ry point is $t=0$ when $\phi'(t) = \cos(t) - 1 = 0$. And $\phi'''(0) = -\cos(0) = 0$, then $p = 3$. Thus $J_n(n) \sim \underbrace{\frac{1}{\pi}\frac{\sqrt{3}}{2}}_{\mathrm{Re}\left(e^{-i\frac{\pi}{6}}\right)}\left(\frac{6}{n}\right)^{\frac{1}{3}}\frac{\Gamma\left(\frac{1}{3}\right)}{3}$

- $J_n(x) = \int_0^1 \cos(n\pi t - x\sin(\pi t))\,dt = \mathrm{Re}\left\{\int_0^1 e^{n\pi i t} e^{-ix(\sin\pi t)}\,dt\right\}$

Where $f(t) = e^{n\pi i t}$ and $\phi(t) = -\sin(\pi t)$ in the interval $[0,1]$.

- $\phi'(t) = -\pi\cos(\pi t) = 0 \longrightarrow t = \frac{1}{2}$
- $\phi''(1/2) = \pi^2 \sin(\pi/2) = \pi^2 > 0$ then $p = 2$.

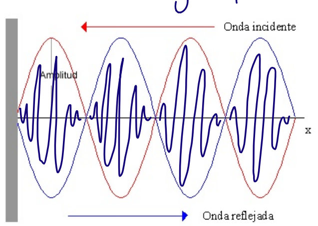
Thus

$$I(x) \sim \mathrm{Re}\left\{e^{\frac{i\pi n}{2}} e^{i(-x+\frac{\pi}{4})}\left(\frac{2!}{x\pi^2}\right)\frac{\Gamma(1/2)}{2}\right\} = \frac{1}{\sqrt{2\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \text{ as } x\to\infty.$$

3. Waves and Klein Gordon equation. Let $u(x,t) = a e^{i(kx-wt)}$ be an harmonic waves, where $k :=$ wave number and $w :=$ frequency. For a dispersive media we will consider $w = w(k)$. A continuous superposition of waves is

$u(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{i(kx-wt)}\,dk$. We want to apply the stationary phase principle for de case $t\to\infty$ and fixed radio $\frac{x}{t} := v$.

1. Consider the group velocity $V_g := -\left(\frac{\partial a_0(x,t)/\partial t}{\partial a_0(x,t)/\partial x}\right) = \frac{W_m}{K_m}$



$$= \frac{(1/2)(W_2 - W_1)}{(1/2)(k_2 - k_1)} = \frac{\Delta w}{\Delta k}.$$

In the apropiate limite $V_g = \frac{dw}{dk}$.

We want to consider stationary points s.t $w'(k) = 0$ (Lable one such point as ξ)

Therefore $u(x,t) \sim \left[a(\xi)e^{ix\xi}\right] e^{-iw(\xi)t + \frac{i\pi}{4}\operatorname{sgn}(w''(\xi))}\sqrt{\frac{\pi}{2t|w''(t)|}}$.

Now, we consider the Klein Gordon equation $u_{tt} - \gamma^2 u_{xx} + c^2 u = 0$ with $t>0$ and $x \in \mathbb{R}$, s.t $u(x,0)$ and $u_t(x,0)$ are specified.

Let $\mathcal{U}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, u(x,t)\, dx$ be the Fourier Transformation in x,

and the inverse transform is $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \mathcal{U}(k,t)\, dt$. Therefore, its

derivatives are

$$u_{tt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^2 \mathcal{U}(k,t)}{\partial t^2}\, dk,$$

$$u_{xx} = \frac{-k^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk}\, \mathcal{U}(k,t)\, dk.$$

Thus $\frac{\partial^2 \mathcal{U}(k,t)}{\partial t} + (\gamma^2 k^2 + c^2)\, \mathcal{U}(k,t) = 0$ with solutions

$$\mathcal{U}(k,t) = a_+(k)\, e^{i\sqrt{\gamma^2k^2+c^2}\, t} + a_-(k)\, e^{-i\sqrt{\gamma^2k^2+c^2}\, t}$$

are fixed by initial conditions

Now, define $w_\pm = \pm\sqrt{\gamma^2 k^2 + c^2}$ we will have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_+(k)\, e^{i(w_+(k)t - kx)}\, dk \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_-(k)\, e^{i(w_-(k)t - kx)}\, dk,$$

both integrals are of the forme previously seen. Thus we already know its assymptotic behaviour.

4. Steepest Decent Method :

Let $I(\lambda) = \int_c f(z)\, e^{\lambda \varphi(z)}\, dz$ be a complex integral with $f(z)$ and $\varphi(z)$ are analytic funtions where $\lambda :=$ Large positive parameter, and $c :=$ contour of integral.

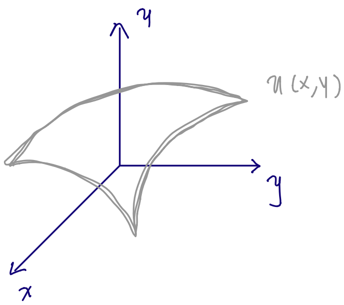
Idea : Deform the contour of integration C into a new contour C^* s.t :

1. C^* passes through one or more zeros of $\varphi(z)$
2. $\mathrm{Im}(\varphi(z)) = k$ (constant) in C^*

Write $z := x+iy$, $\varphi(z) = u(x,y) + i\,v(x,y)$, and remember that if φ is analytic :

- The Cauchy-Riemann is followed
- $\nabla^2 u = 0 \longrightarrow u$ is harmonic "$\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$"

Suppose we consider a 3-dimentional space with coordinates (x, y, u), then $u = u(x,y)$ define a surface:



Suppose that there exist $z_0 = x_0 + y_0 i$ s.t $\varphi'(z_0) = 0$ (z_0 will be stationary). And following Cauchy-Riemann: $\varphi'(z) = u_x + i v_x = u_x - i u_y$, so at $z_0 = 0$ $\varphi'(z_0) = 0 = u_x - i u_y$, thus at z_0

$$u_x(x_0, y_0) = 0 = u_y(x_0, y_0)$$

That is (x_0, y_0) is a critical point for the real function $u(x,y)$.

Theorem: Let $u(x,y)$ be a harmonic function on a bounded region Ω, then an isolated critical point of u in Ω can not give a relative maximun or minimun of u.

Proof: Let p be a icp and suppose that p gives us a maximun for u. Thus $u(q) \leq u(p)$ $\forall\ q \in V_p \subseteq \Omega$. Now, for $\varepsilon > 0$ we consider the level surface $S = \{ q \in B_p \mid u(q) = u(p) - \varepsilon \}$ where B_p is a close ball about p, s.t $\nabla u(q) \neq 0$ and $u(q) \geq u(p) - \varepsilon$ for $p \neq q$.

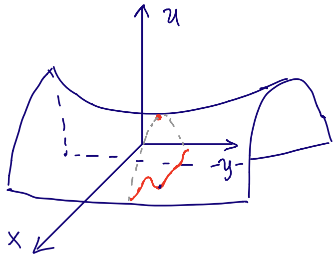
Thus, the outer unit normal to S is given by $\hat{n} = -\frac{\vec{\nabla u}}{|\nabla u|}$ at each point of S.

direction of greatest increase

Then $\iint_S \nabla u \cdot n \, ds < 0$, but by the divergence theorem

$$\iint_S \nabla u \cdot n \, ds = \iiint_E \nabla^2 u \, dv = 0 \quad (E \text{ is a region bounded by } S).$$

so p don't denote a local max, and it must be a saddle point.



In our notation $p = (x_0, y_0)$. In order to get de condition $v(x,y) = cte$ on C^* we will consider a curve γ s.t points in γ are parametrised by a parameter $s \in \mathbb{R}$ as $x = x(s)$, $y = y(s)$, $u = u(s) = u(x(s), y(s))$. In particular, one may consider s as the arc-length of the curve γ, so $dx^2 + dy^2 = ds^2$ and $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$.

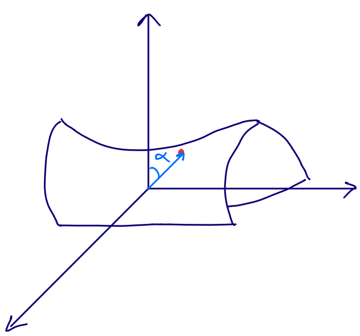
Define a parameter θ by introducing $\cos\theta = \frac{dx}{ds}$ and $\sin\theta = \frac{dy}{ds}$. Thus

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = u_x \cos\theta + u_y \sin\theta.$$

The steepness of the curve $(x(s), y(s), u(s))$ is measured w.r.t the u-axis by introducing an angle α:

$$\cos\alpha = \frac{\frac{du}{ds}}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{du}{ds}\right)^2}} = \frac{\frac{du}{ds}}{\sqrt{1 + \left(\frac{du}{ds}\right)^2}},$$

so



$$\frac{d}{d\theta}\cos\alpha = \frac{\left(1 + \left(\frac{du}{ds}\right)^2\right) - \left(\frac{du}{ds}\right)^2}{\left(\sqrt{1 + \left(\frac{du}{ds}\right)^2}\right)^3} \cdot \frac{d}{d\theta}\left(\frac{du}{ds}\right)$$

$$= \frac{-u_x \sin\theta + u_y \cos\theta}{\left(1 + \left(\frac{du}{ds}\right)^2\right)^{3/2}}$$

Then, as we need $\frac{d}{d\theta}\cos\alpha = 0$ we have using Cauchy-Riemann

$$0 = -(v_y \sin\theta + v_x \cos\theta) = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} = \frac{dv}{ds},$$

then $v = cte$.

Now, getting the second derivative of $\cos\alpha$:

$$\frac{d^2}{d\theta^2}\cos\alpha = -\underbrace{\left[\frac{1+\left(\frac{du}{ds}\right)^2+3\left[\frac{d}{d\theta}\left(\frac{du}{ds}\right)\right]^2}{\left[1+\left(\frac{du}{ds}\right)^2\right]^{3/2}}\right]}_{\text{Bigger than } 0\ (>0)}\frac{du}{ds} \quad \text{because} \quad \frac{d^2}{d\theta^2}\left(\frac{du}{ds}\right)=-\frac{du}{ds}$$

therefore the sign of $\frac{d^2}{d\theta}\cos\alpha$ only depends of the sign of $\frac{du}{ds}$, for that reason $\cos(\alpha)$ has an absolute maximun when $\frac{dv}{ds}=0$ & $\frac{du}{ds}>0$ and $\cos(\alpha)$ has an absolute minimun when $\frac{dv}{ds}=0$ & $\frac{du}{ds}<0$.

* Conclution :

- For any (x,y,u) s.t $v(x,y)=cte$ we will have that the tangent vector to $c(s)=(x(s),y(s),u(s))$ where there is a maximun or minimun inclination (steepness) w.r.t the u-axis, give as the steepest ascent or descent on S.

Definition : Domains where

- $u(x,y)>u(x_0,y_0)$ are called hills
- $u(x,y)<u(x_0,y_0)$ are called Valleys

Also, for saddle points $u(x,y)=u(x_0,y_0)$.

Suppose z_0 be a saddle point of order $m-1$ $(m\geq 2)$, i.e, $\varphi'(z_0)=\dots=\varphi^{(m-1)}(z_0)=0$.

And take $\varphi^{(m)}(z_0)=ae^{i\phi}(z_0)$ where $a>0$ and ϕ is real valued. Also $z=z_0+re^{i\theta}$, then $\varphi(z)=\varphi(z_0)+\frac{1}{m!}ae^{i\phi}(re^{i\theta})^m+\dots = \varphi(z_0)+\frac{a}{m!}r^m e^{i(\theta m+\phi)}+\dots$

In consequence, near z_0 $\varphi(z)=u(x,y)+iv(x,y)$ we can find

$$u(x,y) = u(x_0,y_0)+\frac{r^m}{m!}a\cos(m\theta+\phi)+\dots$$

$$v(x,y) = v(x_0,y_0)+\frac{r^m}{m!}a\sin(m\theta+\phi)+\dots$$

- The directions where $V=cte$ are given by $\cos(m\theta+\phi)=0$

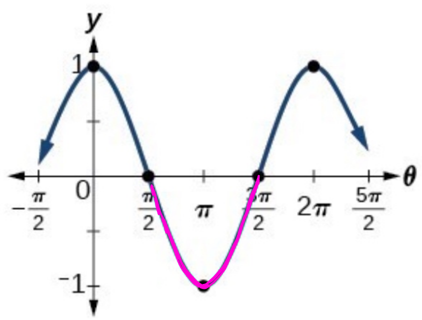
$$m\theta+\phi=(2k+1)\frac{\pi}{2}, \quad k\in\mathbb{Z}$$

then $\theta = -\frac{\phi}{m} + \frac{(2k+1)}{m}\left(\frac{\pi}{2}\right)$ with $k = 0, 1, \dots, m-1$.

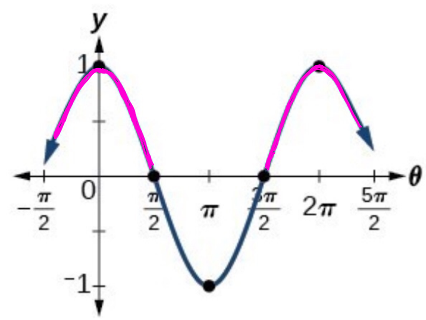
Analogusly, for $v = cte$ $\sin(m\theta + \phi) = 0 \rightarrow m\theta + \phi = k\pi \rightarrow \theta = -\frac{\phi}{m} - k\frac{\pi}{m}$ with $k = 0, 1, \dots, m-1$.

Thus, the valleys are when $\cos(\theta m + \phi) < 0$ and the hills are when $\cos(\theta m + \phi) > 0$.

VALLEYS



HILLS



Ex:

1. $\varphi(z) = z - \frac{z^3}{3}$.

First start with the derivatives of φ:

- $\varphi'(z) = 1 - z^2 \rightarrow 1 - z^2 = 0 \rightarrow z = \pm 1$
- $\varphi''(z) = -2z$ so for $\varphi''(1) = -2$ and $\varphi''(-1) = 2$.

So the critical point $z_0 = \pm 1$ is of order $m = 2$. Also we know that the valley exists when $\cos(\theta m + \phi) < 0$ and $\varphi''(z) = a e^{i\phi}$, then

- $\varphi''(1) = 2e^{i\pi}$ and $\varphi''(-1) = 2e^{i0}$.

The steepst desent is when $\theta m + \phi = (2k+1)\pi$ with $k = 0, 1$, so

$\theta = -\frac{\pi}{2} + (2k+1)\frac{\pi}{2}$ when $\boxed{\theta_0 = \frac{(\pi - \pi)}{2} = 0}$ and $\boxed{\theta_1 = \frac{(3\pi - \pi)}{2} = \pi}$ (*)

$\theta = -\frac{0}{2} + (2k+1)\frac{\pi}{2}$ when $\boxed{\theta_0 = \frac{\pi}{2}}$ and $\boxed{\theta_1 = \frac{3\pi}{2}}$ (**)

Let $\varphi(x,y) = u(x,y) + i\,v(x,y)$ where $z = x + iy$, then

- $\varphi(x+iy) = (x+iy) - \frac{(x+iy)^3}{3}$

$$= x + iy - \frac{1}{3}\left(x^3 + x^2(iy) + x(iy)^2 + (yi)^3\right)$$

$$= x + iy - \frac{1}{3}\left(x^3 + ix^2y - xy^2 - iy^3\right)$$

$$= x - \frac{x^3}{3} - \frac{xy^2}{3} + i\left(y - \frac{x^2y}{3} + \frac{y^3}{3}\right)$$

$$\begin{cases} U(x,y) = x\left(1 - \frac{y^2}{3}\right) - \frac{x^3}{3} \\ V(x,y) = y\left(1 - \frac{x^2}{3}\right) + \frac{y^3}{3} \end{cases}$$

Evaluating in the critical points $(x = \pm 1, 0)$

- $V(1,0) = 0$ for $z_0 = 1$.
- $V(-1,0) = 0$ for $z_0 = -1$.

$(*)$ $\theta_0 = 0$ and $\theta_1 = \pi$. $\cos(\theta_i m + \pi)$
- $\cos(0+\pi) = -1$
- $\cos(\pi+\pi) = \cos(2\pi) = -1$

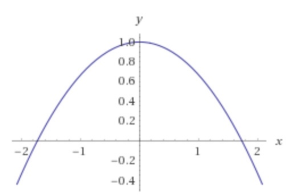
$(**)$ $\theta_0 = \frac{\pi}{2}$ and $\theta_1 = \frac{3\pi}{2}$. $\cos(\theta_i m + 0)$
- $\cos(\pi/2) = 0$
- $\cos(3\pi/2) = 0$

In a neighbourhood of $z_0 = 1$ the curves of steepest ascent/descent are

a) $y = 0$

b) $1 - \frac{x^2}{3} - \frac{y^2}{3} = 0 \to 3 - x^2 - y^2 = 0 \to 3 - x^2 = y^2$

Then $*$ $U(x,y)\big|_{y=0} = x - \frac{x^3}{3} = x\left(1 - \frac{x^2}{3}\right)$



Steepest Descent

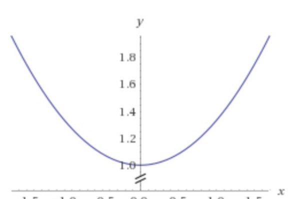
and then the term $\left(1 - \frac{x^2}{3}\right)$ is ...

$*$ $U(x,y)\Big|_{1-\frac{x^2}{3}+y^2=0} = x\left(1 - \frac{(3-x^2)}{3}\right) - \frac{x^3}{3} = x\left[1 - 1 + \frac{x^2}{3}\right] - \frac{x^3}{3} = 0$

Analogusly for $z_0 = -1$. The curves of steepest ascent/descent are

$*$ $U(x,y)\Big|_{y=0} = x\left(1 - \frac{x^2}{3}\right)$

with $Re(z_0) = -1$.



Steepest Ascent.

2. $\varphi(z) = \cosh(z) - \frac{z^2}{2}$

We start with the φ derivatives:

- $\varphi'(z) = \sinh(z) - z$ and it is zero when $z=0$, then it is the critical point
- $\varphi''(z) = \cosh(z) - 1$, $\varphi^{(3)}(z) = \sinh(z)$ and $\varphi^{(4)}(z) = \cosh(z)$ in $z=1$

$\varphi^{(4)}(0) = 0$ then φ is the order $m=4$.

Therefore $\varphi^{(4)} = \cosh(z) e^{i\phi}$ and $\phi = 0$. Then the directions where $V = cte$ are present when $\cos(m\theta + \phi) = 0$, so $m\theta + \phi = (2k+1)\pi$ the steepest descent with $k = 0, 1, 2, 3$, and

- $m\theta_1 = (2(0)+1)\pi \rightarrow \boxed{\theta_1 = \frac{\pi}{4}} \rightarrow \cos(\pi) = -1$
- $\theta_2 = \frac{(2+1)}{4}\pi \rightarrow \boxed{\theta_2 = \frac{3\pi}{4}} \rightarrow \cos(3\pi) = -1$
- $\boxed{\theta_3 = \frac{5}{4}\pi} \rightarrow \cos(5\pi) = -1$ - $\boxed{\theta_4 = \frac{7}{4}\pi} \rightarrow \cos(7\pi) = -1.$

Let $z = x+iy$, then
$$\begin{aligned}\varphi(z) &= \cosh(x+iy) - \frac{(x+iy)^2}{2}\\ &= \cosh(x)\cos(y) + i\sinh(x)\sin(y) - \frac{(x^2 + 2ixy - y^2)}{2}\\ &= \cosh(x)\cos(y) - \frac{(x^2-y^2)}{2} + i\left[\sinh(x)\sin(y) - xy\right],\end{aligned}$$

finally
$$u(x,y) := \cosh(x)\cos(y) - \frac{(x^2-y^2)}{2},$$
$$v(x,y) := \sinh(x)\sin(y) - xy.$$

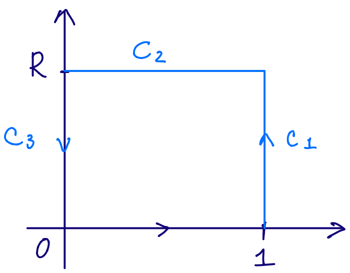
Evaluating in the critical point $z_0 = 0 \rightarrow x = y = 0$

- $V(0,0) = \sinh(0)\sin(0) - 0 = 0$
- $U(0,0) = \cosh(0)\cos(0) - 0 = 1.$

3. Let the integral $I(\lambda) := \int_0^1 e^{i\lambda z}\log(z)dz$ as $\lambda \rightarrow \infty$.

Here $\varphi(z) = iz = i(x+iy) = -y + ix$ then $u(x,y) = -y$ and $v(x,y) = x$, and $\varphi'(z) = i \neq 0$ there are not critical points (saddle points). Then the steepest descent or ascent paths are given by

$- \, V(x,y) = x = cte \qquad - \, u(x,y) = -y$ which is descent for $y>0$.
For that reason when $y>0$, $x = cte$ and they are the steepest descent paths.



We deform the path as no continuous steepest descent path passes through the two endpoints.
Therefore $\displaystyle\int_0^1 e^{i\lambda z}\log(z)\,dz = \int_{C_1+C_2+C_3} e^{i\lambda z}\log(z)\,dz$

* For c_2: We will take $z = x+iR$ then

$$\int_{C_2} e^{i\lambda(x+iR)}\log(x+iR)\,dx = e^{-\lambda R}\int_{C_2} e^{i\lambda x}\log(x+iR)\,dx \, , \text{ and}$$

$$\left|\int_{C_2} e^{i\lambda z}\log(z)\,dz\right| \leq e^{-R\lambda}\int_0^1 |\log(x+iR)|\,dx \text{ then } \left|\int_{C_2} e^{i\lambda z}\log(z)\,dz\right| \to 0 \text{ when}$$

$R \to \infty$.

* For c_1: We will take $z = 1+iy \to dz = i\,dy$, then

$$\int_{C_1} e^{i\lambda z}\log(z)\,dz = \int_0^R i\log(1+iy)e^{i\lambda(1+iy)}\,dy = ie^{i\lambda}\int_0^R e^{-y\lambda}\log(1+iy)\,dy$$

* For C_3: We will take $z = iy \to dz = i\,dy$, then

$$\int_{C_3} e^{i\lambda z}\log(z)\,dz = -i\int_0^R e^{-y\lambda}\log(iy)\,dy \overset{R\to\infty}{=} -i\int_0^\infty e^{-\lambda y}\,\mathrm{Log}(iy)\,dy$$

Thus

$$I(\lambda) = ie^{i\lambda}\int_0^R e^{-y\lambda}\log(1+iy)\,dy - i\int_0^\infty e^{-\lambda y}\,\mathrm{Log}(iy)\,dy = ie^{i\lambda}I_1(\lambda) - i\,I_2(\lambda)$$

$$* \quad I_2(\lambda) = i\int_0^{\infty}\left(\log(y) + i\frac{\pi}{2}\right)e^{-\lambda y}\,dy = i\int_0^{\infty}\log(y)e^{-\lambda y}\,dy - \frac{\pi}{2}\int_0^{\infty}e^{-\lambda y}\,dy$$

$$= -\frac{\pi}{2\lambda} + i\int_0^{\infty}\log(y)e^{-\lambda y}\,dy$$

$$y = \frac{\omega}{\lambda} \rightarrow dy = \frac{d\omega}{\lambda}$$

$$= -\frac{\pi}{2\lambda} + \frac{i}{\lambda}\underbrace{\int_0^{\infty}e^{-\omega}\log(\omega)\,d\omega}_{\gamma := \text{Euler-Mascheroni Constant}} - \frac{i}{\lambda}\log(\lambda)$$

So $I_2(\lambda) = -\frac{1}{\lambda}\left[\frac{\pi}{2} + i\log(\lambda) + i\gamma\right]$.

R

$$* \quad I_1(\lambda) = ie^{i\lambda}\int_0 \left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(iy)^n}{n}\right]e^{-\lambda y}\,dy \sim e^{i\lambda}\sum_{n=1}^{\infty}\left(\frac{(-i)^{n+1}\Gamma(n+1)}{n\,\lambda^{n+1}}\right)$$

Thus $I(\lambda) \sim -i\frac{\log(\lambda)}{\lambda} - \frac{\pi}{2\lambda} - \frac{i\gamma}{\lambda} - e^{i\lambda}\sum_{n=1}^{\infty}\left(\frac{(-i)^{n+1}\Gamma(n+1)}{n\,\lambda^{n+1}}\right)$