Introduction to Moderm Harmonic Analysis
(Fourier Theory applied to groups)

Syllabus:

Bibliography:

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2. G. Folland. A course in abstract harmonic analysis. CRC press 2016.

3. W. Rudin, Fourier Analysis in Groups. Interscience publishers, 1967.

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1. ABELIAN FINITE GROUPS

Definition: A group $\langle G, \cdot \rangle$ is a set G close under the binary operation "$\cdot : G \times G \longrightarrow G$" such that for all $a, b, c \in G$

1. $\cdot$ is associative operation
2. There is $e \in G$ st $e \cdot a = a \cdot e = a$ (Identity)
3. For all $a \in G$ there is $a' \in G$ s.t $a \cdot a' = a' \cdot a = e$

If the operation is conmutative, the group $\langle G, \cdot \rangle$ is an abelian group.

Ex:

1. $\langle U, \cdot \rangle$ with $U = \{ z \in \mathbb{C} \mid |z| = 1 \}$
2. $\langle U_n, \cdot \rangle$ with $U_n = \{ z \in \mathbb{C} \mid z^n = 1 \}$
3. $\langle \mathbb{Z}_n, + \rangle$ with $\mathbb{Z}_n = \{ b \mid a \equiv b \mod (n) \Leftrightarrow (a-b) | n \}$

Let $H \subseteq G$ be a subset of G, close under the binary operation of G, then $\langle H, \cdot_G \rangle$ is called the sub-group of G and it is denoted by $H \leq G$.

Ex:

1. $\langle \mathbb{Z}, + \rangle \leq \langle \mathbb{R}, + \rangle$
2. $\langle \mathbb{Q}^+, + \rangle \nleq \langle \mathbb{R}, + \rangle$, but $\mathbb{Q}^+ \subseteq \mathbb{R}$.

Theorem: Let G be a group, and $a \in G$. The subgroup $H \leq G$ defined by $H := \{ a^n | n \in \mathbb{Z} \}$ is the smalest subgroup of G containing a.

Proof: To prove that $H \leq G$, H has to be closed under $\cdot_G$, i.e, if $e \in G \rightarrow e \in H$ and for all $a \in H$, $a^{-1} \in H$. By properties of powers and as $n \in \langle \mathbb{Z}, + \rangle$, H is a subgroup of G.

Definition: Let $\langle G, \circ \rangle$ be a set and $a \in G$. Let $\{a^n \mid n \in \mathbb{Z}\} \leq G$, it is called cyclic subgroup of G, and it is generated by a, and denotes by $\langle a \rangle$. And $\langle G, \circ \rangle$ is a cyclic group if there is an element $a \in G$ s.t $\langle a \rangle = G$

Ex:

1. $\mathbb{Z}_4$ is a cyclic group because $\langle 1 \rangle = \langle 3 \rangle = \mathbb{Z}_4$
2. $\langle \mathbb{Z}, + \rangle = \langle 1 \rangle = \langle -1 \rangle$

Definition: Let A be a FINITE ABELIAN group. The group A is a cyclic group if there is $\tau \in A$ s.t $\langle \tau \rangle = A$, and $n = |A|$.

Fundamental theorem of Abelian Finite groups: Every finite abelian group can be written by a product of cyclic groups.

Definition: Let A be a finite abelian group (FAG). A character $\chi : A \longrightarrow \mathbb{T}$ of A is an homomorphism s.t $\chi(ab) = \chi(a)\chi(b)$, and $\mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}$. And the set of all characters of A is denoted by $\hat{A}$.

Lemma: $\langle \hat{A}, \cdot \rangle$ is a group with a binary operation

$$\cdot : \begin{array}{l} \hat{A} \times \hat{A} \longrightarrow \hat{A} \\ (\chi, \eta) \longmapsto \chi\eta \end{array} \qquad \text{and} \quad \chi\eta(a) = \chi(a)\eta(a)$$

This group is called the dual group of the Pontryaging dual group of A.

Proof: Let $\chi, \eta \in \hat{A}$ and $a, b \in A$, then
$$\begin{aligned} \chi\eta(ab) &= \chi(ab)\eta(ab) \\ &= \chi(a)\eta(a)\,\chi(b)\eta(b) \\ &= \chi\eta(a)\chi\eta(b) \end{aligned}$$

And in the same way $\chi^{-1} \in \hat{A}$ where $\chi^{-1}(a) = \chi(a)^{-1}$. Then $\hat{A}$ is a subgroup of all maps from A to $\mathbb{T}$.

Lemma: Let A be a cyclic group of orden n. Fix the generator of A, $\langle \tau \rangle = A$. The characters of the cyclic group are given by

$$\eta_\ell(\tau^k) = e^{2\pi i k\ell/n}, \quad \text{with } k \in \mathbb{Z} \text{ for } \ell = 0, \dots, n-1.$$

And the group $\hat{A}$ is a ciclic group of order n.

Proof: Let η be a character of A, i.e, $\eta(\tau) = t \in \mathbb{T}$ s.t $t^n = \eta(\tau^n) = 1$. (root of unity). Then there is $\ell = 0, \dots, n-1$ s.t $\eta_\ell(\tau) = e^{\frac{i2\pi\ell}{n}}$. Now, for all $k \in \mathbb{Z}$, $\eta_\ell(\tau^k) = e^{\frac{i2\pi\ell k}{n}}$.

Theorem: Let A be an abelian finite group. There is a canonic isomorphism from $\hat{A}$ to $\hat{\hat{A}}$ (the bidual group), gave by $a \mapsto \delta_a$ where $\delta_a : \hat{A} \longrightarrow \mathbb{T}$ s.t $\chi \mapsto \delta_a(\chi) = \chi(a)$

Proof:

① ¿Is $A \to \hat{\hat{A}}$ an homomorphism? Let $a, b \in A$, $\delta_{ab}(\chi) = \chi(ab) = \chi(a)\chi(b) = \delta_a(\chi)\delta_b(\chi)$.

② ¿Is $A \to \hat{\hat{A}}$ an isomorphism?

Lemma: Let A be a finite abelian group, and be $a \in A$. Presume that $\chi(a) = 1$ for all $\chi \in \hat{A}$, then $a = 1$.

Proof: As A is a fag, we can take A_i, A_j as cyclic groups of A then if it is true for $A_i \times A_j$ is true for A. So, let $(a_0, b_0) \in A_i \times A_j$ with $\chi(a_0, b_0) = 1$. for all $\chi \in \hat{A}$. The map $\chi(a,b) = \chi(a)$ is a character of $A_i \times A_j$ (in the same way $\chi(a,b) = \chi(b)$ is a character of $A_i \times A_j$), then $\chi(a_0, b_0) = \chi(a_0) = 1 \longrightarrow a_0 = 1$.

Hence, by the lemma $A \longrightarrow \hat{\hat{A}}$ is an injective map. Now we to proof $|A| = |\hat{A}|$. Hw: Prove this equality.

Now, let A be a finite abelian group. The Hilbert Space $\ell^2(A)$ is the space of all maps from A to $\mathbb{C}$ ($f: A \longrightarrow \mathbb{C}$). In partiular $\eta: A \longrightarrow \mathbb{T} \subseteq \mathbb{C}$ is an element of $\ell^2(A)$.

Let S be an arbitrary set, then $\ell^2(S) = \{ f: S \to \mathbb{C} \mid \|f\|^2 = \sum_{s \in S} |f(s)|^2 < \infty \}$.

Forthermore, $\ell^2(S)$ with the product $\langle f,g \rangle = \sum f(s)\overline{g(s)}$ where $f,g \in \ell^2(S)$ is a Hilbert Space.

Lemma: Sea $\eta, \chi \in \hat{A}$. Entonces $\langle \chi, \eta \rangle = \begin{cases} |A| & \text{si } \chi = \eta \\ 0 & \text{si } \chi \neq \eta \end{cases}$

Proof: We will start withe special case when $\chi = \eta$.

Therefore $\langle \chi, \eta \rangle = \sum_{a \in A} \chi(a)\overline{\eta(a)} = \sum_{a \in A} \chi(a)\overline{\chi(a)} = \sum_{a \in A} |\chi(a)|^2 = \sum_{a \in A} 1 = |A|$

Now, we will assume that $\chi \neq \eta$, then $\alpha = \chi\eta^{-1} \neq 1$

$\langle \chi, \eta \rangle = \sum_{a \in A} \chi(a)\overline{\eta(a)} = \sum_{a \in A} \chi(a)\eta^{-1}(a) = \sum_{a \in A} \alpha(a)$. And let $b \in A$ be an element such that $\alpha(b) \neq 1$, so $\langle \chi, \eta \rangle \alpha(b) = \sum_{a \in A} \alpha(a)\alpha(b) = \sum_{a \in A} \alpha(ab)$.

Now, we can replace "a" by "ab^{-1}" and get $\sum_{ab^{-1} \in A} \alpha(ab^{-1}b) = \sum_{ab^{-1} \in A} \alpha(a) = \langle \chi, \eta \rangle$.

Forthermore $\langle \chi, \eta \rangle - \langle \chi, \eta \rangle \alpha(b) = (\alpha(b) - 1)\langle \chi, \eta \rangle = 0$. Finally $\alpha(b) - 1 \neq 0$, so $\langle \chi, \eta \rangle = 0$.

Definition (Fourier Transformation): Let A be a finite abelian group, and $f \in \ell^2(A)$. The fourier transformation of f is map $\hat{f}: \hat{A} \to \mathbb{C}$ such that

$$\boxed{\hat{f}(\chi) = \frac{1}{\sqrt{|A|}} \langle f, \chi \rangle = \frac{1}{\sqrt{|A|}} \sum_{a \in A} f(a)\overline{\chi(a)}}$$

Note: In the case of $\mathbb{R}$, e^{ixy} needs a normalization factor and $e^{i2\pi xy}$ doesn't needs a normalization factor, and it is an element of the dual space.

Theorem: Let $f \longmapsto \hat{f}$ be a map. It is an isomorphism of Hilbert spaces $\ell^2(A) \longrightarrow \ell^2(\hat{A})$, and also $\ell^2(\hat{A}) \longrightarrow \ell^2(\hat{\hat{A}})$ is an isomorphism where $\hat{\hat{f}}(\delta_a) = f(a^{-1})$.

Proof: Let $f, g \in \ell^2(A)$, then

$$\langle \hat{f}, \hat{g} \rangle = \sum_{\chi \in \hat{A}} \hat{f}(\chi)\overline{\hat{g}(\chi)} = \sum_{\chi \in \hat{A}} \frac{1}{\sqrt{|A|}} \langle f, \chi \rangle \frac{1}{\sqrt{|A|}} \overline{\langle g, \chi \rangle}$$

$$= \frac{1}{|A|} \sum_{\chi \in \hat{A}} \sum_{a \in A} \sum_{b \in A} f(a)\overline{\chi(a)}\,\overline{g(b)}\chi(b)$$

$$= \frac{1}{|A|} \sum_{a,b \in A} f(a)\overline{g(b)} \sum_{\chi \in \hat{A}} \overline{\delta_a(\chi)}\delta_b(\chi)$$

$$= \frac{1}{|A|} \sum_{a,b \in A} f(a)\overline{g(b)} \langle \delta_a, \delta_b \rangle = \langle f, g \rangle \text{ when } a = b.$$

Now,
$$\hat{\hat{f}}(\delta_a) = \frac{1}{\sqrt{A}} \sum_{\chi \in \hat{A}} \hat{f}(\chi)\overline{\delta_a(\chi)} = \frac{1}{|A|} \sum_{\chi \in \hat{A}} \left(\sum_{b \in A} f(b)\overline{\chi(b)} \right) \overline{\delta_a(\chi)}$$

$$= \frac{1}{|A|} \sum_{\chi \in \hat{A}} \sum_{b \in A} f(b)\overline{\chi(b)}\,\overline{\chi(a)},$$

and changing $b \leftrightarrow b^{-1}$

$$\hat{\hat{f}}(\delta_a) = \frac{1}{|A|} \sum_{\chi \in \hat{A}} \sum_{b \in A} f(b^{-1})\chi(b)\overline{\chi(a)}$$

$$= \frac{1}{|A|} \sum_{b \in A} f(b^{-1}) \langle \delta_b, \delta_a \rangle = f(a^{-1})$$

Definition: Let A be a finite abelian group, and let $f, g \in \ell^2(A)$. Then the convolution product is defined by

$$\boxed{(f * g)(a) = \frac{1}{\sqrt{|A|}} \sum_{b \in A} f(b)g(b^{-1}a)}$$

with $a, b \in A$.

Theorem: Let $f, g \in \ell^2(A)$. Then $\widehat{(f * g)} = \hat{f}\hat{g}$.

Proof: $$\widehat{(f * g)}(\chi) = \frac{1}{\sqrt{|A|}} \sum_{b \in A} (f * g)(b)\overline{\chi(b)}$$

$$= \frac{1}{|A|} \sum_{b \in A} \sum_{a \in A} f(a) g(a^{-1}b) \overline{\chi(b)}$$

Changing $b \leftrightarrow ab$:

$$(\widehat{f * g})(\chi) = \frac{1}{|A|} \sum_{b \in A} \sum_{a \in A} f(a) g(a^{-1}ab) \overline{\chi(ab)}$$

$$= \frac{1}{|A|} \sum_{b \in A} \sum_{a \in A} f(a) g(b) \overline{\chi(a)} \, \overline{\chi(b)} = \hat{f}(\chi) \hat{g}(\chi)$$

2. LOCAL COMPACT ABELIAN GROUPS:

Definition: An Abelian metrizable group is an abelian group A with a metric, thus with the induced topology by the metric such that

* The product

$$A \times A \longrightarrow A \quad \text{s.t} \quad (x,y) \mapsto xy$$

* The investion

$$A \longrightarrow A \quad \text{s.t} \quad x \mapsto x^{-1}$$

are continuous maps, i.e, let $\{x_n\} \to x$ and $\{y_n\} \to y$ convergent sequences then $\{x_n\}\{y_n\} \longrightarrow xy$ and $\{x_n^{-1}\} \longrightarrow x^{-1}$.

Ex :

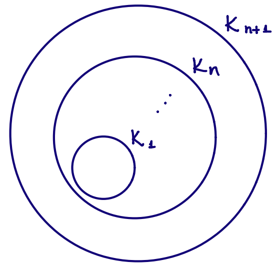
1. Each set with discrete metric is metrizable. Sea X be a set and

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

2. Let $\mathbb{R}^{\times} := \mathbb{R} / \{0\}$. The set $(\mathbb{R}, +)$ and $(\mathbb{R}^{\times}, \cdot)$ with the usual topology in $\mathbb{R}$ are metrizable groups. So if $\{x_n\} \to x$ and $\{y_n\} \to y$ then $\{x_n\} + \{y_n\} \longrightarrow x+y$ and $\{-x_n\} \longrightarrow -x$.

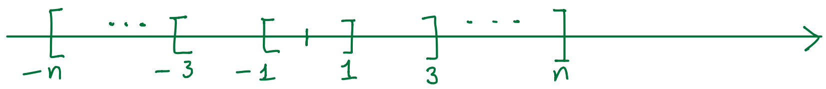
Definition: Let X be a metrizable space. It is called σ-compact if there is a sequence $K_n \subset K_{n+1}$ of Compact subsets, such that $X = \bigcup_n K_n$

A metric space is compact if for all $\{x_n\}$ in X, it has a convergent subsequence.



Ex:

1. $\mathbb{R}$ is σ-compact, so $\mathbb{R} = \bigcup_n [-n, n]$ wit $n \in \mathbb{N}$ and



Definition: A space X is locally compact if for each $x \in X$ there is a compact neighborhood, i.e, for each metric in X and $x \in X$ there is $r > 0$ s.t $\overline{B_r(x)} = \{ y \in X \mid d(x,y) \leq r \}$ is compact.

" A abelian σ-compact group and locally compact is a Local Compact Abelian Group (LCA) "

Ex:

1. Let G be a numerable abelian group with a discrete metric.
2. $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$

Lemma: A LCA group has a dense numerable subset

Proof: It is a result from σ-compactness. Let $A = \bigcup_{n \in \mathbb{N}} K_n$ and we choose a metric for A, then $K_1 \subseteq \bigcup_{n=1}^{m} (B_1)_n$. Let $a_1, \dots, a_{r_1}$ be centers of each ball $(B_1)_n$, and $K_2 \subseteq \bigcup_{n=1}^{m} (B_{1/2})_n$ with $a_{r_1+1}, \dots, a_{r_2}$ be centers. In the same way $K_j \subseteq \bigcup_{n=1}^{m} (B_{1/j})_n$ with $a_{r_{j-1}}, \dots, a_{r_j}$ be their centers. So the sequence is dense in A.

Remark: A LCA group A. satifies:

1. Metrizable
2. σ-compact
3. Locally compact

And a character of a LCA group is a continuous homomorphism $\chi : A \longrightarrow \mathbb{T}$ and it is denoted by $\hat{A}$.

Proposition: Los caracteres de los siguientes groups are:

1. $(\mathbb{Z}, +)$, $k \longmapsto e^{2\pi i k x}$ with $x \in \mathbb{R}/\mathbb{Z}$
2. $(\mathbb{R}/\mathbb{Z}, \cdot)$, $x \longmapsto e^{2\pi i k x}$ with $k \in \mathbb{Z}$
3. $(\mathbb{R}, \cdot)$, $x \longmapsto e^{2\pi x y}$ with $y \in \mathbb{R}$

Proof:

1. Let $\varphi : \mathbb{Z} \longrightarrow \mathbb{T}$ be a character such that $\varphi(1) = e^{2\pi i x}$ with $x \in \mathbb{R}/\mathbb{Z}$ and then $k \in \mathbb{Z}$, $\varphi(k) = (\varphi(1))^k = e^{2\pi i x k}$.

3. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{T}$ be a character. By continuity of φ there is $\varepsilon > 0$ such that $\varphi([-\varepsilon, \varepsilon]) \subseteq \{Re(z) > 0\}$. Let y be an element of $\left[\frac{-1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$ such that $\varphi(\varepsilon) = e^{2\pi i \varepsilon y}$.

Now, we set $\varphi\left(\frac{\varepsilon}{2}\right) = e^{\pi i x y}$. To prove it, let's notice that

$$\varphi\left(\frac{\varepsilon}{2}\right)^2 = e^{(2\pi i \frac{\varepsilon}{2} y)2} = \varphi(\varepsilon) = e^{2\pi i \varepsilon y}$$

$$\varphi\left(\frac{\varepsilon}{2}\right)^2 = \varphi(\varepsilon) = e^{2\pi i \varepsilon y}$$

So $\varphi\left(\frac{\varepsilon}{2}\right) = \pm \varphi(\varepsilon) = \pm e^{2\pi i \varepsilon y}$.

Iterating the last argument $\varphi\left(\frac{\varepsilon}{2^n}\right) = e^{2\pi i \frac{\varepsilon}{2^n} y}$, and for each $k \in \mathbb{Z}$ $\varphi\left(k \frac{\varepsilon}{2^n}\right) = e^{2\pi i k \frac{\varepsilon}{2^n} y}$. And, because φ is continuous it is follow for all real number, by

Lemma Let X and Y be metrizable spaces and let f, g be continuous functions of X and Y. If f and g be closed in a dense set $D \subseteq X$ then $f = g$.

3. It is inmediatlely because the characters in $\mathbb{R}/\mathbb{Z}$ are the $\mathbb{R}$ characters

but send $\mathbb{Z}$ to 1.

Remark: If A is a LCA group then I can define finite integrals

Ex: $\mathbb{R}$.

Objetive: $\hat{A}$ is a LCA, i.e, let $K_n \subseteq K_{n+1}$ be a sequence of compact sets then $\hat{A} = \bigcup_n K_n$.

Taking a compact cover $A = \bigcup_{n\in\mathbb{N}} K_n$ para $\eta, \chi \in \hat{A}$ y $n \in \mathbb{N}$. Let $\hat{d}_n(\chi,\eta) = \sup_{x\in K_n} |\chi(x) - \eta(x)|$ be the metric of a compact set K_n, now

$\hat{d}(\chi,\eta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{d}_n(\chi,\eta)$ be the metric of all $\hat{A}$.

Lemma: $\hat{d}(\chi,\eta)$ is a metric in $\hat{A}$.

Proof:

1. if $\hat{d}(\chi,\chi) = 0$ because $|\chi(x) - \chi(x)| = 0$

2. Let $\chi, \eta, \alpha \in \hat{A}$.

$$\hat{d}_n(\chi,\eta) = \sup_{n\in\mathbb{N}} |\chi(x) - \eta(x)| = \sup_{n\in\mathbb{N}} |\chi(x) + \alpha(x) - \alpha(x) - \eta(x)|$$

$$\leq \sup_{n\in\mathbb{N}} |\chi(x) - \alpha(x)| + \sup_{n\in\mathbb{N}} |\alpha(x) - \eta(x)| = \hat{d}_n(\chi,\alpha) + \hat{d}_n(\alpha,\eta).$$

$$\text{Now, } \hat{d}(\chi,\eta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{d}_n(\chi,\eta) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} [\hat{d}_n(\chi,\alpha) + \hat{d}_n(\alpha,\eta)]$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{d}_n(\chi,\alpha) + \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{d}_n(\alpha,\eta)$$

$$= \hat{d}(\chi,\alpha) + \hat{d}(\alpha,\eta)$$

Theorem: With the metric $\hat{d}$ the goup $\hat{A}$ is an abelian group. Let $\{\chi_n\}_{n\geq 0}$ be a sequence, it converges with respect to the metric if and only if it converges locally and uniformlly. With the topology induced by this metric $\hat{A}$ becames in a LCA.

Proof : To prove $\Rightarrow$ The group operations are continuous.

Let $\{\chi_n\}_{n\geq 0}$ and $\{\eta_n\}_{n\geq 0}$ be two convergent sequences in $\hat{A}$ to χ and η, respectivectly. Then for every $n \in \mathbb{N}$

$$\hat{d_n}(\chi_n\eta_n, \chi\eta) = \sup_{n\in\mathbb{N}} |\chi_n\eta_n(x) - \chi\eta(x)|$$

$$= \sup_{n\in\mathbb{N}} |\chi_n(x)\eta_n(x) - \chi(x)\eta(x) + \chi(x)\eta_n(x) - \chi(x)\eta_n(x)|$$

$$\leq \sup_{n\in\mathbb{N}} |\chi_n(x) - \chi(x)||\eta_n(x)| + \sup_{n\in\mathbb{N}} |\eta_n(x) - \eta(x)||\chi(x)|$$

$$= \hat{d_n}(\chi_n - \chi)\, \cancel{|\eta_n(x)|}^{1} + \hat{d_n}(\eta_n - \eta)\, \cancel{|\chi(x)|}^{1},$$

and multiplying by $\frac{1}{2^n}$ we conclude that

$$\hat{d}(\chi_n\eta_n, \chi\eta) \leq \sum_{n=1}^{\infty} \frac{1}{2^n}[\hat{d_n}(\chi_n - \chi) + \hat{d_n}(\eta_n - \eta)] \longrightarrow 0$$

Thus, the product is a continuous operation. And in the same way the inverse is continuous.

Proposition (Pontryaging Duality):

1. If A is a compact group then $\hat{A}$ is a discrete group
2. If A is a discrete group then $\hat{A}$ is a compact group.

Proof :

1. Let A be a compact group, the we can choose a compact cover $K_1 = K_2 = \dots = A$ and the metric $d(\chi, \eta) = \sup_{x\in A} |\chi(x) - \eta(x)|$.

To prove $\Rightarrow$ $\hat{A}$ is a discrete group, i.e, if $d(\chi, \eta) \leq \sqrt{2}$ it means that $\alpha(A) = \{ z \in \mathbb{C} \mid Re(z) \geq 0 \}$

Now, since $\alpha(A) < \mathbb{T}$ then $\alpha(A) = \{1\}$, so $\alpha = 1$ and $\eta = \chi$.

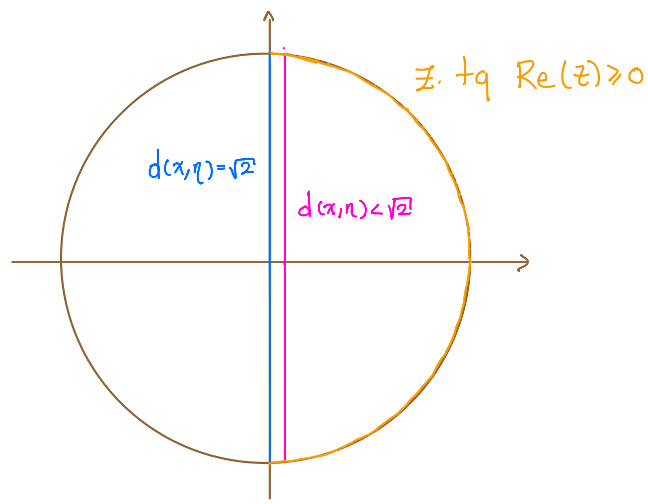
Nota :

$$|z-1|^2 = (z-1)(\bar{z}-1) \leq 2$$

then

$$2 - 2Re(z) \leq 2 \rightarrow Re(z) \geq 0$$

2. Let A be a discrete group and A be σ-compact, then A is númerable. Now, let $(a_k)_{k\in\mathbb{N}}$ be a numeration of A. Let $\{X_j\}_{j\geq n}$ be a sequence in $\hat{A}$. It is easy to prove that there exist a uniform locally convergent subsequence $\{X_{j_i}\}_{i\geq 0}$ of $\{X_j\}_{j\geq 0}$, thus $\hat{A}$ is compact.



Ex:

1. Let $G = GL_n(\mathbb{R})$. Since $GL_n(\mathbb{R}) \subseteq Mat_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ we can consider the usual topology of $\mathbb{R}^{n^2}$, then G is LC and its operations are continuous

3. Groups Integrals (Haar's Measure)

Let f be a not negative continuous function with compact support in $\mathbb{R}$, i.e, $supp(f) = \{x \in X \mid f(x) \neq 0\}$.

The Riemann integral of f is given by: Let $n\in\mathbb{N}$ and let $\mathbb{1}_n$ be the characteristic function in the interval

$$I = \left[\frac{-1}{2n}, \frac{1}{2n}\right], \text{ i.e, } \mathbb{1}_n(x) = \begin{cases} 1 & \text{si } x \in I \\ 0 & \text{si } x \notin I \end{cases},$$

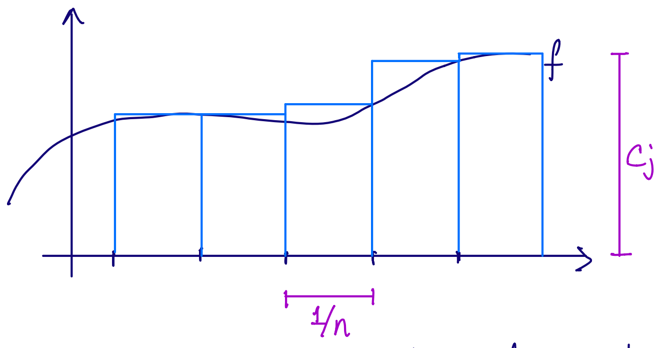
then there is $x_1, \dots, x_n \in \mathbb{R}$ and $c_1, \dots, c_m > 0$ such that $f(x) \leq \sum_{j=1}^{m} c_j \mathbb{1}_n(x, x_j)$, it means that f is dominated by the sum.

Let

$$(f : \mathbb{1}_n) := \inf\left\{ \sum_{j=1}^{M} c_j \;\middle|\; c_1, \dots, c_m > 0 \;\&\; \exists\, x_1, \dots, x_m \in \mathbb{R} \text{ tq } f(x) \leq \sum_{j=1}^{m} c_j \mathbb{1}_n(x - x_j) \right\},$$

thus $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \left(\frac{(f:\mathbb{1}_n)}{n} \right)$, where $\{(f:\mathbb{1}_n)\}$ are the heights, and

n is the base given by $\left\{\frac{1}{n}\right\}$ for each rectangle.



$(f : \mathbb{1}_n)$ represents the minimum of compact set which cover f.

In a set $\mathcal{G}$, the interval can be replaced $\left[\frac{-1}{2n}, \frac{1}{2n}\right]$ by an arbitrary set arround the identity, but ¿ What will the meaning of $\frac{1}{n}$? :

let f be a characteristic function in the inteval $[0,1]$ and $(f_n : \mathbb{1}_n) = n$. Then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \frac{(f : \mathbb{1}_n)}{(f_0 : \mathbb{1}_n)}$$

, where $(f_0 : \mathbb{1}_n)$ is the minimum number of compact sets which cover f_0. And whit this idea we can define for any group the next:

$$\boxed{\int_{-\infty}^{\infty} f(x)dx = \lim_{\{U\}\to\{e\}} \frac{(f : \mathbb{1}_U)}{(f_0 : \mathbb{1}_U)}.}$$

This integral is known as The Haar's Measure.

Now, let $C_c(\mathcal{G}) = \{ f : \mathcal{G} \to \mathbb{C}$ con soporte compacto $\}$ be a complex vectorial space.

Note : $*$ $\overline{C_c(\mathcal{G})} = C^0(\mathcal{G})$

$*$ If g is a continuous function, then $\lim_{n\to\infty} f_n = g$ with $f_n \in C_c(\mathcal{G})$

Definition :

1. Let V be a complex vector space. The lineal map $L : V \to \mathbb{C}$, where $\mathbb{C}$ is a scalar field, is called V-linear functional o V-linear form.

2. We can say that $f \in C_c(\mathcal{G})$ is not negative if $f(x) \geq 0$ for all $x \in \mathcal{G}$, and we can write $f \geq 0$.

Ex:

1. A $C_c(\varphi)$ linear functional is the integral if $f \geq 0$ implies $I(f) \geq 0$

2. Let $x \in \varphi$ and $\int_{-\infty}^{\infty} \delta_x(f)\,dx = f(x)$ with $f \in C_c(\varphi)$. Then δ_x is called the Dirac's Delta.

Note: If it is easy which integral will be used, you will write $I(f) = \int_{\varphi} f(x)\,dx$. Also, if f, g be real function in $C_c(\varphi)$ and $f - g \geq 0$ then $I(f) \geq I(g)$.

Lemma: For every integral in φ, it is followed that

$$\left| \int f(x)\,dx \right| \leq \int |f(x)|\,dx.$$

Proof: Let $f_{\pm} = \max(\pm f, 0)$ be a function in $C_c(\varphi)$, and $f_{\pm} \geq 0$. Then $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Therefore

$$\left| \int_{\varphi} f(x)\,dx \right| = \left| \int_{\varphi} f_+(x)\,dx - \int_{\varphi} f_-(x)\,dx \right|$$

$$\leq \left| \int_{\varphi} f_+(x)\,dx \right| + \left| \int_{\varphi} f_-(x)\,dx \right|$$

$$= \int_{\varphi} f_+(x)\,dx + \int_{\varphi} f_-(x)\,dx = \int_{\varphi} |f(x)|\,dx$$

Definition: Let $s \in \varphi$ and $f \in C_c(\varphi)$. For $x \in \varphi(x)$ we can define the left transformation given by $L_s f(x) = f(s^{-1}x)$ for $s \in \varphi$, so $(L_s f)(x) \in C_c(\varphi)$. Also, it follows $(L_s(L_t f))(x) = (L_{st} f)(x)$ with $s, t \in \varphi$, and $(L_e f)(x) = f(e^{-1}x) = f(x)$

$$(L_s(L_t f))(x) = (L_t f)(s^{-1}x) = f(t^{-1}s^{-1}x) = f((st)^{-1}x) = (L_{st} f)(x).$$

Definition: Let $I : C_c(G) \longrightarrow \mathbb{C}$, be a map, is called Left invariant or invariant if $I(L_s f) = I(f)$ for all $f \in C_c(G)$ and $s \in G$. And, with the integral notation $\int_G f(yx)\,dx = \int_G f(x)\,dx$.

Ex:

1. Let $\mathbb{R}$ be the real numbers group, so

$$I : C_c(\mathbb{R}) \longrightarrow \mathbb{C}$$
$$f \longmapsto I(f) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(a+x)\,dx.$$

Haar's Theorem: Let G be a group. If I and I' are invariant integrals in the group then there is $c>0$ such that $I = cI'$.

Every invariant integral is called Haar's integral.

Corollary: For every invariant integral I and for all function $g \in C_c(G)$ with $g \geq 0$ and $I(g) = 0$ then $g = 0$.

Proof: Let $g \in C_c(G)$ s.t $g > 0$. To prove $\rightarrow$ $I(g) \neq 0$.

Let $f \in C_c(G)$ s.t $f \geq 0$ and $I(f) \neq 0$. Since $g \neq 0$ there exist $c_1, \dots, c_n > 0$ and $x_1, \dots, x_n \in G$ s.t $f \leq \sum_{i=1}^{n} c_i L_{x_i} g$. Then

$0 < I(f) \leq \sum_{i=1}^{n} c_i I(L_{x_i} g) = \sum_{i=1}^{n} c_i I(g)$, so $0 < I(g)$.

Lemma: The map $C_c(G)$ is a pre-Hilbert space with inner product $\langle f, g \rangle = \int_G f(x)\overline{g(x)}\,dx$.

Proof: To prove $\rightarrow$
1. $\mathbb{C}$ - lineal
2. $\langle u, w \rangle = \overline{\langle w, u \rangle}$ Conjugate Symmetry.
3. Positive defined.

1. By it is a linear functional.

2. $\overline{\langle g, f \rangle} = \overline{\int_G g(x)\overline{f(x)}\,dx} = \int_G f(x)\overline{g(x)}\,dx = \langle f, g \rangle$

3. Let $\langle f, f\rangle = 0$ then $\int_{\mathcal{G}} f(x)\overline{f(x)}\,dx = \int_{\mathcal{G}} |f(x)|^2 dx = 0$, so $|f|^2 \in C_c(\mathcal{G})$ and by the last corollary $|f|^2 = 0$. thus $f = 0$.

Note: The $C_c(\mathcal{G})$ completation is a Hilbert Space, and H is called

$$L^2(\mathcal{G}) := \left\{ f \in X \text{ (measure space)} \;\middle|\; \int_X |f|^2 d\mu \right\}.$$

This completation does not depend on Haar's measure.

Ex:

1. In $\mathbb{R}$, $I(f) = \int_{-\infty}^{\infty} f(x)\,dx$

2. In $\mathbb{R}/\mathbb{Z}$, $I(f) = \int_0^1 f(x)\,dx$

3. In $\mathbb{R}_+^{\times}$, $I(f) = \int_0^{\infty} f(x)\,\frac{dx}{x}$

4. In $GL_n(\mathbb{R})$, $I(f) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \frac{d a_{11} \cdots d a_{nn}}{|\det(a)|^n}$

Remark: Let $\mathcal{G}$ and H be LC, metrizable and σ-compact groups. The cartesian product $\mathcal{G} \times H$ is a group of the same type, then it has a Haar's measure.

Theorem (Fubini) Let $I_{\mathcal{G}}(g) = \int_{\mathcal{G}} g(x)\,dx$ be a Haar's Integral in $\mathcal{G}$. If $f \in C_c(\mathcal{G}\times H)$, then $y \longmapsto I(f(\cdot, y)) = \int_{\mathcal{G}} f(x,y)\,dx \in C_c(H)$. And, let $I_H(h) = \int_H h(y)\,dy$ be a Haar's Integral in $\mathcal{G}\times H$, it is given by

$$I(f) = \int_H \int_{\mathcal{G}} f(x,y)\,dx\,dy = \int_{\mathcal{G}} \int_H f(x,y)\,dy\,dx.$$

Definition: Let A be a LCA group, with the integral $\int_A dx$. Let $\hat{A}$ be the dual group of A.

For $f \in L^1_{bc}(A) := \left\{ f \text{ continuous \& bounded} \;\middle|\; \|f\|_1 = \int_{\mathcal{G}} |f(x)|\,dx < \infty \right\}$,

$$\boxed{\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\, dx \quad \text{with } \chi \in \hat{A}.}$$

Ex:

1. If $x \in \mathbb{R}$, φ_x be the character of x $\left(\varphi_x(\varphi) = e^{2\pi i x y}\right)$ and $f \in L^1_{bc}(\mathbb{R})$ then $\hat{f}(\varphi_x) = \int_{-\infty}^{\infty} f(y)\overline{\varphi_x(y)}\, dy = \int_{-\infty}^{\infty} f(y) e^{-2\pi i x y}\, dy = \hat{f}(x)$.

Theorem: Let $f, g \in L^1_{bc}(A)$. The there is the integral $f * g = \int_A f(xy^{-1}) g(y)\, dy$ for al $x \in A$, then $f * g \in L^1_{bc}(A)$ and $(\widehat{f * g})(\chi) = \hat{f}(\chi)\, \hat{g}(\chi)$ for all $\chi \in \hat{A}$.

Proof: We can suppose $|f(x)| \le c$ for all $x \in A$, then

$$f * g = \int_C |f(xy^{-1}) g(y)| dy \le c \int_C |g(y)|\, dy = c \|g\|_1,$$ because there is the integral $f * g$ and it is bounded.

Now, to prove that $f * g$ is continuous, so let $x_0 \in A$ and $|f(x)|, |g(x)| \le c$ for all $x \in A$ and suppose $y \ne 0$. For a $\varepsilon > 0$ there is a function $\varphi \in C_c^+(A)$ s.t $\varphi \le |g|$ and $\int_A |g(y)| - \varphi(y)\, dy < \frac{\varepsilon}{4C}$.

In a compact the function f is continuous uniform function, i.e., there is V_e s.t $x \in V_e(x_0)$ and $y \in \operatorname{Supp}(\varphi)$ implies that $|f(xy^{-1}) - x_0(y^{-1})| < \frac{\varepsilon}{2\|g\|_1}$

so if $x \in V_e(x_0)$ $\int_A |f(xy^{-1}) - f(x_0 y^{-1})| \varphi(y)\, dy \le \frac{e}{2\|g\|_1} \int_A \varphi(y)\, dy \le \frac{\varepsilon}{2}$.

In the othe hand $\int_A |f(xy^{-1}) - f(x_0 y^{-1})| \left(|g(y)| - \varphi(y)\right) dy \le$
$2c \int_A \left(|g(y)| - \varphi(y)\right) dy$.

Therefore $x \in x_0 V_e$, $|f * g(x) - f * g(x_0)| = \left| \int_A \left(f(xy^{-1}) - f(x_0 y^{-1})\right) g(y)\, dy \right|$

$$\le \int_A \left|f(xy^{-1}) - f(x_0 y^{-1})\right| |g(y)|\, dy = \int_A |f(xy^{-1}) - f(x_0 y^{-1})| \left(|g(y)| - \varphi(y) + \varphi(y)\right) dy$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

And $\|f*g\|_1 = \int_A |f*g(x)|\,dx = \int_A \left| \int_A f(xy^{-1})g(y)\,dy \right| dx$

$$\leq \int_A \int_A |f(xy^{-1})||g(y)|\,dx\,dy$$

$$= \int_A |f(x)|\,dx \int_A |g(y)|\,dy = \|f\|_1 \|g\|_1$$

Then, the Fourier Transformation

$$\widehat{(f*g)}(x) = \int_A (f*g)(x)\overline{\chi(x)}\,dx = \int_A \int_A f(xy^{-1})g(y)\overline{\chi(x)}\,dy\,dx$$

$$= \int_A \int_A f(x)g(y)\overline{\chi(xy)}\,dx\,dy$$

We change $x \longmapsto xy$

$$= \int_A f(x)\overline{\chi(x)}\,dx \int_A g(y)\overline{\chi(y)}\,dy$$

$$= \hat{f}(x)\,\hat{g}(x)$$

Lemma: Let A be a compact abelian group. If we fixed the Haar's measure $\int_A 1\,dx = 1$, then each two characters $\chi, \eta \in \hat{A}$

$$\int \chi(x)\overline{\eta(x)}\,dx = \begin{cases} 1 & \text{si } \chi = \eta \\ 0 & \text{si } \chi \neq \eta \end{cases}$$

Proof:

* If $\chi = \eta$ then $\chi(x)\overline{\eta(x)} = 1$.
* If $\chi \neq \eta$, then $\alpha = \chi\overline{\eta} = \chi\eta^{-1} \neq 1$, i.e., there is $a \in A$ s.t $\alpha(a) \neq 1$. Then

$$\alpha(a)\int_A \alpha(x)\,dx = \int_A \alpha(ax)\,dx = \int_A \alpha(x)\,dx$$

Thus

$$\alpha(a)\int_A \alpha(x)\,dx = \int_A \alpha(x)\,dx$$

$$[\alpha(a) - 1]\int_A \alpha(x)\,dx = 0 \longrightarrow \int_A \alpha(x)\,dx = 0$$

Theorem: Let A be LCA group, then there is a unique Haar's measure in $\hat{A}$

such that for all $f \in L^1_{loc}(A)$, $\|f\|_2 = \|\hat{f}\|_2$, i.e, for each $f \in L^1_{loc}(A)$ the Fourier transformation $\hat{f} \in L^1_{loc}(A)$ then there is an homomorphism $\underbrace{L^2(A)}_{\text{Funciones cuadrado integrables}} \longrightarrow L_2(\hat{A})$.

Note: This theorem shows how the harmonic analysis generalizes the Fourier theory

Ex: Let $\mathbb{R}/\mathbb{Z}$ and $f \in L^1_{loc}(\mathbb{R}/\mathbb{Z})$, then

$$\int_0^1 |f(x)|^2 dx = \|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{k\in\mathbb{Z}} |\hat{f}(t)|^2 = \sum_{k\in\mathbb{Z}} |\underbrace{c_k(f)}_{\text{Fourier Serie coefficients.}}|^2$$

$$\hat{f}(t) = \int_{\mathbb{R}/\mathbb{Z}} f(y) e^{-2\pi i x y} dy$$

"It is given by the Riemann- Lebesgue Theorem"

Proof (From Folan):

* Suppose that A is a discrete group $\int_A f(x)\,dx = \sum_{a\in A} f(a)$. For $\chi \in \hat{A}$, $\hat{f}(\chi) = \sum_{a\in A} f(a)\overline{\chi(a)}$ and we choose in $\hat{A}$ the normal Haar's Integral $\int_A 1\, da = 1$.

We need a Lemma to continue with the proof:

Lemma: For each $g \in L^1_{loc}(A)$, the Fourier Transformation $\hat{g} \in C(\hat{A}) = L^1_{loc}(\hat{A})$ and we have for each $a \in A$, $\hat{\hat{g}}(\delta_a) = g(a^{-1})$

Lemma proof:

$$\hat{\hat{g}}(\delta_a) = \int_{\hat{A}} \hat{g}(\chi)\overline{\delta_a(\chi)}\, d\chi = \int_{\hat{A}} \sum_{b\in A} g(b)\, \overline{\delta_b(\chi)}\; \overline{\delta_a(\chi)}\, dx$$

$$(b \leftrightarrow b^{-1}) \qquad = \int_{\hat{A}} \sum_{b\in A} g(b^{-1})\, \delta_b(\chi)\overline{\delta_a(\chi)}\, d\chi$$

$$= \sum_{b\in A} g(b^{-1}) \int_{\hat{A}} \delta_b(\chi)\overline{\delta_a(\chi)}\, d\chi \underset{\text{when } a=b.}{=} g(a^{-1})$$

Go back with the theorem proof:

Let $f \in L^1_{loc}(A)$ and $\tilde{f}(\chi) = \overline{f(\chi^{-1})}$. Let $g = \tilde{f} * f$ and $g(x) = \int_A \overline{f(yx^{-1})} f(y)\,dy$, so $g(e) = \|f\|_2^2$ with $e \in A$ (identity). By the Fourier Transformation Theorem of convolution $\hat{g}(\chi) = \hat{\tilde{f}}(\chi)\hat{f}(\chi) = \overline{\hat{f}(\chi)}\hat{f}(\chi) = |\hat{f}(\chi)|^2$. Therefore

$$\|f\|_2^2 = g(e) = \hat{\hat{g}}(\delta_e) = \int_{\hat{A}} \hat{g}(\chi)\overline{\chi(e)}\,d\chi = \int_{\hat{A}} |\hat{f}(\chi)|^2 d\chi = \|\hat{f}\|_2^2.$$

4. Non-Commutative Groups (Matrix Group):

- General lineal Group and $\mathcal{U}(n)$:

Let $n \in \mathbb{N}$ and $Mat_n(\mathbb{C})$ be the vector space of complex $n \times n$ matrices. Define the norm $\|A\|_1 = \sum_{i,j=1} |a_{ij}|$ with $A = (a_{ij})$, and it gives rise to a metric $d_1(A,B) = \|A - B\|_1$.

On the other side $Mat_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. And taking a natural inner product given by the euclidean norm $\|A\|_2 := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$ with its respectively metric

$$d_2(A,B) = \|A - B\|_2.$$

Lemma: A sequence of matrices $A^{(k)} = (a_{ij}^{(k)})$ converges if and only if for each pair of indexes ij, $\{(a_{ij})^p\}_{p=0}^{k} \longrightarrow b_{ij} \in \mathbb{C}$. In the same way for d_2. Thus the metrics d_1 and d_2 are equivalents.

Proof:

Suppose that $A^{(k)} = (a_{ij}^{(k)}) \longrightarrow A = (a_{ij}) \in Mat_n(\mathbb{C})$ by d_1, it means that for every $\varepsilon > 0$ there is $K_0 \in \mathbb{N}$ s.t $K_0 \leq k$ for all k, then $\|A^{(k)} - A\|_j < \varepsilon$.

Now, let $i_0, j_0 \in \{1, \dots, n\}$ then $k \geq K_0$, $|a_{ij}^{(k)} - a_{ij}| < \frac{\varepsilon}{n^2}$. And let's choose $K_0 \in \mathbb{N}$ s.t $K_0 = \max\{K_0(i,j)\}$ then $k \geq K_0$ and

$$\|A^{(k)} - A\|_1 = \sum_{ij} |a_{ij}^{(k)} - a_{ij}| < \sum_{ij} \frac{\varepsilon}{n^2} = \varepsilon$$

thus $A^{(k)} \longrightarrow K$.