Embrechts et al. 2002; Fang and Fang 2002) which represents implicitly the dependence structure in a multivariate t-distribution. It has recently received much attention in the context of modeling multivariate financial time series. It has also been advocated that its empirical fit is in general superior to that of the dependence structure of the multivariate normal distribution, given by the Gaussian copula (see, e.g., Mashal and Zeevi 2002; Breymann et al. 2003). This is explained by the ability of the t-copula to successfully capture the extreme values dependence phenomenon, which is generally perceived in financial data return and especially in cryptocurrencies.

Markowitz original approach provides a fundamental basis for portfolio selection in a single-period model of investment. In this model of investment, at the beginning of a selected period, the investors make once and for all allocation decisions which remain unchanged until the end of the period, disregarding the market behavior during that locked period. This is why single-period models lead to what is named myopic policies.

Because markets are risky, as changes arise in financial markets and create imbalances in the portfolio allocations, investors need to respond to these changes by rebalancing/realigning their portfolios. This leads to what is called multi-period models. The problem of multi-period portfolio allocation has been studied by Hakansson (1971), Sahalia and Brandt (2001), Calafiore (2007), and many others. Boudt et al. (2012) make use of the differential evolution (DE) algorithm in single- and multi-period settings to numerically solve portfolio optimization problems under complex constraints and objectives. DE was introduced by Storn and Price (1997) and they found that DE was more efficient than genetic algorithms and simulated annealing.

It has become a powerful and flexible tool to solve optimization problems arising in finance. Similar to classic genetic algorithms, DE algorithm is an evolutionary technique which can be used to solve global optimization problems. This algorithm has shown remarkable performance on continuous numerical problems (Price et al. 2006) and optimizing portfolios under non-convex settings; see Ardia et al. (2011), Krink and Paterlini (2011), Krink et al. (2009), Maringer and Oyewumi (2007), Yollin (2009). A combination of this remarkable algorithm with t-copula could produce a powerful tool for portfolio optimization. We aim in this study in developing a models including DE and t-copula, and evaluate its impact on a cryptocurrency portfolio.

In the literature, GARCH models have been applied to analyze and understand the dynamics of cryptocurrencies price movement. For example, Dyhrberg (2016b) used the asymmetric GARCH methodology to explore the Bitcoin hedging capabilities. Including GARCH methodology in portfolio selection and optimization models could also add value to the outcome of the process.

In this regard, we investigate in this paper the performance of the GARCH-differential evolution (GARCH-DE) and GARCH-differential evolution t-copula (GARCH-DE-t-copula) models and compare them to the existing DE model in single- and multi-period optimizations. As this is an innovative approach, we will also focus on presenting a clear detailed methodological approach prior to its implementation. These models are performed on a portfolio consisting of 5 cryptocurrencies, namely Bitcoin , Ripple , Litecoin, Dash, and Dogecoin representing at the time of this writing, nearly 50% market share of the cryptocurrency market.

Our optimization problem is subject to the minimization of the conditional value at risk (CVaR) introduced by Rockafellar and Uryasev (2000) as a coherent risk measure.

Contrasting our two approaches in multi-period portfolio optimization under the above-mentioned risk measure, we are aiming in this way at identifying under which methodology, cryptocurrency portfolios may be more profitable or risky than other. Though innovative, our approach is closely related to that of Bekiros et al. (2015) who estimate the multivariate dependence using pair vine copula to optimize a portfolio consisting of the Australian mining stocks, subject to the minimization of five risk measures including CVaR. Since the extreme value distribution (EVT) is able to accurately model tails risk, we use in this study the generalized pareto distribution (GPD)-based t-copula (GARCH-DE-t-copula) instead of pair vine copula-GARCH used by Bekiros et al. (2015), to explicitly capture the tail events. This model, as we will see in the analysis section, demonstrates the ability to control the portfolio risk while guaranteeing high returns compared to DE-GARCH and the traditional DE models. We believe it will be a useful tool in the portfolio managers community.
The rest of the paper is organized as follows: Sect. 2 presents the new optimization models. Section 3 discusses the empirical findings, and Sect. 4 concludes the work.

2 Optimization periods mathematical formalism

In this section, we review the mathematical formalism of portfolio optimization methods used in this current study, i.e., single-period optimization, multi-period optimization, DE, GARCH-DE, GJR-GARCH, GARCH-DE-t-copula.

2.1 Single-period optimization

In portfolio optimization processes, the main challenge resides in designing a proper model that empirically best fits the data and, at the same time, is feasible and robust enough to generate simulation-based inference for risk evaluation.

The basic mean–variance optimization can be formulated as follows (see Markowitz 1952):

$$\begin{aligned}
&\min_{\omega} \sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_{ik} \omega_i \omega_k \\
&\text{subject to} \\
&\sum_{i=1}^{n} \omega_i = 1 \\
&\sum_{i=1}^{n} E[r_i]\omega_i = \mu_p \\
&\omega_i \geq 0, \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{1}$$

where ω_i are portfolio weights, r_i is the rate of return of asset i and $E[r_i]$ its expectation, $\sigma_{ik} = cov(r_i, r_k)$ is the covariance between r_i and r_k, μ_p is the portfolio expected return.

Among various risk measures, value at risk (VaR) is a popular measure of risk that represents the percentile of the loss distribution with a specified confidence level. Let $\omega \in \mathbb{R}^n$ denote a portfolio vector indication, a proportion of investment of a given budget in each of the n financial assets. Let $\alpha \in (0, 1)$ and $f(\omega, \mathbf{r})$ denoting, respectively, a confidence level and a loss function for the portfolio ω and the return vector $\mathbf{r} \in \mathbb{R}^n$. Then, the VaR function, $\xi(\omega, \alpha)$, results to the smallest number satisfying $\psi(\omega, \xi(\omega, \alpha))) = \alpha$, where $\psi(\omega, \xi) = \Pr[f(\omega, \mathbf{r}) \leq \xi]$ is the probability that the loss $f(\omega, \mathbf{r})$ does not exceed the threshold value ξ. However, VaR does not satisfy the sub-additivity axiom. Furthermore, VaR is nondifferentiable as well as non-convex when using scenarios analysis. Hence, it is difficult to find a global minimum using conventional optimization techniques.

Alternatively, conditional VaR (CVaR), introduced by Rockafellar and Uryasev (2000) is a coherent risk measure with more interesting features such as sub-additivity and convexity. Moreover, it is more appropriate to the loss function of the tail distribution. CVaR is given by

$$\psi_\alpha(\omega) = (1 - \alpha)^{-1} \int_{f(\omega, \mathbf{r}) > \xi(\omega, \alpha)} f(\omega, \mathbf{r}) p(\mathbf{r}) \, \mathrm{d}\mathbf{r}. \tag{2}$$

To avoid complications resulting from the implicitly defined function $\xi(\omega, \alpha)$, Rockafellar and Uryasev (2000) provide an alternative function given by

$$F_\alpha(\omega, \xi) = \xi + (1 - \alpha)^{-1} \int_{f(\omega, \mathbf{r}) > \xi} [f(\omega, \mathbf{r}) - \xi] p(\mathbf{r}) \, \mathrm{d}\mathbf{r}, \tag{3}$$

for which they show the minimum of CVaR can be found by minimizing $F_\alpha(\omega, \xi)$ with respect to (ω, ξ).

Given return data r_j for $j = 1, \ldots, n$, $F_\alpha(\omega, \xi)$ can be approximated by

$$\tilde{F}_\alpha(\omega, \xi) = \xi + [(1-\alpha)n]^{-1} \sum_{j=1}^{n} \max\{f_j(\omega) - \xi, 0\}, \tag{4}$$

where $f_j(\omega) = f(\omega, r_j)$.

In this study, our three models are designed to solve the following CVaR-optimization problem:

$$\begin{aligned} &\min_{\omega} \xi + [(1-\alpha)n]^{-1} \sum_{i=1}^{n} \max\{f_i(\omega) - \xi, 0\} \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \sum_{i=1}^{n} E \quad [r_i]\omega_i = \mu_p \\ \omega_i \geq 0, \quad i = 1, \ldots, n. \end{cases} \end{aligned} \tag{5}$$

We intend to find the portfolio that minimizes CVaR under 90% confidence level subject to the following weight constraints: Weights must sum to 1 ($\sum_i^n \omega_i = 1$) and no short-selling is allowed ($\omega_i \geq 0$).

2.2 Multi-period optimization

In multi-period portfolio optimization, the portfolio optimization problem is to choose a sequence of transactions/trades to perform over a chosen set of periods. One of the advantages of multi-period portfolio optimization is its ability to naturally handle multiple return estimates on different time scales (see for example, Gârleanu and Pedersen 2013; Nystrup et al. 2016).

Consider a universe of n assets $\{a_1, a_2, \ldots, a_n\}$ and an investment planning horizon that extends T periods of equal duration δ ($\delta = 1$ month or $\delta = 1$ quarter). Let $s_i(t)$ be the dollar value of the total wealth portion invested in asset a_i at time t. Let $s(t) = [s_1(t) \ \ldots \ s_n(t)]^T$, then the total wealth invested at time t is given by

$$v(t) = \sum_{i=1}^{n} s_i(t) = \mathbf{1}^T s(t), \tag{6}$$

where $\mathbf{1}$ denotes a $n \times 1$ column matrix of ones.

The investor has the opportunity at the end of each period to adjust the portfolio composition. Let $u(t) = [u_1(t) \ \ldots \ u_n(t)]^T$ be the vector of adjustments. A value of $u_i(t) > 0$ means that the value of asset a_i is increased by $u_i(t)$ dollars (by buying more of the asset a_i), whereas $u_i(t) < 0$ means that the value of asset a_i is decreased by $u_i(t)$ dollars (by selling part or all of the asset a_i).

Let $s^{+}(t)$ be the portfolio composition after the adjustment $u(t)$ is made at time t.

$$s^{+}(t) = s(t) + u(t) \tag{7}$$

Without loss of generality, we assume in this study, a self-financing portfolio, i.e., $\sum_{i=1}^{n} u_i(t) = 0$, for all t. The weight corresponding to asset a_i at time t is $\omega_i(t) = s_i(t)/v(t)$ and $\omega(t) = [\omega_1(t) \cdots \omega_n(t)]^T$ is the vector of weights with $\omega_i(t) \geq 0$ and $\sum_{i=1}^{n} \omega_i(t) = 1$.

Let $p_i(t)$ be the price of the asset a_i at time t. Let $r_i(t)$ be the log-return given by $r_i(t) = \ln[\frac{p_i(t+1)}{p_i(t)}]$, then $r(t) = [r_1(t) \dots r_n(t)]^T$ will denote the return vector.

Portfolio dynamics Let $\omega(0)$ be the initial portfolio allocation at time $t = 0$. If at time $t = 0$ transactions are conducted on the market, then the portfolio will be adjusted, either by increasing or decreasing the amount invested in each asset. So, the re-balanced portfolio will be

$$\omega^{+}(0) = \omega(0) + \widetilde{u}(0), \tag{8}$$

where $\widetilde{u}_i(t) = \dfrac{u_i(t)}{v(t)}$ and $\widetilde{u}(t) = [\widetilde{u}_1(t) \cdots \widetilde{u}_n(t)]^T$.

Assume that the portfolio remains unchanged for the first period of time δ. The portfolio allocation at the end of the first period is $\omega(1) = R(1)\omega^{+}(0) = R(1)\omega(0) + R(1)\widetilde{u}(0)$, where $R(t) = \text{diag}(r(t))$ is a diagonal matrix of asset returns over the interval period $[t-1, t]$, for $t \geq 1$. Iteratively, the composition of the portfolio at the end of period $t+1$ is:

$$\omega(t+1) = R(t+1)\omega(t) + R(t+1)\widetilde{u}(t), \quad t = 0, 1, \dots, T-1. \tag{9}$$

Since the asset returns $r_i(t)$ are random, the iterative equations in (9) define a stochastic process $\omega(t),\ t = 1, \dots, T$

$$\begin{aligned} \omega(t) = \psi(1,t)\omega(0) + [\psi(1,t)\, \psi(2,t) \cdots \psi(t-1,t)\, \psi(t,t)][\widetilde{u}(0)\, \widetilde{u}(1) \\ \cdots \widetilde{u}(t-2)\, \widetilde{u}(t-1)]^T, \end{aligned} \tag{10}$$

where $\psi(k,t),\ k \leq t$, is the compounded return matrix from the start of period k to the end of period t:

$\psi(k,t) = R(t)R(t-1)\cdots R(k)$, $\psi(t,t) = R(t)$, so that
$\psi(k,t+1) = R(t+1)\psi(k,t)$. From (10), we have the weights constraint

$$\psi^T(1,t)\omega(0) + \sum_{j=1}^{t} \psi^T(j,t)\widetilde{u}(j-1) = 1, \tag{11}$$

where $\psi^T(j,t) = 1^T\psi(j,t)$.

The multi-period optimization problem is formulated as follows:

$$\min_{\widetilde{u}_1(t),\ldots,\widetilde{u}_n(t)} \sum_{t=1}^{T} \lambda(t)(\xi(\omega(t),\alpha)+[(1-\alpha)n]^{-1}\sum_{i=1}^{n}\max\{f_i(\omega(t))-\xi(\omega(t),\alpha),0\})$$

subject to

$$\begin{cases} \omega(t)=\psi(1,t)\omega(0)+[\psi(1,t)\,\psi(2,t)\,\cdots\,\psi(t-1,t)\psi(t,t)] \\ \qquad\qquad [\widetilde{u}(0)\,\widetilde{u}(1)\,\cdots\,\widetilde{u}(t-2)\,\widetilde{u}(t-1)]^T; \\ \psi^T(1,t)\omega(0)+\sum_{j=1}^{t}\psi^T(j,t)\widetilde{u}(j-1)=1; \\ \lambda(t)\geq 0; \\ \sum_{i=1}^{n}E[r_i(t)]\omega_i(t)\geq 0, \end{cases} \tag{12}$$

where $r_i(t)$ are the returns or the standardized residuals filtered from GARCH model.

2.3 Optimization methods mathematical formalism

2.3.1 Differential evolution (DE)

Differential evolution (DE), introduced by Storn and Price (1997), is a stochastic, population-based evolutionary algorithm for solving nonlinear optimization problems. This algorithm uses biology-inspired operations of **initialization**, **mutation**, **recombination**, and **selection** on a population to minimize an objective function through successive generations (see Holland 1975). Similar to other evolutionary algorithms to solve optimization problems, DE uses alteration and selection operators to evolve a population of candidate solutions.

Let N denote the population size. To create the initial generation, the optimal guess for N is made either by using values input by the user or random values selected between lower and upper bounds (choosing by the user).

Consider the optimization problem (5) and let $\xi+[(1-\alpha)n]^{-1}\sum_{i=1}^{n}\max\{f_i(\omega)-\xi,0\}=h(\omega)$ where $\omega=\{\omega_1,\omega_2,\ldots,\omega_n\}$.

Given the population

$$\omega_{ki}^{g}=\{\omega_{k1}^{g},\omega_{k2}^{g},\ldots,\omega_{kn}^{g}, \tag{13}$$

where g is the generation and $k=1,2,\ldots,N$. The process is achieved through the following stages:

1) *Initial population*
   The initial population is randomly generated as

$$\omega_{ki}=\omega_{ki}^{L}+\text{rand}()(\omega_{ki}^{U}-\omega_{ki}^{L}), \tag{14}$$

   where ω_i^L and ω_i^U represent the lower and upper bounds of ω_i, respectively, and $i=1,2,\ldots,n$.

2) *Mutation*
The differential mutation is accomplished as follows: A random selection of three members of the population $\omega^g_{r_1k}$, $\omega^g_{r_2k}$, and $\omega^g_{r_3k}$ to create an initial mutant vector parameter $\mathbf{u}^{g+1}_k$, called donor vector, which is generated as

$$\mathbf{u}^{g+1}_k = \omega^g_{r_1k} + F(\omega^g_{r_2k} - \omega^g_{r_3k}), \tag{15}$$

where F is the scale vector and $k = 1, 2, \ldots, N$.
3) *Recombination*
Let ω^g_{ki} denote the target vector.
From the target vector and the donor vector, a trial vector $\mathbf{v}^{g+1}_{ki}$ is selected as follows:

$$\mathbf{v}^{g+1}_{ki} = \begin{cases} \mathbf{u}^{g+1}_{ki}, & \text{if rand()} \leq C_p \text{ or } i = I_{\text{rand}} \quad i = 1, 2, \ldots, n; \\ \omega^g_{ki}, & \text{if rand()} > C_p \text{ and } i \neq I_{\text{rand}} \quad k = 1, 2, \ldots, N, \end{cases} \tag{16}$$

where I_{rand} is a random integer in $[1, n]$ and C_p the recombination probability.
4) *Selection*
At this stage, the target vector is compared with the trial vector and the one with the smallest function value is the candidate for the next generation

$$\omega^{g+1}_{ki} = \begin{cases} \mathbf{v}^{g+1}_{ki}, & \text{if } h(\mathbf{v}^{g+1}_{ki}) < h(\omega^g_{ki}); \\ \omega^g_{ki}, & \text{otherwise,} \end{cases} \tag{17}$$

where $k = 1, 2, \ldots, N$.

2.3.2 GARCH-differential evolution (GARCH-DE)

Chu et al. (2017) fit twelve GARCH models to each of the seven most popular cryptocurrencies and realize that the IGARCH (Integrated GARCH) of Engle and Bollerslev (1986) and the GJR-GARCH of Glosten et al. (1993) provide the best fits, in terms of modeling the volatility in the largest and most popular cryptocurrencies. In this study, we use the GJR-GARCH (1,1).
The implementation of the GARCH-DE is as follows.

Let r_t be the log-return at time t.

a) Fit the mean model ARMA(1,1) and the variance model GJR-GARCH(1,1) of the log-returns as follows:
   i) The mean model

$$r_t = \mu + \theta_1(r_{t-1} - \mu) + \theta_2\varepsilon_{t-1} + \varepsilon_t, \tag{18}$$

where $\varepsilon_t = \sigma_t h_t$, $h_t \sim N(0, 1)$.

ii) The variance model

$$\sigma_t^2 = \omega + \alpha_1 h_{t-1}^2 + \gamma_1 I_{t-1} h_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \tag{19}$$

Alberg et al. (2011) show that the GARCH models with fat-tail distributions are relatively better suited for analyzing returns on stocks. So, student t-distribution has been chosen for our analysis.

b) From the predicted log-returns $\widehat{r_t} = \mu + \theta_1(r_{t-1} - \mu)$, obtain the residuals

$$D_t = r_t - \widehat{r}_t. \tag{20}$$

c) Solve the following optimization problem using DE

$$\begin{aligned} &\min_{\omega} \xi + [(1-\alpha)n]^{-1} \sum_{i=1}^{n} \max\{f_i(\omega) - \xi, 0\} \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \sum_{i=1}^{n} E[d_i]\omega_i \geq 0 \\ \omega_i \geq 0, \quad i = 1, \ldots, n, \end{cases} \end{aligned} \tag{21}$$

where $\alpha = 0.1$, $f_i(\omega) = f(\omega, d_i)$ with d_i as the standardized residuals.

2.3.3 GARCH-differential evolution *t*-copula (GARCH-DE-*t*-copula)

Modeling statistical dependence using linear correlation is deeply embedded in financial risk management practice in such a way that many practitioners are not aware of any other alternatives. Its attractiveness is its simplicity and the fact that, in terms of dependence in an elliptical world, the correlations provide us with all information we need to know. But, it is just one measure of dependence among others. It does have some limitations. For example, it is not invariant to transformations of variables (e.g., the correlation between two random variables X and Y is not generally the same as the one between $\ln(X)$ and $\ln(Y)$). Moreover, correlation is not defined in some circumstances, especially when the variance of one of the variables is not finite or when the two variables are not cointegrated. In other circumstances, especially in the tails, it tells us very little about the dependence. So, linear correlation will often appear to be limited or even of no use in non-elliptical situations. Copulas provide a better way to model dependence.

Statistically, given a collection of marginal distributions, copula is a function that joins these marginals to form the multivariate distribution that captures how they all move together. Conversely, copula is able to take a multivariate distribution and separate its dependence structure from the marginal distribution functions.

t-copula theory The t-copula (see, e.g., Embrechts et al. 2002; Fang and Fang 2002) is the copula type of the multivariate t-distribution used in representing the dependence structure. Much attention has been given to t-copula, especially in modeling financial time series and it has been shown that its empirical fit is generally superior to that of Gaussian copula, the dependence structure of the multivariate normal distribution (see, e.g., Mashal and Zeevi 2002; Breymann et al. 2003). This is due to the ability of the t-copula to better capture the dependency of fat tails displayed by financial data.

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be a n-dimensional random vector. The vector $\mathbf{Y}$ is said to follow a multivariate t distribution (non-singular), denoted $\mathbf{Y} \sim t_n(d, \mu, \Sigma)$, if its density has the form

$$f(\mathbf{y}) = \frac{\Gamma(\frac{d+n}{2})}{\Gamma(\frac{d}{2}\sqrt{(\pi d)^n|\Sigma|})} \left(1 + \frac{(\mathbf{y}-\mu)'\Sigma^{-1}(\mathbf{y}-\mu)}{d}\right)^{-\frac{d+n}{2}}, \tag{22}$$

where d is the degrees of freedom, μ is the mean vector and Σ is a positive-definite dispersion matrix. It is to note here that in these parameterizations $\text{cov}(\mathbf{Y}) = \frac{d}{d-2}\Sigma$.

Given that marginal distributions of financial return data are generally not normally distributed, one can use the Sklar's Theorem (Sklar 1959) to associate these distributions with a copula. Recent developments emerged several types of copulas from two families: Elliptical and Archimedean copula. In our study, we use the functional forms of Student t-copula (i.e., t-copula) which derives from multivariate elliptical distributions. It is to be noted that the copula is invariant under a standardization of the marginal distributions. As such, the copula of a t-distribution $t_n(d, \mu, \Sigma)$ is the same as that of a $t_n(d, 0, P)$ distribution, where P is the correlation matrix implied by the scatter matrix Σ. The unique t-copula is thus given by

$$C^t_{d,P}(\mathbf{u}) = \int_{-\infty}^{t_d^{-1}(u_1)} \cdots \int_{-\infty}^{t_d^{-1}(u_n)} \frac{\Gamma\left(\frac{d+n}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\sqrt{(\pi d)^n|P|}} \left(1 + \frac{\mathbf{x}'P^{-1}\mathbf{x}}{d}\right)^{\frac{d+n}{2}} d\mathbf{x}, \tag{23}$$

where t_d^{-1} denotes the quantile function of a standard univariate t_d distribution. In the bivariate case, the notation $C^t_{d,\rho}$ is used, where ρ is the off-diagonal entry of P. In contrast, the unique copula of a multivariate Gaussian distribution may be thought as the limit of t-copula as d tends to infinity. It is denoted by C_P^{Ga} (see, Embrechts et al. 2002).

To simulate the t-copula, a multivariate t-distributed random vector $\mathbf{X} \sim t_n(d, 0, P)$ is generated using the representation $X \stackrel{n}{=} \mu + \sqrt{W}\mathbf{Z}$, (where $\mathbf{Z} \sim N_n(0, \Sigma)$ and W is independent of $\mathbf{Z}$ and has an inverse gamma distribution $W \sim I_g(\frac{d}{2}, \frac{d}{2})$), and then return a vector $U = (t_d(X_1), \ldots, t_d(X_n))'$, where t_d denotes the distribution function of a standard univariate t. The density of the t-copula can be obtained from (4) and has the following form

$$c^t_{d,P}(\mathbf{u}) = \frac{f_{d,P}(t_d^{-1}(u_1), \ldots, t_d^{-1}(u_n))}{\prod_{i=1}^n f_d(t_d^{-1}(u_i))}, \qquad u \in (0, 1)^n, \tag{24}$$

where $f_{d,P}$ is the joint density function of a $t_n(d, 0, P)$-distributed random vector and f_d represents the density function of the univariate standard t-distribution with d degrees of freedom.

The GARCH-differential evolution t-copula (GARCH-DE-t-copula) method is implemented as follows:

1) Obtain the standardized residuals d_i from GJR-GARCH.
2) Simulate using t-copula sample data c_i from the standardized residuals d_i.
3) Solve the following optimization problem using DE

$$\begin{aligned}
&\min_{\omega} \xi + [(1-\alpha)n]^{-1} \sum_{i=1}^{n} \max\{f_i(\omega) - \xi, 0\} \\
&\text{subject to} \\
&\begin{cases} \sum_{i=1}^{n} \omega_i = 1 \\ \sum_{i=1}^{n} E[c_i]\omega_i \geq 0 \\ \omega_i \geq 0, \quad i = 1, \ldots, n. \end{cases}
\end{aligned} \tag{25}$$

where $\alpha = 0.1$ and $f_i(\omega) = f(\omega, c_i)$.

3 Results and analysis

3.1 Data and preliminary analysis

The data consist of the daily returns (100 times the difference in logarithms of Crypt/USD exchange rates) of 5 cryptocurrencies representing at the time of this writing close to 50% market share of the cryptocurrency market and trading as early as 2014 with at least $300 millions market capitalization. The data spanning the period March 01, 2014–February 28, 2018 comprises the following cryptocurrency assets: Bitcoin (BTC) , Ripple (XRP) , Litecoin (LTC), Dash (DASH), and Dogecoin (DOGE). These assets exhibit evidence of high volatility clustering (see Fig. 1) and the assumptions of serial correlation, non-normality, and arch effect could not be rejected across the return series (Table 1). The skewness and kurtosis test results in Table 1 and values in Table 2 point to the leptokurtic skewed type of distribution for these returns, suggesting that large fluctuations are more likely on the fat tails. Dash appears to be the riskiest among the 5 cryptocurrencies and as expected offers the highest return. Though Bitcoin is the least risky, it offers higher return than Litecoin and Dogecoin (see Fig. 2). More interestingly, they all display a significantly positive correlation (see Table 3).

3.2 Empirical findings

To account for the observed characteristics of the return series in modeling their true dependence, we consider a multivariate t-distribution.

3.2.1 Dependence estimation

The dependence properties are of great importance for portfolio selection and/or risk evaluation. Though the Pearson's correlation coefficient[1] could provide a substantial

[1] Pearson's correlation coefficients provide the degree of linear relationship between two variables.

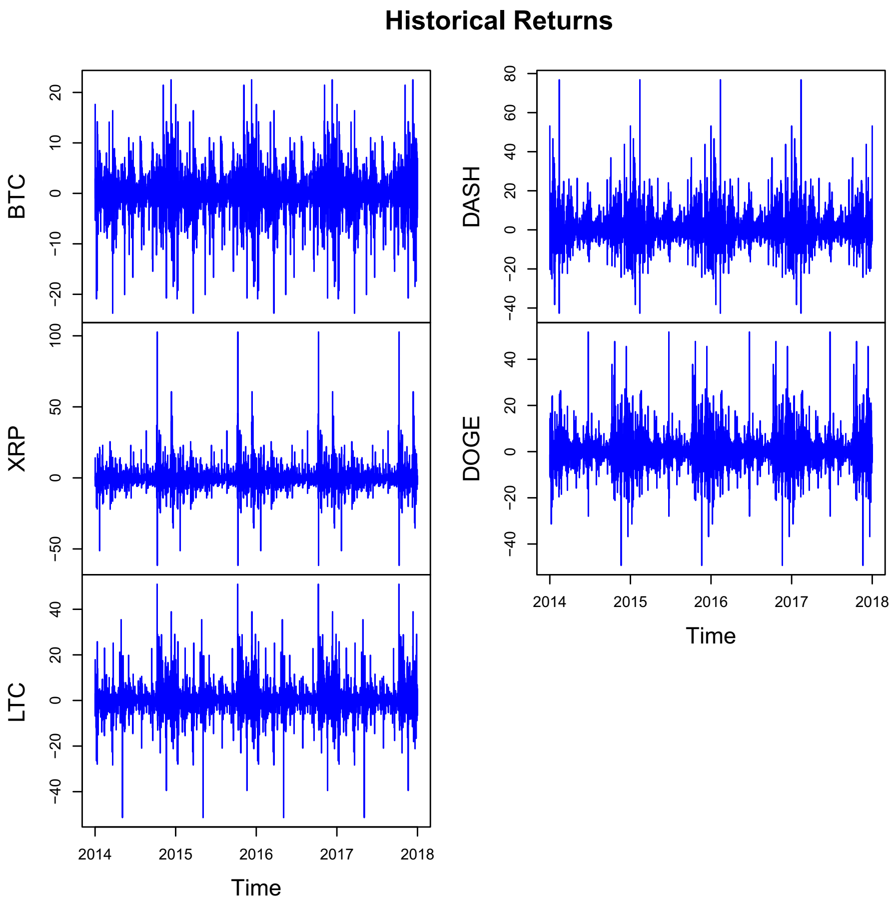

Fig. 1 Historical returns

Table 1 Multivariate diagnostic tests

	Skewness test	Kurtosis test	Normality JB-test	Serial correlation Portmanteau test	Arch effect Arch test
χ^2	2410.5***	103,850***	106,260***	303.89***	4241.3***
p value	$2.2e^{-16}$	$2.2e^{-16}$	$2.2e^{-16}$	$2.969e^{-06}$	$2.2e^{-16}$

*** statistically significant at the 0.01 level

statistical power even for distributions with moderate skewness or excess kurtosis, its sensitivity to extreme values makes it a less powerful statistical test for distributions with extreme skewness or excess kurtosis (where the data with extreme values are more likely). Thus, we choose Student t-copula with extreme value distribution in assessing the dependence structure of the sample returns. We estimate the three main

Table 2 Fat tails parameters

	α	κ	ω	δ	ς	ξ
BTC	2.909	2.986	3.068	− 0.009	− 0.388	6.339
XRP	2.221	2.056	1.914	0.036	2.452	37.633
LTC	2.446	2.358	2.276	0.015	0.645	13.875
DASH	3.072	2.742	2.476	0.039	1.454	12.755
DOGE	2.801	2.646	2.507	0.021	0.749	10.214

ς is the skewness parameter, ξ is the kurtosis parameter, α is the left tail parameter, ω is the right tail parameter, κ is the harmonic mean of α and ω and describes a global tail parameter and δ is the distortion parameter between the right tail parameter ω and the left tail parameter α, satisfying the inequality $-\kappa < d < \kappa$. A negative value $\delta < 0$ (resp. positive value $\delta > 0$) implies $\alpha < \omega$ (resp. $\alpha > \omega$) and indicates that the left tail (resp. the right tail) is heavier than the right tail (resp. the left tail)

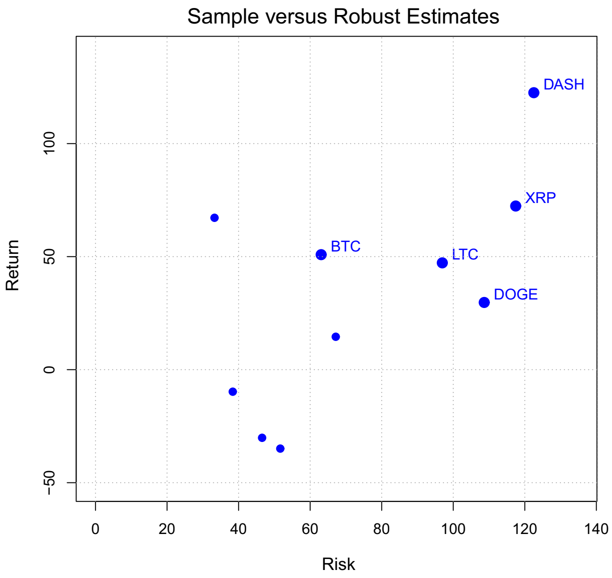

Fig. 2 Risk vs. return

measures of dependence, Pearson, Spearman, and Kendall.[2] The obtained coefficients are displayed in Table 3.

Following Table 3,[3] except for Dash, which exhibits a positive weak relationship with Bitcoin and Ripple according to Pearson's measure, there is a moderate posi-

[2] Kendall is a nonparametric test that measures the strength of dependence between two variables. It is given by: $\tau = \frac{n_c - n_d}{\frac{1}{2}n(n-1)}$, where n_c is the number of concordant (ordered in the same way) pairs and n_d is the number of discordant (ordered differently) pairs.

[3] Although there are no absolute standards, many analysts view coefficients, in absolute values, of less than 0.25 as describing weak relationships, coefficients between 0.25 and 0.50 as moderate relationships, and those greater than 0.50 as strong relationships.

Table 3 Correlation coefficients

	Pearson	Spearman	Kendall
BTC vs. XRP	0.42	0.41	0.27
BTC vs. LTC	0.43	0.42	0.28
BTC vs. DASH	0.17	0.42	0.29
BTC vs. DOGE	0.41	0.43	0.29
XRP vs. LTC	0.34	0.40	0.26
XRP vs. DASH	0.17	0.42	0.28
XRP vs. DOGE	0.50	0.41	0.25
LTC vs. DASH	0.34	0.42	0.27
LTC vs. DOGE	0.41	0.43	0.31
DASH vs. DOGE	0.31	0.42	0.28

tive relationship across the studied cryptocurrencies. This suggests that the prices of these cryptocurrencies move in the same direction. The dependence structure between Dogecoin and Ripple seems to be the strongest across the three measures.

3.3 Portfolio optimization

Let's recall here that in both our methods, the heuristic search algorithm, differential evolution (DE) is used. The efficiency and reliability of different heuristic optimization techniques in portfolio choice problems have been explored in Maringer and Oyewumi (2007) and Maringer and Meyer (2008). In the competition of simulated annealing, threshold accepting, and stochastic differential equations, they find DE to be well suited for non-convex portfolio optimization. It has shown efficient convergence to one (presumably global) optimum. Therefore, we incorporate DE into our portfolio optimization problems.

3.3.1 Single-period optimization

We first implement the single-period optimization with GARCH-DE and GARCH-DE-t-copula. The weights obtained for the two methods are displayed in Table 4. We observe that GARCH-DE-t-copula seems to allow a well-diversified portfolio through a well-controlled risk unlike GARCH-DE. This is not surprising, given the ability of the generalized pareto distribution to model extreme events (in the heavy tails) and the t-copula which can better capture the complex dependence structure between the analyzed variables. Moreover, the GARCH-DE-t-copula outperforms the GARCH-DE in terms of returns. It is important to note here that, while GARCH-DE allocates the largest weight to DASH, it appears in GARCH-DE-t-copula with the smallest weight.

Table 4 Optimal weights and target return for single-period optimization of DE, GARCH-DE, and GARCH-DE-t-copula models

	DE	GARCH-DE	GARCH-DE-t-copula
Optimal weights			
Bitcoin	0.494	0.000	0.150
Ripple	0.014	0.350	0.210
Litecoin	0.480	0.000	0.224
Dash	0.008	0.638	0.090
Dogecoin	0.004	0.012	0.326
Target return			
Return	0.418	0.5949	1.004
Risk measure			
CVaR	1	1	0.4488

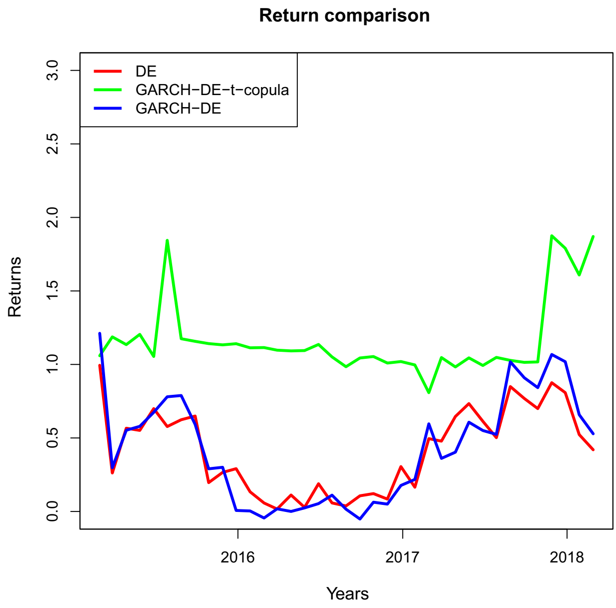

Fig. 3 Return comparison of the three methods

3.3.2 Multi-period optimization

In our multi-period optimization, the 12 first months are used as training period after which the portfolio is re-balanced monthly. There are in total 37 rebalancing periods, starting on February 28, 2015, and ending February 27, 2018.

The multi-period optimization using GARCH-DE-t-copula outperforms both the one using DE and GARCH-DE as illustrated in Fig. 3 and Table 5. Across the entire investment period, the returns given by GARCH-DE-t-copula are higher than that of

Table 5 Weights, CVaR, and returns from multi-period optimization

Periods	BTC			XRP			LTC			DASH			DOGE			CVaR			Returns		
	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE
2015/02/28	0.036	0.030	0.288	0.316	0.032	0.000	0.494	0.000	0.010	0.148	0.000	0.682	0.006	0.000	0.020	0.0125	1	1	1.059	1.212	0.994
2015/03/31	0.002	0.426	0.302	0.242	0.028	0.008	0.084	0.000	0.130	0.388	0.536	0.480	0.284	0.010	0.080	0.126	1	1	1.188	0.297	0.262
2015/04/30	0.096	0.330	0.880	0.416	0.000	0.000	0.088	0.000	0.082	0.400	0.670	0.038	0.000	0.000	0.000	0.063	1	1	1.135	0.553	0.566
2015/05/31	0.026	0.066	0.006	0.212	0.000	0.004	0.080	0.044	0.030	0.680	0.862	0.928	0.002	0.028	0.032	0.016	1	1	1.205	0.579	0.551
2015/06/30	0.06	0.186	0.092	0.290	0.000	0.000	0.142	0.054	0.060	0.322	0.742	0.848	0.186	0.018	0.000	0.041	1	1	1.054	0.674	0.700
2015/07/31	0.000	0.028	0.200	0.016	0.004	0.030	0.030	0.000	0.000	0.936	0.950	0.760	0.018	0.018	0.010	1	1	1	1.844	0.780	0.578
2015/08/31	0.038	0.040	0.506	0.134	0.004	0.000	0.394	0.022	0.016	0.010	0.934	0.478	0.424	0.000	0.000	0.339	1	1	1.175	0.789	0.624
2015/09/30	0.000	0.354	0.208	0.056	0.002	0.004	0.302	0.014	0.012	0.000	0.628	0.772	0.642	0.002	0.004	0.315	1	1	1.157	0.592	0.649
2015/10/31	0.038	0.068	0.854	0.140	0.000	0.000	0.400	0.034	0.008	0.002	0.898	0.124	0.420	0.000	0.014	0.294	1	1	1.142	0.290	0.196
2015/11/30	0.016	0.060	0.960	0.210	0.000	0.000	0.366	0.062	0.000	0.002	0.866	0.028	0.406	0.012	0.012	0.310	1	1	1.133	0.301	0.266
2015/12/31	0.016	0.342	0.954	0.332	0.000	0.000	0.290	0.308	0.034	0.006	0.312	0.012	0.356	0.038	0.000	0.287	1	1	1.141	0.007	0.291
2016/01/31	0.024	0.882	0.884	0.402	0.000	0.002	0.188	0.112	0.100	0.008	0.006	0.012	0.378	0.000	0.002	0.280	1	1	1.113	0.004	0.134
2016/02/29	0.000	0.888	0.752	0.240	0.018	0.008	0.308	0.024	0.150	0.000	0.068	0.090	0.452	0.002	0.000	0.231	1	1	1.115	-0.044	0.058
2016/03/31	0.000	0.912	0.702	0.110	0.000	0.008	0.316	0.034	0.000	0.000	0.054	0.248	0.574	0.000	0.042	0.209	1	1	1.097	0.018	0.017
2016/04/30	0.024	0.886	0.832	0.004	0.014	0.016	0.438	0.002	0.148	0.000	0.098	0.000	0.534	0.000	0.004	0.223	1	1	1.092	0.0003	0.112
2016/05/31	0.012	0.686	0.714	0.194	0.012	0.010	0.320	0.242	0.146	0.014	0.058	0.000	0.460	0.002	0.130	0.173	1	1	1.094	0.024	0.028
2016/06/30	0.004	0.264	0.904	0.116	0.000	0.014	0.384	0.388	0.034	0.088	0.346	0.042	0.408	0.002	0.006	0.275	1	1	1.136	0.053	0.189
2016/07/31	0.000	0.220	0.850	0.096	0.000	0.000	0.312	0.002	0.000	0.000	0.774	0.078	0.592	0.004	0.072	0.186	1	1	1.051	0.111	0.058

Table 5 continued

Periods	BTC			XRP			LTC			DASH			DOGE			CVaR			Returns		
	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE	GDE-t	GDE	DE
2016/08/31	0.096	0.012	0.884	0.098	0.002	0.000	0.388	0.098	0.036	0.006	0.886	0.010	0.412	0.002	0.070	0.185	1	1	0.985	0.016	0.038
2016/09/30	0.030	0.844	0.966	0.164	0.006	0.006	0.430	0.138	0.010	0.000	0.000	0.002	0.376	0.012	0.016	0.202	1	1	1.044	-0.052	0.106
2016/10/31	0.008	0.766	0.702	0.194	0.006	0.000	0.434	0.194	0.086	0.000	0.032	0.210	0.364	0.002	0.002	0.208	1	1	1.054	0.063	0.122
2016/11/30	0.092	0.168	0.780	0.266	0.014	0.016	0.296	0.816	0.024	0.070	0.000	0.082	0.276	0.002	0.098	0.216	1	1	1.010	0.050	0.084
2016/12/31	0.002	0.302	0.738	0.012	0.000	0.002	0.530	0.678	0.252	0.002	0.000	0.000	0.454	0.020	0.008	0.288	1	1	1.020	0.177	0.305
2017/01/31	0.000	0.806	0.854	0.244	0.018	0.074	0.302	0.168	0.000	0.022	0.008	0.000	0.432	0.000	0.072	0.220	1	1	0.997	0.218	0.165
2017/02/28	0.212	0.020	0.958	0.178	0.000	0.006	0.118	0.004	0.032	0.044	0.954	0.004	0.448	0.022	0.000	0.164	1	1	0.808	0.596	0.495
2017/03/31	0.002	0.640	0.904	0.018	0.000	0.006	0.436	0.156	0.004	0.046	0.198	0.070	0.498	0.006	0.012	0.410	1	1	1.047	0.361	0.478
2017/04/30	0.094	0.430	0.978	0.262	0.000	0.000	0.316	0.306	0.000	0.032	0.264	0.022	0.296	0.000	0.000	0.368	1	1	0.983	0.403	0.646
2017/05/31	0.006	0.702	0.932	0.060	0.020	0.000	0.534	0.248	0.016	0.000	0.030	0.050	0.400	0.000	0.002	0.400	1	1	1.045	0.607	0.734
2017/06/30	0.000	0.898	0.810	0.186	0.002	0.002	0.226	0.096	0.116	0.002	0.004	0.072	0.586	0.000	0.000	0.303	1	1	0.993	0.551	0.613
2017/07/31	0.018	0.274	0.864	0.208	0.016	0.002	0.414	0.702	0.066	0.008	0.008	0.068	0.352	0.000	0.000	0.324	1	1	1.048	0.524	0.502
2017/08/31	0.014	0.016	0.034	0.176	0.022	0.000	0.412	0.952	0.934	0.014	0.000	0.018	0.384	0.010	0.018	0.341	1	1	1.028	1.018	0.850
2017/09/30	0.010	0.176	0.198	0.202	0.002	0.000	0.252	0.790	0.764	0.002	0.032	0.034	0.534	0.000	0.004	0.304	1	1	1.014	0.910	0.769
2017/10/31	0.020	0.062	0.212	0.040	0.002	0.002	0.358	0.890	0.766	0.042	0.028	0.018	0.540	0.018	0.002	0.328	1	1	1.017	0.842	0.700
2017/11/30	0.000	0.124	0.216	0.002	0.000	0.000	0.018	0.856	0.720	0.976	0.020	0.000	0.004	0.000	0.064	1	1	1	1.875	1.0678	0.876
2017/12/31	0.002	0.000	0.000	0.032	0.000	0.052	0.124	0.936	0.934	0.842	0.064	0.004	0.000	0.000	0.010	1	1	1	1.790	1.019	0.808
2018/01/31	0.050	0.018	0.966	0.026	0.034	0.000	0.356	0.916	0.018	0.562	0.000	0.004	0.006	0.032	0.012	1	1	1	1.608	0.658	0.523
2018/02/27	0.000	0.000	0.406	0.026	0.030	0.014	0.000	0.942	0.572	0.974	0.006	0.004	0.000	0.022	0.004	1	1	1	1.870	0.528	0.420

DE and GARCH-DE. Let's also stretch out that, with the exception of 5 periods, the returns in the multi-period GARCH-DE-t-copula are all greater than in single period and nearly its double in some periods. So, in portfolio optimization, the power of the DE algorithm is potentially increased when combined with t-copula. This innovative approach will surely be welcome among portfolio managers.
Through our analysis, among the 5 studied cryptoassets, it appears that the weights assigned to Bitcoin, the leading cryptocurrency, although the lowest in our portfolio, remain relatively constant across all rebalancing periods. So, long-term investors should consider including Bitcoin in their portfolio. Weights allocation reveals that DASH and DOGE move in opposite direction and share the highest weights in our portfolio. It is therefore advisable not to have DASH and DOGE in the same portfolio, because whatever will be gained from one will quickly disappear in the loss incurred simultaneously by the other.

Previous studies have indicated the importance for portfolio managers to include cryptocurrencies in their portfolio. Differential evolution algorithm uses biology-inspired operations of initialization, mutation, recombination, and selection on a population to minimize an objective function through successive generations. Its association in this study to t-copula and GARCH to derive GARCH-DE-t-copula has shown remarkable results in multi-period settings in terms of portfolio risk control and returns compared to GARCH-DE and DE models. The comparison with other known portfolio optimization models will be carried out in our future study.

4 Conclusion

Cryptocurrencies are new types of assets on the financial market with trading volumes reaching billions of dollars a day and market capitalizations reaching hundreds of billions of dollars. Highly volatile, they present investors with great opportunity of high returns. Though cryptocurrencies are still in their infancy, recent years have seen some savvy individuals making significant amounts of money by speculating on cryptocurrencies. It is thus important to develop portfolio optimization methods to assist cryptocurrency investors in controlling their risk exposure while maximizing their returns. This study has confirmed the power of regular rebalancing of portfolio assets to adapt to market changes through GARCH-differential evolution t-copula method (GARCH-DE-t-copula). Due to the high volatility that characterizes cryptocurrencies, the modeling of the tail dependence through t-copula and extreme value distribution (GPD) has shown significant positive impact on the returns of the portfolio across all multi-period optimization periods and also in the control of risk.

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References

Alberg, D., Shalit, H., Yosef, R.: Estimating stock market volatility using asymmetric GARCH models. Appl. Financ. Econ. **18**(15), 1201–1208 (2011)

Alvarez-Ramirez, J., Rodriguez, E., Ibarra-Valdez, C.: Long-range correlations and asymmetry in the Bitcoin market. Phys. A. Stat. Mech. Appl. **492**, 948–955 (2018)
Ardia, D., Mullen, K.M., Peterson, B.G., Ulrich, J.: DEoptim: Differential Evolution Optimization in R. R package version 2.1-0 (2011). http://CRAN.R-project.org/package=DEoptim. Accessed 19 Dec 2016
Bekiros, D., Hernandez, J.A., Hammoudeh, S., Nguyen, D.K.: Multivariate dependence risk and portfolio optimization: an application to mining stock portfolios. Int. J. Miner. Policy Econ. **46**(2), 111 (2015)
Breymann, W., Dias, A., Embrechts, P.: Dependence structures for multivariate high-frequency data in finance. Quant. Finance **3**, 114 (2003)
Boudt, K., Carl, P., Peterson, B.G.: PortfolioAnalytics: Portfolio Analysis, Including Numeric Methods for Optimization Of Portfolios, R package version 0.8 (2012)
Bouri, E., Azzi, G., Dyhrberg, A.H.: On the return–volatility relationship in the Bitcoin market around the price crash of 2013, economics: the open-access. Open-Assess. E-J. **11**, 116 (2017)
Brunnermeier, M.K.: Deciphering the 2007–08 liquidity and credit crunch. J. Econ. Perspect. **23**(1), 77100 (2009)
Calafiore, G.: Ambiguous risk measures and optimal robust portfolios. SIAM J. Control Optimi. **18**(3), 853–877 (2007)
Chu, J., Chan, S., Nadarajah, S., Osterrieder, J.: GARCH modelling of cryptocurrencies. J. Risk Financ. Manag. **10**(17), 1–15 (2017)
Dyhrberg, A.H.: Bitcoin, gold and the dollar a GARCH volatility analysis. Finance Res. Lett. **16**, 85–92 (2016)
Dyhrberg, A.H.: Hedging capabilities of bitcoin Is it the virtual gold? Finance Res. Lett. **16**, 139–144 (2016)
Embrechts, P., McNeil, A., Straumann, D.: Correlation and dependence in risk management: properties and pitfalls. In: Dempster, M., Moffatt, H. (eds.) Risk management: value at risk and beyond, pp. 176–223. Cambridge University Press, Cambridge, UK (2002)
Engle, R.F., Bollerslev, T.: Modelling the persistence of conditional variances. Econom. Rev. **5**, 150 (1986)
Fang, H., Fang, K.: The metaelliptical distributions with given marginals. J. Multivar. Anal. **82**, 116 (2002)
Florackis, C., Kontonikas, A., Kostakis, A.: Stock market liquidity and macro-liquidity shocks: evidence from the 2007–2009 financial crisis. J. Int. Money Finance **44**, 97–117 (2014)
Gârleanu, N., Pedersen, L.: Dynamic trading with predictable returns and transaction costs. J. Finance **68**(6), 2309–2340 (2013)
Glosten, L.R., Jagannathan, R., Runkle, D.E.: Relationship between the expected value and the volatility of the nominal excess return on stocks. J. Finance **48**(5), 1779–1801 (1993)
Hakansson, N.: Multi-period mean–variance analysis: toward a general theory of portfolio choice. J. Finance **26**(4), 857–884 (1971)
Hagströmer, B., Binner, J.M.: Stock portfolio selection with fullscale optimization and differential evolution. Appl. Financ. Econ. **19**(19), 1559–1571 (2009)
Holland, J.H.: Adaptation in Natural Artificial Systems. University of Michigan Press, Ann Arbor (1975)
Krink, T., Paterlini, S.: Multiobjective optimization using differential evolution for real-world portfolio optimization. Comput. Manag. Sci. **8**(1), 157–179 (2011)
Krink, T., Mittnik, S., Paterlini, S.: Differential evolution and combinatorial search for constrained index-tracking. Ann. Oper. Res. **172**, 153–176 (2009)
Low, R.K.Y., Alcock, J., Faff, R., Brailsford, T.: Canonical vine copulas in the context of modern portfolio management: are they worth it? J. Bank. Finance **37**(8), 3085–3099 (2013)
Maringer, D.G., Oyewumi, O.: Index tracking with constrained portfolios. Intell. Syst. Account. Finance Manag. **15**(12), 57–71 (2007)
Maringer, D.G., Meyer, M.: Smooth transition autoregressive models: new approaches to the model selection problem. Stud. Nonlinear Dyn. Econom. **12**(1), 1–19 (2008)
Markowitz, H.: Portfolio selection. J. Finance **7**(1), 77–91 (1952)
Mashal, R., Zeevi, A.: Beyond correlation: extreme comovements between financial assets. Columbia Graduate School of Business (2002)
Moshirian, F.: The global financial crisis and the evolution of markets institutions and regulation. J. Bank. Finance **35**, 502–511 (2011)
Nystrup, P., Madsen, H., Lindstrm, E.: Dynamic Portfolio Optimization Across Hidden Market Regimes. Technical University of Denmark, Lyngby (2016)
Osterrieder, J., Lorenz, J.: A statistical risk assessment of Bitcoin and its extreme tail behavior. Ann. Financ. Econ. **12**(01), 17500031–175000319 (2017)
Phillip, A., Chan, J., Peiris, S.: A new look at cryptocurrencies. Econ. Lett. **163**, 69 (2018)

Price, K., Storn, R.M., Lampinen, J.A.: Differential Evolution: A Practical Approach to Global Optimization. Springer, Berlin (2006)
Rockafellar, R.T., Uryasev, S.: Optimization of conditional value-at-risk. J. Risk **2**, 493–517 (2000)
Rockafellar, R.T., Uryasev, S.: Conditional value-at-risk for general loss distributions. J. Bank. Finance **26**(7), 1443–1471 (2002)
Sahalia, V., Brandt, W.: Variable selection for portfolio choice. J. Finance **56**(4), 1297–1351 (2001)
Sklar, A.: Fonctions de répartition à n dimensions et leurs marges. Publications de lInstitut de Statistique de lUniversité de Paris **8**, 229–231 (1959)
Storn, R., Price, K.: Differential evolution a simple and efficient heuristic for global optimization over continuous spaces. J. Glob. Optim. **11**(4), 341–359 (1997)
Yollin, G.: R Tools for Portfolio Optimization. In: Presentation at R/Finance Conference 2009 (2009)

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