FIRST EXAM.
INTEGRAL ASYMPTOTIC APROXIMATION

1. For $\lambda >> 1$, find the leading term of the integral $\int_0^{\pi} e^{i\lambda\cos(t)}\,dt$

Using the Riemann-Lebesgue for stationary points :

- $\varphi(t) = \cos(t) \longrightarrow \varphi'(t) = -\sin(t)$ where $\varphi'(t_0) = 0$ when $t_0 = 0, \pi$

Then the integral is defined between two stationary points, so

$$I(\lambda) = \underbrace{\int_0^{\pi/2} e^{i\lambda\cos(t)}\,dt}_{I_1(\lambda)} + \underbrace{\int_{\pi/2}^{\pi} e^{i\lambda\cos(t)}\,dt}_{I_2(\lambda)}$$

- $\varphi''(t) = -\cos(t)$ which in $t_0 = 0$, $\varphi''(0) = -1$ and in $t_0 = \pi$,

$$\varphi''(\pi) = 1.$$

Then $p = 2$.

Each integral has the following leading term :

$$I_1(\lambda) = e^{i(\lambda+\frac{\pi}{4})}\left[\frac{2!}{\lambda}\right]^{1/2}\frac{\Gamma(1/2)}{2}$$

$$I_2(\lambda) = e^{i(-\lambda-\frac{\pi}{4})}\left[\frac{2!}{\lambda}\right]^{1/2}\frac{\Gamma(1/2)}{2}$$

Then

$$\int_0^{\pi} e^{i\lambda\cos(t)}\,dt \sim \sqrt{\frac{2}{\lambda}}\,\frac{\sqrt{\pi}}{2}\left(e^{i(\lambda+\frac{\pi}{4})} + e^{-i(\lambda+\frac{\pi}{4})}\right) = \sqrt{\frac{2\pi}{\lambda}}\cos\left(\lambda+\frac{\pi}{4}\right)$$

3. The integral $F(x,p) = \int_x^{\infty} t^{-p} e^{it}\,dt$ converges for all positive x if $p>0$. Prove that the complete $x\to\infty$ asymptotic expansion of $F(x,p)$ is given by

$$F(x,p) \sim \frac{i e^{ix}}{x^p}\sum_{n=0}^{\infty}\frac{\Gamma(p+n)}{\Gamma(p)(ix)^n}.$$

$$\mathcal{F}(x,p) = \frac{1}{i}\int_x^\infty t^{-p}\frac{d}{dt}e^{it}\,dt = \frac{1}{it^p}e^{it}\Big|_x^\infty - \frac{1}{i}\int_x^\infty e^{it}\frac{d}{dt}t^{-p}\,dt$$

$$= \frac{-e^{ix}}{ix^p} + \frac{p}{i}\int_x^\infty t^{-(p+1)}e^{it}\,dt.$$

$$= \frac{-e^{ix}}{ix^p} + \frac{p}{i}\left[\frac{1}{i}\int_x^\infty t^{-(p+1)}\frac{d}{dt}e^{it}\,dt\right]$$

$$= \frac{-e^{ix}}{ix^p} + \frac{p}{i^2}\left[\frac{e^{it}}{t^{p+1}}\Big|_x^\infty + (p+1)\int_x^\infty t^{-(p+2)}e^{it}\,dt\right]$$

$$= \frac{-e^{ix}}{ix^p} - \frac{p\,e^{ix}}{i^2x^{p+1}} + \frac{p}{i^2}(p+1)\int_x^\infty t^{-(p+2)}e^{it}\,dt$$

$$= -\frac{e^{ix}}{ix^p}\left[1 + \frac{p}{ix} + \frac{p(p+1)}{i}\int_x^\infty t^{-(p+2)}e^{it}\,dt\right]$$

Continuing with the integration by part integration

$$\mathcal{F}(x,p) = -\frac{e^{ix}}{ix^p}\left[1 + \frac{p}{ix} + \frac{p(p+1)}{i^2x^2} + \dots\right] = \frac{ie^{ix}}{x^p}\sum_{n=0}^{\infty}\frac{\Gamma(p+n)}{\Gamma(p)(ix)^n}$$

2. $I(x) := \int_0^1 e^{ixt^3}\,dt$

Let $\varphi(t) = it^3$ and $t = x+iy$, so $\varphi(x,y) = i\,(x+iy)^3$

$$= i\,(x^3 + iy(x^2) + (iy)^2x + (iy)^3)$$

$$= ix^3 - yx^2 - iy^2x + y^3$$

finally $\varphi(x,y) = u(x,y) + iv(x,y) = [y(y^2-x^2)] + i[x(x^2-y^2)]$.

The critical points of $\varphi(t)$ are given by

- $\varphi'(t) = 3it^2 \rightarrow t_0 = 0$ and $(x=0, y=0)$

The saddle order of $t_0 = 0$ is $m = 3$ and is given by

- $\varphi''(t) = 6it$ at $t_0 = 0$ $\varphi''(t_0) = 0$
- $\varphi'''(t) = 6i \neq 0 \longrightarrow \phi = \frac{\pi}{2}$

The valleys are given by $m\theta + \phi = (2k+1)\pi \longrightarrow \theta_i = \frac{(2k+1)\pi - \phi}{m}$

- $\theta_0 = \frac{1}{3}\left(\pi - \frac{\pi}{2}\right) = \frac{1}{3}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{6}} \longrightarrow \cos(\pi) = -1$
- $\theta_1 = \frac{1}{3}\left(3\pi - \frac{\pi}{2}\right) = \frac{1}{3}\left(\frac{5\pi}{2}\right) = \frac{5}{6}\pi \longrightarrow \cos(3\pi) = -1$
- $\theta_2 = \frac{1}{3}\left(5\pi - \frac{\pi}{2}\right) = \frac{1}{3}\left(\frac{9}{2}\pi\right) = \frac{9}{6}\pi = \frac{3\pi}{2} \longrightarrow \cos(2\pi) = 1.$

As the main contribution comes from a neighbourhood of the saddle point $t_0 = 0$

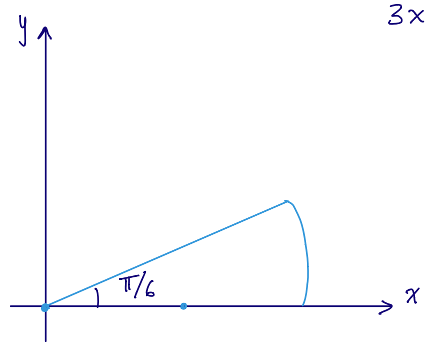
$$I(x) = \int_0^\infty e^{ixt^3} dt$$

Change variable to convert the integrand from oscillatory to exponentially decreasing $(t = i^{1/3} z,\ t^3 = iz^3,\ dt = i^{1/3} dz)$

$$I(x) = i^{1/3} \int_0^\infty e^{-xz^3} dz$$

And changing again $\tau = xz^3$, $d\tau = 3xz^2 dz \longrightarrow dz = \frac{\tau^{-2/3}}{3x^{1/3}} d\tau$, so

$$I(x) = \frac{e^{i\pi/6}}{3x^{1/3}} \int_0^\infty \tau^{-2/3} e^{-\tau} d\tau$$
$$= \frac{e^{i\pi/6}}{3x^{1/3}} \int_0^\infty \tau^{\frac{1}{3}-1} e^{-\tau} d\tau$$
$$= \frac{e^{\frac{i\pi}{6}}}{3x^{1/3}} \Gamma\left(\frac{1}{3}\right).$$



4. The beta function is defined by $B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$

Suppose that y is fixed and $x \to \infty$. Show that $\frac{\Gamma(x)}{\Gamma(x+y)} \sim \frac{1}{x^y}$ as $x \to \infty$.

Considering $t = e^{-z}$ then

$$B(x,y) = \int_\infty^0 (e^{-z})^{x-1}(1-e^{-z})^{y-1}(-e^{-z})\,dz$$

$$= \int_0^\infty e^{-zx}(1-e^{-z})^{y-1}dz \sim \int_0^\infty e^{-zx}[1-(1-z)]^{y-1}\,dz$$

$$= \int_0^\infty e^{-zx} z^{y-1} dz$$

changing $\tau = xz$, then

$$B(x,y) \sim \int_0^\infty e^{-\tau}\left(\frac{\tau}{x}\right)^{y-1}\frac{d\tau}{x} = \frac{1}{x^y}\int_0^\infty e^{-\tau}\tau^{y-1}d\tau = \frac{\Gamma(y)}{x^y}$$

Since $B(x,y) = \dfrac{\Gamma(x)}{\Gamma(x+y)}$ then $\dfrac{\Gamma(x)}{\Gamma(x+y)} \sim \dfrac{1}{x^y}$.