Study of the laplace-beltrami operator auto-spaces.

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Abstract

It is proposed to analyze the behavior of the self-spaces in spherical base figures with small deformations, by means of harmonic functions. In this article the process of construction of the graph on the deformed and previously sampled surface is implemented, with the purpose of generating an analysis on the graph in this type of surfaces.

Keywords: Auto-spaces, Harmonic functions, deformations of spheres, graph, $\varepsilon - net$.

1 Introduction

Geometric modeling has been used in recent years as a tool to understand how the parts that make up a system can be represented by geometric entities such as lines, polygons or circles Villa-Ochoa J (2017). The discrete versions of the Laplace Operator, especially the Beltrami Laplace Operator, allow the development of new tools in this field Sorkine (2008), since they allow the evaluation of light deformations in a sphere of S^2. For this purpose, Laplace or Laplacian is a differential operator defined as the divergence of the gradient of a function in Euclidean space, where it is theoretically defined in a smooth continuous domain called a manifold, the manifolds are often discrete units of polygons composed of geometric figures which in turn represent three-dimensional objects of the real world Fernández (2015). This operator has been very important in fields such as numerical analysis, given its implications in the simulation process and in partial differential geometric equations, with application in several areas such as Biomedical, synthetic and differential geometric modeling, where the discrete laplacian meshes of composite polygons play a leading role, thus demonstrating the quality, predictability and flexibility of the proposed operator Fernández (2015) Keppel (1975) Liu L (2008).

This work intends to continue with the developments obtained in the study with the Laplacian operator, evaluating the behavior of the operator's auto-spaces after small deformations of s^2 Figure 1, with the intention of calculating an approximation of the discrete differential geometry supported by its combination process in finite sets Carmo (1976), which for this case was the discrete representation of the surface of the variety corresponding to a uniform sampling. To achieve this, a graphical representation of the Laplacian auto-spaces with small non-isometric perturbations was implemented in Matlab software with an option chosen from the literature for uniform sampling JE Freund (2020), since it is contemplated as a simple statistical assumption to evaluate various aspects of the theory. We also stratified sampling of 2 manifolds over the variety, thus generating a graph representing a graphical discretization of the laplace-beltrami.

The topic of the project is fully theoretical and experimental so the construction of the problem statement is related to an academic interest of the tutor in charge Carlos Cadavid, who proposed this research in order to analyze the behavior of these auto-spaces and reach a conjecture proposed by him on the influence of the approximation to other dimensions through the technique used in the project and graphically described in Figure 2. Although the mathematical bases that support the development of the technique are not sufficiently used, product of a shortage of information and therefore a low applicability, as it is just being implemented for certain areas of science such as biomedical, modeling, simulation, mechanical physics and others, In this work, we intend to give new contributions of the use of the Laplacian Operator as a new option to the manipulation of non-isometric auto-spaces with manifold of slight deformations and hope to have the freedom to sculpt all kinds of figures while preserving the expected results.

The research is aimed at an academic population with theoretical and technical interests in stereographic tastes on figures in different dimensions, thus generating a contribution to the literature of this level of complexity, since the abstract information is high, so the execution procedure of the technique is visualized by means of software that uses the process of deformation, sampling and discretization in a simpler and experimental way.

For the purposes of the exercise, the deformed object was the sphere, because due to its geometric characteristics, the smooth deformation presents greater difficulty being a challenge for the exercise. The results obtained from this work seek to contribute to the analysis and elaboration of the graph, by means of multiple experiments with the laplacian of the graph, so studying what happens with the self spaces of the laplacian operator in a variety of s^2, would explain this approximation to higher dimensions such as s^4 and show if the conjecture is correct for all types of small deformed objects, to finally understand the behavior of these spaces in their composition and thus be able to make a conjecture about the relationship between the behavior of the eigenvalues of the Laplacian in s^3 and their correspondence in s^2 by means of stereographic projections of the plane.

2 State of the art

For years the vision of mathematics is not only stamped on paper, but with technological advances and the need for expansion to other worlds requires that the human mind does not remain in a 2D plane but expands to the many possibilities presented by computer graphics and processing power that has the new computers, so the spherical harmonics have been studied extensively and have been applied to solve a wide range of problems in science and engineering. Interest in approximations and numerical methods for problems on spheres has grown steadily Kendall Atkinson (2010).

When developing the computational procedure it is very evident the steps to follow for the construction of the laplacian of the auto spaces so the first part to be developed are the light deformations of the object of study, in this case the spheres, which are elaborated with a mechanism mentioned by Sheldon Axler & Ramey (2020) which proposes the use of the Harmonic Functions to obtain this result.

After the deformations of these spheres, sampling is performed with Monte Carlo techniques arise in image synthesis primarily as a means to solve integration problems. Integration over domains of two or higher dimensions is ubiquitous in image synthesis.

The Monte Carlo method stems from a very natural and immediate connection between integration and expectation. Every integral, in both deterministic and probabilistic contexts, can be viewed as the expected value (mean) of a random variable; by averaging over many samples of the random variable, we may thereby approximate the integral. However, there are infinitely many random variables that can be associated with any given integral; of course, some are better than others. One attribute that makes some random variables better than others for the purpose of integration is the ease with which they can be sampled. In general, it tend to construct Monte Carlo methods using only those random variables with convenient and efficient sampling procedures. But there is also a competing attribute. One of the maxims of Monte Carlo integration is that the probability density function of the random variable should mimic the integrand as closely as possible. The closer the match, the smaller the variance of the random variable, and the more reliable (and efficient) the estimator. In the limit, when the samples are generated with a density that is exactly proportional to the (positive) integrand, the variance of the estimator is identically zero Arvo (2001).

The justification of the discretization of the graph can be approximated by the Laplace Bertrami operand through the Laplacian of the proximity graph mentioned in the article Dmitri burago (2014), where it mentions that traditionally, discretization in Riemannian geometry is associated with triangulations and other polyhedral approximations. This approach works perfectly well in dimension two but meets a number of obstacles in higher dimensions. It is now clear, due to works of Cheeger, Petrunin, Panov and others that in dimensions beyond three polyhedral structures are too rigid to serve as discrete models of Riemannian spaces with curvature bounds. In some applications, we get a Riemannian manifold as a cloud of points with approximate distances between them. This issue will be addressed elsewhere. For triangulations, even the problem of determining whether a given simplicial complex is a topological manifold is algorithmically undecidable. In a few papers, you will try to discuss approximating Riemannian manifolds by graphs, of course with additional structures and various boundedness conditions. Here you are show that the spectrum of a suitable graph Laplacian gives a reasonable approximation to the spectrum of the Riemannian Laplace–Beltrami operator. Let us note that we look at the problem from the viewpoint of spectral (Riemannian) geometry. On the other hand, similar problems of course have been receiving a lot of attention from numerical analysts.

3 Statement of the problem

In the last two decades three-dimensional modeling methods used by artists have been evolving and developing rapidly thanks to the use of vector operators of differential geometry such as the Laplacian operator. This operator allows modeling the behavior of complex applications such as noise reduction, enhancement, remeshing, UV mapping, posing and skeletonization, among others, in a simple way. The Laplacian operator is theoretically defined in a continuous and smooth domain, named manifold. In practice manifolds are often approximated by discrete polygon meshes composed by triangles and quadrangles which represent the real world three-dimensional objects with which the artists work. In these meshes, spectral structure is calculated using a discrete Laplacian operator, i.e. the discrete version of the Laplacian operator given by Pinkall in 1993 works only with meshes composed of triangles, and Xiong's in 2011 works exclusively with quadrangles Fernández (2015).

3.1 Problem statement

In mathematics, the Laplace operator or Laplacian is a differential operator given by the divergence of the gradient of a function on Euclidean space. It is usually denoted by the symbols $\nabla \cdot \nabla, \nabla^2$ (where ∇ is the nabla operator), or Δ . In a Cartesian coordinate system, the Laplacian is given by the sum of second partial derivatives of the function with respect to each independent variable. In other coordinate systems, such as cylindrical and spherical coordinates, the Laplacian also has a useful form. Informally, the Laplacian $\Delta f(p)$ of a function f at a point p measures by how much the average value of f over small spheres or balls centered at p deviates from $f(p)$ Medina (2019).

The use of Laplace operators in differential geometry is of utmost importance in the development of the project because in many cases there is a series of second order linear elliptic differential operators that are part of them, as in the case of the Laplace-Beltrami operator which is defined on sub varieties in Euclidean space and, even more generally, on Riemannian and pseudo-Riemannian varieties. a manifold is often approximated by a discrete mesh, it is therefore necessary to define a discrete Laplacian operator that acts on functions defined by such meshes. Considering a compact manifold M of dimension m isometrically embedded in a Eu- clidean space $\mathbb{R}^d$. Given a twice continuously differentiable function $f \in C^2(M)$, let $\Delta_M f$ denote the gradient vector field of f on M. The Laplace-Beltrami operator Δ_M of f is defined as the divergence of the gradient; that is Fernández (2015)

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}&=0 \\ \nabla_{\mathrm{M}}^2 f=\Delta_{\mathrm{M}} f&=0 \\ \Delta_{\mathrm{M}} f&=\operatorname{div}\left(\nabla_{\mathrm{M}} f\right) \end{aligned}$$

For this case we make use of the discretization of the graph by means of the Laplacian operator mentioned above, which is the initial objective of the elaboration of this study focused on the behavior of the auto-spaces of the Laplacian operator in a two-dimensional manifold with light trans-deformations at S^2.

It is important to mention that the research being carried out is a sequence of projects that are contributed by different students under the tutor's care, so the general objective of this does not change, but the approach that each student wants to contribute to the project does. The construction and analysis of the graph Laplacian are the key points of the contribution to the study of non-isometric auto-spaces.

3.2 Sampling of pseudo-spheres

The process of studying the behavior of the auto-spaces of the Laplacian operator after small deformations is realized by means of the necessary computational tools that allow the theoretical and graphical description of the construction of the discretized graph of the Beltrami Laplacian operator in a two-dimensional manifold, so the main structure of its composition are the spherical deformations realized with the support of the contributions mentioned in Sheldon Axler & Ramey (2020) which are light in terms of the natural structure of a sphere and is represented in Figure 1, which makes a simile of the type of perturbations that are expected to arrive with the use of the harmonic functions of Laplace. After this development a uniform sampling is performed within the surface of this object hoping that the amount of points of a piece of area are equal compared to others, this sampling is supported by the technique mentioned in Arvo (2001) using a larger surface than the one of interest, as evidenced in the left part of Figure 2, so it is sampled considering an equity in all types of areas no matter what kind of deformations may have the figure.

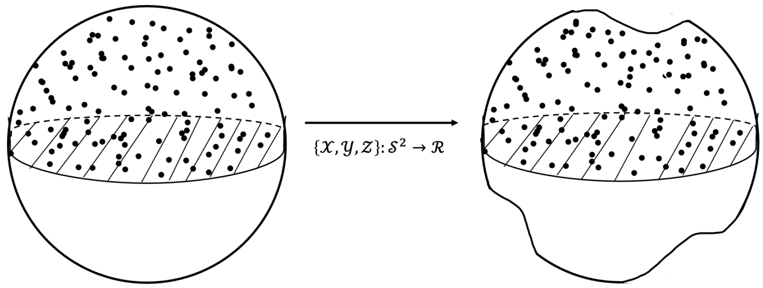

Figure 1: Graphical representation of the light deformations of a sphere $\mathbb{R}^3$ or simply S^2

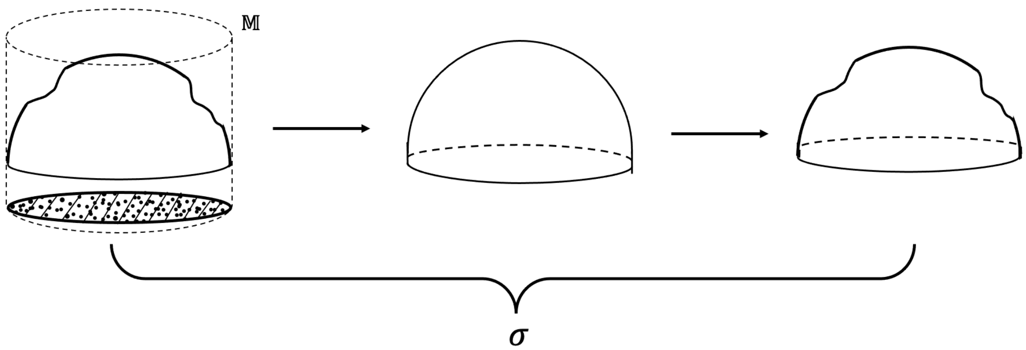

Figure 2: Graphic representation of the procedure to follow for the development of the idea.

The use of Beltrani's Laplaciado operator is approached with the following construction: For $f, g \in C^\infty(M)$ is defined as Medina (2019)

$$(f, g) = \int_M f(x)\overline{g(x)}dv$$

The following defines the Laplacian operator on functions

$$\Delta : C^\infty(M) \to C^\infty(M)$$

on a Riemannian variety M, is defined by the following formula

$$\Delta = -\delta d$$

Analogously the laplace operator is defined on p-forms $\Omega^p(M)$ as

$$\Delta = -(d\delta + \delta d) : \Omega^p(M) \to \Omega^p(M)$$

So far the construction of the two previous ideas has been completed, so the focus of the project is directed to the construction of the graph and the analysis of the Laplacian operator from the discretization supported by the approximations to the Riemannian Laplacian mentioned in Dmitri burago (2014). Finally, it is desired to draw conclusions about the behavior of these auto spaces and their manipulation with this method and possibly to get an idea of how the approximation would be in higher dimensions and to generate an environment of concerns and new questions.

3.3 Graph construction

Here we will show that the spectrum of a suitable graph Laplacian gives a reasonable approximation to the spectrum of the Riemannian Laplace-Beltrami operator. The key difference with finite element and similar methods is that in the construction the vertex set is an arbitrary lattice provided it is sufficiently dense. There are no local regularity constraints and only very approximate data are used.

Then we associate to this approximation a (sparse) matrix in the most straightforward way. In particular, it is describe some way of assigning to a given $\varepsilon - net$ on a Riemannian manifold a proper graph approximation. Once the matrix is constructed, its eigenvalues turn out to be very good approximations to those of the Riemannian Laplacian.

Let M^n be a compact Riemannian manifold (without boundary) and $X \subset M$ a finite $\varepsilon - net$. The geodesic distance between $x, y \in M$ is denoted by $d(x, y)$ or simply $|xy|$. Given such X and $\rho > 0$, one constructs a proximity graph $\Gamma = \Gamma(X, \rho)$: the set of vertices of the graph is X, and two vertices are connected by an edge if and only if $d(x, y) < \rho$. This set-up, it is assume that $\varepsilon << \rho$ and ρ is sufficiently small so that $\rho - balls$ in M are (bi-Lipschitz) close to Euclidean. In addition, it is assign weights to vertices and edges of Γ as explained below. Then there is a graph Laplacian operator associated with this structure. Our goal is to approximate the eigenvalues $\lambda(M)$ the Laplace–Beltrami operator on M by eigenvalues (Γ) of the graph Laplacian. This kind of problems were studied before. Fujiwara (1995) showed that, if X is an $\varepsilon - separated\ \varepsilon - net$ and $\rho = 5\varepsilon$, then the eigenvalues of (unweighted) graph Laplacian of the proximity graph after proper normalization satisfy

$$C_n^{-1}\lambda_k(M) \leq \lambda_k(\Gamma) \leq C_n\lambda_k(M)$$

where $C_n > 0$ is a constant depending only on $n = \dim M$. M. Belkin (2007) considered random, uniformly distributed nets in M and showed that, for a suitable choice of edge weights (depending on distances), the spectrum of the resulting graph Laplacian converges to the spectrum of M in the probability sense.

The construction. Let $\varepsilon > 0$ and $X = \{x_i\}_{i=1}^N$ be a finite ε-net in M. We denote by $B_r(x)$ the closed metric ball of radius r centered at $x \in M$. We assume that X is equipped with a discrete measure $\mu = \sum \mu_i \delta_{x_i}$ which approximates the volume of M in the following sense.

Definition 1.1. A measure μ on X is an ε-approximation of volume vol on M if there exist a partition of M into measurable subsets $V_i, i = 1, \ldots, N$, such that $V_i \subset B_\varepsilon(x_i)$ and $\mathrm{vol}(V_i) = \mu_i$ for every i.

In this case we also say that the pair $(X, \mu)\varepsilon$-approximates $(M, \mathrm{vol}\ )$.

Every ε-net X in M can be equipped with such a measure. For example, let $\{V_i\}$ be the Voronoi decomposition of M with respect to X and $\mu_i = \mathrm{vol}(V_i)$. In particular, it is show that this definition is naturally related to weak convergence of measures.

Consider the space $L^2(X) = L^2(X, \mu)$, that is the N-dimensional space of functions from X to $\mathbb{R}$ equipped with the following inner product:

$$\langle u, v\rangle = \langle u, v\rangle_{L^2(X)} = \sum \mu_i u(x_i) v(x_i)$$

or, equivalently, with a Euclidean norm given by

$$\|u\|^2 = \|u\|^2_{L^2(X)} = \sum \mu_i |u(x_i)|^2$$

We think of $L^2(X)$ as a finite-dimensional approximation to $L^2(M)$. For the sake of brevity, we omit the index $L^2(X)$ in most formulae in the paper.

We define the following weighted graph $\Gamma = \Gamma(X, \mu, \rho)$. The set of vertices is X two vertices $x, y \in X$ are connected by an edge if and only if $d(x, y) < \rho$. We write $x \sim y$ for $x, y \in X$ if they are connected by an edge. Both vertices and edges are weighted. The weight of a vertex x_i is μ_i. To an edge $e_{ij} = (x_i, x_j)$ we associate a weight $w\left(e_{ij}\right) = w_{ij}$ given by

$$w_{ij} = \frac{2(n+2)}{\nu_n \rho^{n+2}} \mu_i \mu_j$$

where ν_n is the volume of the unit ball in $\mathbb{R}^n$. Note that $w_{ij} = w_{ji}$. We approximate the Riemannian Laplace-Beltrami operator $\Delta = \Delta_M$ by the weighted graph Laplacian $\Delta_\Gamma : L^2(X) \to L^2(X)$ defined by

$$\begin{aligned}\left(\Delta_\Gamma u\right)\left(x_i\right) &= \frac{1}{\mu_i} \sum_{j: x_j \sim x_i} w_{ij}\left(u\left(x_j\right) - u\left(x_i\right)\right) \\ &= \frac{2(n+2)}{\nu_n \rho^{n+2}} \sum_{j: x_j \sim x_i} \mu_j\left(u\left(x_j\right) - u\left(x_i\right)\right)\end{aligned}$$

The motivation behind this formula is the following. If u is a discretization of a smooth function $f: M \to \mathbb{R}$, then the latter sum is the discretization of an integral over the ball $B_\rho\left(x_i\right)$:

$$\sum_{j: x_j \sim x_i} \mu_j\left(u\left(x_j\right) - u\left(x_i\right)\right) \approx \int_{B_\rho\left(x_i\right)}\left(f(x) - f\left(x_i\right)\right) d x$$

and the normalization constant is chosen so that the normalized integral approaches $\Delta f\left(x_i\right)$ as $\rho \to 0$, see Section 2.3. It follows that the graph Laplacian of the discretization of f approximates Δf if $\varepsilon \ll \rho \ll 1$.

4 Results

The base example to confirm both good sampling and graph construction is a clean sphere with zero coefficients in the harmonic function.

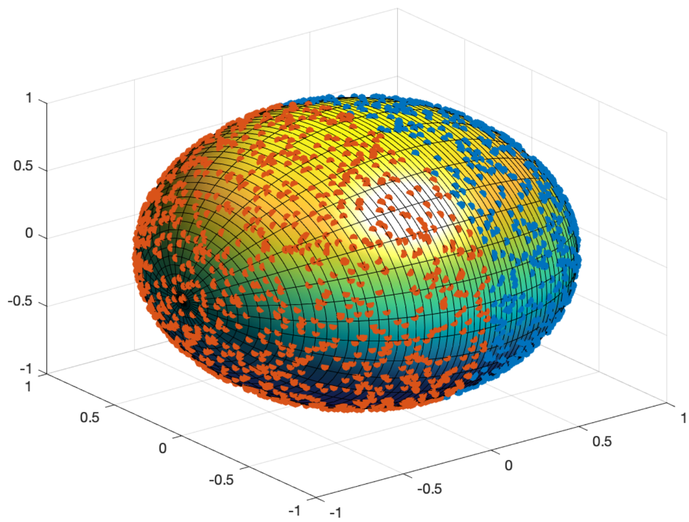

Figure 3: Surface of the sampled sphere without any deformation

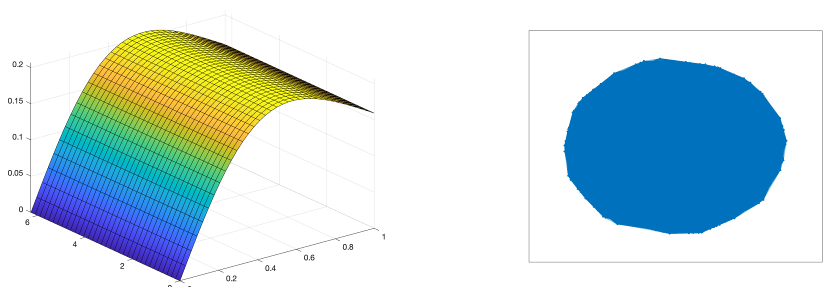

Figure 4: Graph constructed with the nodes obtained by the sampling surface

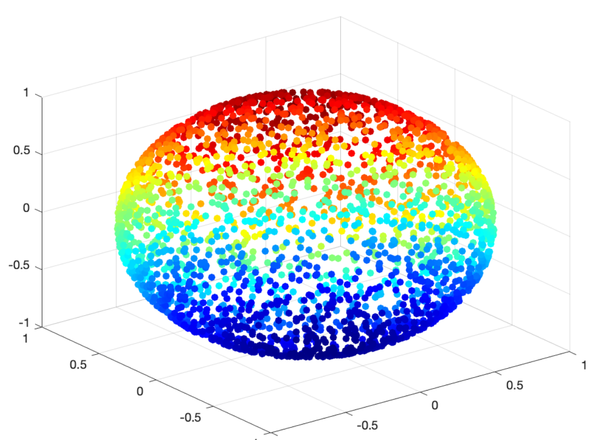

Figure 5: Graph on the surface of the sphere sampled with 1000 points

In the previous figures an initial example of the process generated by the algorithm is made where we conclude a very good sampling for the case in which no deformations are found. It can be noticed that in the figure 4 the graph may seem to be complete, but by analyzing the assignment of the nodes we conclude that the graph has different degrees for each node, so we know that it is well constructed finally obtaining the figure 5 colored by means of the eigenvalues assigned by the plotting.

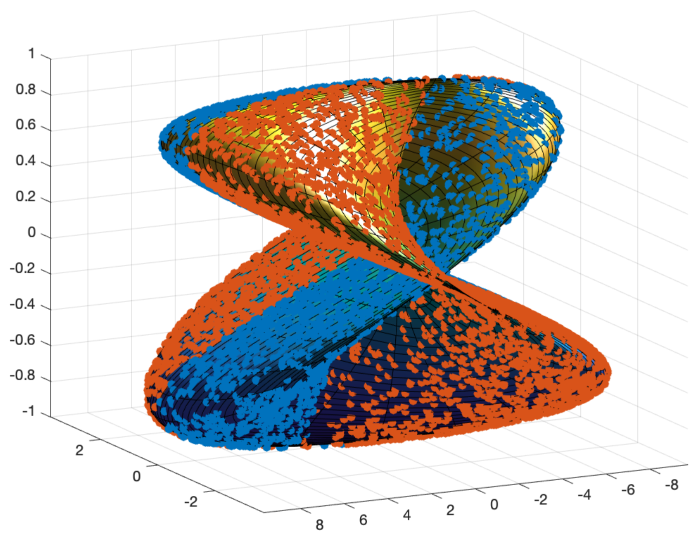

Figure 6: Surface of the sampled sphere without any deformation

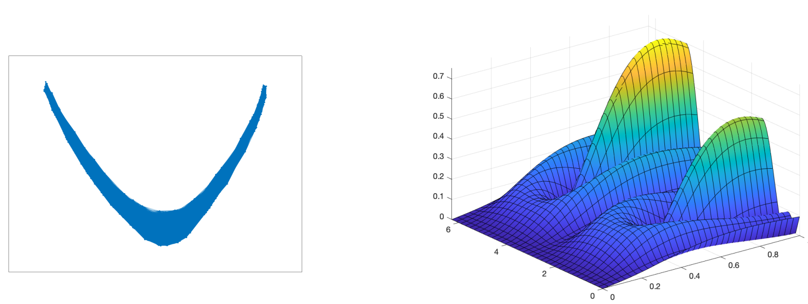

Figure 7: Graph constructed with the nodes obtained by the sampling surface

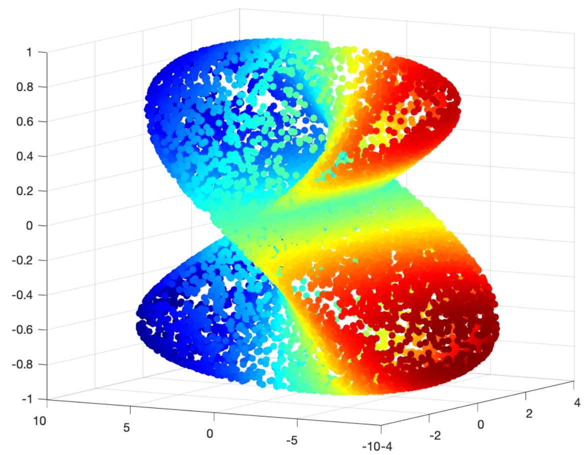

Figure 8: Graph on the surface of the sphere sampled with 10000 points

in the figures 6 a bigger change in the morphology of the sphere is observed so that the sampling and the construction of the graph in it generates that the method has a greater effectiveness being this a viable sample that the construction of the graph by means of the $\varepsilon - net$ is a reliable way to cross star surfaces uniformly.
Again a graph is observed in the figure 5 very unexpected on the sampling nodes but the analysis of the degree of the graph is made to conform its assignment between the nodes that is not a complete graph.

5 Conclusions and future research

We do not yet proceed to make an appropriate choice of edge weights(as a function of distances), because the density of the lattice may vary from region to region and these weights determine a discrete measure on X and we essentially require that X approximates M as a metric measure space, so it is proposed as future work to consider the problem of weights as an indispensable factor for the assessment of the quality of the network.

The difficulties that may arise during the development of the project are linked to the mathematical construction as rigorous and extensive to human understanding as is the analytical and practical construction of the epsilon-net, which if X is a separate $\varepsilon - net$ and $\rho = 5\varepsilon$, then the eigenvalues of the Laplacian (unweighted) graph of the proximity graph are adequate after a normalization, being these concepts very little studied yet, which makes the analysis and study of these self-spaces difficult. Another limitation is the knowledge of discrete differential geometry because it is an area of theoretical mathematics that is not yet so developed and could encounter behaviors that have never been studied before or are not yet defined.

Due to the sampling and graph construction, it is observed that for a sampling of many points, the computational time is exponential for the generation of the distance matrix, adjacency matrix and graph with the $\varepsilon - net$ so a future work would be to look for a more efficient way to generate these matrices that the computational time is less.

6 Intellectual property

According to the internal regulation on intellectual property within Universidad EAFIT, the results of this research practice are product of *Alejandra Ossa Yepes* and *Carlos Alberto Cadavid Moreno* y *Daniel Rojas Diaz* .

In case further products, beside academic articles, that could be generated from this work, the intellectual property distribution related to them will be directed under the current regulation of this matter determined by EAFIT (2018)

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