Digital processing of analog signals

Dr. Ing. Rodrigo Gonzalez

rodralez@frm.utn.edu.ar

Técnicas Digitales III

Universidad Tecnológica Nacional,
Facultad Regional Mendoza.

Summary

Periodic sampling

The discrete-time representation of a continuous-time signal is obtained through periodic sampling from a continuous-time signal $x_c(t)$ according to,

$$x[n] = x_c(nT), \qquad -\infty < n < \infty\,, \tag{1}$$

where T is the sampling period, and $f_s = 1/T$ is the sampling frequency, or $\Omega_s = 2\pi/T$ in radians/s

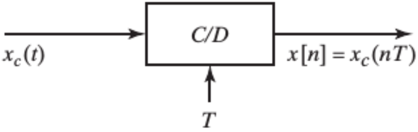


Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.

Sampling process

It is convenient to represent the sampling process mathematically in the two stages.

1. An impulse train $s(t)$ is multiplied by a continuous-time signal $x_c(t)$.
2. The continuous-time signal $x_s(t)$ is transformed to a discrete-time sequence $x[n]$.

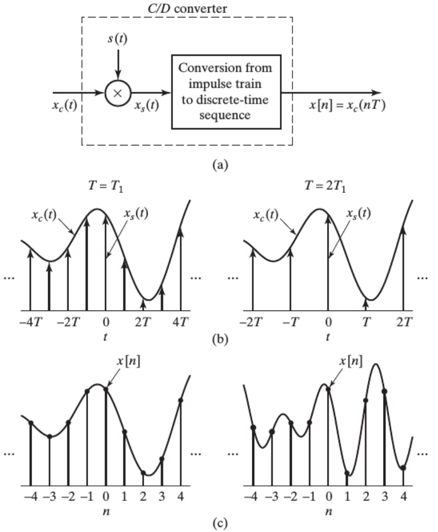


Figure 4.2 Sampling with a periodic impulse train, followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.

Frequency-domain representation of sampling

$x_s(t)$ is obtained multiplying $x_c(t)$ by a periodic impulse train $s(t)$,

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)\,, \tag{2}$$

$$x_s(t) = x_c(t)\, s(t)\,, \tag{3}$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)\,, \tag{4}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t - nT) \qquad \text{by sifting property.} \tag{5}$$

The Fourier transform of the periodic impulse train $s(t)$ is the periodic impulse train,

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \qquad \text{where } \Omega_s = \frac{2\pi}{T}\,. \tag{6}$$

The Fourier transform of $x_s(t)$ is the continuous-variable convolution of $X_c(j\Omega)$ and $S(j\Omega)$,

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)\,, \tag{7}$$

$$X_s(j\Omega) = \frac{1}{T} X_c[j(\Omega - k\Omega_s)]\,. \tag{8}$$

Frequency-domain representation of sampling, II

- Fourier transform of $x_s(t)$ consists of periodically repeated copies of $X_c(j\Omega)$
- These copies are shifted by integer multiples of the sampling frequency.
- It is evident that

  $\Omega_s - \Omega_N \geq \Omega_N$, or,

  $\Omega_s \geq 2\Omega_N$

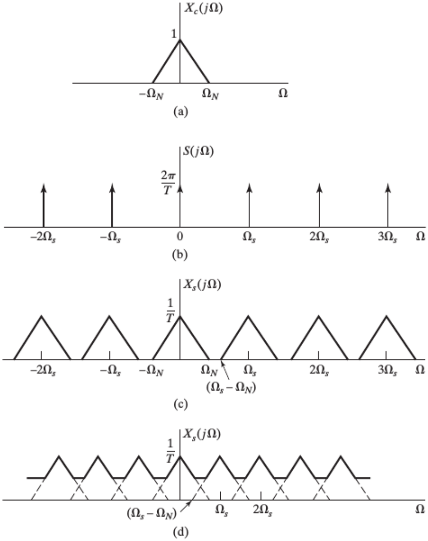


Figure 4.3 Frequency-domain representation of sampling in the time domain. (a) Spectrum of the original signal. (b) Fourier transform of the sampling function. (c) Fourier transform of the sampled signal with $\Omega_S > 2\Omega_N$. (d) Fourier transform of the sampled signal with $\Omega_S < 2\Omega_N$.

Let $x_c(t)$ be a bandlimited signal with,

$$X_c(j\Omega) = 0 \;\text{ para }\; |\Omega| \geq \Omega_N\,. \tag{9}$$

Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm1, \pm2, ...$ si

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N\,. \tag{10}$$

The frequency Ω_N is commonly referred to as the **Nyquist frequency**, and the frequency $2\Omega_N$ as the **Nyquist rate**.

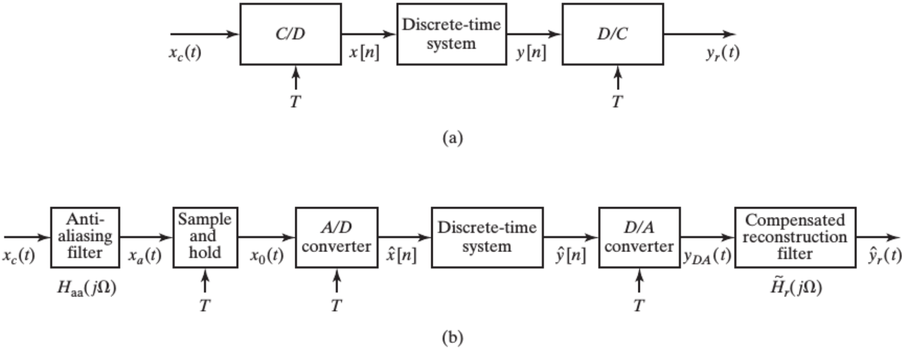


Figure 4.47 (a) Discrete-time filtering of continuous-time signals. (b) Digital processing of analog signals.

Aliasing prefiltering, motivation

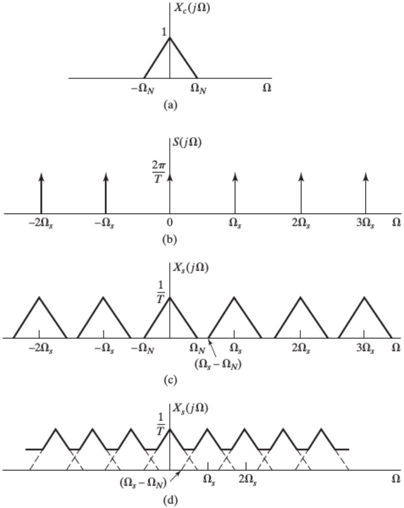


Figure 4.3 Frequency-domain representation of sampling in the time domain. (a) Spectrum of the original signal. (b) Fourier transform of the sampling function. (c) Fourier transform of the sampled signal with $\Omega_S > 2\Omega_N$. (d) Fourier transform of the sampled signal with $\Omega_S < 2\Omega_N$.

Aliasing prefiltering, example

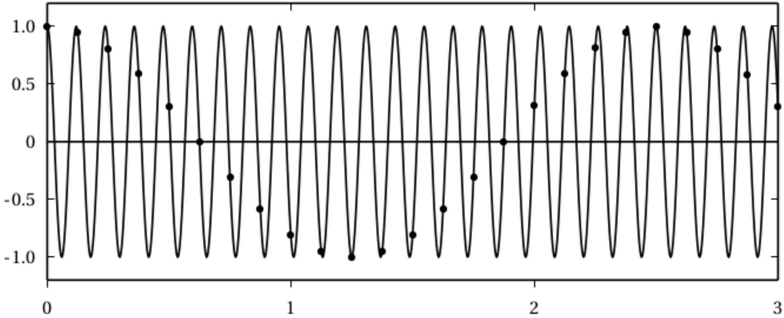


Figure 9.8 Example of aliasing: a sinusoid at 8400 Hz, $x(t) = \cos(2\pi \cdot 8400t)$ (solid line) is sampled at $F_s = 8000$ Hz. The sampled values (dots) are indistinguishable from those of at 400 Hz sinusoid sampled at F_s.

Aliasing prefiltering

- Even if the signal is naturally bandlimited (as music), wideband additive noise may fill in the higher frequency range, and as a result of sampling, these noise components would be aliased into the low-frequency band.
- (Play video).

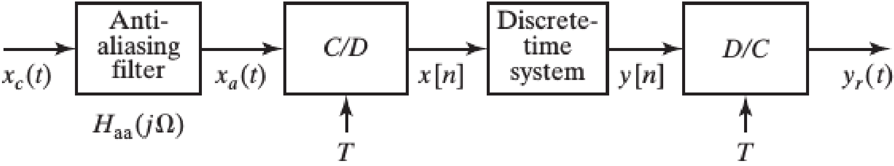


Figure 4.48 Use of prefiltering to avoid aliasing.

Oversampling

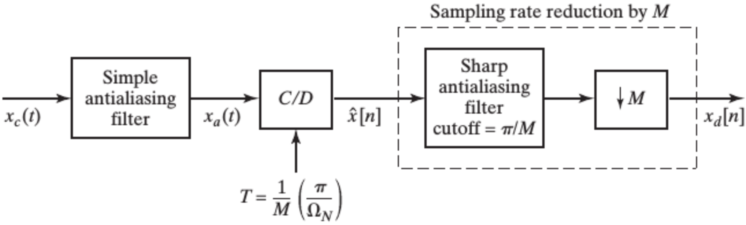


Figure 4.49 Using oversampled A/D conversion to simplify a continuous-time antialiasing filter.

Oversampling frequency response

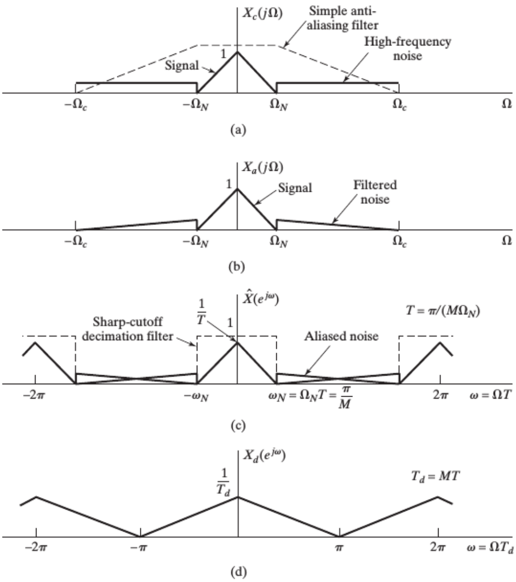
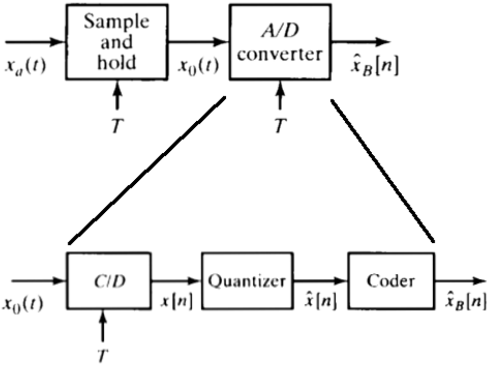


Figure 4.50 Use of oversampling followed by decimation in C/D conversion.

A/D Conversion Stages

- The A/D converter is a physical device that converts a voltage or current amplitude at its input into a binary code representing a quantized amplitude value closest to the amplitude of the input.
- The sample-and-hold stage can be a zero-order-hold.



Quantizer

- Uniformly spaced quantizer.
- The number of quantization levels will be a power of two (2^N).

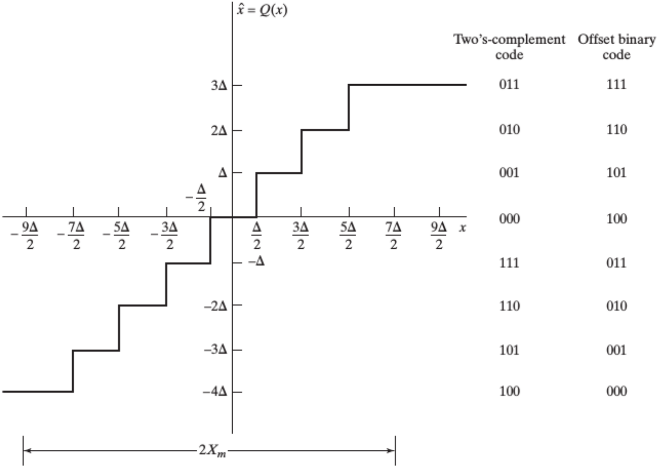


Figure 4.54 Typical quantizer for A/D conversion.

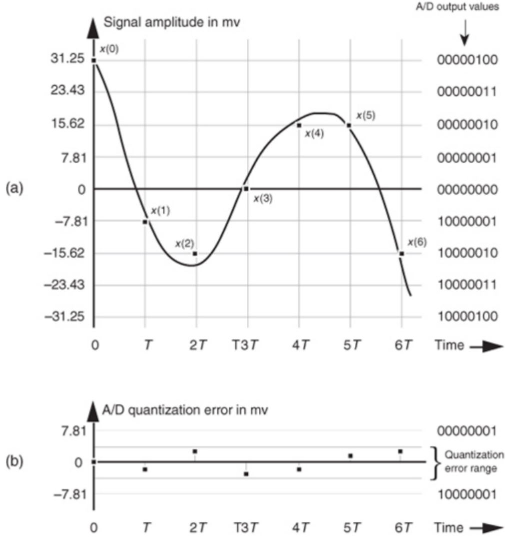

A/D output values
Signal amplitude in mv
31.25
23.43
15.62
7.81
0
–7.81
–15.62
–23.43
–31.25
00000100
00000011
00000010
00000001
00000000
10000001
10000010
10000011
10000100
(a)
x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
0
T
2T
T3T
4T
5T
6T
Time
A/D quantization error in mv
7.81
0
–7.81
(b)
00000001
10000001
Quantization
error range

Figure: 12.1 [2]

Quantizer Error example, II

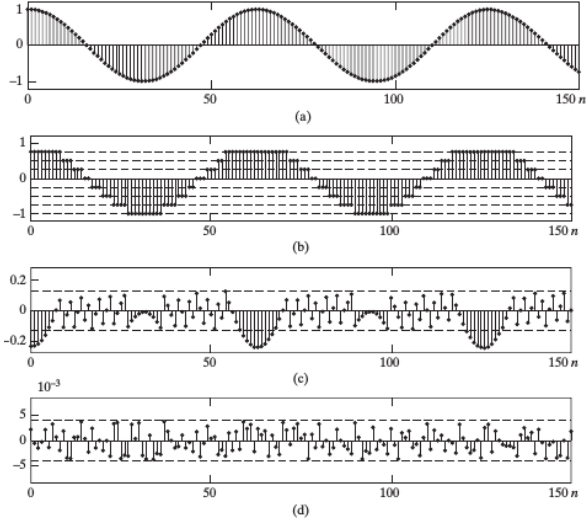


Figure 4.57 (*continued*) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a). (d) Quantization error sequence for 8-bit quantization of the signal in (a).

Quantizer Error model

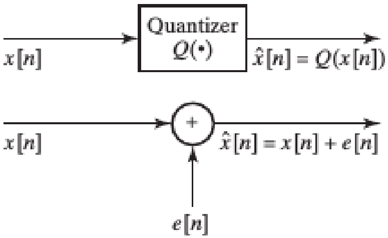


Figure 4.56 Additive noise model for quantizer.

ADC Signal-to-Noise relationship

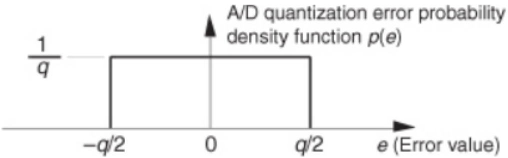
The precision of the quantizer is given by [2]:

$$q = \frac{\text{full voltage range}}{2^{\text{word lenght}}} = \frac{2V_p}{2^B}. \quad [mV] \tag{11}$$

- $SNR = (P_{signal})/(P_{noise})$ relates two powers.
- Since q is defined as a random variable, its power cannot be represented explicitly.
- A statistical version of SNR is used,

$$SNR_{ADC} = 10 \cdot \log_{10}\left(\frac{\text{input signal variance}}{\text{A/D quantization noise variance}}\right), \quad [\text{dB}] \tag{12}$$

$$= 10 \cdot \log_{10}\left(\frac{\sigma^2_{signal}}{\sigma^2_{ADC}}\right). \tag{13}$$



$$\sigma^2_{ADC} = \int_{-q/2}^{q/2} (e-\mu)^2 p(e) de = \int_{-q/2}^{q/2} e^2 p(e) de = \frac{1}{q}\int_{-q/2}^{q/2} e^2 de = \frac{q^2}{12}\,, \tag{14}$$

$$\sigma^2_{ADC} = \left(\frac{2V_p}{2^B}\right)^2 \cdot \frac{1}{12} = \boxed{\frac{V_p^2}{3\cdot 2^{2B}}}\,, \tag{15}$$

$$LF = \frac{rms_{signal}}{V_p} = \frac{\sigma_{signal}}{V_p} \implies \sigma^2_{signal} = \boxed{LF^2 \cdot V_p^2}\,, \quad \text{(Load Factor)} \tag{16}$$

$$SNR_{ADC} = 10\cdot\log_{10}\left[\left(LF^2\cdot V_p^2\right)\cdot\frac{3\cdot 2^{2B}}{V_p^2}\right] = 10\cdot\log_{10}\left[\left(LF^2\cdot 3\cdot 2^{2B}\right)\right]\,, \tag{17}$$

$$= 10\cdot\left[\log_{10}(LF^2) + \log_{10}(3) + 2\log_{10}(2)\cdot B\right]\,, \tag{18}$$

$$= 20\cdot\log_{10}(LF) + 4.77 + 6.02\cdot B\,. \quad \text{[dB]} \tag{19}$$

ADC Signal-to-Noise relationship considerations

$$
\begin{aligned}
SNR_{ADC} &= 20 \cdot \log_{10}(LF) + 4.77 + 6.02 \cdot B \,, \quad [\mathrm{dB}] \\
&= 20 \cdot \log_{10}\left(\frac{rms_{signal}}{V_p}\right) + 4.77 + 6.02 \cdot B \,. \quad [\mathrm{dB}]
\end{aligned}
$$

Considerations about LF:

- Ideally, if $rms_{signal} >> V_p$, SNR_{ADC} increases, but this will produce a severe distortion in the sampling signal (saturation).
- On the other hand, if $rms_{signal} << V_p$, SNR_{ADC} decreases.
- Eq. 19 was obtained for an ideal ADC. Other sources of error should be taken into account.
- Moreover, it was considered that ADC's $V_{MAX} = V_p$.
- Therefore, calculated SNR_{ADC} should be increased by 3 or 6 dB.

Considerations about numbers of bits:

- SNR_{ADC} increases 6 dB by each bit in ADC's quantizer.
- So, the more bits the better, isn't it?

ADC Signal-to-Noise relationship for a sinusoidal signal

For a sinusoidal signal, $rms_{signal} = V_p/\sqrt{2}$.

$$SNR_{ADC} = 20 \cdot \log_{10}\left(\frac{rms_{signal}}{V_p}\right) + 4.77 + 6.02 \cdot B\,, \tag{20}$$

$$= 20 \cdot \log_{10}\left(\frac{V_p/\sqrt{2}}{V_p}\right) + 4.77 + 6.02 \cdot B\,. \tag{21}$$

Thus, the maximum SNR_{ADC} is,

$$SNR_{ADC} = 20 \cdot \log_{10}\left(1/\sqrt{2}\right) + 4.77 + 6.02 \cdot B\,, \tag{22}$$

$$= -3.01 + 4.77 + 6.02 \cdot B\,, \tag{23}$$

$$= 1.76 + 6.02 \cdot B\,. \quad \text{[dB]} \tag{24}$$

ADC resolution for a particular signal

Consider the following example:

- The SNR for an audio output amplifier is 110 dB.
- A 24-bits ADC is chosen to sample the output amplifier (professional audio).

$$SNR_{ADC} = 1.76 + 6.02 \cdot 24 - 3 = 143.24 \text{ dB}\,.$$

- How many bits are used to measure noise? $(143 - 110)/6 \simeq 5.5$ bits!.
- In a control loop, picking a bad ADC resolution could lead to a catastrophic scenario.

Summary: the number of bits in an ADC must match the SNR of the signal to be sampled.

- Rule of thumb: the ADC resolution should be choose in order to provide 6 dB (1 bit) above the SNR of the signal to be sample.
- Additional bits (noisy bits) can be eliminated by right shifting (In C: `adc_read »= 5`).

Ideal D/C Converter

A bandlimited signal can be reconstructed from a sequence of samples using ideal lowpass filtering.

In terms of Fourier transforms, the reconstruction is represented as,

$$X_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega)\,. \tag{25}$$

where $X(e^{j\Omega T})$ X is the DTFT of the sequence of samples and $X_r(j\Omega)$ is the Fourier transform of the reconstructed continuous-time signal.

The ideal reconstruction filter is

$$H_r(j\Omega) = \begin{cases} T & , |\Omega| < \pi/T \\ 0 & , |\Omega| > \pi/T \end{cases}\,. \tag{26}$$

A physically realizable counterpart to the ideal D/C converter is a digital-to-analog converter (D/A converter) followed by an analog lowpass filter.

A D/A converter takes a sequence of binary code words $\hat{x}_B[n]$ as its input and produces a continuous-time output of the form

$$x_{DA}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n] h_0(t - nT) \,. \tag{27}$$

where h_0 is the impulse response of the zero-order hold (ZOH) given by,

$$h_0(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \,. \tag{28}$$

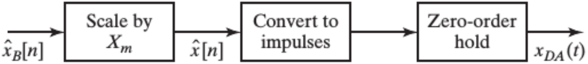


Figure 4.62 Block diagram of D/A converter.

$x_{DA}(t)$ must take into account the quantization noise $e_0(t)$ as,

$$x_{DA}(t) = x_0(t) + e_0(t)\,, \tag{29}$$

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t - nT)\,, \tag{30}$$

$$e_0(t) = \sum_{n=-\infty}^{\infty} e[n]h_0(t - nT)\,. \tag{31}$$

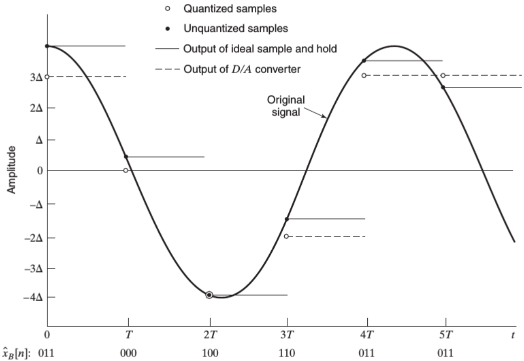


Figure 4.55 Sampling, quantization, coding, and D/A conversion with a 3-bit quantizer.

Fourier transform of $x_0(t)$ is,

$$X_0(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_0(j\Omega) e^{-j\Omega n T}, \tag{32}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n} \right) H_0(j\Omega), \tag{33}$$

$$= X(e^{j\Omega T}) H_0(j\Omega), \tag{34}$$

$$= \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - \frac{2\pi k}{T} \right) \right) \right] H_0(j\Omega). \tag{35}$$

The signal component $x_0(t)$ is related to the input signal $x_a(t)$ since $x[n] = x_a(nT)$.

If $X_a(j\Omega)$ is bandlimited to frequencies below π/T, the shifted copies of $X_a(j\Omega)$ do not overlap in Eq. 35.

If we define a compensated reconstruction filter as,

$$\tilde{H}_r(j\Omega) = \frac{H_r(j\Omega)}{H_0(j\Omega)}, \qquad (36)$$

where,

$$H_0(j\Omega) = \frac{2\sin(\Omega T/2)}{\Omega} e^{-j\Omega T/2}, \qquad (37)$$

$$H_r(j\Omega) = \begin{cases} T & , |\Omega| < \pi/T \\ 0 & , |\Omega| > \pi/T \end{cases}. \qquad (38)$$

Therefore,

$$\tilde{H}_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2} & , |\Omega| < \pi/T \\ 0 & , |\Omega| > \pi/T. \end{cases} \qquad (39)$$

Zero-order hold $|H_0(j\Omega)|$ drops -4 dB at $\Omega = \pi/T$.

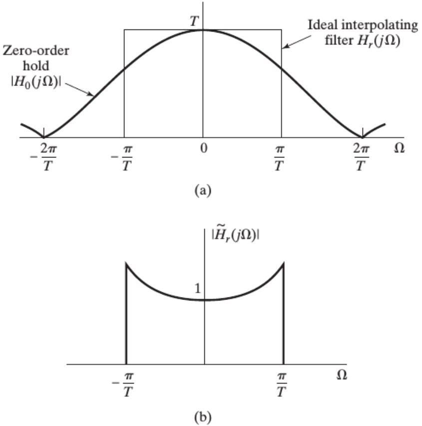


Figure 4.63 (a) Frequency response of zero-order hold compared with ideal interpolating filter. (b) Ideal compensated reconstruction filter for use with a zero-order-hold output.

In the frequency domain, zero-order holder translates into a multiplication of the real signal spectrum by a sinc function!

- Increase signal rate: upsample.
  - At 80 % of Nyquist frequency, the output amplitude is attenuated by 2.42dB.
  - In CD players, data sampling rate is 44.1 kHz. This rate is upsampled by a factor of 8 to 352.8 kHz. By doing so, the need for correction of the ZOH passband distortion is effectively eliminated.
- Pre-Equalizing: inverse sinc, digital filtering.
- Post-Equalizing: analog filtering.

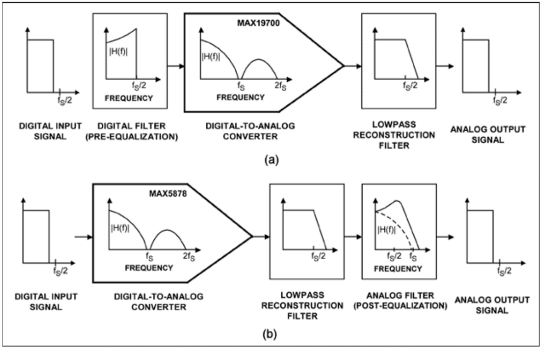


*Figure 4. A pre-equalization digital filter is used to cancel the effect of sinc rolloff in a DAC (**a**). As an alternative, you can use a post-equalization analog filter for the same purpose (**b**).*

1 Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing, 3rd Ed*. Prentice Hall. 2010. Sections 4.1, 4.2, 4.3 and 4.8.

2 Richard G. Lyons. *Understanding Digital Signal Processing, 3rd Ed*. Prentice Hill. 2010. Section 12.3.1.

3 Paolo Prandoni and Martin Vetterli. Signal processing for communications. Taylor and Francis Group, LLC. 2008. Section 9.6.

4 Maxim Integrated. Equalizing Techniques Flatten DAC Frequency Response. Application Note 3853. August 2012.