Proposition: With the topology given by the last metric the group $GL_n(\mathbb{C})$ is a locally compact group, i.e, it is metrizable, σ-compact and locally compact group.

Proof:

1. The product between two elements and the inverse of any element are given by entry rational functions, so since polynomials are continuous functions then $GL_n(\mathbb{C})$ is a topological group.

Since $GL_n(\mathbb{C})$ is an open subset of a locally compact group ($Mat_n(\mathbb{C})$) then it is locally compact.

$GL_n(\mathbb{C})$ is an open set $\leftrightarrow$ $\det(A \in GL_n(\mathbb{C})) \neq 0$

$\leftrightarrow$ $\det(A) > 0$ or $\det(A) < 0$

$\det(A)$ = polynomial $:= f$ (continuous $\leftrightarrow$ $f^{-1}(I)$ abierto &

$I = (\det(A) < 0) \cup (\det(A) > 0)$

Furthermore, $Mat_n(\mathbb{C})$ is σ-compact then

$$K_n = \{ a \in GL_n(\mathbb{C}) : \|a_1\| \leq n_1, \ \|a^{-1}\|_1 \leq n \}.$$

Definition: Let $A \in Mat_n(\mathbb{C})$. The matrix $A^* = \overline{A^T} = (\overline{a_{ji}})$ is called the adjoin matrix. And $\mathcal{U}(n) := \{ g \in Mat_n(\mathbb{C}) \mid g^* g = 1 \}$

Lemma: $\mathcal{U}(n)$ is a compact subgroup of $GL_n(\mathbb{C})$

Proof: We have that $\mathcal{U}(n) \leq GL_n(\mathbb{C})$. <u>To prove</u> $\rightarrow$ $\mathcal{U}(n)$ is compact, i.e, it is closed and bounded in the set $Mat_n(\mathbb{C})$.

Let $\{g_i\}_{i=0}^{\infty}$ be a sequence in $\mathcal{U}(n)$ s.t $\{g_i\} \rightarrow g \in Mat_n(\mathbb{C})$, then for each $i \in \mathbb{N}$ $\lim_{i\to\infty} g_i^* g_i = 1$ and $\lim_{i\to\infty} g_i^* g_i = \left(\lim_{i\to\infty} g_i^*\right)\left(\lim_{i\to\infty} g_i\right)$

$$= g^* \cdot g$$

thus $g^* \cdot g = 1$, and $\mathcal{U}(n)$ is a closed subset of $\mathrm{Mat}_n(\mathbb{C})$.

Now, $\mathcal{U}(n)$ is bounded set because for each $a \in \mathrm{Mat}_n(\mathbb{C})$

$$\mathrm{Tr}(a^* a) = \sum_{i,j=1}^{n} (a^* a)_{ij} = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj}^* a_{jk} = \sum_{k=1}^{n} \sum_{j=1}^{n} \overline{a_{jk}}\, a_{jk}$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{jk}|^2 = \| a \|_2^2$$

Therefore $\| a \|_2 = \sqrt{\mathrm{Tr}(a^* a)} = \sqrt{n}$, then $\mathcal{U}(n)$ is bounded.

Definition: Let G be a topological group (metrizable). Let $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ be a Hilbert Space. Let $GL(\mathcal{V}) := \{ M \in \mathrm{Mat}_n(\mathbb{C}) \mid \det(M) \neq 0 \}$. A representation $\eta : G \longrightarrow GL(\mathcal{V})$ is a group homomorphism such that the binary operation $G \times \mathcal{V} \rightarrow \mathcal{V}$ s.t $(x, v) \rightarrow \eta(x) v$ is a continuous operation. And a unitary representation is a representation such that for all $x \in G$ the operator $\eta(x)$ is $\mathcal{V}$-unitary, i.e, $\langle \eta(x) v, \eta(x) w \rangle = \langle v, w \rangle$ for all $v, w \in \mathcal{V}$.

Definition: A closed subspace $W \subseteq \mathcal{V}$ is called invariant for η if $\eta(x) w \subseteq W$ for all $x \in G$ and $w \in W$. The representation η of G is called Irreducible representation if there is not own invariant closed subspaces, i.e, the unique invariant subspaces are $\emptyset$ and $\mathcal{V}$.

Ex:

1. The identity map $\rho : \mathcal{U}(n) \longrightarrow GL(\mathbb{C}^n) = GL_n(\mathbb{C})$.

Lemma: The identity map is irreducible.

Proof: By definition $\mathcal{U}(n)$ are the linear operators which acts on $\mathbb{C}^n$, s.t they are unitary with respect to inner product $\langle u, w \rangle = u^T w^T$. Let $V \subseteq \mathbb{C}^n$ be a non-own subspace, i.e, $V \neq \emptyset$ and $V \neq \mathcal{U}(n)$. And Let $W = V^T$

be the orthogonal complement, i.e, $W = \{ w \in \mathbb{C}^n \mid \langle w, v \rangle = 0 \ \forall v \in V \}$

Then $\mathbb{C}^n = V \oplus W$.

Furthermore, Let $\{e_1, \dots, e_\ell\}$ be a orthonormal base of V, and $\{e_{\ell+1}, \dots, e_n\}$ be a base of W. Then the operator T, gave by $T(e_1) = e_{\ell+1}$, $T(e_{\ell+1}) = e_1$ and $T(e_j) = e_j$ where $j \neq 1, \ell+1$, is unitary, so $T \in \mathcal{U}(n)$ but it does not let V stable, i.e, T is unitary iff for each orthonormal base $\{e_i\}$, $T(\{e_i\})$ is orthonormal. Thus by last argument ρ is a irreducible representation.

Remark: Let $\sum_{r=0}^{\infty} A_r$ be a serie in $\mathrm{Mat}_n(\mathbb{C})$. It converges if the sequence $\{ S_k = \sum_{r=0}^{k} A_r \}$ converges.

Proposition: For each $A \in \mathrm{Mat}_n(\mathbb{C})$ the serie $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges and it defines an element of $GL_n(\mathbb{C})$. If $A, B \in \mathrm{Mat}_n(\mathbb{C})$ and $AB = BA$, then $e^A e^B = e^{A+B}$. In partiaular $e^{-A} = e^{A^{-1}}$.

Proof: We remember that $\|A\|_1 = \sum_{i,j}^{n} |a_{ij}|$ si $A = (a_{ij})$. To prove the proposition we need two lemmas:

Lemma 1: Let $A, B \in \mathrm{Mat}_n(\mathbb{C})$, they follow that $\|AB\|_1 \leq \|A\|_1 \|B\|_1$. In particular for $j \in \mathbb{N}$ $\|A^j\|_1 \leq \|A\|_1^j$.

Proof (Lemma 1): Let $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathrm{Mat}_n(\mathbb{C})$. Then

$$\|AB\|_1 = \sum_{i,j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \leq \sum_{i,j,k=1}^{n} |a_{ik} b_{kj}| \leq \sum_{i,j,k=1}^{n} |a_{ik}||b_{kj}| = \|A\|_1 \|B\|_1.$$

Lemma 2: Let $(A_\ell)_{\ell \geq 0}$ be a sequence of $\mathrm{Mat}_n(\mathbb{C})$. We suppose that $\sum_{\ell=0}^{\infty} \|A_\ell\| < \infty$, then the serie $\sum_{\ell=0}^{\infty} A_\ell$ converges. (Condition of Banach Spaces)

Proof (Lemma 2): Let $B_k = \sum_{\ell=0}^{k} A_\ell$ in $\mathrm{Mat}_n(\mathbb{C})$. To prove $\rightarrow$ The sequence $(B_k)_{k \geq 0}$ converges, i.e., it is a Cauchy sequence.

The sequence $b_k = \sum_{\ell=0}^{k} \|A_\ell\|_1$ converges in $\mathbb{R}$, for that reazon b_k is a Chauchy sequence. So that for a $\varepsilon > 0$ there is k_0 s.t for $m \geq k \geq k_0$ it follows that

$$\varepsilon > |b_m - b_k| = \sum_{p=0}^{m} \|A_p\|_1 - \sum_{\ell=0}^{k} \|A_\ell\|_1$$

$$= \sum_{p=0}^{m} \sum_{\ell=0}^{k} \|A_p\|_1 - \|A_\ell\|_1$$

??

$$= \sum_{r=k+1}^{m} \|A_r\|_1 \geq \left\| \sum_{r=k+1}^{m} A_r \right\|_1 = \|B_m - B_k\|_1.$$

Then $(B_k)_{k\geq 0}$ is a Cauchy sequence in $Mat_n(\mathbb{C})$ thus it converges

HW 3: Prove that for each $\{M_\ell\}$ in $Mat_n(\mathbb{C})$ which is a Cauchy sequence then it converges with respet to the norms $\|\cdot\|_1$ or $\|\cdot\|_2$.

Continuing with the preposition proof we have to prove that $\sum_{r=0}^{\infty} \frac{\|A^r\|_1}{r!} < \infty$.

Thus $\sum_{r=0}^{\infty} \frac{\|A^r\|_1}{r!} \leq \sum_{r=0}^{\infty} \frac{\|A\|_1^r}{r!} < \infty$ because the exponetial in $\mathbb{R}$ converges.

Let $A, B \in Mat_n(\mathbb{C})$ and $AB = BA$, so

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} A^r B^{k-r} = \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{\cancel{k!}\, A^r B^{k-r}}{\cancel{k!}\, r!(k-r)!}$$

$$= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{A^r B^{k-r}}{r!(k-r)!} = \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} \frac{A^k B^u}{k!\, u!} = \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{r=0}^{\infty} \frac{B^u}{u!}$$

¿Criterio de Sumación de Cauchy ?

$$= e^A e^B \qquad \boxed{u = k - r.}$$

Proposition: For each $A \in Mat_n(\mathbb{C})$ the $\det(e^A) = e^{tr(A)}$

Proof: Let $S \in GL_n(\mathbb{C})$ then $\det(e^{SAS^{-1}}) = \det(S e^A S) = \det(e^A)$, and $e^{tr(SAS^{-1})} = e^{tr(A)}$, so they are invariants by conjugation. By Jordan theorem all nxn matrix can conjugate with a triangle matrix.

Suppose $A = \begin{bmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$. For $r \geq 0$ $A^r = \begin{bmatrix} a_{11}^r & & * \\ & \ddots & \\ 0 & & a_{nn}^r \end{bmatrix}$ then $e^A = \begin{bmatrix} e^{a_{11}} & & * \\ & \ddots & \\ 0 & & e^{a_{nn}} \end{bmatrix}$

so $\det(e^A) = e^{a_{11}} e^{a_{22}} \cdots e^{a_{nn}} = e^{a_{11}+a_{22}+\cdots+a_{nn}} = e^{tr(A)}$.

Definition: Let $G \subseteq GL_n(\mathbb{C})$ be a closed subset. The Lie Algebra of G is defined as $Lie(G) = \{ X \in Mat_n(\mathbb{C}) : e^{tX} \in G \ \forall t \in \mathbb{R} \}$.

Ex:

1. Special Linear Group:

$SL_n(\mathbb{C}) = \{ M \in Mat_n(\mathbb{C}) \mid \det(M) = 1 \}$ and its Lie Algebra is

$Lie(SL_n(\mathbb{C})) = \mathfrak{sl}_n(\mathbb{C}) = \{ M \in Mat_n(\mathbb{C}) \mid tr(M) = 0 \}$

2. Lie Algebra of the unitary group:

$Lie(U(n)) = \mathfrak{u}(n) = \{ M \in Mat_n(\mathbb{C}) \mid M^* = -M \}$ with $M^* = \overline{M^t}$.

Proposition: Let G be a closed subgroup of $GL_n(\mathbb{C})$. Then $Lie(G)$ is a real vector subspace of $Mat_n(\mathbb{C})$. If $X, Y \in Lie(G)$ then $[X,Y] = XY - YX$, it is called Lie parentesis of X and Y. Let $\pi : G \longrightarrow GL(V)$ be a representation of finite dimension, then for X in $Lie(G)$ the map, $t \longmapsto \pi(e^{tX})$ for $t \in \mathbb{R}$, is infinitely diffentiable. Let $\pi(X) = \frac{d}{dt}\Big|_{t=0} \pi(e^{tX}) \in End(V)$. The mapping $X \longmapsto \pi(X)$ satisfies

$\pi([X,Y]) = [\pi(X), \pi(Y)]$.

Proof: In Do Carmo or Helgason of Differential Geometry.

Definition: A subgroup $G < GL_n(\mathbb{C})$ is called path connected if all points $x, y \in G$ can be joined with a curve, i.e., there is $\gamma : [0,1] \to G$ s.t $\gamma(0) = x$ and $\gamma(1) = y$.

Ex:

1 $\mathbb{R}^\times = GL_1(\mathbb{R})$ it is not path connected.

Definition: A representation of the algebra $\pi : Lie(G) \longrightarrow End(V)$ is called $*$-reprepresentation if $X \in Lie(G)$ and $\pi(X)^* = \pi(-X)$

- **Representations of $SU(2)$:**

Definition: The set $SU(2)$ is define as:

$$SU(2) = \{ A \in Mat_2(\mathbb{C}) : A^*A = 1 \ \& \ \det(A) = 1 \}$$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ \middle| \ a, b \in \mathbb{C} \ \& \ |a|^2 + |b|^2 = 1 \right\}$$

Lemma: Let K be a compact and metrizable group, and Let ρ be a representation over a finite dimension Hilbert space $(V, \langle \cdot, \cdot \rangle)$, $\rho : K \longrightarrow GL_n(V)$. Then, there is an element $S \in GL_n(V)$ s.t the representation $S\rho S^{-1}$ is an unitary representation.

Proof: Suppose that we can prove that there is other inner product $(\cdot, \cdot)$ in V s.t $\rho : K \longrightarrow GL(V)$ is an unitary representation w.r.t a $(\cdot, \cdot)$.

Remember: All inner product in V takes the form $(v, w) = \langle Sv, Sw \rangle$ for any $S \in GL(V)$.

Then $S\rho S^{-1}$ is unitary, that is, let $x \in K$ s.t

$$(\rho(x)v, \rho(x)w) = (v, w)$$

$$\langle S\rho(x)v, S\rho(x)w \rangle = \langle Sv, Sw \rangle$$

considering $v = S^{-1}v'$, $w = S^{-1}w'$

$$\langle S\rho(x)S^{-1}v', S\rho(x)S^{-1}w' \rangle = \langle SS^{-1}v', SS^{-1}w' \rangle$$

$$\langle Mv', Mw' \rangle = \langle v', w' \rangle \quad \text{taking } M = S\rho(x)S^{-1}.$$

The only thing that we have to do is to prove that the product exists. For $v, w \in V$, $(v, w) = \int_K \langle \rho(k^{-1})v, \rho(k^{-1})w \rangle \, dk$, then:

$$(\rho(k_0)v, \rho(k_0)w) = \int_K \langle \rho(k^{-1})\rho(k_0)v, \rho(k^{-1})\rho(k_0)w \rangle \, dk$$

$$= \int_K \langle \rho(k^{-1})v, \rho(k^{-1})w \rangle \, dk$$

Because it is a Haar's measure.

$$= (v, w)$$

The Lie Algebra of $SU(2)$ is

$$su(2) := \{ A \in Mat_2(\mathbb{C}) : A^* = -A \ \& \ tr(A) = 0 \}$$

If we fixed a $su(2)$ base

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } X_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which satisfies

- $[X_1, X_2] = X_3$

$$= X_1X_2 - X_2X_1 = \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$= \frac{1}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] = \frac{1}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = X_3.$$

- $[X_2, X_3] = X_4$
- $[X_3, X_1] = X_2$

Let $\pi : su(2) \longrightarrow End(V)$ (n-dimension space) and Let Lie Algebra $*$-representation. Let $L_j = \pi(x_j) \in End(V)$ with $j = 1, 2, 3$ s.t

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1 \text{ and } [L_3, L_1] = L_2.$$

Furthermore, $L_j^* = \pi(x_j)^* = \pi(x_j^*) = \pi(-x_j) = -L_j$, so each L_j is anti-hermitic matrix, then L_j is a diagonalizable matrix.

Note: Let $A : V \longrightarrow V$ s.t $AA^* = A^*A$ be normal, then A is a diagonalizable matrix

Then for each $m \in \mathbb{C}$ $V_m = \{ v \in V \mid L_1 v = imv \}$. By the spectral theorem

$V = \bigoplus_{im \in Spect(L_1)} V_m$, where $spect(L_1)$ is the spectrum of L_1 or eigenvalues set.

Let $L_+ = L_2 - iL_3$ and $L_- = L_2 + iL_3$ be operators s.t $[L_1, L_\pm] = \pm i L_\pm$

Proposition: The operator $L_\pm : V_m \longrightarrow V_{m\pm 1}$. In particular, if $m \in \operatorname{Spec}(L_i)$, then $L_+ = 0$ in V_m or $i(m+i) \in \operatorname{Spec}(L_1)$

Proof: Let $v \in V_m$ then
$$\begin{aligned} L_1(L_+ v) &= L_+ L_1 v + i L v \\ &= i(\mu+1) L_+ v \end{aligned}$$

then $L_+ V_m \subseteq V_{m+1}$. In the similar way, $L_- V_m \subseteq V_{m-1}$.

Let $C = L_1^2 + L_2^2 + L_3^3$ s.t $CL_j = L_j C$ for $j = 1, 2, 3$.

Schur Lemma: If $\pi : \mathfrak{su}(2) \longrightarrow \operatorname{End}(V)$ is an irreducible representation, there is $\lambda \in \mathbb{C}$ s.t $C = \lambda \operatorname{Id}$.

Proof: Let λ be a C-eigenvalue, since $L_j C = C L_j$ then the λ-eigenspace V_λ is invariant, so $V_\lambda = V$.

Proposition: let $\pi : \mathfrak{su}(2) \longrightarrow \operatorname{End}(V)$ be unreducible representation, then the Lie spectrum is a sequence $\{ i\mu_0, i(\mu_0+1), \dots, i(\mu_0+K) = i\mu_1 \}$ with $L_+ = V_{\mu_0+j} \longrightarrow V_{\mu_0}$ be an isomorphism for $0 \leq j \leq K-1$ and $L_- : V_{\mu_1 - j} \rightarrow V_{\mu_1-(j+1)}$ be an isomorphism for $0 \leq j \leq K-1$.

$$\begin{tikzcd} \cdot\mu_1 \arrow[r, shift left, "L_+"] & \cdot\mu_{0+1} \arrow[l, shift left, "L_-"] \arrow[r, shift left, "L_+"] & \cdot\mu_{0+2} \arrow[l, shift left, "L_-"] & \cdots & \cdot\mu_{1-1} \arrow[r, shift left, "L_+"] & \cdot\mu_1 \arrow[l, shift left, "L_-"] \end{tikzcd}$$

The spaces V_{μ_0+j} are unidimensional for $j = 0, 1, \dots, K$ and $\dim(V) = K+1$ and $\mu_0 = -\frac{K}{2}$, $\mu_1 = \frac{K}{2}$.

Proof: For a lemma before we can assume that the representation is unitary, then
$$\begin{aligned} L_- L_+ &= (L_2 + iL_3)(L_2 - iL_3) = L_2^2 + L_3^2 + i[L_3, L_2] = C - L_1^2 - iL_1 \\ &= \lambda - L_1^2 - iL_1 \end{aligned}$$

and $L_+L_- = C - L_1^2 + iL_1 = \lambda - L_1^2 + iL_1$. In V_μ $L_+L_- = \lambda + \mu(\mu-1)$ and $L_-L_+ = \lambda + \mu(\mu+1)$. Since L_2 and L_3 be skew-symmetry $L_+^* = (L_2 - iL_3)^* = -L_-$ then L_+L_- and L_-L_+ are Hermitic. By the lemma $\operatorname{Ker}(L_-) = \operatorname{Ker}(L_+L_-)$ and $\operatorname{Ker}(L_+) = \operatorname{Ker}(L_-L_+)$.

Lemma: Let V be a finite dimensional Hilbert Space and A be a linear operator. Then $\operatorname{Ker} A = \operatorname{Ker} A^*A$.

Proof: Let $v \in V$, then for

$$v \in \operatorname{Ker}(A) \Longleftrightarrow Av = 0 \Longleftrightarrow \langle Av, Aw \rangle = 0 \quad \forall\, w \in V$$

$$\Longleftrightarrow \langle A^*Av, w \rangle = 0 \quad \forall\, w \in V$$

$$\Longleftrightarrow A^*Av = 0 \longrightarrow v \in \operatorname{Ker}(A^*A)$$

Continuing with the proposition proof:

Let $\mu_0, \mu_0+1, \ldots, \mu_0+K \equiv \mu_1$ be a maximum length sequence with $V_{\mu_0+j} \neq 0$ and $j = 0, \ldots, K$. Subsequently, $L_+ V_{\mu_0+K} = 0$ and

$0 = \lambda + \mu_1(\mu_1+1) = \lambda + \mu_0(\mu_0 - 1)$ or $\mu_0(\mu_0-1) = -\lambda = \mu_1(\mu_1+1)$. Then

$-\mu_0 = \mu_0(2K+1) + K(K+1)$ and $\mu_0 = -K/2$.

The space $V = V_{\mu_0} \oplus V_{\mu_0+1} \oplus \ldots \oplus V_{\mu_0+K}$ is preserved under L_1, L_2, L_3, i.e, bie $\mathcal{L}ie(SU(2))$, so V is an invariant space. Since π is a unreducible representation all V_{μ_0+j} are one-dimensional spaces.

- <u>Peter–Weyl's Theorem:</u>

Theorem (PW): Let K be a compact group. The K-unreducible representations form an orthogonal base of $L^2(K)$.

Lemma: Let (π, V_π) be a unitary representation with finite dimension of a Locally compact group G. Then $\pi = \eta_1 \oplus \ldots \oplus \eta_n$ where η_i

Remember that :

* If M is a unitary representation then its matrix is unitary
* Every locally compact group has a unitary representation.

Proof :

By induction we will prove the V_π dimension:

If V_π is 1-dimensional then π is unreducible. and the Theorem is proved. Now, we will suppose that it is true for all spaces K s.t $\dim(K) < \dim(V_\pi)$ then :

1. V_π is unreducible then the theorem is proved.
2. V_π is not unreducible then it has an invariant representation W, let $W^\perp = \{ v \in V_\pi \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}$ so $W^\perp$ is invariant and $V_\pi = W \oplus W^\perp$ with $\dim(W) \leq \dim(V_\pi)$ and $\dim(W^\perp) \leq \dim(V_\pi)$, thus W and $W^\perp$ can be browken down into unreducible representations and V_π can be browken down into unreducible representation too.

Let K be a matrix compact group $(K \leq GL_n(\mathbb{C}))$. Let τ and γ be unreducible K-representations of finite dimension. Let $H = \{ T : V_\gamma \longrightarrow V_\tau \text{ linear map} \}$ and we define a new K-representation η, s.t $\eta(k)T = \tau(k) T \gamma(k^{-1})$. And, let

$\mathrm{Hom}_K(V_\gamma, V_\tau) = \{ T : V_\gamma \longrightarrow V_\tau \text{ linear map} \mid T\gamma(k) = \tau(k) T \ \forall k \in K \}$

They are called Intertwiner.

Lemma : The $\dim(\mathrm{Hom}_K(V_\gamma, V_\tau)) \leq 1$.

Definition : Two representations η and τ are isomorphs if there is $T \in \mathrm{Hom}_K(V_\gamma, V_\tau)$.

Definition: Let (τ, V_τ). And let $\{e_i\}_{i=1}^{n}$ be a base of V, and let $\tau_{ij}(K) = \langle \tau(K)e_i, e_j \rangle$. The map $\tau_{ij} : K \to \mathbb{C}$ is called the (i,j)-esimo coeficient of the matrix τ.

Theorem: Let $\tau \neq \gamma$ be two non-isomorphic representations then $\int_K \tau_{ij}(K)\overline{\gamma_{rs}(K)}\, dk = 0 \ \forall i,j,r,s$. Also $\int_K \tau_{ij}(K)\overline{\tau_{rs}(K)} = 0$ when $i \neq r$ and $j \neq s$. But, if $i=r$ and $j=s$ $\int_K \tau_{ij}(K)\overline{\tau_{ij}(K)}\, dk = \frac{1}{\dim(V_\tau)}$

The last case implies that the family $\{\sqrt{\dim(V_\tau)}\,\tau_{ij}\}$ forms a base of $L^2(K)$.

Proof: We assume $\tau \neq \gamma$. Then for each $T \in \mathrm{Hom}_K(V_\tau, V_\gamma)$ we have that $S = \int_K \tau(K)\,T\,\gamma(K^{-1})\, dK$ satisfies $\tau(K)S = S\gamma(K)$ and $S=0$. Let $e_1, \dots, e_n$ be a base of V_γ, and $f_1, \dots, f_m$ be a base of V_τ.

Let $T \in \mathrm{Hom}_K(V_\tau, V_\gamma)$ given by E_{ij} with $(E_{k\ell}) = 1$ and $(E_{ij}) = 0$ when $i \neq k$ and $j = \ell$. Then

$$\tau(K)\,T\,\gamma(K) = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1m} \\ \vdots & \ddots & \vdots \\ \tau_{m1} & \cdots & \tau_{mm} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \overline{\gamma_{11}} & \cdots & \overline{\gamma_{1n}} \\ \vdots & \ddots & \vdots \\ \overline{\gamma_{n1}} & \cdots & \overline{\gamma_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \tau_{1i}\overline{\gamma_{ij}} & \cdots & \tau_{1i}\overline{\gamma_{nj}} \\ \vdots & \ddots & \vdots \\ \tau_{ni}\overline{\gamma_{i}} & \cdots & \tau_{mi}\overline{\gamma_{nj}} \end{bmatrix} = 0_{m \times n}$$

Now, for $\gamma = \tau$ we have $\int_K \tau(K)\,T\,\gamma(k^{-1})\, dK = \lambda\, Id$ for each $\lambda \in \mathbb{C}$. Let $T := id$ and remenbering τ is a unitary representation $\tau(K)\tau^*(K) = id$ then $\sum_r \tau_{ir}\overline{\tau_{jr}(K)} = \delta_{ij}$ so $\sum_r \int_K |\tau_{ir}(K)|^2\, dk = 1$. For with, when $T = E_{ij}$ it is followed $\int_K |\tau_{ri}(K)|^2\, dk = \int_K |\tau_{r'i}(K)|^2\, dK$ for r and r'.

Finally $\int_K |\tau_{ri}(k)|^2 dk = \int_K |\tau_{ri}(k^{-1})|^2 dk = \int_K |\tau_{ir}(k)|^2 dk$

Hw: $\int_K f(k)\, dk = \int_K f(k^{-1})\, dk$

Thus $\int_K |\tau_{ir}(k)|^2 dk = \int_K |\tau_{ir'}(k)|^2 dk$ ..

Remark: Stone-Weierstrass Theorem. Let X be a compact metrizable space. In $C(X)$ a norm defined by $\|f\|_\infty = \sup_{x \in X} |f(x)|$ and a metric defined by $d(f,g) = \|f-g\|_\infty$ form an Algebra.

Let A be a subalgebra of $C(X)$ s.t:

1. A is a closed set under the complex $(f \in A, \bar{f} \in A)$
2. A separates the points, i.e., $\forall\ x, y \in X$, $x \neq y$ $\exists\ f \in A$ s.t $f(x) \neq f(y)$.

Then A is dense in $C(X)$

Continuing with the theorem prove. Let A be the map generating by $\tau_{ij} \in C(K)$. A is closed under the conjugation and $\bar{\tau}$ is a K-representation.

Now, A separates points because K is a matrix group and it has an injective unitary representation. By the Stone-Wei-erstrass theorem A is dense in $C(K)$ but $C(K)$ is dense in $L^2(K)$, so A is dense in $L^2(K)$.

5. Representations Theory :

- Group of characters : The character, related to a matrix representation, is the function χ s.t $\chi(a) = \mathrm{Tr}\, D(a) = \sum_{i=1}^{n} D(a)_{ii}$.

Let $\chi'(a)$ be a character of $D'(a) = S D(a) S^{-1}$ then

$\chi'(a) = \mathrm{Tr}\left[S D(a) S^{-1}\right] = \mathrm{Tr}\left[S S^{-1} D(a)\right] = \mathrm{Tr}\, D(a) = \chi(a)$.

Proposition : The characters satisfy the orthogonal property:

If $\chi^{\ell}(a)$ of the representation $D^{\ell}(a)$ $\int \overline{\chi^{i}(g)}\,\chi^{j}(g)\,dg = \Omega\delta_{ij}$ where $\Omega = \mathrm{Vol}\ \varphi = \int_{\varphi} dg$.

Proof of equality :

$$\int \overline{\chi^{i}(g)}\,\chi^{j}(g)\,dg = \sum_{k,\ell} \int \overline{D^{i}(g)_{kk}}\, D^{j}(g)_{\ell\ell}\, dg$$

$$= \sum_{k,\ell} \int \overline{D^{i}(g)}_{kk}\, D^{j}(g)_{\ell\ell}\, dg$$

$$= \sum_{k} \frac{\Omega}{n_i}\,\delta_{ij} = \Omega\,\delta_{ij}$$

By the Peter-Weyl's theorem
$\Downarrow$

$\int D^{\ell}(g)_{im}\,\overline{D^{k}(g)_{jm}}\,dg = \frac{\Omega}{n_\ell}\,\delta_{ij}\,\delta_{mn}\,\delta_{\ell k}$
donde $n_\ell = \dim(D^{\ell}(a))$

- Direct Representations Product :

Definition Let $D^1(a)$ and $D^2(a)$ be two distinct representations with dimensions n and m respectively. The direct product is defined as the space L of all tensors A with componets a_{ij} with $1 \le i \le m$ and $1 \le j \le n$.

Ex :

1. Let $A\alpha + B\beta$ be a tensor linear combination, where A and B are tensors. It is defined as a component tensor $\alpha a_{ij} + \beta b_{ij}$ where a_{ij} are the A components and b_{ij} are the B components.

Then the representation product is the set of transformation $D(a)$

defined as : $[D(a)A]_{ij} = \sum_{k=1}^{m} \sum_{l=1}^{n} D^1(a)_{ik} D^2(a)_{jl} a_{kl}$

Proposition : $D(a)$ be a representation.

Proof :

$$[D(a)D(b)A]_{ij} = \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{n} D^1(a)_{ik} D^2(a)_{jl} \cdot D^1(b)_{sk} D^2(b)_{lt} a_{st}$$

$$= \sum_{s=1}^{m} \sum_{t=1}^{n} D^1(ab)_{is} D^2(ab)_{jt} a_{st} = [D(ab)A]_{ij}$$

Note : The notation in some books is $D(a)_{ij,kl} = D^1(a)_{ik} D^2(a)_{jl}$ with $i,j = 1,\dots,m$ and $k,l = 1,\dots,n$. Or $D(a) = D^1(a) \times D^2(a)$. And

$$\chi(a) = \sum_{i,j} D(a)_{ijij} = \sum_{i} \sum_{j} D^1(a)_{ii} D^2(a)_{jj} = \chi^1(a)\chi^2(a)$$

- Function set representation :

Remark : If the group G can be taken as a transformation set on the space S, it is possible to construct representations in the function space with domine S.

Definition : Let

$$\begin{aligned} G &\longrightarrow S \\ x &\longmapsto ax \end{aligned}$$

and let

$$H = C(S) = \{ f : f \text{ is continuous in } S \}.$$

Then $D(a)$ can be defined in H as : Let $f \in H$, $(D(a)f)(x) = f(a^{-1}x)$

Moreover, it is continuous and linear representation.

$$D(a)[\alpha f + \beta g](x) = \alpha f(a^{-1}x) + \beta g(a^{-1}x)$$

$$= \alpha D(a) f + \beta D(a) g$$

The representation will be denoted $D(b)f = f_b$. Then $D(a)f(x) = f_b(a^{-1}x)$.

In the other hand $f_b(y) = f(b^{-1}y)$, so $y = a^{-1}x$. Thus

$$f_b(y) = f_b(a^{-1}x) = f(b^{-1}a^{-1}x) = f((ab)^{-1}x) = (D(ab)f)(x), \text{ then}$$

$D(ab) = D(a)D(b)$.

Let $H_\nu := \{ f : f$ is an homogeneous function of degree ν in $S\}$, then $f(cx) = c^\nu f(x)$ with $H_\nu \subseteq H$.

The generalization in more variables $[D(a)f](x,y) = f(a^{-1}x, b^{-1}y)$

Definition : Let $h(x,y) = c_{ij}\, e^i(x) f^i(y)$ with $c_{ij} \in \mathbb{C}$ be an equivalence representation of $D^1 \times D^2$.

Let $\{e^i(x), f^i(y)\}$ be a base of H then

$$e^m(a^{-1}x)f^n(a^{-1}y) = \sum_i D^1(a)_{im}\, e^i(x) \sum_j D^2(a)_{jn} f^j(y)$$

- Plane Rotation : Let $R(\theta)$ be a rotation in the plane by an angle θ with $0 \leq \theta \leq 2\pi$ defined by

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x\cos(\theta) - y\sin(\theta),\, x\sin(\theta) + y\cos(\theta))$$

Where $D(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

And R_θ satisfies the next properties :

1. $R_\theta R_\phi = R_{\theta+\phi}$ with $0 \leq \theta + \phi \leq 2\pi$
2. $R_\theta R_\phi = R_{\theta+\phi-2\pi}$ with $2\pi \leq \phi + \theta$

The representation $D(\theta)$ is not an irreducible representation because the group is an abelian group, and

"Every irreducible representation of an abelian group has a dimension equal to 1"

Diagonalazing $D(\theta)$ we have : $SDS^{-1} = D^1(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ with

$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, where both $e^{i\theta}$ and $e^{-i\theta}$ are invariant subspaces and irreducible representations of $U(1)$.

Now, by the Peter-Weyl's theorem and by other theorem to prove it we have:

$$\boxed{\int_0^{2\pi} e^{in\theta} e^{im\theta} d\theta = 2\pi \delta_{nm} \text{ . If } f \in L^2(R(\theta)) \text{ then } f = \sum_{l=-\infty}^{\infty} b_l e^{il\theta}}$$

Fourier Theorem.

where $2\pi = \mathrm{Vol}(\mathcal{U}(1))$, $m \neq n$, and $\{e^{il\theta}\}$ is a base of $L^2(R_\theta)$

- **Especial Unitary Group ($SU(2)$):**

Definition: The especial unitary group is defined as the set
$SU(2) := \{ A \in \mathrm{Mat}_2(\mathbb{C}) : AA^* = 1 \ \& \ \det(A) = 1 \}$.

Properties: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SU(2)$ then

1. $|a_{11}|^2 + |a_{22}|^2 = 1$
2. $|a_{12}|^2 + |a_{21}|^2 = 1$
3. $\overline{a_{11}}\, a_{21} + \overline{a_{12}}\, a_{22} = 0$
4. $a_{11}a_{22} - a_{12}a_{21} = 1$

The idea is to give a parametrization of $SU(2)$ with some angles which satisfy the last equations, so:

* The property 1 is satisfied if $a_{11} = \cos(\theta) e^{i\phi}$ and $a_{12} = i \sin(\theta) e^{i\varphi}$ with $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq 2\pi$ and $0 \leq \varphi \leq 2\pi$.

* The properties 2 and 3 are satisfied if $a_{21} = -\overline{a_{12}}\, e^{i\alpha}$ and $a_{22} = \overline{a_{11}}\, e^{i\alpha}$ with $\alpha \in \mathbb{R}$.

* The last property is satisfied if $e^{i\alpha} = 1$.

Thus, each $A \in SU(2)$ can be parametrized in the next way:

$$A = \begin{pmatrix} \cos\theta\, e^{i\phi} & i\sin\theta\, e^{i\varphi} \\ i\sin\theta\, e^{-i\varphi} & \cos\theta\, e^{-i\phi} \end{pmatrix} \quad \text{if } \theta = 0 \ \& \ \theta \neq \frac{\pi}{2}.$$

Let $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ and and the set of functions $e^m(z) = \dfrac{Z_1^{\,j+m} Z_2^{\,j-m}}{[(j+m)!(j-m)!]^{1/2}}$ with

$m = -j, -j+1, \dots, j$ and it generate homogeneous polynomials of degree $2j$. It is because there are $2j+1$ values, and the representations will be indicated by m (dimension).

By the definition of function representation :

$$\sum_i D(a)_{ij}\, e^i = e^n(A^{-1}z)$$

$$= \frac{1}{[(j+m)!\,(j-m)!]^{1/2}} \left[\cos\theta\, e^{i\phi} z_1 - i\sin\theta\, e^{i\varphi} z_2\right]^{j+m} \left[-i\sin\theta\, e^{-i\varphi} z_1 + \cos\theta\, e^{i\phi} z_2\right]^{j-m}$$

$$= \sum_{s,t} \frac{[(j+m)!\,(j-m)!]^{\frac{1}{2}}}{s!\,(j+m-s)!\,t!\,(j-n-t)!} (\cos\theta\, e^{i\phi} z_1)^s (-i\sin\theta\, e^{i\varphi} z_2)^{j+m-s} (-i\sin\theta\, e^{-i\varphi} z_1)^t (\cos\theta\, e^{i\phi} z_2)^{j-m-t}$$

$$= \sum_{s,t} (-i)^{j+m-s+t} \frac{[(j+m)!\,(j-m)!]^{1/2}}{s!\,(j+m-1)!\,t!\,(j-m-t)!} \cos\theta^{j-m+s-t} \sin\theta^{j+m-s+t} e^{i(j-m-s-t)\phi} e^{i(j+n-s-t)\varphi} z_1^{s+t} z_2^{2j-s-t}$$

where $0 \le s \le j+m$ and $0 \le t \le j-m$.

Then the coeficients of the last sum are the representations of matrixes in $SU(2)$:

$$\boxed{D^j(A)_{mn} = i^{m-n} \sum_t (-1)^t \frac{[(j+m)!(j-m)!(j+m)!(j-m)!]^{1/2}}{(j+n-t)!(t+n-m)!\,t!\,(j-n-t)!} \cos\theta^{2j+m-n-2t} \sin\theta^{2t+n-m} e^{i(m-n)\phi + i(n-m)\varphi}}$$

Now, we will prove that the representation is unitary and irreducible :

<u>To prove</u> $\rightarrow$ $f(w,z) = \sum_{m=-j}^{j} \overline{e^m(w)}\, e^m(z)$ is invariant under $f(Aw, Az)$.

$$f(w,z) = \sum_{m=-j}^{j} \frac{\overline{w_1}^{\,j+m}\, \overline{w_2}^{\,j-m}\, z_1^{j+m}\, z_2^{j-m}}{(j+m)!\,(j-m)!} \underset{\substack{\downarrow \\ \text{Binomio} \\ \text{Theorem}}}{=} \frac{[\overline{w_1} z_1 + \overline{w_2} z_2]^{2j}}{(2j)!} = \frac{(w,z)^{2j}}{(2j)!}$$

Now, $f(Aw, Az) = \dfrac{(Aw, Az)^{2j}}{(2j)!} = \dfrac{(w,z)^{2j}}{(2j)!} = f(w,z)$.

<u>To prove</u> $\rightarrow$ For all $N \in SU(2)$ and $M D^j(A) = D^j(A) M$, then $M = \lambda I$ and in consequence A is irreducible. H.W $\rightarrow$ Step by Step.

- Direct Product Reduction: (Irreducible representation product)

The representation product of $D^j(A)$ and $D^k(A)$ can be denoted by $D(A)$. Assume that $j \geq k$, and we will prove that $D(A)$ when it is reduced it has the representation $D^\ell(A)$ once if $j-k \leq \ell \leq j+k$ a d $j+k-\ell \in \mathbb{Z}$.

It can be prove using the characters

$$A \in SU(2) \longmapsto \begin{pmatrix} e^{i\rho} & 0 \\ 0 & e^{-i\rho} \end{pmatrix} \quad \text{with } 0 \leq \rho \leq \pi$$

Defining $\omega = e^{2i\rho}$. As $D^j(0,\phi,\psi) = e^{-2im\phi}\delta_{mn}$ is a diagonal matrix.

By the thing: "The characters of the set $SU(2)$ are the trace of the representation", we have:

$$\chi^j = \omega^{2j} + \omega^{2j-2} + \dots + \omega^{-2j+2} + \omega^{-2j},$$

then

$$\chi^\ell(\rho) = \sum_{m=-\ell}^{\ell} \omega^n = \omega^\ell \sum_{n=0}^{2\ell} \omega^n = (1-\omega)^{-1}(\omega^{-\ell} - \omega^{\ell+1})$$

$$= (1-\omega)^{-2}(\omega^{-j} - \omega^{j-1})(\omega^{-k} - \omega^{k+1})$$

$$= \chi^j(\rho)\, \chi^n(\rho)$$

Thus, the character of the representation product is the character product. The product can be construct in the function space of two variables generated by $e^m(x)e^n(y)$, where $-j \leq m \leq j$ and $-k \leq n \leq k$, in the following way:

$$f_{\ell,p}(x,y) = \sum_{m,n} [\ell]^{1/2} \underbrace{\begin{bmatrix} j & k & \ell \\ m & n & p \end{bmatrix}}_{3j\text{-symbol}} e^m(x)e^n(y) \quad \text{and} \quad [\ell] = 2\ell+1.$$

If we replace x and y with $A^{-1}x$ and $A^{-1}y$ we have:

$$\sum_{m'n'} D^j(A)_{m'm} D^k(A)_{n'n}\, e^{m'}(x)\, e^{n'}(y) = \sum_{\ell p p'} [\ell]^{\frac{1}{2}} \begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix} D^\ell_{pp'}(A) f_{\ell p'}(x,y)$$

Then

$$[\ell]\,\overline{\begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix}}\begin{pmatrix} j & k & \ell \\ m' & n' & p' \end{pmatrix} = \frac{[\ell]}{\Omega}\int D^{\ell}(A)_{pp'}\, D^{j}(A)_{m'm}\, D^{k}(A)_{n'n}\, dA.$$