Efficiency of Heart Pumping Behavior

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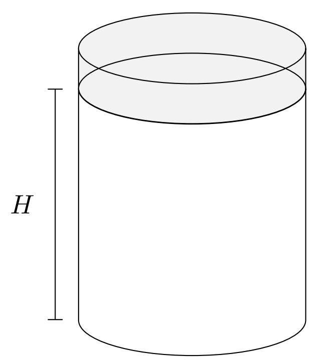
Abstract

In this paper, we model fluid behavior of blood in heart and compare it with an imaginary heart with sinusoid movements.

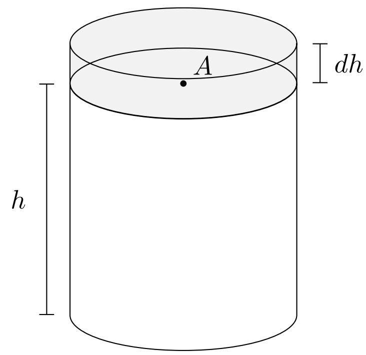
1 Model of Heart

Here, we model the heart as a simple piston which tries to lift a definite amount of a liquid:



The cylinder has a surface of A, the bottom of liquid is exposed to the air with pressure P_0, and the heart is the top cylinder which has a periodic move of $y(H,t) = x(t)$. At stability, consider the height of the liquid is H by which $P(H) > 0$.

Consider $y(h,t)$ as the position of liquid molecules which were at position h at the rest, at the moment t, *i.e.*, $y(h,0) = h$. We write equations for the molecules of $[h, h+dh)$:



$$dm = \rho(h,t)Ady = \rho(h,0)Adh$$

$$dF = P(y)A - P(y+dy)A - dm.g = -dm.g - \frac{\partial P(y,t)}{\partial y}Ady$$

$$\Rightarrow \ddot{y}(h,t) = \frac{dF}{dm} = -g - \frac{1}{\rho(h,t)}\frac{\partial P(y,t)}{\partial y}$$

$$= -g - \frac{1}{\rho(h,0)}\frac{\partial P(h,t)}{\partial h}$$

We write $y(h,t) = h + \epsilon(h,t)$. So, we will have:

$$\begin{cases} \ddot{\epsilon}(h,t) = & -g - \frac{1}{\rho(h,0)}\frac{\partial P(h,t)}{\partial h} \\ \epsilon(h,0) = & 0 \\ \epsilon(H,t) = & \hat{x}(t) = x(t) - H \end{cases}$$

Also, another formula is necessary. That is, the relationship between P and ρ which we express it as $P(h,t) = f(\rho(h,t))$. So, we add this equation to our system:

$$\frac{\partial P(h,t)}{\partial h} = f'(\rho(h,t))\frac{\partial \rho(h,t)}{\partial h} = f'(\rho(h,t))\frac{\partial}{\partial h}\left(\frac{1}{y'}\rho(h,0)\right)$$

2 Solution

We assume that $P = \alpha\rho$ (!, like gases). After simplification of equations we will have:

$$\begin{cases} \ddot{\epsilon}(h,t) = & -g - \frac{\alpha}{\rho(h,0)}\frac{\partial}{\partial h}\left(\frac{1}{y'}\rho(h,0)\right) = -g - \frac{\alpha}{y'}\frac{\rho'}{\rho} + \alpha\frac{y''}{y'^2} \\ \epsilon(h,0) = & 0 \\ \epsilon(H,t) = & \hat{x}(t) = x(t) - H \end{cases}$$

In stability, $y = h$, we have:

$$0 = -g - \frac{\alpha}{y'}\frac{\rho'}{\rho} + \alpha\frac{y''}{y'^2} = -g - \alpha\frac{\rho'}{\rho} \quad \Rightarrow \quad \rho(h,0) = \rho_0 e^{-gh/\alpha}$$

Therefore in our main equation:

$$\ddot{\epsilon}(h,t) = -g + \frac{g}{1+\epsilon'} + \alpha\frac{\epsilon''}{(1+\epsilon')^2} = \alpha\frac{\epsilon''}{(1+\epsilon')^2} - g\frac{\epsilon'}{1+\epsilon'}$$

Here, we use an approximation. We consider $\epsilon << h$ and $\epsilon' << 1$, and as the first order we will have:

$$\ddot{\epsilon}(h,t) = \alpha\epsilon'' - g\epsilon'$$

Or in another words:

$$\frac{\partial^2}{\partial t^2}\epsilon = \alpha\frac{\partial^2}{\partial h^2}\epsilon - g\frac{\partial}{\partial h}\epsilon$$

which is a linear equation. By separation, we will have:

$$\epsilon(h,t) = \int_{\omega\in\Omega} A(\omega)\sin(\omega t)\psi_\omega(h)$$

which satisfies $\epsilon(h,0) = 0$. Consider $F(\omega)$ as the Fourier coefficients of $\hat{x}(t)$. Therefore:

$$A(\omega) = \frac{F(\omega)}{\psi_\omega(H)}$$

Consequently:

$$\epsilon(h,t) = \int_{\omega\in\Omega} F(\omega)\sin(\omega t)\frac{\psi_\omega(h)}{\psi_\omega(H)}$$

Now, our main equations indicates the following:

$$-\omega^2\psi_\omega(h) = \alpha\psi_\omega''(h) - g\psi_\omega'(h) \quad \text{or} \quad \alpha\psi_\omega''(h) - g\psi_\omega'(h) + \omega^2\psi_\omega(h) = 0$$

Using Laplace transformation by putting $\psi_\omega(h) = e^{sh}$, we get:

$$\alpha s^2 - gs + \omega^2 = 0$$

which implies that $s = \frac{g}{2\alpha} \pm \sqrt{\frac{g^2}{4\alpha^2} - \frac{\omega^2}{\alpha}}$. By this, we get:

$$\psi_\omega(h) = e^{\frac{gh}{2\alpha}}\left(A_\omega\sin(\frac{\sqrt{4\alpha\omega^2 - g^2}}{2\alpha}h) + B_\omega\cos(\frac{\sqrt{4\alpha\omega^2 - g^2}}{2\alpha}h)\right)$$

To have an idea of α, if $PV = nRT$ was held, we would have $\alpha = \frac{RT}{M}$. In $T = 37^\circ \approx 310K$, $R = 8.314\frac{J}{K.mol}$. For water, $M = 18\frac{g}{mol}$. Therefore, α should be something like $143\frac{J}{g}$. So, $\bar{\omega} = 0.08\sqrt{\omega^2 - 0.168}$.

$$\psi_\omega(h) = e^{0.034h}\left(A_\omega\sin(0.08h\sqrt{\omega^2 - 0.168}) + B_\omega\cos(0.08h\sqrt{\omega^2 - 0.168})\right)$$

To know A_ω and B_ω, we need to know $\dot{\epsilon}(h,0)$ and its Fourier coefficients:

$$\dot{\epsilon}(h,0) = \int_{\omega\in\Omega} \omega F(\omega)\frac{\psi_\omega(h)}{\psi_\omega(H)}$$

3 Most Efficient Pumpage

In this section, we find the most effective heart movement in a region of $[-\Delta, +\Delta]$ (*i.e.*, $-\Delta \leq \epsilon(H,t) \leq \Delta$). Here, we define the efficiency as the amplitude of movements in $h = 0$ with respect to Δ:

$$\eta = \frac{\max\{|\epsilon(0,t)|\}}{\Delta}$$

Also, heart beating is periodic behavior with a frequency of f_0. Therefore, Ω contains frequencies of $2\pi k f_0$ for $k \in \mathbb{N}$. Hence:

$$\epsilon(H,t) = \sum_{k=1}^{\infty} F_k \sin(2\pi k f_0)$$
$$\epsilon(0,t) = \sum_{k=1}^{\infty} F_k \frac{B_k}{\psi_k(H)} \sin(2\pi k f_0)$$

In fact, this is like a frequency responce of:

$$H_k = \frac{B_k}{\psi_k(H)} = \frac{e^{-0.034H}}{\frac{A_k}{B_k}\sin(0.08H\sqrt{4\pi^2k^2f_0^2 - 0.168}) + \cos(0.08H\sqrt{4\pi^2k^2f_0^2 - 0.168})}$$

Finally, it will be understood that sharp movements help efficiency :))