- Laplace Type Integrals :

Let $I(x) := \int_a^b e^{-x\phi(x)} f(t)\,dt$ with $(a,b) \subseteq \mathbb{R}$, x be a large positive parameter, and f, ϕ be continuous functions. Assume, for simplicity, ϕ has a unique minimum in $[a,b]$ which occurs at $t=a$.

Theorem (Erdélyi): For $I(x) = \int_a^b e^{-x\phi(t)} f(t)\,dt$ assume :

1. $\phi(t) > \phi(a)$ for all $t \in (a,b)$, and for every $\delta > 0$ the infimum of $\phi(t) - \phi(a)$ in $[a+\delta, b)$ is positive. (ϕ es creciente)

2. $\phi'(t)$ and $f(t)$ are continuous in a nbd of $t=a$ (except possible at $t=a$)

3. Assume ϕ and f may be Taylor expanded as

$$\phi(t) = \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}, \quad f(t) = f(b) + \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}, \text{ and}$$

$$\phi'(t) = \sum_{k=0}^{\infty} a_k (k+\alpha)(t-a)^{k+\alpha-1},$$ where $\alpha > 0$, $Re(\beta) > 0$ and $a_0 \neq 0 \neq b_0$

4. The integral $I(x)$ converges (absolutly) for $x \to \infty$

Then, $\boxed{I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{C_n}{x^{(n+\beta)/\alpha}} \text{ as } x \to \infty}$ where $C_n = C_n(a_n, b_n)$

Proof: By 1 and 2 there is a number $c \in (a,b)$ s.t $\phi(t)$ and $f(t)$ are continuous in $(a,c]$. Also, it is known that $\phi'(c) = \lim_{c \to a+} \frac{\phi(c) - \phi(a)}{c-a} > 0$.

Now, define $y := \phi(t) - \phi(a)$, $Y := \phi(c) - \phi(a)$, and $dy = \phi(t)\,dt \to dt = \frac{dy}{\phi(t)}$, so

$$\underbrace{e^{x\phi(a)} \int_a^c e^{-x\phi(t)} f(t)\,dt}_{\text{Looks Like } I(x)} = \int_a^c e^{-x(\phi(t)-\phi(a))} f(t)\,dt = \int_{\phi(a)-\phi(a)}^{\phi(c)-\phi(a)} e^{-xy} \frac{f(t)}{\phi(t)}\,dt$$

$$= \underbrace{\int_0^Y e^{-xy} g(y)\,dy}_{\text{Wat son- Like.}}$$

where $f(t) = g(y)\frac{dy}{dt}$ or $g(y) = \frac{f(t)}{\phi(t)}$. And, also $y = \phi(t) - \phi(a) = \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$

by 3. as $t \to a^+$.

Series Inversion:

If $y = a_1 x + a_2 x^2 + a_3 x^3 + \dots$ with $a_0 = 0$ we may invert x in terms of a serie expansion in y:

$$X = A_1 y + A_2 y^2 + A_3 y^3 + \dots \text{ with } a_0 = 0.$$

Then

$$y = a_0 (A_1 y + A_2 y^2 + A_3 y^3 + \dots) + a_2 (A_1 y + A_2 y_2 + A_3 y^3 + \dots) + \dots$$

so

$$y = a_1 A_1 y + (a_1 A_2 + a_2 A_1) y^2 + (a_1 A_3 + 2 a_2 A_1 A_2 + a_3 A_3) y^3 + \dots$$

Thus, $a_1 A_1 = 1 \longrightarrow A_1 = \frac{1}{a_1}$

$a_2 A_2 + a_2 A_1^2 = 0 \longrightarrow A_2 = -\frac{a_2}{a_3}$

$\dots \longrightarrow a_3 = a_1^{-5}(2 a_2^2 a_1 - a_3)$

And so on ...

In our case:

$$y = \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha} \text{ as } t \to a^k,$$

then

$$(t-a) = \sum_{k=1}^{\infty} d_k \, y^{\frac{k}{\alpha}} \text{ as } y \to 0^+,$$

Doing the procedure of inversion series we got: $d_1 = \frac{1}{a_0^{1/\alpha}}$, $d_2 = \frac{-a_1}{\alpha a_0^{1+(2/\alpha)}}$, ...

Remember that $g(y) \frac{dy}{dt} = f(t)$, $g(y) \frac{d}{dt} \left(\sum_{j=0}^{\infty} a_j (t-a)^{j+\alpha} \right) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$,

then $g(y) \sum_{j=0}^{\infty} a_j (j+\alpha)(t-a)^{j+\alpha-1} = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$, and equating the same powers $j+\alpha-1 = k+\beta-1 \to j = k+\beta-\alpha$

$$\underline{\underline{g(y)}} \sum_{k=0}^{\infty} \underline{\underline{a_{k+\beta-\alpha} (k+\beta)}} (t-a)^{k+\beta-1} = \sum_{k=0}^{\infty} \underline{\underline{b_k}} (t-a)^{k+\beta-1}$$

so $g(y) \sim \sum_{k=0}^{\infty} \underline{\underline{c_k}} \, y^{\left(\frac{k+\beta}{\alpha} - 1\right)} = \sum_{k=0}^{\infty} \underline{\underline{c_k}} \, y^{p_k - 1}$. Thus, using the watson's lemma

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \underline{\underline{c_n}} \frac{\Gamma\left(\frac{k+\beta}{\alpha}\right)}{x^{(k+\beta)/\alpha}} \quad \text{as } x \to \infty.$$

Theorem: If $f, g \in C^\infty(\mathbb{C})$ at $z \in \mathbb{C}$, then $\frac{d^n}{dz^n} f(z) g(z) = \sum_{k=0}^{\infty} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z)$,

Proof: By induction, for $n=1$ and by the standard Leibniz's rule

$$\frac{d}{dz} f(z) g(z) = f'(z) g(z) + f(z) g(z).$$

Assume by n and try for $n+1$ we have

$$\frac{d^{n+1}}{dz^{n+1}} f(z) g(z) = \frac{d}{dz} \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n)}(z) g^{(n)}(z) \right) = \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dz} \left(f^{(n)}(z) g^{(n)}(z) \right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[f^{(n+1)}(z) g^{(n-k)}(z) + f^{(n)}(z) g^{(n+1-k)}(z) \right]$$

$$= f(z) g^{(n+1)}(z) + \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k+1)}(z) + \sum_{k=0}^{n-1} \binom{n}{k} f^{(k+1)}(z) g^{(n-k)}(z) + f^{(n+1)}(z) g(z)$$

now, changing in the second sum $j=k+1$:

$$\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) = f(z)g^{(n+1)}(z) + \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k} + \binom{n}{k-1}\right]}_{\frac{n!}{(k-1)!(n-k)!}\left[\frac{n+1}{(n-k+1)k}\right] = \binom{n+1}{k}} f^{(k)}(z)\, g^{(n-k+1)}(z) + f^{(n+1)}(z)g(z)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(z)\, g^{(n+1-k)}(z).$$

Continuing with the Erdérlyi theorem, the c_n term will be given by :

Now, consider $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$, then

$$f(z)g(z) = \sum_{n=0}^{\infty} \underset{\substack{\downarrow \\ \text{derivative } n.}}{c_n} (z-z_0)^n \longrightarrow \boxed{\begin{aligned} c_n &= \sum_{n=0} \frac{1}{n!}\left(\frac{d^n}{dz^n}\Big|_{z=z_0} f(z)g(z)\right)(z-z_0)^n \\ &= \sum_{n=0} \underbrace{\frac{f^{(n)}(z_0)}{k!}\,\frac{g^{(n-k)}(z_0)}{(n-k)!}}_{a_n b_{n-k}} (z-z_0)^n \end{aligned}}$$

Going back to $g(y) = \frac{f(t)}{\phi(t)}$ in the theorem proof $g(y)\phi'(t) = f(t)$.

Then, in general if we have $\frac{f(z)}{g(z)} = h(z)$,

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = \left[\sum_{k=0}^{\infty} c_k (z-z_0)^n\right]\left[\sum_{j=0}^{\infty} b_j (z-z_0)^j\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_k b_{n-k} (z-z_0)^k$$

thus, $a_n = \sum_{k=0}^{n} c_k b_{n-k}$.

For $n=0$, $a_0 = c_0 b_0 \longrightarrow c_0 = \frac{a_0}{b_0}$

For $n \geq 1$ $a_n = \sum_{k=0}^{n} c_k b_{n-k} = c_n b_0 + \sum_{k=0}^{n-1} c_k b_{n-k} \longrightarrow$

Recurence Relation

$$\boxed{c_n = \frac{\left[a_n - \sum_{k=0}^{n-1} c_n b_{n-k}\right]}{b_0}}$$

Finally is necesary to check that the remaining part (c,b) is negligible :

$\varepsilon := \inf_{c \leq t \leq b} (\phi(t) - \phi(a))$. Assume x_0 s.t $I(x_0)$ is absolutely convergent, and

assume $x \geq x_0$ $\quad x(\phi(t) - \phi(a)) = (x-x_0)(\phi(t)-\phi(a)) + x_0(\phi(t) - \phi(a))$

$$\geq (x-x_0)\varepsilon + x_0(\phi(t)-\phi(a))$$

Then $\int_c^b e^{-x(\phi(t)-\phi(a))} f(t)\,dt \le \int_c^b e^{-(x-x_0)\epsilon} f(t)\,dt + \int_c^b e^{-x_0(\phi(t)-\phi(a))} f(t)\,dt$. As the part containing x_0 are convergent, so $\left| e^{x\phi(a)} \int_c^b e^{-x\phi(t)} f(t)\,dt \right| \le K e^{-\epsilon x}$ where K is given by

$$K = e^{x_0(\epsilon + \phi(a))} \int_c^b e^{-x_0\phi(t)} |f(t)|\,dt$$ is an appropiate constant

Therefore, most of the contribution to the asymptotic behaviour of $I(x)$ comes from the interval $(a, c]$.

Theorem (Perron's Formula) The coefficients c_n are explicitly given by

$$\boxed{c_n = \frac{1}{\alpha n!} \left[\frac{d^n}{dx^n} \left\{ f(t) \left(\frac{(t-a)}{\phi(t)-\phi(a)} \right)^{\left(\frac{n+\beta}{\alpha}\right)} \right\} \right]_{t=a}}$$

where $f(t) \sim \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$, and $\phi(t) \sim \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$.

Proof: First, define ℓ as the level of the first non-vanishing coefficient in this last expression (Apart from a_0), and define

$$\phi_\ell(t) = \frac{\phi(t) - \phi(a) - a_0(t-a)^\alpha}{(t-a)^{\alpha+1}} \sim \sum_{k=0}^{\infty} a_{k+\ell}(t-a)^k \text{ as } t \to a^+.$$ Then

$$I(x) = \int_a^b e^{-x\phi(t)} f(t)\,dt = \int_a^b e^{-x\left(\phi_\ell(t)(t-a)^{\alpha+1} + \phi(a) + a_0(t-a)^\alpha\right)} f(t)\,dt$$

$$= e^{-x\phi(a)} \int_a^b e^{-x a_0 (t-a)^\alpha} h(x,t)\,dt$$

$z := (t-a)x^{\frac{1}{\alpha}}$

$$= \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \int_0^{x^{1/\alpha}(b-a)} e^{-a_0 z^\alpha} \underbrace{h(x, x^{-1/\alpha} z + a)}_{\left\{ e^{-z^\alpha s^\ell \phi_\ell(s+a)} f(s+a) \right\}} dz \qquad s := x^{-\frac{1}{\alpha}} z$$

Now, using Taylor series $h(x, s+a) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dw^k} \left(e^{-z^\alpha w^\ell \phi_\ell(w+a)}\right)_{w=0} s^k$ as $s \to 0^+$, and considering

$$f(x) = \sum_{k=0}^{\infty} b_k (x-a)^{k+\beta+1}$$

$$h(x, s+a) \sim \sum_{k=0}^{\infty} \sum_{m=0}^{n} \frac{b_{k-m}}{m!} \frac{d^m}{dw^m} \left[e^{-z^\alpha w^\ell \phi_\ell(w+a)} \right]_{w=0} s^{k+\beta-1}.$$

Then

$$I(x) \sim \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{d^k}{dw^k} \int_0^{x^{1/\alpha}(b-a)} e^{-(a_0+w^\ell \phi_\ell(w+a)) z^\alpha} s^{n+\beta-1} \right]_{w=0} dz$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{d^k}{dw^k} \int_0^{x^{1/\alpha}(b-a)} e^{-(a_0+w^\ell \phi_\ell(w+a)) z^\alpha} z^{(n+\beta-1)} \right]_{w=0} \frac{dz}{x^{(n+\beta)/\alpha}}.$$

Change $\rho := (a_0+w^\ell \phi_\ell(w+a)) z^\alpha$,

$$dz = \frac{d\rho}{\alpha}\left[\frac{\rho^{1/\alpha-1}}{\alpha[a_0+w^\ell \phi_\ell(w+a)]^{1/\alpha}}\right] \quad \& \quad z^{n+\beta-1} = \frac{\rho^{(n+\beta-1)/\alpha}}{(a_0+w^\ell\phi_\ell(w+a))^{(n+\beta-1)/\alpha}}.$$

Thus,

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{dw^k} \left[\int_0^\infty e^{-\rho} \rho^{\left(\frac{n+\beta}{\alpha}\right)-1} d\rho\right] \left[\frac{1}{\alpha(a_0+w^\ell\phi_\ell(w+a))^{(n+\beta)/\alpha}}\right]_{w=0} \frac{1}{x^{\left(\frac{n+\beta}{\alpha}\right)}}$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{d^k}{dw^k}\left(\frac{1}{[a_0+w^\ell\phi_\ell(w+a)]^{(n+\beta)/\alpha}}\right)_{w=0}\right] \frac{1}{x^{(n+\beta)/\alpha}}$$

Using the definition of ϕ_ℓ

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{dw^k} \left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}} \Bigg|_{w=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

before we knew $I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n}{x^{(n+\beta)/\alpha}}$. Therefore

$$c_n = \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{b_{n-k}}{k!} \frac{d^k}{dt^k} \left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}} \Bigg|_{t=a}$$

$$= \frac{1}{\alpha n!} \frac{d^n}{dt^n} \left\{ f(t) \left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}} \Bigg|_{t=a} \right\}$$

* Summarize (Erdélyi Theorem): Let the integral $I(x) = \int_a^b f(t) e^{-x\phi(t)} dt$ has a assymtotic aproximation:

$$\boxed{I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n}{x^{(n+\beta)\alpha}} \quad \text{as } x \to \infty}$$

con $\underline{\underline{C_n}} = \frac{1}{\alpha n!}\left[\frac{d}{dt^n}\left\{f(t)\left(\frac{(t-\alpha)^\alpha}{\phi(t)-\phi(a)}\right)^{\frac{n+\beta}{\alpha}}\right\}\right]_{t=a}$ if $\phi(t)=\phi(a)+\sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$,

and $f(t)=\sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$

Ex:

1. (Gamma Function) $\Gamma(\lambda+1)\int_0^\infty e^{-t} t^\lambda dt$ for $\lambda>0$.

It starts with a change of coordinates : $\begin{cases} t = \lambda(1+x) \\ dt = \lambda dx \end{cases}$, so

$$\Gamma(\lambda+1) = \int_{-1}^{\infty} e^{-\lambda(1+x)} [\lambda(1+x)]^\lambda \lambda dx \text{ for } \lambda>0$$

$$= \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda x} (1+x)^\lambda dx = \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda(x-\log(1+x))} dx,$$

then $$\frac{\Gamma(\lambda+1)}{\lambda^{\lambda+1}e^{-\lambda}} = \int_0^\infty e^{-\lambda x}(1+x)^\lambda + \int_{-1}^{0} e^{-\lambda(x-\log(1+x))} dx$$

$$= \underbrace{\int_0^\infty e^{-\lambda(x-\log(1+x))} dx}_{I_1(\lambda)} + \underbrace{\int_0^1 e^{-\lambda(-x-\log(1+x))} dx}_{I_2(\lambda)} \quad (x \mapsto -x)$$

- For $I_1(\lambda)$:

  * $\phi(x) = x - \log(1+x)$ and $f(x)=1$, so $k=0$ and $\beta=1$. Because $\log(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$, $\phi(x) = x - \log(1+x) \sim \frac{x^2}{2} - \frac{x^3}{3} + \ldots$ Thus, $\alpha = 2$. Finally

$$I_1(\lambda) \sim e^{-\lambda\phi(0)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right)\frac{C_n}{\lambda^{(n+1)/2}} \text{ as } \lambda\to\infty.$$

The minimum of $\phi(x)$ here is zero

And $$C_n = \frac{1}{2n!}\frac{d^n}{dx^n}\left[\frac{x^2}{x-\log(1+x)}\right]^{\left(\frac{n+1}{2}\right)}\Bigg|_{x=0}.$$

- For $I_2(\lambda)$:
  - $\phi(x) = -(x+\log(1+x)) \sim \frac{x^2}{2}+\frac{x^3}{2}+\dots$ for $x \to 0^+$, so $\alpha = 2$.
  - $f(x) = 1 \longrightarrow \beta = 1$.

then
$$I_2(x) \sim e^{-\lambda(\phi(0))} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{(-1)^n C_n}{\lambda^{(n+1)/2}}$$

With the same C_n as before. Therefore

$$\Gamma(\lambda+1) \sim \lambda^{\lambda+1} e^{-\lambda} \left[\sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{\lambda^{(n+1)/2}} (1+(-1)^n) \right]$$
$$= 2\lambda^{\lambda+1} e^{-\lambda} \sum_{n=0}^{\infty} \Gamma\left(n+\frac{1}{2}\right) \frac{C_{2n}}{\lambda^{n+\frac{1}{2}}} = \sqrt{2\pi}\, \lambda^{\lambda+\frac{1}{2}} e^{-\lambda} \sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n},$$

where $\gamma_n = \sqrt{\frac{2}{\pi}}\, \Gamma\left(n+\frac{1}{2}\right) C_{2n}$, and it is know as The Sterling Coefficients.

$$\Gamma(\lambda) \sim \sqrt{2\pi}\, \lambda^{\lambda-\frac{1}{2}} e^{-\lambda} \left(1+\frac{1}{12x}+\dots\right) \text{ as } \lambda \to \infty.$$

2. Legendre Polynomials $P_m(x)$ with $x>1$.

$$P_m(x) = \frac{1}{\pi}\int_0^{\pi} (x+\cos(t)\sqrt{x^2-1})^m \, dt .$$

Change $x = \cosh\theta$ with $\theta>0$

$$P_m(\theta) = \frac{1}{\pi}\int_0^{\pi} (\cosh(\theta)+\cos(t)\sinh(\theta))^m \, dt$$
$$= \frac{1}{\pi}\int_0^{\pi} e^{\theta m} \left[1-\sin^2\left(\frac{t}{2}\right)(1-e^{-2\theta})\right]^m dt$$
$$= \frac{1}{\pi} e^{m\theta} \int_0^{\pi} e^{-m\left(-\log[1-\sin^2(t/2)(1-e^{-2\theta})]\right)} dt$$

$$\boxed{\begin{aligned} &\cosh(\theta)+\cos(t)\sinh(\theta) = \\ &\frac{e^{\theta}+e^{-\theta}}{2} + \cos t\left(\frac{e^{\theta}-e^{-\theta}}{2}\right) = \\ &e^{\theta}\cos^2\left(\frac{t}{2}\right) + e^{-\theta}\sin^2\left(\frac{t}{2}\right) = \\ &e^{\theta}\left[1-\sin^2\left(\frac{t}{2}\right)(1-e^{-2\theta})\right] \end{aligned}}$$

where $\begin{cases} \phi(t) = -\log[1-\sin^2(t/2)(1-e^{-2\theta})] \\ f(t) = 1 \end{cases}$ we have $\beta = 1$.

Also, we have $\phi(t) = \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^{k}}{k} \sin^{2k}\left(\frac{t}{2}\right)$

$$= \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^{k}}{k} \left(\frac{t}{2} - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right)^{2k}$$

$$= (1-e^{-2\theta})\left(\frac{t^2}{4} - \frac{2t^4}{12} + \dots\right) + \dots$$

then $\alpha = 2$. And applying the theorem

$$P_m(\cosh\theta) \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{c_n}{m^{(n+1)/2}} \quad \text{where}$$

$$c_n = \frac{1}{2n!} \frac{d^n}{dt^n}\left[\frac{t^2}{-\log\left[1-\sin^2\left(\frac{t}{2}\right)(1-e^{-2\theta})\right]}\right]^{1/2}\Bigg|_{t=0}.$$

Other way to get the coefficients is: $f(t) = 1 \longrightarrow b_0 = 1$ and $b_j = 0$ for all $j \neq 0$. Now, we want $g(t)\phi'(t) = f(t)$, then

$$\sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = \left[\sum_{j=0}^{\infty} c_j (t-a)^j\right]\left[\sum_{l=0}^{\infty} a_l (l+\alpha)(t-a)^{l+\alpha-1}\right], \text{ so}$$

$$b_n = \sum_{k=0}^{n} c_k \, a_{n-k}^{1/\alpha} (n-k+\alpha)^{1/\alpha}.$$

For $n=0$ $\quad b_0 = c_0 a^{1/2}\sqrt{2} \longrightarrow c_0 = \frac{1}{\sqrt{1-e^{-2\theta}}}$,

for $n=1$ $\quad b_1 = c_0 a_1^{1/2}\sqrt{3} + c_1 a_0^{1/2}\sqrt{2} \rightarrow c_1 = \frac{-\cancelto{0}{c_0 a_1^{1/2}}\sqrt{3}}{a_0^{1/2}\sqrt{2}} = 0$, and

for $n=2$ $\quad c_2 = -\frac{1}{\sqrt{2}}$.

3. Stationary Phase Method:

Definition: A stationary point of a differentiable function is a point on the graph when the function derivative is zero. $(f'(c) = 0)$

Consider $\phi(t)$ in Laplace integral such that $Re(\phi(t)) = 0$, then $\phi(t) = i\varphi(t)$ with $\varphi(t)$ be a real function. The Generalised Fourier Integral is

$$\boxed{I(x) = \int_a^b f(t) e^{ix\varphi(t)} dt.}$$

Ex:

1. $$I(x) = \int_0^1 \sqrt{t}\, e^{ixt} dt = \frac{1}{ix}\sqrt{t}\, e^{ixt}\Big|_0^1 \underbrace{- \frac{1}{2ix}\int_0^1 \frac{1}{\sqrt{t}} e^{ixt} dt}_{I_1(x)} \quad \text{as } x \to \infty$$

So applying a wich rotation $(s \mapsto is)$, and taking $s = xt$ and $ds = xdt$.

$$I_1(x) = \frac{1}{2ix}\int_0^x \sqrt{\frac{x}{s}}\, e^{is} \frac{ds}{x} = \frac{1}{2ix^{3/2}} \int_0^x s^{-1/2} e^{is} ds = \frac{i}{2x^{3/2}} \int_0^x \frac{e^{-s}}{i^{1/2}} s^{\frac{1}{2}-1}\, i\, ds$$

$$I_1(x) \sim \frac{i\sqrt{\pi}}{2x^{3/2}} e^{i\frac{\pi}{2}} \quad \text{as } x \to \infty$$

Riemann Lebesgue lemma: Let $f(t)$ be continuous in (a,b). Then $I(x) = \int_a^b f(t) e^{ixt} dt = O\left(\frac{1}{x}\right)$ as $x \to \infty$, provided that the integral $\int_a^b |f(t)| dt$ converges.

Proof: Let $I(x) = \int_a^{a+\frac{\pi}{x}} f(t) e^{ixt} dt + \int_{a+\frac{\pi}{x}}^b f(t) e^{ixt} dt$, and $I(x) = \int_a^{b-\pi/x} f(t) e^{ixt} dt + \int_{b-\frac{\pi}{x}}^b f(t) e^{ixt} dt$ be different ways to write the integral.

Now, chaning $t' = t - \frac{\pi}{x}$ we have that $\int_{a+\frac{\pi}{x}}^b f(t) e^{ixt} dt = \int_a^{b-\frac{\pi}{x}} f\left(t - \frac{\pi}{x}\right) e^{ix(t-\frac{\pi}{x})} dt$ then

$$2I(x) = \int_a^{a+\frac{\pi}{x}} f(t) e^{ixt} dt + \int_a^{b-\frac{\pi}{x}} [f(t) + e^{i\pi} f(t + \pi/x)] e^{ixt} dt + \int_{b-\pi/x}^b f(t) e^{ixt} dt$$

as $x \to \infty$.

- Mean Value Theorem: ($f(t)$ is continuous an bounded on $[a,b]$)

$$\int_a^b f(t)\,dt = f'(c)(b-a) \text{ for some real number } c \in [a,b].$$

By the m.v.t $\displaystyle\int_a^{a+\frac{\pi}{x}} f(t)\,e^{ixt}\,dt = f'\left(\frac{\pi}{x}\right)\left(\frac{\pi}{x}\right) \sim \mathcal{O}\left(\frac{1}{x}\right)$ and

$$\int_{b-\frac{\pi}{x}}^{b} f(t)\,e^{ixt}\,dt = f'\left(\frac{\pi}{x}\right)\left(\frac{\pi}{x}\right) \sim \mathcal{O}\left(\frac{1}{x}\right).$$

Finally, as $f(t)$ is continuous for all $t \in [a,b]$

$$\lim_{x\to\infty} \int_a^{b-\frac{\pi}{x}} [f(t) - f(t+\pi/x)]\,e^{ixt}\,dt = 0$$

Therefore $I(x) \sim \mathcal{O}\left(\frac{1}{x}\right)$

One way extended the Riemann-Lebesgue Lemma to generalised Fourier Integrals as long as $|f(t)|$ is integrable, $\varphi(t)$ continuous differenciable and $\varphi'(t) \neq 0$. Take $I(x) = \displaystyle\int_a^b f(t)\,e^{ix\varphi(t)} = \frac{1}{ix}\int_a^b \frac{f(t)}{\varphi'(t)} \frac{d}{dt}\left(e^{ix\varphi(t)}\right) dt$

$$= \frac{1}{ix}\frac{f(t)}{\varphi'(t)}\,e^{ix\varphi(t)}\Bigg|_a^b - \underbrace{\frac{1}{ix}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\varphi'(t)}\right) e^{ixt}\,dt}_{\text{It vanishes more rapidly than } \frac{1}{x} \text{ as } x\to\infty}$$

then $\boxed{I(x) \sim \frac{1}{ix}\frac{f(t)}{\varphi'(t)}\,e^{ix\varphi(t)}\Bigg|_a^b \quad \text{as} \quad x \to \infty.}$

Note : This method does not work for stationary points because $\varphi'(t) = 0$.

The method of stationary phase will give the asymptotic behaviour of generalised Fourier Integrals with stationary points.

* Steps to follow :

Choose the interval such that $\varphi'(a) = 0$ and $\varphi'(t) \neq 0$ for $a < t \leq b$, so

$$I(x) = \int_a^b f(t)\,e^{ix\varphi(t)}\,dt = \int_a^{a+\varepsilon} f(t)\,e^{ix\varphi(t)}\,dt + \int_{a+\varepsilon}^b f(x)\,e^{ix\varphi(t)}\,dt$$ with $\varepsilon > 0$ be a small parameter. Then

$$\int_{a+\varepsilon}^b f(t)\,e^{ix\varphi(t)}\,dt \sim \frac{1}{ix}\frac{f(t)}{\varphi'(t)}\,e^{ix\varphi(t)}\Bigg|_{a+\varepsilon}^b \sim \frac{1}{x} \quad \text{as } x\to\infty.$$

For the first integral, we can change $f(t) \longmapsto f(a)$ and $\varphi(t) \longmapsto \varphi(a) + \frac{\varphi^{(p)}(a)(t-a)^p}{p!}$ where $\varphi'(a) = \varphi''(a) = \dots = \varphi^{(p-1)}(a) = 0$. Now, for Laplace, the leading contribution comes from a neighbourhood of the stationary points, then

$$I_1(x) = \int_a^{a+\varepsilon} f(t) e^{ix\varphi(t)} dt = \int_a^{a+\varepsilon} f(a) \exp\left\{ix\left[\varphi(a) + \frac{1}{p!}\varphi^{(p)}(a)(t-a)\right]\right\} dt.$$

Next, replacing $\varepsilon \to \infty$ this will introduce error terms $\mathcal{O}\left(\frac{1}{x}\right)$. Let $s = t-a$ & $ds = dt$ so $I_1(x) = f(a) e^{ix\varphi(a)} \int_0^\infty \exp\left\{\frac{ix}{p!}\varphi^{(p)}(a) s^p\right\} ds$. Define $u := \mp \frac{is^p x \varphi^{(p)}(a)}{p!}$

where $\begin{cases} - & \text{if } \varphi^{(p)}(a) > 0 \\ + & \text{if } \varphi^{(p)}(a) < 0 \end{cases}$, and $s = \left\{e^{\pm i\frac{\pi}{2}}\left[\frac{p!\,u}{x\varphi^{(p)}(a)}\right]\right\}$ with

$ds = e^{\pm\frac{i\pi}{2p}}\left(\frac{p!}{x|\varphi^{(p)}(a)|}\right)^{1/p} \frac{1}{p} u^{\frac{1}{p}-1} du$. Then the integral becomes

$$I_1(x) = f(a) e^{i(x\varphi(a) \pm \pi/2p)}\left(\frac{p!}{x|\varphi^{(p)}(a)|}\right)^{1/p}\frac{1}{p}\int_0^\infty e^{-u} u^{1/p-1} du$$

Finally, $\boxed{I_1(x) \sim f(a) e^{i(x\varphi(a) \pm \pi/2p)}\left(\frac{p!}{x|\varphi^{(p)}(a)|}\right)^{1/p}\frac{\Gamma(1/p)}{p}}$ Riemann-Lebesgue for Stationary Points

Ex

1. $I(x) = \int_0^{\pi/2} e^{ix\cos t} dt$. Then $\begin{cases} f(t) = 1 \\ \varphi(t) = \cos(t) \end{cases}$ with stationary points in

$\varphi'(t) = -\sin(t)$ when $t = 0$. And $p = 2$, because $\varphi''(t) \neq 0$. Then

$$I(x) \sim e^{i(x+\pi/4)}\left(\frac{2}{x}\right)^{1/2}\frac{\Gamma(1/2)}{2} = e^{i(x+\frac{\pi}{4})}\sqrt{\frac{\pi}{2x}}$$

2. Bessel Functions :

- $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin(t) - nt)\, dt$. Find the leading behaviour of $J_n(n)$

as $n \to \infty$.

$J_n(n) = \mathrm{Re}\left(\frac{1}{n}\int_0^\pi e^{in[\sin(t)-t]} dt\right)$ then $\begin{cases} f(t) = 1 \\ \phi(t) = \sin(t) - t \end{cases}$, where stationa-