

Introdução ao Método dos Elementos Finitos

Competências Transferíveis 2

Sessão 06

Robert Valente (robertt@ua.pt)

Introdução ao Método dos Elementos Finitos

Tópicos para hoje

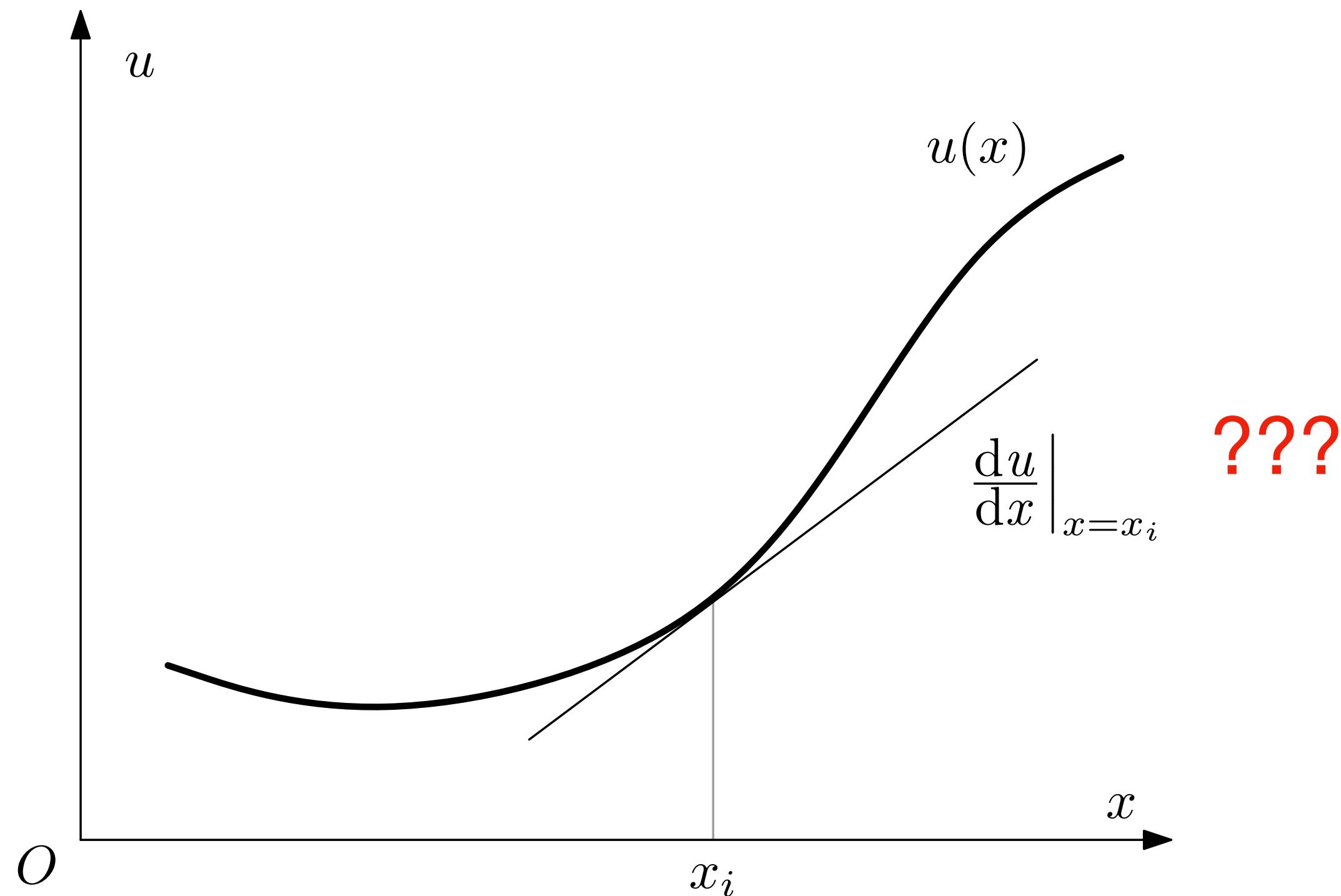
algumas abordagens alternativas e/ou avançadas

Método das Diferenças Finitas

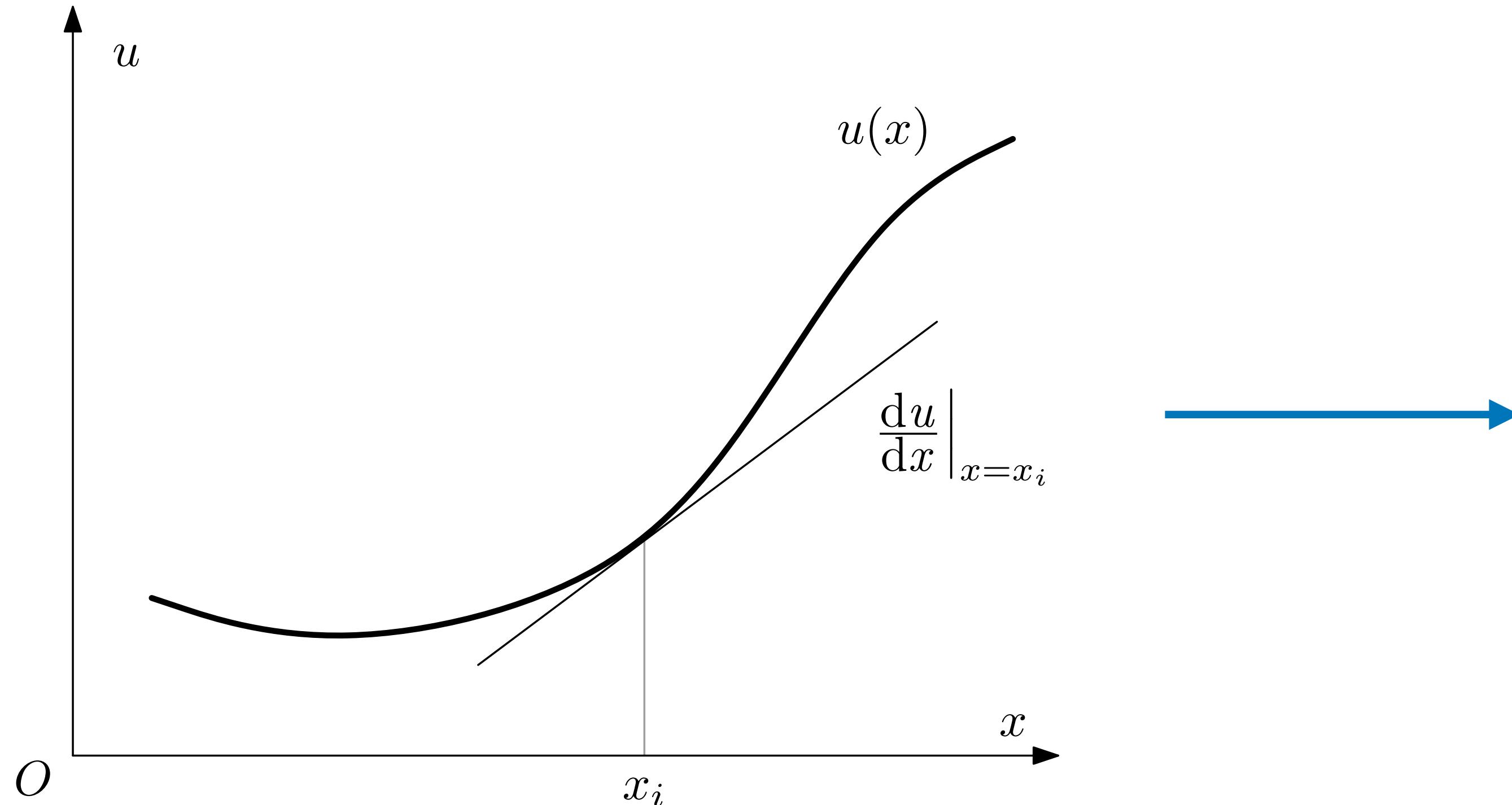
Método das Diferenças Finitas (MDF)

- Técnica de modelação e obtenção de uma solução aproximada para uma equação diferencial
- Fácil implementação, bastando conhecer uma equação diferencial para o fenómeno físico que se quer modelar
- Transformação de um conjunto de equações diferenciais num conjunto de equações algébricas
- Bastante prático para geometrias simples que possam ser discretizadas com “malhas” regulares
- Análise estrutural, transferência de calor, etc.

Método das Diferenças Finitas (MDF) - 1D

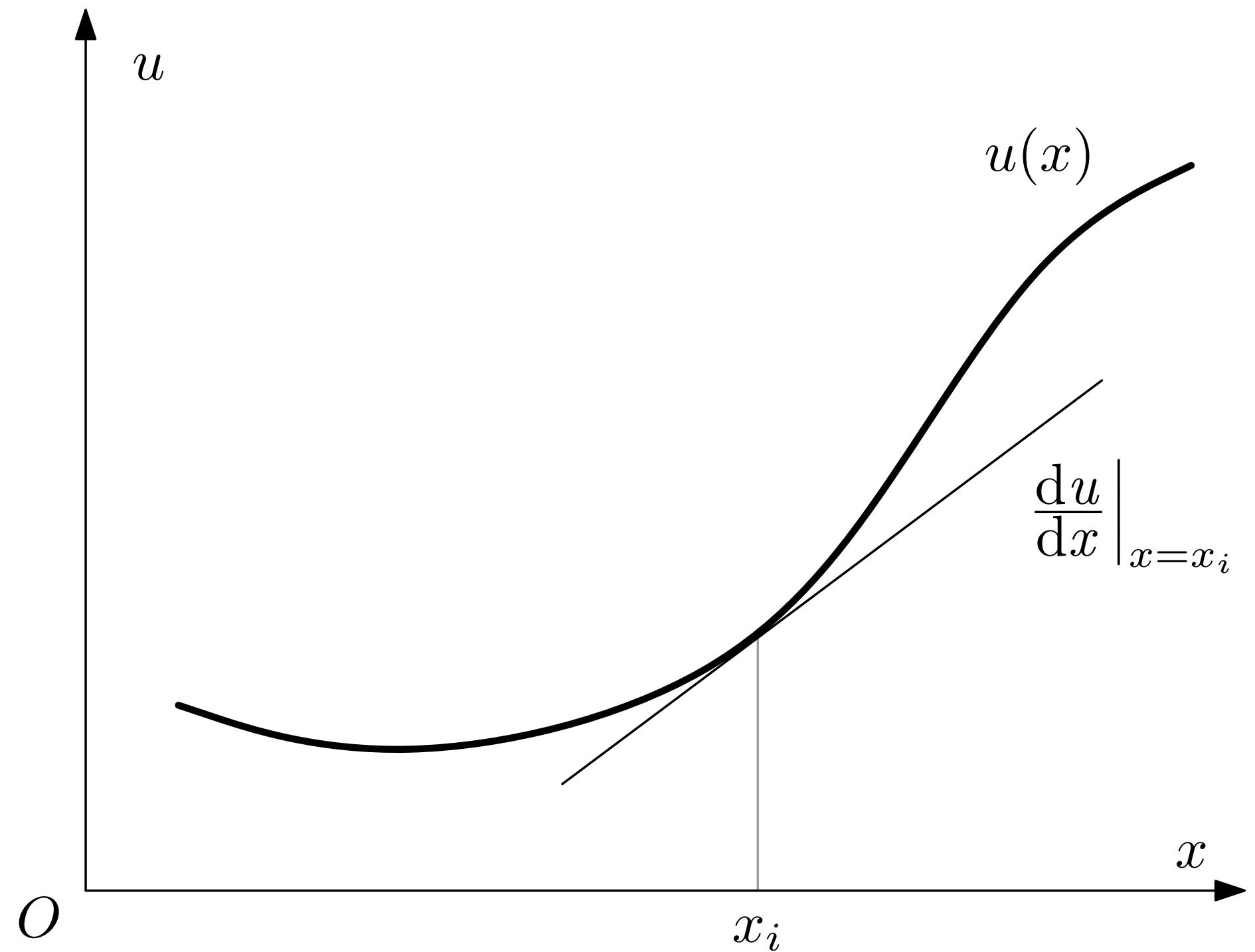


Método das Diferenças Finitas (MDF) - 1D



$$\frac{du}{dx} = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

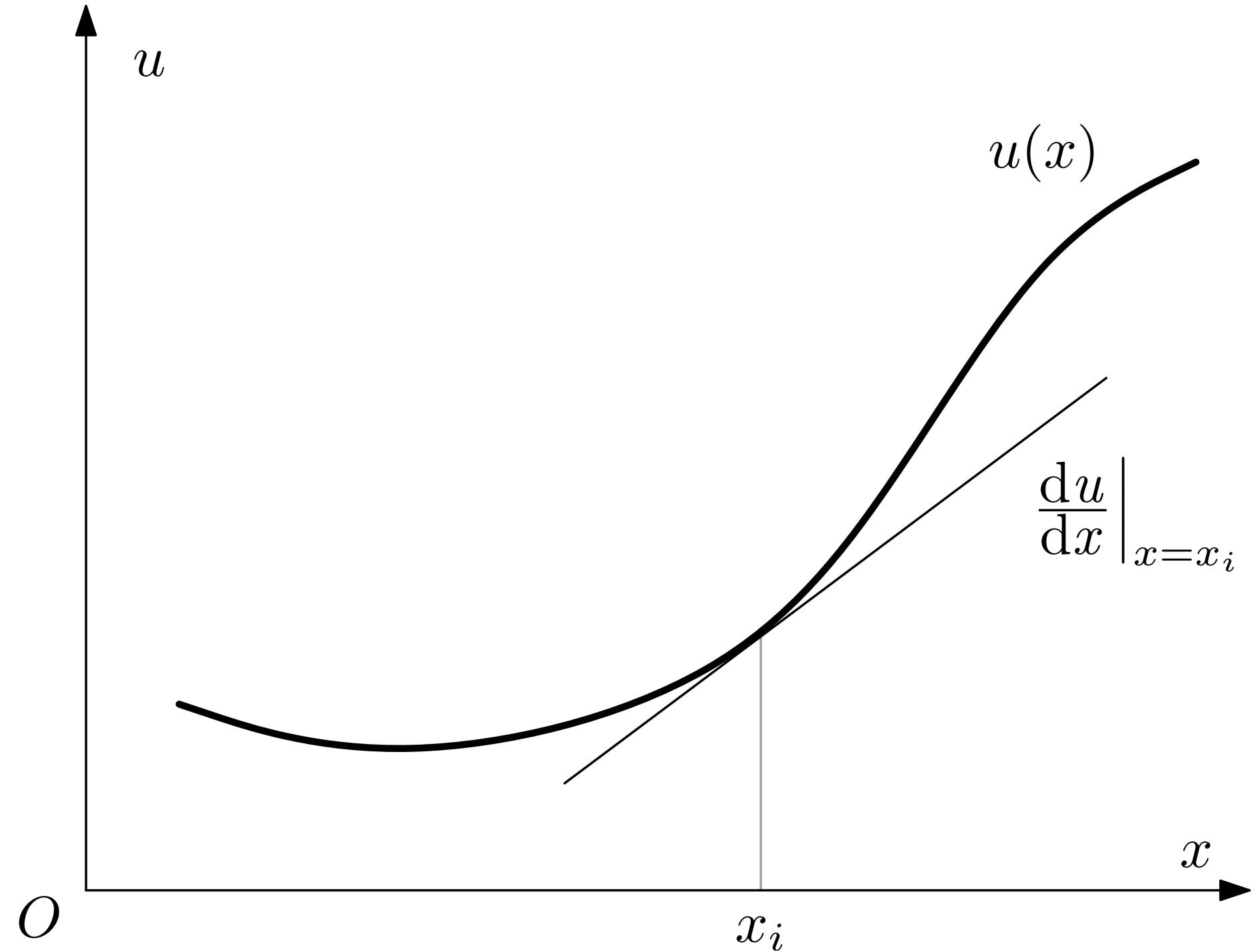
Método das Diferenças Finitas (MDF) - 1D



$$\frac{du}{dx} = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

$$u(x + \Delta x) = u(x) + \Delta x \frac{du}{dx} + \frac{\Delta x^2}{2!} \frac{d^2 u}{dx^2} + \frac{\Delta x^3}{3!} \frac{d^3 u}{dx^3} + \frac{\Delta x^4}{4!} \frac{d^4 u}{dx^4} + \dots$$

Método das Diferenças Finitas (MDF) - 1D

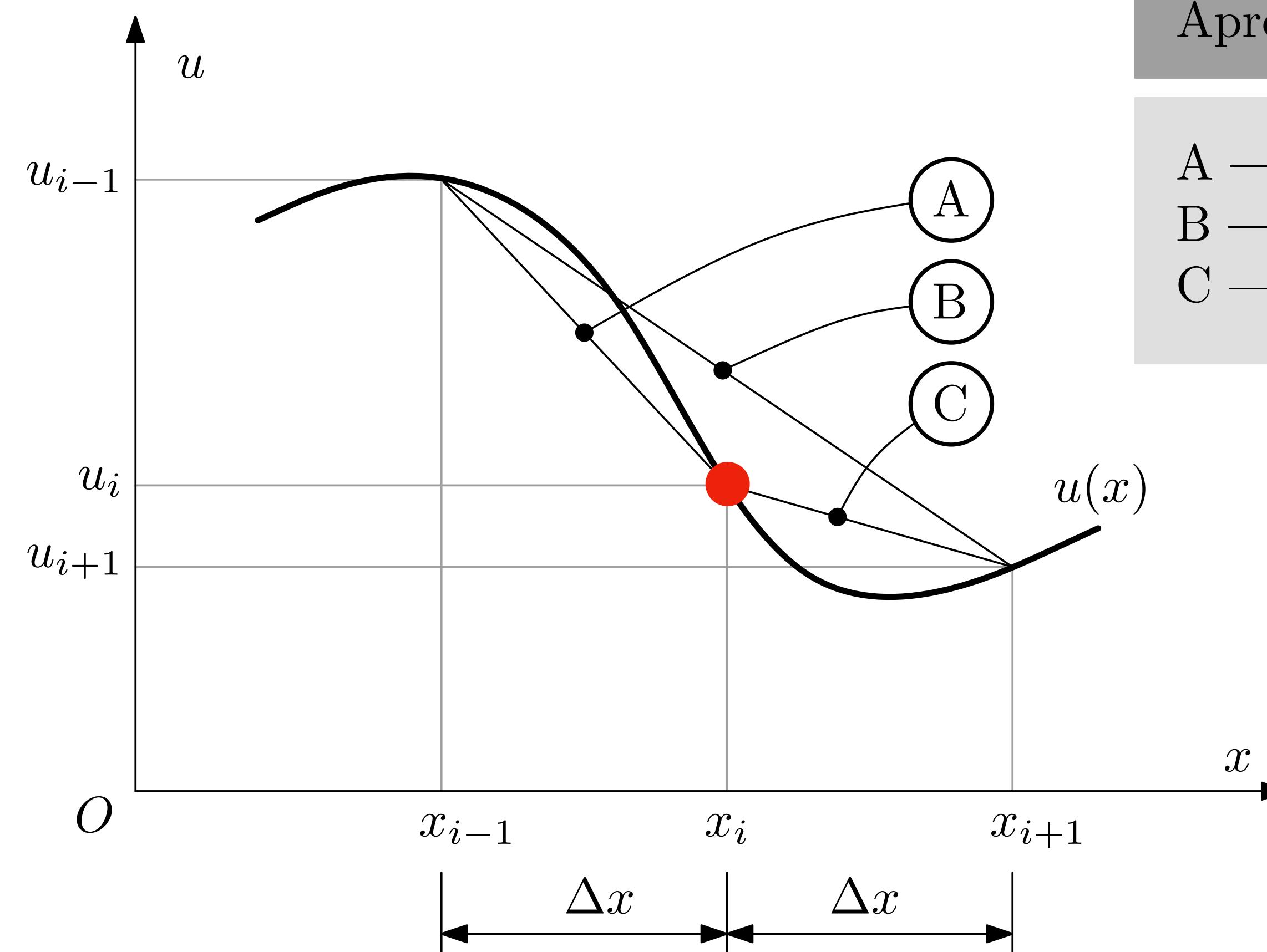


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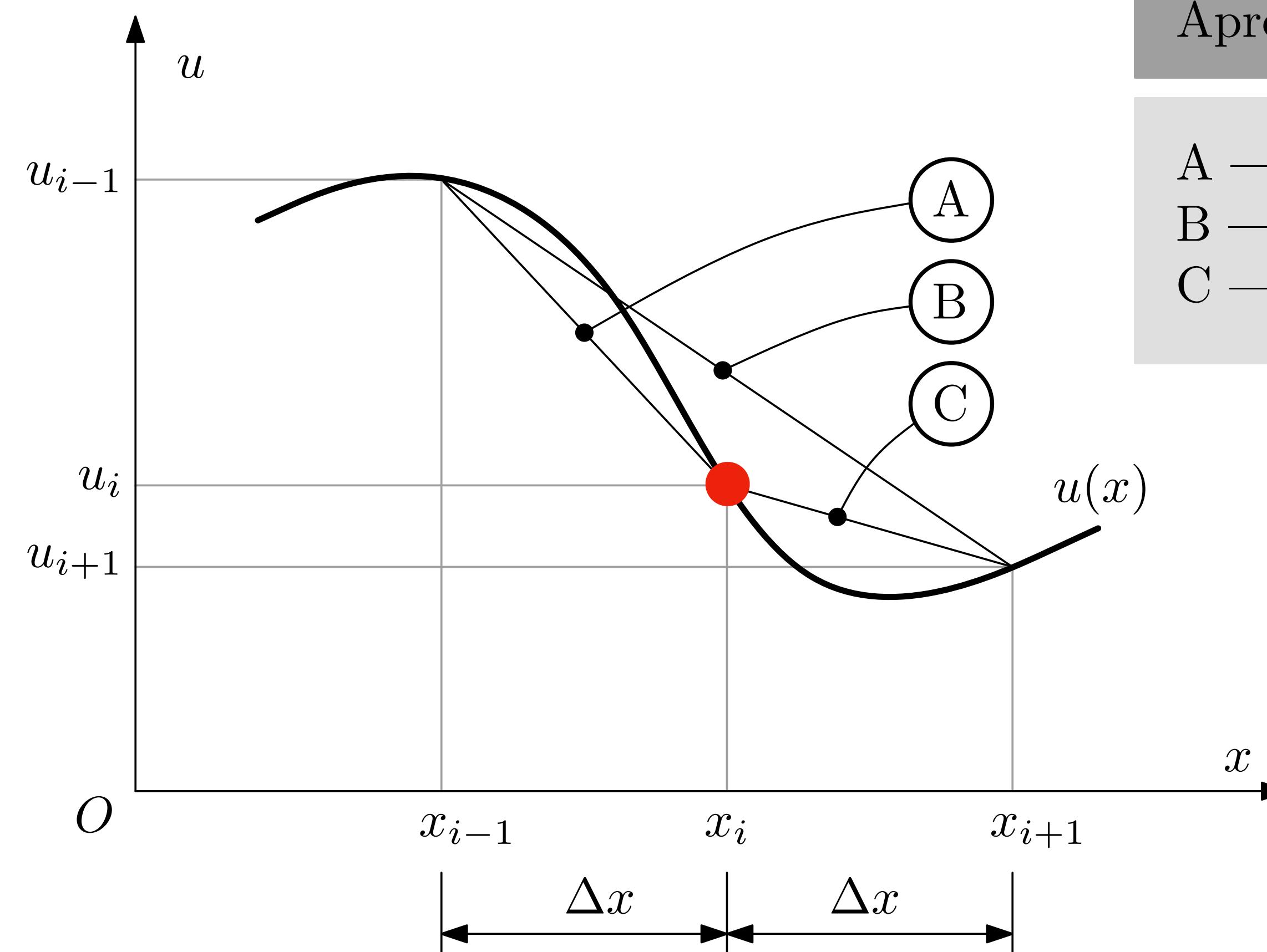
Método das Diferenças Finitas (MDF) - 1D



Aproximações por diferenças finitas

- A — Diferença finita atrás
- B — Diferença finita central
- C — Diferença finita à frente

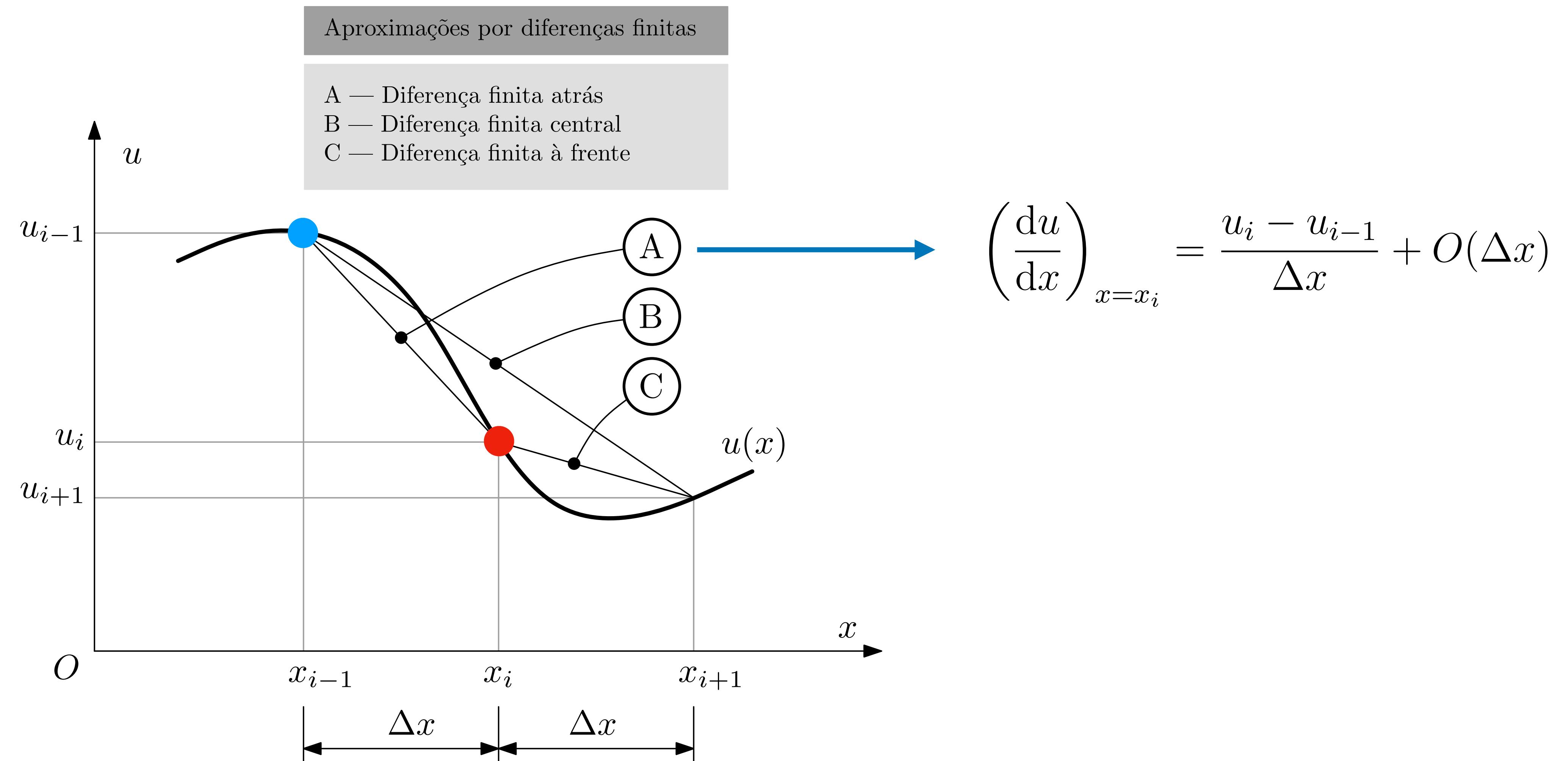
Método das Diferenças Finitas (MDF) - 1D



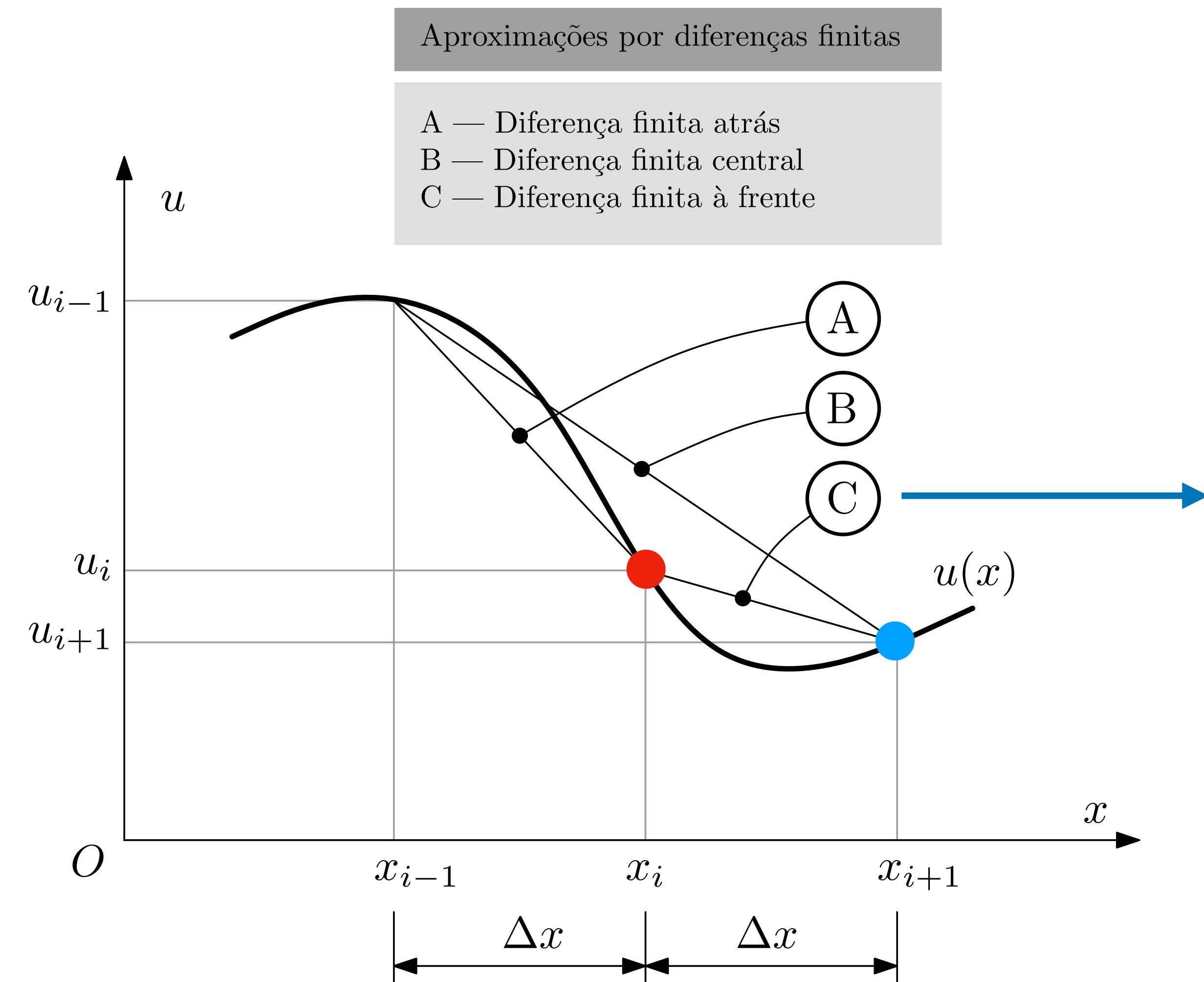
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Método das Diferenças Finitas (MDF) - 1D

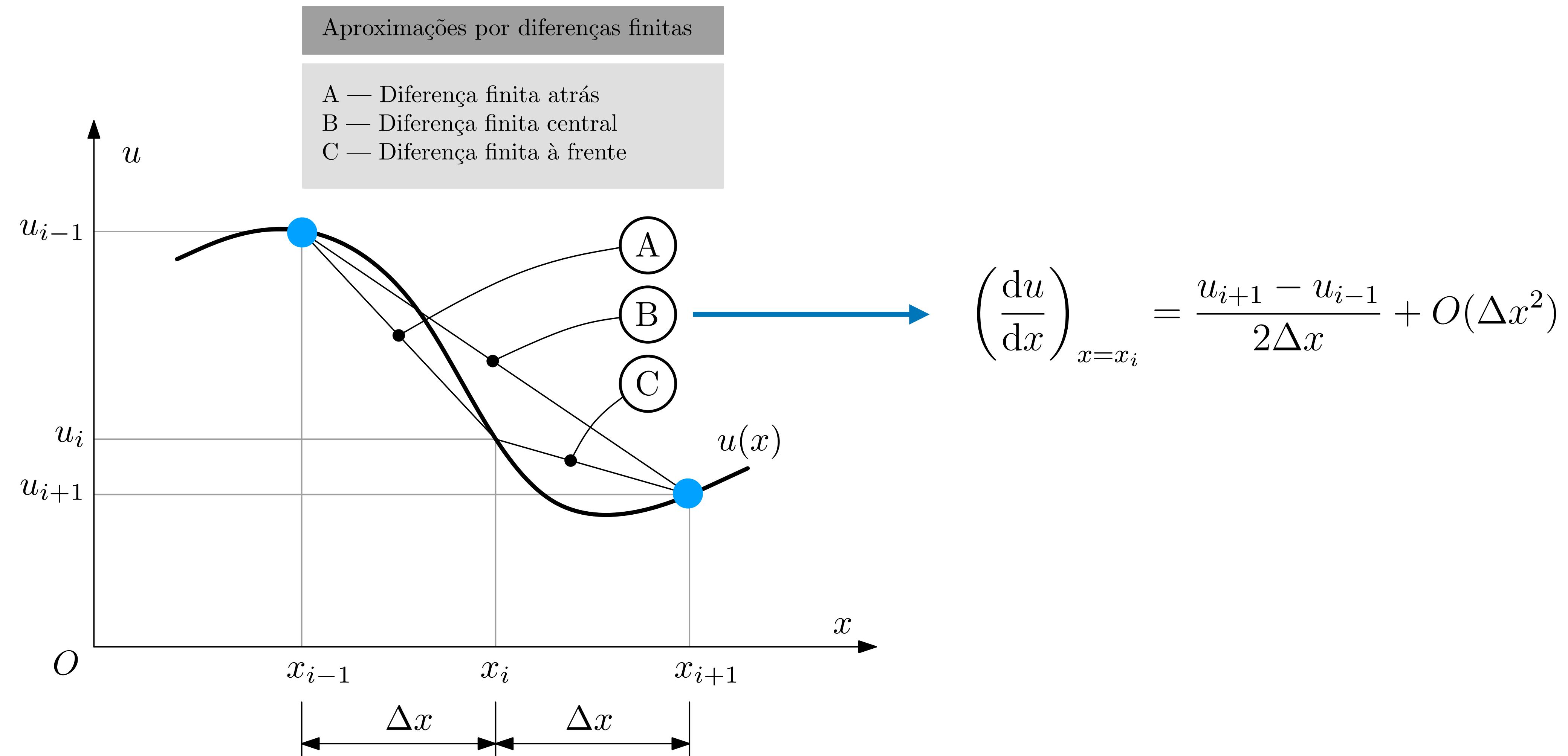


Método das Diferenças Finitas (MDF) - 1D

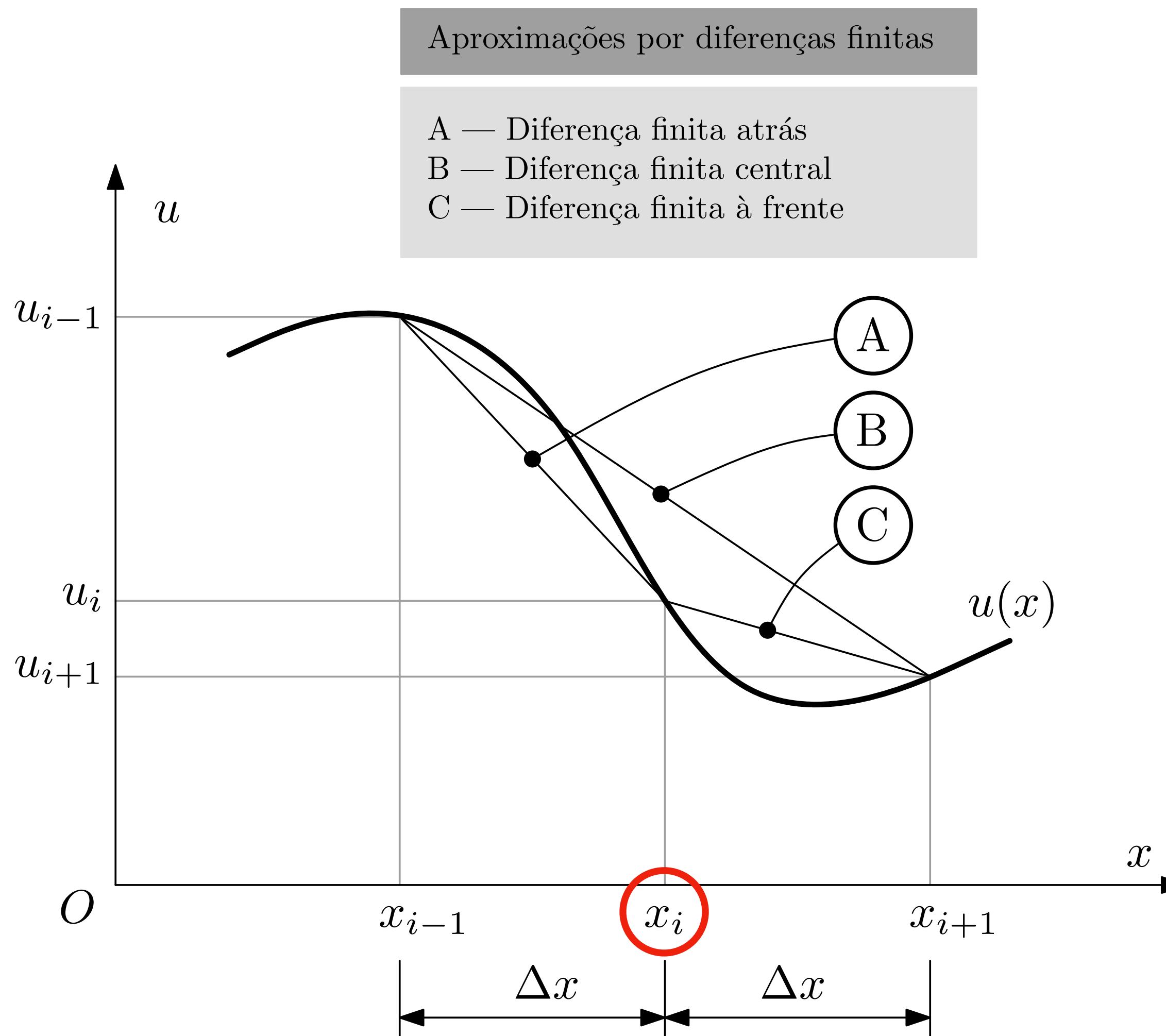


$$\left(\frac{du}{dx} \right)_{x=x_i} = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x)$$

Método das Diferenças Finitas (MDF) - 1D



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$$\left(\frac{du}{dx} \right)_{x=x_i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

$$\frac{1}{\Delta x} \begin{bmatrix} -1 & 1 \end{bmatrix} \rightarrow \text{Diferença finita atrás}$$

$$\frac{1}{\Delta x} \begin{bmatrix} 1 & -1 \end{bmatrix} \rightarrow \text{Diferença finita à frente}$$

$$\frac{1}{2\Delta x} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \rightarrow \text{Diferença finita central}$$

Método das Diferenças Finitas (MDF) - 1D

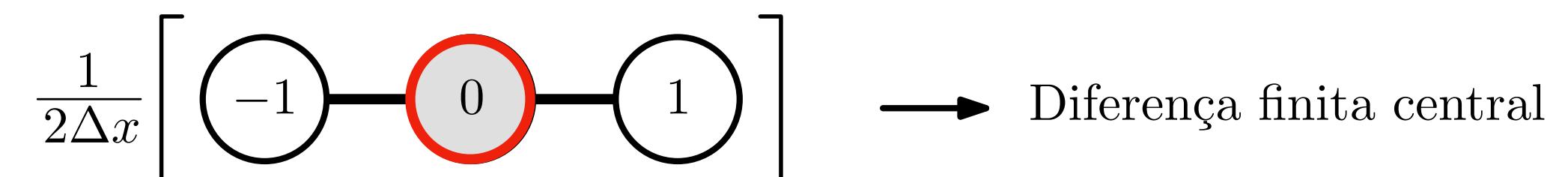
derivadas de ordem mais elevada...

$$\begin{aligned}\left(\frac{d^2u}{dx^2}\right)_{x=x_i} &= \frac{1}{\Delta x} \left[\left(\frac{du}{dx}\right)_{x=x_{i+1}} - \left(\frac{du}{dx}\right)_{x=x_i} \right] = \\ &= \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2).\end{aligned}$$

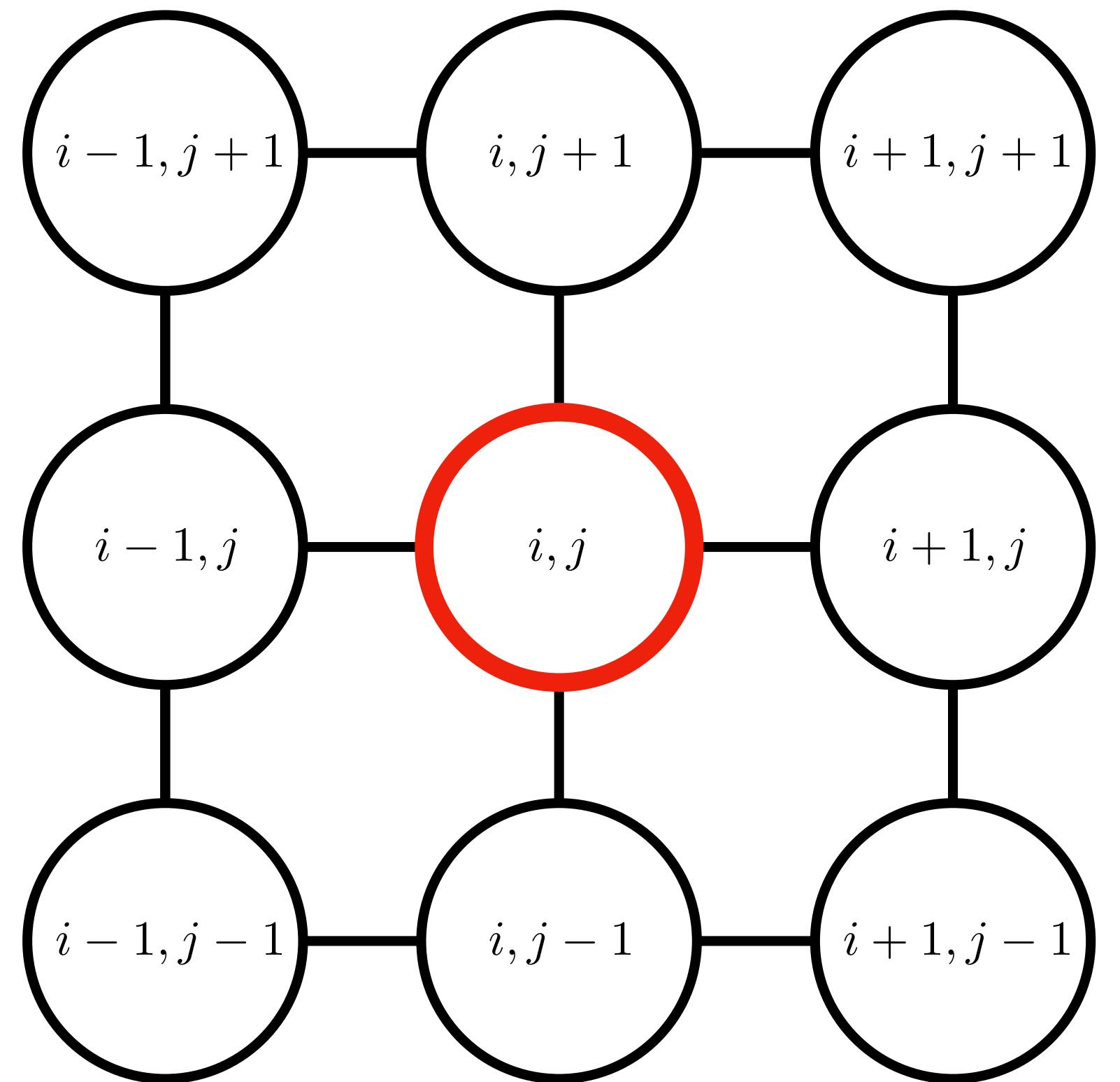
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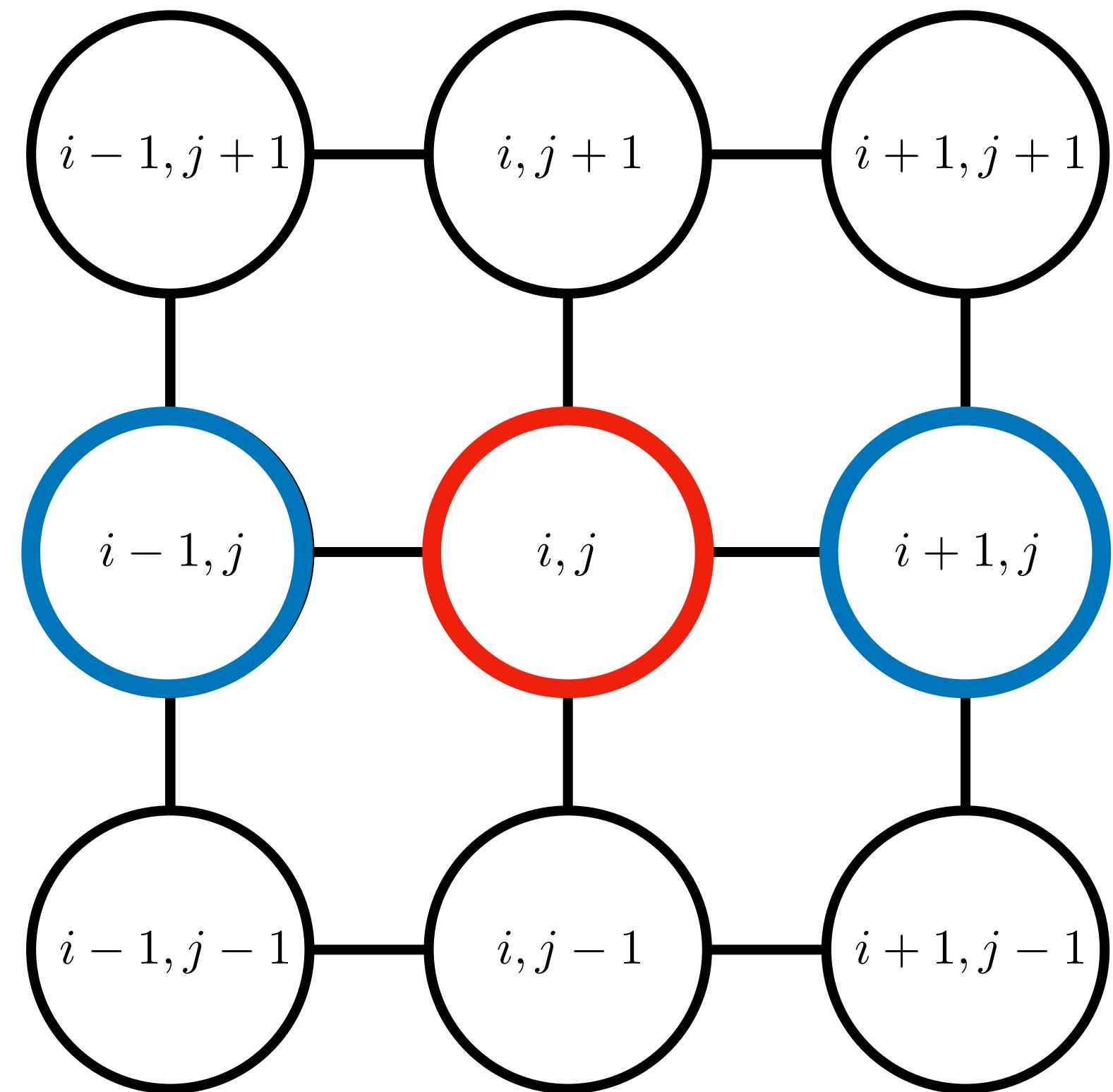
Método das Diferenças Finitas (MDF) - 2D



$$\left(\frac{du}{dx} \right)_{x=x_i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

$$\frac{1}{2\Delta x} \left[(-1, 0, 1) \right] \rightarrow \text{Diferença finita central}$$

Método das Diferenças Finitas (MDF) - 2D

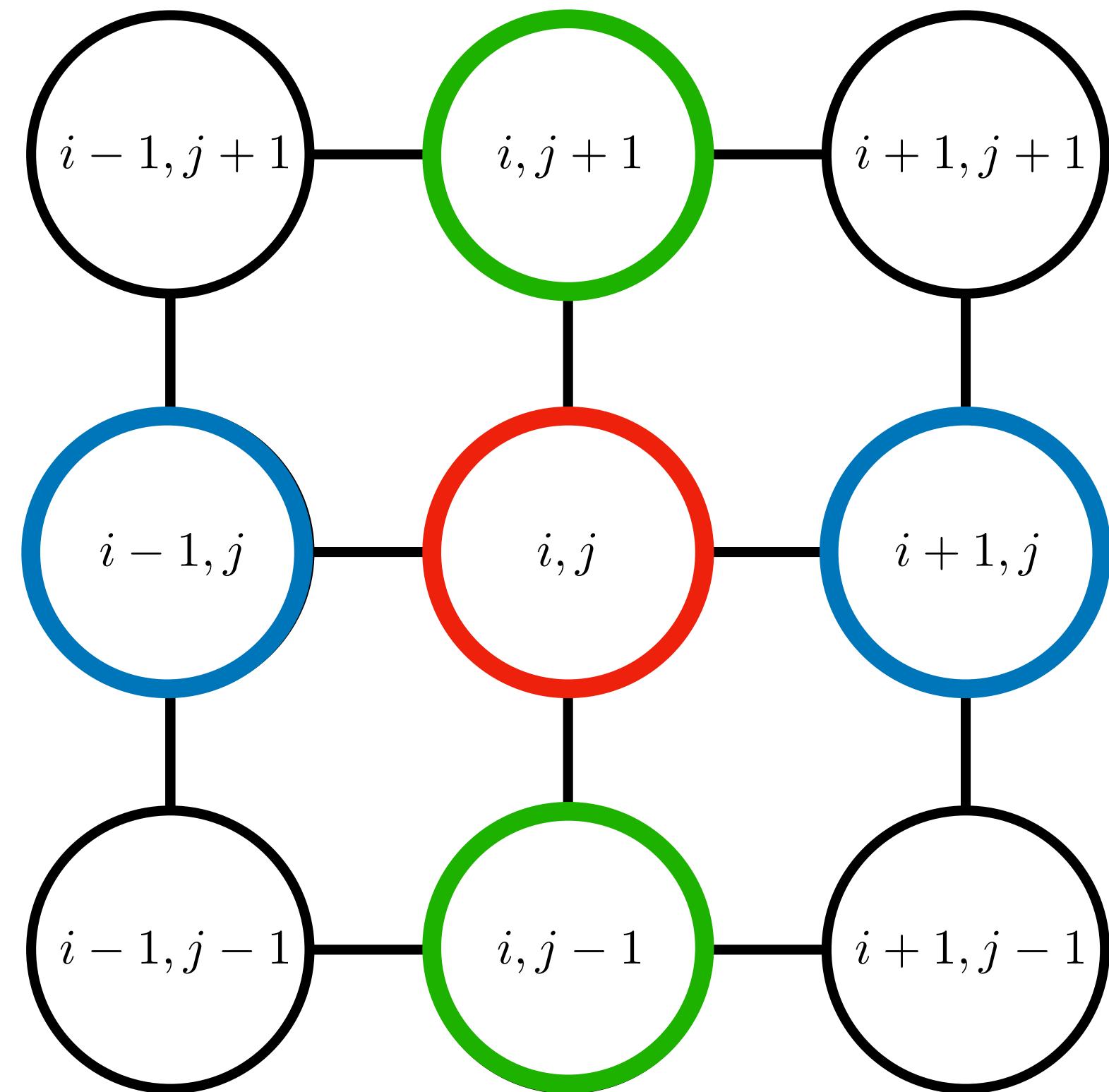


$$\left\{ \begin{array}{l} \left(\frac{\partial u}{\partial x} \right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2) \\ \left(\frac{\partial u}{\partial y} \right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} + O(\Delta y^2) \end{array} \right.$$

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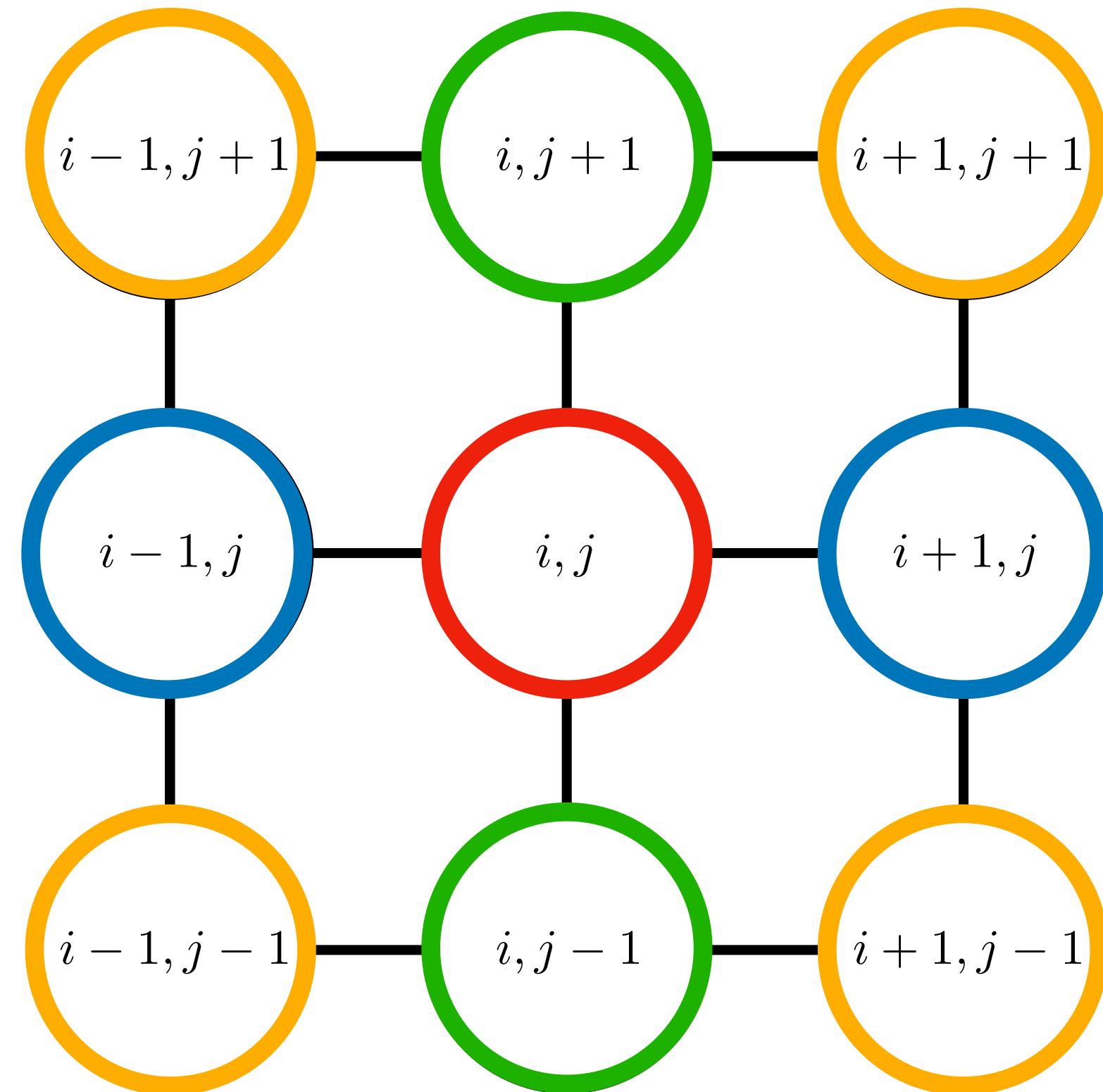


$$\left\{ \begin{array}{l} \left(\frac{\partial u}{\partial x} \right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2) \\ \left(\frac{\partial u}{\partial y} \right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} + O(\Delta y^2) \end{array} \right.$$

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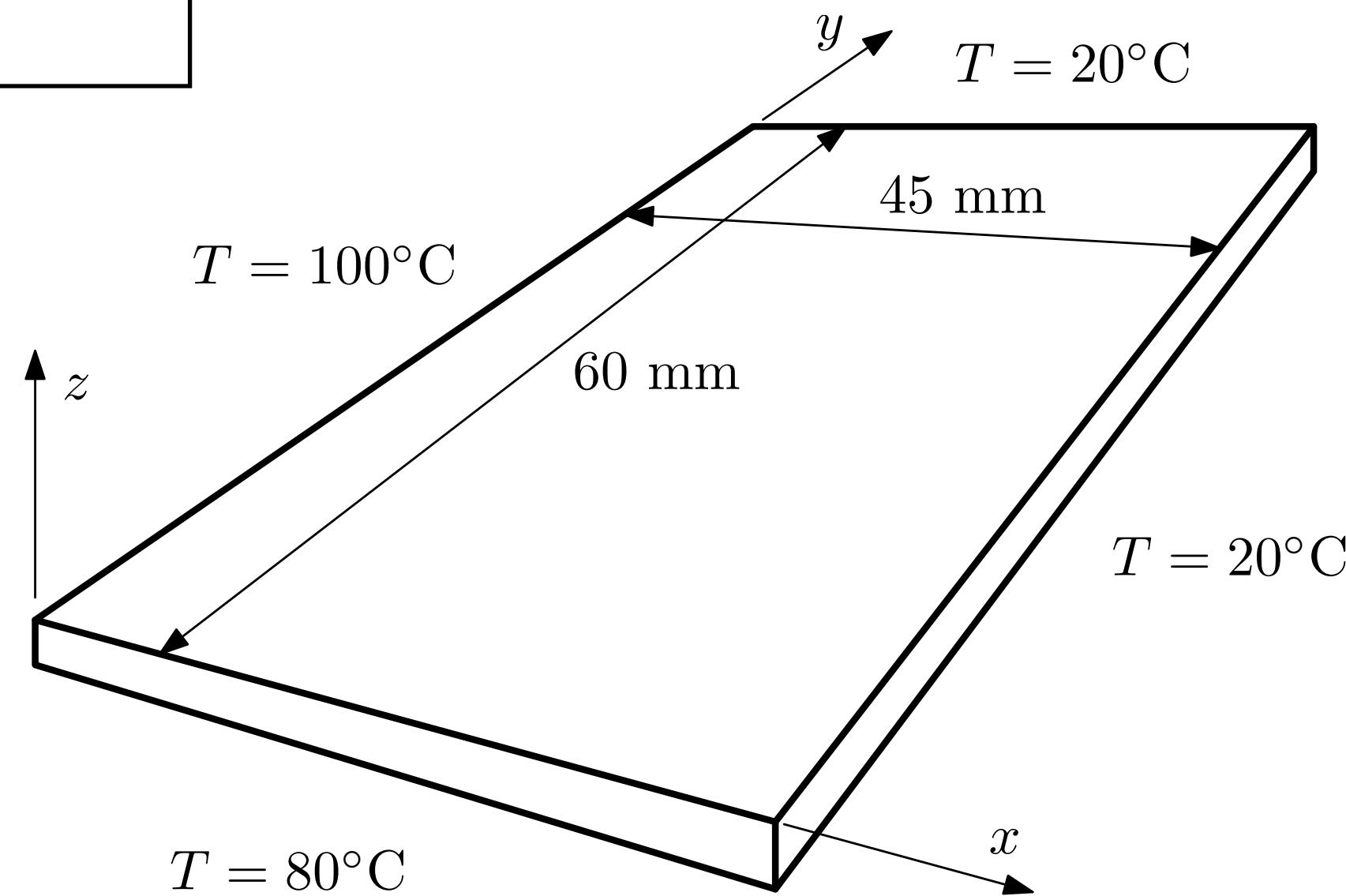
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$$\begin{cases} \left(\frac{\partial u}{\partial x} \right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2) \\ \left(\frac{\partial u}{\partial y} \right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} + O(\Delta y^2) \end{cases} .$$

$$\begin{aligned} \left(\frac{\partial^2 u}{\partial x \partial y} \right)_{u_{i,j}} &= \left(\frac{\partial^2 u}{\partial y \partial x} \right)_{u_{i,j}} = \\ &= \frac{1}{4\Delta x \Delta y} (u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}) \end{aligned}$$

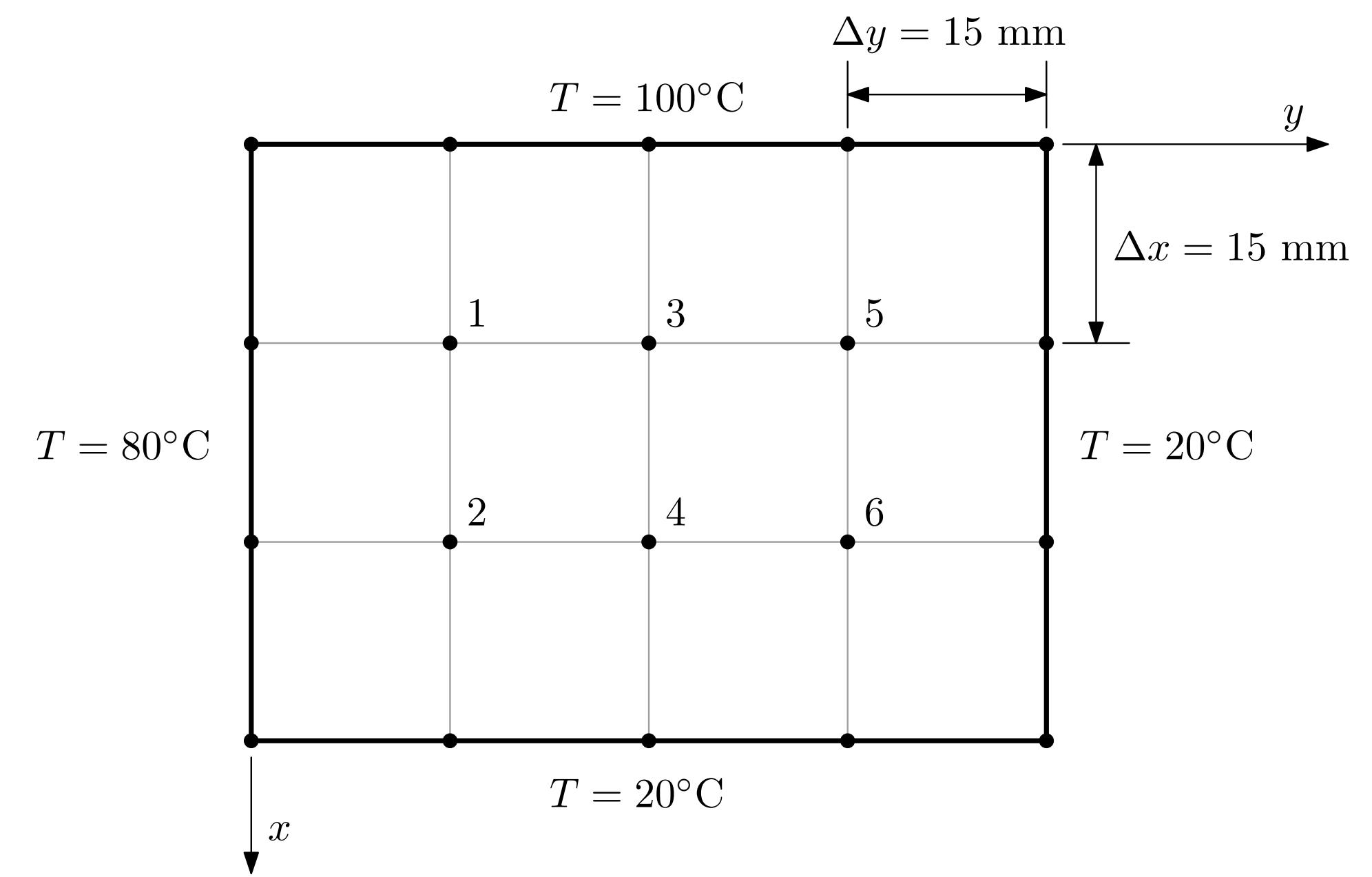
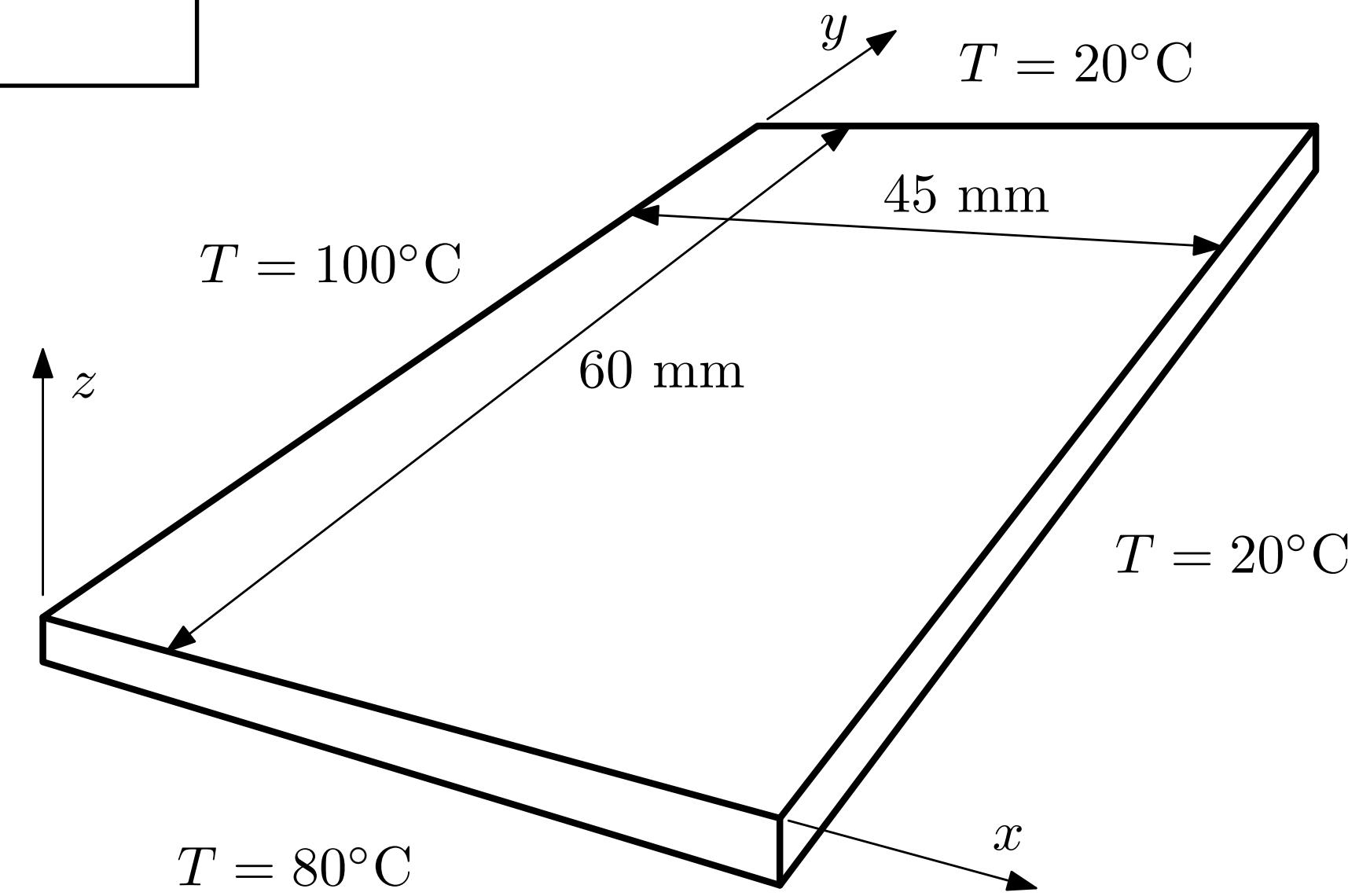
Exemplo - Distribuição de temperaturas numa placa (2D)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



Exemplo - Distribuição de temperaturas numa placa (2D)

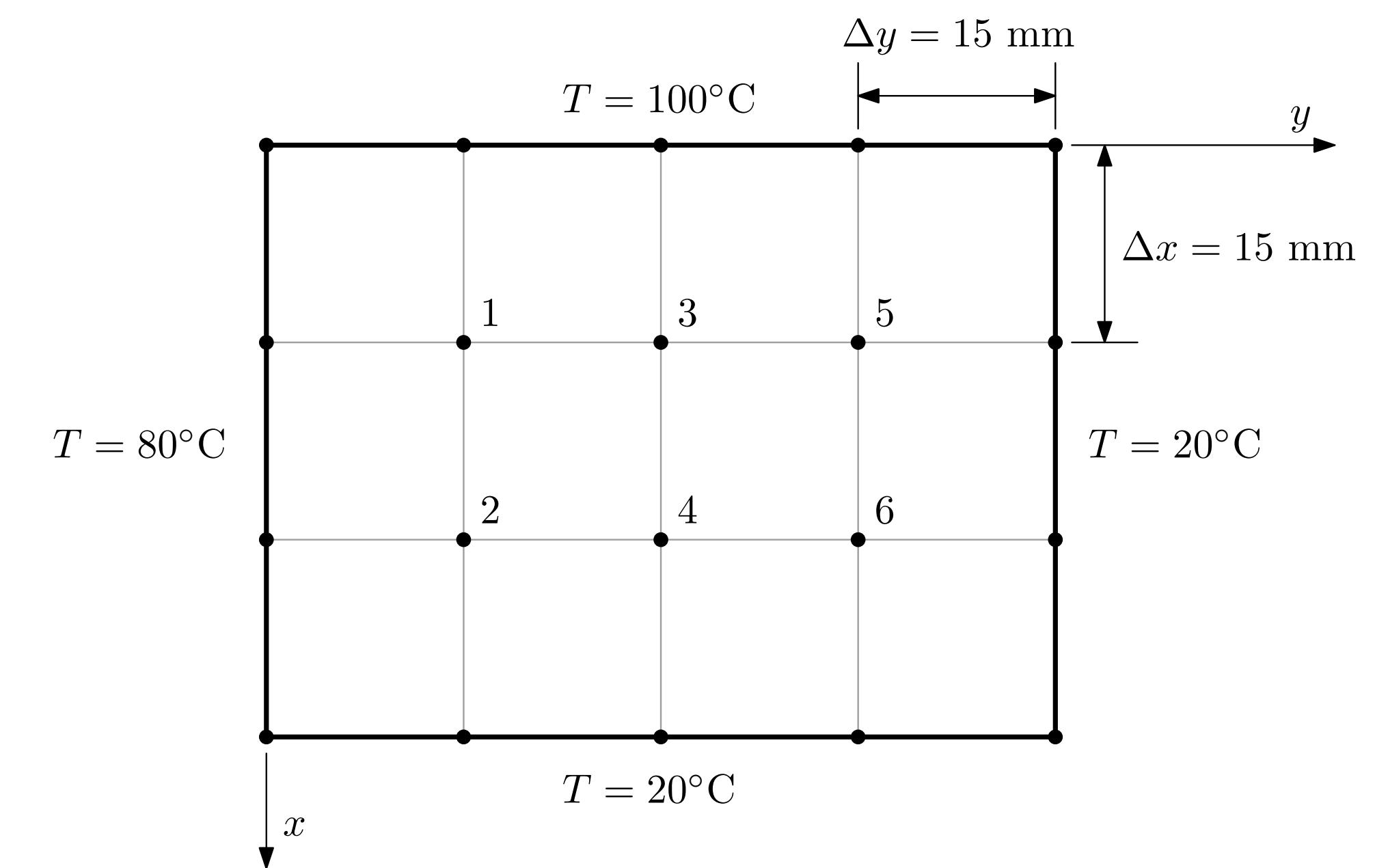
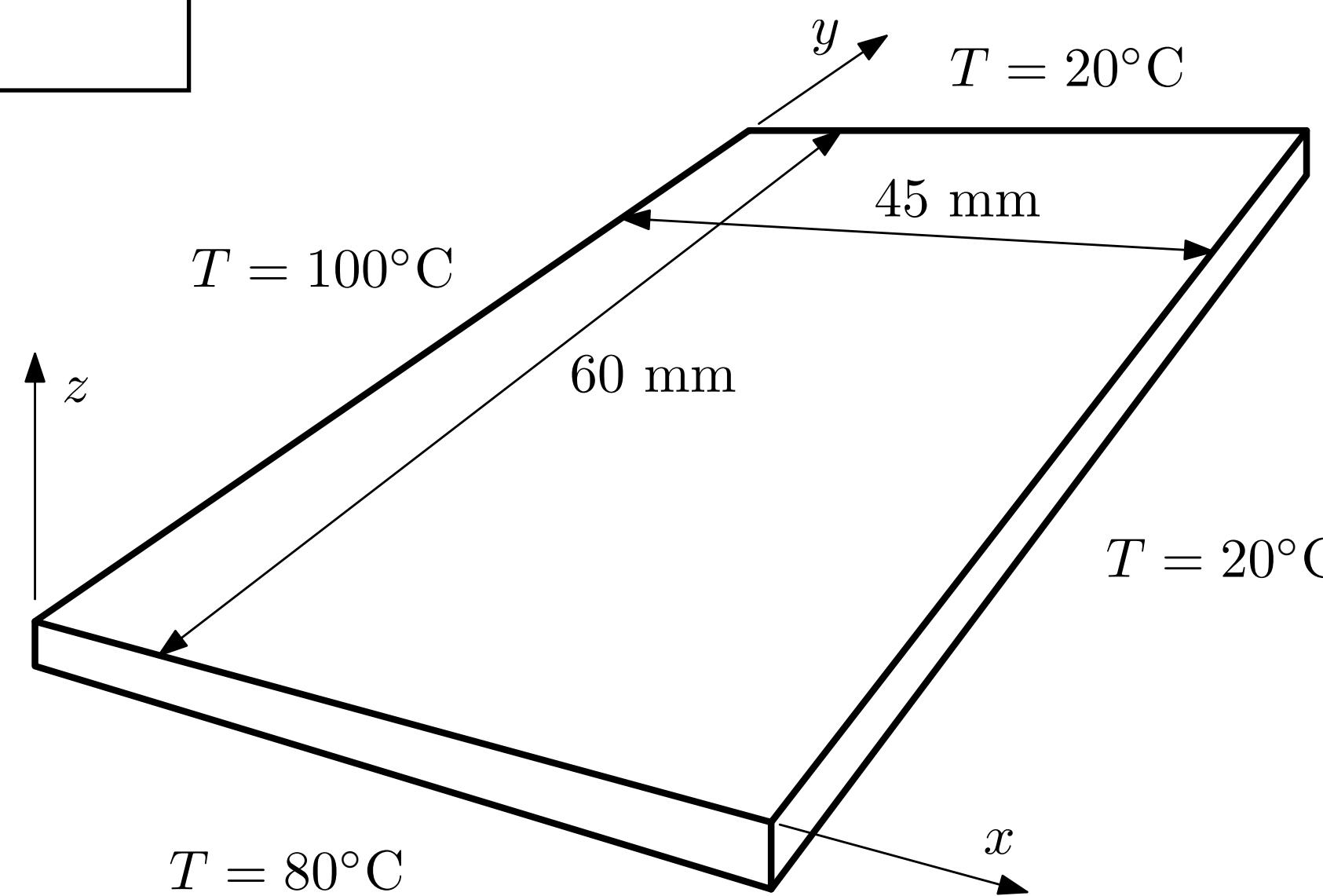
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



$$\left\{ \begin{array}{l} \left(\frac{\partial^2 T}{\partial x^2} \right)_{i,j} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} \\ \left(\frac{\partial^2 T}{\partial y^2} \right)_{i,j} = \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} \end{array} \right.$$

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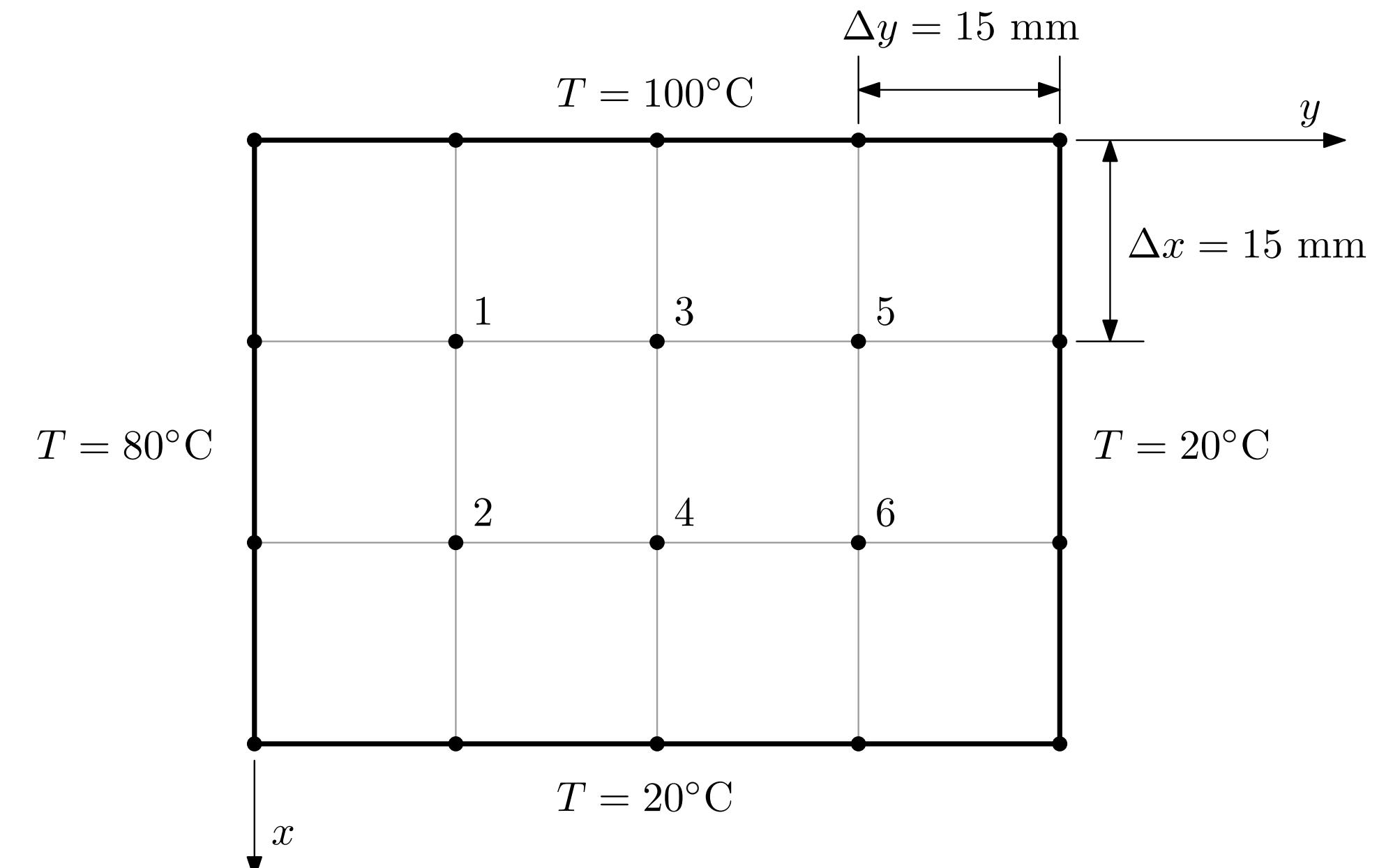
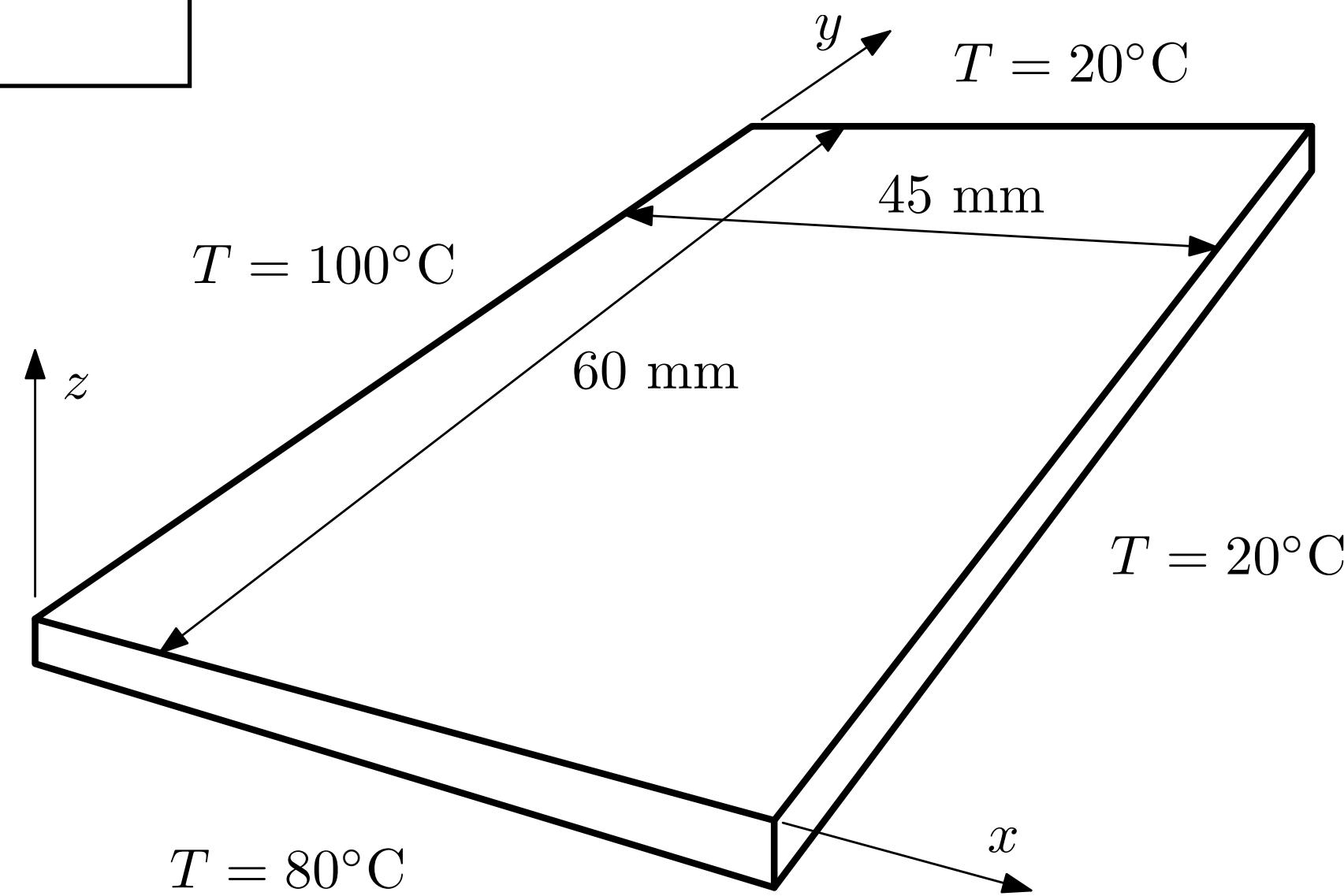


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$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} = 0$$

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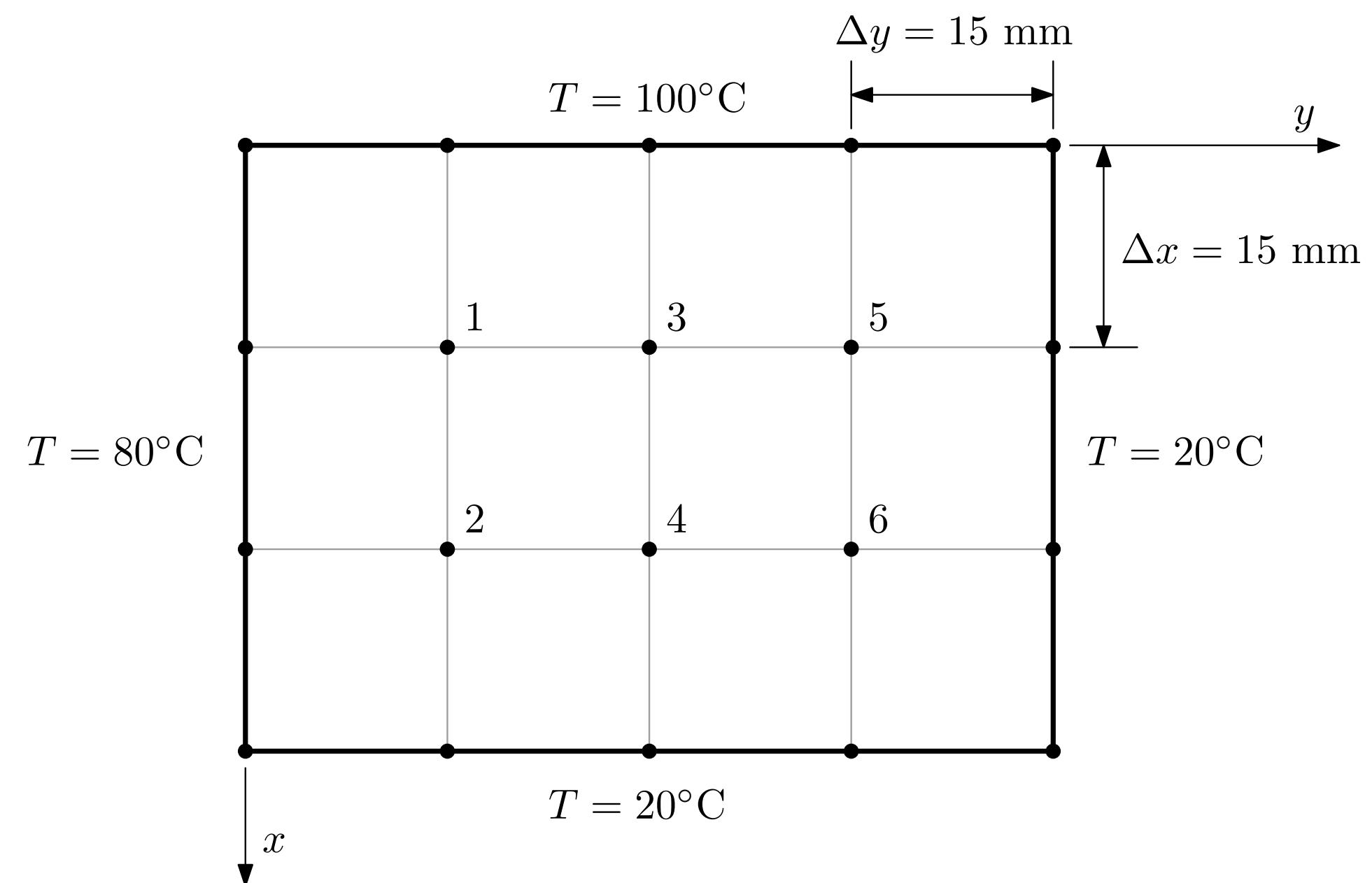


$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} = 0$$

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$$T_{i-1,j} + T_{i+1,j} - 4T_{i,j} + T_{i,j-1} + T_{i,j+1} = 0$$

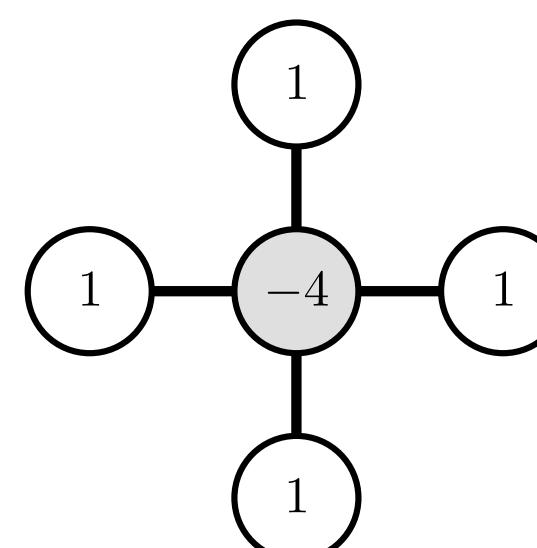
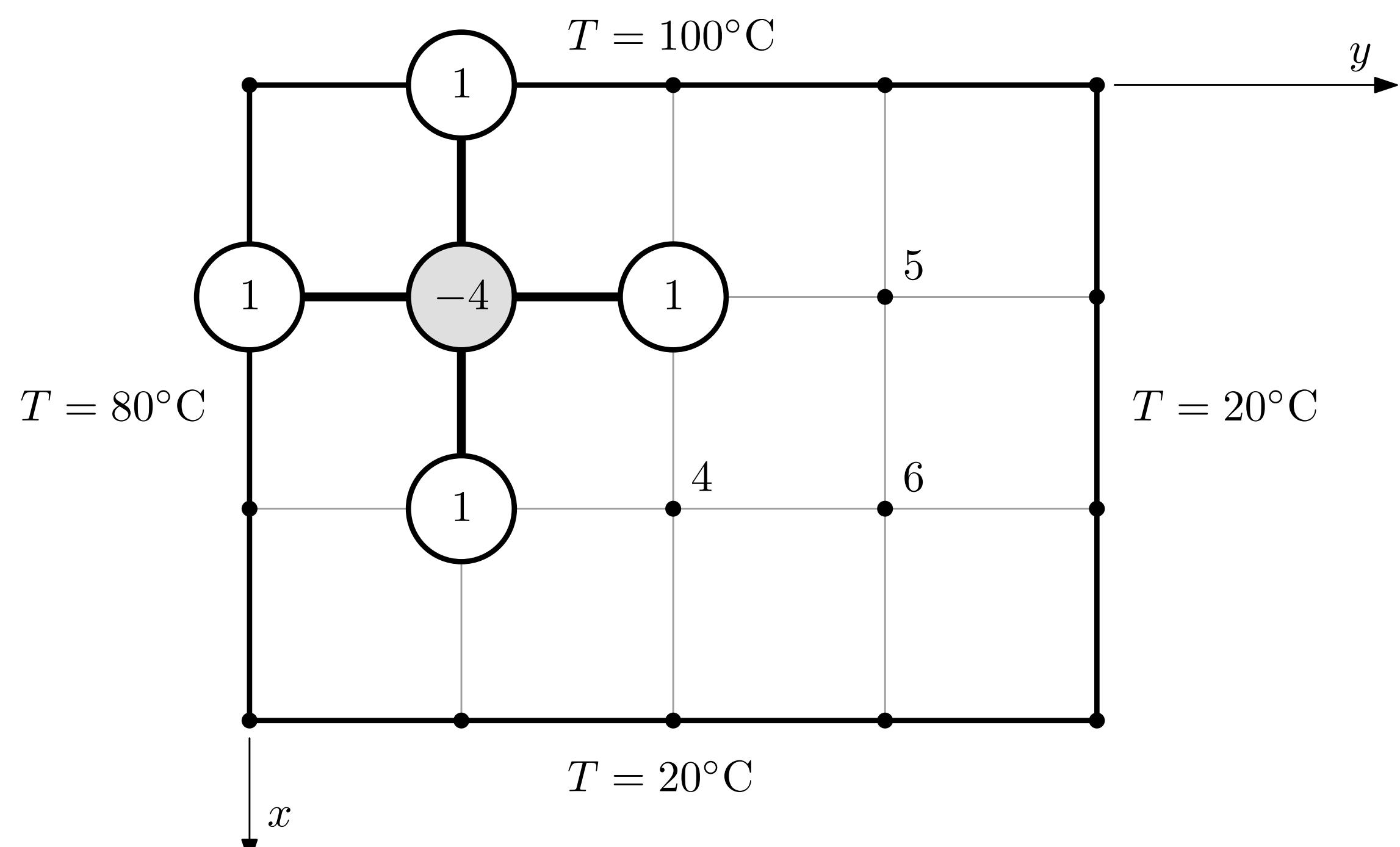
Exemplo - Distribuição de temperaturas numa placa (2D)



$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} = 0$$

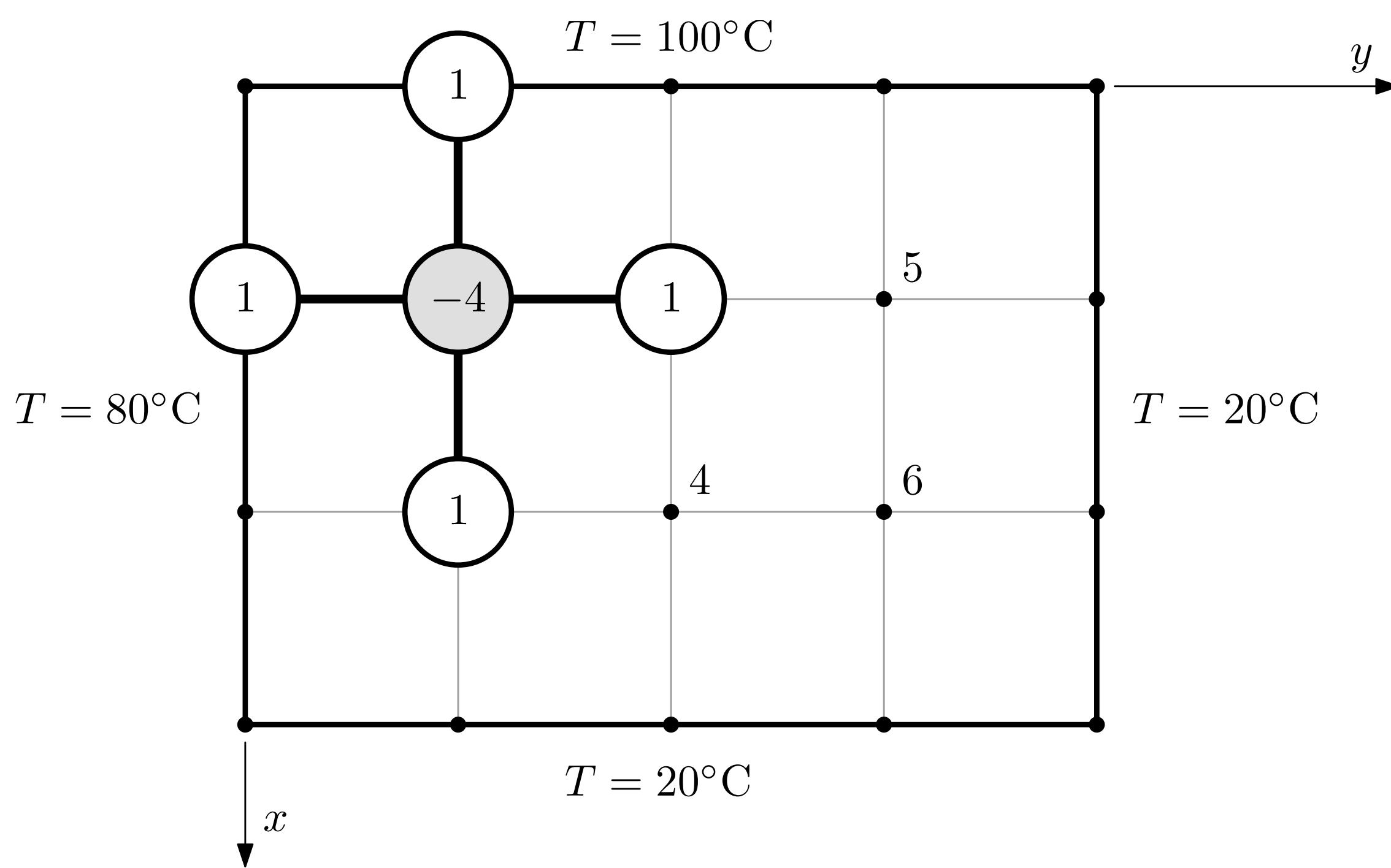
$$T_{i-1,j} + T_{i+1,j} - 4T_{i,j} + T_{i,j-1} + T_{i,j+1} = 0$$

$$-4T_1 + T_2 + T_3 + 80 + 100 = 0.$$

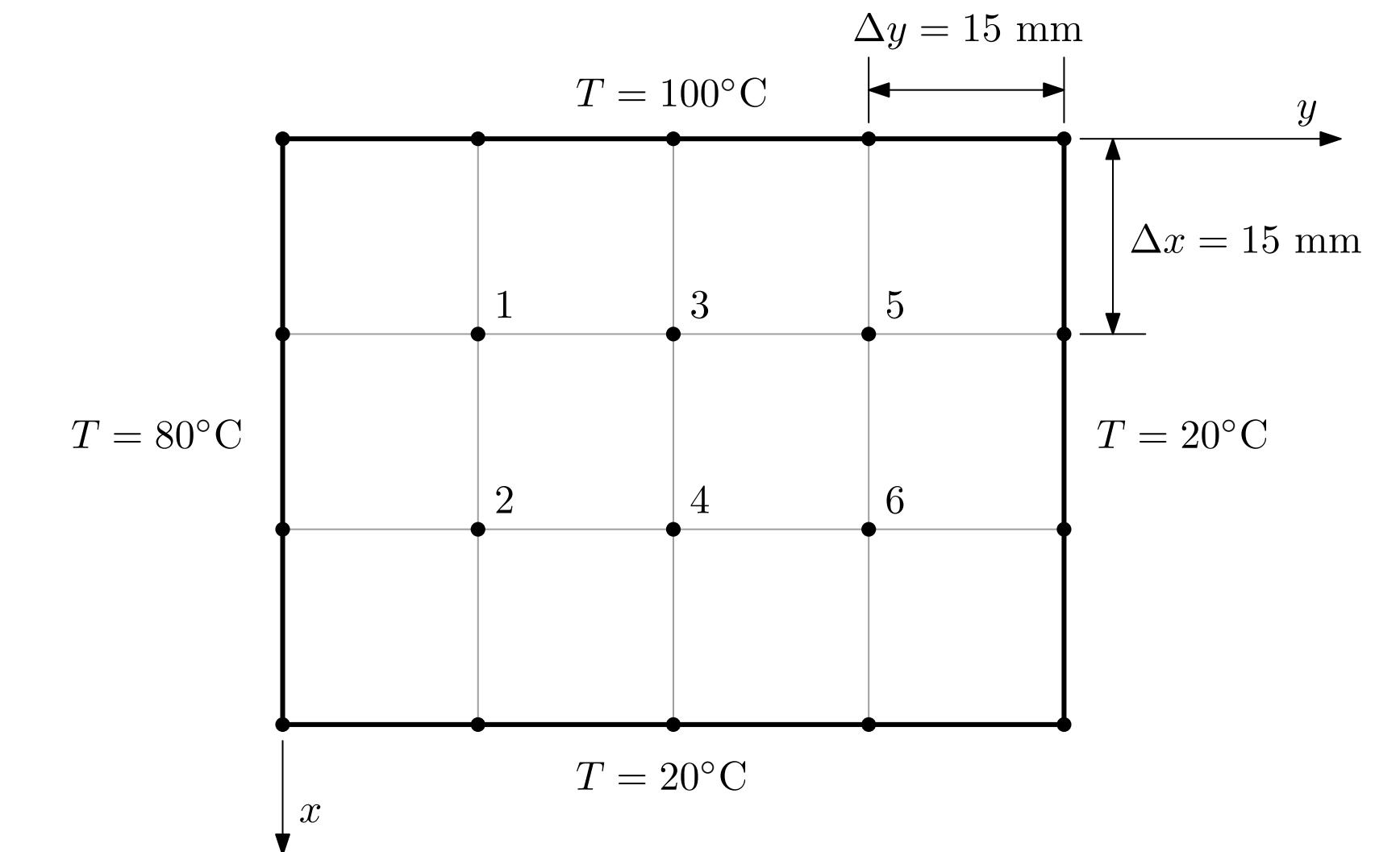


Exemplo - Distribuição de temperaturas numa placa (2D)

$$-4T_1 + T_2 + T_3 + 80 + 100 = 0.$$

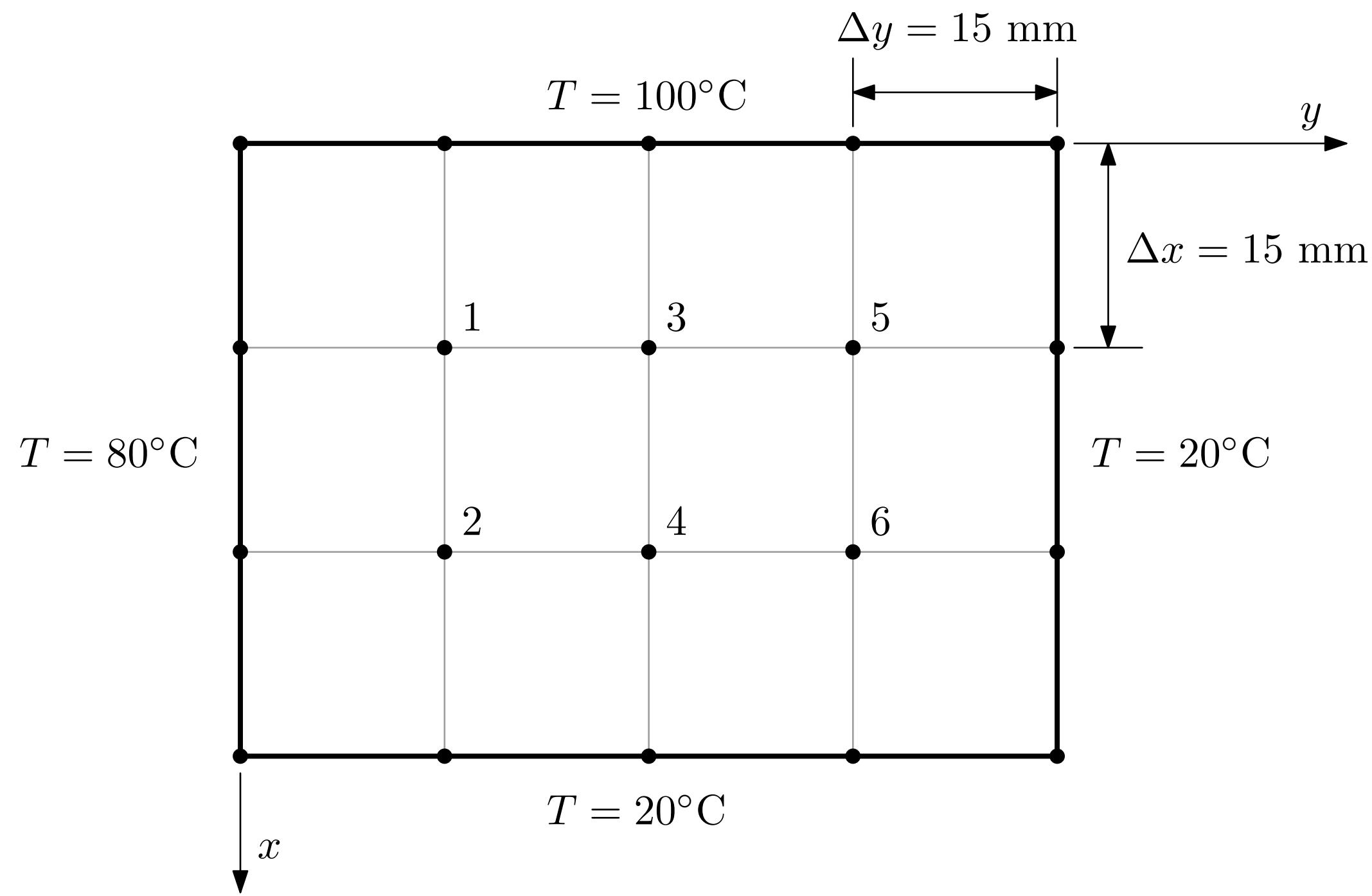


$$T_{i-1,j} + T_{i+1,j} - 4T_{i,j} + T_{i,j-1} + T_{i,j+1} = 0$$

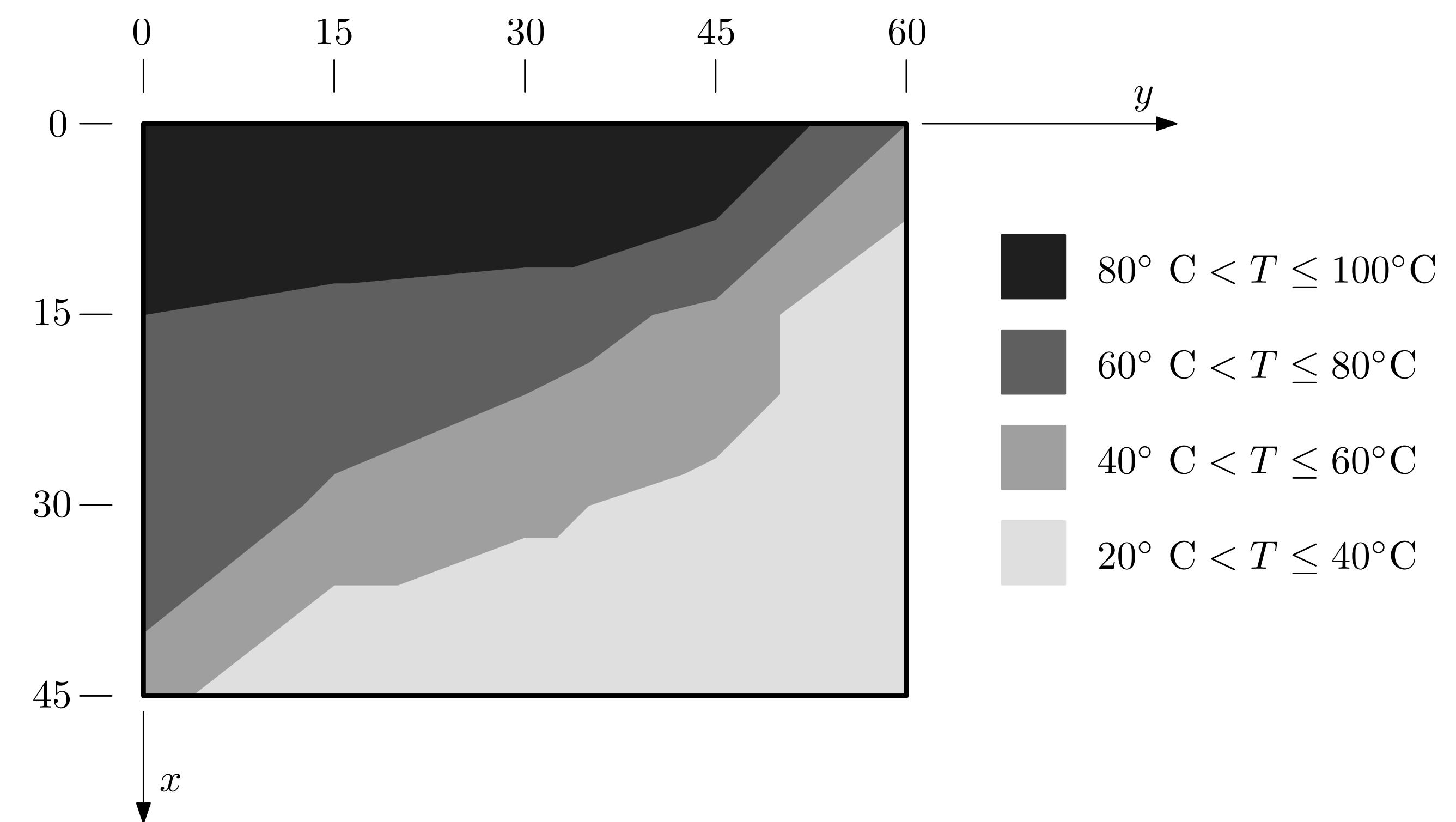


$$\left\{ \begin{array}{l} \text{Nó 1} \Rightarrow -4T_1 + T_2 + T_3 + 80 + 100 = 0 \\ \text{Nó 2} \Rightarrow -4T_2 + T_1 + 20 + 80 + T_4 = 0 \\ \text{Nó 3} \Rightarrow -4T_3 + T_4 + T_5 + T_1 + 100 = 0 \\ \text{Nó 4} \Rightarrow -4T_4 + T_3 + 20 + T_6 + T_2 = 0 \\ \text{Nó 5} \Rightarrow -4T_5 + T_6 + 100 + T_3 + 20 = 0 \\ \text{Nó 6} \Rightarrow -4T_6 + T_5 + 20 + 20 + T_4 = 0 \end{array} \right.$$

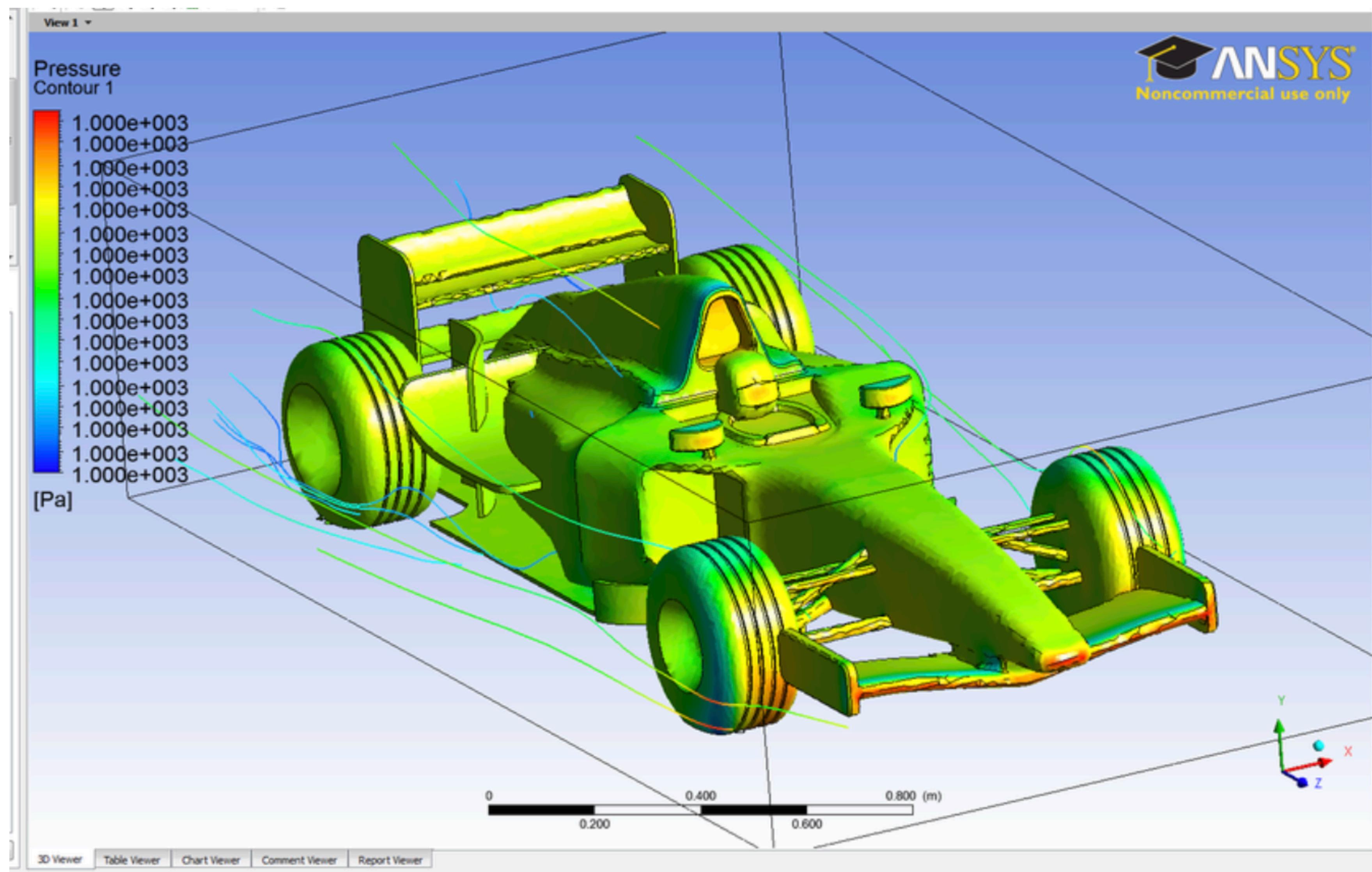
Exemplo - Distribuição de temperaturas numa placa (2D)



$$\left\{ \begin{array}{l} T_1 = 76, 15^\circ\text{C} \\ T_2 = 55, 28^\circ\text{C} \\ T_3 = 69, 32^\circ\text{C} \\ T_4 = 44, 97^\circ\text{C} \\ T_5 = 56, 15^\circ\text{C} \\ T_6 = 35, 28^\circ\text{C} \end{array} \right.$$

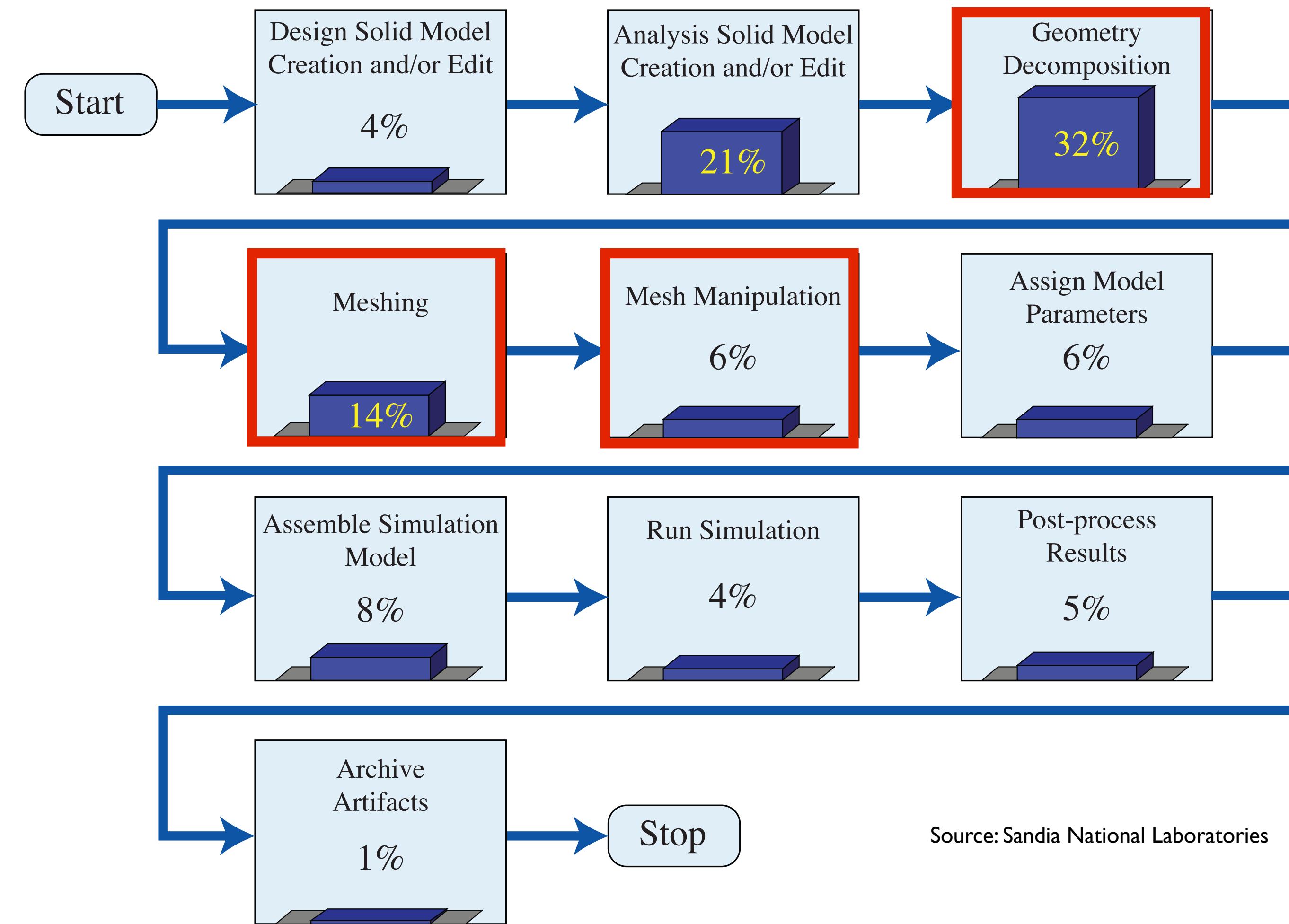


Exemplo - Distribuição de pressões / arrasto sobre um volume (CFD)



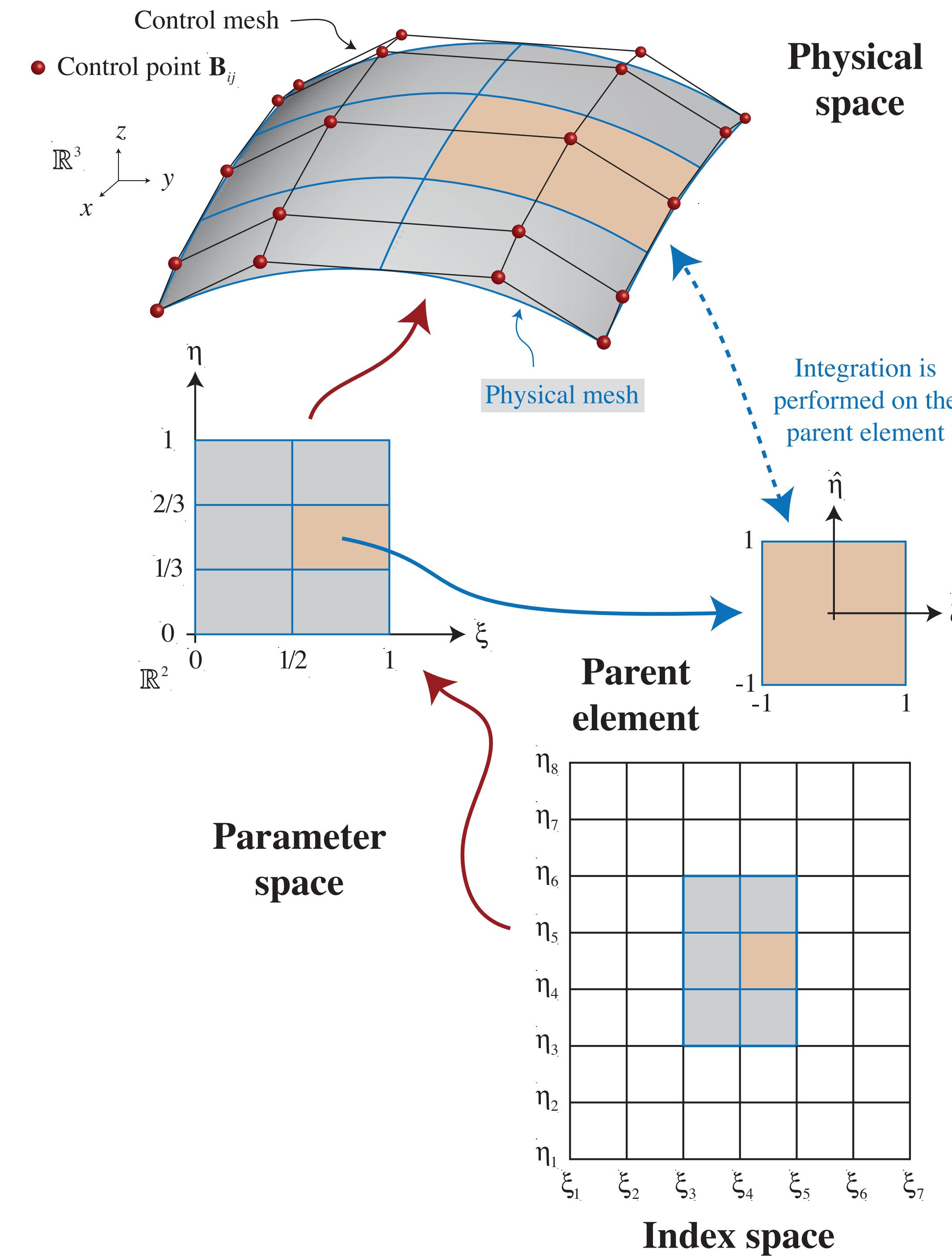
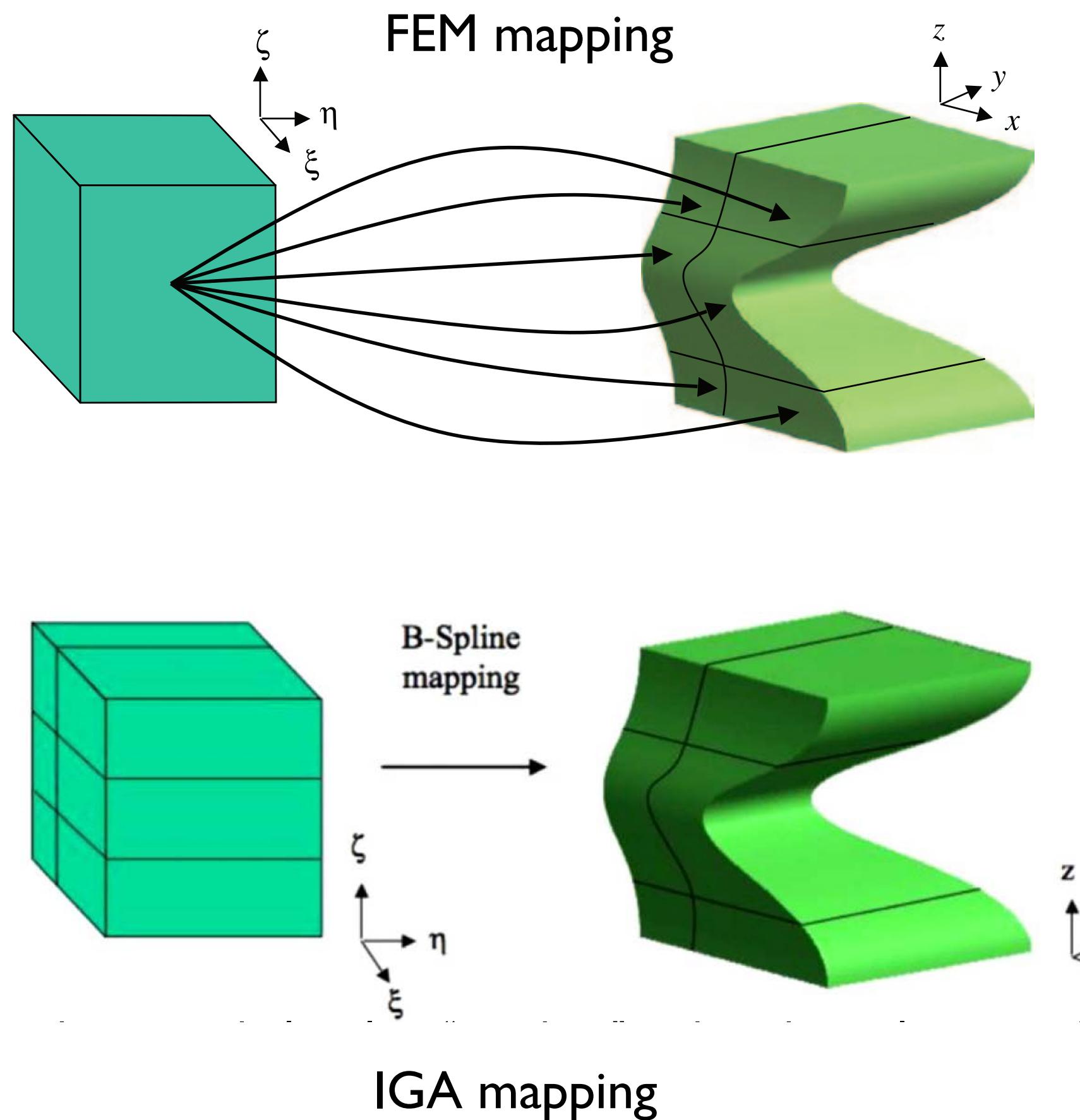
Isogeometric Analysis (IgA)

Motivation: a smooth transition between CAD and numerical simulation

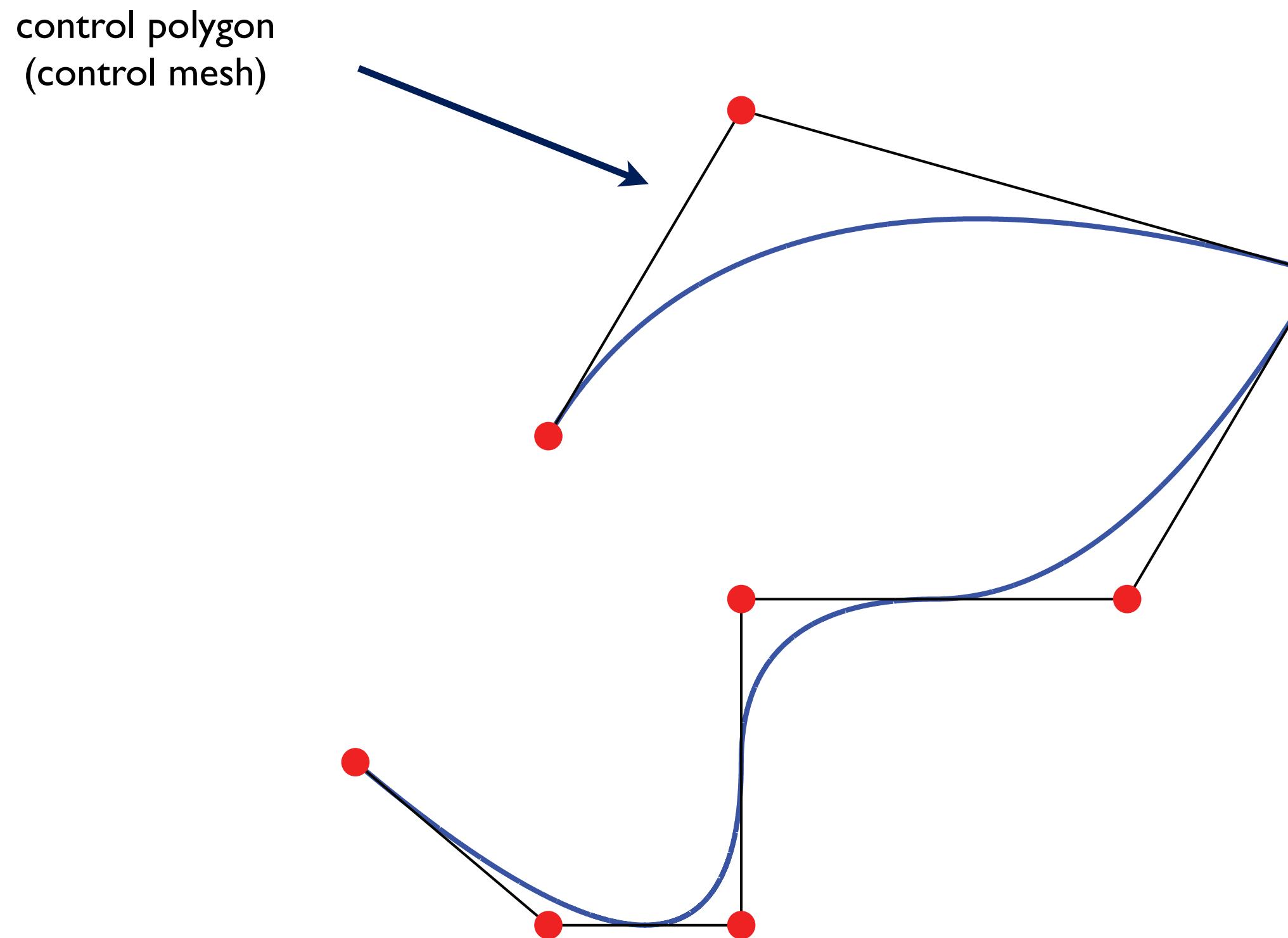


Source: Sandia National Laboratories

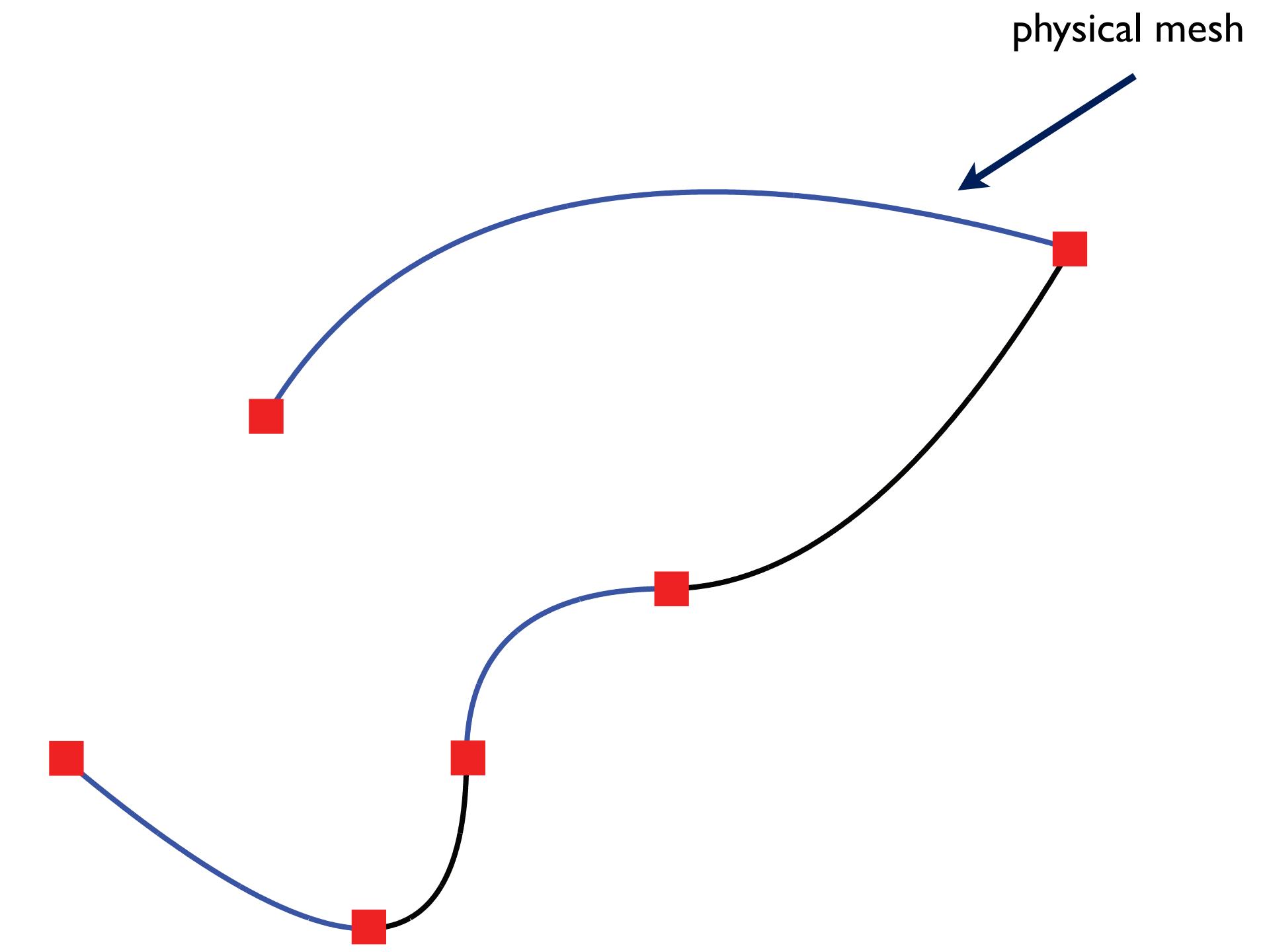
other potential applications with advantages over FEM: refinement, contact, buckling, optimization, etc.



por exemplo, criar nós e elementos IgA sobre uma curva genérica



(a) Curve and control points

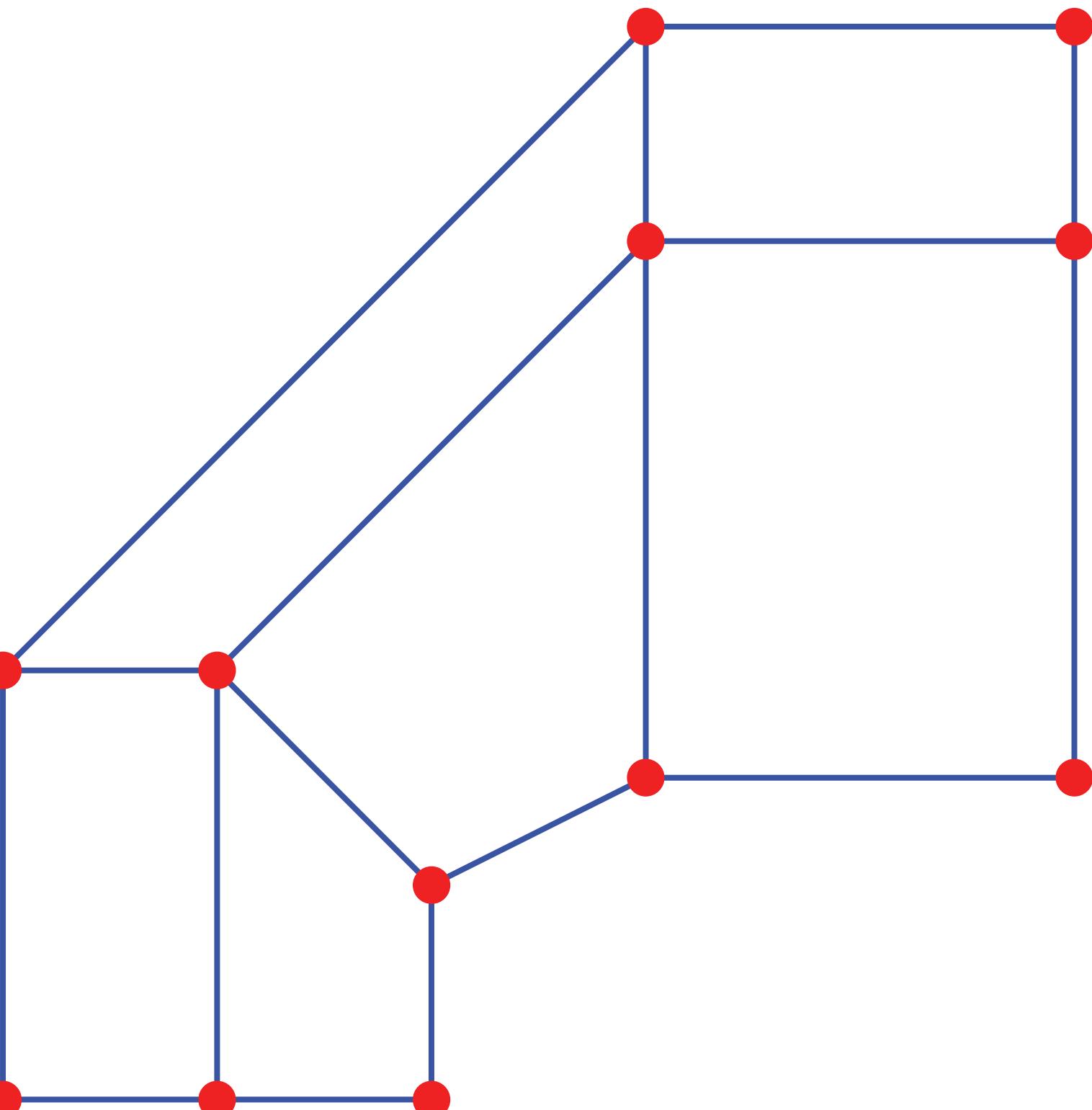


(b) Curve and mesh denoted by knot locations

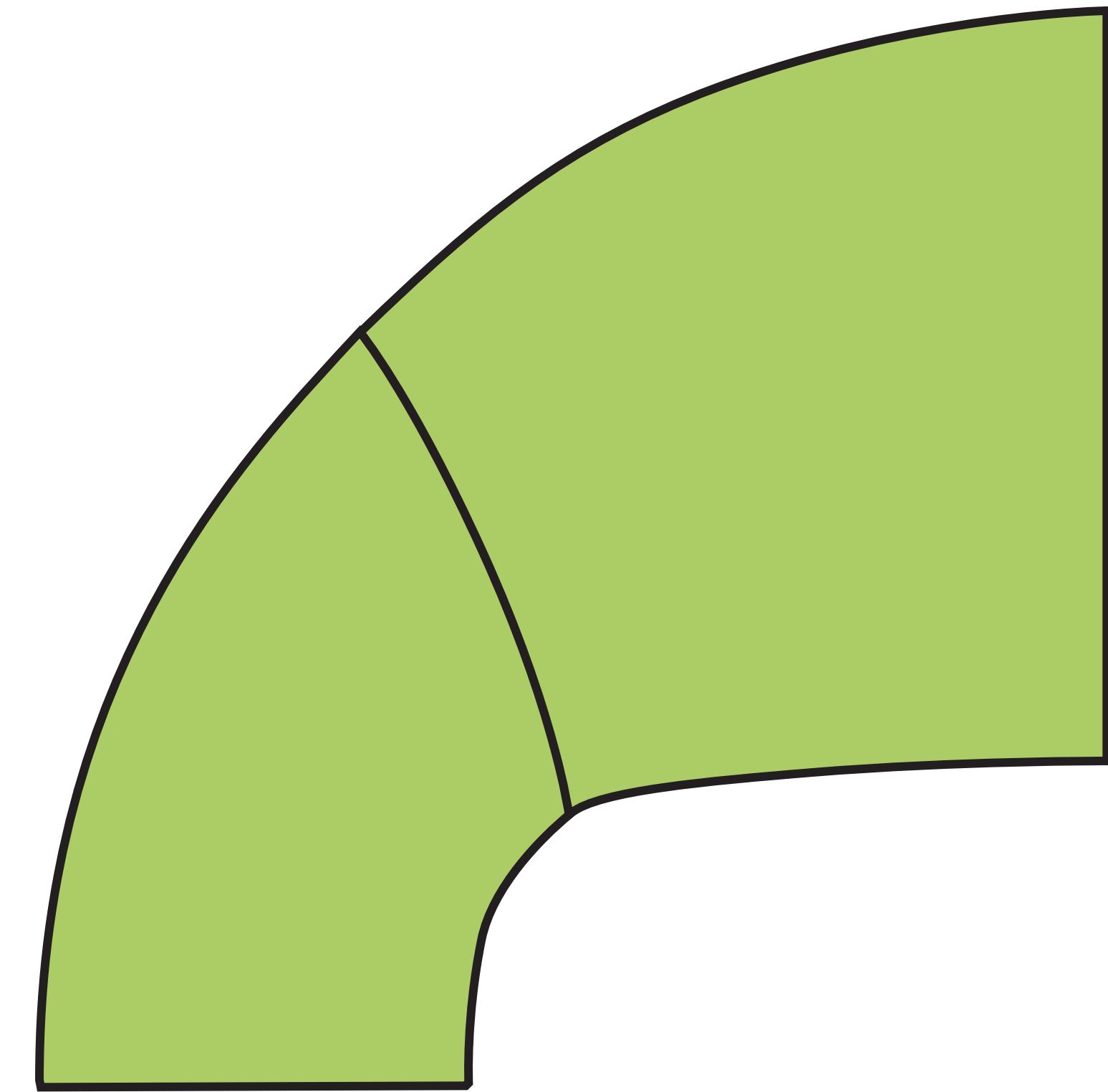
ou então, criar nós e elementos IgA sobre uma superfície genérica

control points
(control mesh)

i	j	$\mathbf{B}_{i,j}$
1	1	(0, 0)
1	2	(-1, 0)
1	3	(-2, 0)
2	1	(0, 1)
2	2	(-1, 2)
2	3	(-2, 2)
3	1	(1, 1.5)
3	2	(1, 4)
3	3	(1, 5)
4	1	(3, 1.5)
4	2	(3, 4)
4	3	(3, 5)

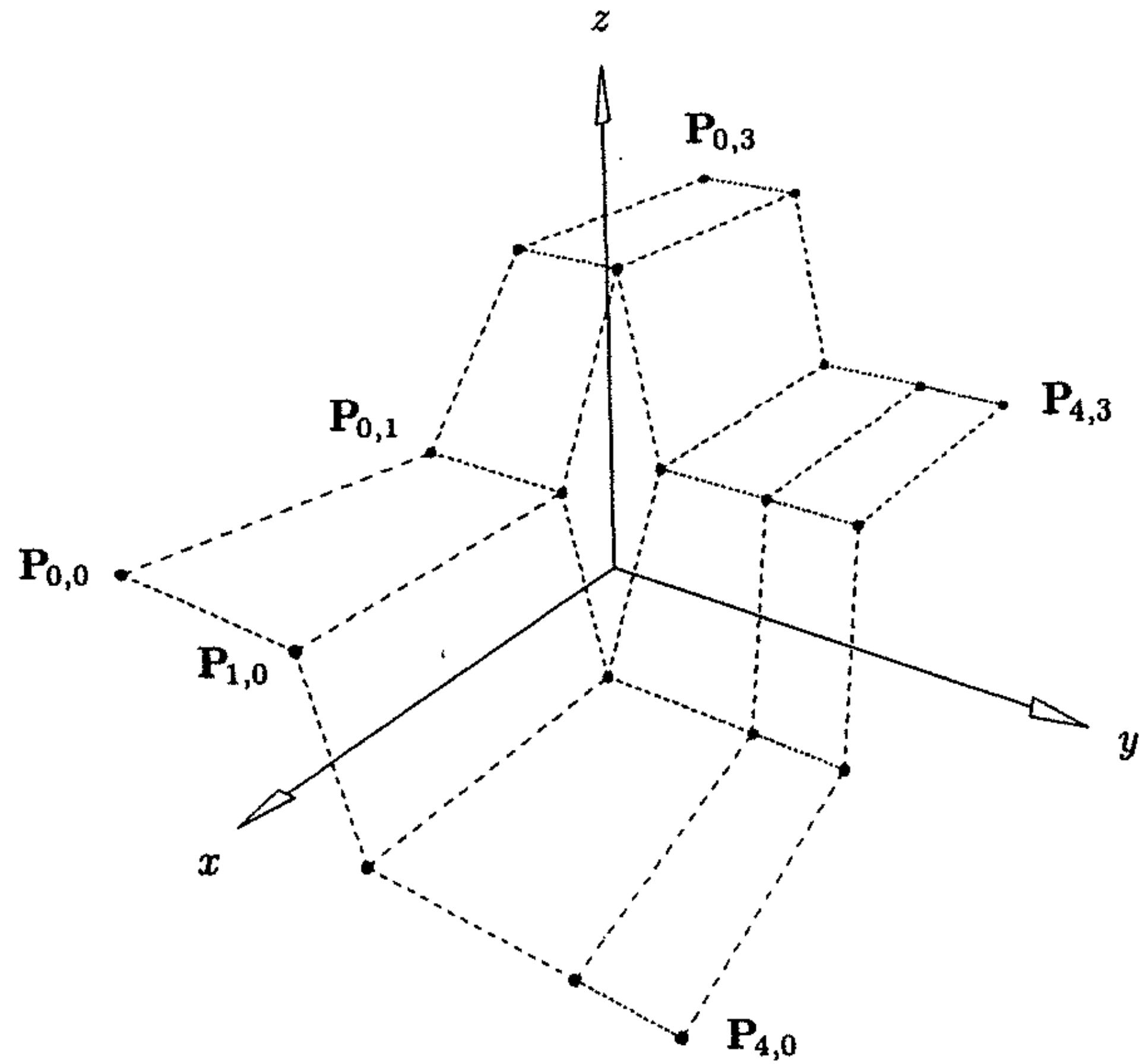


Control net

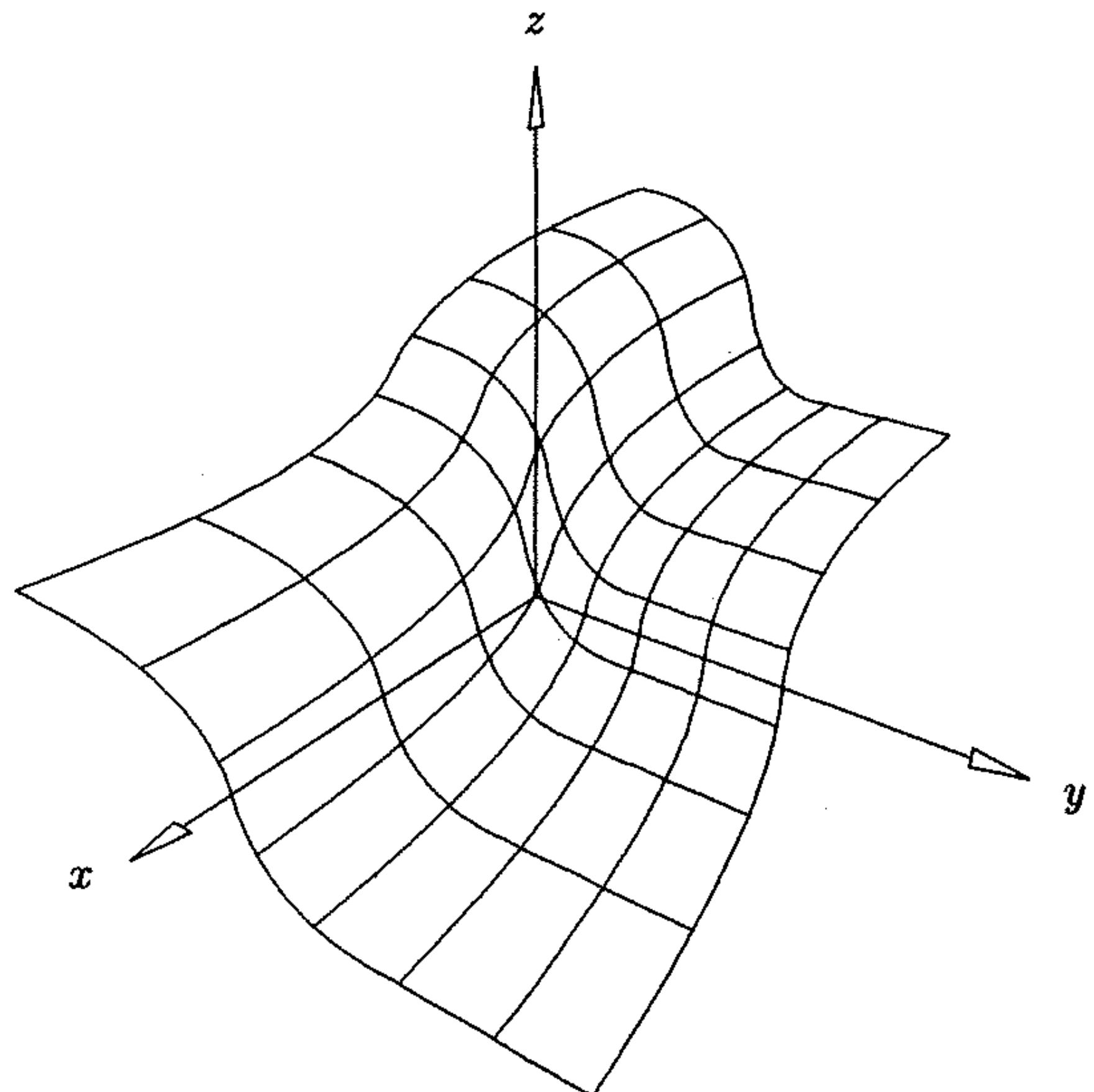
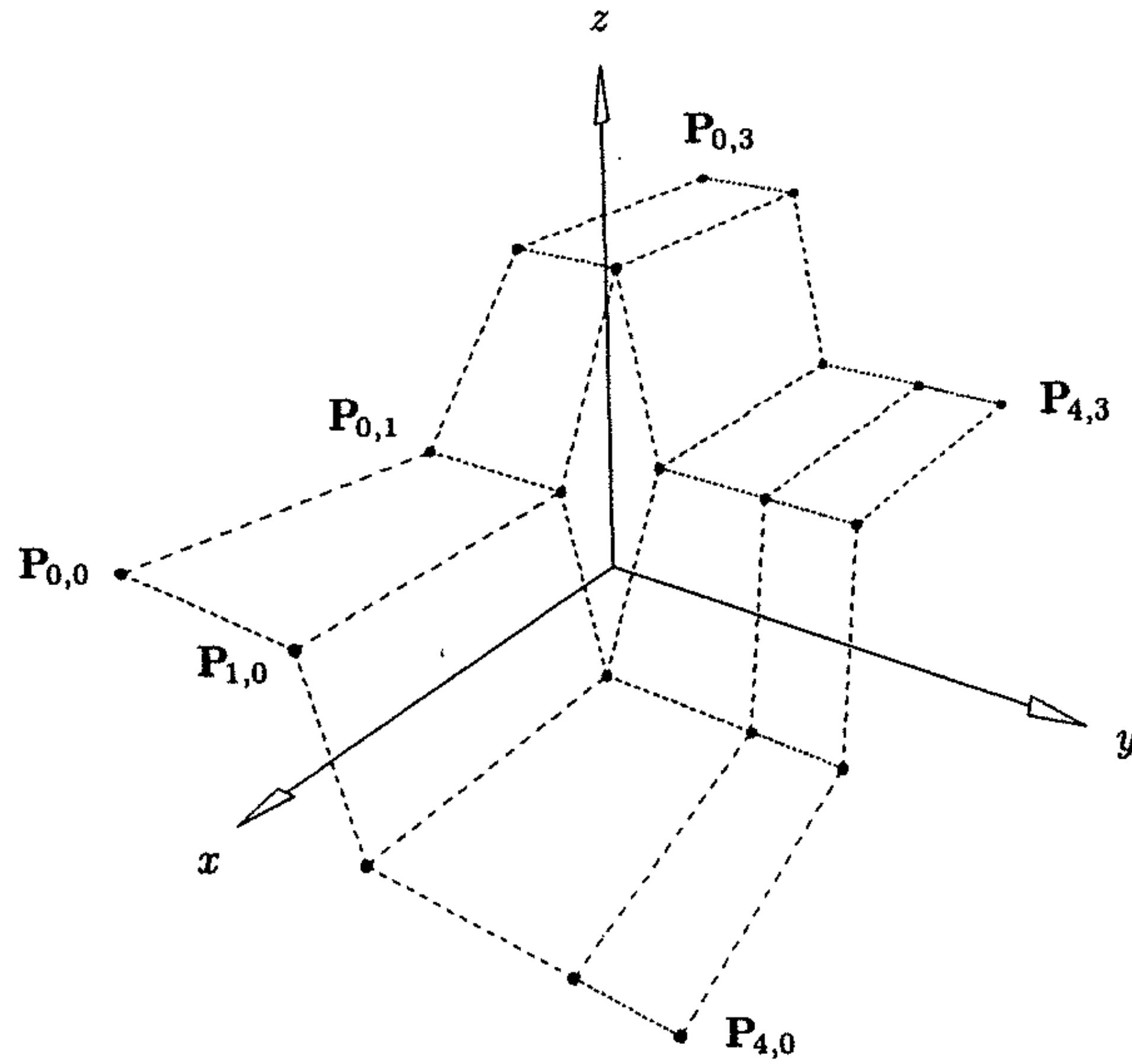


Mesh

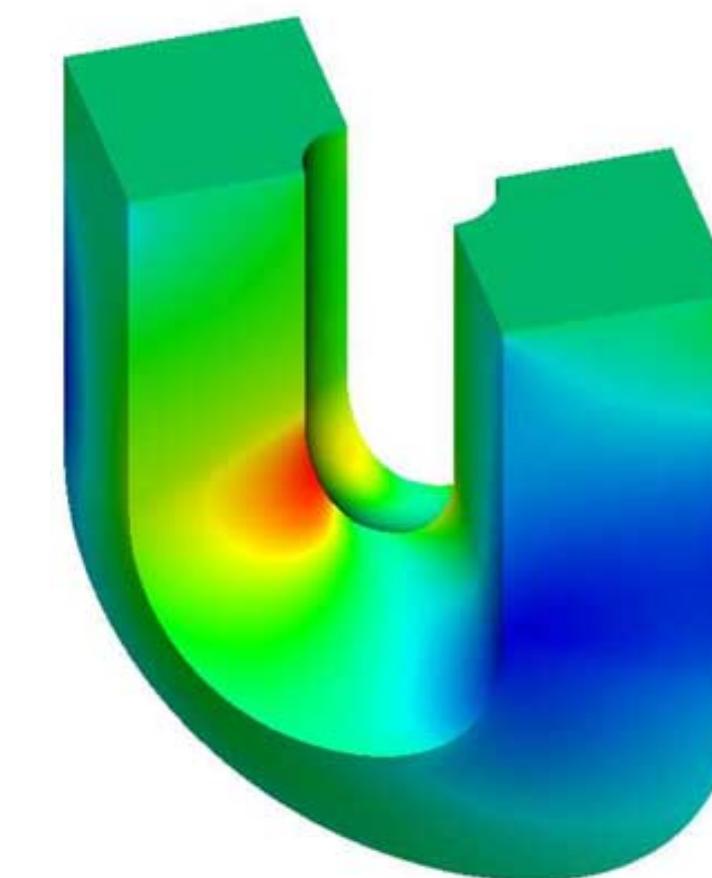
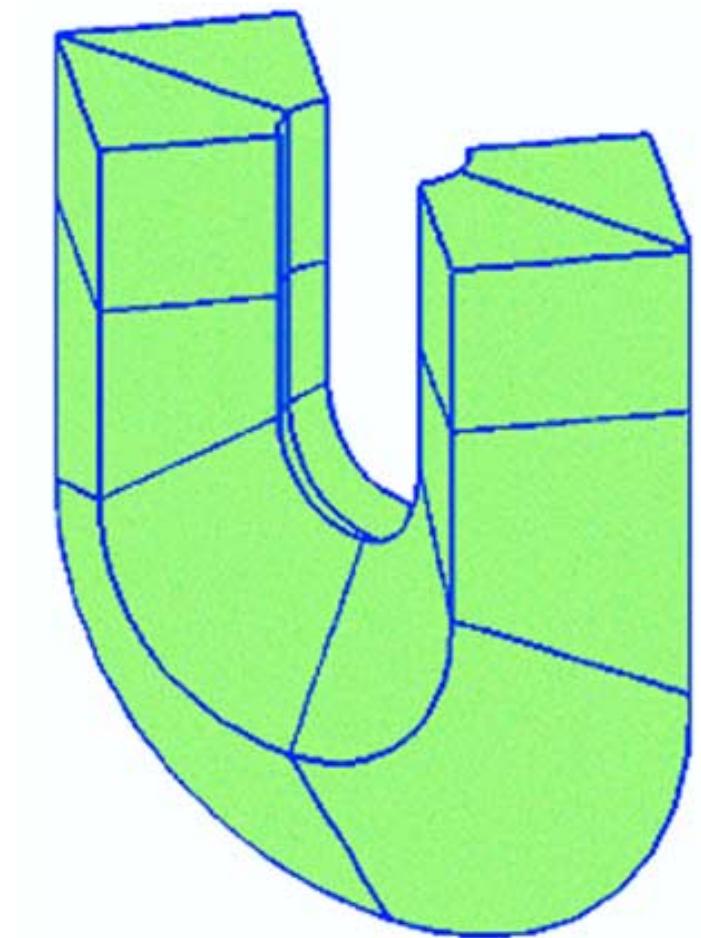
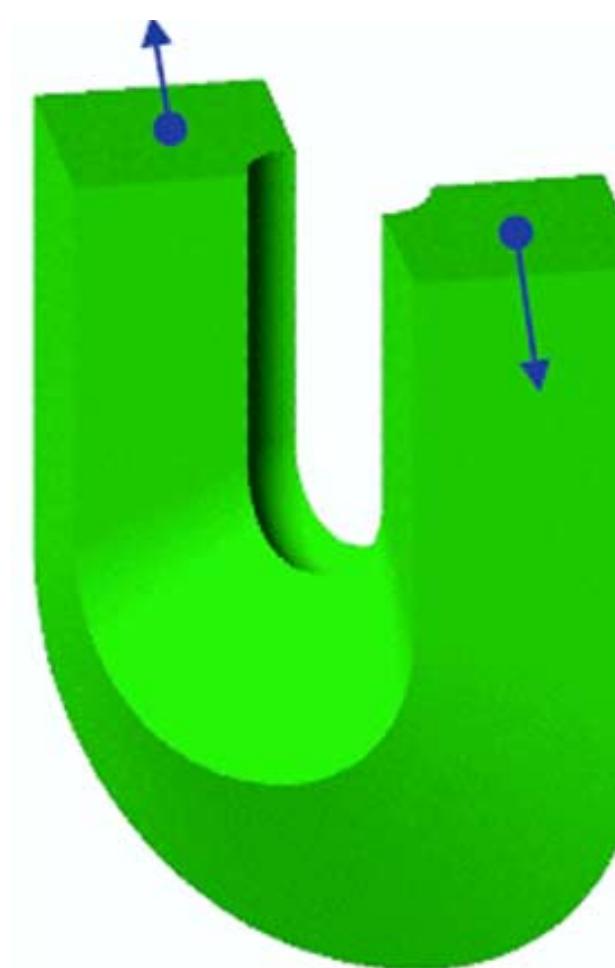
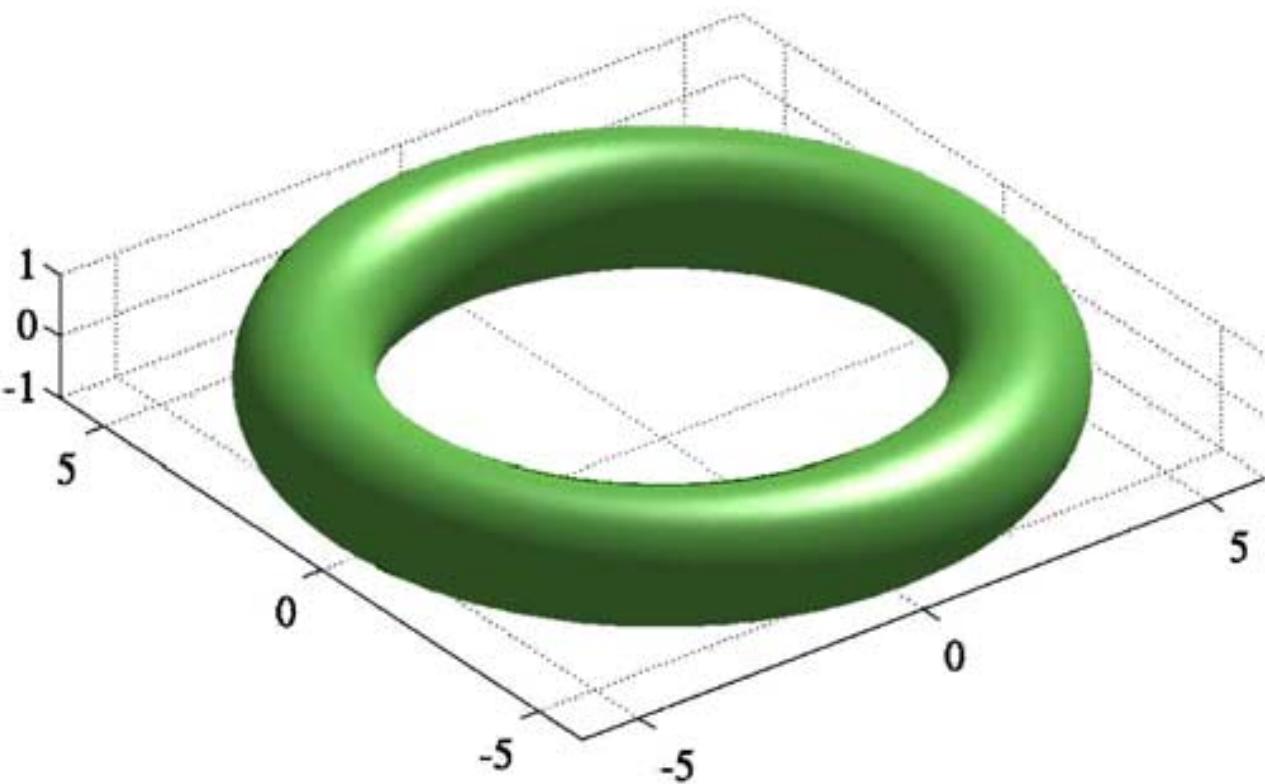
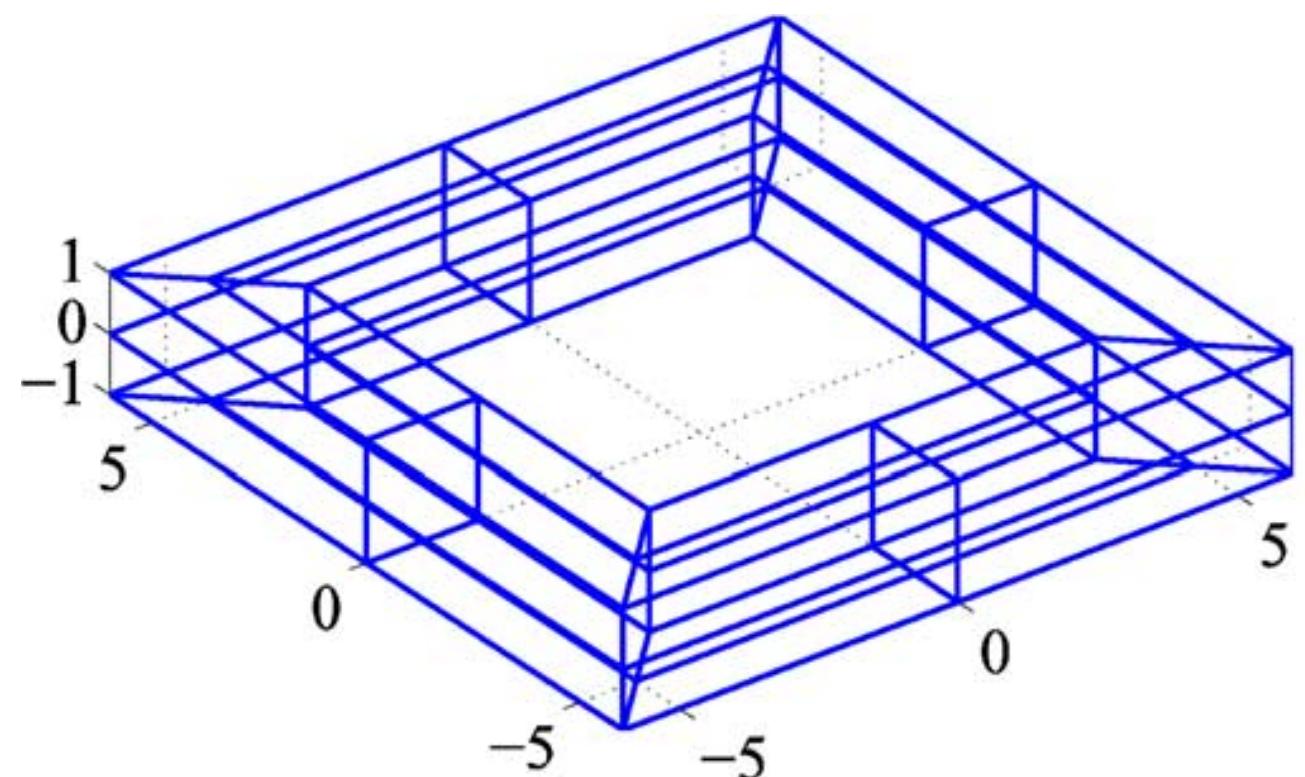
ou então, criar nós e elementos IgA sobre uma superfície genérica



ou então, criar nós e elementos IgA sobre uma superfície genérica



ou ainda, criar nós e elementos IgA sobre um volume



Patient-Specific Vascular NURBS Modeling for Isogeometric Analysis of Blood Flow[★]

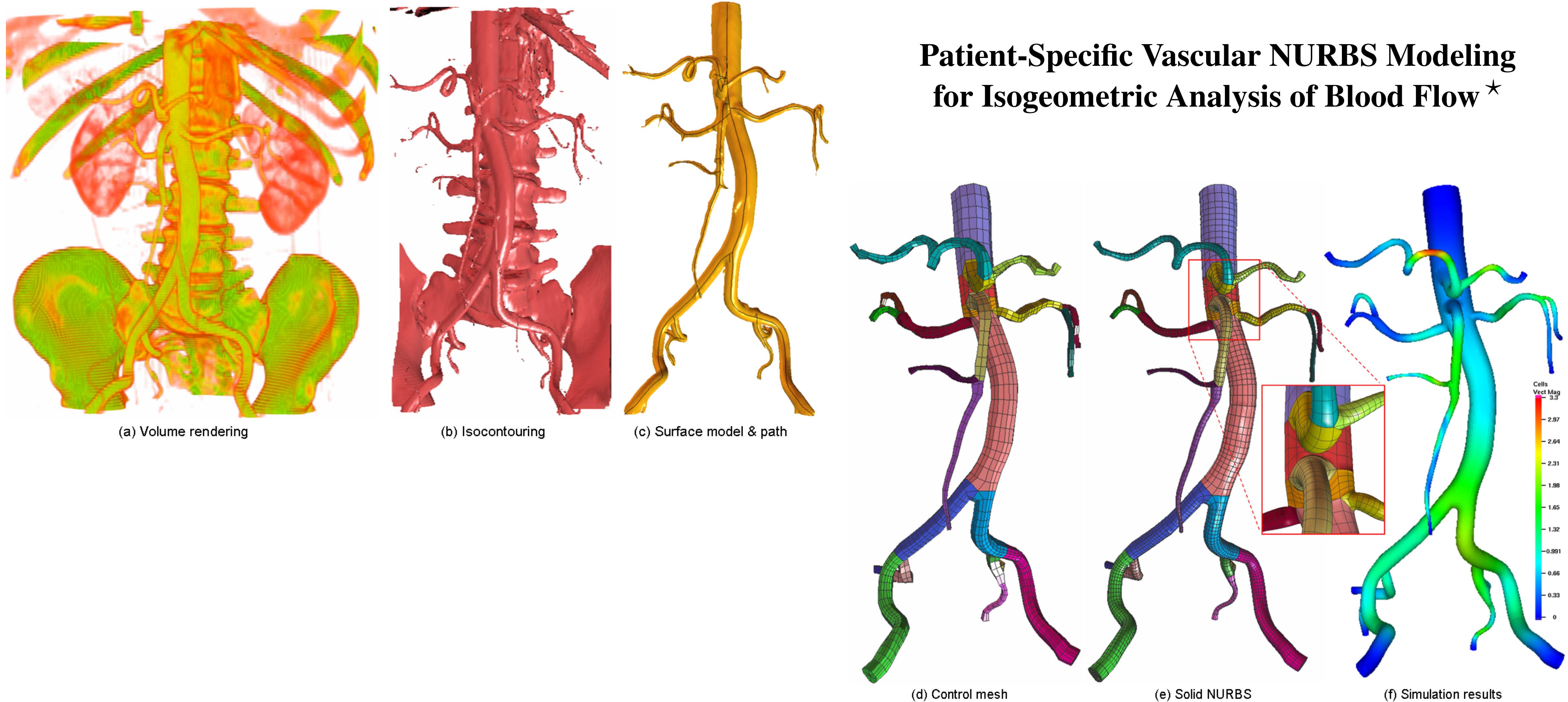


Fig. 1. The abdominal aorta model is divided into 26 patches, and each color represents one different patch. (a) - volume rendering result; (b) - isocontouring result; (c) - surface model and its path after removing unnecessary components; (d) - control mesh; (e) - solid NURBS mesh after refinement (73,314 elements); (f) - fluid-structure interaction simulation results: contours of the arterial wall velocity (cm/s) during late systole plotted on the current configuration. Only major branches are kept in (d-f).

Introdução ao Método dos Elementos Finitos

Competências Transferíveis 2

Sessão 06

Robert Valente (robertt@ua.pt)