Advanced Data Structures

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# Random Binary Search Trees: An empirical analysis

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## 1 Introduction

Let $T$ be a binary tree with subtrees $T_l$ and $T_r$. We say that $T$ is a *binary search tree* (BST) if it is either an empty binary tree or it contains at least one element $x$ as its root such that

- $T_l$ and $T_r$ are also BSTs.
- $\forall y \in T_l, y < x$ and $\forall z \in T_r, z > x$.

Although it is well known that, in the worst case, a BST behaves like a linked list (with the height of the tree being $\Theta(n)$), in this report, we focus on *random BSTs* of size $n$.

By *random BSTs*, we mean the following: Given a universe of keys $U$ with $|U| = n$, we construct the BST by inserting each element of $U$ exactly once, choosing the insertion order uniformly at random.

## 2 Analysis of the Average Cost of Insertions

### 2.1 Theoretical Study

Let us first analyze the expected cost of inserting an element $u \in U$ into our BST $T$. For that, we will consider this cost as the cost of searching for $u$ in our BST, which is valid since we can assume that, if $u$ does not exist in $T$, our search terminates in any empty subtree with identical probability.

Let $I_n$ be the expected cost of the insertion of a key $x$ in a random BST of size $n$. Let, also, be $I_{n,q}$ be the expected cost of the insertion of a key $x$ in a random BST which root is the $q-$th smallest element. Then, the expected cost of $I_n$ is

$$\begin{aligned}
I_n &= \frac{1}{n}\sum_{q=1}^{n} I_{n,q} \\
&= 1 + \frac{1}{n}\sum_{k=1}^{n}(\frac{1}{n}0 + \frac{k-1}{n}I_{k-1} + \frac{n-k}{n}I_{n-k}) \\
&= 1 + \frac{1}{n}\sum_{k=0}^{n-1}(\frac{k}{n}I_k + \frac{n-k-1}{n}I_{n-k-1}) \\
&= 1 + \frac{1}{n^2}\sum_{k=0}^{n-1}(kI_k + (n-k-1)I_{n-k-1}) \\
&= 1 + \frac{2}{n^2}\sum_{k=0}^{n-1} kI_k
\end{aligned}$$

We can solve this recurrence using the continuous master theorem. The continuous master theorem solves recurrences of the form

$$F_n = t_n + \sum_{0 \le j < n} w_{n,j} F_j$$

with $t_n = \Theta(n^a (\log n)^b)$. We proceed as follows:

- Determine the values of $a$ and $b$: Since $t_n = \Theta(1)$, it is straightforward to see that $a = b = 0$.
- Provide a shape function for the weights $w_{n,j}$: We use the following trick to determine the shape function:

$$w(z) = \lim_{n\to\infty} n \cdot w_{n,z\cdot n} = n \cdot \frac{2zn}{n^2} = 2z.$$

- Determine the value of

$$\mathcal{H} = 1 - \int_0^1 w(z) z^a dz.$$

  Substituting the values, we obtain:

$$\mathcal{H} = 1 - \int_0^1 2z dz = 1 - (1 - 0) = 0.$$

- Since $\mathcal{H} = 0$, we need to compute

$$\mathcal{H}' = -(b+1) \int_0^1 w(z) z^a \ln z \, dz.$$

  Substituting the known values,

$$\mathcal{H}' = -1 \int_0^1 2z \ln z \, dz.$$

  This integral can be solved using integration by parts. For the purpose of applying the theorem, we skip the detailed calculation, giving the result:

$$\mathcal{H}' = -(x^2 \ln x - \frac{x^2}{2})\Big|_0^1 = \frac{1}{2}.$$

Since $\mathcal{H} = 0$ and $\mathcal{H}' \neq 0$, we use the result

$$F_n = \frac{t_n}{\mathcal{H}'} \ln n + o(t_n \log n).$$

Substituting the values, we obtain

$$I_n = 2 \ln n + o(\log n).$$

Thus, the expected cost of an insertion into a random binary search tree is bounded by $O(\log n)$.

## 2.2 Experimental Study

Once we have theoretical results on the expected cost of an insertion in a random BST, we can provide experimental results to assess how closely they match the theoretical predictions. For this, we will conduct the following experiment:

1. We create a random BST of size $n$ by generating $n$ random keys in the interval $[0, 1]$.
2. After constructing the BST, we generate $q = 2 \cdot n$ random numbers in the interval $[0, 1]$.
3. For each generated value, we perform a `find` operation in the BST, counting the number of nodes traversed during the operation.
4. We sum up the total number of nodes traversed across all $q$ search operations and compute the average.
5. We repeat all previous steps with 20 different seeds and compute the final average.

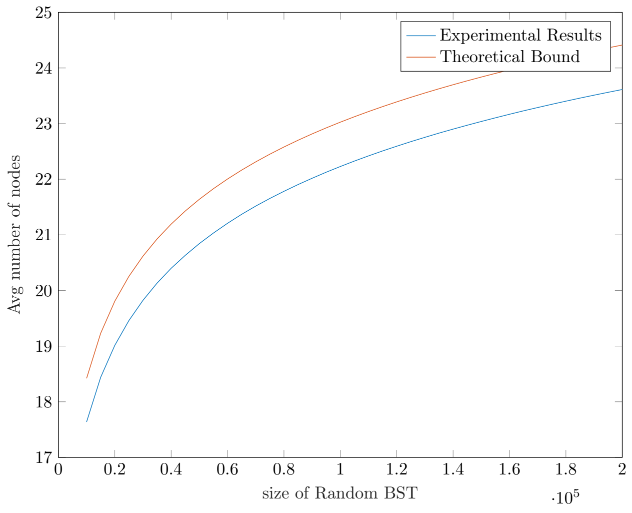


Figure 1: Plot average number of nodes visited respect theoretical bound

6. We repeat the entire experiment for different values of $n$.

I conducted the experiment previously explained with values of $n$ ranging from $10,000$ to $200,000$ in increments of $5,000$, using seeds to generate random numbers from 1989 (in honor of the year of the first publication of the book *Introduction to Algorithms*, which was a great resource for refreshing my knowledge of BSTs and expected cost) to 2008. You can execute exactly the same experiment by just executing the script `execInsertion.sh`. Figure 1 provides a plot of the values obtained from this experiment, as well as the theoretical bound derived using the continuous master theorem, indicating that the theoretical bound always bounds the experimental results. In fact, the shapes of both plots appear to be the same, suggesting that the only difference between the two functions is a constant factor. Indeed, by applying the continuous master theorem, we know that this difference is $o(\log n)$, but a more detailed analysis, presented in Appendix A shows that this difference is actually $O(1)$.

This constant factor is clearly visible in Table 1, where the column *Difference* (computed as the difference between the theoretical value obtained using Roura's theorem and the experimental value) consistently yields basically a constant difference.

| $n$ | Experimental | Theoretical | Difference |
|---|---|---|---|
| 10000 | 17.6373 | 18.4207 | 0.78335 |
| 15000 | 18.4401 | 19.2316 | 0.79148 |
| 20000 | 19.0132 | 19.807 | 0.79379 |
| 25000 | 19.4589 | 20.2533 | 0.79437 |
| 30000 | 19.8232 | 20.6179 | 0.79466 |
| 35000 | 20.1327 | 20.9262 | 0.79355 |
| 40000 | 20.3994 | 21.1933 | 0.79389 |
| 45000 | 20.6318 | 21.4288 | 0.79707 |
| 50000 | 20.8435 | 21.6396 | 0.79606 |
| 55000 | 21.0332 | 21.8302 | 0.79698 |
| 60000 | 21.2081 | 22.0042 | 0.79606 |
| 65000 | 21.3695 | 22.1643 | 0.79484 |
| 70000 | 21.5175 | 22.3125 | 0.79501 |
| 75000 | 21.6545 | 22.4505 | 0.79602 |
| 80000 | 21.7826 | 22.5796 | 0.79696 |
| 85000 | 21.9042 | 22.7008 | 0.79657 |
| 90000 | 22.0176 | 22.8151 | 0.79753 |
| 95000 | 22.1266 | 22.9233 | 0.79666 |
| 100000 | 22.2288 | 23.0259 | 0.7971 |
| 105000 | 22.3268 | 23.1234 | 0.79667 |
| 110000 | 22.4196 | 23.2165 | 0.79691 |
| 115000 | 22.5088 | 23.3054 | 0.79659 |
| 120000 | 22.5934 | 23.3905 | 0.79713 |
| 125000 | 22.6754 | 23.4721 | 0.7967 |
| 130000 | 22.7532 | 23.5506 | 0.79738 |
| 135000 | 22.8291 | 23.6261 | 0.79692 |
| 140000 | 22.901 | 23.6988 | 0.7978 |
| 145000 | 22.9706 | 23.769 | 0.79836 |
| 150000 | 23.0382 | 23.8368 | 0.79857 |
| 155000 | 23.1038 | 23.9024 | 0.79857 |
| 160000 | 23.1678 | 23.9659 | 0.79808 |
| 165000 | 23.2297 | 24.0274 | 0.79766 |
| 170000 | 23.2888 | 24.0871 | 0.79832 |
| 175000 | 23.3468 | 24.1451 | 0.79826 |
| 180000 | 23.4032 | 24.2014 | 0.79818 |
| 185000 | 23.4583 | 24.2562 | 0.79793 |
| 190000 | 23.5112 | 24.3096 | 0.79838 |
| 195000 | 23.5627 | 24.3615 | 0.79878 |
| 200000 | 23.6128 | 24.4121 | 0.79932 |

Table 1: Experimental vs Theoretical results on the expected cost of insertion

# 3 Analysis of Internal Path Length

## 3.1 Theoretical Study

Let $T$ be a BST of size $n$ with root $r$. We define the *Internal Path Length* of $T$ as the sum of all distances between every node of the tree and the root. More formally:

$$\text{IPL}(T) = \sum_{v \in V(T)} d(r, v).$$

This metric should not be surprising, as we can estimate the distance between the root and every node by averaging the IPL of such a tree. Before establishing this relationship, let us first derive a recurrence for the IPL of a tree, solve this recurrence, and later analyze the obtained relation.

Under the assumption of having a random BST, given $n$ keys, we consider every permutation among the $n!$ possible ones to be equally likely. Therefore, the sizes of the subtrees $T_l$ and $T_r$ will be completely random, provided that their combined sizes sum to $n-1$. Consequently, in our recurrence, we must consider every possible size distribution of the subtrees as equally likely.

Moreover, given the IPL of $T_l$ and $T_r$, since we introduce a new root at the top of the tree, every node will need to traverse one additional edge to reach the new root. This results in adding 1 to the previous distances with respect to their original root. As a consequence, we must add the number of edges in the tree to the sum of the previous IPLs of both subtrees.

Hence, creating the following recurrence:

$$\begin{aligned} IPL_n &= n - 1 + \frac{1}{n}\sum_{k=0}^{n-1} IPL_k + IPL_{n-1-k} \\ &= n - 1 + \frac{1}{n}\sum_{k=0}^{n-1} 2 \cdot IPL_k \\ &= n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} IPL_k \end{aligned}$$

We can, again, use the continuous master theorem in order to solve this recurrence. Again we need to determinate the following values:

- Determine the values of $a$ and $b$: Since $t_n = \Theta(n)$, it is straightforward to see that $a = 1$ and $b = 0$.
- Provide a shape function for the weights $w_{n,j}$: We use the following trick to determine the shape function:

$$w(z) = \lim_{n\to\infty} n \cdot w_{n,z\cdot n} = n \cdot \frac{2}{n} = 2.$$

- Determine the value of

$$\mathcal{H} = 1 - \int_0^1 w(z) z^a dz.$$

Substituting the values, we obtain:

$$\mathcal{H} = 1 - \int_0^1 2z dz = 1 - (1 - 0) = 0$$

- Since $\mathcal{H} < 0$ we are in the case

$$\mathcal{H}' = -(b+1) \int_0^1 w(z) z^a \ln z \, dz.$$

Substituting the known values,

$$\mathcal{H}' = -1 \int_0^1 2z \ln z \, dz.$$

This integral can be solved using integration by parts. For the purpose of applying the theorem, we skip the detailed calculation, giving the result:

$$\mathcal{H}' = -(x^2 \ln x - \frac{x^2}{2})\Big|_0^1 = \frac{1}{2}.$$

Since $\mathcal{H} = 0$ and $\mathcal{H}' \neq 0$, we use the result

$$F_n = \frac{t_n}{\mathcal{H}'} \ln n + o(t_n \log n).$$

Substituting the values, we obtain

$$I_n = 2(n-1) \ln n + o(n \log n) = 2n \ln n + o(n \log n)$$

Thus, the expected internal path length in a BST is bounded by $O(n \log n)$.

The IPL recurrence can be handy to obtain the expected number of nodes needed to traverse the BST to perform an insertion operation. Having a more acurrate solution of this recurrence can give us a more acurrate bound not only for the IPL but also for the insertion expected cost. Such solution can be found in Appendix A

## 3.2 Experimental Study

Once we have theoretical results on the expected internal path length in a random BST, we can provide experimental results to assess how closely they match the theoretical predictions. For this, we will conduct the following experiment:

1. We create a random BST of size $n$ by generating $n$ random keys in the interval $[0, 1]$.
2. We calculate the internal path length of the random BST by doing a Breadth-first search algorithm
3. We repeat all previous steps with 20 different seeds and compute the final average.
4. We repeat the entire experiment for different values of $n$.

Again, I conducted the experiment with the same value of $n$ and the same *seeds* as subsection 2.2. Figure 2 presents, as before, a plot comparing the expected values obtained from the continuous master theorem result with the experimental results. This time, however, the two curves do not exhibit similar behavior as observed previously. It appears that the theoretical bound increases faster than the experimental result, unlike the previous case. A more numerical perspective is provided in Table 2, where

| $n$ | Experimental | Theoretical | Difference |
|---|---|---|---|
| 10000 | 156245.35 | 184206.8074 | 27961.4574 |
| 15000 | 246662.85 | 288474.1644 | 41811.3144 |
| 20000 | 340332.35 | 396139.5021 | 55807.1521 |
| 25000 | 436473.15 | 506331.5552 | 69858.4052 |
| 30000 | 534628.65 | 618537.1596 | 83908.5096 |
| 35000 | 634383.5 | 732417.2338 | 98033.7338 |
| 40000 | 735610.7 | 847730.7786 | 112120.0786 |
| 45000 | 838045.5 | 964297.5992 | 126252.0992 |
| 50000 | 941527.05 | 1081977.8284 | 140450.7784 |
| 55000 | 1046067.1 | 1200659.7311 | 154592.6311 |
| 60000 | 1151365.35 | 1320251.9809 | 168886.6309 |
| 65000 | 1257575.2 | 1440678.5314 | 183103.3314 |
| 70000 | 1364456.25 | 1561875.0729 | 197418.8229 |
| 75000 | 1472133.3 | 1683786.5089 | 211653.2089 |
| 80000 | 1580482.2 | 1806365.1062 | 225882.9062 |
| 85000 | 1689287.05 | 1929569.111 | 240282.061 |
| 90000 | 1798726 | 2053361.6909 | 254635.6909 |
| 95000 | 1908744.3 | 2177710.1124 | 268965.8124 |
| 100000 | 2019274.8 | 2302585.093 | 283310.293 |
| 105000 | 2130221.85 | 2427960.2821 | 297738.4321 |
| 110000 | 2241658.35 | 2553811.8419 | 312153.4919 |
| 115000 | 2353417.6 | 2680118.1037 | 326700.5037 |
| 120000 | 2465595.3 | 2806859.2852 | 341263.9852 |
| 125000 | 2578168 | 2934017.2541 | 355849.2541 |
| 130000 | 2691101.95 | 3061575.3297 | 370473.3797 |
| 135000 | 2804431.8 | 3189518.1155 | 385086.3155 |
| 140000 | 2918219.4 | 3317831.3564 | 399611.9564 |
| 145000 | 3032270.35 | 3446501.8162 | 414231.4662 |
| 150000 | 3146588.05 | 3575517.1719 | 428929.1219 |
| 155000 | 3261131.6 | 3704865.9227 | 443734.3227 |
| 160000 | 3376159.9 | 3834537.3101 | 458377.4101 |
| 165000 | 3491454.05 | 3964521.2485 | 473067.1985 |
| 170000 | 3607009.65 | 4094808.2635 | 487798.6135 |
| 175000 | 3722815.9 | 4225389.4385 | 502573.5385 |
| 180000 | 3838806.45 | 4356256.3668 | 517449.9168 |
| 185000 | 3955127.75 | 4487401.1085 | 532273.3585 |
| 190000 | 4071748.65 | 4618816.1534 | 547067.5034 |
| 195000 | 4188621.75 | 4750494.3866 | 561872.6366 |
| 200000 | 4305674.95 | 4882429.0582 | 576754.1082 |

Table 2: Experimental vs Theoretical results on the expected Internal Path Length

we observe that the difference is no longer constant. In fact, the discrepancy increases as the input size grows.

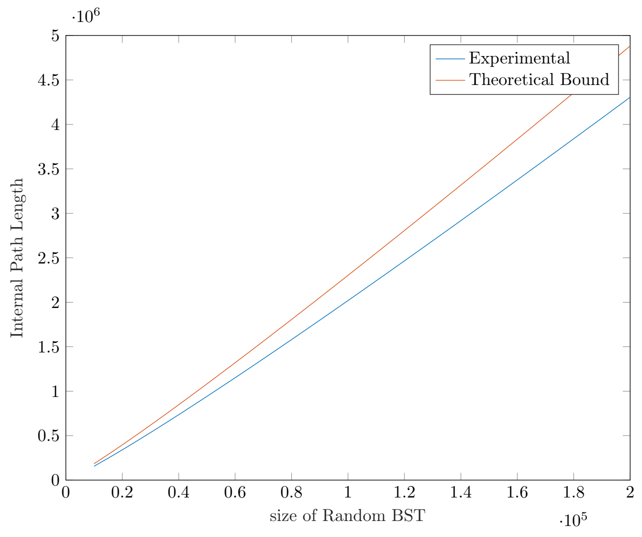


Figure 2: Plot average IPL respect the theoretical bound

We know that there is a factor $o(n \log n)$ (in fact, with a more detailed solution of the recurrence, see Appendix A, we know that this $o(n \log n)$ is $O(n)$) so it must be taken into account in our study. As a result, even with a high number of nodes in a random BST, this theoretical bound may still be influenced by the $o(n \log n)$ factor until we reach a significantly larger number of nodes. To analyze this effect, we provide Figure 3, where we divide the experimental data by $n \log n$ to observe when this $o(n \log n)$ factor stabilizes. As seen in the figure, even for large input sizes (over 200,000 nodes), this factor still plays a significant role in our analysis.

So, that $o(n \log n)$ factor must be the reason why it is difficult to predict a more accurate theoretical

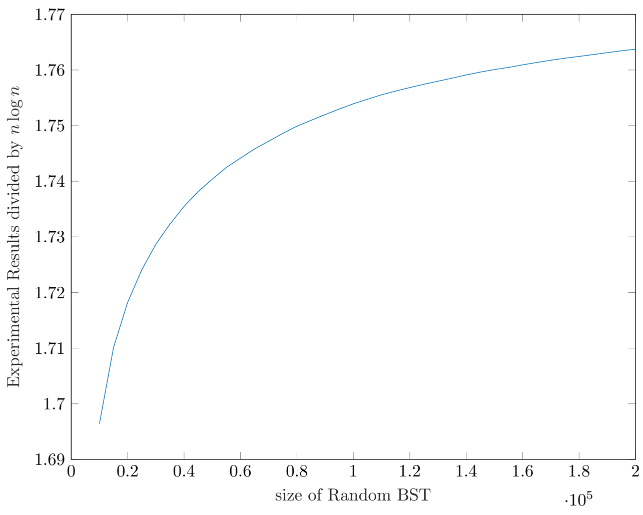


Figure 3: Plot experimental result divided by $n \log n$

solution to the IPL: because this factor is very influential even in instances with more than $100,000$ nodes, and we need to reach an even larger size of random BSTs to make this factor smaller than the actual bound.

## 4 Analysis of Interleaver Insertions and Deletions

Knuth, in Volume 3 of his well-known book *The Art of Computer Programming* [1], mentioned that random insertions and deletions indeed *destroy* the randomness of a BST. Knott[2] and Culberson & Munro[3] noticed that, after doing such operations, while the structure of a tree is probabilistically the same, the distribution of keys is not. Eppinger[4] demonstrated experimentally that the path length, although initially decreasing slightly, eventually increases and stabilizes after performing a quadratic number of deletions and insertions. Many other articles have also appeared in the literature[5, 6]

The following experiment aims to support the observations made by Eppinger and Knuth:

1. We create a random BST of size $n$ by generating $n$ random keys in the interval $[0, 1]$.
2. We perform a quadratic number of insertions and deletions as follows: we alternately insert a random element from the interval $[0, 1]$ and delete a random element from the BST.
3. We compute the internal path length of the BST using a Breadth-First Search algorithm.
4. We repeat all the previous steps with 20 different seeds and compute the final average.
5. We repeat the entire experiment for different values of $n$.

This time, as we are doing a quadratic number of steps, because of memory and time performance I have decided to conduct this experient with Random BSTs of size 1000 to 2000 in increases of 50 elements (same seeds as section 2.2). Figure 4 provides a plot of the final IPL with and without doing this alternation between deletions and insertion. As expected, we see that, indeed, the IPL of a tree increases if we perform this operations, indicating that we are unbalancing our random BST (hence, destroying our randomness).

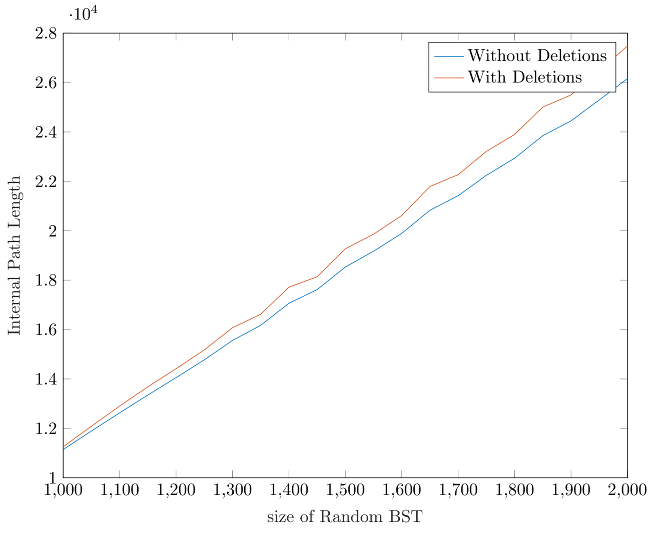


Figure 4: Plot average IPL with and without alternating insertions and deletions

Table 3 provides a more numerical view of such plot. As we see, the difference in IPL gets bigger as we increases the size of $n$, definitely destroying the initial randomness that we created.

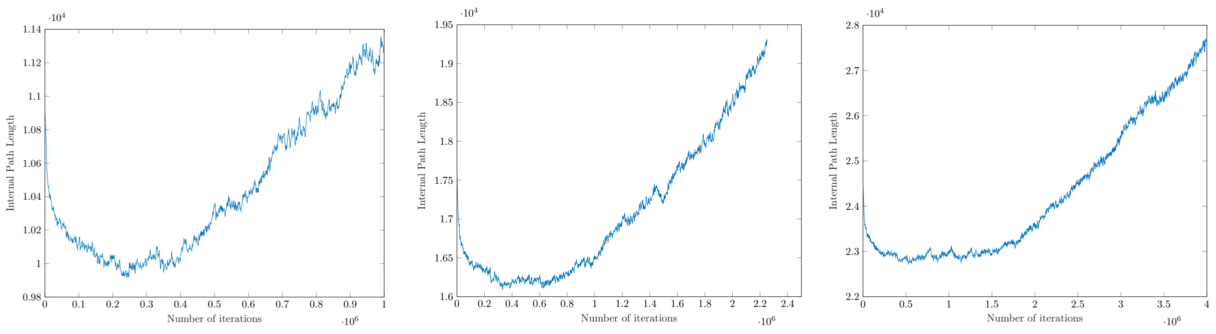
| $n$ | No Deletions | Deletions | Difference |
|---|---|---|---|
| 1000 | 11145.85 | 11252.15 | 106.3 |
| 1050 | 11890.2 | 12086.95 | 196.75 |
| 1100 | 12627.65 | 12900.5 | 272.85 |
| 1150 | 13348.65 | 13680.2 | 331.55 |
| 1200 | 14053.475 | 14414.25 | 360.775 |
| 1250 | 14773.025 | 15172.7 | 399.675 |
| 1300 | 15557.95 | 16072.8 | 514.85 |
| 1350 | 16171.5 | 16617.6 | 446.1 |
| 1400 | 17062.5 | 17715.25 | 652.75 |
| 1450 | 17615.975 | 18136.7 | 520.725 |
| 1500 | 18530.425 | 19268.7 | 738.275 |
| 1550 | 19169.875 | 19860.2 | 690.325 |
| 1600 | 19894.3 | 20618.35 | 724.05 |
| 1650 | 20830.225 | 21792 | 961.775 |
| 1700 | 21419.875 | 22272.1 | 852.225 |
| 1750 | 22243.3 | 23213 | 969.7 |
| 1800 | 22937.55 | 23899.25 | 961.7 |
| 1850 | 23848.45 | 25009.05 | 1160.6 |
| 1900 | 24447.525 | 25495.45 | 1047.925 |
| 1950 | 25295.475 | 26473.65 | 1178.175 |
| 2000 | 26161.125 | 27483.5 | 1322.375 |

Table 3: Difference of IPL after doing deletions

For a more detailed view of Eppinger's work we provide the following experiment:

1. We create a random BST of size $n$ by generating $n$ random keys in the interval $[0, 1]$.
2. We perform a quadratic number of insertions and deletions as follows: we alternately insert a random element from the interval $[0, 1]$ and delete a random element from the BST.
3. We compute the internal path length of the BST using a Breadth-First Search algorithm every $n$ iterations.
4. We repeat all the previous steps with 20 different seeds and compute the final average.
5. We repeat the entire experiment for different values of $n \in \{1000, 1500, 2000\}$.

Figure 5 provides a plot of such experiment. Indeed, we see what it was said: The IPL decrease slightly at the beginning but, after that, it starts increasing more than what we had at the beginning. Again, indicating that we are unbalancing our tree and losing our randomness.



(a) Evolution of internal path length by doing insertions and deletions with $n = 1000$

(b) Evolution of internal path length by doing insertions and deletions with $n = 1500$

(c) Evolution of internal path length by doing insertions and deletions with $n = 2000$

Figure 5: Evolution of IPLs by doing insertions and deletions

## 5 Conclusion

Regarding the theoretical analysis of recurrences, although the continuous master theorem is a powerful tool for solving recurrences, it may not always provide a more accurate bound than solving them explicitly.

Regarding the practical part, randomness is a powerful tool for keeping a random BST balanced, but we must be careful in certain situations.

Last but not least, I would like to mention that this last section challenged my intuition: I was expecting that performing random insertions and deletions would not affect the structure of our tree, as every subtree is equally likely.

Special thanks to Conrado's slides on Data Structures and Algorithms, which, along with the book *Introduction to Algorithms*, were a great resource for refreshing my knowledge and assisting me when I struggled with certain calculations!

## A More accurate solutions to the previous recurrences

Let us consider the previous recurrence for the IPL of a Random BST:

$$I_n = n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} I_k$$

We have to try to put this recurrence in terms of just one call instead of multiple calls of it. For that we need to firstly see that:

$$I_{n+1} = n + \frac{2}{n+1}\sum_{k=0}^{n} I_k$$

Now we need to subtract carefully both recurrences. For that, let us first multiply each recurrence by $n$ and $n+1$:

$$\begin{aligned}(n+1)I_{n+1} &= (n+1)n + 2\sum_{k=0}^{n} I_k \\ nI_n &= n^2 - n + 2\sum_{k=0}^{n-1} I_k\end{aligned}$$

Now we are ready to subtract both recurrences:

$$\begin{aligned}(n+1)IPL_{n+1} - nIPL_n &= (n+1)n + 2IPL_n - n^2 + n \\ (n+1)IPL_{n+1} &= n^2 + n + 2IPL_n - n^2 + n + nIPL_n \\ (n+1)IPL_{n+1} &= 2n + 2IPL_n + nIPL_n = 2n + (2+n)IPL_n \\ IPL_{n+1} &= \frac{2n}{n+1} + \frac{2+n}{n+1}IPL_n \\ &= \frac{2n}{n+1} + \frac{2(2+n)(n-1)}{(n+1)n} + \frac{(2+n)(n+1)}{(n+1)n}IPL_{n-1} \\ &= \frac{2n}{n+1} + \frac{2(2+n)(n-1)}{(n+1)n} + \frac{2+n}{n}(\frac{2(n-2)}{n-1} + \frac{n}{n-1}IPL_{n-2}) \\ &= \frac{2n}{n+1} + 2(n+2)\sum_{i=1}^{n}\frac{n-i}{(n-i+1)(n-i+2)} \\ &= \frac{2n}{n+1} + 2(n+2)\sum_{i=1}^{n}\frac{i}{(i+1)(i+2)} \\ &= \frac{2n}{n+1} + 2(n+2)\sum_{i=1}^{n}\frac{2}{i+2} - \frac{1}{i+1} \\ &= \frac{2n}{n+1} + 2(n+2)(\frac{2}{n+2} + \frac{1}{n+1} + H_n - 2) \\ &= 2nH_n - 4n + 4H_n + O(1)\end{aligned}$$

We know that harmonic numbers grow similarly to the natural logarithm, so $H_n = \ln n + O(1)$. Hence, $I_n = 2n\ln n + O(n)$. Using the IPL, we can estimate the average number of nodes required for an insertion by averaging the IPL over the number of nodes. This gives the following estimation:

$$I_n = 1 + \frac{IPL_n}{n} = 2\ln n + O(1).$$

## B Main file implementation

We have created three different main files for each different section. Both will require as an input the size of the BST as well as a seed for generating random numbers. Then a random BST is going to be created and it will perform a single execution of the required experiment.

### B.1 Average Cost of Insertions

The provided `Makefile` can be used to compile this code. To do so, execute the following command in the terminal: `make insertion`. This will compile the main file along with the BST class.

To run the executable, the size and the seed must also be provided via the command line, following the format:

`./insertion n seed`.

```
#include <iostream>
#include "bst.hh"
using namespace std;

int main(int argc, char** argv) {

    unsigned int n = atoi(argv[1]);
    unsigned int seed = atoi(argv[2]);

    BST t;

    srand(seed);
    for (unsigned int i = 0; i < n; ++i) t.insert(((double)rand()) / RAND_MAX);


    unsigned int q = 2*n;
    unsigned int tpl = 0;
    for (unsigned int i = 0; i < q; ++i) tpl += t.find(((double)rand()) / RAND_MAX);

    cout << n << "," << float(tpl) / q << endl;
}
```

### B.2 Analysis of Internal Path Length

The provided `Makefile` can be used to compile this code. To do so, execute the following command in the terminal: `make ipl`. This will compile the main file along with the BST class.

To run the executable, the size and the seed must also be provided via the command line, following the format:

`./ipl n seed`.

```
#include <iostream>
#include "bst.hh"
using namespace std;

int main(int argc, char** argv) {
    unsigned int n = atoi(argv[1]);
    unsigned int seed = atoi(argv[2]);

    BST t;

    srand(seed);
    for (unsigned int i = 0; i < n; ++i) t.insert(((double)rand()) / RAND_MAX);

    cout << n << "," << t.ipl() << endl;
}
```

## B.3 Analysis of Interleaved Insertions and Deletions

The provided `Makefile` can be used to compile this code. In this case, we either want to obtain the IPL of the final BST or track instances of the IPL to observe how it evolves during the procedure. To compute the final IPL, compile the code using `make delete`. Otherwise, to track the IPL evolution, compile using `make deleteSingle`.

To execute the program, we also need to specify the size and the seed via the command line, following the format:
`./delete n seed` or `./deleteSingle n seed`.

```
#include <iostream>
#include <random>
#include "bst.hh"
using namespace std;

int main(int argc, char** argv) {

    unsigned int n = atoi(argv[1]);
    unsigned int seed = atoi(argv[2]);

    float *v = (float *) malloc(n*n*sizeof(float));

    BST t;
    unsigned int lastElem = n - 1;

    mt19937 generator(seed);
    uniform_real_distribution<float> distribution(0.0, 1.0);
    for (unsigned int i = 0; i < n; ++i) {
        v[i] = distribution(generator);
        t.insert(v[i]);
    }

    unsigned int erasedElems = 0;
    srand(seed);
    bool ins = true;
    unsigned int count = 0;
    for (unsigned int i = 0; i < n*n; ++i) {

        if (ins) {
            ++lastElem;
            v[lastElem] = distribution(generator);
            t.insert(v[lastElem]);
        }

        else {
            unsigned int cand = rand() % (lastElem - erasedElems + 1) + erasedElems;
            float aux = v[cand];
            v[cand] = v[erasedElems];
            v[erasedElems] = aux;
            t.erase(aux);
            ++erasedElems;
        }

        ins = !ins;
        ++count;
#ifdef SINGLE
        if (count == n) {
            count = 0;
            cout << i << "," << t.ipl() << endl;
        }
#endif
    }

#ifndef SINGLE
    cout << n << "," << t.ipl() << endl;
#endif
    free(v);
}
```

## C Binary Search Tree implementation

A C++ implementation has been developed for efficiency purposes in the BST. A C++ class has been created to represent the Binary Search Tree data structure (refer to `bst.hh` for a more detailed exploration of different methods for the class). It primarily consists of a node that stores its value along with two pointers to its respective subtrees. In C++, this can be achieved by using a struct that holds two pointers to its own type.

### C.1 Find operation

```
unsigned int BST::find(float x, node* n) {
    if (n == nullptr) return 0;
    if (n -> x == x) return 1;
    if (n -> x > x) return 1 + find(x, n->l);
    else return 1 + find(x, n->r);
}
```

### C.2 Insert operation

```
BST::node* BST::insert(float x, node* n) {

    if (n == nullptr) {
        node* newNode = new node;
        newNode -> x = x;
        return newNode;
    }

    else if (n -> x == x) return n;

    else {
        if (n -> x > x) n -> l = insert(x, n -> l);
        else n -> r = insert(x, n -> r);
        return n;
    }
}
```

## C.3 Erase Operation

```
BST::node* BST::erase(node* n, float x) {

    if (n == nullptr) return nullptr;

    if (n -> x < x) {
        n -> r = erase(n -> r, x);
        return n;
    }

    if (n -> x > x) {
        n -> l = erase(n -> l, x);
        return n;
    }

    if (n -> l == nullptr) {
        node* aux = n -> r;
        delete n;
        return aux;
    }

    if (n -> r == nullptr) {
        node* aux = n -> l;
        delete n;
        return aux;
    }

    float val = 0;
    n -> r  = eraseSuccessor(n -> r, val);
    n -> x = val;
    return n;
}

BST::node* BST::eraseSuccessor(node* n, float& x) {

    if (n -> l == nullptr) {
        node* aux = n -> r;
        x = n -> x;
        delete n;
        return aux;
    }

    else {
        n -> l = eraseSuccessor(n -> l, x);
        return n;
    }
}
```

## C.4 Internal Path Length calculation

```
unsigned int BST::ipl() {
    queue<pair<node*,unsigned int>> q;
    unsigned int iplLocal = 0;
    q.push({root,0});

    while (not q.empty()) {
        pair<node*, unsigned int> p = q.front();
        q.pop();

        if (p.first != nullptr) {
            unsigned int d = p.second;
            iplLocal += d;
            q.push({p.first -> l, d+1});
            q.push({p.first -> r, d+1});
        }
    }

    return iplLocal;
}
```

## References

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