Home Assignment 1, SF1861 Optimization, 2019

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Questions about the home assignment that may be of general interest should be posed in the discussion forum on canvas, so that everyone can get the same info.
This assignment should be done in groups of 2 persons each. Note that it is important for the results to be registered on both students, that you join one of the predefined groups for home assignment 1, and submit your report through the group. If you are looking for someone to cooperate with, we recommend that you use the discussion forum to find a coworker.
If the report is handed in before, or on, April 10, the home assignment points will be transformed to bonus points for the exams of this year.

Some groups may be choosen for separate oral presentation. This is notified by email, so make sure to regularly check for it. A correct solution and clear report will grant 4 home assignment points.

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We have used matrix nr: A1 (see question 1 below)

This home assignment will focus on linear algebra and aims to familiarize some concepts like linear subspaces, row spaces, nullspaces, orthogonal complements and bases of subspaces. These concepts will be used later in the course to describe feasible directions for problems with linear constraints, tangent planes of nonlinear constraints *et.c.* Furthermore, some network flow problems are considered.
The theory needed to complete the home assignment will be presented in lectures 1-5 and exercise session 1-2.
It is allowed/recommended to use computer aids to perform your calculations. We will offer some hints about useful Matlab commands, but you are welcome to use other software.

1. Select a 7×5–matrix $\mathbf{A} = \mathbf{A}_k$ on page 4, using the k defined next.

Let $d \in \{1, 2, \ldots, 31\}$ be the day of the month that the **oldest** member of the group is born on, and let $k \in \{0, 1, \ldots, 14\}$ be the rest that is obtained when dividing d

with 15. That is

$$k = \begin{cases} d & \text{if } k \le 14, \\ d - 15 & \text{if } 15 \le k \le 29, \\ d - 30 & \text{if } 30 \le k. \end{cases}$$

(Example:
If the oldest group member is born on the 8:th, choose matrix $\mathbf{A}_8$.
If the oldest group member is born on the 19:th, choose matrix $\mathbf{A}_4$.)

All answers should be given with two decimals, unless it is an integer. Do **not** use format `rat` in matlab.

(*a*) Transform the matrix $\mathbf{A}$ to a reduced staircase form using Gauss-Jordans method, and fill in the matrix below.
In Matlab the function `rref` may be useful.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 1.5 & -1.25 \\ 0 & 1 & 0 & -0.67 & 11.33 \\ 0 & 0 & 1 & 0.17 & -4.08 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ & & & & \\ & & & & \end{bmatrix}.$$

Given this matrix $\mathbf{T}$, determine by hand, without software, two matrices $\mathbf{V}_1$ and $\mathbf{V}_2$, such that the columns in $\mathbf{V}_1$ forms a base for the subspace $\mathcal{R}(\mathbf{A})$, and the columns in $\mathbf{V}_2$ forms a base for the subspace $\mathcal{N}(\mathbf{A})$.
Enter the values below: (cancel the entries you do not need)

$$\mathbf{V}_1 = \begin{bmatrix} 5 & 1 & 1 & & \\ 6 & 3 & 6 & & \\ 2 & 3 & 6 & & \\ 7 & 4 & 7 & & \\ 1 & 2 & 5 & & \\ 1 & 1 & 1 & & \\ 6 & 2 & 2 & & \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} -1.5 & 1.25 & & & \\ 0.67 & -11.33 & & & \\ -0.17 & 4.08 & & & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & & & \\ & & & & \end{bmatrix}$$

(*b*) Transform the matrix $\mathbf{A}^\mathsf{T}$ to reduced staircase form again and fill in the matrix below.

$$\tilde{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given this matrix $\tilde{\mathbf{T}}$, determine by hand, without software, two matrices $\mathbf{W}_1$ and $\mathbf{W}_2$ such that the columns in $\mathbf{W}_1$ forms a base for the subspace $\mathcal{R}(\mathbf{A}^\mathsf{T})$, and the columns in $\mathbf{W}_2$ forms a base for the subspace $\mathcal{N}(\mathbf{A}^\mathsf{T})$.
Enter the values below: (cancel the entries you do not need)

$$\mathbf{W}_1 = \begin{bmatrix} 5 & 6 & 2 & & \\ 1 & 3 & 3 & & \\ 1 & 6 & 6 & & \\ 7 & 8 & 2 & & \\ 1 & 2 & 7 & & \\ & & & & \\ & & & & \end{bmatrix}, \quad \mathbf{W}_2 = \begin{bmatrix} -1 & 1 & -1 & -2 & \\ 0 & -1 & 1 & 1 & \\ -1 & 0 & -1 & -1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{bmatrix}$$

(*c*) We know that $\mathcal{R}(\mathbf{A}^\mathsf{T})$ and $\mathcal{N}(\mathbf{A})$ are the orthogonal complements of each others in $I\!R^5$, so an arbitrary vector in $I\!R^5$ can be written as a sum of a vector in the row space of $\mathbf{A}^\mathsf{T}$ and a vector in the nullspace of $\mathbf{A}$.

Let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^\mathsf{T}$ where x_i = the i:th digit in the oldest group members 10-digit personal number. Given this vector $\mathbf{x}$, determine two vectors $\mathbf{y}$ and $\mathbf{z}$ such that $\mathbf{y} \in \mathcal{R}(\mathbf{A}^\mathsf{T})$, $\mathbf{z} \in \mathcal{N}(\mathbf{A})$ and $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

Hint: Use the matrices $\mathbf{W}_1$ and $\mathbf{V}_2$ above (but not the rounded numbers): Each vector $\mathbf{y} \in \mathcal{R}(\mathbf{A}^\mathsf{T})$ can be written on the form $\mathbf{y} = \mathbf{W}_1\mathbf{t}$ for some vector $\mathbf{t}$, and every vector $\mathbf{z} \in \mathcal{N}(\mathbf{A})$ can be written on the form $\mathbf{z} = \mathbf{V}_2\,\mathbf{s}$ for some vector $\mathbf{s}$. Determine $\mathbf{t}$ and $\mathbf{s}$ (using software) from the equation system $\mathbf{W}_1\mathbf{t} + \mathbf{V}_2\,\mathbf{s} = \mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} 8 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3.66 \\ 0.68 \\ 0.71 \\ 5.15 \\ 0.30 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 4.34 \\ 1.32 \\ -0.71 \\ -3.15 \\ -0.30 \end{bmatrix}, \quad \mathbf{y}^\mathsf{T}\mathbf{z} = \text{..0..}$$

(*d*) Now let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^\mathsf{T}$, where x_i = is the i:th digit in the oldest group members 10-digit personal number. Given this vector $\mathbf{x}$, determine two vectors $\mathbf{y}$ and $\mathbf{z}$ such that $\mathbf{y} \in \mathcal{R}(\mathbf{A})$, $\mathbf{z} \in \mathcal{N}(\mathbf{A}^\mathsf{T})$ and $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

$$\mathbf{x} = \begin{bmatrix} 8 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 4.38 \\ -1.25 \\ 1.25 \\ -0.38 \\ 0.38 \\ 1.88 \\ -2.75 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 3.63 \\ 3.25 \\ -1.25 \\ 2.38 \\ -0.38 \\ -0.88 \\ 2.75 \end{bmatrix}, \quad \mathbf{y}^\mathsf{T}\mathbf{z} = \text{..0..}$$

$$\mathbf{A}_0 = \begin{pmatrix} 4 & 1 & 1 & 7 & 1 \\ 5 & 3 & 6 & 8 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 6 & 4 & 7 & 9 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 5 & 2 & 2 & 8 & 7 \end{pmatrix} \quad \mathbf{A}_1 = \begin{pmatrix} 5 & 1 & 1 & 7 & 1 \\ 6 & 3 & 6 & 8 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 7 & 4 & 7 & 9 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 6 & 2 & 2 & 8 & 7 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 6 & 1 & 1 & 7 & 1 \\ 7 & 3 & 6 & 8 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 8 & 4 & 7 & 9 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 7 & 2 & 2 & 8 & 7 \end{pmatrix}$$

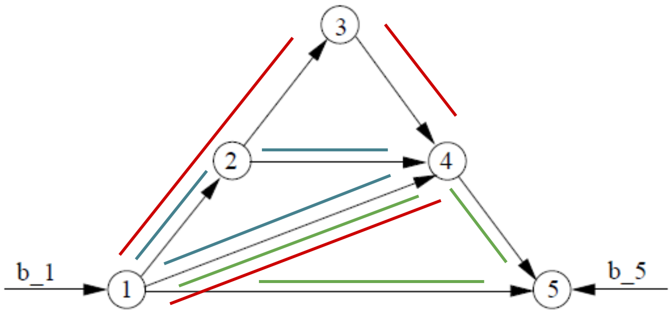
$$\mathbf{A}_3 = \begin{pmatrix} 2 & 1 & 1 & 5 & 1 \\ 3 & 3 & 6 & 6 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 4 & 4 & 7 & 7 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 3 & 2 & 2 & 6 & 7 \end{pmatrix} \quad \mathbf{A}_4 = \begin{pmatrix} 3 & 1 & 1 & 5 & 1 \\ 4 & 3 & 6 & 6 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 5 & 4 & 7 & 7 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 4 & 2 & 2 & 6 & 7 \end{pmatrix} \quad \mathbf{A}_5 = \begin{pmatrix} 4 & 1 & 1 & 5 & 1 \\ 5 & 3 & 6 & 6 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 6 & 4 & 7 & 7 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 5 & 2 & 2 & 6 & 7 \end{pmatrix}$$

$$\mathbf{A}_6 = \begin{pmatrix} 2 & 1 & 1 & 6 & 1 \\ 3 & 3 & 6 & 7 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 4 & 4 & 7 & 8 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 3 & 2 & 2 & 7 & 7 \end{pmatrix} \quad \mathbf{A}_7 = \begin{pmatrix} 3 & 1 & 1 & 6 & 1 \\ 4 & 3 & 6 & 7 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 5 & 4 & 7 & 8 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 4 & 2 & 2 & 7 & 7 \end{pmatrix} \quad \mathbf{A}_8 = \begin{pmatrix} 4 & 1 & 1 & 6 & 1 \\ 5 & 3 & 6 & 7 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 6 & 4 & 7 & 8 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 5 & 2 & 2 & 7 & 7 \end{pmatrix}$$

$$\mathbf{A}_9 = \begin{pmatrix} 5 & 1 & 1 & 6 & 1 \\ 6 & 3 & 6 & 7 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 7 & 4 & 7 & 8 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 6 & 2 & 2 & 7 & 7 \end{pmatrix} \quad \mathbf{A}_{10} = \begin{pmatrix} 2 & 1 & 1 & 7 & 1 \\ 3 & 3 & 6 & 8 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 4 & 4 & 7 & 9 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 3 & 2 & 2 & 8 & 7 \end{pmatrix} \quad \mathbf{A}_{11} = \begin{pmatrix} 3 & 1 & 1 & 7 & 1 \\ 4 & 3 & 6 & 8 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 5 & 4 & 7 & 9 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 4 & 2 & 2 & 8 & 7 \end{pmatrix}$$

$$\mathbf{A}_{12} = \begin{pmatrix} 2 & 1 & 1 & 3 & 1 \\ 3 & 3 & 6 & 4 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 4 & 4 & 7 & 5 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 3 & 2 & 2 & 4 & 7 \end{pmatrix} \quad \mathbf{A}_{13} = \begin{pmatrix} 2 & 1 & 1 & 4 & 1 \\ 3 & 3 & 6 & 5 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 4 & 4 & 7 & 6 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 3 & 2 & 2 & 5 & 7 \end{pmatrix} \quad \mathbf{A}_{14} = \begin{pmatrix} 3 & 1 & 1 & 4 & 1 \\ 4 & 3 & 6 & 5 & 2 \\ 2 & 3 & 6 & 2 & 7 \\ 5 & 4 & 7 & 6 & 8 \\ 1 & 2 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 6 \\ 4 & 2 & 2 & 5 & 7 \end{pmatrix}$$

2. In this assignment, a network defined by the figure below is studied. We let x_{ij} denote the flow in the arc (i, j) and the external flow to the nodes is denoted b_i, for $i = 1, \dots, 5$. In the figure we have only included two such external flows. We do not require that the above flows x_{ij} are positive, if, for example, x_{12} is negative it means that the flow in arc $(1, 2)$ goes from node 2 to node 1. In the same way, the flow b_1 is interpreted as an external flow in to node 1 if $b_1 > 0$ and flow out if $b_1 < 0$.



The flow balance in the five nodes are described by the equation $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_{12} \\ x_{14} \\ x_{15} \\ x_{23} \\ x_{24} \\ x_{34} \\ x_{45} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

(a) Transform $\mathbf{A}$ to a reduced staircase form, and by hand, without software, find a base $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for the nullspace $\mathcal{N}(\mathbf{A})$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) These basis vectors corresponds to so called "cyclic flows", *i.e.*, flows that have neither a source or a sink, but rather just circulate in the network. Which cyclic flows do your bases correspond to? Draw these in the network graph above, *using three different colors*!

(c) Transform $\mathbf{A}^\mathsf{T}$ to a reduced staircase form, and determine by hand, without software, a base for the nullspace $\mathcal{N}(\mathbf{A}^\mathsf{T})$. What is that base? . {(1, 1, 1, 1, 1)}.

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(d) Let $\mathbf{B}$ be the matrix you get by cancelling the last row in the matrix $\mathbf{A}$. Repeat the (a)-part and (c)-part with $\mathbf{B}$ instead of $\mathbf{A}$.
Answer to (a)-part: . ...

. ...

. ...

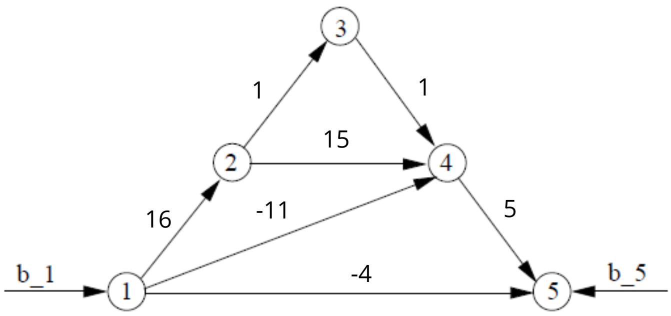
Answer to (c)-part: . ...

. ...

(e) Let $\mathbf{b} = (1, 0, 0, 0, -1)^\mathsf{T}$ and $\mathbf{x}_0 = (0, 0, 1, 0, 0, 0, 0)^\mathsf{T}$. Is $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$? Answer: ....
Let $t_1, \ldots, t_6$ be the first six digits of the youngest group members 10-digit personal number, and determine (with or without software)

$$\mathbf{x} = \mathbf{x}_0 + (t_1{+}t_2)\mathbf{v}_1 + (t_3{+}t_4)\mathbf{v}_2 + (t_5{+}t_6)\mathbf{v}_3.$$

Write the flow that corresponds to this vector $\mathbf{x}$ in the network graph below, *i.e.*, write on each arc in the graph how much flow that goes through it (with sign). Does it hold for your vector $\mathbf{x}$ that $\mathbf{A}\mathbf{x} = \mathbf{b}$? Answer:.................



Prove that with $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as above it holds that the vector

$$\mathbf{x} = \mathbf{x}_0 + (t_1{+}t_2)\mathbf{v}_1 + (t_3{+}t_4)\mathbf{v}_2 + (t_5{+}t_6)\mathbf{v}_3$$

satisfies $\mathbf{A}\mathbf{x} = \mathbf{b}$ for all values of $t_1, \ldots, t_6$, even non-integer and negative. Test first numerically that it seems to hold.

Proof: (use only these rows!) ....

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Good luck!