An Experimental Guided Approach to the Metric Dimension on Different Graph Families

Alex Herrero Bravo
Director: Antoni Lozano Boixadors
Department of Computer Science

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat d'Informàtica de Barcelona

Defense Date: January 26, 2024

Table of Contents

Introduction

Are graphs important in Computer Science?

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

- k-Coloring

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

- k-Coloring
- Vertex Cover

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

- k-Coloring
- Vertex Cover
- Clique

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

- k-Coloring
- Vertex Cover
- Clique
- Dominating Set

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

- k-Coloring
- Vertex Cover
- Clique
- Dominating Set
- And many more in the literature!

Hint: Want to win \$1M? Solve one of these problems in polynomial time!

Introduction

Are graphs important in Computer Science? The P vs NP question relies on solving or proving no polynomial algorithm exists for various graph problems:

- k-Coloring
- Vertex Cover
- Clique
- Dominating Set
- Metric Dimension

Metric Dimensions appears on Garey and Johnson's Book *Computers and Intractability: A Guide to the Theory of NP-Completeness*!

Introduction

Slater 1975, Harary-Melter 1976

1. The Metric Representation of a vertex $u \in V$ respect a subset $W \subseteq V$ where $W = \{w_1, w_2, \ldots, w_n\}$ is an ordered tuple defined as $r(u|W) = (d(w_1, u), d(w_2, u), \ldots, d(w_n, u))$

Introduction

Slater 1975, Harary-Melter 1976

1. The Metric Representation of a vertex $u \in V$ respect a subset $W \subseteq V$ where $W = \{w_1, w_2, \ldots, w_n\}$ is an ordered tuple defined as $r(u|W) = (d(w_1, u), d(w_2, u), \ldots, d(w_n, u))$
2. A set $W \subseteq V$ is called Resolving Set if and only if $\forall i, j \in V, \quad \exists x \in W : d(x, i) \neq d(x, j)$

Introduction

Slater 1975, Harary-Melter 1976

1. The Metric Representation of a vertex $u \in V$ respect a subset $W \subseteq V$ where $W = \{w_1, w_2, \ldots, w_n\}$ is an ordered tuple defined as $r(u|W) = (d(w_1, u), d(w_2, u), \ldots, d(w_n, u))$
2. A set $W \subseteq V$ is called Resolving Set if and only if $\forall i, j \in V, \quad \exists x \in W : d(x, i) \neq d(x, j)$

NP-Completeness

Given an arbitrary graph $G = (V, E)$ and an integer k, deciding whether $\beta(G) \leq k$ is NP-complete.

Proof: By a reduction from 3-SAT[a].

[a] Khuller, Raghavachari, and Rosenfeld, "Landmarks in graphs".

Introduction

Slater 1975, Harary-Melter 1976

1. The Metric Representation of a vertex $u \in V$ respect a subset $W \subseteq V$ where $W = \{w_1, w_2, \ldots, w_n\}$ is an ordered tuple defined as $r(u|W) = (d(w_1, u), d(w_2, u), \ldots, d(w_n, u))$
2. A set $W \subseteq V$ is called Resolving Set if and only if $\forall i, j \in V, \quad \exists x \in W : d(x, i) \neq d(x, j)$
3. The Metric Dimension of a graph G is the cardinality of the smallest **Resolving Set** of G, denoted as $\beta(G)$.

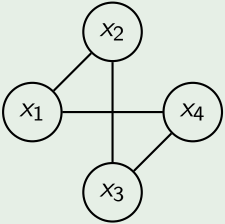
NP-Completeness

Given an arbitrary graph $G = (V, E)$ and an integer k, deciding whether $\beta(G) \leq k$ is NP-Complete.
Proof: By a reduction from 3-SAT[a].

[a] Khuller, Raghavachari, and Rosenfeld, "Landmarks in graphs".

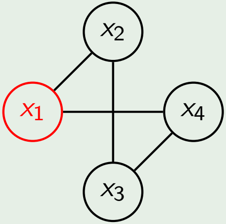
Introduction

Example (Metric Dimension)



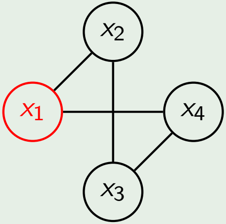
Introduction

Example (Metric Dimension)



Introduction

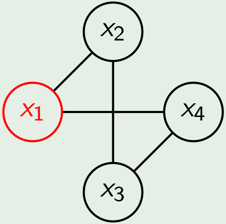
Example (Metric Dimension)



$$r(x_2|\{x_1\}) = (d(x_1, x_2)) = (1)$$

Introduction

Example (Metric Dimension)

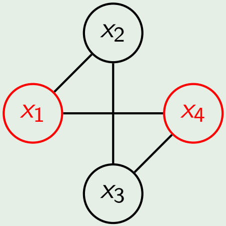


$$r(x_2|\{x_1\}) = (d(x_1, x_2)) = (1)$$
$$r(x_4|\{x_1\}) = (d(x_1, x_4)) = (1)$$

$r(x_2|\{x_1\})$ and $r(x_4|\{x_1\})$ has to be different.

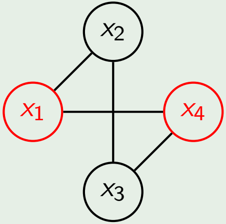
Introduction

Example (Metric Dimension)



Introduction

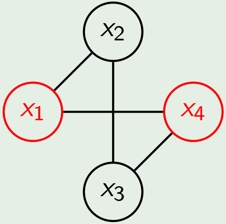
Example (Metric Dimension)



$$r(x_2|\{x_1, x_4\}) = (d(x_1, x_2), d(x_4, x_2)) = (1, 2)$$

Introduction

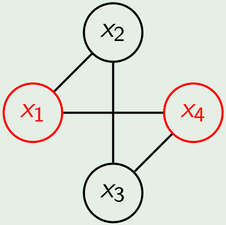
Example (Metric Dimension)



$$r(x_2|\{x_1, x_4\}) = (d(x_1, x_2), d(x_4, x_2)) = (1, 2)$$
$$r(x_3|\{x_1, x_4\}) = (d(x_1, x_3), d(x_4, x_3)) = (2, 1)$$

Introduction

Example (Metric Dimension)

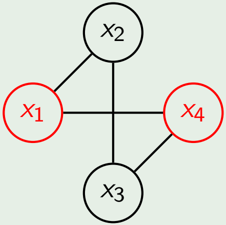


$$r(x_2|\{x_1, x_4\}) = (d(x_1, x_2), d(x_4, x_2)) = (1, 2)$$
$$r(x_3|\{x_1, x_4\}) = (d(x_1, x_3), d(x_4, x_3)) = (2, 1)$$
$$r(x_1|\{x_1, x_4\}) = (d(x_1, x_1), d(x_4, x_1)) = (0, 1)$$
$$r(x_4|\{x_1, x_4\}) = (d(x_1, x_4), d(x_4, x_4)) = (1, 0)$$

Metric Representations are all unique!

Introduction

Example (Metric Dimension)



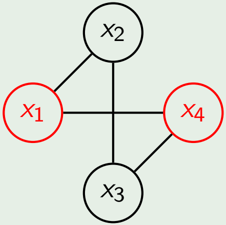
$$r(x_2|\{x_1, x_4\}) = (d(x_1, x_2), d(x_4, x_2)) = (1, 2)$$
$$r(x_3|\{x_1, x_4\}) = (d(x_1, x_3), d(x_4, x_3)) = (2, 1)$$
$$r(x_1|\{x_1, x_4\}) = (d(x_1, x_1), d(x_4, x_1)) = (0, 1)$$
$$r(x_4|\{x_1, x_4\}) = (d(x_1, x_4), d(x_4, x_4)) = (1, 0)$$

Metric Representations are all unique!

$W = \{x_1, x_4\}$ is a Resolving Set.

Introduction

Example (Metric Dimension)



$$r(x_2|\{x_1, x_4\}) = (d(x_1, x_2), d(x_4, x_2)) = (1, 2)$$
$$r(x_3|\{x_1, x_4\}) = (d(x_1, x_3), d(x_4, x_3)) = (2, 1)$$
$$r(x_1|\{x_1, x_4\}) = (d(x_1, x_1), d(x_4, x_1)) = (0, 1)$$
$$r(x_4|\{x_1, x_4\}) = (d(x_1, x_4), d(x_4, x_4)) = (1, 0)$$

Metric Representations are all unique!

$W = \{x_1, x_4\}$ is a Resolving Set. $\beta(G) = 2$

Goal of this project: **Contribute** to the Metric Dimension problem.

There are two ways to contribute:

Goal of this project: **Contribute** to the Metric Dimension problem.

There are two ways to contribute:

- Purely theoretical work: Initially, this was our preferred approach. However, due to time constraints and my skill level, there was a possibility that we might not discover anything particularly interesting.

Goal of this project: **Contribute** to the Metric Dimension problem.

There are two ways to contribute:

- Purely theoretical work: Initially, this was our preferred approach. However, due to time constraints and my skill level, there was a possibility that we might not discover anything particularly interesting.
- Using experiments: Experiments could **guide** us to the theoretical part. In the worst case, our experiments could serve as a starting point for other researchers.

Goal of this project: **Contribute** to the Metric Dimension problem.

There are two ways to contribute:

- Purely theoretical work: Initially, this was our preferred approach. However, due to time constraints and my skill level, there was a possibility that we might not discover anything particularly interesting.
- Using experiments: Experiments could **guide** us to the theoretical part. In the worst case, our experiments could serve as a starting point for other researchers.

Opting for experiments is the safest option! We can perform theoretical work based on experimental results and have substantial findings to present.

Experiments could help us in

- Improving bounds
- Conjecture Testing
- Study New Graphs Families
- Stimulate future research

How?

How? Use *Integer Linear Programming solvers*!

Integer Linear Programming

$$
\begin{aligned}
\text{Maximize or Minimize:} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{Subject to:} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \\
& x_i \in \mathbb{Z}, \quad i = 1, 2, \ldots, n
\end{aligned}
$$

Integer Linear Programming is NP-Complete, but highly optimized software is available for solving ILP instances.

Summary of the project

- **Theoretical Work:**
  - Graph Family: Tournaments
  - Graph Family: Bicyclic graphs
- **Experimental Work:**
  - Integer Linear Programming vs Weighted Max-SAT
  - Graph Family: Hypercube graphs

Outline

Tournaments

Definition

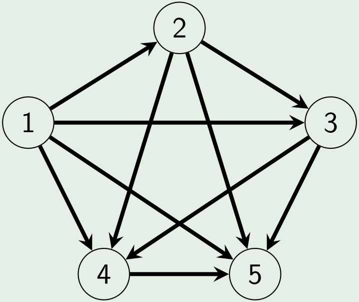
A Tournament $T = (V, E)$ is a directed graph where each pair of vertices are connected by one arc.

Tournaments

Definition

A Tournament $T = (V, E)$ is a directed graph where each pair of vertices are connected by one arc.

Example



Related Studies

Chartrand, Raines, and Zhang (2001)

There is no constant positive k that bounds the Metric Dimension of Tournaments.[a]

[a] Gary Chartrand, Michael Raines, and Ping Zhang. "On the dimension of oriented graphs". In: *Utilitas Mathematica* 60 (Nov. 2001).

Related Studies

Chartrand, Raines, and Zhang (2001)

There is no constant positive k that bounds the Metric Dimension of Tournaments.[a]

[a] Chartrand, Raines, and Zhang, "On the dimension of oriented graphs".

A. Lozano (2013)

For a **strong** tournament T, the optimal upper bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.[a]

[a] Lozano, "Symmetry Breaking in Tournaments".

Our results

Our results

A. Herrero and A. Lozano (2023)

For a **tournament** T, the optimal bound is given by
$\beta(T) \leq \lfloor n/2 \rfloor$.

Our results

A. Herrero and A. Lozano (2023)

For a **tournament** T, the optimal bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.

A. Herrero and A. Lozano (2023)

Characterization of tournaments with $\beta(T) = 1$.

Source: Tournament samples taken from `https://users.cecs.anu.edu.au/~bdm/data/` by Professor McKay.

Metric Dimension values by myself.

n	$\beta(G) = 1$	$\beta(G) = 2$	$\beta(G) = 3$	$\beta(G) = 4$	$\beta(G) = 5$	$\beta(G) = 6$	$\beta(G) = 7$	$\beta(G) = 8$
2	1	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0
4	2	2	0	0	0	0	0	0
5	2	10	0	0	0	0	0	0
6	2	49	5	0	0	0	0	0
7	2	348	106	0	0	0	0	0
8	2	2581	4286	11	0	0	0	0
9	2	16809	174188	537	0	0	0	0

Table: Metric Dimension values for tournaments of different sizes (self-elaborated)

Characterization

A. Herrero and A. Lozano (2023)

Characterization of tournaments with $\beta(T) = 1$.

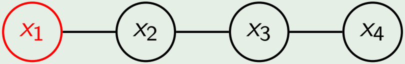
Characterization

Characterization

Well-known result in Metric Dimension

Let G be a simple graph. Then $\beta(G) = 1 \iff G \cong P_n$.

Example (Metric Dimension of Paths)

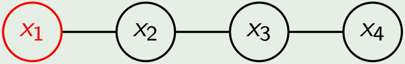


Characterization

Well-known result in Metric Dimension

Let G be a simple graph. Then $\beta(G) = 1 \iff G \cong P_n$.

Example (Metric Dimension of Paths)



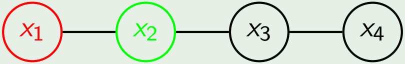
$$r(x_1|\{x_1\}) = (d(x_1, x_1)) = (0)$$

Characterization

Well-known result in Metric Dimension

Let G be a simple graph. Then $\beta(G) = 1 \iff G \cong P_n$.

Example (Metric Dimension of Paths)



$$r(x_1|\{x_1\}) = (d(x_1, x_1)) = (0)$$

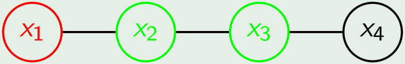
$$r(x_2|\{x_1\}) = (d(x_1, x_2)) = (1)$$

Characterization

Well-known result in Metric Dimension

Let G be a simple graph. Then $\beta(G) = 1 \iff G \cong P_n$.

Example (Metric Dimension of Paths)



$$r(x_1|\{x_1\}) = (d(x_1, x_1)) = (0)$$

$$r(x_2|\{x_1\}) = (d(x_1, x_2)) = (1)$$

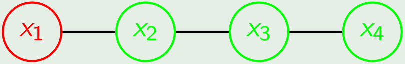
$$r(x_3|\{x_1\}) = (d(x_1, x_3)) = (2)$$

Characterization

Well-known result in Metric Dimension

Let G be a simple graph. Then $\beta(G) = 1 \iff G \cong P_n$.

Example (Metric Dimension of Paths)

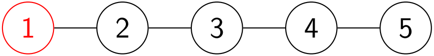


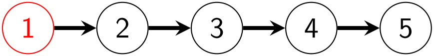
$$r(x_1|\{x_1\}) = (d(x_1, x_1)) = (0)$$

$$r(x_2|\{x_1\}) = (d(x_1, x_2)) = (1)$$

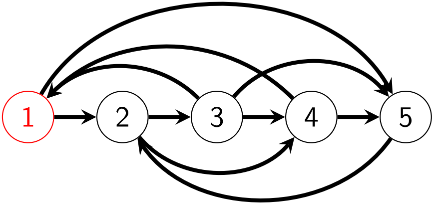
$$r(x_3|\{x_1\}) = (d(x_1, x_3)) = (2)$$

$$r(x_4|\{x_1\}) = (d(x_1, x_4)) = (3)$$


1
2
3
4
5



- Tournaments are directed graphs



- Tournaments are directed graphs
- Every pair of vertices is connected by one arc

Find the orientation to have the same property of paths!

Let's define $G_1 = (V, E)$ as a tournament with $V = \{0, 1, \ldots, n-1\}$.

Let's define $G_1 = (V, E)$ as a tournament with $V = \{0, 1, \ldots, n-1\}$.

Every arc (i, j) must adhere to one of these **two** rules:

Let's define $G_1 = (V, E)$ as a tournament with $V = \{0, 1, \ldots, n-1\}$.

Every arc (i, j) must adhere to one of these **two** rules:

- $j - i = 1 \rightarrow$ First Rule.

Let's define $G_1 = (V, E)$ as a tournament with $V = \{0, 1, \ldots, n-1\}$.

Every arc (i, j) must adhere to one of these **two** rules:

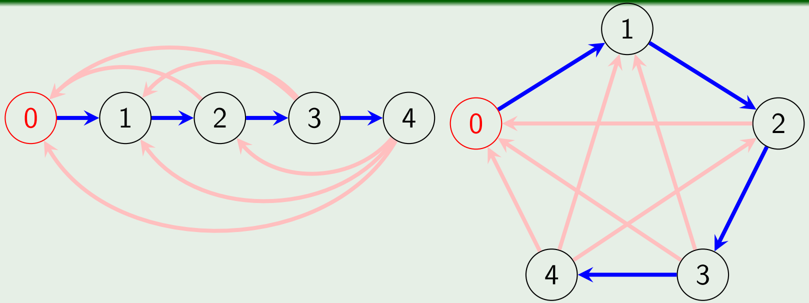
- $j - i = 1 \rightarrow$ First Rule.
- $i - j \geq 2 \rightarrow$ Second Rule.
- $E = \{(i, j) : j - i = 1 \vee i - j \geq 2\}$

Let's define $G_1 = (V, E)$ as a tournament with $V = \{0, 1, \ldots, n-1\}$.

Every arc (i, j) must adhere to one of these **two** rules:

- $j - i = 1 \rightarrow$ First Rule.
- $i - j \geq 2 \rightarrow$ Second Rule.
- $E = \{(i, j) : j - i = 1 \lor i - j \geq 2\}$

Example



Lemma

$$d(0, k) = k, k \in V \text{ in } G_1.$$

Proof:

Lemma

$$d(0, k) = k, k \in V \text{ in } G_1.$$

Proof:

- Clearly $d(0, k) \leq k$ because the existence of a path.

Lemma

$$d(0,k) = k, k \in V \text{ in } G_1.$$

Proof:

- Clearly $d(0,k) \leq k$ because the existence of a path.
- Suppose we have a shortest path from 0 to k of the form $v_0, v_1, \ldots, v_t$ where $v_0 = 0$ and $v_t = k$. By the definition of G_1 we have $v_{i+1} \leq v_i + 1$ for $0 \leq i < t$. Therefore $v_t \leq v_{t-1} + 1 \leq v_{t-2} + 2 \leq \ldots \leq v_0 + t = t$. Since $v_t = k$ we conclude that $d(0,k) \geq k$.

Lemma

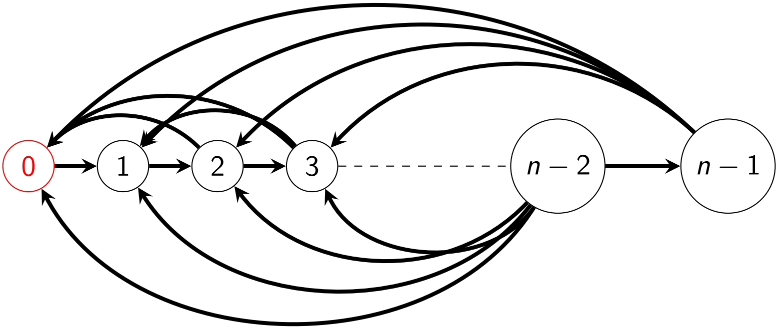
$$d(0, k) = k, k \in V \text{ in } G_1.$$

Proof:

- Clearly $d(0, k) \leq k$ because the existence of a path.
- Suppose we have a shortest path from 0 to k of the form $v_0, v_1, \ldots, v_t$ where $v_0 = 0$ and $v_t = k$. By the definition of G_1 we have $v_{i+1} \leq v_i + 1$ for $0 \leq i < t$. Therefore $v_t \leq v_{t-1} + 1 \leq v_{t-2} + 2 \leq \ldots \leq v_0 + t = t$. Since $v_t = k$ we conclude that $d(0, k) \geq k$.

Corollary

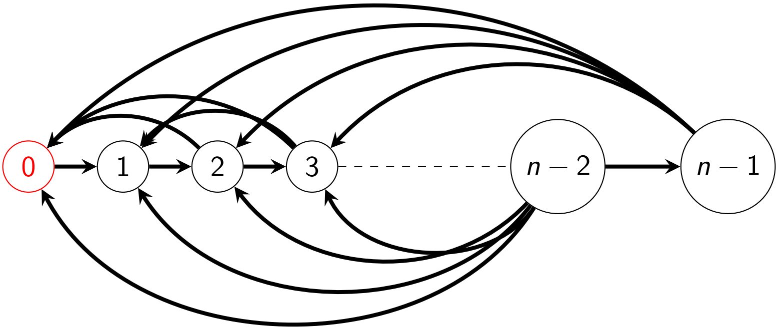
$$\beta(G_1) = 1$$



$$d(0,k) = k$$

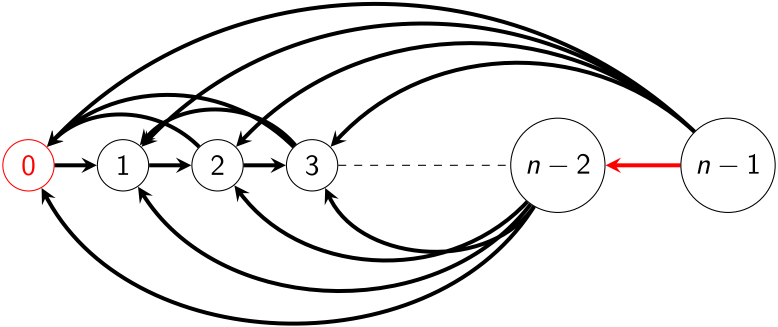
First Graph

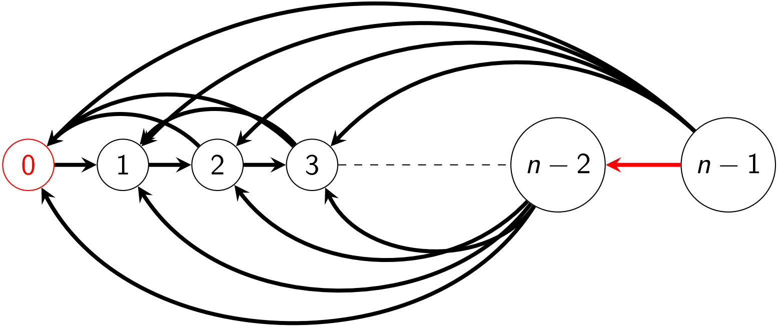
$V = \{0, 1, \ldots, n-1\}$
$E_1 = \{(i,j) : j - i = 1 \lor i - j \geq 2\}$
$G_1 = (V, E_1)$



Second Graph

$V = \{0, 1, \ldots, n-1\}$
$E_2 = (\{(i,j) : j - i = 1 \vee i - j \geq 2\} - \{(n-2, n-1)\}) \cup \{(n-1, n-2)\}$
$G_2 = (V, E_2)$





$$d(0,k) = \begin{cases} k & \text{if } k \in V - \{n-1\} \\ \infty & \text{if } k = n-1 \end{cases}$$

A. Herrero and A. Lozano (2023)

Characterization of tournaments with $\beta(T) = 1$.

A. Herrero and A. Lozano (2023)

Characterization of tournaments with $\beta(T) = 1$.

Characterization

Let T be a tournament, then $\beta(T) = 1 \iff T \cong G_1 \vee T \cong G_2$.

Consult **Section 12.1.2** for more details!

Optimal upper bound

A. Lozano (2013)

For a **strong tournament** T, the optimal upper bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.[a]

[a] Lozano, "Symmetry Breaking in Tournaments".

Optimal upper bound

A. Lozano (2013)

For a **strong tournament** T, the optimal upper bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.[a]

[a] Lozano, "Symmetry Breaking in Tournaments".

A. Herrero and A. Lozano (2023)

For a **tournament** T, the optimal bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.

Optimal upper bound

Optimal upper bound

A.Lozano (2013)

- For every tournament T exists a subset $S \subseteq V$ called *anchor* such that $|S| \leq \lfloor n/2 \rfloor$.

Optimal upper bound

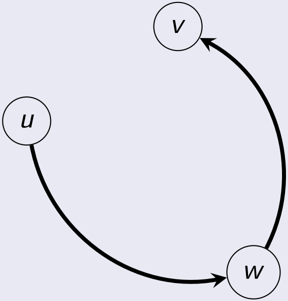
A.Lozano (2013)

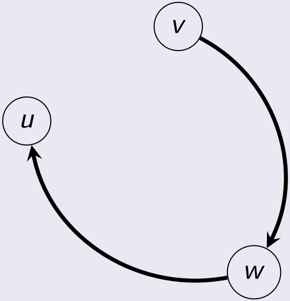
- For every tournament T exists a subset $S \subseteq V$ called *anchor* such that $|S| \leq \lfloor n/2 \rfloor$.
- An anchor S is a subset of vertices such that $\forall u, v \in V(T) - S \quad \exists w \in S : uw, wv \in E(T) \vee vw, wu \in E(T)$.

Optimal upper bound

A.Lozano (2013)

- For every tournament T exists a subset $S \subseteq V$ called *anchor* such that $|S| \leq \lfloor n/2 \rfloor$.
- An anchor S is a subset of vertices such that $\forall u, v \in V(T) - S \quad \exists w \in S : uw, wv \in E(T) \vee vw, wu \in E(T)$.





Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

- Consider an anchor S.

Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

- Consider an anchor S.
- For every
  $i, j \in V - S, i \neq j, \quad \exists w \in S : d(w, i) = 1 \wedge d(w, j) \neq 1$

Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

- Consider an anchor S.
- For every $i, j \in V - S, i \neq j, \quad \exists w \in S : d(w, i) = 1 \wedge d(w, j) \neq 1$
- S is a Resolving Set.

Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

- Consider an anchor S.
- For every $i, j \in V - S, i \neq j, \quad \exists w \in S : d(w, i) = 1 \wedge d(w, j) \neq 1$
- S is a Resolving Set.
- $|S| \leq \lfloor n/2 \rfloor$.

Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

- Consider an anchor S.
- For every $i, j \in V - S, i \neq j, \quad \exists w \in S : d(w, i) = 1 \wedge d(w, j) \neq 1$
- S is a Resolving Set.
- $|S| \leq \lfloor n/2 \rfloor$.
- $\beta(T) \leq \lfloor n/2 \rfloor$.

Lozano's proof only considered finite distances!

Updating optimal bound

Let T be a tournament. Then $\beta(T) \leq \lfloor n/2 \rfloor$. **Proof:**

- Consider an anchor S.
- For every $i, j \in V - S, i \neq j, \quad \exists w \in S : d(w, i) = 1 \wedge d(w, j) \neq 1$
- S is a Resolving Set.
- $|S| \leq \lfloor n/2 \rfloor$.
- $\beta(T) \leq \lfloor n/2 \rfloor$.

A. Herrero and A. Lozano (2023)

For a tournament T, the optimal bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.

Summary on Tournaments

A. Herrero and A. Lozano (2023)

Characterization of tournaments with $\beta(T) = 1$.

A. Herrero and A. Lozano (2023)

For a tournament T, the optimal bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.

Outline

Bicyclic Graphs

Bicyclic Graphs definition

A simple connected graph G is said to be bicyclic if $|E(G)| = n+1$.

Bicyclic Graphs

Bicyclic Graphs definition

A simple connected graph G is said to be bicyclic if $|E(G)| = n+1$.

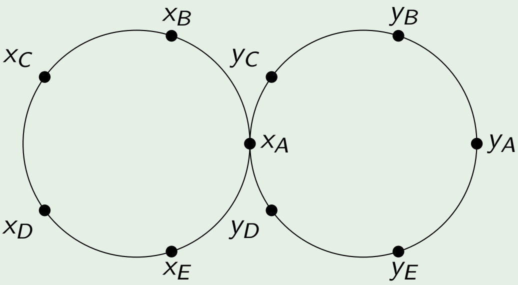
Actual Studies on the Metric Dimension

"Metric Dimension of Bicyclic Graphs" by Khan et al. (2023).

Khan et al. studied the Metric Dimension of bicyclic graphs without vertices of degree 1 and classified them into three types:

- Type I: Two disjoint cycles C_n and C_m sharing a single vertex.

Example (Type I)

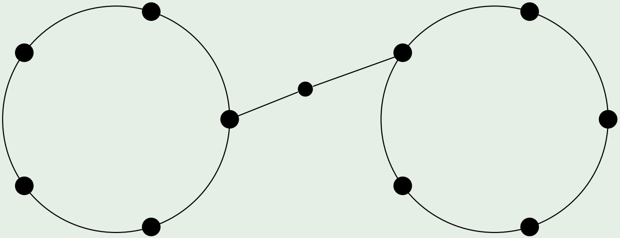


C_5 with C_6 sharing vertex x_a.

Khan et al. studied the Metric Dimension of bicyclic graphs without vertices of degree 1 and classified them into three types:

- Type II: Two disjoint cycles C_n and C_m joined by a path P_r connecting any vertex from C_n to any vertex of C_m

Example (Type II)

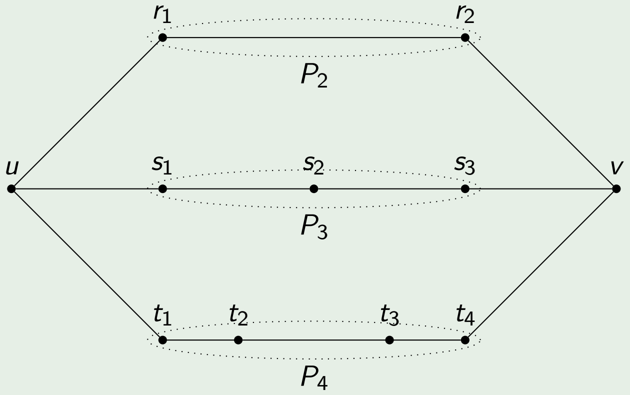


C_5 with C_5 connected by a path P_3.

Khan et al. studied the Metric Dimension of bicyclic graphs without vertices of degree 1 and classified them into three types:

- Type III: Three disjoint paths P_r, P_s, P_t and two vertices u, v that connect the beginning and the end of the paths.

Example (Type III)



Currrent Work:

Currrent Work:

- Type I: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓

Currrent Work:

- Type I: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓
- Type II: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓

Currrent Work:

- Type I: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓
- Type II: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓
- Type III: Not proven. In fact, one open question is, "**Study the Metric Dimension of Type III Bicycle**, providing a proof that they have a constant Metric Dimension."

Currrent Work:

- Type I: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓
- Type II: Proven by Khan et al. (Metric Dimension is 2 or 3) ✓
- Type III: Not proven. In fact, one open question is, "**Study the Metric Dimension of Type III Bicycle**, providing a proof that they have a constant Metric Dimension."

Goal: Proof Metric Dimension on Type III Bicyclic Graphs.

The idea of how embedding a graph: By Mercè Mora et al.[1]

Lemma

Let G be a graph with $\beta(G) = 2$, then G can be *embedded* in a *strong product* of paths $P_n \boxtimes P_n$ of order n.

[1] Mercè Mora et al. "Metric dimension of maximal outerplanar graphs". In: (2019). arXiv: 1903.11933 [math.CO].

The idea of how embedding a graph: By Mercè Mora et al.[1]

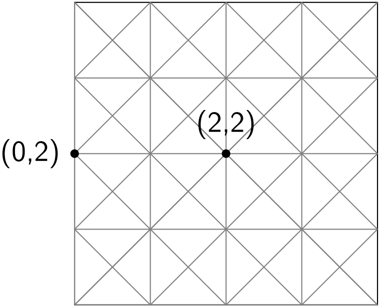
Lemma

Let G be a graph with $\beta(G) = 2$, then G can be *embedded* in a *strong product* of paths $P_n \boxtimes P_n$ of order n.

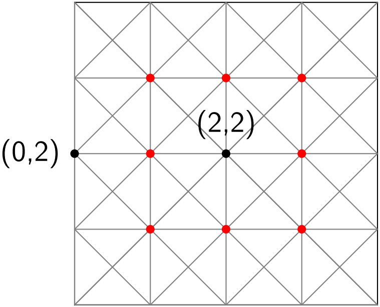
$$V(P_n \boxtimes P_n) = [0, \ldots, n-1] \times [0, \ldots, n-1]$$

$$E(P_n \boxtimes P_n) = (i, j) \sim (i', j') : |i - i'| \leq 1 \land |j - j'| \leq 1$$

[1] Mora et al., "Metric dimension of maximal outerplanar graphs".



$P_5 \boxtimes P_5$



$P_5 \boxtimes P_5$

Proposition

If $x_1x_2 \in E$ and $d(x_0, x_1) = d$ for some $x_0 \in V$, then $d(x_0, x_2) \in \{d-1, d, d+1\}$

$$x_0 \text{ - - - - - - - - - - - - - } x_3 \text{ —— } x_1 \text{ —— } x_2 \qquad d$$

$$x_0 \text{ - - - - - - - - - - - - - } x_2 \text{ —— } x_1 \qquad d-1$$

$$x_0 \text{ - - - - - - - - - - - - - } x_1 \text{ —— } x_2 \qquad d+1$$

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.
- $u \sim v$ and $d(x, u) = d$ then $d(x, v) \in \{d, d + 1, d - 1\}$

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.
- $u \sim v$ and $d(x, u) = d$ then $d(x, v) \in \{d, d + 1, d - 1\}$
- $u \sim v$ and $d(y, u) = d'$ then $d(y, v) \in \{d', d' + 1, d' - 1\}$

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.
- $u \sim v$ and $d(x, u) = d$ then $d(x, v) \in \{d, d + 1, d - 1\}$
- $u \sim v$ and $d(y, u) = d'$ then $d(y, v) \in \{d', d' + 1, d' - 1\}$
- $|d(x, v) - d(x, u)| \leq 1 \wedge |d(y, v) - d(y, u)| \leq 1$.

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.
- $u \sim v$ and $d(x, u) = d$ then $d(x, v) \in \{d, d + 1, d - 1\}$
- $u \sim v$ and $d(y, u) = d'$ then $d(y, v) \in \{d', d' + 1, d' - 1\}$
- $|d(x, v) - d(x, u)| \leq 1 \wedge |d(y, v) - d(y, u)| \leq 1$.
- $E(P_n \boxtimes P_n) = (i, j) \sim (i', j') : |i - i'| \leq 1 \wedge |j - j'| \leq 1$

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

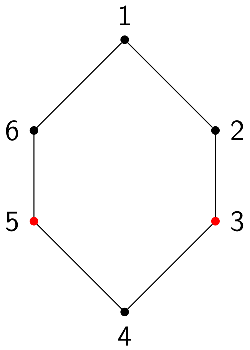
- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.
- $u \sim v$ and $d(x, u) = d$ then $d(x, v) \in \{d, d + 1, d - 1\}$
- $u \sim v$ and $d(y, u) = d'$ then $d(y, v) \in \{d', d' + 1, d' - 1\}$
- $|d(x, v) - d(x, u)| \leq 1 \wedge |d(y, v) - d(y, u)| \leq 1$.
- $E(P_n \boxtimes P_n) = (i, j) \sim (i', j') : |i - i'| \leq 1 \wedge |j - j'| \leq 1$
- $V(G^*) = r(v|W) = (d(x, v), d(y, v))$ for every $v \in V$.

Why embedding on $P_n \boxtimes P_n$? Consider the graph $G = (V, E)$ with $W = \{x, y\}$ as the Resolving Set to be embbeded on $P_n \boxtimes P_n$. Let $u, v \in V$ s.t $uv \in E$. Let G^* be the embedded graph.

Isomorphism

- $r(u|W) = (d(x, u), d(y, u))$ and $r(v|W) = (d(x, v), d(y, v))$.
- $u \sim v$ and $d(x, u) = d$ then $d(x, v) \in \{d, d + 1, d - 1\}$
- $u \sim v$ and $d(y, u) = d'$ then $d(y, v) \in \{d', d' + 1, d' - 1\}$
- $|d(x, v) - d(x, u)| \leq 1 \wedge |d(y, v) - d(y, u)| \leq 1$.
- $E(P_n \boxtimes P_n) = (i, j) \sim (i', j') : |i - i'| \leq 1 \wedge |j - j'| \leq 1$
- $V(G^*) = r(v|W) = (d(x, v), d(y, v))$ for every $v \in V$.
- $E(G^*) = r(v|W)r(u|W) : uv \in E$.

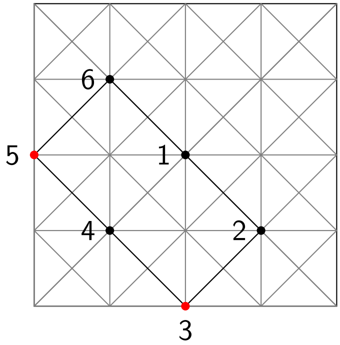
Graph Embedding Example



x_i	$d(5, x_i)$	$d(3, x_i)$	$r(x_i\|W)$
1	2	2	$(2, 2)$
2	3	1	$(3, 1)$
3	2	0	$(2, 0)$
4	1	1	$(1, 1)$
5	0	2	$(0, 2)$
6	1	3	$(1, 3)$

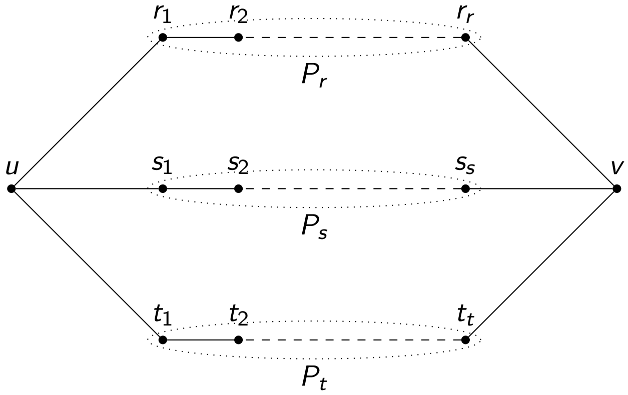
C_6 with $R = \{5, 3\}$ as the Resolving Set.

Graph Embedding Example



x_i	$d(5, x_i)$	$d(3, x_i)$	$r(x_i \mid W)$
1	2	2	(2, 2)
2	3	1	(3, 1)
3	2	0	(2, 0)
4	1	1	(1, 1)
5	0	2	(0, 2)
6	1	3	(1, 3)

An embedding of C_6 in $P_5 \boxtimes P_5$.


r_1
r_2
r_r
P_r
u
s_1
s_2
s_s
v
P_s
t_1
t_2
t_t
P_t

Subcases to prove

- $1 \leq r < s \leq t$ all with same parity
- $1 \leq r < s \leq t$ s, t different parity
- $r = 0, s, t \geq 1$
- $s = r, t = r + k, k \geq 1$ and $k \neq 2$.
- $1 \leq r < s \leq t$ s, t same parity and r different one.
- $r = s = t$
- $r = s, t = r + 2$.

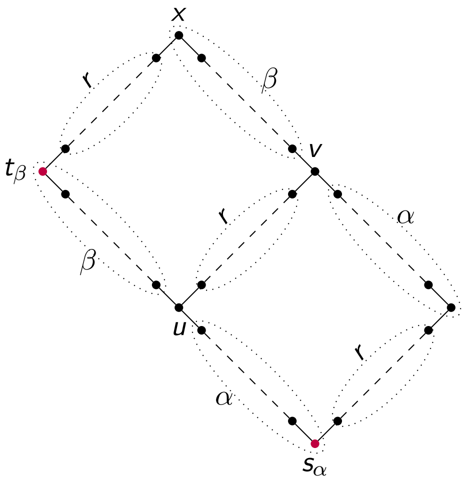
Subcases to prove

- $1 \leq r < s \leq t$ all with same parity ✓
- $1 \leq r < s \leq t$ s, t different parity ✓
- $r = 0, s, t \geq 1$ ✓
- $s = r, t = r + k, k \geq 1$ and $k \neq 2$. ✓
- $1 \leq r < s \leq t$ s, t same parity and r different one.
- $r = s = t$
- $r = s, t = r + 2$.

Our results

Although we couldn't prove them all we only left 3/7 subcases.

$1 \leq r < s \leq t$ same parity



$$\begin{cases} s = r + 2\alpha, \alpha \geq 1 \\ t = r + 2\beta, \beta \geq 1 \end{cases}$$

Main Idea of Proof

Proof: $R = \{s_\alpha, t_\beta\}$ is a Resolving Set.

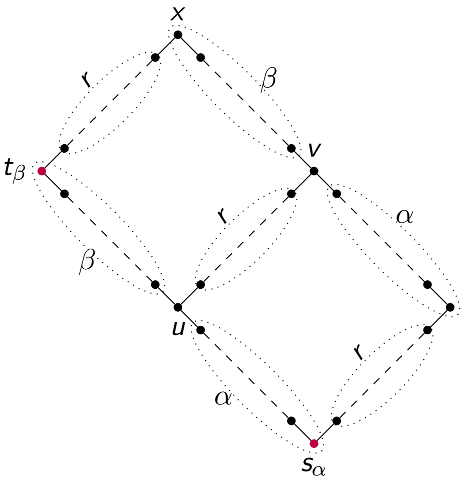
Main idea of proof: Let $i, j \in V$, $i \neq j$, and consider $d(s_\alpha, i) = d(s_\alpha, j)$. Then prove $d(t_\beta, i) \neq d(t_\beta, j)$.

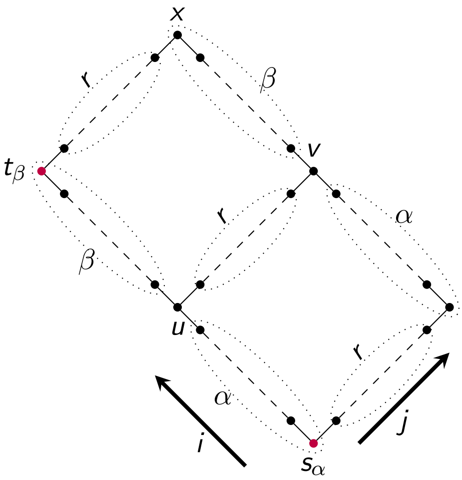
Main Idea of Proof

Proof: $R = \{s_\alpha, t_\beta\}$ is a Resolving Set.

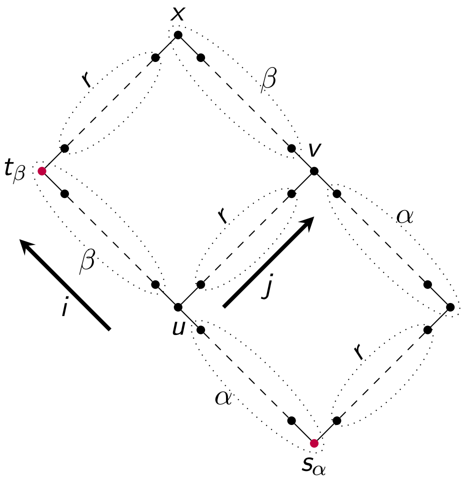
Main idea of proof: Let $i, j \in V$, $i \neq j$, and consider $d(s_\alpha, i) = d(s_\alpha, j)$. Then prove $d(t_\beta, i) \neq d(t_\beta, j)$.

Why? If $i \neq j$ and $d(s_\alpha, i) = d(s_\alpha, j)$, we can consider the shortest paths $s_\alpha - i$ and $s_\alpha - j$. At one point, the paths will split. Now *guess* where the vertices will be.


x
r
β
v
t_β
r
α
β
u
r
α
s_α

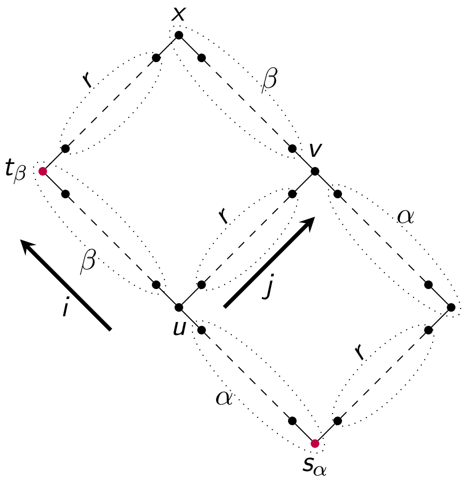


Paths split at s_α



Paths split at u

Path $t_\beta - j$ doesn't pass through x



- $d(t_\beta, j) = d(t_\beta, u) + d(u, j)$
- $d(t_\beta, i) = |d(t_\beta, u) - d(u, i)|$

Path $t_\beta - j$ doesn't pass through x

If $d(t_\beta, i) = d(t_\beta, u) - d(u, i)$

$$
\begin{aligned}
& d(t_\beta, i) = d(t_\beta, j) \\
\xrightarrow{\text{substitute}}\ & d(t_\beta, u) - d(u, i) = d(t_\beta, u) + d(u, j) \\
\xrightarrow{\text{simplify}}\ & -d(u, i) = d(u, j)
\end{aligned}
$$

This is only possible if and only if $u = i = j$ but $i \neq j$, contradiction.

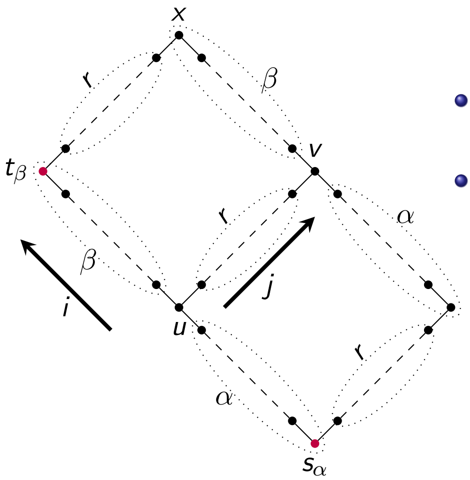
Path $t_\beta - j$ doesn't pass through x

If $d(t_\beta, i) = d(u, i) - d(t_\beta, u)$

$$
\begin{aligned}
& d(t_\beta, i) = d(t_\beta, j) \\
\xrightarrow{\text{substitute}}\ & d(u, i) - d(t_\beta, u) = d(t_\beta, u) + d(u, j) \\
\xrightarrow{\text{simplify}}\ & -d(t_\beta, u) = d(t_\beta, u)
\end{aligned}
$$

This is not possible because $d(t_\beta, u) = \beta \geq 1$, contradiction.

Path $t_\beta - j$ pass through x



- $d(t_\beta, j) =$
  $d(t_\beta, x) + d(x, u) - d(u, j)$
- $d(t_\beta, i) = |d(t_\beta, u) - d(u, i)|$

Path $t_\beta - j$ pass through x

If $d(t_\beta, i) = d(t_\beta, u) - d(u, i)$

$$
\begin{aligned}
& d(t_\beta, i) = d(t_\beta, j) \\
\xrightarrow{\text{substitute}}\ & d(t_\beta, u) - d(u, i) = d(t_\beta, x) + d(x, u) - d(u, j) \\
\xrightarrow{d(u,i)=d(u,j) \text{ and simplify}}\ & d(t_\beta, u) = d(t_\beta, x) + d(x, u) \\
\xrightarrow[d(x,u)=\beta+r+1]{d(t_\beta,u)=\beta \quad d(t_\beta,x)=r+1}\ & \beta = r + 1 + \beta + r + 1 \\
\xrightarrow{\text{simplify}}\ & 0 = 2r + 2
\end{aligned}
$$

Contradiction, because $2r + 2 \geq 1$.

Path $t_\beta - j$ pass through x

If $d(t_\beta, i) = d(u, i) - d(t_\beta, u)$

$$\begin{aligned} d(t_\beta, i) &= d(t_\beta, j) \\ \xrightarrow{\text{substitute}} d(u, i) - d(t_\beta, u) &= d(t_\beta, x) + d(x, u) - d(u, j) \\ \xrightarrow{\text{reorder}} d(u, i) + d(u, j) &= d(t_\beta, x) + d(x, u) + d(t_\beta, u) \\ \xrightarrow[d(x,u)=\beta+r+1 \quad d(t_\beta,u)=\beta]{d(u,i)=d(u,j) \quad d(t_\beta,x)=r+1} 2d(u, i) &= r + 1 + \beta + r + 1 + \beta = 2r + 2\beta + 2 \\ \xrightarrow{\div 2} d(u, i) &= d(u, j) = r + \beta + 1. \end{aligned}$$

But this is possible if and only if $i = j = x$. But $i \neq j$, contradiction.

Subcases to prove

- $1 \leq r < s \leq t$ all with same parity ✓, $\beta(G) = 2$
- $1 \leq r < s \leq t$ s, t different parity ✓, $\beta(G) = 2$
- $r = 0, s, t \geq 1$ ✓, $\beta(G) = 2$
- $s = r, t = r + k, k \geq 1$ and $k \neq 2$ ✓, $\beta(G) = 2$
- $1 \leq r < s \leq t$ s, t same parity and r different one.
- $r = s = t$
- $r = s, t = r + 2$.

Our results

Although we couldn't prove them all we only left 3/7 subcases.

Consult **Section 13.2.1** for more details!

For the unproven cases we conducted some experiments, here are the results:

Subcases	Resolving Set
$r = s = t$ and $r \geq 2$	$R = \{2r, 5 + 3 \cdot (r - 2), 6 + 3 \cdot (r - 2)\}$
$r = r, s = r, t = r + 2$ and $r \geq 4$	$R = \{2r, 3r, 3r + 1\}$
$1 \leq r < s \leq t$ s, t same parity and r different one.	No significant Resolving Set, but $\beta(G) = 2$ in all cases

For the unproven cases we conducted some experiments, here are the results:

Subcases	Resolving Set
$r = s = t$ and $r \geq 2$	$R = \{2r, 5 + 3 \cdot (r-2), 6 + 3 \cdot (r-2)\}$
$r = r, s = r, t = r + 2$ and $r \geq 4$	$R = \{2r, 3r, 3r+1\}$
$1 \leq r < s \leq t$ s, t same parity and r different one.	No significant Resolving Set, but $\beta(G) = 2$ in all cases

Conjecture

Let G be a Type III bicyclic graph, then $\beta(G) = 2$ or $\beta(G) = 3$.

Outline

ILP formulation

Objective Function:

$$\text{Minimize:} \quad \sum_{i=1}^{n} x_i$$

Subject to:

$$\text{Constraint 1:} \quad \sum_{v \in A(i,j)} v \geq 1 \text{ for every } 1 \leq i < j \leq n$$

$$\text{Constraint 2:} \quad x_i \in \{0, 1\} \quad \text{ for every } 1 \leq i \leq n$$

Where:

$$A(i,j) = \{x_k \in V : \ d(x_k, x_i) \neq d(x_k, x_j)\}$$

$$x_i = 1 \iff x_i \in \text{Resolving Set}$$

SAT Problem

Given a boolean formula in CNF form, find a model that satisfies the formula.

Decisional version of SAT is NP-complete: **Cook-Levin Theorem**.

SAT Problem

Given a boolean formula in CNF form, find a model that satisfies the formula.

Decisional version of SAT is NP-complete: **Cook-Levin Theorem**.

Example (SAT Problem)

$$F = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_4 \vee \neg x_1 \vee x_7 \vee x_2)$$

F is SAT, $x_2 = 1$ is enough for satisfying the formula.

$$\boxed{\begin{array}{c}\text{Constraint 1: } \displaystyle\sum_{v \in A(i,j)} v \geq 1 \text{ for every } 1 \leq i < j \leq n \\ \text{Constraint 2: } x_i \in \{0,1\} \text{ for every } 1 \leq i \leq n\end{array}}$$

$$\boxed{\begin{array}{c}\text{Constraint 1: } \displaystyle\sum_{v \in A(i,j)} v \geq 1 \text{ for every } 1 \leq i < j \leq n \\ \text{Constraint 2: } x_i \in \{0,1\} \text{ for every } 1 \leq i \leq n\end{array}}$$

SAT to ILP

ILP to SAT

$$\boxed{\begin{array}{c}\text{Constraint 1: } \displaystyle\bigvee_{v \in A(i,j)} v \text{ for every } 1 \leq i < j \leq n \\ \text{Constraint 2: Inherent from SAT nature}\end{array}}$$

Example

$$\boxed{\begin{gathered} x_0 + x_1 + x_2 + x_3 \geq 1 \\ x_0 + x_3 \geq 1 \\ x_1 + x_2 \geq 1 \\ x_0, x_1, x_2, x_3 \in \{0, 1\} \end{gathered}}$$

Example

ILP to SAT

$$\begin{gathered} x_0 + x_1 + x_2 + x_3 \geq 1 \\ x_0 + x_3 \geq 1 \\ x_1 + x_2 \geq 1 \\ x_0, x_1, x_2, x_3 \in \{0, 1\} \end{gathered}$$

$$\begin{gathered} (x_0 \vee x_1 \vee x_2 \vee x_3) \wedge \\ (x_0 \vee x_3) \wedge (x_1 \vee x_2) \end{gathered}$$

SAT to ILP

SAT Problem

Given a boolean formula in CNF form, find a model that satisfies the formula.
Decisional version of SAT is NP-complete (**Cook-Levin Theorem**).

SAT Problem

Given a boolean formula in CNF form, find a model that satisfies the formula.
Decisional version of SAT is NP-complete (**Cook-Levin Theorem**).

Weighted Max-SAT Problem

Given a boolean formula in CNF and a weight for every clause, find a model that minimizes the total weight.
Decisional version of WMax-SAT is also NP-complete.

$$\text{Minimize: } \sum_{i=1}^{n} x_i$$
$$\text{Constraint 1: } \sum_{v \in A(i,j)} v \geq 1 \text{ for every } 1 \leq i < j \leq n$$
$$\text{Constraint 2: } x_i \in \{0,1\} \text{ for every } 1 \leq i \leq n$$

WMSAT to ILP ↑ ↓ ILP to WMSAT

$$\text{Minimize section: } \bigwedge_{i=1}^{n} (\neg x_i, 1)$$
$$\text{Constraint 1: } (\bigvee_{v \in A(i,j)} v, \infty) \text{ for every } 1 \leq i < j \leq n$$
$$\text{Constraint 2: Inherent from SAT nature}$$

Example

ILP to WMSAT

$$\begin{array}{c} \text{Minimize: } x_0 + x_1 + x_2 + x_3 \\ \text{Subject To:} \\ x_0 + x_3 \geq 1 \\ x_1 + x_2 \geq 1 \\ x_0 + x_1 + x_2 + x_3 \geq 1 \\ x_0, x_1, x_2, x_3 \in \{0, 1\} \end{array}$$

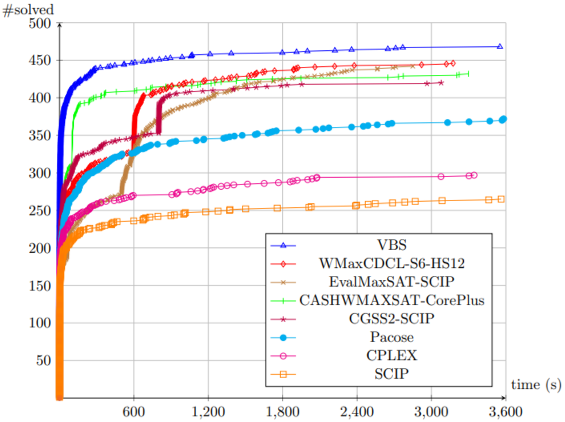
$$\begin{array}{c} (\neg x_0, 1) \wedge (\neg x_1, 1) \wedge (\neg x_2, 1) \\ \wedge(\neg x_3, 1) \wedge (x_0 \vee x_3, \infty) \\ \wedge(x_1 \vee x_2, \infty) \\ \wedge(x_0 \vee x_1 \vee x_2 \vee x_3, \infty) \end{array}$$

WMSAT to ILP

Solvers from Max-SAT competition are free!

How well do ILP solvers perform by themselves?

Weighted



- ILP solvers by themselves are not competitive with MaxSAT solvers

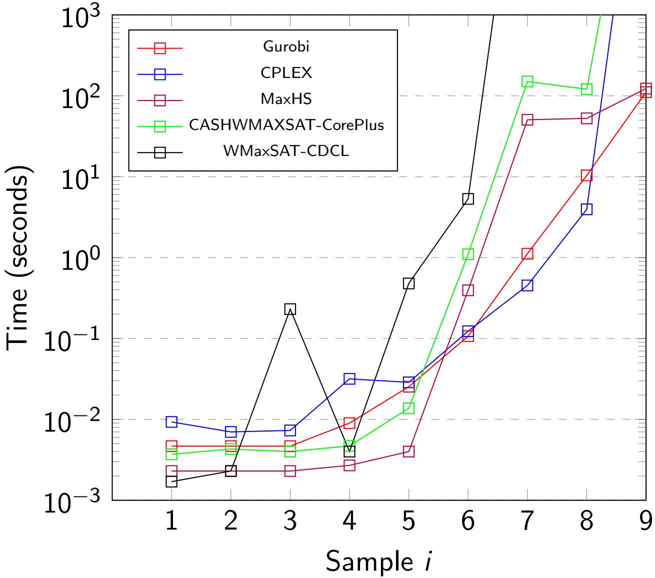
Goal: See which solver performs better on the Metric Dimension problem. Participants for my comparison:

- CPLEX (ILP)
- Gurobi (ILP) → Personal recommendation from Enric!
- MaxHS (WMax-SAT) → Solves *Hitting Sets* (thanks to Jordi Coll, creator of this solver)
- WMaxCDCL (WMax-SAT)
- CASHWMAXSAT-CorePlus (WMax-SAT)

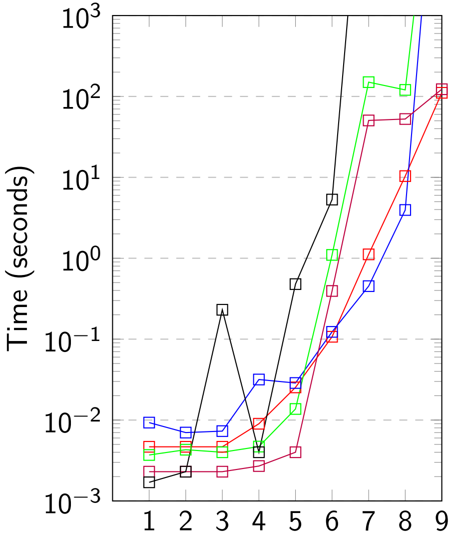
Samples: Metric Dimension of Hypercubes.

Q_d	**#Variables**	**#Constraint 1**
Q_1	2	1
Q_2	4	6
Q_3	8	28
Q_4	16	120
Q_5	32	496
Q_6	64	2016
Q_7	128	8128
Q_8	256	32640
Q_9	512	130816

Results


Gurobi
CPLEX
MaxHS
CASHWMAXSAT-CorePlus
WMaxSAT-CDCL
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
Time (seconds)
1
2
3
4
5
6
7
8
9
Sample i

Results



Q_d	**Winner**
Q_1	WMaxCDCL
Q_2	WMaxCDCL/MaxHS
Q_3	MaxHS
Q_4	MaxHS
Q_5	MaxHS
Q_6	Gurobi/CPLEX
Q_7	CPLEX
Q_8	CPLEX
Q_9	Gurobi/MaxHS

Q_d	**Winner**
Q_1	WMaxCDCL
Q_2	WMaxCDCL/MaxHS
Q_3	MaxHS
Q_4	MaxHS
Q_5	MaxHS
Q_6	Gurobi/CPLEX
Q_7	CPLEX
Q_8	CPLEX
Q_9	Gurobi/MaxHS

In fact Gurobi/MaxHS were the unique solvers to solve Q_9 without a *Timeout*.

Goal: See which solver performs better on the Metric Dimension problem.

Our Results

A more in-depth study would contribute to this section. ILP vs Hitting Sets would be an interesting experiment. But WMaxSAT not always performs better than an ILP solver.

Outline

$$Q_d = \underbrace{K_2 \times K_2 \times \ldots \times K_2}_{d \text{ times}}$$

$$\beta(Q_d) = \begin{cases} d & \text{if } d = 1,2,3,4 \\ d-1 & \text{if } d = 5,6,7 \\ d-2 & \text{if } d = 8,9 \\ d-3 & \text{if } d = 10,11 \\ d-4 & \text{if } d = 12,13 \\ d-5 & \text{if } d = 14,15,16 \\ d-6 & \text{if } d = 17 \end{cases}$$

Actual known values[2] (since 2013!)

[2] A.F. Beardon. "Resolving the hypercube". In: *Discrete Applied Mathematics* 161.13 (2013), pp. 1882–1887. ISSN: 0166-218X. DOI: https://doi.org/10.1016/j.dam.2013.02.012. URL: https://www.sciencedirect.com/science/article/pii/S0166218X13000644.

Goal: Calculate new Hypercube dimension.

Goal: Calculate new Hypercube dimension.
Not possible: Even with the support of Daniel Jiménez (AC Department), I rejected the idea. But something interesting appeared...

Q_d	**#Variables**	**#Constraint**	**#Different Constraints**	**Ratio different constraints**
Q_1	2	1	1	100%
Q_2	4	6	3	50%
Q_3	8	28	7	25%
Q_4	16	120	21	17.5%
Q_5	32	496	61	12.30%
Q_6	64	2016	183	9.05%
Q_7	128	8128	547	6.73%
Q_8	256	32640	1641	5.03%
Q_9	512	130816	4921	3.76%

Q_d	**Expected #Constraints using interpolation**
Q_{10}	14251
Q_{11}	38655
Q_{12}	97021
Q_{13}	225525
Q_{14}	488671
Q_{15}	994971
Q_{16}	1918417
Q_{17}	3527025
Q_{18}	6219859
Q_{19}	10574071
Q_{20}	17403621

Summary on Hypercubes

Our Results

Curious behaviour in the number of constraints generated; many of these restrictions can be ignored because they are repeated.

Summary on Tournaments

A. Herrero and A. Lozano (2023)

Characterization of tournaments with $\beta(G) = 1$.

A. Herrero and A. Lozano (2023)

For a **tournament** T, the optimal bound is given by $\beta(T) \leq \lfloor n/2 \rfloor$.

Summary on Type III

- $1 \leq r < s \leq t$ all with same parity $(\beta(G) = 2)$ ✓
- $1 \leq r < s \leq t$ s, t different parity $(\beta(G) = 2)$✓
- $r = 0, s, t \geq 1$ $(\beta(G) = 2)$✓
- $s = r, t = r + k, k \geq 1$ and $k \neq 2$. $(\beta(G) = 2)$✓
- $1 \leq r < s \leq t$ s, t same parity and r different one. $(\beta(G) = 2)$
- $r = s = t$ $(\beta(G) = 3)$
- $r = s, t = r + 2$ $(\beta(G) = 3)$

Our results

Although we couldn't prove them all we only left 3/7 subcases.

Conjecture

Let G be a Type III bicyclic graph, then $\beta(G) = 2$ or $\beta(G) = 3$.

Summary on ILP vs WMaxSAT

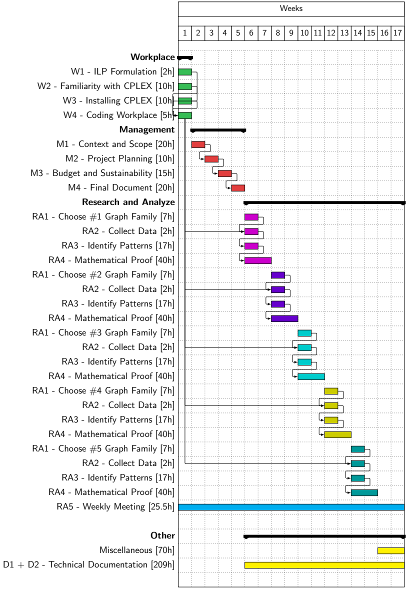
Our Results

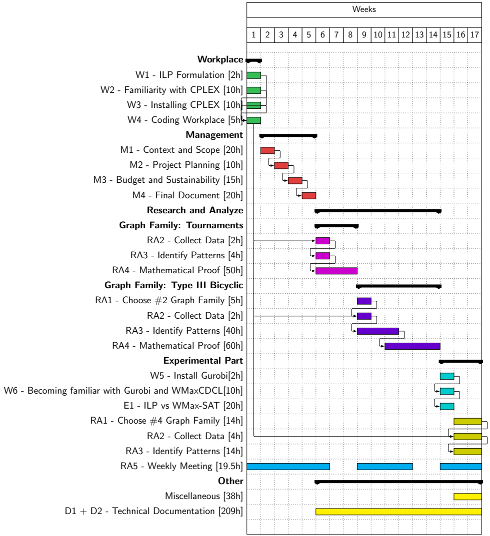
A more in-depth study would contribute to this section. ILP vs Hitting Sets would be an interesting experiment. But WMaxSAT not always performs better than a ILP solver.

Summary on Hypercubes

Our Results

Curious behaviour in the number of constraints generated; many of these restrictions can be ignored because they are repeated.


Weeks
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
Workplace
W1 - ILP Formulation [2h]
W2 - Familiarity with CPLEX [10h]
W3 - Installing CPLEX [10h]
W4 - Coding Workplace [5h]
Management
M1 - Context and Scope [20h]
M2 - Project Planning [10h]
M3 - Budget and Sustainability [15h]
M4 - Final Document [20h]
Research and Analyze
RA1 - Choose #1 Graph Family [7h]
RA2 - Collect Data [2h]
RA3 - Identify Patterns [17h]
RA4 - Mathematical Proof [40h]
RA1 - Choose #2 Graph Family [7h]
RA2 - Collect Data [2h]
RA3 - Identify Patterns [17h]
RA4 - Mathematical Proof [40h]
RA1 - Choose #3 Graph Family [7h]
RA2 - Collect Data [2h]
RA3 - Identify Patterns [17h]
RA4 - Mathematical Proof [40h]
RA1 - Choose #4 Graph Family [7h]
RA2 - Collect Data [2h]
RA3 - Identify Patterns [17h]
RA4 - Mathematical Proof [40h]
RA1 - Choose #5 Graph Family [7h]
RA2 - Collect Data [2h]
RA3 - Identify Patterns [17h]
RA4 - Mathematical Proof [40h]
RA5 - Weekly Meeting [25.5h]
Other
Miscellaneous [70h]
D1 + D2 - Technical Documentation [209h]


Weeks
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
Workplace
W1 - ILP Formulation [2h]
W2 - Familiarity with CPLEX [10h]
W3 - Installing CPLEX [10h]
W4 - Coding Workplace [5h]
Management
M1 - Context and Scope [20h]
M2 - Project Planning [10h]
M3 - Budget and Sustainability [15h]
M4 - Final Document [20h]
Research and Analyze
Graph Family: Tournaments
RA2 - Collect Data [2h]
RA3 - Identify Patterns [4h]
RA4 - Mathematical Proof [50h]
Graph Family: Type III Bicyclic
RA1 - Choose #2 Graph Family [5h]
RA2 - Collect Data [2h]
RA3 - Identify Patterns [40h]
RA4 - Mathematical Proof [60h]
Experimental Part
W5 - Install Gurobi[2h]
W6 - Becoming familiar with Gurobi and WMaxCDCL[10h]
E1 - ILP vs WMax-SAT [20h]
RA1 - Choose #4 Graph Family [14h]
RA2 - Collect Data [4h]
RA3 - Identify Patterns [14h]
RA5 - Weekly Meeting [19.5h]
Other
Miscellaneous [38h]
D1 + D2 - Technical Documentation [209h]

Expected Time: 726.5h vs Real Time: 586.5h
This caused reductions in the Budget!

Summary

Source	Expected Cost	Real Cost	Cost Deviation
Hardware	76.66€	61.89€	14.77€
Software	0€	0€	0€
Human Resources	19699.82€	15780.45€	3919.37€
Indirect Costs	52.64€	37.54€	15.10€
Contingency	2974.37€	2386.47€	587.70€
Unexpected Costs	1699.58€	0€	1699.58€
Total	24503.07€	18266.35€	6236.72€

Also a review on the Sustainability of the project had appeared!
More sustainable than expected :-)

Special Thanks

- Antoni Lozano for his $\Theta(2^{n!})$ wisdom!
- Mercè Mora for the idea for proving Type III Bicyclic
- Enric Rodríguez for the idea of the experiment.
- To the friends I've made during my time studying this degree.
- Who is currently reading this :D

An Experimental Guided Approach to the Metric Dimension on Different Graph Families

Alex Herrero Bravo
Director: Antoni Lozano Boixadors
Department of Computer Science

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat d'Informàtica de Barcelona

Defense Date: January 26, 2024