

Finite representation of real numbers

Floating-point numbers

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Técnicas Digitales III

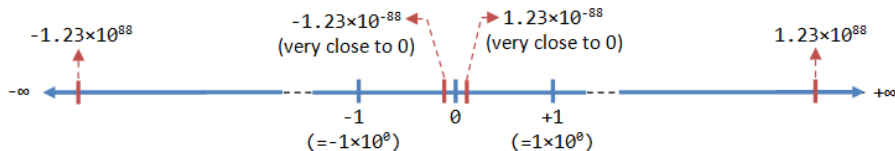
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Floating-point Representation

A floating-point number can represent a very large or a very small value, positive and negative.



Floating-point Numbers (Decimal)

A floating-point number is typically expressed in the scientific notation in the form of

$$(-1)^S \times F \times r^E,$$

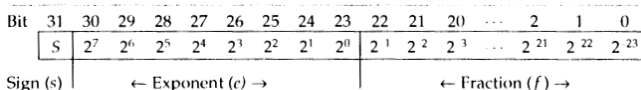
where,

- S , sign bit.
- F , fraction.
- E , exponent.
- r , certain radix. $r = 2$ for binary; $r = 10$ for decimal.

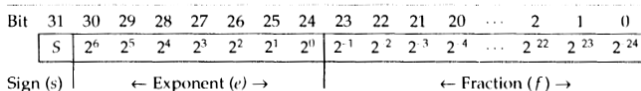
Standards

Modern computers adopt IEEE 754-2008 standard for representing floating-point numbers.

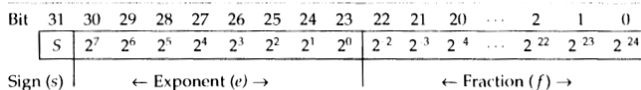
IEEE Standard P754 Format



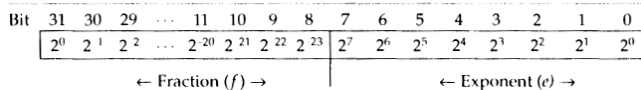
IBM Format



DEC (Digital Equipment Corp.) Format



MIL-STD 1750A Format

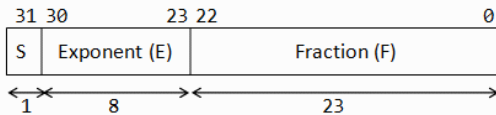


IEEE 754-2008 standard

IEEE 754-2008 standard defines several formats.

Parameter	Binary formats ($B = 2$)				Decimal formats ($B = 10$)		
	Binary 16	Binary 32	Binary 64	Binary 128	Decimal 132	Decimal 164	Decimal 128
p , digits	10 + 1	23 + 1	52 + 1	112 + 1	7	16	34
e_{max}	+15	+127	+1023	+16383	+96	+384	+16,383
e_{min}	-14	-126	-1022	-16382	-95	-383	-16,382
Common name	Half precision	Single precision	Double precision	Quadruple precision			

IEEE-754 32-bit Single-Precision

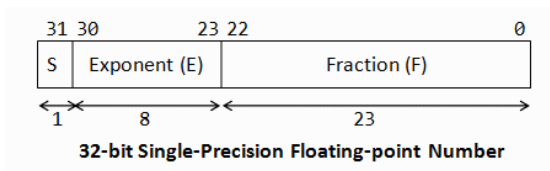


32-bit Single-Precision Floating-point Number

$$(-1)^S \times F \times r^{(E - bias)}$$

- S, sign bit. 0 for positive numbers and 1 for negative numbers.
- E, 8-bits exponent.
- We need to represent both positive and negative exponents.
- $E = [1, 254]$, $bias = 127$; $-126 \leq E - bias \leq 127$.
- $E = 0$ and $E = 255$ are reserved.
- F, 23-bits fraction.

Normalized Form



- Representation of a floating point number may not be unique:
 $11.01_2 = 1.101_2 \times 2^1 = 110.1_2 \times 2^{-1}$.
- Therefore, the fractional part F is normalized.
- $1.F$, implicit leading 1.

Example 1

Represent 3215.020002_{10}

Decimal Value Entered: 3215.020002

Single precision (32 bits):

Binary: Status: normal

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
0	100 0101 0	1.100 1000 1111 0000 0101 0010
0: + 1: -	Decimal value of exponent field and exponent 138 - 127 = 11	Decimal value of the significand 1.5698340

Hexadecimal: 4548F052 Decimal: 3215.0200

<http://babbage.cs.qc.cuny.edu/IEEE-754.old/Decimal.html>

Example 2

Represent $3215.020002_{10} \times 2 = 6430.040004_{10}$

Decimal Value Entered:

Single precision (32 bits):

Binary: Status:

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
<input type="text" value="0"/>	<input type="text" value="10001011"/>	<input type="text" value="1.10010001111000001010010"/>
0: + 1: -	Decimal value of exponent field and exponent <input type="text" value="139"/> - 127 = <input type="text" value="12"/>	Decimal value of the significand <input type="text" value="1.5698340"/>

Hexadecimal: Decimal:

Example 3

Represent $3215.020002_{10}/4 = 803.7550005_{10}$

Decimal Value Entered:

Single precision (32 bits):

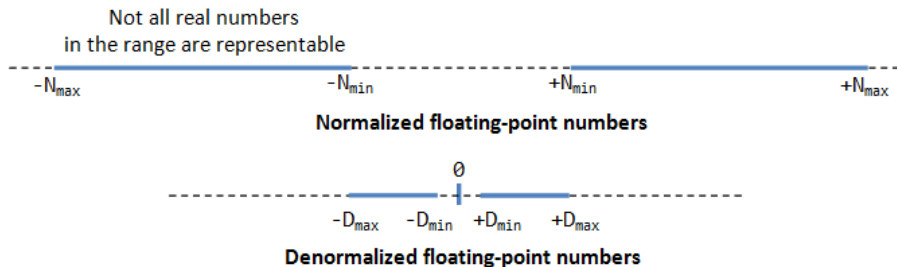
Binary: Status:

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
<input type="text" value="0"/>	<input type="text" value="10001000"/>	<input type="text" value="1.10010001111000001010010"/>
0: + 1: -	Decimal value of exponent field and exponent <input type="text" value="136"/> - 127 = <input type="text" value="9"/>	Decimal value of the significand <input type="text" value="1.5698340"/>

Hexadecimal: Decimal:

Floating-point numbers are auto-scaled!

De-normalized Form



- Normalized form has a serious problem.
- With an implicit leading 1 for the fraction, it cannot represent the number zero!
- De-normalized form is devised to represent zero and small numbers.
- $E = 0 \Rightarrow 0.F$, implicit leading 0.

Example

Represent $-3.4E-39_{10}$

Decimal Value Entered:

Single precision (32 bits):

Binary: Status:

Bit 31 Sign Bit <input type="text" value="1"/> 0: + 1: -	Bits 30 - 23 Exponent Field <input type="text" value="00000000"/> Decimal value of exponent field and exponent <input type="text" value="0"/> - 127 = <input type="text" value="-127"/>	Bits 22 - 0 Significand <input type="text" value="0.1001010000010111010001"/> Decimal value of the significand <input type="text" value="0.5784800"/>
--	--	--

Hexadecimal: Decimal:

Special values

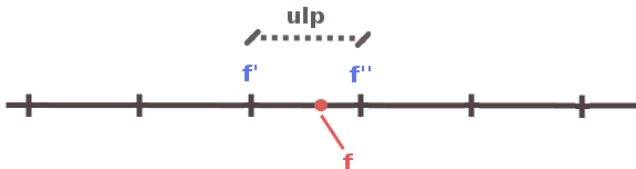
- **Zero**: $E = 0, F = 0$. Two representations: **+0** ($S = 0$) and **-0** ($S = 1$).
- **Inf** (Infinity): $E = 0xFF, F = 0$. Two representations: **+Inf** ($S = 0$) and **-Inf** ($S = 1$).
- **NaN** (Not a Number): $E = 0xFF, F \neq 0$. A value that cannot be represented as a real number (e.g. $0/0$).

MATLAB

```

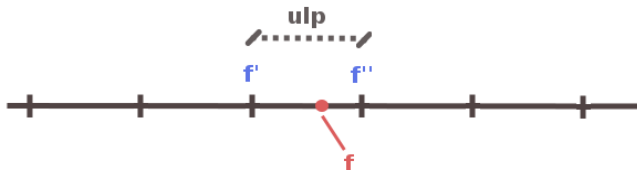
1 » a = 1/0
2 » ans = Inf
3 » b = -1/0
4 » ans = -Inf
5 » c = 0/0
6 » ans = NaN
    
```

Rounding schemes



- ulp (unit of least precision, $\epsilon_{ps}()$).
- f , significant, $f = 1.F$.
- f' and f'' being two successive multiples of ulp .
- Assume that $f' < f < f''$, $f'' = f' + ulp$,
- Then, the rounding function $round(f)$ associates to f either f' or f'' , according to some rounding strategy.

Rounding schemes II



Rounding schemes are:

- *Truncation* (also called *round toward 0* or *chopping*):
 - if f is positive, $\text{round}(f) = f'$.
 - if f is negative, $\text{round}(-f) = f''$.
- *Round toward plus infinity*: $\text{round}(f) = f''$.
- *Round toward minus infinity*: $\text{round}(f) = f'$.
- *Round to nearest* (default):
 - if $f < f' + \text{ulp}/2$, $\text{round}(f) = f'$.
 - if $f > f' + \text{ulp}/2$, $\text{round}(f) = f''$.

Dynamic range

$$DR_{dB} \approx 6.02 \cdot 2^{b_E}$$

where b_E is the number of bits of E .

For single precision (32-bits):

$$DR_{dB} \approx 6.02 \cdot 2^8 \approx 1541 \text{ dB}$$

For fixed-point Q31 (32-bits):

$$DR_{dB} \approx 6.02 \cdot 31 \approx 186 \text{ dB}$$

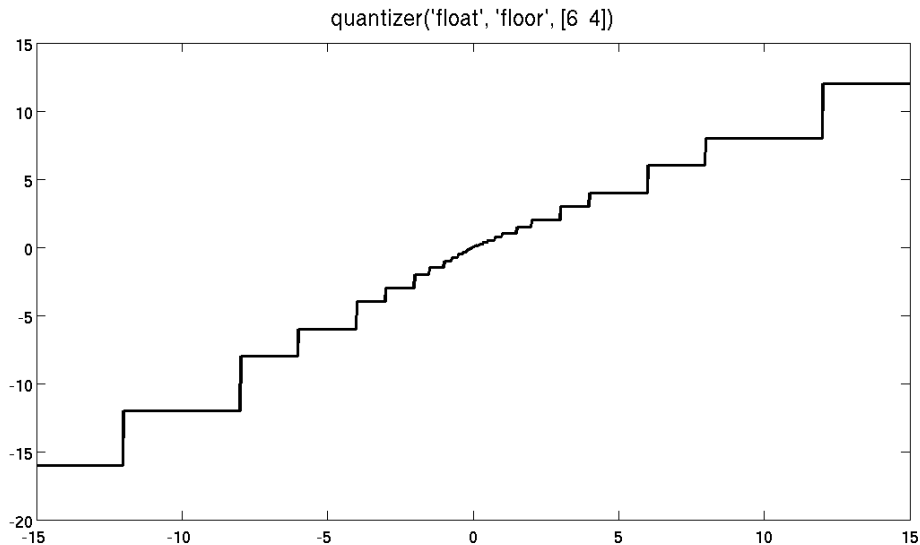
Precision

- Precision is not constant throughout floating point numbers' range.
- As the numbers get larger, the precision gets worse.

MATLAB

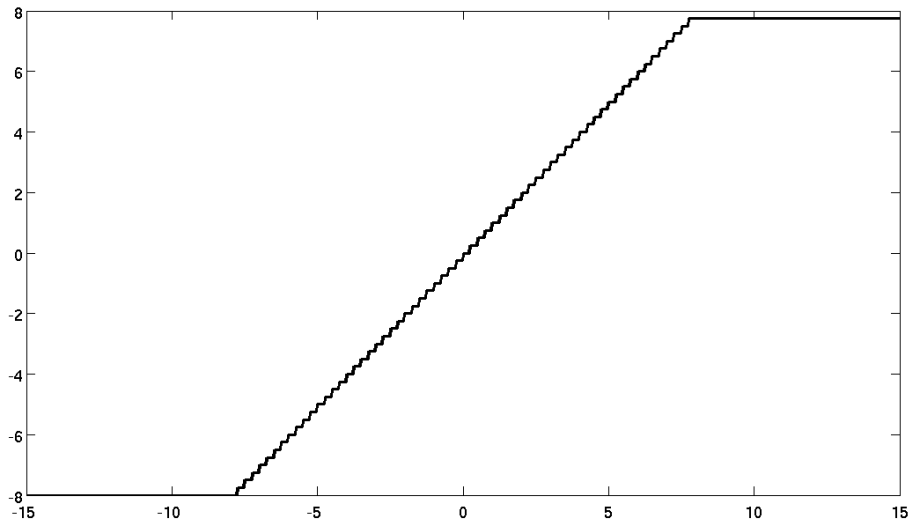
```
1 » u = linspace(-15,15,1000);  
2 » q = quantizer([6 4],'float'); % [wordlength exponentlength]  
3 » y1 = quantize(q,u);  
4 » plot(u,y1); title(tostring(q))  
5 »  
6 » q = quantizer('fixed',[6 2]); % [wordlength fractionlength]  
7 » y2 = quantize(q,u);  
8 » plot(u,y2); title(tostring(q))
```

Precision II



Precision III

quantizer('fixed', 'floor', 'saturate', [6 2])



Precision IV

`eps(x)` returns the positive distance from `abs(x)` to the next larger in magnitude floating point number of the same precision.

MATLAB

```
❶ » e1 = eps(single(1))  
❷ » e1 = 1.1920929e-07  
❸ » e2 = eps(single(1e1))  
❹ » e2 = 9.5367432e-07  
❺ » e3 = eps(single(1e10))  
❻ » e3 = 1024  
❼ » t = single(1e10) + single(1300)  
❽ » t = 10000001024.00
```

Sum of two floating-point positive numbers

$$\begin{aligned} n &= n_1 + n_2 = 1.F \times r^{(E-bias)}, \\ n_1 &= 1.F_1 \times r^{(E_1-bias)}, \\ n_2 &= 1.F_2 \times r^{(E_2-bias)}. \end{aligned}$$

- if $E_1 \geq E_2$ then,

$$E = E_1, \quad F = F_1 + (F_2 \gg (E_1 - E_2))$$

- else,

$$E = E_2, \quad F = (F_1 \gg (E_2 - E_1)) + F_2$$

- if $F \geq r$ then, (first normalization)

$$E = E + 1, \quad F = F \gg 1$$

- $F = \text{round}(F)$

- if $F \geq r$ then, (second normalization)

$$E = E + 1, \quad F = F \gg 1$$

Example 1

$$n = 1e10 + 1300,$$

$$1e10 = (-1)^0 \times 1.00101010000001011111001 \times r^{(160-127)},$$

$$1300 = (-1)^0 \times 1.111000000000000000000000 \times r^{(131-127)}.$$

- if $160 \geq 131$ then,

$$E = 160,$$

$$F = 1.00101010000001011111001 + (1.111000000000000000000000 \gg 29)$$

$$E = 160, F = 1.00101010000001011111001 + (0.000000000000000000000000)$$

$$E = 160, F = 1.00101010000001011111001$$

$$n = (-1)^0 \times 1.00101010000001011111001 \times r^{(160-127)},$$

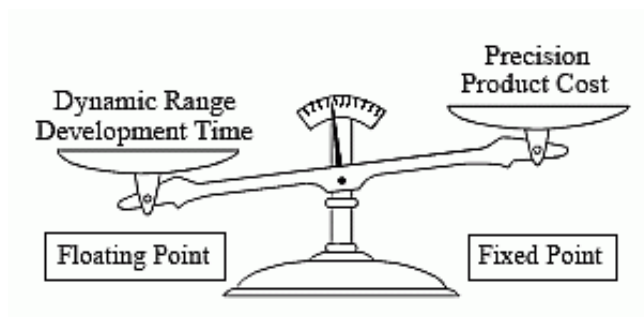
Example 2

- In floating-point processors scaling the data increases dynamic range, but scaling does not improve precision, and in fact degrades performance.
- When calculations involve large and small numbers at the same time, the loss of precision affects the small number and the result.

MATLAB

```
1 >> (2^53 + 1) - 2^53
2 >> ans = 0
3 >> x=1, t = tan(x) - sin(x)/cos(x)
4 >> t = 2.2204e-16 % eps(1)
```

Fixed-point vs floating-point



Bibliography

- 1 Jean-Pierre Deschamps, Gustavo D. Sutter, and Enrique Cantó. Guide to FPGA Implementation of Arithmetic Functions, Chapter 12.