Infinite impulse response filtering

Dr. Ing. Rodrigo Gonzalez

rodralez@frm.utn.edu.ar

Técnicas Digitales III

Universidad Tecnológica Nacional,
Facultad Regional Mendoza.

Resumen

Classification of discrete filters

Table: Classification of discrete filters

	Finite impulse response (FIR)	Infinite impulse response (IIR)
Filtering in time domain	Moving average	Leaky Integrator
Filtering in frequency domain	Windowed Filters Equiripple Minimax	Bilinear z-transform

Leaky integrator filter

The MA filter equation,

$$y[n] = x[n] * h[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \,, \tag{1}$$

$$y[n] = \frac{1}{M} \left[\sum_{k=1}^{M-1} x[n-k] + x[n] \right] . \tag{2}$$

Since,

$$y[n-1] = \frac{1}{M-1} \left[\sum_{k=1}^{M-1} x[n-k] \right] . \tag{3}$$

Then,

$$y[n] = \frac{1}{M} x[n] + \frac{M-1}{M} y[n-1] \,. \tag{4}$$

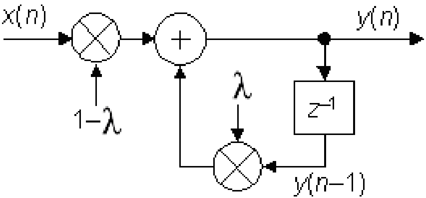
Defining $\lambda = \frac{M-1}{M}$,

$$y[n] = \lambda \, y[n-1] + (1-\lambda) \, x[n] \,. \tag{5}$$

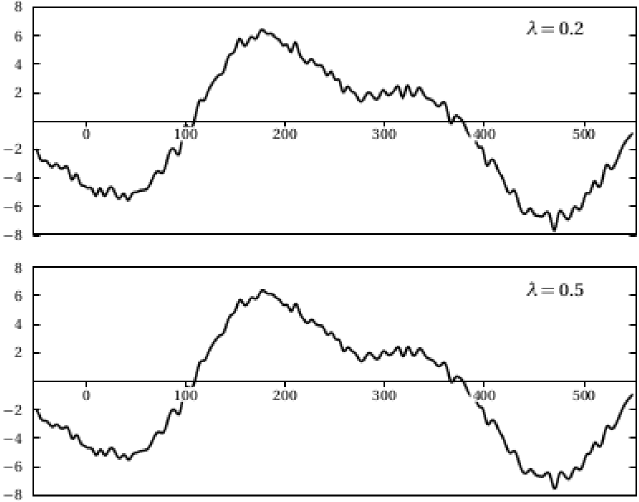
It can be seen that the leaky integrator filter is an IIR filter. Why?

$$y[n] = \lambda\, y[n-1] + (1-\lambda)\, x[n]\,.$$

- No longer a convolution.
- Instead, a *constant coefficient difference equation*. Initial conditions must be set.
- The new system is LTI [2].
- System is stable for $|\lambda| < 1$.
- The value of λ (which is the pole of the system) determines the smoothing power of the filter.

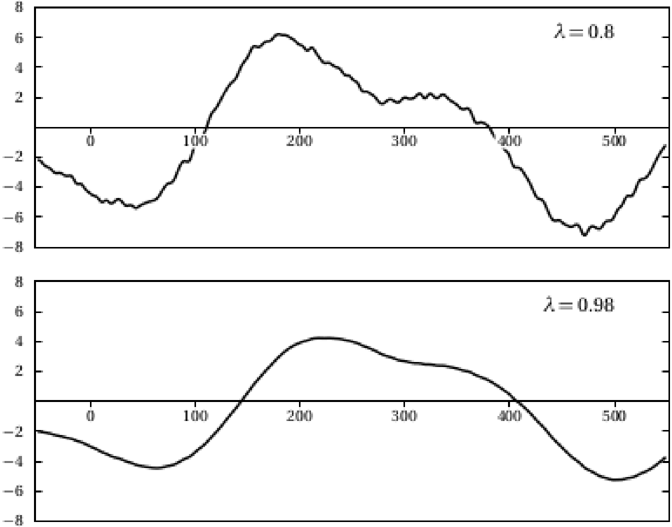


Noise Reduction

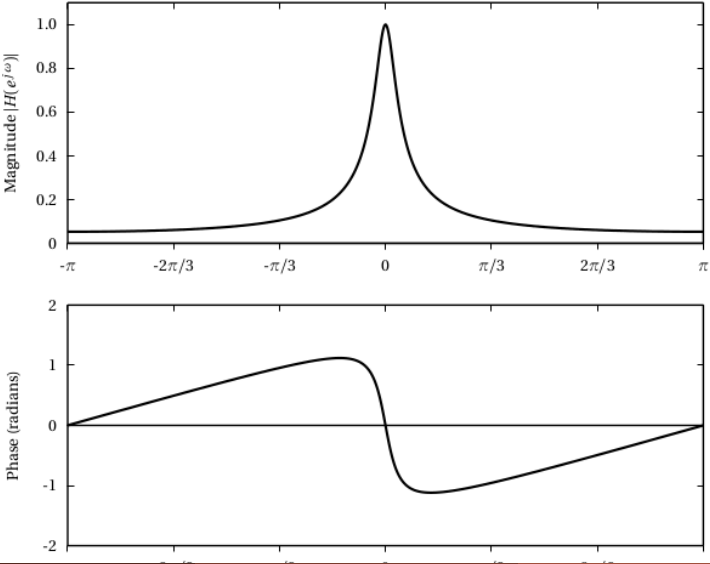


Noise Reduction

- Note how the signal is delayed as λ grows.



Magnitude and phase response of the leaky integrator for $\lambda = 0.9$.



Bilinear transform

The technique is an algebraic transformation between variables s and z.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right). \tag{6}$$

Solving for z,

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}. \tag{7}$$

Doing $s = j\Omega$, where Ω is the analog frequency, $-\infty, < \Omega < \infty$,

$$z = \frac{1 + (T_d/2)j\Omega}{1 - (T_d/2)j\Omega}. \tag{8}$$

The relationship between Ω and ω, the "digital" frequency, $-\pi, < \omega < \pi$, can be found by replacing $z = e^{j\omega}$ in Eq. 6,

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j\sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = j\frac{2}{T_d}\tan(\omega/2). \tag{9}$$

Bilinear transform (2)

Real and imaginary parts on both sides of Eq. 9 are,

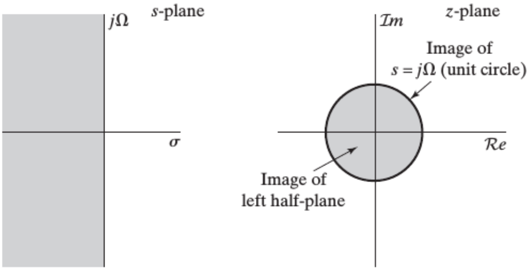
$$\sigma = 0\,, \tag{10}$$

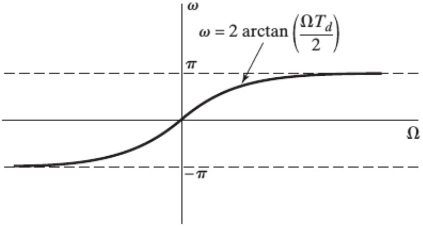
$$\Omega = \frac{2}{T_d}\tan(\omega/2)\,. \tag{11}$$

Or,

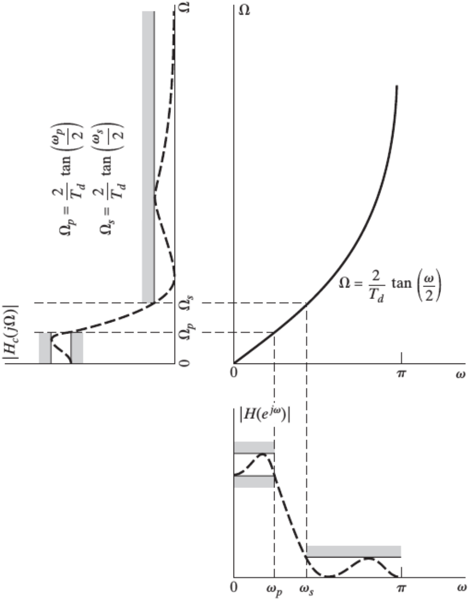
$$\omega = \arctan(\Omega T_d/2)\,. \tag{12}$$

Bilinear transform, Map from s to z



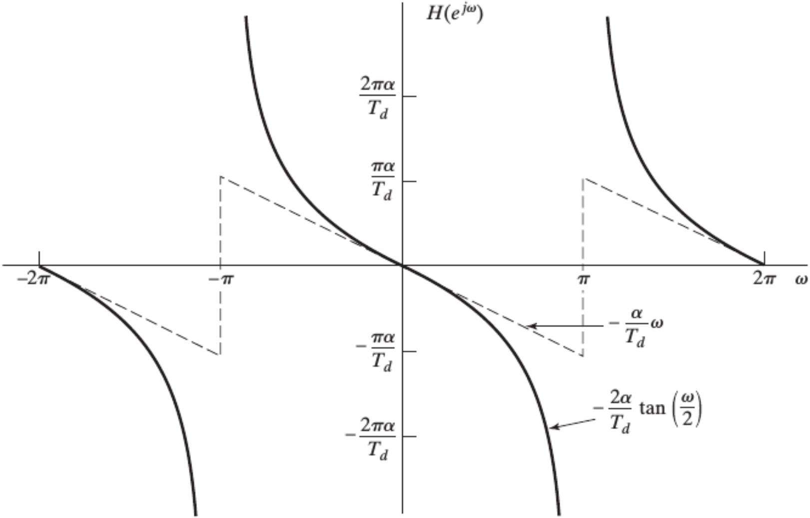


- Non-linear compression of the frequency axis.
- The design of discrete-time filters using the bilinear transformation is useful only when this compression can be tolerated or compensated for.



Bilinear transform, Phase response

Suppose a continuous-time filter with linear phase response. The nonlinear warping of the frequency axis introduced by the bilinear transformation will not preserve linearity in phase response.



Design a digital filter equivalent of a 2nd order Butterworth low-pass filter with a cut-off frequency $f_c = 100$ Hz and a sampling frequency $f_s = 1000$ samples/sec. Derive the finite difference equation and draw the realisation structure of the filter. Given that the analogue prototype of the frequency-domain transfer function $H(s)$ for a Butterworth filter is:

$$H(s) = \frac{1}{s^2 + \sqrt{2} \cdot s + 1}$$

The normalised cut-off frequency of the digital filter is given by the following equation:

$$\omega = \frac{2\pi f_c}{f_s} = \frac{2\pi 100}{1000} = 0.628$$

Now determine the equivalent analogue filter cut-off frequency ω_{ac}, using the pre-warping function of Equation 5.9. The value of K is immaterial so let $K = 1$.

$$\Omega = K \cdot \tan\left(\frac{\omega}{2}\right) = 1 \cdot \tan\left(\frac{0.628}{2}\right)$$

$$\Omega = 0.325 \quad rads/\sec$$

Now denormalise the frequency-domain transfer function $H(s)$ of the Butterworth filter, with the corresponding low-pass to low-pass frequency transformation of Equation 5.10. Hence the transfer function of the Butterworth filter becomes:

$$H(s) = \frac{1}{\left[\dfrac{s}{0.325}\right]^2 + \sqrt{2}\cdot\left[\dfrac{s}{0.325}\right] + 1}$$

Next, convert the analogue filter into an equivalent digital filter by applying the bilinear z-transform. This is achieved by making a substitution for s in the transfer function.

$$s = \frac{z-1}{z+1} \equiv \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{1}{\dfrac{1}{0.325^2}\cdot\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]^2 + \dfrac{\sqrt{2}}{0.325}\cdot\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right] + 1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.067 + 0.135z^{-1} + 0.067z^{-2}}{1 - 1.1429z^{-1} + 0.4127z^{-2}}$$

The finite difference equation of the filter is found by inverting the transfer function.

$$y(n) = 1.1429y(n-1) - 0.4127y(n-2) + 0.067x(n) + 0.135x(n-1) + 0.067x(n-2)$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + ... + b_{N-1} z^{N-1}}{1 + a_1 z^{-1} + ... + a_{M-1} z^{M-1}}$$

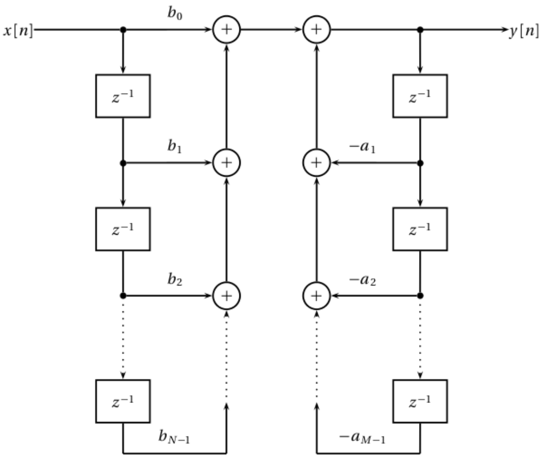


Figure 7.24 Direct Form implementation of an IIR filter.

Direct form I IIR implementation inverted

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

By the commutative properties of the z-transform, we can invert the order of computation to turn the Direct Form I structure into a new structure.

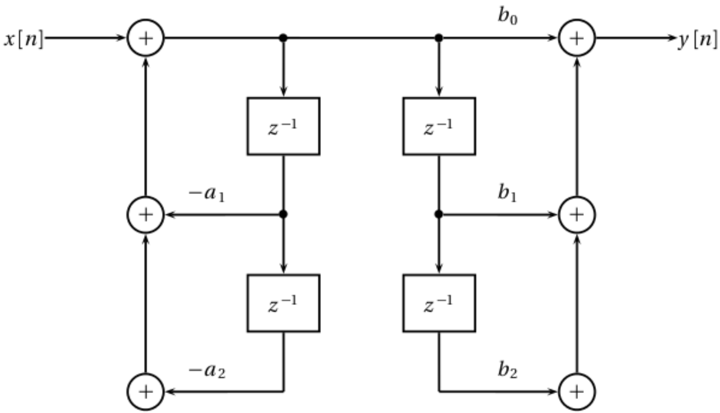


Figure 7.25 Direct form I with inverted order.

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

We can then combine the parallel delays together. This implementation is called Direct Form II; its obvious advantage is the reduced number of the required delay elements (hence of memory storage).

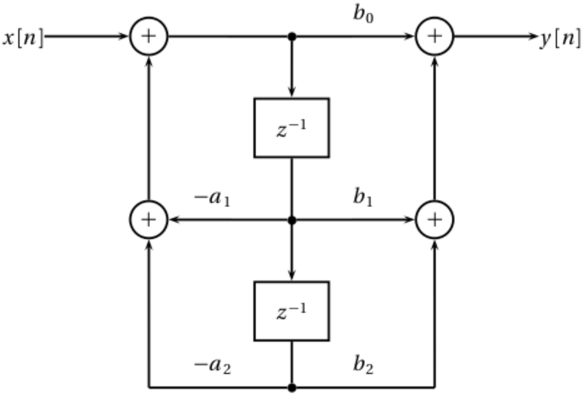


Figure 7.26 Direct Form II implementation of a second-order section.

The cascade structure of N second-order sections is much less sensitive to quantization than the previous Direct form II of order $2 \cdot N$.

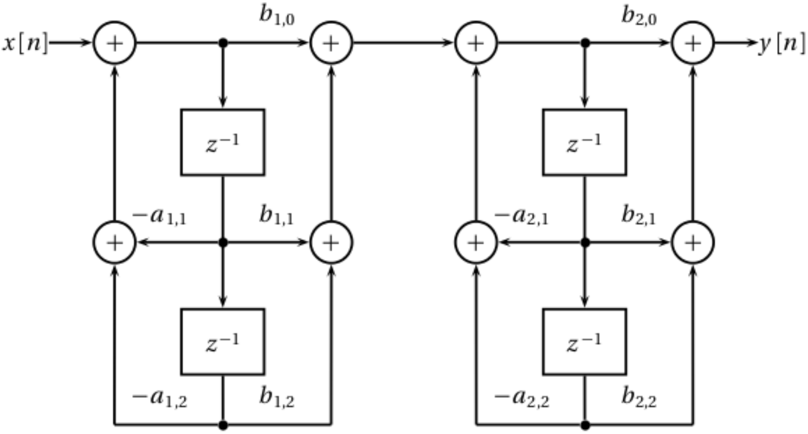


Figure 7.27 4th order IIR: cascade implementation.

FIR vs IIR

FIR, pros:

- Unconditional stability (no poles).
- Precise control of the phase response and, in particular, exact linear phase.
- Optimal algorithmic design procedures.
- Robustness with respect to finite numerical precision hardware.

FIR, cons:

- Longer input-output delay.
- Higher computational cost with respect to IIR solutions.

IIR, pros:

- Lower computational cost with respect to an FIR with similar behavior.
- Shorter input-output delay.
- Compact representation.

IIR, cons:

- Stability is not guaranteed.
- Phase response is difficult to control.
- Design is complex in the general case.
- Sensitive to numerical precision.

1 Alan V. Oppenheim and Ronald W. Schafer. *Discrete-time signal processing, 3rd Ed*. Prentice Hall. 2010. Sections 7.2 and 7.3.

2 Paolo Prandoni and Martin Vetterli. Signal processing for communications. Taylor and Francis Group, LLC. 2008. Sections 5.3.2, 7.3, and 7.4.2.

3 Oliver Hinton. Digital Signal Processing Resources for EEE305 Course. Chapter 5. www.staff.ncl.ac.uk/oliver.hinton/eee305/