Model of Evolution in Sexual Reproduction

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Abstract

In this paper, we model a female-choice based evolution in organisms with sexual reproduction. We study populations in which females are choosers, and males are doers.

1   Evolution!

First of all, we should represent an organism. In the context of sexual reproduction, it is enough to model an organism with the features perceivable by the sensory system of the opposite sex . For instance, if they have eyes, color and shape and movements would be of those features, or if they have olfactory systems, the smells are counted as perceivable features. We introduce the following abstraction functions (for ease, we address each organism in the population by an id):

$$h : \text{organism} \rightarrow \text{perceivable features}$$

$$\vec{\mathbf{x}}_j^{(p)} = h(organism_j) = \text{perceivable features of the organism } j$$

(Superscript (p) declares perceivable.) Each female in the population, has a distinct *Female Desire Function* for the features of a male:

$$\text{Female id: } i, \qquad \text{Male id: } j$$

$$f_i : \text{perceivable features} \rightarrow \text{degree of desire of female } i$$

$$d_{ij} = f_i(\vec{\mathbf{x}}_j^{(p)}) = \text{ desire of female } i \text{ for male } j$$

As commitment is not bold in most the organism (!, especially for the simple ones), we consider the mate model as *Sequential Acceptance or Rejection*, in which a productive female deicides to accept or reject a male when confronting him regardless of what she has seen before on other males (in other words, the expectations do not rise or fall through time). Consequently, we can consider a *Suitability Threshold* for each female in the population, by which a female accepts or rejects a male:

$$t_i : \text{ suitability threshold of female } i$$

$$\begin{cases} \text{accepts} & d_{ij} \geq t_i \\ \text{rejects} & d_{ij} < t_i \end{cases}$$

So the accepting function is $\mathbb{1}_{d_{ij} \geq t_i}$. We are free to have any interaction model in the population. A simple model could be a statistical contact between males and females, like what we have in statistical mechanics: For simplification, we
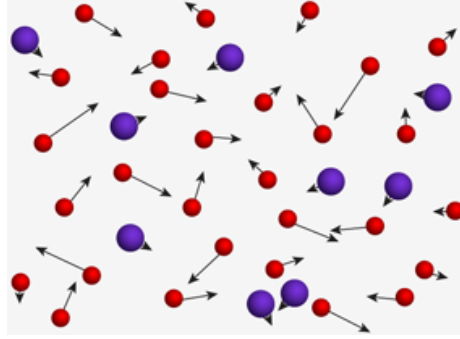
ignore purposeful seeking for mates in our model.

We introduce the parameter β_j as the probability of a male being accepted by a female. A constant multiply of β_j can represent the mating rate of the male j, by assuming a constant n as the number of mating seasons in this organism that makes the mating rate to be $n.\beta_j$. As our random model, we assume that a male meets a random female (uniform random selection). Therefore, β_j will be the probability that for a random female, we have $d_{ij} \geq t_i$:

$$\beta_j = \mathbb{E}_{\mathcal{D}_i}[\mathbb{1}_{d_{ij} \geq t_i}] = \mathbb{P}_{\mathcal{D}_i}[f_i(\vec{\mathbf{x}}_j^{(p)}) \geq t_i]$$

in which $\mathcal{D}_i$ is the distribution of females. This is the key parameter in our evolutionary mode.

2 Natural Pressure

In this section, we take natural evolutional advantages into account. Consider a population in which only males are responsible for protecting the family and the children. Assume function α to be the natural fitness of a male:

$$\alpha : \text{features} \rightarrow \text{evolutionary fitness}$$
$$\alpha(\vec{\mathbf{x}}_j) = \text{evolutionary fitness of male } j$$

Therefore, a contact between a male j and a female i will result to a new living generation with the rate of $\alpha(\vec{\mathbf{x}}_j).\mathbb{1}_{d_{ij} \geq t_i}$. The average rate of reproduction in of a male is as follows:

$$\mathbb{E}_{\mathcal{D}_i}[\alpha(\vec{\mathbf{x}}_j).\mathbb{1}_{d_{ij} \geq t_i}] = \alpha(\vec{\mathbf{x}}_j).\mathbb{P}_{\mathcal{D}_i}[f_i(\vec{\mathbf{x}}_j^{(p)}) \geq t_i] = \alpha_j \beta_j$$

And for a female, the average rate of reproducing a new generation is as follows:

$$\gamma_i = \mathbb{E}_{\mathcal{D}_j}[\alpha(\vec{\mathbf{x}}_j).\mathbb{1}_{d_{ij} \geq t_i}] = \mathbb{E}_{\mathcal{D}_j}[\alpha(\vec{\mathbf{x}}_j).\mathbb{1}_{f_i(\vec{\mathbf{x}}_j^{(p)}) \geq t_i}]$$

in which $\vec{\mathbf{x}}_j \sim \mathcal{D}_j$ is the distribution of males.

3   Generations Dynamic

Now, we have everything to predict the evolutional phenomena in a single generation. The *Evolutionary Fitness* for males is clear, which is the desire of a female corresponding to his perceivable features. But what is the evolutionary fitness for a female? This is more complicated. In our model, horny males seek any female to satisfy their sexual desires without refusing any female (However, this *Male-Choice* behavior could be simply added to our model). In this world, females are immune from being old maids. However, there exists a hidden *evolutionary fitness* for them, which is the *ability of their sons to be chosen by a girl*! This is directly related to the quality of a female's desired function, f, as the offspring of a female can be extinct if her selection function does not select naturally-good males. In fact, there are two different evolutionary forces on males and females. For males, they should fit the females' desires, and for females, they should possess a right desire function to choose appropriate males, who can result in naturally good offspring. Mathematically understanding this phenomenon is the subject of the next section.

In this part, we provide a quantitative approach to understand the evolutionary forces on males and females. Consider $\vec{\mathbf{x}}_j$ as *full features* of male j, including the features affecting the children, and similarly, $\vec{\mathbf{y}}_i$ as *full features* of female i (which also defines the choice function $f_i = f_{y_i}$). Then we introduce the representation of the sons and the daughters born from male j and female i:

$$s : \text{Males} \ \times \ \text{Females} \ \rightarrow \ \text{Males}$$
$$g : \text{Males} \ \times \ \text{Females} \ \rightarrow \ \text{Females}$$
$$\vec{\mathbf{s}}_{ij} = s(\vec{\mathbf{x}}_j, \vec{\mathbf{y}}_i)$$
$$\vec{\mathbf{g}}_{ij} = g(\vec{\mathbf{x}}_j, \vec{\mathbf{y}}_i)$$

However, here we a use a more simplified model in which males reproduce themselves with a rate of $\propto \alpha_j \beta_j$ and males with a rate of $\propto \gamma_i$ (see the previous section). In the model above, this is like that there is no mutation and daughters are identical to mothers and sons are identical to fathers. We will show that females learn to choose males with greater α_j, *i.e.*, f_i will narrow down to select males with great α.

4   Comparison to Non-sexual Reproduction

In a population with non-sexual reproduction, the reproduction rate for each organism is related to $\alpha(\vec{\mathbf{x}})$ (we don't need a separation between males and females). Similar to sexual reproduction, the population will converge to organisms with higher $\alpha(\vec{\mathbf{x}})$. In this section, we compare the convergence rate to the same apex between these two kinds of reproduction.
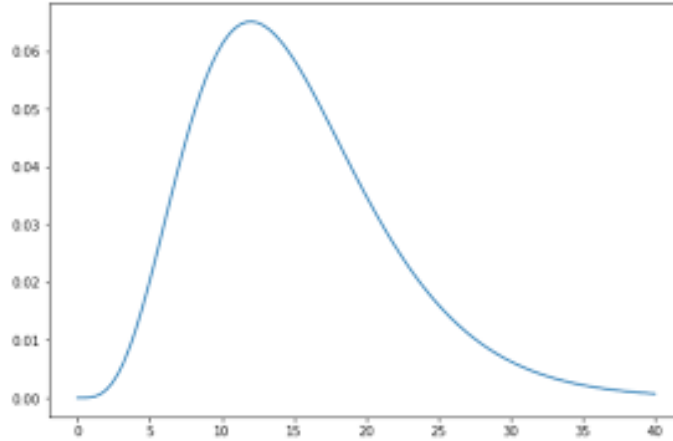
5 Real Example

Here we bring the example of the fiddler crab, with strange males possessing a single big claw and one smaller one.

Taking the predators and environment into account, consider that a big claw

provides a fitness function as below:

which the horizontal axis represents the survival percentage of his children

(related to the protection power of a crab). We call $\vec{\mathbf{x}}_j = l_j$ as the male's claw size, the perceivable feature, and $\alpha(l_j)$ as this percentage. Recall our model, a female always find someone to mate. Therefore, a female would have some evolutionary advantage over another female if her husband has a greater $\alpha_j = \alpha(l_j)$. So for generations of females we have:

$$maximize \quad \alpha(l_j)$$

However, a male should do more things than a female. If a male does not get accepted by any female, he will lose the chance of reproduction. Therefore, the

evolutionary advantage of a male depends on both α_j and the acceptance rate, β_j, which is the probability of facing a female for whom $d_{ij} \geq t_i$. Consequently, for generations of males, we have:

$$maximize \quad \alpha_j.\beta_j$$

As females, during generations, evolve their Desire Function, f_i, to choose males with higher α_j, at the apex of the population, we have $\beta_j \propto \alpha_j$.

Return to our model, we assume the females *Desire Function* as an interval selection:

$$f_i(\vec{\mathbf{x}}_j) = \mathbb{1}_{l_j \in [a_i, b_i]}$$

and a threshold of 0.5 to make a binary acceptance function (in reality, $b_i = \infty$ and females utilize a minimum selection process but our model is more generalized). For β_j we will have:

$$\beta_j = \mathbb{P}_{\mathcal{D}_i}[a_i \leq l_j \leq b_i] = \mathbb{E}_{\mathcal{D}_i}[\mathbb{1}_{l_j \in [a_i, b_i]}]$$

Therefore, males who have l_j's inside the interval of the majority of females will have a greater chance of being accepted. If this β_j is not on the same page with the natural fitness, α_j, it would be wasteful for females to have this desire function. Consequently, females who have endowed reproduction chance to bad males would lose their children. Eventually, females with a better sense of selection for greater α_j will reproduce more, and the same thing happens for males possessing a better α score, resulting in convergence to the natural optimal point at the apex (a population of males with maximum α_j and the females preferring the same characteristic).

6   Suggestions for Simulation

A simple but comprehensive simulation can be done by parameters α_j and β_j. In each time slice, we have a population of $\frac{N}{2}$ males and $\frac{N}{2}$ females, referenced by $j_1, j_1, \cdots, j_{\frac{N}{2}}$ and $i_1, i_1, \cdots, i_{\frac{N}{2}}$. In the next generations, we scale each male by the coefficient of $\frac{1}{Z_{\alpha\beta}} \alpha_j \beta_j$ in which $Z_{\alpha\beta} = \frac{2}{N} \sum_j \alpha_j \beta_j$ acts as a normalizer. For females, we need a new parameter γ_i:

$$\delta_i = \mathbb{P}_{\mathcal{D}_j}[a_i \leq l_j \leq b_i] = \mathbb{E}_{\mathcal{D}_j}[\mathbb{1}_{l_j \in [a_i, b_i]}]$$

in which $\mathcal{D}_j$ is the distribution of males. But the following one is more important:

$$\gamma_i = \mathbb{E}_{\mathcal{D}_j}[\mathbb{1}_{l_j \in [a_i, b_i]}.\alpha(l_j)]$$

(If you remember, the male important parameter, $\alpha_j \beta_j$, was exactly the same expression averaged on $\mathcal{D}_i$.)

Similar to males, in the next generations, we scale each female by the coefficient of $\frac{1}{Z_\gamma} \gamma_i$ in which $Z_\gamma = \frac{2}{N} \sum_i \gamma_i$ acts as a normalizer.

More mathematically and continues, about distrobutions:

$$l \sim Q_t \quad \Rightarrow \quad q_{t+1}(l) \propto \alpha(l)\beta(l)q_t(l)$$

and for females, considering $b_i = \infty$:

$$a \sim G_t \quad \Rightarrow \quad g_{t+1}(a) \propto \gamma(a)g_t(a)$$

Going further:

$$\beta(l) = \mathbb{P}_{a \propto G}[a \leq l] = G(l)$$

in which we have used G as the cumulative distribution of a. For γ:

$$\gamma(a) = \mathbb{E}_{l \propto Q}[\alpha(l)\mathbb{1}_{a \leq l}] = \int_a^\infty \alpha(l)q(l)dl = A_Q(a)$$

Simplification:

$$q_{t+1}(l) \propto \alpha(l)G(l)q_t(l)$$
$$g_{t+1}(a) \propto A_Q(a)g_t(a)$$

In the steady-state, $Q_t = Q_{t+1}$ and $G_t = G_{t+1}$, therefor $\alpha(l)G(l)$ should be a constant with respect to l and $A_Q(a)$ should be a constant with respect to a. It is possible when:

$$g(l) = G'(l) = \frac{d}{dl}\left(\frac{c}{\alpha(l)}\right) = \frac{-c\alpha'(l)}{\alpha^2(l)}$$

which says that α should be descending in order to the existence of a steady-state. If not, $q(l)$ should be a delta function, $\delta(l - l_0)$ $(\alpha(l_0)G(l_0) = 1?)$. Besides:

$$\frac{d}{da}A_Q(a) = 0 = -\alpha(a)q(a)$$

which happens only when $q(l) = \delta(l)$! Again, if this is not held, $g(a)$ should be a delta function like $\delta(a - a_0)$ $(A_Q(a_0) = 1?)$. Having both constraints together, steady-state distributions are $q(l) = \delta(l - l_0)$ and $g(a) = \delta(a - a_0)$ and $a_0 < l_0$ (and $\alpha(l_0) = 1?$). By dynamics, we have $a_0 \geq \alpha_0$ and $l_0 \geq \alpha_0$, for α_0 as the maximal point of α, which tells us that nature has done it's job!