Equivarrant Cohomology

Let G be a topological ~~space~~ acting on a topological space X.

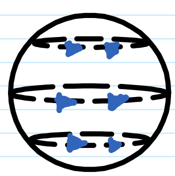
What happen with the cohomology of the quotient space X/G?

Examples: 1. Let $G=\mathbb{Z}$ be the group integers acting on $X=\mathbb{R}$ by translation

$$\mu(n,x) = x+n$$

then $\mathbb{R}/\mathbb{Z} \cong S^1$ and $H^*(X/G, \mathbb{Z}) = H^*(S^1, \mathbb{Z})$

2. Let's consider $S^1 \curvearrowright S^2$ by rotations



$S^1 \curvearrowright S^2$

S^2/S^1

then $H^*(S^2/S^1, \mathbb{Z})$

$\cong H^*([-1,1], \mathbb{Z})$

$\cong H^*(pt, \mathbb{Z})$

Borel Model

Let G be a topological group and X a G-space.

Idea: Get a contractible space E with a free action.

This exists for any topological group (Milnor's construction)

Recall!

In the category of principal G-bundles, there exists the classifying bundle $EG \longrightarrow BG$ such that

$$\left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{principal } G\text{-bundles over } Y \end{array} \right\} \longleftrightarrow \{ f : Y \longrightarrow BG \} / \text{homotopy}$$

- With this space EG in mind, we can consider the action

$$\begin{aligned} \theta : G \times EG \times M &\longrightarrow EG \times M \\ (g, f, m) &\longmapsto (g.f, g.m) \end{aligned}$$

this action is free since $\theta(g, f, m) = (g.f, g.m) = (f, m)$ implies that $g = e$, the identity of G.

- Since EG is contractible, we are going to have that:

$$H^*(EG \times M/_G, \mathbb{Z}) = H^*(M/_G, \mathbb{Z})$$

When this makes sense.

Notation:

$$(M)_G = EG \times_G M = EG \times M/_G$$

Functorial properties

Let $f: M \longrightarrow N$ be a G-equivariant map

$$f(g.m) = g.f(m)$$

We can induces a map

$$f_G: EG \times_G M \longrightarrow EG \times_G N$$

$$f_G[e,m] = [e, f(m)]$$

and this is induces in cohomology

$$f_G^*: H_G^*(N) \longrightarrow H_G^*(M)$$

Propiedades

- $EG \times_G (\ )$ is a covariant functor from the category of G-spaces to the category of topological spaces.

- The equivariant cohomology H_G is a contravariant functor from the category of G-spaces to the category of groups

$$H_G = H \circ EG \times_G (\ ).$$

What happen when we have two contractible spaces with free action on them?

Recall:

- Classifying bundles are unique up to G-homotopy equivalence
- Principal G-bundles with contractible total space are classifying bundles

Prop: If there are two contractible spaces E and E' with free action by G, then

$$H^*(E\times_G M, \mathbb{Z}) = H^*(E'\times_G M, \mathbb{Z}).$$

Proof: Then if we consider the principal bundles

$$E \to E/G \quad \text{and} \quad E' \to E'/G$$

we know that there exist

$$\phi: E \to E' \quad \text{and} \quad \psi: E' \to E \quad \text{equivariantes}$$

such that

$$\phi\circ\psi \simeq id_{E'} \quad \text{and} \quad \psi\circ\phi \simeq id_E$$

then this homotopies are induced $E\times_G M$ and $E'\times_G M$ and we got the result.

Definition

The equivariant cohomology of a smooth manifold is defined as

$$H^*_G(M) = H^*(EG \times_G M, \mathbb{Z}).$$

Examples:

- if $M = pt$ then $EG \times_G pt \simeq EG/G \cong BG$ then $H^*_G(pt) = H^*(BG, \mathbb{Z})$
- If $S^1 \curvearrowright S^2$ by rotations, we can consider the fiber bundle with fiber

$$\begin{array}{ccc} S^2 \longrightarrow & ES^1 \times_{S^1} S^2 & \\ & \downarrow & \\ & \mathbb{C}P^\infty & = BS^1 \end{array}$$

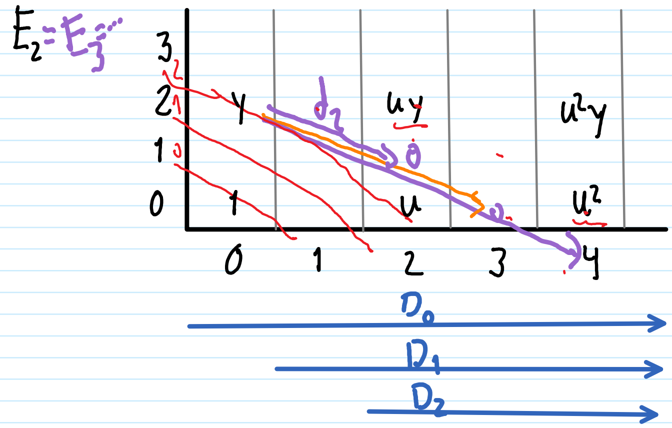
Recall: Spectral sequences (Leray's theorem)

If we have a fiber bundle $\pi: E \longrightarrow B$ with fiber F over a simply connected B. Then there exists a spectral sequence with $E_2^{p,q} = H^p(B) \otimes H^q(F)$ and the filtration by p on the E_2 term induces a filtration $\{D_p \cap H^n\}$ on $H^n = H^n(E)$ such that its successive quotients are $E_\infty^{p,n-p}$

Then for our example

$$E_2 = H^*(\mathbb{C}P^\infty) \otimes_{\mathbb{Z}} H^*(S^2)$$

$$= \mathbb{Z}[u] \otimes \frac{\mathbb{Z}[y]}{(y^2)} \qquad \deg(y) = 2 = \deg(u)$$



Observe that $d_2 = 0 = d_3 = d_4 \cdots$

$d_i = 0 \quad i \geq 4$

Then $E_2 = E_3 = E_4 = \dots = E_\infty$

$$H^2_{S^1}(S^2) = \mathbb{Z}(y) \oplus \mathbb{Z}[u]$$

$$H^4_{S^1}(S^2) = \mathbb{Z}[uy] \oplus \mathbb{Z}[u^2]$$

$$H^{odd}_{S^1}(S^2) = 0$$

Cartan Model

Let G a compact Lie group acting on a smooth manifold M by $\mu: G\times M \longrightarrow M$

Def: An equivariant form of a smooth manifold M is a polynomial function $\alpha: \mathfrak{g} \longrightarrow \Omega^*(M)$ s.t.

$\mathfrak{g} = \langle X_1, \dots, X_n \rangle$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\alpha} & \Omega^*(M) \\ \downarrow{\scriptstyle Ad_g} & & \downarrow{\scriptstyle g} \\ \mathfrak{g} & \longrightarrow & \Omega^*(M) \end{array}$$

$\alpha = \sum w_i\, p(x_1, \dots, x_n)$, $w_i \in \Omega^*(M)$.

that is, $\alpha(Ad_g X) = g\alpha(X)$, where $\mathfrak{g}$ is the Lie algebra of G and $g\alpha$ is given by the pullback in differential forms of $\mu_g: M \longrightarrow M$ $\quad \mu_g(m) = g.m.$

The set of equivariant forms is denoted as $Map(\mathfrak{g}, \Omega^*(M))$

We can induce an action on all polynomial functions in the following way

$$G \times Hom(\mathfrak{g}, \Omega^*(M)) \xrightarrow{v} Hom(\mathfrak{g}, \Omega^*(M))$$

$$(g, \alpha(X)) \longmapsto v(g,\alpha)(X) = g^{-1}\alpha(Ad_g X)$$

for any $X \in \mathfrak{g}$.

Remark:
- α is an equivariant form iff α is invariant for the action v.
- Polynomial function from $\mathfrak{g}$ to $\Omega^*(M)$ can be written as

$$\bigoplus (S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M)) \circlearrowleft G$$

for being equivariant forms we need to be invariant for the action v

that is, $\bigoplus \left(S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M) \right)^G$.

The Cartan differential

Consider $d_G : Map(\mathfrak{g}, \Omega^*(M)) \longrightarrow Map(\mathfrak{g}, \Omega^*(M))$

given by $d_G(\alpha)(X) = d_{dR}(\alpha(X)) - \iota_X \alpha(X)$, where

d_{dR} is the exterior derivative
and

$\iota_X \alpha(X) = \sum a_i \, i_{X_i} \alpha(X)$, for $X = \sum a_i X_i$

Prop: for $\alpha \in Map(\mathfrak{g}, \Omega^*(M))$

- $d_G \alpha \in Map(\mathfrak{g}, \Omega^*(M))$
- $d_G^2 = 0$

Proof:
- $d_G \alpha$ is a polynomial map. d_{dR}. ι
- $d_G \alpha(Ad_g X) = d_{dR}(\alpha(Ad_g X)) - \iota_{(Ad_g X)} \alpha(Ad_g X)$.

$$= d_{dR}(g\alpha(X)) - (g \iota_X g^{-1}) g \alpha(X)$$

$$= g \, d_{dR}(\alpha(X)) - g \iota_X \alpha(X) = g . d_G(\alpha(X))$$

- $d_G^2(\alpha)(X) = \underbrace{d_{dR}^2}_{d^2=0}(\alpha(X)) - (d_{dR}\iota_X \alpha(X) + \iota_X d_{dR}\alpha(X)) + \iota_X^2 \alpha(X)$

$d_{dR}^2 = 0 = \iota_X^2$ $\qquad$ $\iota = 0$

$$= -\mathcal{L}_X \alpha(X)$$

$$= -\frac{d}{dt}\Big|_{t=0} (\exp(tX).\alpha(X)) = -\frac{d}{dt}\Big|_{t=0} \alpha(Ad(\exp(-tX))X)$$

$$= -\frac{d}{dt}\Big|_{t=0} \alpha(e^{-ad(tX)}X)$$

$\frac{d}{dt}\Big|_{t=0}$

$$= -\frac{d}{dt}\Big|_{t=0} \underline{\alpha(X)} \overset{!!}{=} 0$$

Remark: $\underline{\Omega^*_G(M)} = \mathrm{Map}(\mathfrak{g}, \Omega^*(M))$

Another way to see this

We have that these equivariant forms can be written as

$$\Omega^*_G(M) = (S^*(\mathfrak{g}^\vee) \otimes \Omega^*(M))^G$$

Such that

$$\Omega^n_G(M) = \bigoplus_{2k+i=n} (S^k(\mathfrak{g}^\vee) \otimes \Omega^i(M))^G$$

we use this gradation since

d_{dR} increase the degree by one $\uparrow 1$

ι decrease the degree of the form by one increse the degree of the polinomial by one.

if $n = 2k+i$ is the degree of $\alpha \in \Omega^n_G$

then $d_G \alpha$ has as degree $2(k+1)+(i-1)$
$= 2k+2+i-1$
$= n+1$

Ejemplo: • If $M = pt$ then with trivial action by a connected compact Lie group G with dimension n

$$\Omega^*_G(pt) = (S^*(\mathfrak{g}^\vee))^G = (\mathbb{R}[u_1, \dots, u_n])^G$$

where u_i's are generators of $S^*(\mathfrak{g}^\vee)$

then $d_G u_k = \overset{=0}{d_{dR}} u_k - \sum u_i \iota_{X_i} u_k$
$= 0$

then $H^*_G(pt) = S^*(\mathfrak{g}^\vee)^G$

Theorem (Cartan) (1950) [Meinrenken]

If G is compact Lie group acting on a smooth compact manifold M, then the complex of equivariant forms computes the equivariant cohomology in the Borel model.

Example: By the previous example

$$H^*_G(pt,\mathbb{R}) = H^*(EG\times_G pt) \cong H^*(BG) \cong (S^*(\mathfrak{g}^\vee))^G$$