Review Report on Chapter 1: Geometry and Dynamics

Based on Daniel Baumann's Cosmology Lectures (DAMTP, Cambridge)

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Abstract

This report provides an expanded and reformulated discussion of Chapter 1 ("Geometry and Dynamics") from Daniel Baumann's *Cosmology* lecture notes. The content is restructured as a comprehensive review of the geometric foundations of modern cosmology. It includes extended derivations, interpretive commentary, and contextual insights connecting mathematical formalism with observational cosmology. Additionally, illustrative plots of cosmological quantities and light propagation are provided.

1 Introduction

At cosmic scales exceeding hundreds of megaparsecs, the universe displays statistical uniformity. This observation motivates the **Cosmological Principle**, which asserts that the universe is *homogeneous* and *isotropic* when averaged over sufficiently large volumes. These assumptions allow one to determine the general form of spacetime geometry, particle trajectories, and the equations governing cosmic expansion.

Einstein's theory of general relativity (GR) provides the dynamical link between spacetime curvature and energy content. In this review, we derive the metric structure consistent with the cosmological principle, explore the motion of particles and light (kinematics), and then connect geometry with dynamics through the Einstein field equations.

2 Geometry of the Universe

In this section, we describe the large-scale geometric structure of the Universe as implied by the Cosmological Principle. Assuming spatial homogeneity and isotropy, we introduce the general form of maximally symmetric three-dimensional spaces and characterize them by a constant curvature parameter. The discussion in this section is purely geometrical and kinematical; no assumptions are made about the dynamical evolution of the scale factor or the matter content of the Universe.

2.1 The Role of the Metric

The metric tensor $g_{\mu\nu}$ encapsulates the geometric properties of spacetime. It defines the invariant line element

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu, \tag{1}$$

where $X^\mu = (t, x^i)$. In flat (Minkowski) spacetime, $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, while in curved spacetime $g_{\mu\nu}(x^\alpha)$ depends on position and time, reflecting gravitational effects.

2.2 Symmetric Three-Spaces

The large-scale structure of the universe motivates the assumption that each spatial slice of spacetime (at fixed cosmic time t) is both **homogeneous** and **isotropic**. Mathematically, such a space is said to be *maximally symmetric*—it possesses the largest possible number of Killing vectors, namely six for three spatial dimensions (corresponding to three rotations and three translations).

These symmetries restrict the spatial curvature to be **constant everywhere**, leading to only three possible geometries: *flat*, *positively curved*, and *negatively curved* three-spaces. We now derive their metrics in detail.

1. Flat Space ($k = 0$)

The simplest case corresponds to Euclidean three-space, $\mathbb{E}^3$, with the line element

$$d\ell^2 = dx^2 + dy^2 + dz^2 = \delta_{ij}\, dx^i dx^j. \tag{2}$$

This geometry is invariant under spatial translations $x^i \to x^i + a^i$ and rotations $x^i \to R^i{}_j x^j$ where $R^i{}_j$ is an orthogonal matrix satisfying $R^T R = I$. The curvature scalar ${}^{(3)}R = 0$ for this space.

2. Positively Curved Space ($k = +1$)

A homogeneous and isotropic space with constant *positive curvature* can be visualized as a three-dimensional sphere (S^3) embedded in a four-dimensional Euclidean space with coordinates (x_1, x_2, x_3, u).

The embedding condition is

$$x_1^2 + x_2^2 + x_3^2 + u^2 = a^2, \tag{3}$$

where a is the radius of curvature. The induced line element on the hypersurface is

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + du^2, \tag{4}$$

subject to the constraint (3). Differentiating gives $u\, du = -x_i\, dx_i$, hence

$$d\ell^2 = dx_i dx_i + \frac{(x_i dx_i)^2}{a^2 - x_j x_j}. \tag{5}$$

This expression shows explicitly that the metric is invariant under 4-dimensional rotations preserving the constraint surface (S^3).

Switching to spherical coordinates,

$$x_1 = a \sin\chi \sin\theta \cos\phi, \tag{6}$$
$$x_2 = a \sin\chi \sin\theta \sin\phi, \tag{7}$$
$$x_3 = a \sin\chi \cos\theta, \quad u = a\cos\chi, \tag{8}$$

the induced line element becomes

$$d\ell^2 = a^2 \left[d\chi^2 + \sin^2\chi \,(d\theta^2 + \sin^2\theta\, d\phi^2) \right], \tag{9}$$

which makes the isotropy of S^3 manifest. The quantity a here determines the curvature radius, and the spatial curvature scalar is ${}^{(3)}R = +6/a^2$.

3. Negatively Curved Space ($k = -1$)

For constant *negative curvature*, the 3-space can be represented as a hyperboloid (H^3) embedded in 4-dimensional Minkowski space with coordinates (x_1, x_2, x_3, u), where the embedding relation is

$$x_1^2 + x_2^2 + x_3^2 - u^2 = -a^2. \tag{10}$$

The induced metric now reads

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 - du^2, \tag{11}$$

and differentiating the constraint gives $u\,du = x_i\,dx_i$. Substituting this into the metric yields

$$d\ell^2 = dx_i dx_i - \frac{(x_i dx_i)^2}{a^2 + x_j x_j}. \tag{12}$$

Introducing coordinates analogous to the spherical case,

$$x_1 = a \sinh\chi \sin\theta \cos\phi, \tag{13}$$
$$x_2 = a \sinh\chi \sin\theta \sin\phi, \tag{14}$$
$$x_3 = a \sinh\chi \cos\theta, \quad u = a\cosh\chi, \tag{15}$$

we obtain the hyperbolic line element

$$d\ell^2 = a^2 \left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right]. \tag{16}$$

The corresponding curvature scalar is ${}^{(3)}R = -6/a^2$, confirming that the space is uniformly negatively curved.

4. Unified Form and Coordinate Relation

The three geometries can be expressed compactly using the function

$$S_k(\chi) = \begin{cases} \sin\chi, & k = +1, \\ \chi, & k = 0, \\ \sinh\chi, & k = -1. \end{cases}$$

Then, the line element for a maximally symmetric 3-space can be written as

$$d\ell^2 = a^2(t) \left[d\chi^2 + S_k^2(\chi)\,d\Omega^2\right], \tag{17}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$ is the metric on a unit 2-sphere.

The coordinate χ is related to the standard radial coordinate r via

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}},$$

which converts Eq. (17) into the alternative but equivalent form

$$d\ell^2 = a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]. \tag{18}$$

5. Curvature Properties and Rescaling

The 3-dimensional Ricci tensor and scalar corresponding to Eq. (18) satisfy

$$^{(3)}R_{ij} = 2k\,\gamma_{ij}, \qquad ^{(3)}R = 6k,$$

where γ_{ij} denotes the unit curvature metric (the expression within brackets in Eq. (18)).

The metric admits a rescaling symmetry:

$$a \to \lambda a, \qquad r \to r/\lambda, \qquad k \to \lambda^2 k,$$

so that only the combination k/a^2 has physical meaning. It is conventional to normalize the present-day scale factor as $a(t_0) = 1$.

6. Physical Interpretation

The three possible curvatures correspond to geometrically distinct universes:

- **Closed Universe** ($k = +1$): Finite in spatial volume with no boundary; geodesics eventually reconverge, analogous to the 2D surface of a sphere. Parallel lines meet, and the total volume is $V = 2\pi^2 a^3$.
- **Flat Universe** ($k = 0$): Infinite and Euclidean, with parallel lines remaining parallel. This geometry is strongly supported by observations of the cosmic microwave background (CMB).
- **Open Universe** ($k = -1$): Infinite in spatial extent with hyperbolic geometry; parallel lines diverge, and the volume grows exponentially with radius.

Each of these spatial geometries can serve as a time-slice of a four-dimensional spacetime. By allowing the curvature radius a to evolve with cosmic time, we obtain the time-dependent scale factor $a(t)$ that forms the basis of the full Robertson–Walker metric discussed next.

2.3 Spatially Homogeneous and Isotropic Three-Spaces

Any three-dimensional space that is maximally symmetric must have constant curvature k. There are three possibilities:

$$k = \begin{cases} +1, & \text{closed (spherical)} \\ 0, & \text{flat (Euclidean)} \\ -1, & \text{open (hyperbolic).} \end{cases}$$

The general spatial line element can be written as

$$d\ell^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{19}$$

where $a(t)$ is the **scale factor**. Using $d\chi = dr/\sqrt{1-kr^2}$ gives

$$d\ell^2 = a^2(t)\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right], \tag{20}$$

with

$$S_k(\chi) = \begin{cases} \sin\chi, & k = +1, \\ \chi, & k = 0, \\ \sinh\chi, & k = -1. \end{cases}$$

2.4 Scale Factor Evolution (Illustrative Plot)

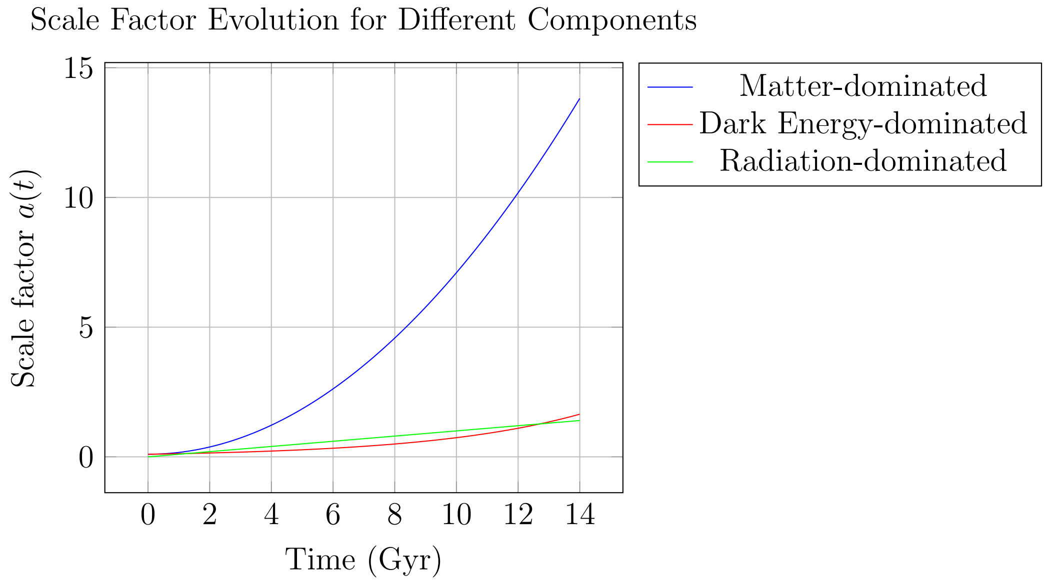

Figure 1: Illustrative evolution of the scale factor $a(t)$ in different cosmological eras.

3 Dynamics of the FRW Universe

The purpose of this section is to develop the theoretical framework that governs the time evolution of the Universe. Starting from Einstein's field equations and the relativistic description of cosmic matter as a perfect fluid, we derive the conservation laws and the Friedmann equations that determine the dynamics of the scale factor. These results are then used to analyze the behavior of different cosmological components and to establish the foundations of the standard Λ*CDM model.*

3.1 Foundations of Cosmological Dynamics

3.1.1 Einstein equation and its role

The dynamics of spacetime in General Relativity is governed by the Einstein equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{21}$$

where $G_{\mu\nu}$ is the Einstein tensor (a local measure of spacetime curvature constructed from the metric and its first and second derivatives), $T_{\mu\nu}$ is the stress–energy tensor describing the matter content, and G is Newton's gravitational constant. Equation (21) states that matter/energy determines curvature, and curvature tells matter how to move.

For a spatially homogeneous and isotropic background (FRW metric), the high degree of symmetry constrains the possible forms of $T_{\mu\nu}$ strongly. We now describe the allowed macroscopic matter content.

3.1.2 Number current and its conservation

A useful simple object is the number current (four-vector) N^μ. For a comoving ensemble of particles ("particles" may mean galaxies, dark matter particles, etc.) the comoving observer measures a proper number density $n(t)$. Isotropy forces the spatial current to vanish in the comoving frame, so

$$N^\mu_{(\text{comov.})} = \big(n(t),\, 0,\, 0,\, 0\big).$$

A general observer moving with four-velocity U^μ measures the boosted current

$$N^\mu = n\, U^\mu, \tag{22}$$

since N^μ must be a vector proportional to the only available vector U^μ and its time component in the comoving frame is n. In particular, for $U^\mu = \gamma(1, v^i)$ the time component $N^0 = \gamma n$ reproduces Lorentz contraction of number density.

Number conservation is expressed covariantly as

$$\nabla_\mu N^\mu = 0, \tag{23}$$

where ∇_μ is the covariant derivative compatible with the metric. In local inertial coordinates this reduces to $\partial_\mu N^\mu = 0$, the usual continuity equation.

Explicit evaluation in an FRW background. In the comoving frame $N^\mu = (n(t), 0, 0, 0)$. Using the covariant divergence in a curved spacetime,

$$\nabla_\mu N^\mu = \frac{1}{\sqrt{-g}}\, \partial_\mu\big(\sqrt{-g}\, N^\mu\big).$$

For the FRW metric $\sqrt{-g} = a^3(t)\sqrt{\gamma}$ where γ is the determinant of the spatial metric γ_{ij} (independent of t). Thus

$$\nabla_\mu N^\mu = \frac{1}{a^3\sqrt{\gamma}}\, \partial_t\big(a^3\sqrt{\gamma}\, n(t)\big) = \frac{1}{a^3}\frac{d}{dt}\big(a^3 n\big).$$

Setting $\nabla_\mu N^\mu = 0$ gives

$$\frac{d}{dt}\big(a^3 n\big) = 0 \quad \Longrightarrow \quad n(t) \propto a^{-3}(t).$$

This is the intuitive result that the proper number density decays as the inverse of the proper volume $\propto a^3$.

3.1.3 Why the stress–energy must be a perfect fluid

Isotropy (rotation invariance about any spatial point) and homogeneity (spatial translation invariance) imply that the coarse-grained stress–energy tensor cannot single out any spatial direction or location. The most general rank-2 tensor that respects these symmetries and is constructed from the metric and a four-velocity U^μ is the perfect fluid form

$$T_{\mu\nu} = (\rho + P)\, U_\mu U_\nu - P\, g_{\mu\nu}, \tag{24}$$

where $\rho(t)$ is the energy density measured in the fluid rest frame and $P(t)$ is the isotropic pressure. In the comoving frame $U^\mu = (1, 0, 0, 0)$ and the components reduce to

$$T^0{}_0 = \rho, \qquad T^i{}_j = -P\, \delta^i{}_j.$$

Any anisotropic stress or heat flux would violate isotropy and therefore cannot appear in the coarse-grained $T_{\mu\nu}$ of the homogeneous FRW background.

3.1.4 Covariant derivative: definition and index rules

The covariant derivative ∇_μ is the unique derivative operator compatible with the metric ($\nabla_\mu g_{\alpha\beta} = 0$) and torsion free (Levi–Civita connection). Its action on tensors is:

- On scalars f: $\nabla_\mu f = \partial_\mu f$.
- On a contravariant vector V^ν:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda,$$

  where $\Gamma^\nu_{\mu\lambda}$ are the Christoffel symbols.
- On a covariant vector ω_ν:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda.$$

- In general, for each upper index add $+\Gamma$-term; for each lower index add a $-\Gamma$-term.

These rules ensure that ∇_μ transforms covariantly under coordinate changes.

3.1.5 Energy–momentum conservation and the fluid continuity equation

Local energy–momentum conservation follows from the contracted Bianchi identity together with Einstein's equation:

$$\nabla_\mu T^{\mu\nu} = 0.$$

For a perfect fluid (24) this equation yields both the energy continuity equation and the Euler (momentum) equation. Projecting along U_ν and orthogonal to U_ν separates these pieces.

Energy (time) component. Contract with U_ν:

$$U_\nu \nabla_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad U_\nu \nabla_\mu \big[(\rho + P) U^\mu U^\nu - P g^{\mu\nu}\big] = 0.$$

Using $U_\nu U^\nu = 1$ and $\nabla_\mu g^{\mu\nu} = 0$ one obtains after straightforward algebra

$$U^\mu \nabla_\mu \rho + (\rho + P) \nabla_\mu U^\mu = 0.$$

In the homogeneous FRW background $U^\mu = (1, 0, 0, 0)$ and $\nabla_\mu U^\mu = 3\dot{a}/a$. Thus

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0.$$

This is the covariant generalisation of energy conservation for a homogeneous isotropic perfect fluid.

Momentum (spatial) components. The projection orthogonal to U^μ yields the fluid Euler equation

$$(\rho + P) U^\mu \nabla_\mu U^\alpha + (g^{\alpha\mu} - U^\alpha U^\mu) \nabla_\mu P = 0,$$

which for a homogeneous background (no spatial gradients of P) reduces to $U^\mu \nabla_\mu U^\alpha = 0$ (geodesic motion of the fluid elements) or trivial identities for the comoving frame.

3.1.6 Interpretation: scaling laws and equations of state

Equation (3.1.5) relates how the energy density evolves with the expansion given the pressure. For a simple barotropic equation of state $P = w\rho$ with constant w,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0 \quad \Rightarrow \quad \rho(a) \propto a^{-3(1+w)}.$$

Important special cases:

- Dust (pressureless matter): $w = 0 \Rightarrow \rho \propto a^{-3}$. This matches the number density scaling $n \propto a^{-3}$ when mass per particle is conserved.
- Radiation (relativistic): $w = \frac{1}{3} \Rightarrow \rho \propto a^{-4}$. The extra factor of a^{-1} compared with matter encodes the redshifting of photon energy.
- Cosmological constant (vacuum energy): $w = -1 \Rightarrow \rho = \text{const}$.

Concluding statement. In an FRW background, symmetry forces matter to behave as a perfect fluid; number conservation gives $n \propto a^{-3}$ and energy conservation gives $\dot{\rho} + 3H(\rho + P) = 0$.

Remarks. The equations above are the basis for deriving the Friedmann equations (which give $\dot{a}(t)$ given ρ and P) once the Einstein tensor $G_{\mu\nu}$ is evaluated on the FRW metric; the combination of the Friedmann equations and the fluid continuity equation fully determines the expansion history $a(t)$ for given matter components and their equations of state.

3.2 Robertson–Walker Metric and Cosmological Dynamics

The Robertson–Walker (RW) metric describing a homogeneous and isotropic universe is

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right]. \tag{25}$$

This metric encodes the **geometry** of the universe. To understand its **dynamics**, we study how the scale factor $a(t)$ evolves with time under Einstein's field equations.

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Step 1. Einstein's Field Equations

Einstein's equations are

$$G_{\mu\nu} = 8\pi G \, T_{\mu\nu}, \tag{26}$$

where

- $G_{\mu\nu}$: the Einstein tensor, representing the curvature (geometry) of spacetime,
- $T_{\mu\nu}$: the energy–momentum tensor, representing the matter and energy content.

By homogeneity and isotropy, the matter content must be a **perfect fluid**:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P \, g_{\mu\nu}, \tag{27}$$

where

- $\rho(t)$: energy density,
- $P(t)$: pressure,
- $U^\mu = (1, 0, 0, 0)$: four–velocity of comoving observers.

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Step 2. Einstein Tensor and Friedmann Equations

Substituting the RW metric into Einstein's equations, we obtain the simplified components of the Einstein tensor:

1. **First Friedmann Equation**

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{28}$$

2. **Second Friedmann (Acceleration) Equation**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}. \tag{29}$$

Here,

- ρ: total energy density (matter + radiation + dark energy),
- P: pressure,
- k: spatial curvature parameter $(+1, 0, -1)$,
- Λ: cosmological constant (dark energy contribution).

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Step 3. Conservation Equation

Energy–momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, yields

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{30}$$

where $H = \dot{a}/a$ is the **Hubble parameter**.

This expresses how the density of each component evolves with expansion:

- Matter $(P = 0)$: $\rho \propto a^{-3}$,
- Radiation $(P = \frac{1}{3}\rho)$: $\rho \propto a^{-4}$,
- Vacuum energy $(P = -\rho)$: $\rho = \text{constant}$.

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Step 4. Physical Interpretation

- The **first Friedmann equation** acts like an energy conservation law, relating the expansion rate to energy density and curvature.
- The **second Friedmann equation** determines whether the universe's expansion accelerates or decelerates.
- Together, they describe the complete **evolution of the scale factor** $a(t)$, and therefore the fate of the universe.

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By substituting the Robertson–Walker metric into Einstein's field equations with a perfect fluid source, we obtain the **Friedmann equations**—the fundamental equations governing cosmological expansion. They describe how $a(t)$ evolves under the influence of matter, radiation, spatial curvature, and dark energy.

3.3 Derivation of the Friedmann Equations from the Robertson–Walker Metric

Step 1. The Robertson–Walker Metric

We start with the general Robertson–Walker (RW) metric:

$$ds^2 = dt^2 - a^2(t)\,\gamma_{ij}\,dx^i dx^j, \tag{31}$$

where

- $a(t)$ is the **scale factor**, and
- γ_{ij} is the three-dimensional metric of a maximally symmetric space with curvature $k = 0, +1, -1$.

In spherical coordinates, this becomes

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]. \tag{32}$$

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Step 2. Stress–Energy Tensor of Matter

By homogeneity and isotropy, the matter content of the universe must take the form of a **perfect fluid**:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P\,g_{\mu\nu}, \tag{33}$$

where

- $\rho(t)$ is the energy density,
- $P(t)$ is the pressure, and
- $U^\mu = (1, 0, 0, 0)$ is the four–velocity of comoving observers.

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Step 3. Computing the Einstein Tensor for the FRW Metric

Einstein's field equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{34}$$

where the Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}. \tag{35}$$

(a) Christoffel Symbols From the RW metric, the nonzero Christoffel symbols are:

$$\Gamma^0_{ij} = a\dot{a}\,\gamma_{ij}, \tag{36}$$

$$\Gamma^i_{0j} = \frac{\dot{a}}{a}\,\delta^i_j, \tag{37}$$

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}(\gamma), \tag{38}$$

where $\tilde{\Gamma}^i_{jk}(\gamma)$ are the Christoffel symbols of the spatial metric γ_{ij}.

(b) Ricci Tensor Components After a straightforward but lengthy computation, we obtain:

$$R_{00} = -3\frac{\ddot{a}}{a}, \tag{39}$$

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\gamma_{ij}. \tag{40}$$

(c) Ricci Scalar The Ricci scalar is given by

$$R = g^{\mu\nu} R_{\mu\nu} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{41}$$

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Step 4. The Einstein Tensor

Substituting into $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$, we find:

(a) Time–Time Component

$$G_{00} = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{42}$$

(b) Space–Space Components

$$G_{ij} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) g_{ij}. \tag{43}$$

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Step 5. Einstein Equations

(a) Time–Time Component Substituting into Einstein's equations gives

$$3\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)=8\pi G\rho+\Lambda. \tag{44}$$

This is the **First Friedmann Equation**:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho-\frac{k}{a^2}+\frac{\Lambda}{3}.} \tag{45}$$

(b) Spatial Components Similarly, the spatial components yield

$$-\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)=-8\pi GP+\Lambda, \tag{46}$$

which simplifies to the **Second Friedmann Equation**:

$$\boxed{\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\left(\rho+\frac{3P}{c^2}\right)+\frac{\Lambda}{3}.} \tag{47}$$

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Step 6. Conservation Equation

From the conservation law $\nabla_\mu T^{\mu\nu}=0$, we obtain

$$\dot{\rho}+3H(\rho+P)=0, \qquad \text{where} \quad H=\frac{\dot{a}}{a}. \tag{48}$$

This determines how the energy density of each component evolves with expansion:

- Matter ($P=0$): $\rho\propto a^{-3}$,
- Radiation ($P=\frac{1}{3}\rho$): $\rho\propto a^{-4}$,
- Vacuum energy ($P=-\rho$): $\rho=\text{constant}$.

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Concluding statement. The Robertson–Walker metric plus a perfect-fluid source reduces Einstein's equations to the Friedmann and continuity equations, which fully determine the expansion history once ρ and P (or an equation of state) are specified. **Note:** A detailed derivation to friedmann equation is in latter sections

3.4 Energy–Momentum Tensor and Conservation in FRW

3.4.1 Symmetry arguments and the perfect fluid form

The FRW background is spatially homogeneous and isotropic. These symmetries strongly constrain the form of any coarse–grained (averaged) stress–energy tensor $T_{\mu\nu}$. Decompose

$T_{\mu\nu}$ into time and spatial parts: a 3-scalar T_{00}, 3-vectors T_{0i}, T_{i0} and a spatial 3-tensor T_{ij}. Isotropy requires the mean value of any 3-vector to vanish, hence

$$T_{0i} = T_{i0} = 0.$$

Isotropy also requires any spatial 3-tensor to be proportional to the spatial metric g_{ij} (there is no preferred spatial direction), so at each spatial point

$$T_{ij}(t, \mathbf{x}) \propto g_{ij}(t, \mathbf{x}).$$

Homogeneity then implies the proportionality coefficient is a function of time only. Denote the energy density by $\rho(t)$ and the isotropic pressure by $P(t)$. In the comoving frame (fluid rest frame) we therefore have

$$T_{00} = \rho(t), \qquad T_{0i} = T_{i0} = 0, \qquad T_{ij} = -P(t)\, g_{ij}(t, \mathbf{x}). \tag{49}$$

It is convenient to display mixed components $T^{\mu}{}_{\nu} = g^{\mu\alpha} T_{\alpha\nu}$. In the comoving frame $U^{\mu} = (1, 0, 0, 0)$ one obtains

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}. \tag{50}$$

More generally, the covariant perfect fluid stress–energy tensor (valid in any frame) is

$$\boxed{T^{\mu\nu} = (\rho + P)\, U^{\mu} U^{\nu} - P\, g^{\mu\nu}, \qquad T^{\mu}{}_{\nu} = (\rho + P) U^{\mu} U_{\nu} - P\, \delta^{\mu}{}_{\nu}} \tag{51}$$

where U^{μ} is the fluid four-velocity, ρ its rest-frame energy density and P the isotropic pressure. Setting $U^{\mu} = (1, 0, 0, 0)$ immediately reproduces (49) and (50).

Evolution of density and pressure in Minkowski spacetime. Before turning to general relativity, it is useful to recall how energy conservation works in flat (Minkowski) spacetime. There, the conservation of energy and momentum is expressed by

$$\partial_{\mu} T^{\mu\nu} = 0.$$

For the energy density, take the $\nu = 0$ component:

$$\partial_{\mu} T^{\mu 0} = 0.$$

Since $T^{00} = \rho$ and $T^{i0} = \pi^{i}$ (the energy flux or momentum density), this becomes

$$\dot{\rho} = -\, \partial_i \pi^{i}.$$

Thus, in Minkowski space the time evolution of the energy density is given by the negative divergence of the energy flux: energy decreases in a region if energy flows out of it. This flat-space expression will generalize to curved spacetime once partial derivatives are replaced by covariant derivatives, leading to $\nabla_{\mu} T^{\mu\nu} = 0$.

3.4.2 Conservation: $\nabla_\mu T^\mu{}_\nu = 0$ and the continuity equation

Energy–momentum conservation in general relativity is expressed as the vanishing covariant divergence

$$\nabla_\mu T^\mu{}_\nu = 0.$$

For a mixed tensor the covariant derivative expands to

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha = 0. \tag{52}$$

We now evaluate the $\nu = 0$ component of (52) in the homogeneous isotropic FRW background. In the comoving frame the only nonzero mixed components are $T^0{}_0 = \rho$ and $T^i{}_j = -P\,\delta^i{}_j$; also the spatial derivatives vanish by homogeneity, $\partial_i T^\mu{}_\nu = 0$. Thus (52) with $\nu = 0$ reduces to

$$\partial_0 T^0{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^\alpha_{\mu 0} T^\mu{}_\alpha = 0.$$

Evaluate each term:

- $\partial_0 T^0{}_0 = \partial_0 \rho \equiv \dot\rho$.
- $\Gamma^\mu_{\mu 0} T^0{}_0 = (\Gamma^0_{00} + \Gamma^i_{i0})\,\rho$. For the FRW metric $\Gamma^0_{00} = 0$ and $\Gamma^i_{i0} = 3\dot a/a$ (sum over $i = 1, 2, 3$), hence
$$\Gamma^\mu_{\mu 0} T^0{}_0 = 3\frac{\dot a}{a}\,\rho.$$
- The last term $-\Gamma^\alpha_{\mu 0} T^\mu{}_\alpha$ is nonzero only when μ, α are spatial indices because $T^0{}_i = T^i{}_0 = 0$. Thus
$$-\Gamma^\alpha_{\mu 0} T^\mu{}_\alpha = -\Gamma^j_{i0} T^i{}_j.$$
For FRW $\Gamma^j_{i0} = (\dot a/a)\,\delta^j{}_i$ and $T^i{}_j = -P\,\delta^i{}_j$. Therefore
$$-\Gamma^j_{i0} T^i{}_j = -\frac{\dot a}{a}\,\delta^j{}_i(-P\,\delta^i{}_j) = +3\frac{\dot a}{a}\,P.$$

Combining the three contributions gives

$$\dot\rho + 3\frac{\dot a}{a}\rho + 3\frac{\dot a}{a}P = 0,$$

or equivalently

$$\boxed{\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0} \tag{53}$$

This is the cosmological energy continuity equation (the $\nu = 0$ component of $\nabla_\mu T^\mu{}_\nu = 0$). It expresses local energy conservation in the expanding FRW background.

3.4.3 Physical interpretation and thermodynamic form

Equation (65) can be given a simple thermodynamic interpretation. Let V be a comoving physical volume; for example if comoving coordinate volume is fixed, the physical volume scales as $V \propto a^3$. Define the total energy in that volume $U = \rho V$. Then

$$\dot U = \dot\rho\,V + \rho\,\dot V.$$

Using $\dot{V}/V = 3\dot{a}/a$ and substituting (65) for $\dot{\rho}$ gives

$$\dot{U} = -P\dot{V},$$

or in differential form

$$\boxed{dU = -P\,dV}$$

which is the first law of thermodynamics for an adiabatically expanding homogeneous fluid (no heat transfer). Thus the continuity equation is equivalent to the statement that the work done by the pressure reduces the energy inside the comoving volume.

3.4.4 Remarks: equations of state and scaling laws

If the fluid obeys a barotropic equation of state $P = w\rho$ with constant w, integrate (65) to obtain

$$\rho(a) \propto a^{-3(1+w)}.$$

Important cases:

- Dust (pressureless matter): $w = 0 \Rightarrow \rho \propto a^{-3}$ (number density diluted by expansion).
- Radiation: $w = 1/3 \Rightarrow \rho \propto a^{-4}$ (extra a^{-1} factor from redshifting of photon energies).
- Vacuum (cosmological constant): $w = -1 \Rightarrow \rho = \text{const}$.

These results, together with the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, lead to the Friedmann equations which determine $a(t)$ given the matter content.

4 Cosmic Inventory: Matter, Radiation and Dark Energy

The purpose of this section is to identify and classify the different components that contribute to the energy content of the Universe and to describe how each component evolves under cosmic expansion. Using the continuity equation and the Friedmann equations derived earlier, we examine matter, radiation, and vacuum energy as distinct cosmological fluids characterized by their equations of state. This framework forms the basis of the standard ΛCDM model and allows direct connection between theoretical dynamics and observational cosmology.

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The large-scale dynamics of the universe is governed not only by geometry but also by the contents of the universe. Different components contribute differently to the energy density and pressure and therefore dilute differently with the expansion. In this section we classify the main components, derive their scaling with the scale factor $a(t)$, and illustrate their relative evolution graphically.

4.1 General derivation from the continuity equation

Begin with the covariant energy–momentum conservation for a homogeneous, isotropic perfect fluid:

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0, \tag{1}$$

where $\rho(t)$ is the energy density and $P(t)$ the pressure in the fluid rest frame. Assume a barotropic equation of state of the form

$$P = w\,\rho, \qquad w = \text{const.}$$

Substitute into (1):

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + w) = 0.$$

This is an ordinary differential equation for $\rho(a)$. Rewrite using $\dot{\rho} = \frac{d\rho}{da}\dot{a}$ to get

$$\frac{d\rho}{da} + \frac{3(1 + w)}{a}\rho = 0.$$

This separates to

$$\frac{d\rho}{\rho} = -3(1 + w)\frac{da}{a}.$$

Integrate from some reference scale a_0 (often chosen $a_0 = 1$) to a:

$$\ln\frac{\rho(a)}{\rho(a_0)} = -3(1 + w)\ln\frac{a}{a_0}.$$

Exponentiating gives the general scaling law

$$\boxed{\rho(a) = \rho(a_0)\left(\frac{a}{a_0}\right)^{-3(1+w)}} \tag{54}$$

or, with the common convention $a_0 \equiv 1$,

$$\rho(a) = \rho_0\, a^{-3(1+w)}.$$

4.2 Important special cases

Apply (54) to the most important cosmological components:

Non-relativistic matter ("dust"): $w = 0$. Then

$$\rho_m(a) = \rho_{m,0}\, a^{-3}.$$

This reflects pure dilution of number density as the comoving volume $\propto a^3$ expands.

Radiation: $w = \frac{1}{3}$. Then

$$\rho_r(a) = \rho_{r,0}\, a^{-4}.$$

The extra factor a^{-1} (compared to matter) encodes the redshifting of individual photon energies: $E \propto a^{-1}$.

Vacuum / cosmological constant / dark energy (perfect vacuum): $w = -1$. Then

$$\rho_\Lambda(a) = \rho_{\Lambda,0} \quad \text{(constant)}.$$

In this case the energy density does not dilute with expansion.

4.3 Vacuum energy and the cosmological constant

Quantum field theory predicts a vacuum energy (zero-point energy) which acts gravitationally like a perfect fluid with

$$T^{\mu\nu}_{\text{vac}} = \rho_{\text{vac}}\, g^{\mu\nu}.$$

Comparing with the perfect-fluid form $T^{\mu\nu} = (\rho + P)U^\mu U^\nu - Pg^{\mu\nu}$ in the fluid rest frame $U^\mu = (1, 0, 0, 0)$, one finds for vacuum

$$P_{\text{vac}} = -\rho_{\text{vac}}.$$

Equivalently, the cosmological constant Λ may be moved to the right-hand side of the Einstein equation as an effective stress tensor:

$$T^{\mu\nu}_{(\Lambda)} \;=\; \frac{\Lambda}{8\pi G}\, g^{\mu\nu} \equiv \rho_\Lambda\, g^{\mu\nu}, \qquad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G}.$$

This component has $w = -1$ and constant energy density.

The cosmological-constant problem. Quantum field theory estimates of the vacuum energy are many orders of magnitude larger than the observed value. A very rough statement is

$$\frac{\rho^{(\text{QFT})}_{\text{vac}}}{\rho_{\text{obs}}} \sim 10^{120},$$

which is a major unsolved problem in theoretical physics .

4.4 Mixtures and dominant components

In the real universe several components coexist (radiation, baryons, dark matter, dark energy). The total energy density is the sum

$$\rho_{\text{tot}}(a) = \rho_r(a) + \rho_m(a) + \rho_\Lambda + \cdots .$$

Because each component dilutes differently with a, the *dominant* component changes with time: at early times radiation ($a \ll 10^{-4}$) dominated, later matter dominated, and today the cosmological-constant–like component appears to dominate.

4.5 Notes and physical interpretation

- The extra factor of a^{-1} for radiation relative to matter is the energy redshift: each photon's energy scales as $E \propto a^{-1}$, while number density scales as a^{-3}, giving a^{-4}.
- Massive non-relativistic particles have $|P| \ll \rho$, hence $w \approx 0$. A collisionless gas of cold particles (cold dark matter, baryons once cooled) follows $\rho \propto a^{-3}$.
- Dark energy (as vacuum energy or a true cosmological constant) has $w = -1$. Its constant density leads to accelerated expansion when it dominates the Friedmann equation.

- In a multi-component universe the Hubble parameter is given by the Friedmann equation
$$H^2(a) = \frac{8\pi G}{3}\sum_i \rho_i(a) - \frac{k}{a^2},$$
and the component that redshifts slowest eventually dominates at late times. For a flat universe with radiation, matter and cosmological constant:
$$H^2(a) = H_0^2\Big[\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}\Big],$$
where $\Omega_{i,0} = \rho_{i,0}/\rho_{\mathrm{crit},0}$ and $\rho_{\mathrm{crit},0} = 3H_0^2/(8\pi G)$.

4.6 Detailed derivation of spacetime curvature for FRW

We work with the FRW line element in cosmic time,

$$ds^2 = dt^2 - a^2(t)\,\gamma_{ij}(x)\,dx^i dx^j,$$

where γ_{ij} is the metric of the maximally symmetric spatial slices (with constant curvature k). Our goal is to compute the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}$.

1. Christoffel symbols (needed ingredients). From the metric one finds the nonzero Christoffel symbols (recall $g_{00} = 1,\ g_{0i} = 0,\ g_{ij} = -a^2\gamma_{ij}$):

$$\Gamma^0_{ij} = a\dot{a}\,\gamma_{ij}, \tag{55}$$
$$\Gamma^i_{0j} = \frac{\dot{a}}{a}\,\delta^i{}_j, \tag{56}$$
$$\Gamma^i_{jk} = {}^{(\gamma)}\Gamma^i{}_{jk}, \tag{57}$$

where ${}^{(\gamma)}\Gamma^i{}_{jk}$ are the Christoffel symbols built from the purely spatial metric γ_{ij}. the components collected in eq. (1.2.41).)

2. Ricci tensor definition. The Ricci tensor is

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda}.$$

We evaluate the relevant components separately.

(a) R_{00}. Set $\mu = \nu = 0$ in the definition:

$$R_{00} = \partial_\lambda\Gamma^\lambda_{00} - \partial_0\Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{00} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{0\lambda}.$$

Using $\Gamma^\lambda_{00} = 0$ for the FRW metric (no Christoffel with two lower time indices), the first and third terms vanish and we are left with

$$R_{00} = -\partial_0\Gamma^\lambda_{0\lambda} - \Gamma^\lambda_{0\rho}\Gamma^\rho_{0\lambda}.$$

Now $\Gamma^\lambda_{0\lambda} = \Gamma^0_{00} + \Gamma^i_{0i} = 0 + 3\frac{\dot a}{a}$, so $\partial_0 \Gamma^\lambda_{0\lambda} = 3\frac{d}{dt}\left(\frac{\dot a}{a}\right)$. Also $\Gamma^\lambda_{0\rho}\Gamma^\rho_{0\lambda} = \Gamma^i_{0j}\Gamma^j_{0i}$ (with nonzero contributions only from spatial indices) and using $\Gamma^i_{0j} = (\dot a/a)\delta^i{}_j$ we find $\Gamma^i_{0j}\Gamma^j_{0i} = 3(\dot a/a)^2$. Therefore

$$R_{00} = -3\frac{d}{dt}\left(\frac{\dot a}{a}\right) - 3\left(\frac{\dot a}{a}\right)^2 = -3\left(\frac{\ddot a}{a}\right).$$

This gives the boxed result

$$\boxed{R_{00} = -3\,\frac{\ddot a}{a}} \tag{58}$$

(b) R_{ij}**.** For the spatial components set $\mu = i,\ \nu = j$:

$$R_{ij} = \partial_\lambda \Gamma^\lambda_{ij} - \partial_j \Gamma^\lambda_{i\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{ij} - \Gamma^\lambda_{j\rho}\Gamma^\rho_{i\lambda}.$$

The strategy is to evaluate this at a convenient point where the spatial coordinates are chosen so that $\gamma_{ij} = \delta_{ij}$ at the point; because R_{ij} is a tensor and the final expression is covariant, the result will hold for general x. We therefore drop terms that vanish at that point and keep the contributions that survive.

Split R_{ij} into two groups labelled (A) and (B) as in the notes:

$$(A) \equiv \partial_0 \Gamma^0_{ij} + \partial_\ell \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{i\ell},$$

$$(B) \equiv \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{ij} - \Gamma^\lambda_{j\rho}\Gamma^\rho_{i\lambda}.$$

Compute (A). Using $\Gamma^0_{ij} = a\dot a\,\gamma_{ij}$ and evaluating at the point where $\gamma_{ij} = \delta_{ij}$,

$$\partial_0 \Gamma^0_{ij} = \partial_0\big(a\dot a\big)\,\delta_{ij} = \big(a\ddot a + \dot a^2\big)\delta_{ij}.$$

The spatial derivative term $\partial_\ell \Gamma^\ell_{ij}$ produces a curvature contribution from the spatial metric γ_{ij}; after evaluating the derivative (see the notes) one gets a term $2k\,\delta_{ij}$. The last term $-\partial_j \Gamma^\ell_{i\ell}$ cancels part of this but leaves the net contribution accounted for below. Combining these results yields

$$(A) = \big(a\ddot a + \dot a^2 + 2k\big)\,\delta_{ij}.$$

Compute (B). Using $\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{ij}$ and the explicit Christoffel components, the dominant contribution at the chosen point is $\Gamma^0_{00}\Gamma^0_{ij} + \Gamma^k_{k0}\Gamma^0_{ij} - \cdots$, and the algebra gives

$$(B) = \dot a^2\,\delta_{ij}.$$

Adding (A) and (B) gives

$$R_{ij}(x=0) = \big(a\ddot a + 2\dot a^2 + 2k\big)\,\delta_{ij}.$$

Promoting this back to general coordinates (replace δ_{ij} by γ_{ij}), and using $g_{ij} = -a^2\gamma_{ij}$, we can write the result in the form

$$\boxed{R_{ij} = -\Big(\frac{\ddot a}{a} + 2\frac{\dot a^2}{a^2} + 2\frac{k}{a^2}\Big)\,g_{ij}} \tag{59}$$

or equivalently

$$R_{ij} = \big(a\ddot a + 2\dot a^2 + 2k\big)\,\gamma_{ij}.$$

3. Ricci scalar R. The Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$. Using $g^{00} = 1$ and $g^{ij} = -\frac{1}{a^2}\gamma^{ij}$, and the results for R_{00} and R_{ij}, we find

$$\begin{aligned} R = R_{00} + g^{ij}R_{ij} &= -3\frac{\ddot{a}}{a} - \frac{1}{a^2}\gamma^{ij}\big(a\ddot{a} + 2\dot{a}^2 + 2k\big)\gamma_{ij} \\ &= -3\frac{\ddot{a}}{a} - \frac{3}{a^2}\big(a\ddot{a} + 2\dot{a}^2 + 2k\big) \\ &= -6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right). \end{aligned}$$

Hence

$$\boxed{R = -6\Big[\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big]} \tag{60}$$

4. Einstein tensor $G_{\mu\nu}$. Finally compute $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}$.

• For the 00 component:

$$G_{00} = R_{00} - \tfrac{1}{2}R\,g_{00} = -3\frac{\ddot{a}}{a} - \tfrac{1}{2}R \cdot 1.$$

Substitute R from (103):

$$G_{00} = -3\frac{\ddot{a}}{a} - \tfrac{1}{2}\left[-6\Big(\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big)\right] = 3\Big(\frac{\dot{a}}{a}\Big)^2 + 3\frac{k}{a^2}.$$

Thus

$$\boxed{G^0{}_0 = 3\Big[\Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big]} \tag{61}$$

(remember $G^0{}_0 = g^{0\mu}G_{\mu 0} = G_{00}$ because $g^{00} = 1$.)

• For the spatial components G_{ij}:

$$G_{ij} = R_{ij} - \tfrac{1}{2}R\,g_{ij}.$$

Using $R_{ij} = -\big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\big)g_{ij}$ and R from above, a short algebraic rearrangement yields

$$G_{ij} = \Big[2\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big]\,g_{ij}.$$

Raising an index we may write

$$\boxed{G^i{}_j = \Big[2\frac{\ddot{a}}{a} + \Big(\frac{\dot{a}}{a}\Big)^2 + \frac{k}{a^2}\Big]\,\delta^i{}_j} \tag{62}$$

These are the nonzero components of the Einstein tensor for the FRW metric and are the left-hand side of the Einstein equations $G_{\mu\nu} = 8\pi G\,T_{\mu\nu}$. When combined with the perfect-fluid stress tensor they produce the Friedmann equations (the familiar equations determining $a(t)$).

Remark on the computation strategy. The computation of R_{ij} is the most tedious step if performed with all indices everywhere; a convenient trick is to evaluate expressions at a point where $\gamma_{ij} = \delta_{ij}$ (so many terms simplify) and then restore covariance by replacing δ_{ij} with γ_{ij} (or g_{ij}) in the final tensorial expression. The tensorial form is then valid at all points.

□

4.7 Physical Implications of Computing $R_{\mu\nu}$, R and $G_{\mu\nu}$ in FLRW

Having derived the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu},$$

for the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, we are now in a position to understand their physical significance. These curvature tensors encode the full geometric content of the FLRW spacetime, and when combined with Einstein's field equations, they yield the dynamical laws governing the evolution of the Universe.

1. Consistency of FLRW with homogeneity and isotropy

The explicit computation shows that the Einstein tensor takes the diagonal form

$$G^{\mu}{}_{\nu} = \mathrm{diag}\left(3H^2 + \frac{3k}{a^2},\ -(2\dot{H} + 3H^2 + \frac{k}{a^2})\delta^i{}_j \right),$$

with all off-diagonal components vanishing. This structure is a direct consequence of the symmetries of the FLRW metric: spatial homogeneity and isotropy forbid any preferred direction, and therefore rule out shear, vorticity, or anisotropic stress on the geometric side of Einstein's equation.

Thus, the computation of the curvature tensors confirms that FLRW spacetime is *maximally symmetric on spatial slices* and self-consistent with cosmology's foundational assumptions.

2. Derivation of the Friedmann equations

Einstein's equation,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

combined with the perfect-fluid stress-energy tensor,

$$T_{\mu\nu} = (\rho + P) U_\mu U_\nu - P\, g_{\mu\nu},$$

implies that only two independent components of the field equations exist: the 00 equation (energy constraint) and the ii equation (acceleration equation). Equating the corresponding components of $G_{\mu\nu}$ and $T_{\mu\nu}$ gives:

First Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}.$$

Second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P).$$

These equations are the *fundamental dynamical laws* of relativistic cosmology. They describe how the scale factor evolves in response to matter, radiation, and dark energy.

3. Energy–momentum conservation from the Bianchi identity

Another key implication arises from the geometric identity

$$\nabla^{\mu} G_{\mu\nu} = 0,$$

which holds for any metric. Substituting Einstein's equation leads to

$$\nabla^{\mu} T_{\mu\nu} = 0,$$

which expresses the conservation of energy and momentum.

When evaluated for a homogeneous fluid in an expanding universe, this becomes the continuity equation:

$$\dot{\rho} + 3H(\rho + P) = 0.$$

This tells us how the energy density evolves with time. For example, matter ($P = 0$) dilutes as $\rho \propto a^{-3}$, radiation ($P = \frac{1}{3}\rho$) as $\rho \propto a^{-4}$, and dark energy ($P = -\rho$) remains constant.

Importantly, this conservation law is *not an additional assumption*: it follows automatically from the geometric structure encoded in $G_{\mu\nu}$.

Summary of Implications

To summarize, computing the curvature tensors of the FLRW metric leads to three central results:

1. The diagonal, isotropic form of $G_{\mu\nu}$ ensures that the FLRW ansatz is consistent with homogeneity and isotropy.
2. Einstein's equation applied to these tensors directly yields the Friedmann equations, the fundamental evolution equations of cosmology.
3. The Bianchi identity implies energy–momentum conservation, producing the continuity equation that governs how ρ and P change with the expansion.

Together, these results fully determine the history of the Universe once the matter content (i.e. the equation of state) is specified.

4.8 Friedmann Equations: Physical Consequences for Cosmic Components

In this section we apply the Friedmann and continuity equations derived earlier to different forms of cosmic matter. The aim is not to re-derive the dynamical equations, but to understand how the Universe evolves once its energy content and equation of state are specified.

The expansion of the Universe is governed by the Friedmann equations,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{63}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}, \tag{64}$$

together with the energy–momentum conservation equation,

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{65}$$

These equations form a closed system once an equation of state

$$P = w\rho \tag{66}$$

is specified for a given cosmic component.

Density Scaling with the Scale Factor

Substituting the equation of state into the continuity equation (65), we obtain

$$\dot{\rho} + 3H(1 + w)\rho = 0, \tag{67}$$

which integrates to

$$\rho(a) \propto a^{-3(1+w)}. \tag{68}$$

This result shows that different components dilute at different rates as the Universe expands:

- **Radiation** ($w = 1/3$): $\rho_r \propto a^{-4}$, where the extra factor of a^{-1} arises from redshifting of photon energies.
- **Matter (dust, CDM)** ($w = 0$): $\rho_m \propto a^{-3}$, reflecting dilution by volume expansion.
- **Vacuum energy** ($w = -1$): $\rho_\Lambda = \text{const}$, corresponding to a constant energy density per unit volume.

Implications for Cosmic Expansion

The acceleration equation (64) reveals that pressure contributes to the gravitational dynamics of the Universe. Components with positive pressure (radiation and matter) lead to decelerated expansion, while a component with sufficiently negative pressure drives accelerated expansion.

At early times, when the scale factor was small, radiation dominated the energy density of the Universe. As the Universe expanded, matter eventually became dominant, enabling the formation of cosmic structures. At late times, the near-constant vacuum energy comes to dominate, resulting in the observed accelerated expansion of the Universe.

These behaviors form the physical basis of the standard ΛCDM cosmological model.

4.9 Friedmann equations: derivation and physical consequences

We now combine the geometric results for the FRW metric, namely the nonzero components of the Einstein tensor,

$$G^0{}_0 = 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2}, \qquad G^i{}_j = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\delta^i{}_j,$$

with Einstein's equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

and the perfect-fluid stress tensor

$$T^\mu{}_\nu = \mathrm{diag}(\rho,\, -P,\, -P,\, -P),$$

to obtain the dynamical equations for the scale factor $a(t)$.

00 component (energy equation). Equating $G^0{}_0$ and $8\pi G\, T^0{}_0 = 8\pi G\, \rho$ yields

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = 8\pi G\, \rho. \tag{69}$$

Rearranging gives the first Friedmann equation

$$\boxed{H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \tag{70}$$

This is an energy (constraint) equation: it relates expansion rate to energy density and spatial curvature.

ii component (acceleration equation). Equating the spatial diagonal components, $G^i{}_j = 8\pi G\, T^i{}_j = -8\pi G\, P\, \delta^i{}_j$, yields

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G\, P.$$

Combining with the first Friedmann equation to eliminate the curvature/ H^2 term gives the acceleration equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)} \tag{71}$$

This displays explicitly that pressure gravitates: positive pressure contributes to deceleration, while sufficiently negative pressure ($P < -\rho/3$) causes acceleration.

Continuity equation from Bianchi identity. The twice-contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ implies $\nabla^\mu T_{\mu\nu} = 0$, which for a homogeneous perfect fluid reduces to

$$\boxed{\dot{\rho} + 3H(\rho + P) = 0}$$

This is not independent of the Friedmann equations (it follows from them) but it is extremely useful for closing the system once an equation of state is specified.

4.10 Special Case: Cold Dark Matter (CDM) / Dust

Cold Dark Matter (CDM), or “dust”, refers to a component of the universe whose pressure is negligible compared to its energy density. Physically, this corresponds to non-relativistic particles whose kinetic energy is small, so that

$$P \simeq 0.$$

In this case, the continuity equation,

$$\dot{\rho} + 3H(\rho + P) = 0,$$

reduces to

$$\dot{\rho}_m + 3H\rho_m = 0,$$

which integrates to the familiar scaling law

$$\rho_m(a) \propto a^{-3}.$$

This expresses the simple fact that as the universe expands, the physical volume scales as $V \propto a^3$, and since particle number is conserved, the matter density must dilute as $\rho_m \propto 1/V$.

Friedmann Equation for a Flat, Matter-Only Universe

For a spatially flat universe ($k = 0$), the first Friedmann equation becomes

$$H^2 = \frac{8\pi G}{3}\,\rho_m(a).$$

Substituting the matter-scaling law, we immediately find

$$H^2 \propto a^{-3}, \qquad \text{or equivalently} \qquad \frac{\dot{a}}{a} = C\,a^{-3/2},$$

where $C = \sqrt{\frac{8\pi G}{3}\rho_{m,0}}$. This differential equation determines the evolution of the scale factor in a matter-dominated universe.

Solving for the Scale Factor

We rewrite the equation as

$$\dot{a} = C\,a^{-1/2}.$$

Separating variables,

$$a^{1/2}\,da = C\,dt,$$

and integrating both sides gives

$$\frac{2}{3}a^{3/2} = Ct + \text{const.}$$

Choosing the Big Bang condition $a(0) = 0$ sets the integration constant to zero. Solving for $a(t)$ yields the well-known analytic solution

$$a(t) \propto t^{2/3}.$$

Using $a(t) = At^{2/3}$, we compute the Hubble parameter:

$$H(t) = \frac{\dot{a}}{a} = \frac{\frac{2}{3}At^{-1/3}}{At^{2/3}} = \frac{2}{3t}.$$

Physical Interpretation

A matter-dominated universe therefore expands as

$$a(t) \propto t^{2/3}, \qquad H(t) = \frac{2}{3t}.$$

From the second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m,$$

we see that since $\rho_m > 0$,

$$\ddot{a} < 0.$$

Thus, the expansion is *decelerating*. This deceleration is a gravitational effect: matter clumps and attracts, slowing the cosmic expansion.

In summary, the CDM/dust universe is characterized by:

- pressureless matter $P = 0$,
- energy density scaling $\rho_m \propto a^{-3}$,
- scale factor evolution $a(t) \propto t^{2/3}$,
- Hubble rate $H(t) = 2/(3t)$,
- decelerating expansion ($\ddot{a} < 0$).

4.11 General ΛCDM model and observational functions

For a multi-component universe (radiation, matter, curvature, cosmological constant), it is convenient to write the Friedmann equation as

$$H^2(a) = H_0^2 \left[\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda\right],$$

where $\Omega_i \equiv \rho_{i,0}/\rho_{\mathrm{crit},0}$ and $\rho_{\mathrm{crit},0} = 3H_0^2/(8\pi G)$. Equivalently in terms of redshift $z = a^{-1} - 1$,

$$\boxed{H(z) = H_0\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}}.$$

Comoving and luminosity distances. The comoving distance to redshift z is

$$\chi(z) = \int_0^z \frac{dz'}{H(z')}.$$

In flat space the luminosity distance is $d_L(z) = (1+z)\,\chi(z)$. The observable distance modulus used for supernovae is

$$\mu(z) = 5\log_{10}\left(\frac{d_L(z)}{10\ \mathrm{pc}}\right).$$

4.12 Plots and comparison with matter-only model

The signature observational test that led to the discovery of cosmic acceleration was the comparison of the distance modulus $\mu(z)$ predicted by a matter-only universe ($\Omega_m = 1$, $\Omega_\Lambda = 0$) with that predicted by a ΛCDM model (e.g. $\Omega_m \approx 0.32$, $\Omega_\Lambda \approx 0.68$). Observationally the Type Ia supernovae are fainter (larger μ) than expected in a matter-only model. This is precisely what is expected if the expansion is accelerating due to dark energy.

5 Single-Component Universe: analytic solutions and conformal time

The purpose of this section is to obtain analytic solutions for the cosmic expansion in idealized universes dominated by a single energy component. Using the Friedmann equations together with a constant equation of state, we derive the time dependence of the scale factor for radiation-, matter-, and vacuum-dominated universes. The introduction of conformal time provides a convenient parametrization of the expansion and facilitates later discussions of horizon scales and light propagation. These simplified models serve as essential benchmarks for understanding the more realistic multi-component cosmological evolution.

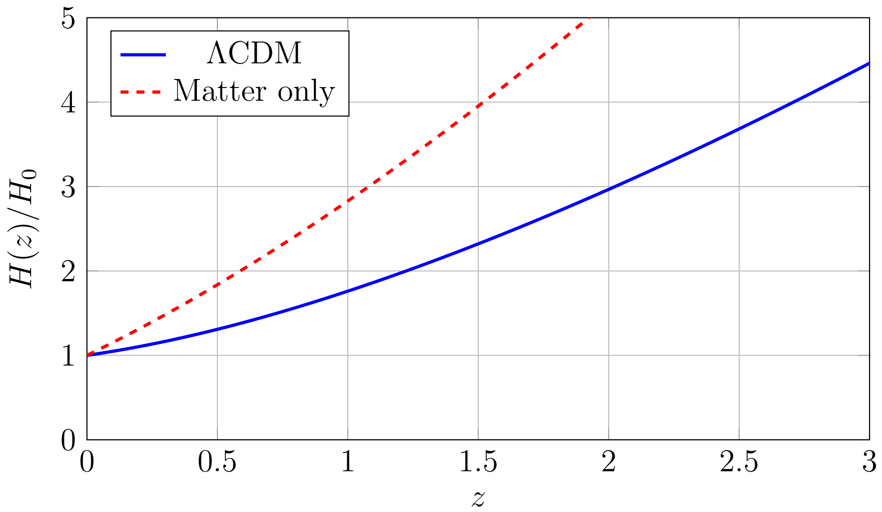

Figure 2: Hubble function $H(z)/H_0$.

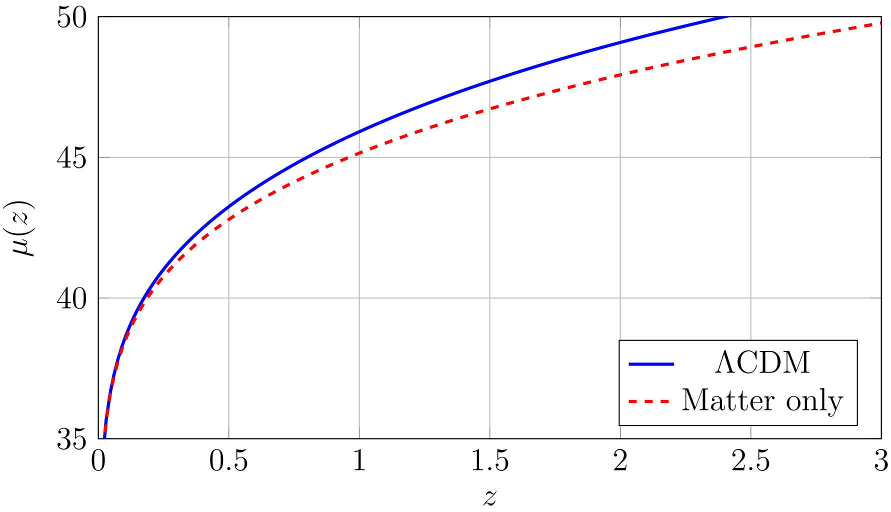

Figure 3: Distance modulus $\mu(z)$: ΛCDM vs. matter-only (analytic approximation).

5.1 Derivation of the single-component solution

Consider a spatially flat, single-component universe where the only energy source has equation of state $P = w_I \rho$ with constant w_I. The Friedmann equation (1.3.135) then reduces to

$$H^2(a) = H_0^2\,\Omega_I\,a^{-3(1+w_I)}, \tag{72}$$

where Ω_I is the fractional density of that component today (or at the chosen normalization $a_0 = 1$). Taking the positive square root and writing $H = \dot{a}/a$,

$$\frac{\dot{a}}{a} = H_0\sqrt{\Omega_I}\,a^{-\frac{3}{2}(1+w_I)}.$$

This is a separable ordinary differential equation. Separate variables:

$$a^{\frac{3}{2}(1+w_I)-1}\,da = H_0\sqrt{\Omega_I}\,dt.$$

Note that

$$\frac{3}{2}(1+w_I) - 1 = \frac{1+3w_I}{2}.$$

Integrate both sides. Two cases appear:

Case $w_I \neq -1$.

$$\int a^{(1+3w_I)/2}\, da = H_0\sqrt{\Omega_I} \int dt.$$

The left integral gives $\dfrac{2}{1+3w_I}\, a^{(1+3w_I)/2}$. Thus

$$\frac{2}{1+3w_I}\, a^{(1+3w_I)/2} = H_0\sqrt{\Omega_I}\, t + C.$$

Choosing the normalization so that the Big Bang occurs at $t = 0$ (i.e. $a \to 0$ as $t \to 0$) sets $C = 0$. Solving for $a(t)$ yields

$$a(t) \propto t^{\frac{2}{3(1+w_I)}}.$$

This is the general power-law result quoted in Eq. (1.3.137) for $w_I \neq -1$.

Case $w_I = -1$. If $w_I = -1$ the r.h.s. of (72) is constant and $\dot{a}/a = H_0\sqrt{\Omega_I} \equiv H$ (a constant). Integrating gives

$$a(t) \propto e^{Ht},$$

the de Sitter (exponential expansion) solution. This is the ΛD line.

5.2 Derivation of conformal-time form

Conformal time τ is defined by $d\tau = dt/a(t)$. Starting from the power-law solution $a(t) \propto t^{\alpha}$ with $\alpha = \frac{2}{3(1+w_I)}$, we compute

$$\tau(t) = \int \frac{dt}{a(t)} \propto \int t^{-\alpha}\, dt = \frac{1}{1-\alpha}\, t^{1-\alpha} + \text{const}.$$

Dropping the integration constant (choose $\tau = 0$ at $t = 0$ if convergence allows), we have

$$\tau \propto t^{1-\alpha}.$$

Invert to get $t \propto \tau^{1/(1-\alpha)}$. Substitute into $a(t) \propto t^{\alpha}$ to obtain

$$a(\tau) \propto \tau^{\alpha/(1-\alpha)}.$$

Compute the exponent:

$$\frac{\alpha}{1-\alpha} = \frac{\frac{2}{3(1+w_I)}}{1-\frac{2}{3(1+w_I)}} = \frac{2}{1+3w_I}.$$

Hence for $w_I \neq -1$,

$$\boxed{a(\tau) \propto \tau^{\frac{2}{1+3w_I}}}$$

which is Eq. (1.3.138) in conformal time. The $w_I = -1$ case is special: for $a(t) \propto e^{Ht}$, $d\tau = dt/a$ integrates to $\tau = -e^{-Ht}/H$ (choose the additive constant so that $\tau \to 0^-$ as $t \to \infty$), and hence $a(\tau) \propto (-\tau)^{-1}$.

5.3 Particular physical cases and interpretation

Using the general formulas above, evaluate the commonly used cases:

Radiation domination (RD): $w = \frac{1}{3}$.

$$a(t) \propto t^{1/2}, \qquad a(\tau) \propto \tau.$$

Energy density: $\rho \propto a^{-4}$. Radiation redshifts: number density $\propto a^{-3}$ and each photon's energy $\propto a^{-1}$.

Matter domination (MD): $w = 0$ (dust / CDM).

$$a(t) \propto t^{2/3}, \qquad a(\tau) \propto \tau^2.$$

Energy density: $\rho \propto a^{-3}$.

Dark-energy / cosmological-constant domination (Λ D): $w = -1$.

$$a(t) \propto e^{Ht}, \qquad a(\tau) \propto (-\tau)^{-1}.$$

Energy density: $\rho = \text{const}$.

These results are summarized in Table 1.

Case	w	$\rho(a)$	$a(t)$ / $a(\tau)$
Radiation (RD)	1/3	a^{-4}	$a(t) \propto t^{1/2}$, $a(\tau) \propto \tau$
Matter (MD)	0	a^{-3}	$a(t) \propto t^{2/3}$, $a(\tau) \propto \tau^2$
Dark energy (ΛD)	-1	a^0	$a(t) \propto e^{Ht}$, $a(\tau) \propto (-\tau)^{-1}$

Table 1: Solutions for a flat single-component universe.

5.4 Physical summary

- For most of cosmic history one component dominates (radiation early, matter later, dark energy today). The dominance changes because components dilute at different rates: radiation $\propto a^{-4}$, matter $\propto a^{-3}$, vacuum $\propto a^0$.
- The value of w directly determines the time dependence of $a(t)$ and the causal structure (e.g. conformal time) of the universe.
- The ΛD solution has a finite conformal time to the future ($\tau \to 0^-$ as $t \to \infty$); this has important consequences for horizon scales and for inflationary/late-time causal structure.

5.5 Two–Component Universe: matter + radiation (detailed derivation)

We consider a spatially flat FRW universe filled with matter (dust, $P_m = 0$) and radiation (relativistic component, $P_r = \frac{1}{3}\rho_r$). It is convenient to work in conformal time τ, defined by $d\tau = dt/a(t)$. Primes denote derivatives with respect to τ: $a' \equiv da/d\tau$, $a'' \equiv d^2a/d\tau^2$.

Friedmann equations in conformal time

Starting from the usual Friedmann equations in cosmic time,

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P),$$

and using $\dot a = a'/a$ and $\ddot a = a''/a^2 - (a')^2/a^3$ (straightforward to verify), one finds the conformal-time form:

$$(a')^2 = \frac{8\pi G}{3}\,\rho\, a^4, \tag{73}$$

$$a'' = \frac{4\pi G}{3}(\rho - 3P)\, a^3. \tag{74}$$

(Exercise: derive the algebraic relations between cosmic- and conformal-time derivatives that were used to obtain (73)–(74).)

Total energy density for matter + radiation

Write the total energy density as $\rho = \rho_m + \rho_r$. Let $a_{\rm eq}$ denote the scale factor at matter–radiation equality, defined by $\rho_m(a_{\rm eq}) = \rho_r(a_{\rm eq})$. Denote the total density at equality by $\rho_{\rm eq} \equiv \rho(a_{\rm eq})$. Since at $a_{\rm eq}$ the two components are equal, $\rho_m(a_{\rm eq}) = \rho_r(a_{\rm eq}) = \frac{1}{2}\rho_{\rm eq}$.

Using the scaling laws $\rho_m \propto a^{-3}$ and $\rho_r \propto a^{-4}$, we may write the total density at general a in terms of equality quantities as

$$\rho(a) \;=\; \rho_m(a) + \rho_r(a) = \frac{\rho_{\rm eq}}{2}\left(\frac{a_{\rm eq}^3}{a^3} + \frac{a_{\rm eq}^4}{a^4}\right). \tag{75}$$

Evaluate the source term in the second conformal equation

In (74) the combination $\rho - 3P$ appears. For the mixture,

$$\rho - 3P = (\rho_m + \rho_r) - 3(P_m + P_r) = \rho_m + \rho_r - 3\left(0 + \frac{1}{3}\rho_r\right) = \rho_m,$$

because radiation satisfies $\rho_r - 3P_r = 0$. Thus the source of the acceleration in conformal time is simply the matter density. Using $\rho_m(a) = \rho_m(a_{\rm eq})(a_{\rm eq}/a)^3$ and $\rho_m(a_{\rm eq}) = \frac{1}{2}\rho_{\rm eq}$, we get

$$\rho_m(a) = \frac{\rho_{\rm eq}}{2}\,\frac{a_{\rm eq}^3}{a^3}.$$

Substitute this into (74) to obtain a remarkably simple ODE:

$$a''(\tau) \;=\; \frac{4\pi G}{3}\,\rho_m(a)\, a^3 = \frac{4\pi G}{3}\left(\frac{\rho_{\rm eq}}{2}\frac{a_{\rm eq}^3}{a^3}\right) a^3 = \underbrace{\frac{2\pi G}{3}\,\rho_{\rm eq}\, a_{\rm eq}^3}_{\equiv K},$$

so $a''(\tau) = K$ with

$$K \;=\; \frac{2\pi G}{3}\,\rho_{\rm eq}\, a_{\rm eq}^3. \tag{76}$$

This is a constant because $\rho_{\rm eq}$ and $a_{\rm eq}$ are fixed reference values.

Integrate the acceleration equation

Integrating the constant second derivative gives

$$a'(\tau) = K\tau + C,$$

and integrating once more,

$$a(\tau) = \tfrac{1}{2}K\tau^2 + C\tau + D, \tag{77}$$

where C and D are integration constants to be determined from the first Friedmann equation and the chosen initial conditions. This is the general quadratic solution; putting back K we may also write

$$a(\tau) = \frac{\pi G}{3}\,\rho_{\rm eq}\,a_{\rm eq}^3\,\tau^2 + C\tau + D. \tag{78}$$

(Compare with the notes: they write the same combination as $\pi G\rho_{\rm eq}a_{\rm eq}^3\tau^2/3$.)

Fix the integration constant D

A natural choice is to set conformal time origin at the Big Bang. Requiring $a(\tau = 0) = 0$ implies

$$a(0) = D = 0.$$

Thus $D = 0$. (This choice corresponds to choosing the origin of τ such that the scale factor vanishes at $\tau = 0$.)

Determine C from the first conformal Friedmann equation

We now use the first conformal-time equation (73),

$$(a')^2 = \frac{8\pi G}{3}\,\rho(a)\,a^4,$$

together with the explicit form of $\rho(a)$ in (75) and the quadratic ansatz (78) (with $D = 0$) to determine C.

Evaluate this identity at $\tau = 0$. Using $a(0) = 0$, the right-hand side simplifies because $a^4\rho(a)$ remains finite as $a \to 0$. Indeed from (75),

$$\rho(a)\,a^4 \;=\; \frac{\rho_{\rm eq}}{2}\left(a_{\rm eq}^3 a + a_{\rm eq}^4\right).$$

At $a = 0$ only the second term survives, so

$$\lim_{\tau\to 0}(a')^2 = \frac{8\pi G}{3}\cdot\frac{\rho_{\rm eq}}{2}\,a_{\rm eq}^4 = \frac{4\pi G}{3}\,\rho_{\rm eq}\,a_{\rm eq}^4.$$

Hence

$$a'(0) = C = \sqrt{\frac{4\pi G}{3}\,\rho_{\rm eq}\,a_{\rm eq}^4}.$$

Therefore the integration constants are

$$D = 0, \qquad C = a_{\rm eq}^2\sqrt{\frac{4\pi G}{3}\,\rho_{\rm eq}}.$$

Substituting C and K into (78) gives the full explicit solution for the scale factor in conformal time:

$$\boxed{a(\tau) = \frac{\pi G}{3}\,\rho_{\rm eq}\,a_{\rm eq}^3\,\tau^2 + a_{\rm eq}^2\sqrt{\frac{4\pi G}{3}\,\rho_{\rm eq}}\;\tau} \tag{79}$$

Dimensionless, suggestive form and the equality timescale

It is useful to cast the solution in dimensionless form. Define the dimensionless conformal time $x \equiv \tau/\tau_\star$ where $\tau_\star$ is the ratio C/K (this choice makes the two terms in $a(\tau)$ comparable near $x \sim 1$). Compute

$$\tau_\star \equiv \frac{C}{K} = \frac{a_{\rm eq}^2\sqrt{\frac{4\pi G}{3}\rho_{\rm eq}}}{\frac{2\pi G}{3}\rho_{\rm eq}a_{\rm eq}^3} = \frac{3}{2}\frac{\sqrt{\frac{4\pi G}{3}\rho_{\rm eq}}}{\pi G\rho_{\rm eq}a_{\rm eq}} = \frac{3}{2\pi G\rho_{\rm eq}a_{\rm eq}}\sqrt{\frac{4\pi G}{3}\rho_{\rm eq}}.$$

One may simplify algebraically; the important point is that $\tau_\star$ is a finite reference conformal time built from $a_{\rm eq}$ and $\rho_{\rm eq}$. Using $x = \tau/\tau_\star$ we can rewrite (79) as

$$a(\tau) = C\tau\left(1 + \frac{K\tau}{2C}\right) = C\tau\big(1 + \tfrac{1}{2}x\big),$$

or (dividing by $a_{\rm eq}$) one obtains a dimensionless shape function showing the transition between radiation and matter domination. The exact algebraic prefactors depend on the chosen definition of $\tau_\star$; the representation above makes it clear that for $\tau \ll \tau_\star$ the linear term dominates, while for $\tau \gg \tau_\star$ the quadratic term dominates.

Recovering the RD and MD limits

Radiation-dominated (RD) limit: $\tau \ll \tau_\star$. When τ is very small the τ-linear term in (79) dominates:

$$a(\tau) \approx C\tau \propto \tau.$$

Using $dt = a\,d\tau$, this corresponds to the known radiation solution $a(t) \propto t^{1/2}$.

Matter-dominated (MD) limit: $\tau \gg \tau_\star$. When τ is large the quadratic term dominates:

$$a(\tau) \approx \frac{\pi G}{3}\rho_{\rm eq}a_{\rm eq}^3\,\tau^2 \propto \tau^2,$$

which corresponds to $a(t) \propto t^{2/3}$ after converting back to cosmic time.

Thus the exact solution (79) interpolates smoothly between the radiation-dominated regime $a \propto \tau$ (or $a \propto t^{1/2}$) and the matter-dominated regime $a \propto \tau^2$ (or $a \propto t^{2/3}$), as required.

Alternative compact form (as often quoted)

Because the combination of prefactors above is somewhat bulky, many texts rewrite the solution in the equivalent compact form

$$a(\tau) \;=\; a_{\rm eq}\left[\left(\frac{\tau}{\tau_\star}\right) + \left(\frac{\tau}{\tau_\star}\right)^2\right], \tag{80}$$

with an appropriate definition of $\tau_\star$ (which collects the numerical factors built from π, G, $\rho_{\rm eq}$, $a_{\rm eq}$); one easily checks that this form is algebraically equivalent to (79) up to the chosen normalization. Using such a form makes the interpolation transparent: for $\tau \ll \tau_\star$, $a \propto \tau$; for $\tau \gg \tau_\star$, $a \propto \tau^2$.

Plotting the exact scale factor vs. conformal time (schematic)

Below is a PGFPlots figure that visualizes the exact analytic solution

$$a(\tau) = A\tau^2 + C\tau,$$

expressed in dimensionless units. For convenience we set $a_{\rm eq} = 1$ and choose units such that $4\pi G\rho_{\rm eq}/3 = 1$. In these units the constants reduce to $A = \frac{1}{3}$ and $C = 1$. This is for visualization only—the analytic derivation remains completely general.

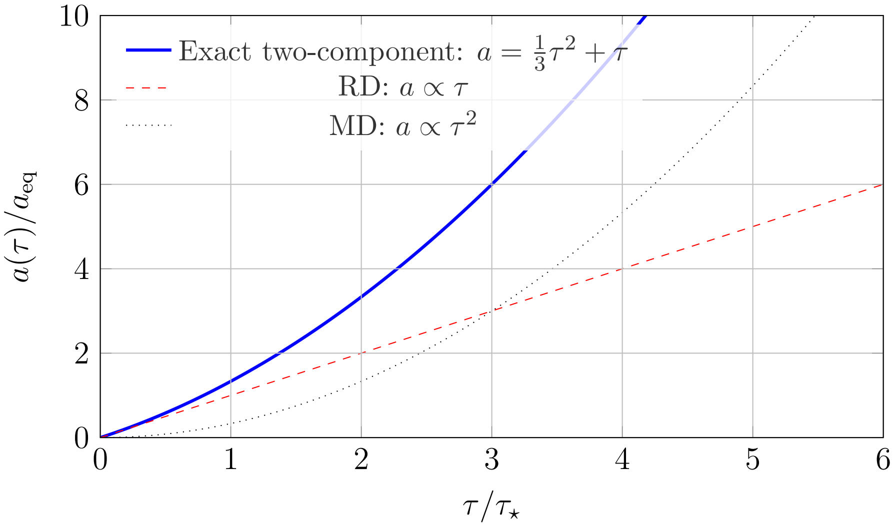

Figure 4: Schematic interpolation of the scale factor between radiation domination (linear in τ) and matter domination (quadratic in τ).

The plot clearly shows that:

- At early conformal times $\tau \to 0$, the τ-term dominates, giving $a(\tau) \propto \tau$, characteristic of radiation domination.
- At late conformal times, the τ^2 term dominates, giving $a(\tau) \propto \tau^2$, characteristic of matter domination.

Remarks

- In conformal time, the second Friedmann equation becomes $a'' = \frac{4\pi G}{3}(\rho - 3P)a^3$. Since radiation satisfies $\rho_r = 3P_r$, it drops out of the source term. Only matter ($P = 0$) contributes, giving a constant a''.
- A constant a'' implies a quadratic solution $a(\tau) = A\tau^2 + C\tau + D$. Imposing the Big Bang condition $a(0) = 0$ fixes $D = 0$.
- The remaining constant C is fixed by the first Friedmann equation evaluated at equality, giving the compact solution displayed above.