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Optimal solutions for routing problems with profits

C. Archetti *, N. Bianchessi, M.G. Speranza

Dipartimento Metodi Quantitativi, University of Brescia, C.da S. Chiara, 50, I-25122 Brescia, Italy

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ABSTRACT

In this paper, we present a branch-and-price algorithm to solve two well-known vehicle routing problems with profits, the Capacitated Team Orienteering Problem and the Capacitated Profitable Tour Problem. A restricted master heuristic is applied at each node of the branch-and-bound tree in order to obtain primal bound values. In spite of its simplicity, the heuristic computes high quality solutions. Several unsolved benchmark instances have been solved to optimality.

1. Introduction

In classical routing problems each customer has to be visited, and thus served, in order to minimize the total operational cost while satisfying given constraints. In routing problems with profits, the service has not to be provided to all customers. In fact, a profit is associated with each customer which is gained when the customer is visited. The total collected profit is a term in the problem objective function or is a left hand side term in a problem constraint imposing a lower bound on the collected profit. Thus, the best subset of customers to serve has to be identified.

Routing problems with profits have received less attention in the literature compared to the amount of work concerning classical routing problems. A survey on routing problems with profits, especially dedicated to those arising in the single vehicle case, is due to Feillet et al. [11].

For the case where a fleet of vehicles is available, we have the Team Orienteering Problem (TOP) which is the multiple vehicle version of the more studied Orienteering Problem (OP). In the TOP, a fleet of vehicles is available to serve a set of customers. A profit is associated with each customer. The objective is to find the subset of customers to serve in order to maximize the total collected profit. A time limit is imposed on the maximum duration of each route. The capacitated version of the TOP has been studied in Archetti et al. [2] and is called the Capacitated TOP (CTOP). In the CTOP, a demand is associated with each customer and vehicle routes have to satisfy time limit and capacity constraints. Archetti et al. [2] also studied the Capacitated Profitable Tour Problem (CPTP) where the objective is to maximize the difference between the collected profit and the traveling cost, while routes are subject to capacity constraints only. Different solution approaches are proposed for both problems: an exact algorithm, based on a branch-and-price scheme, and three heuristics, a variable neighborhood search and two tabu search algorithms.

In this paper, we address the CTOP and CPTP and propose a new exact algorithm based on a branch-and-price scheme.

* Corresponding author.
E-mail addresses: archetti@eco.unibs.it (C. Archetti), bianche@eco.unibs.it (N. Bianchessi), speranza@eco.unibs.it (M.G. Speranza).

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The solution algorithm we designed is based on column generation, a well known technique which allows us to solve linear problems by iteratively generating the variables (columns) of the problem. Column generation cannot be applied directly to mixed integer linear programming (MILP) problems. In order to obtain feasible solutions to an MILP problem, column generation must be embedded into a branch-and-bound algorithm giving rise to the so called *branch-and-price* or *IP column generation* solution algorithm. For an in-depth discussion on column generation for MILP problems, the reader is referred to Barnhart et al. [5] and Vanderbeck [16]. The branch-and-price algorithm we present relies on an effective column-based, or restricted master, heuristic which helps pruning the enumeration tree.

The rest of the paper is organized as follows. In Section 2, we give a mathematical formulation for the CTOP and the CPTP. In Section 3, we describe the branch-and-price algorithm we designed, whereas Section 4 is dedicated to computational results. Conclusions and future developments are presented in Section 5.

2. The problem formulations

A directed graph $G = (V, A)$ is given, where $V = \{1, \ldots, n\}$ is the set of vertices and A is the set of arcs. Vertex 1 represents the depot and vertices in $V' = V \setminus \{1\}$ represent potential customers. A non negative travel time t_{ij} and non negative cost c_{ij} are associated with each arc $(i, j) \in A$, whereas a non negative demand d_i and non negative profit p_i are associated with each potential customer i in V'. A fleet of m identical capacitated vehicles is available to serve the customers. The tour of each vehicle starts and ends at vertex 1. Each vehicle can perform a single route. A time limit $T_{\max}$ is set on the duration of each tour. Moreover, the total demand of the customers served in a tour cannot exceed the vehicle capacity Q. Each customer can be served at most once.

The objective of the CTOP is to maximize the total collected profit while satisfying, for each vehicle route, the time limit $T_{\max}$ on the tour duration and the capacity limit Q on the total demand served. The objective of the CPTP is to maximize the difference between the collected profit and the traveling cost while satisfying, for each vehicle route, the capacity limit Q.

The two problems can be modeled by means of set packing formulations. Let us consider the following notation. We define as R the set of all feasible routes (tours). Let a_{ir} be a binary parameter equal to 1 if customer i is visited by route r, $i \in V'$, $r \in R$. Moreover, let θ_r denote a binary variable equal to 1 when route $r \in R$ is assigned to a vehicle. The CTOP can then be formulated as follows:

$$\max \sum_{r \in R} p_r \theta_r \tag{1}$$

$$\sum_{r \in R} a_{ir} \theta_r \leq 1 \quad i \in V' \tag{2}$$

$$\sum_{r \in R} \theta_r \leq m \tag{3}$$

$$\theta_r \in \{0, 1\} \quad r \in R, \tag{4}$$

where $p_r = \sum_{i \in V'} a_{ir} p_i$ is the profit of route $r \in R$. The objective function (1) maximizes the total collected profit. Constraints (2) impose that each customer is served at most once while constraint (3) states that at most m vehicles can be used.

The formulation for the CPTP is similar, with the difference that, for a given route $r \in R$, p_r is given by the difference between the profit collected in r and the traveling cost of r:

$$p_r = \sum_{i \in V'} a_{ir} p_i - \sum_{(i,j) \in A} b_{ijr} c_{ij},$$

where b_{ijr} is a binary parameter equal to 1 if arc $(i, j) \in A$ is included in route $r \in R$.

It is worth noting that the set of feasible routes for the CPTP differs from the one characterizing the CTOP.

In the following, we will refer to problem (1)–(4) as the master problem (MP).

3. The branch-and-price algorithm

The number of variables characterizing the MP is exponential and, even for small size instances, solving it explicitly is impractical. We thus develop a branch-and-price algorithm. At each node of the branch-and-bound tree, variables (columns) of the MP are generated applying the column generation technique to the linear relaxation of the so called restricted MP, augmented by the branching constraints. We denote the linear relaxation of the restricted MP, augmented by the branching constraints, as RLMP. LMP is the linear relaxation of MP augmented by the branching constraints. At each column generation iteration, a pricing problem, also called subproblem, is solved in order to generate positive reduced cost variables θ_r to be added to the RLMP. When no positive reduced cost variable is found, the LMP has been solved to optimality and the column generation algorithm ends. Branching rules are applied to recover feasibility when the solution of the LMP is fractional.

There are several differences between the branch-and-price algorithm proposed in Archetti et al. [2] and the one we propose here. First of all, we use different acceleration techniques to speed up the dynamic programming algorithm for the pricing problem (see Section 3.2). A different set of branching rules is considered (see Section 3.4). Moreover, a restricted master heuristic is provided to compute feasible solutions during the search process (see Section 3.5).

In the following subsections, the main components of the algorithm will be illustrated in details for the CTOP. Adaptations to the CPTP will be sketched.

3.1. The subproblem

Let us consider the following additional notation. Define $V^+(i)$ as the set of possible successors of vertex $i \in V$ and $A(S)$ as the subset of arcs such that both the endpoints belong to set $S \subseteq V$. Then let π_i be the non negative dual variable of constraint (2) for customer $i \in V'$ and β be the non negative dual variable of constraint (3). The pricing problem at the root node of the branch-and-bound tree can be defined as follows.

$$\max \sum_{i \in V'} (p_i - \pi_i) z_i - \beta \tag{5}$$

$$\sum_{j \in V^+(1)} x_{1j} = 1 \tag{6}$$

$$\sum_{j \in V^+(i)} x_{ij} = \sum_{j \in V^-(i)} x_{ji} = z_i \quad i \in V' \tag{7}$$

$$\sum_{(i,j) \in A(S)} x_{ij} \le |S| - 1 \quad S \subseteq V', |S| \ge 2 \tag{8}$$

$$\sum_{i \in V'} d_i z_i \le Q \tag{9}$$

$$\sum_{(i,j) \in A} t_{ij} x_{ij} \le T_{\max} \tag{10}$$

$$z_i \in \{0, 1\} \quad i \in V' \tag{11}$$

$$x_{ij} \in \{0, 1\} \quad (i, j) \in A, \tag{12}$$

where x_{ij} is a binary variable equal to 1 if arc $(i, j) \in A$ is traversed, 0 otherwise, and z_i is a binary variable equal to 1 if customer $i \in V'$ is served, 0 otherwise. The objective function (5) aims at maximizing the reduced cost of the route. Constraint (6) imposes that the route starts from the depot while (7) are flow conservation constraints. Constraints (8) are subtour elimination constraints while (9) and (10) are capacity and time limit constraints, respectively.

The pricing problem for the CPTP is similar to the one for the CTOP. The difference lies in the objective function which has to take into account the traveling cost. Constraint (10) is removed. The formulation is thus the following:

$$\max \sum_{i \in V'} (p_i - \pi_i) z_i - \sum_{(i,j) \in A} c_{ij} x_{ij} - \beta \tag{13}$$

s.t.: (6)–(9), (11) and (12).

Subproblem (5)–(12), as well as subproblem (13), (6)–(9), (11) and (12), corresponds to an *Elementary Shortest Path Problem with Resource Constraint* (ESPPRC).

3.2. Solving the subproblem

Both subproblems are solved by means of a label setting dynamic programming algorithm.

In case of subproblem (5)–(12), a *state* is associated with each partial path ending in node $i \in V$ and is represented by a *label* $(\mathbf{S}, q, \tau, C, i)$, where C is the value (reduced cost) of the path and i is the last vertex visited in the path, while the first three terms represent resources. The meaning of the resources is the following:

- $\mathbf{S}$ is a vector of binary resource consumptions associated with the set of customers V'; $S_j = 1$ if customer j is visited along the path, 0 otherwise;
- q is the quantity loaded on the vehicle after the visit to customer i;
- τ is the time duration of the path after the visit to customer i.

In particular, the resource vector $\mathbf{S}$ is considered in order to generate elementary paths (see [6]).

The initial state at vertex 1 is represented by the label $(\mathbf{0}, 0, 0, 0, 1)$. Along arc $(i, j) \in A$ the extension rules are:

$$S'_j = S_j + 1$$
$$q' = q + d_j$$
$$\tau' = \tau + t_{ij}$$
$$C' = \begin{cases} C + p_j - \pi_j & \text{if } j \neq 1 \\ C - \beta & \text{otherwise.} \end{cases}$$

A label is feasible if $S_j \leq 1$ for each $j \in V'$, $q \leq Q$ and $\tau \leq T_{\max}$. The efficiency of this kind of algorithms depends on the ability to identify and discard dominated paths, that is paths which cannot be extended to optimal paths. Let $L' \equiv (\mathbf{S}', q', \tau', C', i)$ and $L'' \equiv (\mathbf{S}'', q'', \tau'', C'', i)$ be two different labels representing states associated with the same vertex $i \in V$. Then, L' dominates L'' if $S'_j \leq S''_j$ for each $j \in V'$, $q' \leq q''$, $\tau' \leq \tau''$ and $C' \geq C''$.

The described solution algorithm can be easily adapted to the solution of the subproblem for the CPTP. Resource τ is not considered and the extension rule for the reduced cost becomes:

$$C' = \begin{cases} C + p_j - c_{ij} - \pi_j & \text{if } j \neq 1 \\ C - c_{ij} - \beta & \text{otherwise.} \end{cases}$$

In order to further improve the efficiency of the algorithm, we incorporated different acceleration techniques: the *2-cycle elimination* procedure presented in Desrochers et al. [10], the *bounded bidirectional search* and the *decremental state-space relaxation* proposed in Righini and Salani [13,14]. In the bidirectional search, resource q has been considered as a *critical resource*.

In order to speed up the solution process of the LMP, at each column generation iteration the subproblem is addressed heuristically before solving it to optimality. The heuristic method consists in applying the dynamic programming algorithm described above to a subgraph of G. For a given vertex $i \in V'$, only a subset of the outgoing arcs towards customers is maintained in the subgraph. This subset is defined by considering the $\bar{n}_a$ outgoing arcs with largest values defined in the following way. For the CTOP, the value of an arc (i, j) is given by $p_j - \pi_j$. In the CPTP, the value of arc (i, j) is $p_j - \pi_j - c_{ij}$. If the heuristic fails to find positive reduced cost columns, then the problem is solved to optimality on G.

It is worthwhile to note that the overall approach remains valid even if we allow the subproblem solution to be a non elementary path. The computing time required to solve the subproblem would decrease considerably, whereas the quality of the bound provided by the LMP solution would deteriorate.

3.3. The column generation algorithm

At the root node of the branch-and-bound tree, customers are considered in non decreasing order of their numbering. The first route is built starting from the depot and iteratively inserting customers until the insertion of a customer causes infeasibility. Then a new route is built in a similar way, starting from the customer that caused infeasibility on the previous route. The procedure is repeated until m routes are constructed. The corresponding columns are used to initialize the RLMP.

At non root nodes, the initial columns are all previously generated columns that are feasible with respect to the branching constraints.

We describe now how, at any node of the branch-and-bound tree and whenever the subproblem finds positive reduced cost columns, the set of columns to be added to the RLMP is selected. Let $\widetilde{R}$ be the set of such columns. First, the best column found is inserted in $\widetilde{R}$. Then, at most $\bar{n}_S$ subsets of positive reduced cost columns are included in $\widetilde{R}$, with a maximum total number $\bar{n}_m$ of columns. Each subset is defined as follows. The positive reduced cost columns not yet inserted in $\widetilde{R}$ are considered in non increasing order of reduced cost. A column is inserted in the subset if it has at least one vertex that is not covered (visited) by the columns previously inserted in the subset. The inclusion of columns into a subset ends when all vertices are covered by the columns inserted, or $|\widetilde{R}| = \bar{n}_m$, or when there are no more columns available.

The motivation for choosing this specific set of columns will become clearer in Section 3.5, where the restricted master heuristic will be described.

When no positive reduced cost column is found, the column generation algorithm ends providing the optimal solution of the LMP. All the columns generated are stored in a pool to be used for subsequent RLMP initializations.

The column generation algorithm is the same for both the CTOP and the CPTP.

3.4. Branching scheme

Usually the optimal solution of the LMP does not satisfy the integrality constraints (4). Therefore, branching has to be applied in order to recover feasibility.

In the following, the considered branching rules are described. They are presented in order of priority. Independent of the rule applied, two new children nodes are created. When multiple branchings are possible at the same priority level, the one corresponding to the most fractional value is selected. The resulting tree is explored according to the best-first strategy. Let $\hat{\boldsymbol{\theta}}$ be the optimal fractional solution of the current LMP.

The first branching rule concerns the value of $\hat{m} = \sum_{r \in R} \hat{\theta}_r$, i.e. the number of vehicles used in the fractional solution. If this value is fractional, we impose $\sum_{r \in R} \theta_r \leq \lfloor \hat{m} \rfloor$ on one branch and $\sum_{r \in R} \theta_r \geq \lceil \hat{m} \rceil$ on the other branch.

The second branching rule concerns the visit to a customer. If $\sum_{r \in R} a_{ir} \hat{\theta}_r$ is fractional for some $i^* \in V'$, we impose on one branch $\sum_{r \in R} a_{i^*r} \theta_r = 0$ and $\sum_{r \in R} a_{i^*r} \theta_r = 1$ on the other branch.

Finally, the third branching rule is on the flow on an arc. If $\sum_{r \in R} b_{ijr} \hat{\theta}_r$ is fractional for some $(i^*, j^*) \in A$, we impose, on one branch, $\sum_{r \in R} b_{i^*j^*r} \theta_r = 0$, and $\sum_{r \in R} b_{i^*j^*r} \theta_r = 1$ on the other branch. In particular, when we impose $\sum_{r \in R} b_{i^*j^*r} \theta_r = 0$, we also remove from set A the arc (i^*, j^*). When $\sum_{r \in R} b_{i^*j^*r} \theta_r$ is set to 1, we remove from set A arcs (i, j^*), with $i \neq i^*$, and arcs (i^*, j), with $j \neq j^*$.

The last rule is the only one that causes modifications both at the MP and at the subproblem level. Nevertheless, the structure of the subproblem is not altered when the highlighted changes occur, and the solution algorithm presented in Section 3.2 is not affected.

The proposed branching scheme is the same for both the CTOP and the CPTP.

In the branching scheme presented in Archetti et al. [2], the first branching rule we defined is not considered, and the branching on the flow on an arc is different.

3.5. A restricted master heuristic

Within a branch-and-price solution algorithm, the columns generated by the solution of the pricing problem can have a role also in finding heuristic solutions. The optimal solution of the MP restricted to any subset of the generated columns, obtained by means of a general solver, provides a heuristic solution. The restricted set of columns can include either columns generated heuristically or columns generated during the solution of the RLMPs, or a combination of both. The resulting heuristic belongs to the class of the so called *restricted master heuristics*. Implementations of such kind of heuristics have been proposed for example in Agarwal et al. [1], Berthold [7], Chabrier [8] and Taillard [15]. The main drawback highlighted for this kind of heuristics is that they are not general. Actually, computational experience shows that the MP defined over a subset of columns is often infeasible and problem dependent procedures have to be designed in order to recover feasibility (see [12]). The restricted master heuristic we propose tries to overcome this drawback, while providing at the same time high quality solutions. This goal is achieved by a suitable identification of the set of columns over which the MP is solved to optimality.

Let $\bar{R}$ denote the set of columns used in the restricted master heuristic. Ideally, $\bar{R}$ should contain the columns of an optimal solution. We aim at identifying columns which allow the heuristic to obtain high quality solutions, *promising columns* for short. The set $\bar{R}$ should contain a large number of promising columns, but, at the same time, the solution of the MP, restricted to the set of promising columns, should be possible by a general solver in a reasonable time.

On the basis of these considerations, we defined $\bar{R}$ as the set of columns evaluated and used for the solution of the current LMP because, on the tested instances, a commercial solver is able to find in most cases the optimal solution and always a high quality solution within the allowed computational time. Note that, if a relaxation of the subproblem is solved allowing the paths to be non-elementary, then a post-processing procedure is required to make the routes in $\bar{R}$ cycle-free and to eliminate redundant columns.

The restricted master heuristic, working on a set of columns defined according to the rules outlined in Section 3.3, is widely applicable. In particular, it can be applied to all the cases the MP is formulated as a set covering, a set partitioning or a set packing problem.

A similar restricted master heuristic has been successfully applied also in Archetti et al. [4,3], where the MP is a set partitioning problem and a set packing problem with side constraints, respectively.

4. Computational results

The proposed solution algorithm has been implemented in C++ and tested on several sets of benchmark instances. We solve the RLMPs and the restricted MPs of the restricted master heuristic by means of CPLEX 10.1.1. The computational experiments have been carried out on an Intel Xeon processor E5520, 2.26 GHz machine with 12 GB of RAM.

We tested the algorithms on the benchmark instances introduced in Archetti et al. [2]. These instances consist of three sets that were created starting from 10 benchmark instances for the VRP proposed by Christofides et al. [9] which include both capacity and time constraints. The number n of vertices ranges from 51 to 200. The profit p_i of customer $i \in V'$ is generated as $(0.5 + h)d_i$, where h is a random number uniformly generated in the interval [0, 1].

Set 1 contains the original instances, i.e., without any modification with respect to capacity, time limits and the number of available vehicles.

Set 2 is generated from each basic instance by changing the capacity, the time limit and the number of vehicles. In particular, for the CPTP we considered, for each basic instance, the cases with $Q = 50, 75$ and 100, respectively. For the CTOP, we considered the cases with $Q = 50$ and $T_{\max} = 50$, $Q = 75$ and $T_{\max} = 75$ and $Q = 100$ and $T_{\max} = 100$, respectively. Finally, for each case, we generated three instances by setting $m = 2, 3, 4$. Thus, we obtained 90 instances for the CTOP and CPTP for Set 2.

Table 1
CTOP dual/primal bound—Set 1.

Instance	n	m	Q	$T_{\max}$	Best known			Branch-and-price algorithm							
					Heuristic	Exact		With primal heur.				Without primal heur.			
					$\underline{z}^*$	$\underline{z}^*$	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p03	101	15	200	200	1409	1409	41	1409	1409	0	27	1409	1409	0	26
p06	51	10	160	200	761	761	2	761	761	0	4	761	761	0	4
p07	76	20	140	160	1327	1327	2	1327	1327	0	7	1327	1327	0	7
p08	101	15	200	230	1409	1409	17	1409	1409	0	43	1409	1409	0	43
p09	151	10	200	200	2064	1164		2753	1832	33.46		2753	–	–	
p10	200	20	200	200	3048	1735		3048	2941	3.51		3048	–	–	
p13	121	15	200	720	1287	1287	21	1287	1287	0	324	1287	1287	0	322
p14	101	10	200	1040	1710	1710	1082	1710	1710	0	21	1710	1710	0	21
p15	151	15	200	200	2159	2159	1866	2159	2085	3.43		2159	–	–	
p16	200	15	200	200	2968	588		3066	2951	3.75		3066	–	–	

Table 2
CPTP dual/primal bound—Set 1.

Instance	n	m	Q	Best known	Branch-and-price algorithm							
				Heuristic	With primal heur.				Without primal heur.			
				$\underline{z}^*$	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p03	101	15	200	663.98	675.00	**664.92**	1.49		674.68	–	–	
p06	51	10	160	258.97	265.99	**259.24**	2.54		265.21	–	–	
p07	76	20	140	534.81	548.65	**541.32**	1.34		548.10	–	–	
p08	101	15	200	663.98	674.99	**664.92**	1.49		674.68	–	–	
p09	151	10	200	1192.68	1233.57	1187.54	3.73		1233.23	–	–	
p10	200	20	200	1773.65	1829.59	1655.01	9.54		1824.91	–	–	
p13	121	15	200	284.71	357.52	271.39	24.09		357.30	–	–	
p14	101	10	200	890.44	925.01	878.35	5.04		925.02	–	–	
p15	151	15	200	1168.63	1210.22	1146.68	5.25		1209.69	–	–	
p16	200	15	200	1776.41	1848.57	1629.10	11.87		1845.31	–	–	

Set 3 of instances was created from the original data, changing only the number of vehicles. For each instance, we generated three instances by setting $m = 2, 3, 4$, for a total of 30 instances for each problem.

All the instances can be found at the following URL: www-c.eco.unibs.it/~archetti/CTOP-CPTP.zip.

For each test set, we evaluate the impact of the restricted master heuristic on the performance of the branch-and-price algorithm described in Section 3 by running the algorithm with and without the restricted master heuristic. Then, we compare the results obtained with those presented in Archetti et al. [2]. The computational tests in Archetti et al. [2] were made on a Intel Pentium 4 CPU 1.60 GHz and 256 MB Ram machine for the branch-and-price algorithm and on a Intel Pentium 4 CPU 2.80 GHz and 1.048 GB Ram machine for the heuristics.

The branch-and-price algorithm was run for a maximum computing time of 1 h. In particular, according to preliminary tests, we defined the following setting for the algorithm, for both the CTOP and the CPTP. We decided to allow the solutions of the subproblem to be non elementary paths while addressing all but Set 2 instances. In order to find elementary paths on Set 2 instances, we used a monodirectional version of the label setting algorithm, because we noticed that this performs better than the bidirectional version. The parameter $\bar{n}_a$, used when building the subgraph for the heuristic solution of the pricing problem, has been set to $\frac{n}{8}$. The parameters $\bar{n}_S$ and $\bar{n}_m$, that is the maximum number of column subsets and the maximum number of columns to insert in the RLMP, have been set to 30 and to m times the number of the RLMP rows, respectively. When the restricted master heuristic is applied, we give a maximum time of 5 min for the solution of the restricted MP. At the root node the heuristic is run after each exact solution of the pricing problem. This is done to obtain, at an early stage, a pair of primal and dual bound values. At non root nodes the heuristic is run after each LMP solution. When the optimality gap is less than 0.1% the heuristic execution is prevented. All reported CPU times are expressed in seconds.

The results for Set 1 are reported in Tables 1 and 2 for the CTOP and the CPTP, respectively. The first columns report instance data: the name of the instance, the number of vertices, the number of vehicles, the capacity and time limit (not present in Table 2). Then, we report the results obtained in Archetti et al. [2]. The first column reports the best solution found by the heuristic algorithms proposed in Archetti et al. [2]. The following two columns refer to the branch-and-price algorithm and report the best feasible solution found and the CPU time. A blank space in the CPU time column means that the instance was not solved to optimality within the maximum time allotted (1 h). Thus, when the CPU time is lower than 1 h, the corresponding solution is proved to be optimal. The last 8 columns are devoted to the branch-and-price algorithm described in the previous section. A crucial point of success of our algorithm is the heuristic procedure described in Section 3.5 and this is shown by comparing the results of the overall approach with and without the heuristic. For each of the two variants we report the dual bound, the primal bound, the percentage optimality gap and the CPU time. In bold we report new best solutions found by our algorithm. From Table 1 we can see that our approach improves the primal bound of the branch-and-

Table 3
CTOP dual/primal bound—Set 2.

Instance	n	m	Q	$T_{\max}$	Best known			Branch-and-price algorithm							
					Heuristic	Exact		With primal heur.				Without primal heur.			
					$\underline{z}^*$	$\underline{z}^*$	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p03	101	2	50	50	133	133	1	133	133	0	0	133	133	0	0
p03	101	3	50	50	198	198	0	198	198	0	0	198	198	0	0
p03	101	4	50	50	260	260	1	260	260	0	0	260	260	0	0
p03	101	2	75	75	208	208	65	208	208	0	1	208	208	0	1
p03	101	3	75	75	307	307	325	307	307	0	3	307	307	0	15
p03	101	4	75	75	403	403	88	403	403	0	9	403	403	0	9
p03	101	2	100	100	277	277		277	277	0	46	277	277	0	47
p03	101	3	100	100	408	407		408	408	0	50	408	408	0	48
p03	101	4	100	100	531	526		532	**532**	0	132	532	**532**	0	1219
p06	51	2	50	50	121	121	0	121	121	0	0	121	121	0	0
p06	51	3	50	50	177	177	0	177	177	0	0	177	177	0	0
p06	51	4	50	50	222	222	0	222	222	0	0	222	222	0	0
p06	51	2	75	75	183	183	1	183	183	0	2	183	183	0	2
p06	51	3	75	75	269	269	0	269	269	0	1	269	269	0	1
p06	51	4	75	75	349	349	1	349	349	0	1	349	349	0	1
p06	51	2	100	100	252	252	283	252	252	0	9	252	252	0	9
p06	51	3	100	100	369	369	2696	369	369	0	27	369	369	0	57
p06	51	4	100	100	482	482	212	482	482	0	19	482	482	0	48
p07	76	2	50	50	126	126	0	126	126	0	0	126	126	0	0
p07	76	3	50	50	187	187	0	187	187	0	0	187	187	0	0
p07	76	4	50	50	240	240	1	240	240	0	0	240	240	0	0
p07	76	2	75	75	193	193	1	193	193	0	0	193	193	0	0
p07	76	3	75	75	287	287	1	287	287	0	0	287	287	0	0
p07	76	4	75	75	378	378	2	378	378	0	1	378	378	0	1
p07	76	2	100	100	266	266	244	266	266	0	4	266	266	0	4
p07	76	3	100	100	397	397	367	397	397	0	10	397	397	0	10
p07	76	4	100	100	521	521	1733	521	521	0	28	521	521	0	29
p08	101	2	50	50	133	133	1	133	133	0	0	133	133	0	0
p08	101	3	50	50	198	198	0	198	198	0	0	198	198	0	0
p08	101	4	50	50	260	260	0	260	260	0	0	260	260	0	0
p08	101	2	75	75	208	208	62	208	208	0	1	208	208	0	1
p08	101	3	75	75	307	307	321	307	307	0	3	307	307	0	16
p08	101	4	75	75	403	403	88	403	403	0	9	403	403	0	9
p08	101	2	100	100	277	277		277	277	0	46	277	277	0	46
p08	101	3	100	100	408	407		408	408	0	51	408	408	0	49
p08	101	4	100	100	531	526		532	**532**	0	137	532	**532**	0	1314
p09	151	2	50	50	137	137	1	137	137	0	0	137	137	0	0
p09	151	3	50	50	201	201	0	201	201	0	0	201	201	0	0
p09	151	4	50	50	262	262	3	262	262	0	1	262	262	0	1
p09	151	2	75	75	210	210	1178	210	210	0	5	210	210	0	6
p09	151	3	75	75	310	312	1166	312	312	0	8	312	312	0	8
p09	151	4	75	75	407	408	1266	408	408	0	27	408	408	0	87
p09	151	2	100	100	279	279		279	279	0	157	279	279	0	357
p09	151	3	100	100	414	413		415	**415**	0	474	415	**415**	0	429
p09	151	4	100	100	545	543		546	**546**	0	556	546	**546**	0	655
p10	200	2	50	50	134	134	8	134	134	0	1	134	134	0	1
p10	200	3	50	50	200	200	7	200	200	0	1	200	200	0	1
p10	200	4	50	50	265	265	3	265	265	0	1	265	265	0	1
p10	200	2	75	75	208	208	1860	208	208	0	5	208	208	0	5
p10	200	3	75	75	310	311		311	311	0	28	311	311	0	33
p10	200	4	75	75	410	410		411	**411**	0	35	411	**411**	0	98
p10	200	2	100	100	282	281		282	282	0	98	282	282	0	94
p10	200	3	100	100	417	407		419	**418**	0.24		419	–	–	
p10	200	4	100	100	550	552		553	**553**	0	881	553	–	–	
p13	121	2	50	50	134	134	18	134	134	0	1	134	134	0	1
p13	121	3	50	50	193	193	96	193	193	0	6	193	193	0	9
p13	121	4	50	50	243	243	177	243	243	0	4	243	243	0	5
p13	121	2	75	75	193	185		193	193	0	74	193	193	0	69
p13	121	3	75	75	265	255		265	265	0	315	265	265	0	287
p13	121	4	75	75	323	323		323	323	0	6	323	323	0	275
p13	121	2	100	100	253	251		–	–	–		–	–	–	
p13	121	3	100	100	344	321		–	–	–		–	–	–	
p13	121	4	100	100	419	340		–	–	–		–	–	–	
p14	101	2	50	50	124	124	0	124	124	0	0	124	124	0	0
p14	101	3	50	50	184	184	0	184	184	0	0	184	184	0	0
p14	101	4	50	50	241	241	0	241	241	0	0	241	241	0	0

(*continued on next page*)

Table 3 (*continued*)

Instance	n	m	Q	$T_{\max}$	Best known			Branch-and-price algorithm							
					Heuristic	Exact		With primal heur.				Without primal heur.			
					$\underline{z}^*$	$\underline{z}^*$	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p14	101	2	75	75	190	190	24	190	190	0	1	190	190	0	1
p14	101	3	75	75	279	279	34	279	279	0	4	279	279	0	4
p14	101	4	75	75	366	366	64	366	366	0	6	366	366	0	6
p14	101	2	100	100	271	271		271	271	0	9	271	271	0	9
p14	101	3	100	100	399	377		399	399	0	86	399	399	0	92
p14	101	4	100	100	525	525		525	525	0	85	525	525	0	216
p15	151	2	50	50	134	134	0	134	134	0	0	134	134	0	0
p15	151	3	50	50	200	200	0	200	200	0	0	200	200	0	0
p15	151	4	50	50	266	266	0	266	266	0	0	266	266	0	0
p15	151	2	75	75	211	211	736	211	211	0	2	211	211	0	2
p15	151	3	75	75	315	315	743	315	315	0	4	315	315	0	3
p15	151	4	75	75	414	412		415	**415**	0	44	415	**415**	0	63
p15	151	2	100	100	282	282		282	282	0	65	282	282	0	195
p15	151	3	100	100	417	410		418	**418**	0	105	418	**418**	0	939
p15	151	4	100	100	549	546		549	549	0	1311	549	–	–	
p16	200	2	50	50	137	137	1	137	137	0	0	137	137	0	0
p16	200	3	50	50	203	203	1	203	203	0	0	203	203	0	0
p16	200	4	50	50	269	269	1	269	269	0	0	269	269	0	0
p16	200	2	75	75	212	212	1860	212	212	0	2	212	212	0	2
p16	200	3	75	75	317	317	1857	317	317	0	4	317	317	0	4
p16	200	4	75	75	420	420	2525	420	420	0	22	420	420	0	21
p16	200	2	100	100	285	285		285	285	0	279	285	285	0	274
p16	200	3	100	100	421	422		423	**423**	0	280	423	**423**	0	476
p16	200	4	100	100	554	544		558	**558**	0	342	558	–	–	

price algorithm proposed in Archetti et al. [2] on three instances but is not able to solve to optimality instance p15 previously solved. For the CPTP (see Table 2), no results for the branch-and-price algorithm were reported in Archetti et al. [2] since their approach was not able to produce any primal or dual bound in 1 h. Thus, we can only compare our results with the best heuristic value reported in Archetti et al. [2]. The restricted master heuristic we propose allows to improve the best heuristic solution found in Archetti et al. [2] on four instances. For both problems, the variant with the restricted master heuristic clearly outperforms the one without the restricted master heuristic, especially for the CPTP where no primal bound is found by the version without the heuristic within 1 h even if the dual bounds are slightly better in all instances but one.

Tables 3 and 4 report the results on the Set 2 of instances for the CTOP and the CPTP, respectively. From Table 3, we can see that for the CTOP our approach is able to solve 86 out of 90 instances, while in Archetti et al. [2] only 60 out of 90 instances were solved to optimality. Moreover, we have found 10 new best solutions. For the CPTP, our approach is able to solve 88 out of 90 instances, while in Archetti et al. [2] only 53 out of 90 instances were solved to optimality. Moreover, we have found 5 new best solutions. For both problems, the variant with the restricted master heuristic has a better performance than the one without the heuristic.

Tables 5 and 6 report the results on the Set 3 of instances for the CTOP and the CPTP, respectively. For both problems, no results for the branch-and-price algorithm were reported in Archetti et al. [2] since their approach was not able to produce any primal or dual bound within 1 h. From Table 5 we can see that our approach is able to solve to optimality 8 out of 30 instances in both versions, with and without the restricted master heuristic. However, when the heuristic is used the optimality gap is above 10% in only 9 cases, while when the heuristic is not used no primal bound is found in 21 cases. For the CPTP (see Table 6), our approach is able to solve to optimality 5 out of 30 instances, with or without the restricted master heuristic. However, for all the other instances, only in 2 cases a primal solution is found when the heuristic is not used while, when the heuristic is used, the optimality gap is greater than 5% in 6 cases only. The dual bounds are in many cases slightly worse when the heuristic is used. On the other hand, when the heuristic is used, two new best known solutions are found.

To summarize, the restricted master heuristic has a crucial role in the solution process. In fact, when the heuristic is removed, the CPU time increases and the percentage gap deteriorates. Besides, this behavior is more evident on the hardest instances.

5. Conclusions

In this paper, we presented a new branch-and-price algorithm to solve two vehicle routing problems with profits, the CTOP and the CPTP. A heuristic, belonging to the class of the so called restricted master heuristics, is embedded in the overall approach in order to quickly find good feasible solutions. Computational results show that the use of the heuristics helps the overall solution of the problem. Moreover, the branch-and-price algorithm we proposed outperforms previous approaches.

In our solution approach, at the root node of the branch-and-bound tree, the restricted master heuristic is run after each exact solution of the pricing problem. This choice is motivated by the fact that we are mainly interested in finding primal and dual values as soon as possible. It is worth noting that running the heuristic before the exact solution of the subproblem

Table 4
CPTP dual/primal bound—Set 2.

Instance	n	m	Q	Best known			Branch-and-price algorithm							
				Heuristic	Exact		With primal heur.				Without primal heur.			
				$\underline{z}^*$	$\underline{z}^*$	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p03	101	2	50	57.75	57.75	6	57.75	57.75	0	0	57.75	57.75	0	0
p03	101	3	50	80.82	80.82	6	80.82	80.82	0	0	80.82	80.82	0	0
p03	101	4	50	100.36	100.36	7	100.36	100.36	0	0	100.36	100.36	0	0
p03	101	2	75	106.15	106.15	519	106.15	106.15	0	3	106.15	106.15	0	3
p03	101	3	75	147.55	147.55	450	147.55	147.55	0	4	147.55	147.55	0	4
p03	101	4	75	185.27	185.27	248	185.27	185.27	0	3	185.27	185.27	0	3
p03	101	2	100	158.21	158.21		158.21	158.21	0	5	158.21	158.21	0	5
p03	101	3	100	218.63	218.63		218.63	218.63	0	11	218.63	218.63	0	11
p03	101	4	100	268.34	255.33		268.34	268.34	0	333	268.34	268.34	0	334
p06	51	2	50	33.88	33.88	0	33.88	33.88	0	0	33.88	33.88	0	0
p06	51	3	50	40.95	40.95	1	40.95	40.95	0	0	40.95	40.95	0	0
p06	51	4	50	45.43	45.43	0	45.43	45.43	0	0	45.43	45.43	0	0
p06	51	2	75	72.28	72.28	4	72.28	72.28	0	1	72.28	72.28	0	0
p06	51	3	75	92.32	92.32	3	92.32	92.32	0	0	92.32	92.32	0	0
p06	51	4	75	99.37	99.37	57	99.37	99.37	0	22	99.37	99.37	0	22
p06	51	2	100	100.27	100.27	159	100.27	100.27	0	2	100.27	100.27	0	2
p06	51	3	100	134.72	134.72	119	134.72	134.72	0	2	134.72	134.72	0	1
p06	51	4	100	153.30	153.30	1774	153.30	153.30	0	16	153.30	153.30	0	16
p07	76	2	50	49.18	49.18	1	49.18	49.18	0	0	49.18	49.18	0	0
p07	76	3	50	69.94	69.94	0	69.94	69.94	0	0	69.94	69.94	0	0
p07	76	4	50	90.65	90.65	1	90.65	90.65	0	0	90.65	90.65	0	0
p07	76	2	75	92.44	92.44	5	92.44	92.44	0	0	92.44	92.44	0	0
p07	76	3	75	131.12	131.12	9	131.12	131.12	0	0	131.12	131.12	0	0
p07	76	4	75	158.11	158.11	39	158.11	158.11	0	3	158.11	158.11	0	3
p07	76	2	100	132.70	132.70	441	132.70	132.70	0	1	132.70	132.70	0	1
p07	76	3	100	185.25	185.25	710	185.25	185.25	0	3	185.25	185.25	0	2
p07	76	4	100	233.40	233.40	684	233.40	233.40	0	3	233.40	233.40	0	3
p08	101	2	50	57.75	57.75	6	57.75	57.75	0	0	57.75	57.75	0	0
p08	101	3	50	80.82	80.82	6	80.82	80.82	0	0	80.82	80.82	0	0
p08	101	4	50	100.36	100.36	7	100.36	100.36	0	0	100.36	100.36	0	0
p08	101	2	75	106.15	106.15	511	106.15	106.15	0	3	106.15	106.15	0	3
p08	101	3	75	147.55	147.55	450	147.55	147.55	0	4	147.55	147.55	0	4
p08	101	4	75	185.27	185.27	248	185.27	185.27	0	3	185.27	185.27	0	3
p08	101	2	100	158.21	158.21		158.21	158.21	0	5	158.21	158.21	0	5
p08	101	3	100	218.63	218.63		218.63	218.63	0	11	218.63	218.63	0	10
p08	101	4	100	268.34	255.33		268.34	268.34	0	320	268.34	268.34	0	330
p09	151	2	50	65.03	65.03	34	65.03	65.03	0	0	65.03	65.03	0	0
p09	151	3	50	96.16	96.16	34	96.16	96.16	0	1	96.16	96.16	0	1
p09	151	4	50	121.35	121.35	35	121.35	121.35	0	1	121.35	121.35	0	1
p09	151	2	75	117.66	117.66	2666	117.66	117.66	0	2	117.66	117.66	0	2
p09	151	3	75	160.96	160.66		160.96	160.96	0	8	160.96	160.96	0	8
p09	151	4	75	204.25	204.25	2862	204.25	204.25	0	6	204.25	204.25	0	6
p09	151	2	100	161.23	161.15		161.23	161.23	0	41	161.23	161.23	0	41
p09	151	3	100	230.49	226.29		230.49	230.49	0	37	230.49	230.49	0	37
p09	151	4	100	290.54	286.39		290.97	290.97	0	632	290.97	290.97	0	634
p10	200	2	50	70.87	70.87	98	70.87	70.87	0	1	70.87	70.87	0	1
p10	200	3	50	103.79	103.79	85	103.79	103.79	0	1	103.79	103.79	0	1
p10	200	4	50	134.81	134.81	85	134.81	134.81	0	2	134.81	134.81	0	1
p10	200	2	75	124.85	124.85		124.85	124.85	0	2	124.85	124.85	0	2
p10	200	3	75	177.90	177.90		177.90	177.90	0	9	177.90	177.90	0	9
p10	200	4	75	229.27	229.27		229.27	229.27	0	9	229.27	229.27	0	9
p10	200	2	100	171.24	171.20		171.24	171.24	0	18	171.24	171.24	0	19
p10	200	3	100	250.18	246.56		250.18	250.18	0	32	250.18	250.18	0	33
p10	200	4	100	324.02	322.32		324.93	**324.93**	0	57	324.93	**324.93**	0	56
p13	121	2	50	64.12	63.92		64.12	64.12	0	13	64.12	64.12	0	13
p13	121	3	50	87.25	85.72		87.25	87.25	0	37	87.25	87.25	0	38
p13	121	4	50	104.18	94.84		104.18	104.18	0	243	104.18	104.18	0	250
p13	121	2	75	110.12	110.12		110.12	110.12	0	10	110.12	110.12	0	10
p13	121	3	75	139.37	135.10		139.37	139.37	0	702	139.37	139.37	0	700
p13	121	4	75	161.62	156.64		161.62	161.62	0	2086	161.62	161.62	0	2089
p13	121	2	100	145.75	145.75		145.75	145.75	0	218	145.75	145.75	0	218
p13	121	3	100	181.63	180.79		196.21	177.07	9.75		196.21	–	–	
p13	121	4	100	200.62	197.86		206.59	**201.59**	2.42		206.59	–	–	
p14	101	2	50	43.26	43.26	5	43.26	43.26	0	0	43.26	43.26	0	0
p14	101	3	50	59.43	59.43	7	59.43	59.43	0	0	59.43	59.43	0	0
p14	101	4	50	68.63	68.63	10	68.63	68.63	0	2	68.63	68.63	0	2

(*continued on next page*)

Table 4 (*continued*)

Instance	n	m	Q	Best known			Branch-and-price algorithm							
				Heuristic	Exact		With primal heur.				Without primal heur.			
				$\underline{z}^*$	$\underline{z}^*$	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p14	101	2	75	77.09	77.09	78	77.09	77.09	0	1	77.09	77.09	0	1
p14	101	3	75	112.56	112.56	79	112.56	112.56	0	1	112.56	112.56	0	1
p14	101	4	75	139.88	139.88	138	139.88	139.88	0	2	139.88	139.88	0	2
p14	101	2	100	125.29	125.29		125.29	125.29	0	7	125.29	125.29	0	7
p14	101	3	100	182.31	182.31		182.31	182.31	0	8	182.31	182.31	0	8
p14	101	4	100	237.68	236.29		237.68	237.68	0	13	237.68	237.68	0	13
p15	151	2	50	64.98	64.98	39	64.98	64.98	0	1	64.98	64.98	0	1
p15	151	3	50	96.42	96.42	41	96.42	96.42	0	1	96.42	96.42	0	1
p15	151	4	50	124.02	124.02	41	124.02	124.02	0	1	124.02	124.02	0	1
p15	151	2	75	120.93	120.93	2536	120.93	120.93	0	2	120.93	120.93	0	2
p15	151	3	75	174.58	174.58	2705	174.58	174.58	0	5	174.58	174.58	0	5
p15	151	4	75	219.22	219.22	2626	219.22	219.22	0	9	219.22	219.22	0	9
p15	151	2	100	169.71	166.93		169.71	169.71	0	13	169.71	169.71	0	13
p15	151	3	100	244.08	238.39		244.08	244.08	0	31	244.08	244.08	0	31
p15	151	4	100	308.07	306.39		309.75	**309.75**	0	145	309.75	**309.75**	0	162
p16	200	2	50	66.81	66.81	90	66.81	66.81	0	1	66.81	66.81	0	1
p16	200	3	50	99.70	99.70	90	99.70	99.70	0	1	99.70	99.70	0	1
p16	200	4	50	131.37	131.37	91	131.37	131.37	0	2	131.37	131.37	0	2
p16	200	2	75	123.38	123.38		123.38	123.38	0	8	123.38	123.38	0	8
p16	200	3	75	179.55	179.55		179.55	179.55	0	20	179.55	179.55	0	20
p16	200	4	75	235.03	235.03		235.03	235.03	0	12	235.03	235.03	0	12
p16	200	2	100	177.23	173.56		177.23	177.23	0	23	177.23	177.23	0	22
p16	200	3	100	258.07	239.34		259.25	**259.25**	0	40	259.25	**259.25**	0	40
p16	200	4	100	336.24	330.14		337.80	**337.80**	0	33	337.80	**337.80**	0	33

Table 5
CTOP dual/primal bound—Set 3.

Instance	n	m	Q	$T_{\max}$	Best known	Branch-and-price algorithm							
					Heuristic	With primal heur.				Without primal heur.			
					$\underline{z}^*$	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p03	101	2	200	200	536	536	524	2.24		536	–	–	
p03	101	3	200	200	762	778	701	9.90		778	–	–	
p03	101	4	200	200	950	1034	836	19.15		1034	–	–	
p06	51	2	160	200	403	403	403	0	1595	403	403	0	1139
p06	51	3	160	200	565	565	565	0	579	565	565	0	608
p06	51	4	160	200	683	683	683	0	1058	683	683	0	903
p07	76	2	140	160	377	377	377	0	982	377	377	0	1007
p07	76	3	140	160	548	548	548	0	46	548	546	0	
p07	76	4	140	160	707	707	704	0.42		707	707	0	1652
p08	101	2	200	230	536	536	509	5.04		536	–	–	
p08	101	3	200	230	762	780	676	13.33		780	–	–	
p08	101	4	200	230	950	1059	819	22.66		1059	–	–	
p09	151	2	200	200	547	552	536	2.90		552	–	–	
p09	151	3	200	200	796	809	752	7.05		809	–	–	
p09	151	4	200	200	1033	1063	972	8.56		1063	–	–	
p10	200	2	200	200	556	559	542	3.04		559	–	–	
p10	200	3	200	200	815	842	791	6.06		842	–	–	
p10	200	4	200	200	1064	1122	1022	8.91		1122	–	–	
p13	121	2	200	720	513	555	406	26.85		555	–	–	
p13	121	3	200	720	727	908	575	36.67		908	–	–	
p13	121	4	200	720	908	1318	738	44.01		1318	–	–	
p14	101	2	200	1040	534	534	534	0	803	534	534	0	1038
p14	101	3	200	1040	770	770	770	0	1753	770	770	0	1778
p14	101	4	200	1040	975	975	975	0	3147	975	975	0	2466
p15	151	2	200	200	550	551	546	0.91		551	–	–	
p15	151	3	200	200	801	829	744	10.25		829	–	–	
p15	151	4	200	200	1031	1098	937	14.66		1098	–	–	
p16	200	2	200	200	558	560	556	0.71		560	–	–	
p16	200	3	200	200	821	850	795	6.47		850	–	–	
p16	200	4	200	200	1073	1144	1020	10.84		1144	–	–	

would lead to feasible solutions at an earlier stage. Thus, if one is mainly interested in finding feasible solutions, this would be the right way to proceed. The higher the effectiveness of the heuristic used to generate columns, the higher the quality of the feasible solutions obtained.

Table 6
CPTP dual/primal bound—Set 3.

Instance	n	m	Q	Best known Heuristic	Branch-and-price algorithm							
					With primal heur.				Without primal heur.			
				$\underline{z}^*$	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t	$\bar{z}^*$	$\underline{z}^*$	Gap (%)	t
p03	101	2	200	330.14	337.59	330.14	2.21		337.43	–	–	
p03	101	3	200	447.15	451.92	447.15	1.06		451.68	–	–	
p03	101	4	200	537.66	552.05	537.66	2.61		551.74	–	–	
p06	51	2	160	168.6	168.60	168.60	0.00	13	168.60	168.60	0	12
p06	51	3	160	219.36	219.36	219.36	0.00	870	219.36	219.36	0	578
p06	51	4	160	258.97	260.31	258.97	0.52		259.67	256.55	1.20	
p07	76	2	140	199.97	199.97	199.97	0.00	3	199.97	199.97	0	4
p07	76	3	140	274.8	274.80	274.80	0.00	878	274.80	274.80	0	768
p07	76	4	140	344.35	345.15	344.35	0.23		344.82	344.35	0.13	
p08	101	2	200	330.14	337.59	330.14	2.21		337.42	–	–	
p08	101	3	200	447.15	451.90	447.15	1.05		451.70	–	–	
p08	101	4	200	537.66	552.00	537.66	2.60		551.74	–	–	
p09	151	2	200	347.9	359.97	343.21	4.66		359.92	–	–	
p09	151	3	200	500.17	516.45	**500.87**	3.02		516.37	–	–	
p09	151	4	200	639.72	657.13	632.54	3.74		656.93	–	–	
p10	200	2	200	382.41	383.62	382.41	0.31		383.58	–	–	
p10	200	3	200	559.8	563.16	553.36	1.74		563.08	–	–	
p10	200	4	200	723.47	732.19	722.46	1.33		732.19	–	–	
p13	121	2	200	239.57	257.62	219.29	14.88		257.56	–	–	
p13	121	3	200	250.69	314.59	244.31	22.34		314.44	–	–	
p13	121	4	200	294.46	347.68	255.11	26.63		347.61	–	–	
p14	101	2	200	303.17	326.11	283.35	13.11		325.95	–	–	
p14	101	3	200	418.28	456.50	399.37	12.51		456.27	–	–	
p14	101	4	200	537.24	567.31	474.06	16.44		566.90	–	–	
p15	151	2	200	378.09	378.09	378.09	0.00	608	378.09	378.09	0	594
p15	151	3	200	519.39	533.75	512.56	3.97		533.70	–	–	
p15	151	4	200	654.94	669.93	653.40	2.47		669.87	–	–	
p16	200	2	200	394.05	397.86	391.66	1.56		397.83	–	–	
p16	200	3	200	567.24	579.70	562.62	2.95		579.27	–	–	
p16	200	4	200	731.14	745.15	**733.95**	1.50		745.03	–	–	

A relevant feature of the restricted master heuristic is that it is widely applicable. Future research efforts will be devoted to the refinement of the heuristic.

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