Finite representation of real numbers

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Técnicas Digitales III

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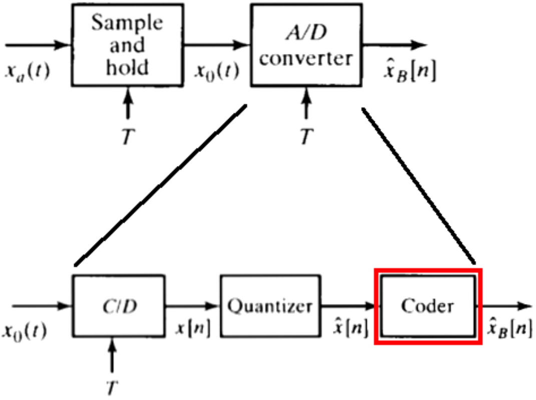
Resumen

ADC stages


Sample
and
hold
$x_a(t)$
$x_0(t)$
T
A/D
converter
T
$\hat{x}_B[n]$
$x_0(t)$
C/D
T
$x[n]$
Quantizer
$\hat{x}[n]$
Coder
$\hat{x}_B[n]$

Representation

Unsigned integers

- An N-bit binary word can represent a total of 2^N separate values.
- Range: 0 to $2^N - 1$
- $n_{10} = 2^{N-1} b_{N-1} + 2^{N-2} b_{N-2} + \cdots + 2^1 b_1 + 2^0 b_0$

2's complement signed integers

- Range: -2^{N-1} to $2^{N-1} - 1$.
- $n_{10} = -b_{N-1} 2^{N-1} + \sum_{i=0}^{N-2} b_i 2^i$

Bit Pattern	Unsigned	2's Complement
0000 0000	0	0
0000 0001	1	1
0000 0010	2	2
•	•	•
•	•	•
0111 1110	126	126
0111 1111	127	127
1000 0000	128	-128
1000 0001	129	-127
•	•	•
•	•	•
1111 1110	254	-2
1111 1111	255	-1

How much bits are needed to represent $-\alpha_{min} \leq \alpha \leq \alpha_{max}$?

$$N = \text{floor}(\log_2(\max([\alpha_{min}, \alpha_{max}])) + 2)$$

Representation, cont'd

$$N = \text{floor}(\log_2(\max([\alpha_{min}, \alpha_{max}])) + 2)$$

MATLAB

```
1 » a_m = 15; a_M = 15;
2 » N = floor (log2 ( max ( [ a_m , a_M] ) ) + 2 );
```

Representation, cont'd

$$N = \text{floor}(\log_2(\max([\alpha_{min}, \alpha_{max}])) + 2)$$

MATLAB

```
1 » a_m = 15; a_M = 15;
2 » N = floor (log2 ( max ( [ a_m , a_M] ) ) + 2 );
3 » N = 5.00
```

"Q" notation

The fractional notation can be applied to the 2's complement notation.

Qm.n

- *m* represents the number of bits to the left of the binary point.
- *n* represents the number of bits to the right of the binary point.
- The weights of bits that are to the right of the binary point are negative powers of 2: $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$ ... , etc.
- The naming convention does not take the MSB of the number (sign bit) into account. A Qm.n notation therefore uses $m + n + 1$ bits.
- Precision: 2^{-n}.
- Range: -2^m to $2^m - 2^{-n}$.

"Q" notation, cont'd

For instance:

- Q0.15 (Q15)
  - 16 bits;
  - Range: -1 to 0.99996948;
  - Precision: 1/32768 (2^{-15}).
- Q3.12
  - 16 bits;
  - Range: -8 to 7.9998;
  - Precision: 1/4096 (2^{-12}).
- Q0.31 (Q31)
  - 32 bits;
  - Range: -1 to 0.999999999534339;
  - Precision: 4.6566129e-10 (2^{-31}).

Precision examples

Format (N.M)		Largest positive value (0x7FFF)	Least negative value (0x8000)	Precision (0x0001)		DR(dB)
1	15	0,999969482421875	-1	3,05176E-05	2^-15	90,30873362
2	14	1,99993896484375	-2	6,10352E-05	2^-14	90,30873362
3	13	3,9998779296875	-4	0,00012207	2^-13	90,30873362
4	12	7,999755859375	-8	0,000244141	2^-12	90,30873362
5	11	15,99951171875	-16	0,000488281	2^-11	90,30873362
6	10	31,99902344	-32	0,000976563	2^-10	90,30873362
7	9	63,99804688	-64	0,001953125	2^-9	90,30873362
8	8	127,9960938	-128	0,00390625	2^-8	90,30873362
9	7	255,9921875	-256	0,0078125	2^-7	90,30873362
10	6	511,984375	-512	0,015625	2^-6	90,30873362
11	5	1023,96875	-1024	0,03125	2^-5	90,30873362
12	4	2047,9375	-2048	0,0625	2^-4	90,30873362
13	3	4095,875	-4096	0,125	2^-3	90,30873362
14	2	8191,75	-8192	0,25	2^-2	90,30873362
15	1	16383,5	-16384	0,5	2^-1	90,30873362
16	0	32767	-32768	1	2^-0	90,30873362

Scale of representation

- Values represented in Qm.n notation can be seen as an integer simply divided by a power-of-two scale factor, 2^n.
- In fact, the scale factor can be an arbitrary scale that is not a power of two.
- Example: 16-bit 2's complement numbers between 8000H and 7FFFH can represent decimal values between –5 and +5, where the scale factor is 5/32768 ($5/2^{15}$).
- It can be said that the scale factor is in "the head of the programmer".

Scale factor examples

Format	Scaling factor ( )	Range in Hex (fractional value)
(1.15)	$2^{15} = 32768$	0x7FFF (0.99) → 0x8000 (–1)
(2.14)	$2^{14} = 16384$	0x7FFF (1.99) → 0x8000 (–2)
(3.13)	$2^{13} = 8192$	0x7FFF (3.99) → 0x8000 (–4)
(4.12)	$2^{12} = 4096$	0x7FFF (7.99) → 0x8000 (–8)
(5.11)	$2^{11} = 2048$	0x7FFF (15.99) → 0x8000 (–16)
(6.10)	$2^{10} = 1024$	0x7FFF (31.99) → 0x8000 (–32)
(7.9)	$2^{9} = 512$	0x7FFF (63.99) → 0x8000 (–64)
(8.8)	$2^{8} = 256$	0x7FFF (127.99) → 0x8000 (–128)
(9.7)	$2^{7} = 128$	0x7FFF (511.99) → 0x8000 (–512)
(10.6)	$2^{6} = 64$	0x7FFF (1023.99) → 0x8000 (–1024)
(11.5)	$2^{5} = 32$	0x7FFF (2047.99) → 0x8000 (–2048)
(12.4)	$2^{4} = 16$	0x7FFF (4095.99) → 0x8000 (–4096)
(13.3)	$2^{3} = 8$	0x7FFF (4095.99) → 0x8000 (–4096)
(14.2)	$2^{2} = 4$	0x7FFF (8191.99) → 0x8000 (–8192)
(15.1)	$2^{1} = 2$	0x7FFF (16383.99) → 0x8000 (–16384)
(16.0)	$2^{0} = 1$(Integer)	0x7FFF (32767) → 0x8000h (–32768)

Dynamic range

Dynamic range is defined as,

$$DR_{db} = 20\ log_{10}\left(\frac{\text{largest possible word value}}{\text{smallest possible word value}}\right) \quad [\text{dB}]$$

For N-bit unsigned integers,

$$DR_{dB} = 20\ log_{10}\left[\frac{2^{(N-1)}}{1}\right] \quad [\text{dB}]$$
$$DR_{dB} = 20\ \left[(N-1)log_{10}(2) - log_{10}(1)\right]$$
$$DR_{dB} = 20\ log_{10}(2)\cdot(N-1)$$
$$DR_{dB} = 6.02\cdot N - 6.02 \quad [\text{dB}]$$

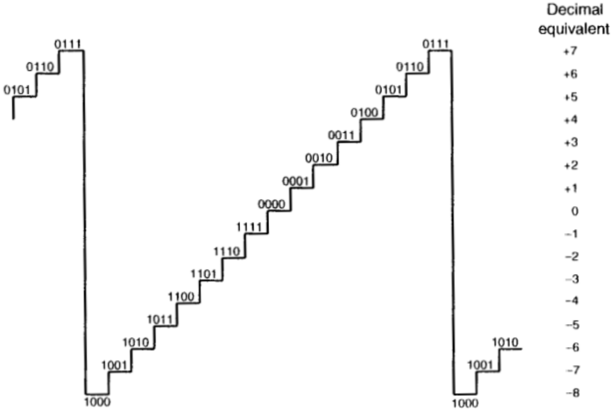
Addition in 2's complement

- Adding two N-bits numbers can produce a N+1 bits result.
- The last two bits of the carry row show if overflow occurs.
- Saving the result in a N+1 word avoids overflows.
- The general rule is the sum of *m* individual *b*-bit can require as many as $b + log_2(m)$.
- Example: 256 8-bits words requires an accumulator whose word lenght is $8 + log_2(256) = 16$
- ¿How many sums are supported by a 40-bits accumulator for 16-bits numbers?

```
 11111 111    (carry)       0111    (carry)
   0000 1111  (15)            0111  (7)
 + 1111 1011  (-5)          + 0011  (3)
 ==================         =============
   0000 1010  (10)            1010  (-6)  invalid!
```

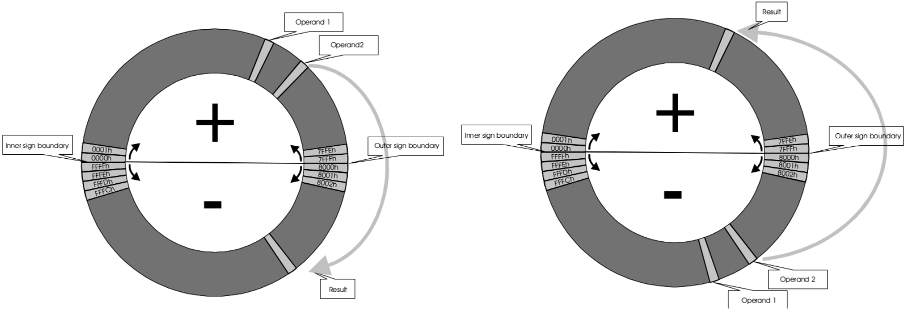
Overflow

- An **overflow** occurs in an N-bit 2's complement notation when a result is greater than $2^{N-1} - 1$.
- An overflow produces a **roll-over** (wrap).
- An **underflow** occurs if a result is less than 2^{-N}.



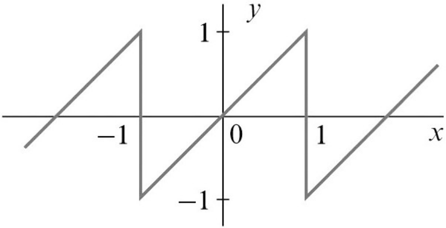
Overflow, cont'd

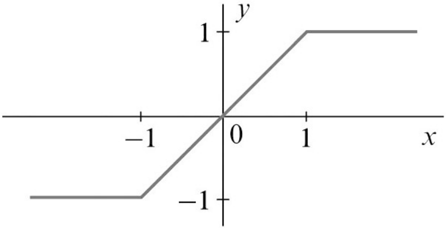
- A roll-over usually has catastrophic consequences on a process.
- Only happen when two very large positive operands, or two very large negative operands are added.
- It can never happen during the addition of a positive operand and a negative operand, whatever their magnitude.



Saturation

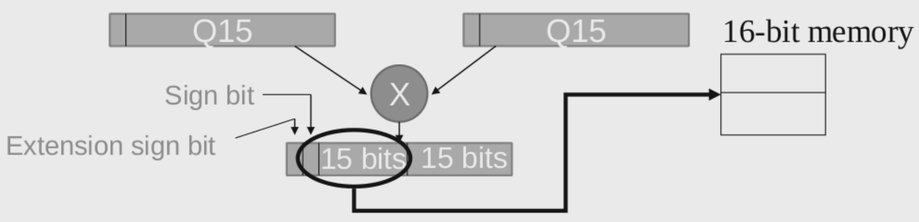
- To avoid a rollover, overflow is detected and the result is saturated to the most positive or most negative value that can be represented.
- This procedure is called **saturation arithmetic**.
- PDSP allows the results to be saturated automatically in hardware (In TI DSP C5505, SATD Bit of ST1_55 register).





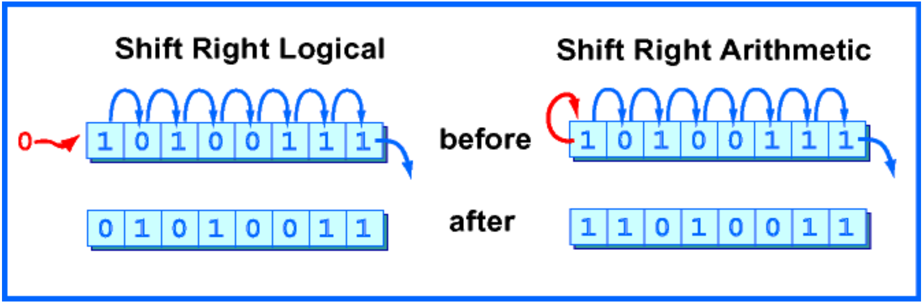
Multiplication in 2's complement

- The product of two N-bit numbers requires 2N bits to contain all possible values.
- But the two MSB are always equal (sign extension bit).
- Therefore, 2N-1 bits are enough to store the result.
- Q15 will not produce an overflow.



Multiplication and division by 2 in 2's complement

- Multiplication: all bits are shifted left by one position.
- Division: all bits are shifted right by one position, however the sign bit must be preserved (**arithmetic shift**).
- Arithmetic shift $\neq$ logical shift.



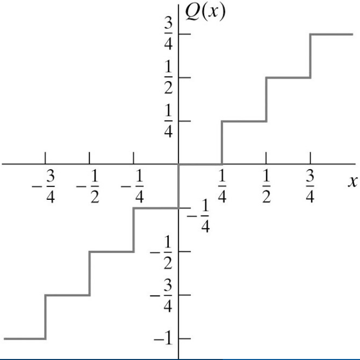
Accumulator

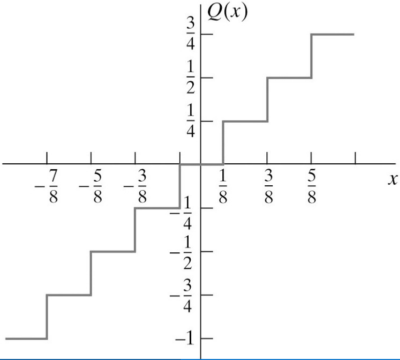
- PDSP have an accumulator with extra bits to avoid overflow during internal calculations (In C5505, 40-bits accumulator).
- Guard bits: extra bits to avoid addition overflows.
- Only round final results to the final data size and format if possible.

b39–b32	b31–b16	b15–b0
G	H	L
Guard bits	High-order bits	Low-order bits

Truncation and roundoff

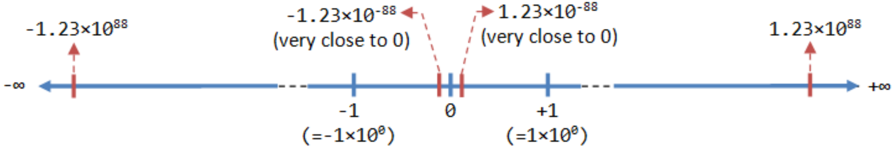
- After multiplication, a 2N-bits number must be stored in memory of N-bits word.
- Truncation: $e = Q[x] - x, \quad -\Delta \leq e < 0,$ $\boxed{\mu = -\frac{\Delta}{2}}$, $\boxed{\sigma^2 = \frac{\Delta}{12}}$.
- Roundoff: $e = Q[x + 0.5] - x, \quad -\Delta/2 < e \leq \Delta/2,$ $\boxed{\mu = 0}$, $\boxed{\sigma^2 = \frac{\Delta}{12}}$.





Number Representation

A floating-point number can represent a very large or a very small value, positive and negative.



Floating-point Numbers (Decimal)

A floating-point number is typically expressed in the scientific notation in the form of

$$(-1)^S \times F \times r^E \,,$$

where,

- S, sign bit.
- F, fraction.
- E, exponent.
- r, certain radix. $r = 2$ for binary; $r = 10$ for decimal.

Standards

IEEE Standard P754 Format

Bit	31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
	S	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	...	2^{-21}	2^{-22}	2^{-23}
	Sign (s)	← Exponent (e) →								← Fraction (f) →						

IBM Format

Bit	31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
	S	2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	...	2^{-22}	2^{-23}	2^{-24}
	Sign (s)	← Exponent (e) →							← Fraction (f) →							

DEC (Digital Equipment Corp.) Format

Bit	31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
	S	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	2^{-2}	2^{-3}	2^{-4}	...	2^{-22}	2^{-23}	2^{-24}
	Sign (s)	← Exponent (e) →								← Fraction (f) →						

MIL-STD 1750A Format

Bit	31	30	29	...	11	10	9	8	7	6	5	4	3	2	1	0
	2^0	2^{-1}	2^{-2}	...	2^{-20}	2^{-21}	2^{-22}	2^{-23}	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
	← Fraction (f) →								← Exponent (e) →							

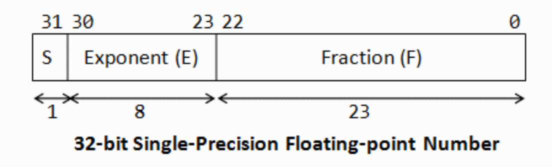
Modern computers adopt IEEE 754-2008 standard for representing floating-point numbers.

IEEE 754-2008 standard

IEEE 754-2008 standard defines several formats.

	Binary formats ($B = 2$)				Decimal formats ($B = 10$)		
Parameter	Binary 16	Binary 32	Binary 64	Binary 128	Decimal 132	Decimal 164	Decimal 128
p, digits	$10 + 1$	$23 + 1$	$52 + 1$	$112 + 1$	7	16	34
e_{max}	+15	+127	+1023	+16383	+96	+384	+16,383
e_{min}	−14	−126	−1022	−16382	−95	−383	−16,382
Common name	Half precision	Single precision	Double precision	Quadruple precision			

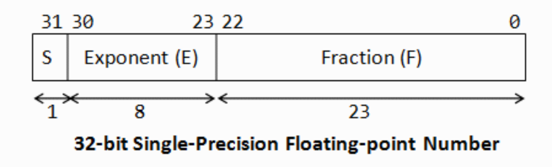
IEEE-754 32-bit Single-Precision



$$(-1)^S \times F \times r^{(E-bias)}$$

- S, sign bit. 0 for positive numbers and 1 for negative numbers.
- E, 8-bits exponent.
- We need to represent both positive and negative exponents.
- $E = [1, 254]$, $bias = 127$; $\quad -126 \leq E - bias \leq 127$.
- $E = 0$ and $E = 255$ are reserved.
- F, 23-bits fraction.

Format



- Representation of a floating point number may not be unique: $11.01_2 = 1.101_2 \times 2^1 = 110.1_2 \times 2^{-1}$.
- Therefore, the fractional part F is normalized.
- $1.F$, implicit leading 1.

Example 1

Represent 3215.020002_{10}

Decimal Value Entered: 3215.020002

Single precision (32 bits):

Binary: *Status:* normal

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
0	100 0101 0	1 .100 1000 1111 0000 0101 0010
0: + 1: -	Decimal value of exponent field and exponent 138 - 127 = 11	Decimal value of the significand 1.5698340

Hexadecimal: 4548F052 *Decimal:* 3215.0200

http://babbage.cs.qc.cuny.edu/IEEE-754.old/Decimal.html

Example 2

Represent $3215.020002_{10} \times 2 = 6430.040004_{10}$

Decimal Value Entered: 6430.040004

Single precision (32 bits):

Binary: *Status:* normal

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
0	10001011	1 .10010001111000001010010
0: + 1: -	Decimal value of exponent field and exponent: 139 - 127 = 12	Decimal value of the significand: 1.5698340

Hexadecimal: 45C8F052 *Decimal:* 6430.0400

Example 3

Represent $3215.020002_{10}/4 = 803.7550005_{10}$

Decimal Value Entered: 803.7550005

Single precision (32 bits):

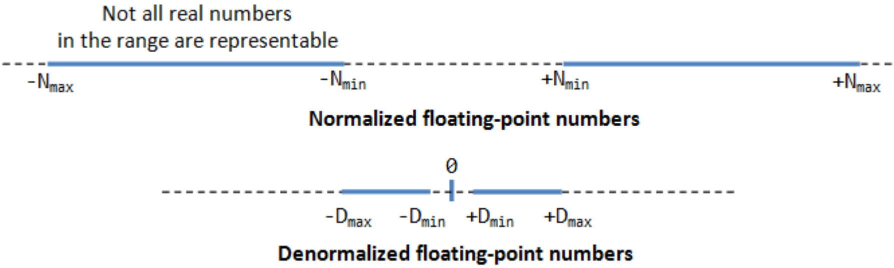
Binary: *Status:* normal

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
0	10001000	1 .10010001111000001010010
0: + 1: -	Decimal value of exponent field and exponent: 136 - 127 = 9	Decimal value of the significand: 1.5698340

Hexadecimal: 4448F052 *Decimal:* 803.75500

Floating-point numbers are auto-scaled!.

Format



- Normalized form has a serious problem, with an implicit leading 1 for the fraction, it cannot represent the number zero!
- De-normalized form was devised to represent zero and small numbers.
- $E = 0 \Rightarrow 0.F$, implicit leading 0.

Example

Represent $-3.4\text{E-}39_{10}$

Decimal Value Entered: -3.4e-39

Single precision (32 bits):

Binary: *Status:* denormalized

Bit 31 Sign Bit	Bits 30 - 23 Exponent Field	Bits 22 - 0 Significand
1	00000000	0.10010100000101110100001
0: + 1: -	Decimal value of exponent field and exponent: 0 - 127 = -127	Decimal value of the significand: 0.5784800

Hexadecimal: 802505D1 *Decimal:* -3.3999999e-39

Special values

- **Zero**: $E = 0$, $F = 0$. Two representations: **+0** ($S = 0$) and **-0** ($S = 1$).
- **Inf** (Infinity): $E = 0\text{xFF}$, $F = 0$. Two representations: **+Inf** ($S = 0$) and **-Inf** ($S = 1$).
- **NaN** (Not a Number): $E = 0\text{xFF}$, $F \neq 0$. A value that cannot be represented as a real number (e.g. 0/0).

MATLAB

```
1 » a = 1/0
2 » ans = Inf
3 » b = -1/0
4 » ans = -Inf
5 » c = 0/0
6 » ans = NaN
```

Rounding schemes

- *ulp* (unit of least precision, `eps()`).
- f, significant, $f = 1.F$.
- f' and f'' being two successive multiples of ulp.
- Assume that $f' < f < f''$, $\;f'' = f' + ulp$,
- Then, the rounding function $round(f)$ associates to f either f' or f'', according to some rounding strategy.

Rounding schemes are:

- *Truncation* (also called *round toward 0* or *chopping*): $round(s) = f'$ if f is positive, $round(-f) = f''$ if f is negative.
- *Round toward plus infinity*: $round(s) = f''$
- *Round toward minus infinity*: $round(s) = f'$
- *Round to nearest* (default): if $f < f' + ulp/2$, $round(f) = f'$, and if $f > f' + ulp/2$, $round(f) = f''$.

Dynamic range

$$DR_{dB} \approx 6.02 \cdot 2^{b_E}$$

where b_E is the number of bits of E.

For single precision (32-bits):

$$DR_{dB} \approx 6.02 \cdot 2^8 \approx 1541\,\mathrm{dB}$$

For fixed-point Q31 (32-bits):

$$DR_{dB} \approx 6.02 \cdot 32 \approx 192\,\mathrm{dB}$$

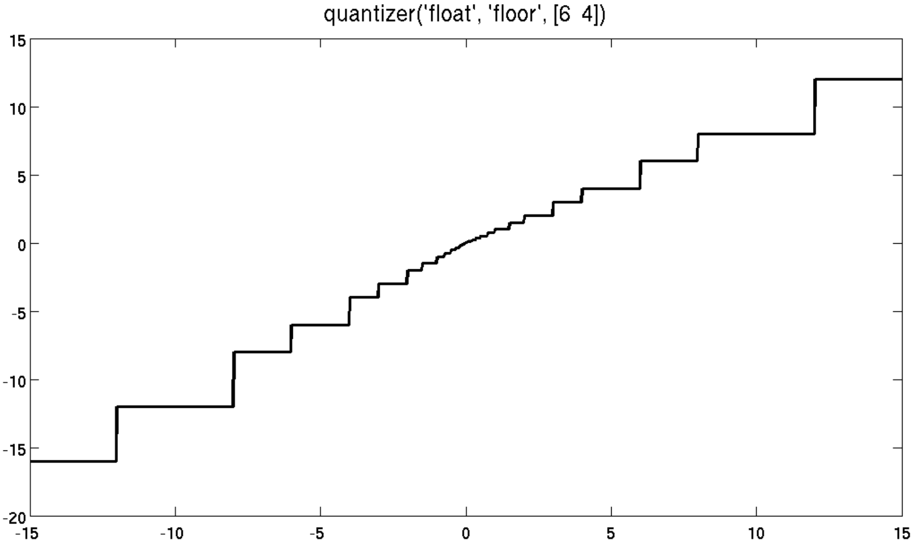
Precision

- Precision is not constant throughout floating point numbers' range.
- As the numbers get larger, the precision gets worse.

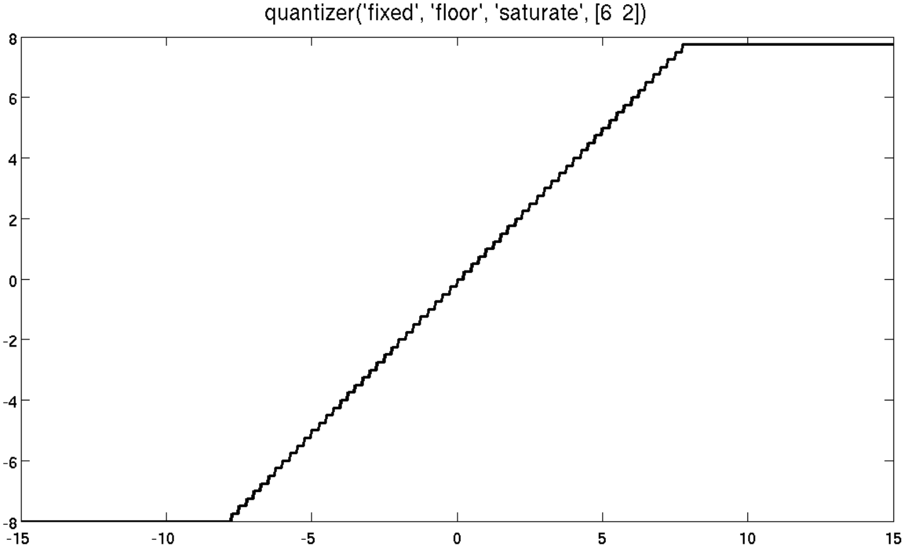
MATLAB

```
» u = linspace(-15,15,1000);
» q = quantizer([6 4],'float');  % [wordlength exponentlength]
» y1 = quantize(q,u);
» plot(u,y1); title(tostring(q))
»
» q = quantizer('fixed',[6 2]); % [wordlength fractionlength]
» y2 = quantize(q,u);
» plot(u,y2); title(tostring(q))
```

Precision, cont'd



Precision, cont'd



Precision, cont'd

`eps(x)` returns the positive distance from `abs(x)` to the next larger in magnitude floating point number of the same precision.

MATLAB

1. » e1 = eps(single(1))
2. » e1 = 1.1920929e-07

Precision, cont'd

`eps(x)` returns the positive distance from `abs(x)` to the next larger in magnitude floating point number of the same precision.

MATLAB

```
1 » e1 = eps(single(1))
2 » e1 = 1.1920929e-07
3 » e2 = eps(single(1e1))
4 » e2 = 9.5367432e-07
```

Precision, cont'd

`eps(x)` returns the positive distance from `abs(x)` to the next larger in magnitude floating point number of the same precision.

MATLAB

```
1 » e1 = eps(single(1))
2 » e1 = 1.1920929e-07
3 » e2 = eps(single(1e1))
4 » e2 = 9.5367432e-07
5 » e3 = eps(single(1e10))
6 » e3 = 1024
```

Precision, cont'd

`eps(x)` returns the positive distance from `abs(x)` to the next larger in magnitude floating point number of the same precision.

MATLAB

```
1 » e1 = eps(single(1))
2 » e1 = 1.1920929e-07
3 » e2 = eps(single(1e1))
4 » e2 = 9.5367432e-07
5 » e3 = eps(single(1e10))
6 » e3 = 1024
7 » t = single(1e10) + single(1300)
8 » t = 1000001024.00
```

Sum of two floating-point positive numbers

$$n = n_1 + n_2 = 1.F \times r^{(E-bias)} ,$$
$$n_1 = 1.F_1 \times r^{(E_1-bias)} ,$$
$$n_2 = 1.F_2 \times r^{(E_2-bias)} .$$

- if $E_1 >= E_2$ then,

$$E = E_1, \quad F = F_1 + (F_2 >> (E_1 - E_2))$$

- else,

$$E = E_2, \quad F = (F_1 >> (E_2 - E_1)) + F_2$$

- if $F >= r$ then, (first normalization)

$$E = E + 1, \quad F = F >> 1$$

- F = round(F)
- if $F >= r$ then, (second normalization)

$$E = E + 1, \quad F = F >> 1$$

Example 1

$$n = 1e10 + 1300\,,$$
$$1e10 = (-1)^0 \times 1.00101010000001011111001 \times r^{(160-127)}\,,$$
$$1300 = (-1)^0 \times 1.11100000000000000000000 \times r^{(131-127)}\,.$$

- if $160 >= 131$ then,

$$E = 160,$$
$$F = 1.00101010000001011111001 + (1.11100000000000000000000 >> 29)$$

$$E = 160, \quad F = 1.00101010000001011111001 + (0.00000000000000000000000)$$

$$E = 160, \quad F = 1.00101010000001011111001$$

$$n = (-1)^0 \times 1.00101010000001011111001 \times r^{(160-127)}\,,$$

Example 2

- In floating-point processors scaling the data increases dynamic range, but scaling does not improve precision, and in fact degrades performance.
- When calculations involve large and small numbers at the same time, the loss of precision affects the small number and the result.

MATLAB

1.
```
» (2^53 + 1) - 2^53
```

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1. » `(2^53 + 1) - 2^53`
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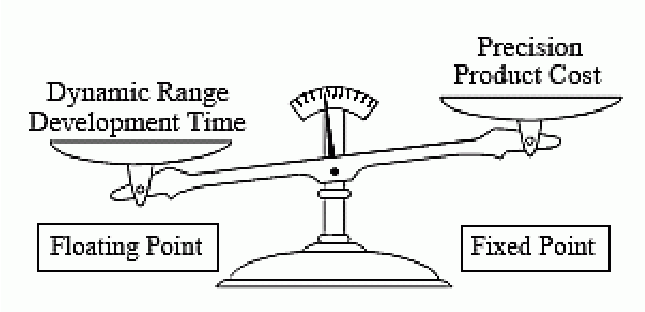
MATLAB

1.
```
» (2^53 + 1) - 2^53
```
2.
```
» ans = 0
```
3.
```
» x=1, t = tan(x) - sin(x)/cos(x)
```
4.
```
» t = 2.2204e-16 % eps(1)
```

Fixed-point vs floating-point



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