Infinite impulse response filtering

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Técnicas Digitales III

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Resumen

1. Classification of discrete filters
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Table: Classification of discrete filters

	Finite impulse response (FIR)	Infinite impulse response (IIR)
Filtering in time domain	Moving average	Leaky Integrator
Filtering in frequency domain	Windowed Filters Equiripple Minimax	Bilinear z-transform

Leaky integrator filter

The MA filter equation,

$$y[n] = x[n] * h[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]\,, \tag{1}$$

$$y[n] = \frac{1}{M} \left[\sum_{k=1}^{M-1} x[n-k] + x[n] \right]\,. \tag{2}$$

Since,

$$y[n-1] = \frac{1}{M-1} \left[\sum_{k=1}^{M-1} x[n-k] \right]\,. \tag{3}$$

Then,

$$y[n] = \frac{1}{M} x[n] + \frac{M-1}{M} y[n-1]\,. \tag{4}$$

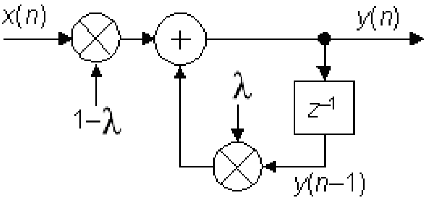
Defining $\lambda = \frac{M-1}{M}$,

$$y[n] = \lambda\, y[n-1] + (1-\lambda)\, x[n]\,. \tag{5}$$

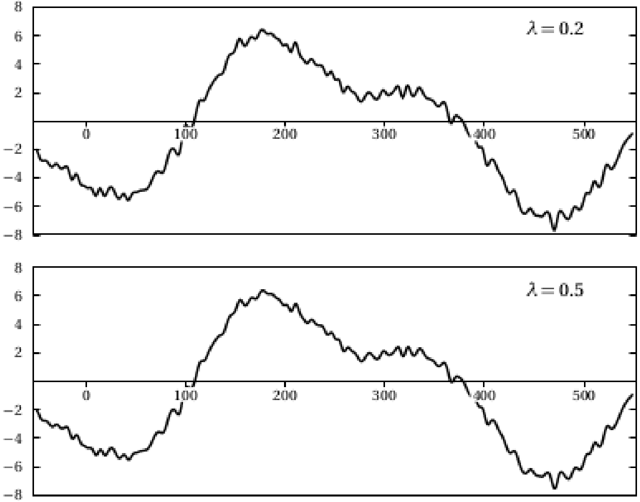
It can be seen that the leaky integrator filter is an IIR filter. Why?

$$y[n] = \lambda\, y[n-1] + (1-\lambda)\, x[n]\,.$$

- No longer a convolution.
- Instead, a *constant coefficient difference equation*. Initial conditions must be set.
- The new system is LTI [2].
- System is stable for $|\lambda| < 1$.
- The value of λ (which is the pole of the system) determines the smoothing power of the filter.

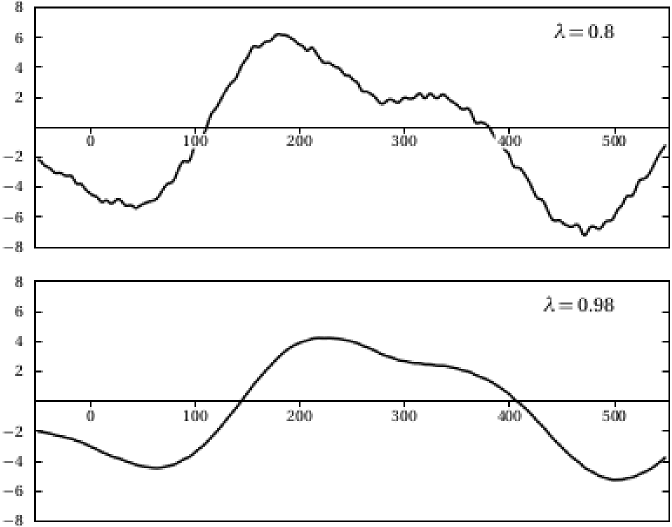


Noise Reduction

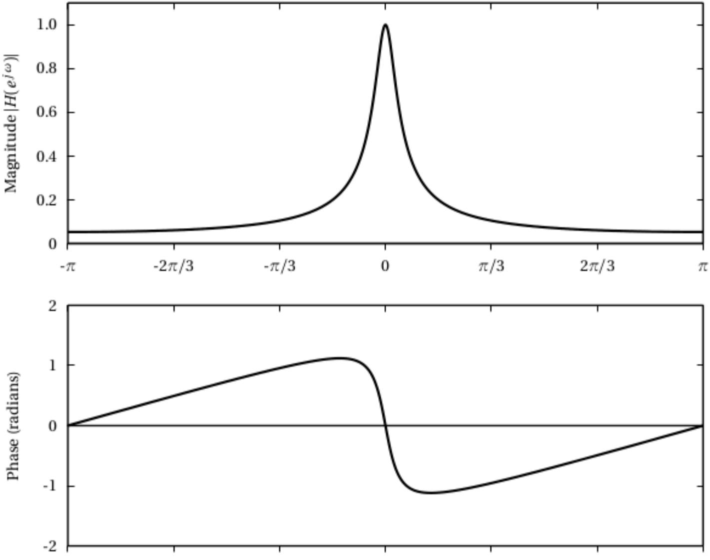


Noise Reduction

- Note how the signal is delayed as λ grows.



Magnitude and phase response of the leaky integrator for $\lambda = 0.9$.



The technique is an algebraic transformation between variables s and z.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right). \tag{6}$$

Solving for z,

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}. \tag{7}$$

Doing $s = j\Omega$, where Ω is the analog frequency, $-\infty, < \Omega < \infty$,

$$z = \frac{1 + (T_d/2)j\Omega}{1 - (T_d/2)j\Omega}. \tag{8}$$

The relationship between Ω and ω, the "digital" frequency, $-\pi, < \omega < \pi$, can be found by replacing $z = e^{j\omega}$ in Eq. 6,

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j\sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = j\frac{2}{T_d}\tan(\omega/2). \tag{9}$$

Bilinear transform (2)

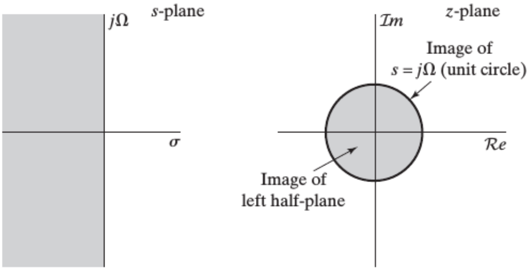
Real and imaginary parts on both sides of Eq. 9 are,

$$\sigma = 0\,, \tag{10}$$

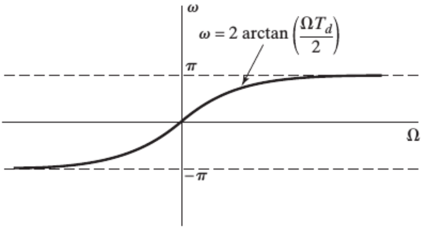
$$\Omega = \frac{2}{T_d}\tan(\omega/2)\,. \tag{11}$$

Or,

$$\omega = \arctan(\Omega T_d/2)\,. \tag{12}$$

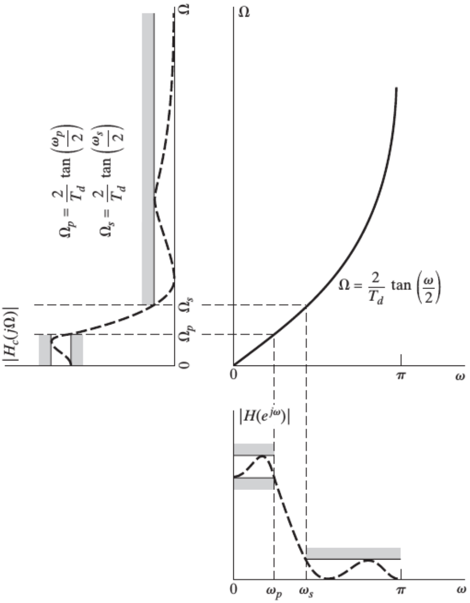


jΩ
s-plane
σ
Im
z-plane
Image of
s = jΩ (unit circle)
Re
Image of
left half-plane

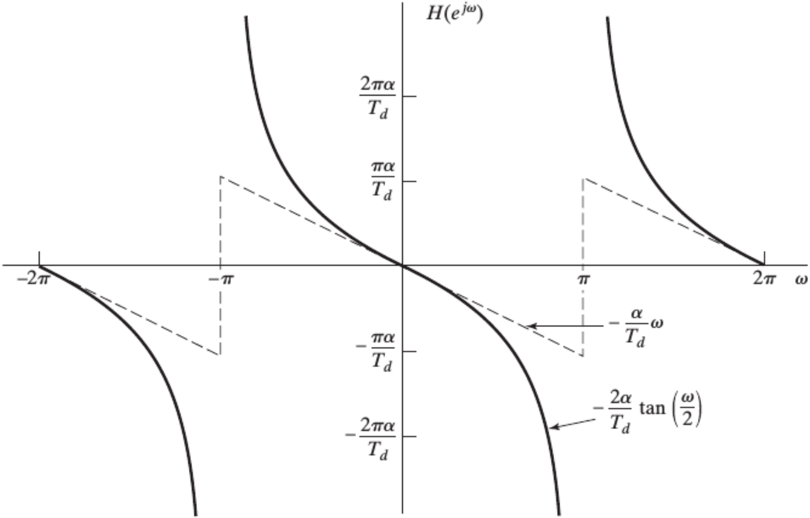


ω
ω = 2 arctan (ΩT_d / 2)
π
Ω
−π

- Non-linear compression of the frequency axis.
- The design of discrete-time filters using the bilinear transformation is useful only when this compression can be tolerated or compensated for.



Suppose a continuous-time filter with linear phase response. The nonlinear warping of the frequency axis introduced by the bilinear transformation will not preserve linearity in phase response.



Example of IIR design using bilinear transform

Design a digital filter equivalent of a 2nd order Butterworth low-pass filter with a cut-off frequency $f_c = 100$ Hz and a sampling frequency $f_s = 1000$ samples/sec. Derive the finite difference equation and draw the realisation structure of the filter. Given that the analogue prototype of the frequency-domain transfer function $H(s)$ for a Butterworth filter is:

$$H(s) = \frac{1}{s^2 + \sqrt{2} \cdot s + 1}$$

The normalised cut-off frequency of the digital filter is given by the following equation:

$$\Omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi 100}{1000} = 0.628$$

Now determine the equivalent analogue filter cut-off frequency ω_{ac}, using the pre-warping function of Equation 5.9. The value of K is immaterial so let $K = 1$.

$$\omega_{ac} = K \cdot \tan\left(\frac{\Omega_c}{2}\right) = 1 \cdot \tan\left(\frac{0.628}{2}\right)$$

$$\omega_{ac} = 0.325 \quad rads/\sec$$

Example of IIR design (2)

Now denormalise the frequency-domain transfer function $H(s)$ of the Butterworth filter, with the corresponding low-pass to low-pass frequency transformation of Equation 5.10. Hence the transfer function of the Butterworth filter becomes:

$$H(s)=\frac{1}{\left[\frac{s}{0.325}\right]^2+\sqrt{2}\cdot\left[\frac{s}{0.325}\right]+1}$$

Next, convert the analogue filter into an equivalent digital filter by applying the bilinear z-transform. This is achieved by making a substitution for s in the transfer function.

$$s=\frac{z-1}{z+1}\equiv\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z)=\frac{1}{\frac{1}{0.325^2}\cdot\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2+\frac{\sqrt{2}}{0.325}\cdot\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1}$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{0.067+0.135z^{-1}+0.067z^{-2}}{1-1.1429z^{-1}+0.4127z^{-2}}$$

The finite difference equation of the filter is found by inverting the transfer function.

$$y(n)=1.1429y(n-1)-0.4127y(n-2)+0.067x(n)+0.135x(n-1)+0.067x(n-2)$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + ... + b_{N-1} z^{N-1}}{1 + a_1 z^{-1} + ... + a_{M-1} z^{M-1}}$$

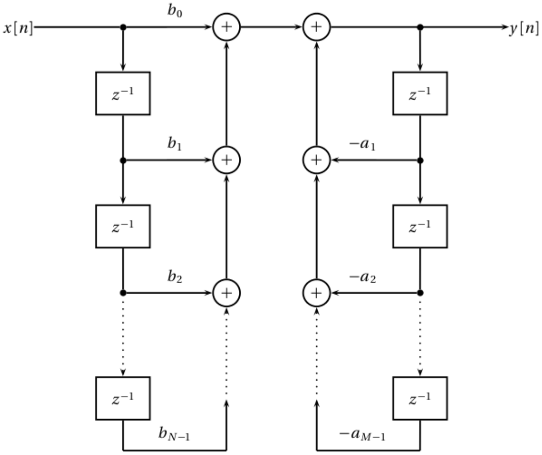


Figure 7.24 Direct Form implementation of an IIR filter.

Direct form I IIR implementation inverted

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

By the commutative properties of the z-transform, we can invert the order of computation to turn the Direct Form I structure into a new structure.

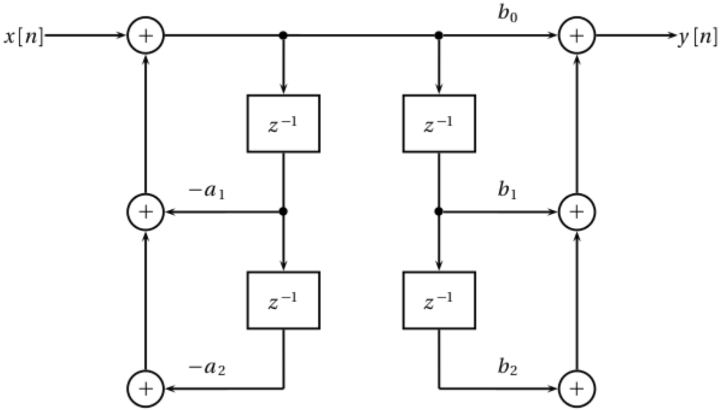


Figure 7.25 Direct form I with inverted order.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

We can then combine the parallel delays together. This implementation is called Direct Form II; its obvious advantage is the reduced number of the required delay elements (hence of memory storage).

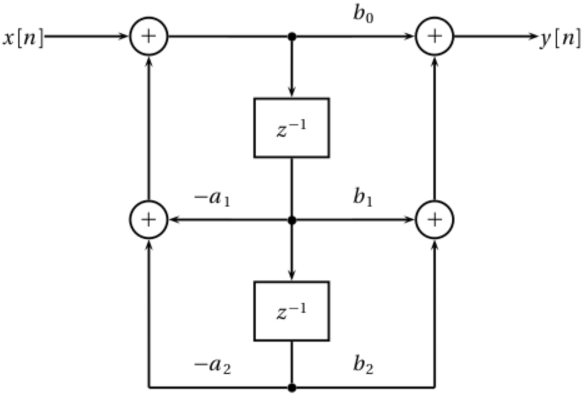


Figure 7.26 Direct Form II implementation of a second-order section.

IIR cascade implementation

The cascade structure of N second-order sections is much less sensitive to quantization than the previous Direct form II of order $2 \cdot N$.

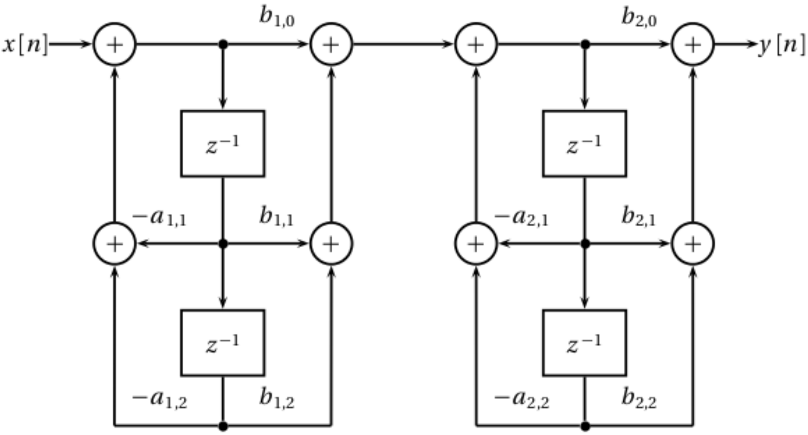


Figure 7.27 4th order IIR: cascade implementation.

FIR vs IIR

FIR, pros:

- Unconditional stability (no poles).
- Precise control of the phase response and, in particular, exact linear phase.
- Optimal algorithmic design procedures.
- Robustness with respect to finite numerical precision hardware.

FIR, cons:

- Longer input-output delay.
- Higher computational cost with respect to IIR solutions.

IIR, pros:

- Lower computational cost with respect to an FIR with similar behavior.
- Shorter input-output delay.
- Compact representation.

IIR, cons:

- Stability is not guaranteed.
- Phase response is difficult to control.
- Design is complex in the general case.
- Sensitive to numerical precision.

Bibliography

1 Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing, 3rd Ed*. Prentice Hall. 2010. Sections 7.2 and 7.3.

2 Paolo Prandoni and Martin Vetterli. Signal processing for communications. Taylor and Francis Group, LLC. 2008. Sections 5.3.2, 7.3, and 7.4.2.

3 Oliver Hinton. Digital Signal Processing Resources for EEE305 Course. Chapter 5. www.staff.ncl.ac.uk/oliver.hinton/eee305/