Asymptotic Aproximation of Integrals :

* Bibliography

- Wong (2001)
- Bender and Orszag (1999) V1.

Let f and g be complex function define un a subset of the complex plane $H \subseteq \mathbb{C}$, and let z_0 be a limit point of H.

Definition :

- $f(z) = \mathcal{O}(g(z))$ means that there is a constant $K > 0$ and a neighbourhood of z_0, U_{z_0} such that $|f(z)| \leq K\,|g(z)|$ for all $z \in H \cap \mathbb{C}$, and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} < \infty .$$ f is said that it is big-oh g, and it is how fast the function grows or declines. (as $z \to z_0$)

- $f(z) = o(g(z))$ as $z \to z_0$ means that for all $\varepsilon > 0$ there exist a neig-bourhood of z_0 with radiour ε $U_{z_0}(\varepsilon)$ such that $|f(z)| \leq \varepsilon\,|g(z)|$ for all $z \in U_{z_0}(\varepsilon) \cap H$.

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0 .$$ f is called litle-oh g, and it is how smaller f is than g.

Note : The notation is known as Landau notation, and physicaly it means that f is the same order of g.

Examples :

1. $f(x) = 4x^3 - 3x^2 + 2x - 1$ as $x \to \infty$
   $\hookrightarrow$ Highest grow rate term

First we have to determine $g(x)$:

$$|f(x)| = |4x^3 - 3x^2 + 2x - 1| \leq |4x^3| + |3x^2| + |2x| + |1|$$
$$\leq 4|x^3| + 3|x^3| + 2|x^3| + |x^3|$$
$$= 10\,|x^3|$$

then $|f(x)| \leq 10|x^3|$. And other way to do that is using the limit definition $\lim\limits_{z \to \infty} \frac{f(x)}{x^3} = \lim\limits_{z \to \infty} \frac{4x^3 - 3x^2 + 2x - 1}{x^3} = 4 < \infty$.

2. $f(x) = 10\log(n) + 5(\log(n))^3 + 7n + 3n^2 + 6n^3$ as $n \to \infty$

First we have to determine $g(x)$:

$$|f(x)| = |10\log(n) + 5[\log(n)]^3 + 7n + 3n^2 + 6n^3|$$

$$\leq 10|\log(n)| + 5|(\log(n))^3| + 7|n| + 3|n^2| + 6|n^3| = 31|n^3|$$

Then $|f(x)| \leq 31|n^3|$.

3. $x^2 = o(x)$ as $x \to 0$

Using the little-oh limit definition $\lim\limits_{x \to 0} \frac{x^2}{x} = \lim\limits_{x \to 0} x = 0$

4. $x - \sin(x) = o(x)$ as $x \to 0$

Then $\lim\limits_{x \to 0} \frac{(x - \sin(x))}{x} = \lim\limits_{x \to 0} \left(1 - \frac{\sin(x)}{x}\right) = 0$

5. $x - \sin(x) \neq o(x^3)$

L'Hopital

Then $\lim\limits_{x \to 0} \frac{x - \sin(x)}{x^3} = \lim\limits_{x \to 0} \frac{1 - \cos(x)}{3x^2} = \lim\limits_{x \to 0} \frac{\sin(x)}{6x} = \frac{1}{6} \lim\limits_{x \to 0} \frac{\sin(x)}{x}$

$= \frac{1}{6} \neq 0$, so $f(x) = x - \sin(x)$ is not little-oh x^3.

Definitions:

- Let $\{\varphi_n\}_{n \geq 0}$ be a sequence of continuous complex functions on $H \subset \mathbb{C}$. We say that $\{\varphi_n\}_{n \geq 0}$ is an Asymptotic Sequence as $z \to z_0$ in H if we have that $\varphi_{n+1}(z) = o(\varphi_n(z))$ or $\lim\limits_{z \to z_0} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = 0$.

- If $\{\varphi_n\}_{n \geq 0}$ is an A.S as $z \to z_0$ we say that $\sum\limits_{n=0}^{\infty} a_n \varphi_n(z)$, with $\{a_n\}_{n \geq 0}$ constants sequence, is an Asymptotic Expression of the function f, if for each $N \geq 0$ we have $f(z) = \sum\limits_{n=1}^{N} a_n \varphi_n(z) + o(\varphi_N(z))$ as $z \to z_0$

In this case we can write $f(z) \sim \sum_{n=1}^{\infty} a_n \varphi_n(z)$ as $z \to z_0$.

This expansion is called Poincaré Type Expanssion.

Methods of integrals asymptotic aproximation :

1. Integration by parts

"Each integration gives as an expansion term, and the error term is given by the integral part."

Ex :

- $I(x) = \int_x^{\infty} e^{-t^4} dt$ as $x \to \infty$

Case ①: $x \longrightarrow 0$.

If I write $e^{-t^4} = 1 - t^4 + \frac{t^8}{2} - t^{12} + O(t^{13})$ then

$$\int_x^{\infty} e^{-t^4} dt = \int_x^{\infty} \left(1 - t^4 + \frac{t^8}{2} - t^{12} + \dots\right) dt = \left(t - \frac{t^5}{5} + \frac{t^9}{9} - \frac{t^{12}}{12} + \dots\right)\Big|_x^{\infty}$$

it gives us an apparent divergent result, so we can write

$$I(x) = I_1 + I_2 = \int_0^{\infty} e^{-t^4} dt + \int_x^{0} e^{-t^4} dt = \int_0^{\infty} e^{-t^4} dt - \int_0^{x} e^{-t^4} dt$$

Then by the substitution $s = t^4$ and $ds = 4t^3 dt$, I_1 can be written as

$$I_1 = \int_0^{\infty} e^{-s} \frac{ds}{4t^3} = \frac{1}{4} \int_0^{\infty} e^{-s} s^{-3/4} ds$$

$$= \Gamma\left(\frac{5}{4}\right)$$

$$\boxed{\begin{array}{l} \Gamma(n) = \int e^{-z} z^{n-1} dz \\ n\Gamma(n) = \Gamma(n+1) \end{array}}$$

Thus, $I(x) = \Gamma\left(\frac{5}{4}\right) - \int_0^x \left(1 - t^4 + \frac{t^8}{2} - \frac{t^{12}}{6} + \dots\right) dt$

$$= \Gamma\left(\frac{5}{4}\right) - \left[x - \frac{x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{78} + \dots\right]$$

Then $I(x) \sim \Gamma\left(\frac{5}{4}\right)$ as $x \to 0$.

Case 2 : $x \to \infty$.

$$I(x) = \int_0^\infty e^{-t^4}\,dt = -\frac{1}{4}\int_x^\infty \frac{1}{t^3}\frac{d}{dt}\left(e^{-t^4}\right)dt$$

$$= -\frac{1}{4}\frac{e^{-t^4}}{t^3}\Big|_x^\infty + \frac{1}{4}\int_x^\infty e^{-t^4}\frac{d}{dt}\left(\frac{1}{t^3}\right)dt$$

$$= \frac{1}{4}\frac{e^{-x^4}}{x^4} - \frac{3}{4}\int_x^\infty \frac{e^{-t^4}}{t^4}\,dt$$

But $\int_x^\infty \frac{1}{t^4}e^{-t^4}dt < \frac{1}{x^4}\int_x^\infty e^{-t^4}dt = \frac{1}{x^4}I(x) << I(x)$ leading behaviour of $I(x)$. Thus $I(x) \sim \frac{1}{4x^3}e^{-x^4}$ as $x \to \infty$

- $I(x) = \int_0^x t^{-\frac{1}{2}} e^{-t}\,dt$ as $x \to \infty$

Remember the Gamma Function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ with $Re(z) > 0$

First, rewriting $I(x) = \underbrace{\int_0^\infty t^{-\frac{1}{2}}e^{-t}\,dt}_{I_1(x)} - \underbrace{\int_x^\infty t^{-\frac{1}{2}}e^{-t}\,dt}_{I_2(x)}$

$z - 1 = -\frac{1}{2} \to z = 1 - \frac{1}{2} = \frac{1}{2}$

$$I_1(x) = \int_0^\infty t^{(\frac{1}{2})-1}e^{-t}\,dt = \Gamma(1/2).$$ Then

$$I(x) = \Gamma(1/2) - \int_x^\infty t^{-\frac{1}{2}}e^{-t}\,dt = \Gamma(1/2) + \int_x^\infty t^{-1/2}\frac{d}{dt}\left(e^{-t}\right)dt$$

$\frac{d}{dt}\left(e^{-t}\right) = -e^{-t}$

$$I(x) = \Gamma(1/2) + \frac{e^{-t}}{\sqrt{t}}\Big|_x^\infty + \int_x^\infty \frac{e^{-t}}{2t^{3/2}}\,dt$$

$$= \Gamma(1/2) + \frac{e^{-x}}{x^{1/2}}$$

Then, by the leading order $I(x) = \sqrt{\pi} - \frac{e^{-x}}{{}^{1/2}}$ as $x \to \infty$. Finally, repeating the integration by parts, it gives us the asymptotic expansion:

$$I(x) = \int_a^b t^{-1/2}e^{-t}\,dt \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2x)^n}\right] \text{ as } x \to \infty.$$

- The Fourier Transformation: $\mathcal{F}(x) = \int_a^b f(t) e^{ixt} dt$ with $f(t) \in C^N$.

$$\mathcal{F}(x) = \int_a^b f(t) \left(\frac{-i}{x}\right) \frac{d}{dt}(e^{ixt}) dt = \left(\frac{-if(t)e^{ixt}}{x}\right)\Bigg|_a^b + \frac{i}{x}\int_a^b \frac{d}{dt} f(t)\, e^{ixt} dt$$

$$= -\frac{i}{x} f(b) e^{ixb} + \frac{i}{x} f(a) e^{ixa} + \frac{i}{x}\int_a^b \frac{df}{dt}(t)\, e^{ixt} dt.$$

We can continuo with the iteration process, multiplying by parts, till $f^{(n)}(t)$ and get:

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} \left(\frac{i}{x}\right)^{n+1} \left(e^{iax} f^{(n)}(a) - e^{ibx} f^{(n)}(b)\right) + \underbrace{\left(\frac{i}{x}\right)^N \int_a^b f^{(N)}(t) e^{ixt} dt}_{\mathcal{O}(1/x^N)}$$

- Stieltjes Transformation of e^t $\quad I(x) = \int_0^\infty \frac{e^{-t}}{1+xt} dt$ as $x \to 0^+$

$$I(x) = -\int_0^\infty \frac{1}{1+xt} \frac{d}{dt}(e^{-t}) dt = 1 - \int_0^\infty \frac{d}{dt}\left(\frac{1}{1+xt}\right) e^{-t} dt = 1 - \int_0^\infty \frac{x}{(1+xt)^2} e^{-t} dt$$

$$= 1 + \int_0^\infty \frac{x}{(1+xt)^2} \frac{d}{dt}(e^{-t}) dt$$

$$= 1 + x\left[\frac{e^{-t}}{(1+xt)^2}\Bigg|_0^\infty \frac{d}{dt}\left(\frac{1}{(1+xt)^2} e^{-t} dt\right)\right] = 1 - x + 2x^2 \int_0^\infty \frac{1}{(1+xt)^3} e^{-t} dt$$

$$= 1 - x + 2x^2 - \dots + (-1)^n (n-1)!\, x^{n-1} + (-1)^n n!\, x^n \int_0^\infty \frac{e^{-t}}{(1+xt)^{n+1}} dt$$

Then $I(x) \sim \sum_{n=0}^{\infty} (-1)^n n!\, x^n$ as $x \to 0^+$

- Laplace Integrals: $\quad I(x) = \int_a^b f(t) e^{x\phi(t)} dt$ as $x \to \infty$.

With f, ϕ be continuous functions.

Let $I(x) = \int_a^b f(t) e^{x\phi(t)} dt = \int_a^b \frac{f(t)}{\phi'(t)} \frac{d}{dt}(e^{x\phi(t)}) dt$

$$= \frac{1}{x} \frac{f(t)}{\phi'(t)} e^{x\phi(t)}\Bigg|_a^b - \int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi(t)}\right) e^{x\phi(t)} dt$$

The leading term is:

$$I(x) \sim \frac{1}{x}\frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x}\frac{f(a)}{\phi'(a)} e^{x\phi(a)} \quad \text{as } x\to\infty.$$

Remark: Integrating by parts again will introduce $\frac{1}{x^2}$ terms, and so on!

Note: The integration by parts method does not work when in the Laplace type integrals we expect to get the expansion of the form $I(x)\sim e^{x\phi(b)}\sum_{n=1}^{N} A_n x^{-n}$ for $x\to\infty$ and it is not happening.

Ex:

1. Gaussian $\int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ as $x\to\infty$

If I integrate by parts

$$I(x) = \left(\frac{e^{-xt^2}}{-2xt}\right)\Bigg|_0^\infty - \frac{1}{x}\int_0^\infty \frac{d}{dt}\left(\frac{-1}{2t}\right)e^{-xt^2}\,dt$$

but the first term is divergent when $t\to 0$, i.e, $\lim_{t\to 0} -\frac{1}{2xte^{xt^2}} = \infty$.

$$\left[\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right]^2 = \left[\int_{-\infty}^{\infty} e^{-ay_1^2}\,dy_1\right]\left[\int_{-\infty}^{\infty} e^{-ay_2^2}\,dy_2\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(y_1^2+y_2^2)}\,dy_1\,dy_2$$
$$= \int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\,dr\,(r\,d\theta)$$
$$= \frac{\pi}{a}\int_0^{\infty} e^{-s}\,ds = \frac{\pi}{a}$$

Remembering:

* $r = y_1^2+y_2^2$ & $d(x,y) = |J|\,d(r,\theta)$
* $s = ar^2$ & $ds = 2a\,dr\,r$

Then

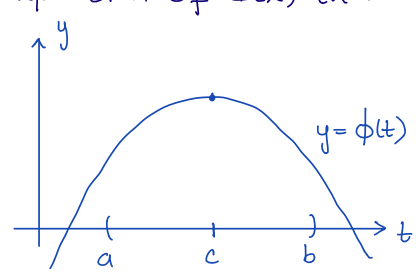
$$\int_{-\infty}^{\infty} e^{-ay^2}\,dy = \sqrt{\frac{\pi}{a}}$$

2. Laplace Method:

We want to study the Laplace Integrals $I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt$, where $f,\phi\in C^1(\mathbb{R})$.

Theorem: Assume $\phi(t)$ attains its maximum on $t=c\in(a,b)$, and $f(c)\neq 0$. Thus, the major contribution to the asymptotic expansion of $I(x)$ $(x\to\infty)$ comes from a neighbourhood of the point $t=c$.

Note: We will consider here ϕ has only a maximum and thus $\phi'(c)=0$ and $\phi''(c)<0$.



Proof: We may Taylor expand $\phi(t)$ and $f(t)$ arround $t=c$

$$\phi(t) = \phi(c) + \cancel{\phi'(c)}^{\,0}(t-c) + \frac{\phi''(c)}{2!}(t-c)^2 + \dots \quad \text{and}$$

$$f(t) = f(c) + f'(c)(t-c) + \frac{f''(c)}{2!}(t-c)^2 + \dots$$

Then

$$I(x) = \int_a^b f(c)\, e^{x[\phi(c) + \frac{\phi''(c)}{2!}(t-c)^2]} dt = f(c) e^{x\phi(c)} \int_a^b f(c)\, e^{\frac{x}{2}(\phi''(c)(t-c)^2)} dt .$$ Taking

$s = t-c$, $ds = dt$ and $a = -\frac{x}{2}\phi''(c)$

$I(x) = f(c) e^{x\phi(c)} \int_a^b e^{-as^2} ds$. Because it represents a Gaussian, we can consider $a, b \longrightarrow \infty$. And, because $t=c$ is a maximun $I(x)$ is well define, then

$$\boxed{I(x) = f(c) e^{x\phi(c)} \int_{-\infty}^{\infty} e^{-as^2} ds = f(c) e^{x\phi(c)} \sqrt{\frac{2\pi}{|\phi''(c)x|}} .}$$

Ex :

1. $I(\lambda) = \int_{-1}^{1} \frac{\sin(t)}{t} e^{-\lambda(\cosh(t))} dt$ as $\lambda \to 0$.

Let $f(t) = \frac{\sin(t)}{t}$ and $\phi(t) = -\cosh(t)$ so $\phi'(t) = -\sinh(t) = 0$ when $t=0$ and $\phi''(t) = -\cosh(t)$ with $\phi''(0) = -1 < 0$. As $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$ so

$$I(\lambda) \sim e^{-\lambda} \sqrt{\frac{2\pi}{|-1\lambda|}} = \sqrt{\frac{2\pi}{|\lambda|}} \left(1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots \right)$$

- Higher order asymtotic aproximation : Taking a higher order aproximaby Taylors expansion

$$I(x) = \int_a^b \left[f(c) + f'(c)(t-c) + \frac{f''(c)}{2!}(t-c)^2 + \dots \right] e^{x\left[\phi(c) + \frac{\phi''(c)}{2!}(t-c)^2\right]} dt$$

and taking $s = t-c$, $I(x) = \int_a^b \left[f(c) + f'(c)s + \frac{f''(c)}{2} c^2 + \dots \right] e^{x\left[\phi(c) + \frac{\phi''(c)}{2}s^2\right]} ds$

$$= f(c) e^{x\phi(c)} \sqrt{\frac{2\pi}{|\phi''(c)x|}} + e^{\phi(c)x} \cancel{\int_a^b f'(c) s\, e^{(x/2)\phi''(c)s^2} ds}^{\,0} + \frac{e^{x\phi(c)}}{2} \int_a^b f''(c) s^2 e^{(x/2)\phi''(c)s^2} ds$$

$+ \dots$ and replacing $u := -\frac{x}{2}\phi''(c)s^2$, $du := -x\phi''(c) s\, ds$, and

$$s = \pm\sqrt{\frac{2u}{|x\phi''(c)|}}, \quad \text{so} \quad \frac{e^{x\phi(c)}}{2}\int_{a^*}^{b^*} f''(c)\, s^2 e^{\left(\frac{x}{2}\right)\phi''(c) s^2}\, ds = \frac{e^{x\phi(c)}}{2}\int_{a^*}^{b^*} \frac{-f''(c)\, s^2 e^{-u}}{x\phi''(c)\, s}\, du$$

$$= \frac{2^{1/2} e^{x\phi(c)}}{[x|\phi''(c)|]^{3/2}} \int_0^\infty u^{1/2} e^{-u}\, du = \frac{2^{1/2} e^{x\phi(c)}}{[|x\phi''(c)|]^{3/2}} \Gamma(3/2) = \sqrt{\frac{\pi}{2}}\, \frac{e^{x\phi(c)}}{[|x\phi''(c)|]^{3/2}}$$

Finally,

$$I(x) = e^{x\phi(c)}\left[f(c)\sqrt{\frac{2\pi}{|x\phi''(c)|}} + \sqrt{\frac{\pi}{2}}\, \frac{1}{[|x\phi''(c)|]^{3/2}} + \dots \right]$$

Watson's Lemma: Let $f(t)$ be a complex valued function of a real variable t, such that:

1. $f \in C^1((0,\infty))$.
2. As $t \to 0^+$, $f(t) \sim \sum_{k=0}^{\infty} a_k t^{\rho_k - 1}$,

   with $0 < Re(\rho_0) < Re(\rho_1) < \dots < \lim_{k\to\infty} Re(\rho_k) = \infty$.
3. For some fixed $c > 0$, $f(t) = \mathcal{O}(e^{ct} t^{\rho_N - 1})$ as $t \to \infty$.

Then $$I(x) := \int_0^\infty e^{-xt} f(t)\, dt \sim \sum_{k=0}^{\infty} \frac{a_k \Gamma(\rho_k)}{x^{\rho_k}} \quad \text{as } x \to \infty$$

Proof: The conditions (1-3) garantee that $I(x)$ converges for $x>0$, and the conditions (2-3) imply

$$\left| f(t) - \sum_{k=0}^{\infty} a_k t^{\rho_k - 1} \right| \le K_N e^{ct} |t^{\rho_N - 1}| \quad \text{for } t > 0.$$

Then $$\left| e^{-xt} f(t) - \sum_{k=0}^{\infty} e^{-xt} a_k t^{\rho_k - 1} \right| \le K_N e^{-(x-c)t} \left| t^{\rho_N - t} \right|$$

$$\left| \int_0^\infty \left(e^{-xt} f(t) - \underbrace{\sum_{k=0}^{\infty} e^{-xt} a_k t^{\rho_k - 1}}_{I_1} \right) dt \right| \le \underbrace{K_N \int_0^\infty e^{-(x-c)t} |t^{\rho_N - t}|\, dt}_{I_2}$$

Now, with a change of variables, $u = xt$, in I_1 we have

$$I_1 = \frac{1}{x^{\rho_k}} \int_0^\infty e^{-u} u^{\rho_k - 1}\, du = \frac{1}{x^{\rho_k}} \Gamma(\rho_k).$$

And with a change of variables, $u=(x-c)t$, in I_2 we have

$$I_2 = \frac{1}{|(x-c)|^{\rho_N}} \int_0^\infty e^{-u} u^{\rho_N - 1} du = \frac{1}{|x-c|^{\rho_N}} \Gamma(Re(\rho_N))$$

Finally, $\left| I(x) - \sum_{k=0}^{N-1} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}} \right| \le K_N \frac{\Gamma(Re(\rho_N))}{|x-c|^{\rho_N}}$. Therefore

$$I(x) := \int_0^\infty e^{-xt} f(t)\, dt = \sum_{k=0}^{N-1} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}} + \mathcal{O}\left(\frac{1}{x^{\rho_N}}\right).$$

Ex:

1. $I(x) = \int_0^5 \frac{e^{-xt}}{1+t^2} dt$ for large x

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots \text{ around } t=0,$$
$$= \sum_{k=0}^\infty (-1)^k t^{2k}$$

And by watson's : 1. Substitute this expansion into the integral

2. Interchange integral and summation

3. Extend from 5 to ∞.

$$I(x) = \int_0^\infty \sum_{k=0}^\infty (-1)^k t^{2k} e^{-xt} dt = \sum_{k=0}^\infty (-1)^k \int_0^\infty e^{-xt} t^{2k} dt$$

From $I(x)$ we have $a_k = (-1)^k$ and $\rho_k = 2k+1$, then

$$I(x) = \sum_{k=0}^{N-1} (-1)^k \frac{\Gamma(2k+1)}{x^{2k+1}} + \mathcal{O}\left(\frac{1}{x^{2k+1}}\right) \quad \text{as } x \to \infty,$$

So $I(x) \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \dots$ as $x \to \infty$.

2. $I(x) = \int_0^{\pi/4} e^{-xt} \sqrt{1+\cos(t)}\, dt$ as $x \to \infty$

$$f(t) = \sqrt{1+\cos(t)} = \sqrt{2} \cos\left(\frac{t}{2}\right) = \sqrt{2} \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \left(\frac{t}{2}\right)^{2n} \text{ then}$$

$$I(x) = \int_0^{\pi/4} \sqrt{2} \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \left(\frac{t}{2}\right)^{2n} e^{-xt} dt = \sum_{n=0}^\infty \frac{\sqrt{2}}{(2n)!} (-1)^n \left(\frac{1}{2}\right)^{2n} \int_0^{\pi/4} e^{-xt} t^{2n} dt.$$

Therefore $a_k = \frac{\sqrt{2}}{(2n)!}(-1)^n\left(\frac{1}{2}\right)^{2n}$ and $\rho_k = 2n+1$.

Finally $I(x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k\sqrt{2}}{(2n)!}\left(\frac{1}{2}\right)^{2n}\frac{\Gamma(2k+1)}{x^{2k+1}} = \sqrt{2}\left(\frac{1}{x}-\frac{1}{4x^3}+\frac{1}{16x^5}\right)+O\left(\frac{1}{x^7}\right)$

as $x \to \infty$.

3. Modify Bessel Function:

$$K_0(x) = \int_1^{\infty}(s^2-1)^{-\frac{1}{2}}e^{-sx}\,ds \quad \text{as } x \to \infty.$$

First, we will take $s = t+1$, then $t^2+2t+1 = s^2$ and $ds = dt$, so

$$K_0(x) = \int_1^{\infty}[t(t+2)]^{-\frac{1}{2}}e^{-(t+1)x}\,dt = e^{-x}\int_1^{\infty}[t(t+2)]^{-\frac{1}{2}}e^{-tx}\,dt$$

Now, $f(t) = [t(t+2)]^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}}\left(\frac{t}{2}+1\right)^{-\frac{1}{2}} = (2t)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\binom{k-\frac{1}{2}}{-\frac{1}{2}}\frac{t^k}{2^k}$

$$= (2t)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k(k-1/2)!}{k!\,(-\frac{1}{2})!}\left(\frac{t}{2}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+1/2)}{k!\,\Gamma(1/2)}\frac{1}{2^{k+\frac{1}{2}}}t^{k-\frac{1}{2}}.$$

Then

$$K_0(x) = e^{-x}\sum_{k=0}^{\infty}\int_1^{\infty}\underbrace{\frac{(-1)^k\Gamma(k+1/2)}{k!\,\Gamma(1/2)\,2^{k+1/2}}}_{a_k}t^{k-1/2}e^{-tx}\,dt$$

a_k and $\rho_k = k-\frac{1}{2}+1 = k+\frac{1}{2}$

Thus

$$K_0(x) \sim e^{-x}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+1/2)}{k!\,\Gamma(1/2)\,2^{k+1/2}}\frac{\Gamma(k+1/2)}{x^{k+1/2}}$$

$$e^{-x}\sum_{k=0}^{\infty}\frac{(-1)^k(\Gamma(k+1/2))^2}{\sqrt{\pi}\,k!\,2^{k+1/2}x^{k+1/2}}$$

$$K_0(x) \sim \frac{\Gamma^2(1/2)}{\sqrt{\pi}\,(2x)^{1/2}} - \frac{\Gamma^2(3/2)}{\sqrt{\pi}\,(2x)^{3/2}} + O\left(\frac{1}{x^{5/2}}\right). \quad \text{as } x \to \infty$$