The Labor Share in a Production Network Economy

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Abstract

In this paper, I provide the first nonparametric decomposition for the variations in the labor share for a production network economy with distortions. This decomposition uncovers the firm-level granular sources of variations for the labor share and segments them in supply and demand shocks. The firms' share of revenue that reaches directly or indirectly labor compensation, or *payment centralities*, act as sufficient statistics that measure the strength of the response of the aggregate labor share to specific supply and demand shocks at the firm level. Using the input-output tables for the United States, I estimate sectoral *payment centralities*, showing that they are positively correlated with sectoral markdowns, negatively correlated with sectoral intermediate input intensity, and have fallen over the last two decades. Variations in sectoral *payment centralities* explain 99.1% of the labor share volatility, and stronger sectoral distortions in a handful of industries (e.g., credit intermediation and computers and electronics) drove the reductions in the labor share in the United States over the XXIst century.

1 Introduction

Modern economies are a complex web of market interactions shaped by the decisions of billions of agents. These economic interactions rely on multilayered networks through which disaggregated flows of goods, services, and payments circulate. Within these complex economies, understanding the granular sources for aggregate variations is essential for a theory of macroeconomic aggregation that relies on realistic microeconomic network connectivity.

In this paper, I utilize a neoclassical representative household model with production networks, distortions, and endogenous labor supply. My main contribution to the theory of macroeconomic aggregation is to provide the first nonparametric decomposition for the variations of the labor share in a production network economy with distortions. This decomposition uncovers the microeconomic granular supply and demand channels that drive variations in the labor share. A set of sufficient statistics quantifies the strength of these channels. I demonstrate that the shares of firms' revenue that directly or indirectly reach labor compensation, or firms' *payment centralities*, play a crucial role in measuring the impact on the labor share in response to microeconomic supply and demand shocks. The underlying intuition behind these payment centralities is that tracing the position of a firm within a specific network structure allows us to comprehend how firms' revenue reaches final expenditure, either as factoral compensation or rents.

Contribution to the Production Network Literature

My labor share decomposition is related to Bigio & La'O (2020), who obtain the first-order variation for the labor wedge around the efficient equilibrium in a production network representative household economy with one type of endogenous labor. The labor wedge measures the effect of distortions on the labor supply decision, and in equilibrium, it equals the labor share. Their article shows that around the efficient equilibrium, the first-order variation for the labor share depends exclusively on distortions, and the sales distribution is a sufficient statistic for these aggregate effects.

My decomposition extends the first-order local approximation for the labor share to any equilibrium with distortions. I show that for distorted equilibria, variations in distortions are no longer the only channel that affects the aggregate labor share, and changes in the economy's demand structure also have to be considered. By demand structure variations, I refer to recompositions in how: (i) the representative household uses their expenditure across firms, (ii) firms segment their costs across labor and intermediate inputs, and (iii) firms segment their costs across intermediate input suppliers. These recompositions might be exogenous or arise endogenously in response to changes in relative prices. The main lessons from my decomposition are that the labor share rises as (i) firm-level distortions fall, more so for large firms with high payment centralities; (ii) as firms shift their costs from intermediate inputs to labor; and (iii) as the representative household or firms shift their expenditure on final or intermediate goods and services from low to high payment centrality firms.

Additionally, I show that the aggregate labor supply's optimal composition satisfies symmetry between labor types in their value-added to labor income ratios (*distortion centralities*). The distortion centralities measure how undervalued labor types are due to distortions. The intuition behind symmetry in distortion centralities as a characterization of the representative household's solution is that when labor income faces distortions, the best a representative household can do is to allow all types of labor to be equally undervalued. I show that when an endogenous labor supply is allowed for, the aggregate TFP decomposition from Baqaee & Farhi (2020) is simplified, and instead of tracking the variations for each component of the labor income distribution, it is sufficient to follow the variation for the aggregate labor share. Hence, the aggregate labor share decomposition introduced by this paper also captures the microeconomic granular sources of variation behind aggregate TFP.

Contribution to the Labor Share Literature

The decline in the labor's share of GDP at the global level and in many countries like the United States since the early 1980s is well documented. For the United States, this decline became sharper in the 2000s. Most of the explanations for this decline go back to the standard within-between accounting decomposition

$$d\,\Gamma = \underbrace{\sum_i \lambda_i \, d\,\Lambda_i}_{\text{Within}} + \underbrace{\sum_i \Lambda_i \, d\,\lambda_i}_{\text{Between}}, \tag{1}$$

where Γ denotes the aggregate labor share, and for firm or industry i, λ_i is the sales share, and Λ_i is the labor's revenue share. For the United States, the literature agrees on two things. First, the within-industry component has predominantly driven the decline of the labor share. Hence, some industry-level labor shares have declined, and the processes of industrial structural transformation captured by the between-industry channel was not the main driver. Second, within industries, the reallocation between firms is the main driver for the decline in the labor share. In other words, it is not that firms' labor cost share is changing but that the market concentration is rising on firms with low labor cost intensity or high markups.

Nevertheless, no consensus exists on the causes underlying the between-firm-within-industry-driven labor share decline. Elsby, Hobijn, & Şahin (2013) attribute the within-industry variations to reductions in the payroll share by those industries that faced the biggest rises in their import exposure. This narrative relies on offshoring the labor-intensive component of the US production through trade integration, which is problematic because it counterfactually implies that the between component will be strong and that labor compensation in labor-abundant countries such as China, Mexico, and India should have risen. In reality, the labor share also declined for these economies. Karabarbounis & Neiman (2014) associate the within-industry reductions to the decrease in the relative price of investment goods, attributed to advances in information technology and the computer age, which has allowed firms to substitute labor for capital. The problem with this explanation is that it requires an elasticity of substitution between capital and labor above 1, when the bulk of the literature suggests an elasticity of below 1 (Hamermesh, 1993; Antras, 2004; Lawrence, 2015; Oberfield & Raval, 2021). Piketty (2014) emphasizes the role of labor market institutions, such as unions and the minimum wage. However, this explanation seems problematic as most countries experienced a decline in the labor share, not only those with deunionization. The issue with the previous three narratives is that they assume that the main driver behind the decline in the labor share was the within-firm-within-industry variations due to trade integration, capital-labor substitutability, or institutional changes. However, Autor, Dorn, Katz, Patterson, & Van Reenen (2020) show that higher market concentration and reallocation of workers towards "superstar firm" with high markups and low labor shares of value added was the main driver behind the within-industry variations.

The previous explanations rely on models with an aggregate production function or firm heterogeneity within a single sector. The first contribution of this paper is to introduce a theory for the variation in the labor share for multisector economies with production networks, distortions, and endogenous labor supply. In its most general form, my model allows for heterogeneity in the production functions and distortions firms face within and between sectors, opening the door for general reallocation across all firms. In the context of a nonparametric constant returns to scale economy, Theorem 1 provides accounting decompositions for the labor share for different types of labor (e.g., the labor share for specific skills, sectors, or firms), and Theorem 2 provide decompositions for the aggregate labor share. While the within-between accounting decomposition from equation (1) provides variations driven by the firm- or industry-level sales share λ_i and the labor share Λ_i, both of which are equilibrium objects, Theorems 1 and 2 provide variations for the labor share driven by changes in firm-level: (i) distortions, (ii) final expenditure intensity, (iii) labor cost intensity, and (iv) intermediate input cost intensity. These results provide analytical expressions with sufficient statistics that account for realistic network connectivity that allows us to empirically study the underlying granular sources of variations for the labor share in terms of the model's microeconomic primitives instead of the model's general equilibrium outcome. Theorem 3 links the variation in the labor share with the aggregate TFP response due to reallocation between firms. These accounting decompositions attempt to fill the gap described by Autor et al. (2020) for an explicit and cleanly identified quantitative macro model that measures how much of the decline in the labor share is due to underlying changes in the competitive, technological, or demand-driven conditions.

The empirical implementation of these accounting decompositions uses the input-output tables from 1997 to 2021 for the United States. I estimate the sectoral payment centralities, and show that they have fallen for this period, which indicates that rising distortions have shrunk the firms' share of revenue that reaches labor compensation. Additionally, sectoral payment centralities are positively correlated with markdowns, negatively correlated with intermediate input cost intensity, and uncorrelated with the sectors' size. Changes in payment centralities explain 99.1% of the aggregate labor share volatility.

During these years, the labor share felt, mainly in the 2000s, and the aggregate effect from higher sectoral distortions drove the reduction in levels and volatility. For instance, without changes in distortions between 1997 and 2021, the aggregate labor share would have been 1.51% higher, and in the absence of changes in distortions between 2001 and 2009, it would have been 4.78% higher. Counterfactual exercises in which I leave out one of the channels at the time, indicate the main granular drivers for this decline. Between 1997 and 2021, the main drivers were higher demand for final goods from the computers and electronics sector (1.34%), lower labor cost intensity in the wholesale trade industry (1.10%), and higher distortions in the credit intermediation sector (0.80%). The sharper reduction on the labor share in the 2000s had as its main drivers, the stronger demand from households from the construction (1.52%), the computers and electronics (0.91%), and the motor vehicle (0.69%) industries, and the stronger distortions in the internet and information services (0.53%), and the computers and electronics industry (0.50%).

The previous non-parametric decompositions allow us to measure the role that each of the granular channels has when we observed the variations in distortions and the economy's demand structure. However, adjustment in the demand structure might arise endogenously in response to other shocks. For this reason, I implement a parametric CES version of the model that evaluates the general equilibrium effects from a manifold of shocks in sectoral technologies and distortions. A parametric version of the model that allows for heterogeneity across sectors in elasticities of substitution between labor and intermediate inputs, and between intermediate inputs, explains 94.31% of the observed changes in the aggregate labor share between 1997 and 2021. Additionally, I find that the magnitude of the variation in the labor share in response to sectoral shocks in distortions correlates with the industrial payment centralities.

Related Literature

This paper relates to the literature on production networks that builds on the canonical multisector models from Hulten (1978) and Long & Plosser (1983). The main emphasis of this literature has been on the propagation of sectoral productivity shocks (Foerster et al., 2011;

Horvath, 1998, 2000; Dupor, 1999; Acemoglu et al., 2012, 2016; Carvalho et al., 2021). However, the same models have been used to study the propagation of sectoral distortions under specific (Basu, 1995; Ciccone, 2002; Yi, 2003; Jones, 2011; Asker et al., 2014) and generic (Jones, 2013; Baqaee, 2018; Liu, 2019; Baqaee & Farhi, 2020; Bigio & La'O, 2020) input-output structures. The literature on production networks belongs to the broader attempt to map the aggregate effects from "granular" microeconomic shocks that follow the seminal work from Gabaix (2011). My model nests all of these environments and shocks as specific cases.

Layout

The structure of the paper is as follows. Section 2 introduces a simple economy that shows how aggregate production function models might inadequately miss labor share variations driven by reallocation between firms. Section 3 introduces the general representative household multisector input-output model with distortions and endogenous labor supply. Section 4 characterizes the equilibrium and the centrality measures. Section 5 presents sufficient statistics for the labor share variations and TFP under a nonparametric environment. Section 6 describes the data and the quantitative implementation that uncovers the granular sources behind the labor share decline. Section 7 introduces a parametric setting that disciplines endogenous variations and evaluates the effects on the labor share from a manifold of shocks. Section 8 concludes.

2 A Tale of Two Firms

This section considers a simple economy with two firms that provide an example of the blindness of aggregate production function models to reallocation effects on the labor share. This model is the most simple case for the general environments introduced in Sections 3 and 7 that generates opposing results with an aggregate production function model.

Let me start by considering an economy with one firm, one good, and a continuum of measure one of symmetric households. Assume that the households own the firm and supply its labor to the firm, and the firm only uses labor to produce the goods the households consume. Households do not save, so all their income is used to acquire consumption goods from the firm. Additionally, assume a reduced form restriction in which the firm's marginal cost is a fraction μ of the price. The cost-share μ is the representative firm's markdown (the inverse of the markup). Hence, the labor share is given by $\Gamma = \mu$. Consequently, a μ increase undoubtedly rises Γ and

$$\frac{\partial\, log\, \Gamma}{d\, log\, \mu} = 1.$$

Let me consider an analogous environment with two firms that produce differentiated goods. Both firms only require labor, i.e., the firm i's output depends linearly on their labor demand $y_i = \ell_i$. The firm i's cost-share is given by μ_i, with $\mu_1 \geq \mu_2$ and $\mu_1 + \mu_2 = 1$. A representative household centralizes the households' decisions and operates with the following normalized CES preferences

$$\frac{Y}{\overline{Y}} = \left(\frac{1}{2} \left(\frac{C_1}{\overline{C}_1} \right)^{\frac{\varrho - 1}{\varrho}} + \frac{1}{2} \left(\frac{C_2}{\overline{C}_2} \right)^{\frac{\varrho - 1}{\varrho}} \right)^{\frac{\varrho}{\varrho - 1}}$$

where ϱ stands for the elasticity of substitution, Y for real GDP, C_i is the household's consumption of goods from firm i, and the overlined variables correspond to the equilibrium values. Under these preferences, households use 50% of their expenditure to acquire goods from each firm. Equilibrium requires $y_1 = C_1$, $y_2 = C_2$, and $1 = \ell_1 + \ell_2$.

When markdowns are symmetric, each firm hires 50% of the labor, i.e., $\ell_1 = \ell_2 = 1/2$. However, as soon as asymmetries in cost-shares arise, firm 2 exploits its stronger monopolistic power by demanding less labor, and in equilibrium, a higher share of workers is allocated to firm 1. To be more precise, the equilibrium labor allocation follows the markdowns, i.e., $\ell_i = \mu_i$ for $i \in \{1, 2\}$. Consequently, relative to an economy with symmetric markdowns, there is an excess of production of goods from firm 1, and under production of goods from firm 2. Hence, the marginal productivity on real GDP from the labor demand of firm one is low, while for firm two, it is high. These wedges in aggregate marginal productivity between firms result from labor misallocation driven by heterogeneous markdowns. Aggregate output could improve by moving workers from firm 1 to firm 2.

The labor share in equilibrium equals the average markdown, i.e., $\Gamma = \frac{1}{2}(\mu_1 + \mu_2) = \frac{1}{2}$. Therefore, irrespective of the combination of markdowns, the labor share will be 50% of aggregate income. The labor share elasticity in response to markdown variations is, to a first-order, given by

$$\frac{\partial \, log \, \Gamma}{\partial \, log \, \mu_1} = \mu_1 + \frac{1}{2} \left(\varrho - 1 \right) \left(\mu_1 - \mu_2 \right); \qquad \frac{\partial \, log \, \Gamma}{\partial \, log \, \mu_2} = \mu_2 - \frac{1}{2} \left(\varrho - 1 \right) \left(\mu_1 - \mu_2 \right).$$

As the markdown for a firm rises, the share of its revenue used to cover labor costs increases, and the labor share augments; this is the first channel in both equations. Additionally, the price of their goods falls, and under preference substitutability (i.e., $\varrho > 1$), the household will shift their expenditure toward the firm that becomes relatively cheaper, forcing labor to relocate toward this firm, making labor compensation more dependent on the firm that increased their cost share; this is the second channel in both equations. An increase in the cost share for the first firm will undoubtedly raise the labor share. When the second firm receives the markdown shock and preferences are highly substitutable (i.e., $\varrho > \left(\mu_1 - \mu_2 \right)^{-1}$), the expenditure shift towards the second firm dominates and the labor share falls.

Both shocks raise the average economy's cost share, but the effect on the labor share depends on the distribution of markdowns and the household's elasticity of substitution. This tale of two firms shows how the representative firm's model lesson of strict monotonicity between markdowns and the labor share does not extend to a model with multiple firms and labor reallocation. Hence, an explanation of the sources of variation for the labor share that relies on an aggregate production model can be affected by its blindness to the reallocation effects on the labor share.

3 General Framework

In this section, I set up a static nonparametric general equilibrium model with constant-returns-to-scale (CRS) for representative household economies with N sectors. Sector $i \in \mathscr{N} = \{1, \cdots, N\}$ consists of two types of firms: (i) a unit mass of monopolistic competitive firms indexed by $z_i \in [0, 1]$ producing differentiated goods, and (ii) a perfectly competitive producer that aggregates the industry's differentiated goods into a uniform sectoral good consumed by the household or used by other firms as intermediate inputs. Firms differ along three dimensions: first, monopolistic firms across sectors operate under different technologies; second, monopolistic firms within sectors have heterogeneous input demand; and third, sectoral aggregators face different distortions. The representative household consumes sectoral goods using the income received from their endogenous supply of different types of labor and rebated profits. $h \in \mathscr{H} = \{1, \cdots, H\}$ identifies the different types of labor the representative household supplies.

3.1 Production

Monopolistic firms within sectors produce differentiated goods using the same technology. The production for firm z_i in sector i follows

$$y_{z_i} = A_i \, Q_i \left(L_{z_i}, X_{z_i} \right), \quad L_{z_i} = A_i^\ell \, Q_i^\ell \left(\left\{ A_{ih}^\ell \, \ell_{z_i h} \right\}_{h \in \mathscr{H}} \right), \quad X_{z_i} = A_i^x \, Q_i^x \left(\left\{ A_{ij}^x \, x_{z_i j} \right\}_{j \in \mathscr{N}} \right), \tag{2}$$

where y_{z_i} stands for output, A_i is the sector-specific Hicks-neutral productivity term. L_{z_i} is the labor composite that depends on the productivity A_i^ℓ. $\ell_{z_i h}$ is the amount of labor of type h and is influenced by the productivity A_{ih}^ℓ. X_{z_i} is the intermediate input composite that depends on the productivity A_i^x. $x_{z_i j}$ is the amount of intermediate input goods purchased from sector j and is influenced by the productivity A_{ij}^x.

The technologies $Q_i : \mathbb{R}_+^2 \to \mathbb{R}_+$, $Q_i^\ell : \mathbb{R}_+^H \to \mathbb{R}_+$, and $Q_i^x : \mathbb{R}_+^N \to \mathbb{R}_+$ are neoclassical and satisfy the following regularity conditions: they are positive, finite, and for the set of labor types and intermediate inputs for which there is effective demand, they are monotonically increasing, twice continuously differentiable, strictly concave, and the Inada conditions hold.

The profits for firms z_i are given by

$$\pi_{z_i} = p_{z_i} y_{z_i} - \underbrace{\sum_{h \in \mathscr{H}} w_h \, \ell_{z_i h}}_{= p_{z_i}^\ell L_{z_i}} - \underbrace{\sum_{j \in \mathscr{N}} p_j \, x_{z_i j},}_{= p_{z_i}^x X_{z_i}} \tag{3}$$

where p_{z_i} is the price of its output, $p_{z_i}^\ell$ is the price for the labor composite, $p_{z_i}^x$ is the price for the intermediate input composite, w_h is the wage received from the supply of labor of type h, and p_j is the market price for the good produced by the competitive aggregator in sector j.

The competitive firm in sector i guarantees a homogeneous good by aggregating sectoral production using the following CES production function

$$y_i = \left(\int {y_{z_i}}^{\mu_i} \, dz_i \right)^{\frac{1}{\mu_i}}, \tag{4}$$

where $\mu_i \leq 1$ stands for the sector-specific markdown, and y_{z_i} represents the demand of goods produced by firm z_i. The aggregator takes prices as given and maximizes profits given by $\bar{\pi}_i = p_i y_i - \int p_{z_i} y_{z_i} \, dz_i$.

3.2 Consumption

There is a continuum of measure one of symmetric infinitesimal households that take prices and wages as given. Consequently, the notation of the model becomes simpler by assuming a representative household with preferences given by the utility function $U\left(Y, L\right)$, where Y stands for real output and L for the aggregate labor supply. Real output $Y = Q_Y\left(\{C_i\}_{i \in \mathscr{N}}\right)$ depends on a composite of final consumption C_i of goods from sector i. The aggregate labor supply $L = F\left(\{L_h\}_{h \in \mathscr{H}}\right)$ depends on a composite of labor types. The utility $U : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfies the following regularity conditions: $U_Y > 0$, $U_L < 0$, twice continuously differentiable, strictly concave, and the Inada conditions hold. The aggregation technologies $Q_Y : \mathbb{R}^N_+ \to \mathbb{R}_+$ and $F : \mathbb{R}^H_+ \to \mathbb{R}_+$ are neoclassical: positive, finite, homogeneous of degree one, and for the set of goods for which there is effective final demand or for the types of labor that there is effective supply, it is monotonically increasing, twice continuously differentiable, strictly concave, and the Inada conditions hold.

The representative household has a budget constraint given by

$$\begin{gathered} E = p_Y \, Y = \sum_{i \in \mathscr{N}} p_i C_i \leq w \, L + \Pi, \\ w \, L = \sum_{h \in \mathscr{H}} w_h \, L_h, \quad \text{and} \quad \Pi = \sum_{i \in \mathscr{N}} \left(\bar{\pi}_i + \int \pi_{z_i} \, dz_i \right). \end{gathered} \tag{5}$$

Nominal GDP is captured by E, and it must not be greater than income; the latter includes labor income $J_h = w_h L_h$, and dividend income Π.

3.3 Market Clearing

The technologies, productivities, markdowns, and ownership distributions are primitives. Monopolistic competition is the only source of market imperfections. These distortions change how income reaches households. Hence, the goods market clearing is given by

$$y_i = C_i + \sum_{j \in \mathscr{N}} x_{ji} \quad \forall i \in \mathscr{N}, \tag{6}$$

where $x_{ji} \equiv \int x_{z_j i} \, dz_j$ is the total amount of intermediate inputs from sector i bought by all monopolistic firms in sector j. Labor market clearing requires $L_h = \ell_h \ \forall h \in \mathscr{H}$, with $\ell_h = \sum_{i \in \mathscr{N}} \int \ell_{z_i h} \, dz_i$.

4 Equilibrium, Centralities, and Information Theory

In this section, I first characterize the equilibrium for this economy and then introduce centrality measures that portray each firm's role in the economy. This section is essential to understanding the first-order approximations that make up this paper's main contribution.

4.1 Equilibrium Characterization

Let $e \equiv (\mathscr{A}, \mu)$ represent the aggregate state, and $\mathscr{E}$ denote the measurable collection of all possible realizations for this state. The matrix $\mathscr{A} \equiv (A, A_\ell, A_x, \underline{A}_\ell, \underline{A}_x)$ collects all productivity measures,[1] and sectoral markdowns are captured by $\mu \equiv (\mu_1, \cdots, \mu_N)'$.

For this economy, a mapping of the realization of the aggregate state to an allocation $\vartheta =$

[1] $A \equiv (A_1, \cdots, A_N)'$, $A_\ell = \left(A_1^\ell, \cdots, A_N^\ell\right)'$, $A_x \equiv (A_1^x, \cdots, A_N^x)'$, $\underline{A}_\ell = \left(\underline{A}_1^\ell, \cdots, \underline{A}_N^\ell\right)'$, $\underline{A}_x = (\underline{A}_1^x, \cdots, \underline{A}_N^x)'$, $\underline{A}_i^\ell = \left(A_{i1}^\ell, \cdots, A_{iH}^\ell\right)'$, and $\underline{A}_i^x = (A_{i1}^x, \cdots, A_{iN}^x)'$.

$(\vartheta(e))_{e\in\mathscr{E}}$ and the price system $\rho = (\rho(e))_{e\in\mathscr{E}}$ is represented by the set of functions

$$\vartheta(e) \equiv \left\{ Y(e), L(e), \{L_h(e)\}_{h\in\mathscr{H}}, \left\{ y_i(e), C_i(e), \left(y_{z_i}(e), \{\ell_{z_ih}(e)\}_{h\in\mathscr{H}}, \{x_{z_ij}(e)\}_{j\in\mathscr{N}} \right)_{z_i\in[0,1]} \right\}_{i\in\mathscr{N}} \right\},$$

$$\rho(e) \equiv \left\{ w(e), p_Y(e), \{w_h(e)\}_{h\in\mathscr{H}}, \left\{ p_i(e), \left(p_{z_i}(e), p^{\ell}_{z_i}(e), p^{x}_{z_i}(e) \right)_{z_i\in[0,1]} \right\}_{i\in\mathscr{N}} \right\}.$$

Definition 1. For any realization of the aggregate state e in the state space $\mathscr{E}$, an equilibrium is the combination of an allocation and a price system (ϑ, ρ) such that:

(i) given wages $\{w_h(e)\}_{h\in\mathscr{H}}$ and prices $\{p_j(e)\}_{j\in\mathscr{N}}$, monopolistically competitive firms' labor $\{\ell_{z_ih}(e)\}_{h\in\mathscr{H}}$ and intermediate input demand $\{x_{z_ij}(e)\}_{j\in\mathscr{N}}$, output $y_{z_i}(e)$, and price $p_{z_i}(e)$ maximize their profits;

(ii) given prices $[p_{z_i}(e)]_{z_i\in[0,1]}$, aggregator firms' good demand $[y_{z_i}(e)]_{z_i\in[0,1]}$, and output $y_i(e)$ maximize their profits;

(iii) given prices $\{p_i(e)\}_{i\in\mathscr{N}}$ and wages $\{w_h(e)\}_{h\in\mathscr{H}}$, the representative household's consumption $\{C_i(e)\}_{i\in\mathscr{N}}$ and labor supply $\{L_h(e)\}_{h\in\mathscr{H}}$ maximize utility while satisfying their budget constraints;

(iv) goods and labor markets clear.

Proposition 1. *The set of functions (ϑ, ρ) are an equilibrium if and only if the following set of conditions are jointly satisfied*

$$\frac{\partial Y(e)/\partial C_j(e)}{\partial Y(e)/\partial C_i(e)} = \mu_i(e) \left(\frac{y_i(e)}{y_{z_i}(e)} \right)^{1-\mu_i(e)} \frac{\partial y_{z_i}(e)}{\partial x_{z_ij}(e)} \quad \forall i,j \in \mathscr{N},\ \forall z_i \in [0,1], \tag{7}$$
$$\textit{and}\ \forall e \in \mathscr{E}\ \textit{such that}\ C_i(e) > 0,\ C_j(e) > 0,\ \textit{and}\ x_{z_ij}(e) > 0,$$

$$-\frac{w_b(e)}{w_h(e)} \frac{U_L(e)}{U_Y(e)} \frac{\partial L(e)/\partial L_h(e)}{\partial Y(e)/\partial C_i(e)} = \mu_i(e) \left(\frac{y_i(e)}{y_{z_i}(e)} \right)^{1-\mu_i(e)} \frac{\partial y_i(e)}{\partial \ell_{ib}(e)} \quad \forall i \in \mathscr{N}, \tag{8}$$
$$\forall z_i \in [0,1],\ \forall h,b \in \mathscr{H},\ \textit{and}\ \forall e \in \mathscr{E}\ \textit{such that}\ C_i(e) > 0,\ \textit{and}\ \ell_{ib}(e) > 0,$$

and resource constraints

$$y_i(e) = C_i(e) + \sum_{j\in\mathscr{N}} \int x_{z_ji}(e)\, dz_j \quad \forall i \in \mathscr{N},$$
$$\textit{and} \qquad L_h(e) = \sum_{i\in\mathscr{N}} \int \ell_{z_ih}(e)\, dz_i \quad \forall h \in \mathscr{H}. \tag{9}$$

Proposition 1 identifies the set of equilibrium allocations. In equation (7), for a firm z_i, the markdown-adjusted marginal productivity from using the good from sector j as an intermediate input has to equate the household's marginal rate of substitution between goods i and j. In equation (8), for a firm z_i, the markdown-adjusted marginal productivity from using labor of type b has to equate the household's wage-adjusted marginal rate of substitution between the consumption of the good from sector i and the supply of labor of type h.

Notice that in the set of conditions captured by equation (8), the only thing that is necessary for the existence of an equilibrium relationship between the labor demand from firm z_i and the labor supply of type h, is the representative household's consumption of the goods supplied by sector i. Whenever firm z_i hires labor of type b, and $b \neq h$, the differential wage adjustment w_b/w_h arises in these equilibrium conditions. This wage ratio is a point of difference with Bigio & La'O's (2020) economy, where they only consider the endogenous supply of one factor. A higher w_b/w_h is isomorphic to an increase in the marginal rate of substitution between consumption of good i and labor supply of type h, and in equilibrium, it requires higher marginal productivity in firm z_i for the labor of type b. Additionally, there is an isomorphism between distortionary markdown increases and positive productivity shocks in equations (7) and (8): both will increase the markdown-adjusted marginal productivities from labor and intermediate goods.

Furthermore, a relevant technicality is that Proposition 1 does not require final consumption in each sector. The usual assumption for this type of proof in the production network literature is that $\forall i \in \mathscr{N}$, the representative household's consumption technology satisfies $\partial Y / \partial C_i > 0$ (see Bigio & La'O (2020) and La'O & Tahbaz-Salehi (2022)). However, this assumption does not match the empirical input-output tables, where it is not uncommon to find sectors for which there is no direct registered final consumption, e.g., oil and gas extraction. The less stringent assumption that I make instead is that $\forall h \in \mathscr{H}$, there $\exists i \in \mathscr{N}$ such that for all the firms in this sector, it is possible to establish a direct or indirect demand of labor of type h.

To make the notation cleaner, the definitions and implementation of the model in the following sections are conditional in a specific aggregate state $e \in \mathscr{E}$, e.g., $\mu(e)$ is portrayed by μ. Finally, I will abstract from within sector firm heterogeneity by imposing the assumption of symmetry, i.e., $\ell_{ih} = \ell_{z_ih}$, and $x_{ij} = x_{z_ij}$ $\forall z_i \in [0,1]$, $\forall i, j \in \mathscr{N}$ and $\forall h \in \mathscr{H}$.[2] For this reason, I will refer indistinguishably to firm z_i as firm i.

4.2 Measures of Centrality

For the following measures, downstream or cost centrality refers to the propagation of costs from the supply of labor or intermediate inputs through supply chains, and upstream or revenue centrality refers to the propagation of money flows from the demand for labor and goods through payment chains. Table I summarizes the direct centralities and Table II the network centralities.

4.2.1 Direct Centralities

The vectors $\omega_\ell \equiv \left(\omega_1^\ell, \cdots, \omega_N^\ell\right)'$ and $\omega_x \equiv \left(\omega_1^x, \cdots, \omega_N^x\right)'$ portray the direct cost centralities from composites. Its elements $\omega_i^\ell \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_i^\ell} = \frac{p_i^\ell L_i}{c_i(\vartheta,\rho)}$ and $\omega_i^x \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_i^x} = \frac{p_i^x X_i}{c_i(\vartheta,\rho)}$ capture respectively firm i's cost elasticities to p_i^ℓ and p_i^x, and in equilibrium they equal the cost share of the labor and intermediate input composites. For this reason, $\omega_i^\ell + \omega_i^x = 1$.

The matrices $\widetilde{\Omega}_\ell$ and $\widetilde{\Omega}_x$ depict direct labor and intermediate input downstream centralities. Its elements $\widetilde{\Omega}_{ih}^\ell \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log w_h} = \frac{w_h \ell_{ih}}{c_i(\vartheta,\rho)}$ and $\widetilde{\Omega}_{ij}^x \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_j} = \frac{p_j x_{ij}}{c_i(\vartheta,\rho)}$ capture respectively firm i's cost elasticities to w_h and p_j, and in equilibrium they equal the cost share of the labor type h and the good from firm j. The fact that $\sum_{h\in\mathscr{H}} \widetilde{\Omega}_{ih}^\ell + \sum_{j\in\mathscr{N}} \widetilde{\Omega}_{ij}^x = 1$ indicate that all costs come from labor or intermediate inputs.

Using these definitions, I obtain the labor network $\alpha \equiv diag\left(\omega_\ell\right)^{-1} \widetilde{\Omega}_\ell$ and the input-output network $\mathscr{W} \equiv diag\left(\omega_x\right)^{-1} \widetilde{\Omega}_x$, where $diag$ stands for the diagonal operator. Its elements $\alpha_{ih} \equiv \frac{\partial \log p_i^\ell L_i}{\partial \log w_h} = \frac{w_h \ell_{ih}}{p_i^\ell L_i}$ and $\omega_{ij} \equiv \frac{\partial \log p_i^x X_i}{\partial \log p_j} = \frac{p_j x_{ij}}{p_i^x X_i}$ capture respectively firm i's composite cost elasticities to w_h and p_j, and in equilibrium they equal the corresponding composites' cost share of the labor supplied by households of type h and the good from firm j. Notice that $\sum_{h\in\mathscr{H}} \alpha_{ih} = 1$ and $\sum_{j\in\mathscr{N}} \omega_{ij} = 1$.

From here, I can define the revenue-based upstream centrality matrices $\Omega_\ell \equiv diag\left(\mu\right) \widetilde{\Omega}_\ell$

[2] As a consequence $y_i = y_{z_i}$, $p_i = p_{z_i}$, $L_i = L_{z_i}$, and $X_i = X_{z_i}$.

and $\Omega_x \equiv diag\,(\mu)\,\widetilde{\Omega}_x$. Since $\mu_i \in (0,1]$ $\forall i \in \mathscr{N}$, $\widetilde{\Omega}_\ell \succcurlyeq \Omega_\ell$ and $\widetilde{\Omega}_x \succcurlyeq \Omega_x$, where $\succcurlyeq$ stands for elementwise greater than or equal to. Its elements $\Omega^\ell_{ih} \equiv \frac{\partial\, log\, S_i}{\partial\, log\, w_h} = \frac{w_h\,\ell_{ih}}{S_i}$ and $\Omega^x_{ij} \equiv \frac{\partial\, log\, S_i}{\partial\, log\, p_j} = \frac{p_j\,x_{ij}}{S_i}$ capture respectively the elasticities of firm i's sales to w_h and p_j, and in equilibrium they equal the sales share of payments for labor type h and goods from firm j. Additionally, $\Omega^\pi_i = \frac{\pi_i}{S_i}$ portrays the equilibrium sales share of firm i's profits rebated back to the hosusehold. The fact that $\sum_{h\in\mathscr{H}} \Omega^\ell_{ih} + \sum_{j\in\mathscr{N}} \Omega^x_{ij} + \Omega^\pi_i = 1$ indicate that all revenue generated by firm i ends as payments for labor, intermediate inputs, or dividends.

Finally, for the representative household, the consumption vector $\beta = (\beta_1, \cdots, \beta_N)'$ contains elements $\beta_i \equiv \frac{\partial\, log\, E}{\partial\, log\, p_i} = \frac{p_i\,C_i}{E}$ that capture the final expenditure elasticity to p_i, and in equilibrium they equal the expenditure share on the good supplied by firm i. For this reason $\sum_{i\in\mathscr{N}} \beta_i = 1$.

4.2.2 Network Adjusted Centralities

The firm-to-firm downstream centrality matrix or cost-based Leontief inverse matrix is given by $\widetilde{\Psi}_x \equiv \left(I - \widetilde{\Omega}_x\right)^{-1} \equiv \sum_{q=0}^{\infty} \widetilde{\Omega}^q_x$. Its element $\widetilde{\psi}^x_{ij}$ captures the centrality of intermediate inputs supplied by firm j on the costs of firm i. Similarly, I define the firm-to-firm upstream centrality matrix or revenue-based Leontief inverse matrix $\Psi_x \equiv (I - \Omega_x)^{-1} \equiv \sum_{q=0}^{\infty} \Omega^q_x$, where its element ψ^x_{ij} represents the revenue share from firm i that through the payment of intermediate input reaches sales of firm j.

The firm-to-consumer downstream centrality vector or cost-based sales Domar weight is given by $\widetilde{\lambda} \equiv \widetilde{\Psi}'_x\,\beta$. Its element $\widetilde{\lambda}_i \equiv \sum_{j\in\mathscr{N}} \beta_j\,\widetilde{\psi}^x_{ji}$ captures all direct or indirect paths through which the costs of firm i can reach final expenditure. Likewise, $\lambda \equiv \Psi'_x\,\beta$ captures the consumer-to-firm upstream centrality vector or revenue-based sales Domar weight. Its element $\lambda_i \equiv \sum_{j\in\mathscr{N}} \beta_j\,\psi^x_{ji} = S_i/E$ captures all direct or indirect paths through which final expenditure reaches revenue for firm i, and in equilibrium it coincides with the ratio of sales to GDP.

The worker-to-firm downstream centrality matrix is given by $\widetilde{\Psi}_\ell \equiv \widetilde{\Psi}_x\,\widetilde{\Omega}_\ell$. All costs for a firm can be traced back through the production network to some original labor cost and for this reason $\sum_{h\in\mathscr{H}} \widetilde{\psi}^\ell_{ih} = 1$. As a consequence, $\widetilde{\psi}^\ell_{ih}$ is the value-added share by labor of type h on the production process of firm i. In the same way, I define the firm-to-worker upstream centrality matrix $\Psi_\ell \equiv \Psi_x\, \Omega_\ell$, where the element ψ^ℓ_{ih} represents the revenue share from firm i that reaches labor compensation for type h labor. Firm i's *payment centrality* $\psi^\ell_i = \sum_{h \in \mathscr{H}} \psi^\ell_{ih}$ captures the share of revenue from a firm that reaches labor income, and this serves as a measure for how higher distortions modify the upstream flow of revenue from firm i.

The worker-to-consumer downstream centrality vector or cost-based factor Domar weight is given by $\widetilde{\Lambda} \equiv \widetilde{\Psi}'_\ell\, \beta$. Given that $\sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h = 1$, its element $\widetilde{\Lambda}_h$ representes the aggregate value-added by the labor of type h. Similarly, $\Lambda \equiv \Psi'_\ell\, \beta$ portrays the consumer-to-worker upstream centrality vector or revenue-based factor Domar weight. Its element $\Lambda_h \equiv \sum_{i \in \mathscr{N}} \beta_i\, \psi^\ell_{ih} = J_h/E$ captures all direct or indirect paths through which final expenditure reaches labor compensation for type h labor, and in equilibrium it coincides with the ratio of labor income to GDP.

Cost-based centralities are greater than or equal to revenue-based centralities, i.e., $\widetilde{\Psi}_x \succcurlyeq \Psi_x$, $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$, $\widetilde{\lambda} \succcurlyeq \lambda$, and $\widetilde{\Lambda} \succcurlyeq \Lambda$. For this reason, for labor of type h, $\delta_h = \widetilde{\Lambda}_h/\Lambda_h \geq 1$ is a measure of *distortion centrality* that captures how undervalued this type of work is. The larger the distortion centrality, the further away compensation for labor is from the value-added it creates. Types of labor mainly supplied to sectors operating in heavily distorted supply chains will have high distortion centralities, and a higher share of the value added will reach final expenditure via profits.

Finally, in equilibrium, the revenue-based Domar weights and the profit shares satisfy $\sum_{h \in \mathscr{H}} \Lambda_h + \sum_{i \in \mathscr{N}} \Omega^\pi_i\, \lambda_i = 1$.

4.2.3 A Diagrammatic Recap

Figure I illustrates the centrality measures for a vertical economy with three firms, $f, i, j \in \mathscr{N}$, and one type of labor $h \in \mathscr{H}$. Labor h is supplied to firm j ($\widetilde{\Omega}^\ell_{jh}$ and Ω^ℓ_{jh}), firm j supplies intermediate inputs to firm i ($\widetilde{\Omega}^x_{ij}$ and Ω^x_{ij}), firm i supplies intermediate inputs to firm f ($\widetilde{\Omega}^x_{fi}$ and Ω^x_{fi}), and firm f supplies final goods (β_f). Firm f does not demand intermediate inputs from firm j, but it has exposure to j's costs through the demand and supply of intermediate inputs from firm i; the indirect linkages between f and j are captured by $\widetilde{\psi}^x_{fj}$ and ψ^x_{fj}. Firm i does not demand labor, but it has exposure to labor costs through the demand of labor and supply of intermediate inputs from firm j; the indirect linkages between i and h are captured by $\widetilde{\psi}^{\ell}_{ib}$ and ψ^{ℓ}_{ib}. The representative household does not demand goods from firm i, but it is exposure to i's cost through the demand of intermediate inputs and supply of final goods from firm f; the indirect linkages between the representative household and i are captured by $\widetilde{\lambda}_i$ and λ_i. Finally, the household's income comes from the labor compensation and profits from firm j, which are captured by Ω^{ℓ}_{jh} and Ω^{π}_{j}.

5 Labor Income Shares and Aggregate Output

In this section, I derive the nonparametric ex-post sufficient statistics that characterize the first-order variations in prices, labor income shares, value-added shares, and aggregate TFP, in response to supply and demand shocks. I call these measures ex-post because they assume that the necessary variations are observable and do not depend on underlying model primitives. First, I present the price variation in response to shocks and show that these effects propagate downstream through the cost of intermediate and final goods. Second, I characterize the first-order variation for the labor income and value-added shares. Third, I decompose the first-order response for aggregate TFP and establish a connection with the labor income share variations that allows me to identify the granular sources behind the aggregate effects arising from the reallocation of factors and resources across firms. I decompose these granular effects into variations of (i) distortions, (ii) household expenditure patterns, and (iii) firms' labor and intermediate input cost composition.

Definition 2. Supply and Demand Shocks: For supply shocks, I refer to variations in (i) productivities, (ii) markdowns, and (iii) wages. For demand shocks, I refer to variations in cost composition patterns in (i) final expenditure - $d\,\beta$, (ii) labor costs - $d\,\widetilde{\Omega}_{\ell}$, and (iii) intermediate input costs - $d\,\widetilde{\Omega}_{x}$.

5.1 Prices

Proposition 2 captures the network-adjusted response of prices to supply shocks. These shocks propagate downstream through the costs of intermediate inputs and final goods, and the cost-based firm-to-firm and firm-to-consumer centrality measures capture their magnitude.

Proposition 2. *The change in sector i's prices and the GDP deflator are, to a first-order,*

$$
\begin{aligned}
d\log p_i^{\ell} &= -d\log A_i^{\ell} - \sum_{h\in\mathscr{H}} \alpha_{ih}\left(d\log A_{ih}^{\ell} - d\log w_h\right),\\
d\log p_i^{x} &= -d\log A_i^{x} - \sum_{j\in\mathscr{N}} \omega_{ij}\ \left(d\log A_{ij}^{x} - d\log p_j\right),\\
d\log p_i &= -\sum_{j\in\mathscr{N}} \widetilde{\psi}_{ij}^{x}\left(d\log\mathscr{A}_j + d\log\mu_j\right) + \sum_{h\in\mathscr{H}} \widetilde{\psi}_{ih}^{\ell}\, d\log w_h,\\
d\log p_Y &= -\sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\left(d\log\mathscr{A}_i + d\log\mu_i\right) + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h\, d\log w_h,
\end{aligned}
$$

where $d\log\mathscr{A}_i = d\log A_i + \omega_i^{\ell}\, d\log A_i^{\ell} + \omega_i^{x}\, d\log A_i^{x} + \sum_{h\in\mathscr{H}} \widetilde{\Omega}_{ih}^{\ell}\, d\log A_{ih}^{\ell} + \sum_{j\in\mathscr{N}} \widetilde{\Omega}_{ij}^{x}\, d\log A_{ij}^{x}$.

First, prices are only directly influenced by supply shocks. Second, non-Hicks neutral productivity shocks directly influence firms' composite bundle prices. Third, firm i's compound measure of productivity $d\log\mathscr{A}_i$ incorporates Hicks-neutral, labor-specific, and input-specific augmenting productivity shocks, and its effect on prices across all firms and households is isomorphic to an increase in the markdown for firm i. Finally, labor costs directly affect the labor bundle price that propagates through the supply of intermediate inputs to other firms and finally reaches consumption bundle prices.

5.2 Labor Income and Value-Added

Theorem 1 decomposes the endogenous variation of the labor income and value-added shares.

Theorem 1. *The variation of Λ_h and $\widetilde{\Lambda}_h$ are, to a first-order,*

$$
d\Lambda_h = \overbrace{\sum_{i\in\mathscr{N}} \psi_{ih}^{\ell}\,\lambda_i\, d\log\mu_i}^{\substack{\textit{Competitive}\\ \textit{Income}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \psi_{ih}^{\ell}\, d\beta_i}^{\substack{\textit{Final Demand}\\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i\,\lambda_i\, d\widetilde{\Omega}_{ih}^{\ell}}^{\substack{\textit{Labor Demand}\\ \textit{Recomposition}_h}} + \overbrace{\sum_{j\in\mathscr{N}} \psi_{jh}^{\ell} \sum_{i\in\mathscr{N}} \mu_i\,\lambda_i\, d\widetilde{\Omega}_{ij}^{x}}^{\substack{\textit{Intermediate Demand}\\ \textit{Recomposition}_h}}, \tag{10}
$$

$$
d\widetilde{\Lambda}_h = \overbrace{\sum_{j\in\mathscr{N}} \widetilde{\psi}_{jh}^{\ell}\, d\beta_j}^{\substack{\textit{Final Demand}\\ \textit{Value-Added}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\, d\widetilde{\Omega}_{ih}^{\ell}}^{\substack{\textit{Labor Demand}\\ \textit{Value-Added}_h}} + \overbrace{\sum_{j\in\mathscr{N}} \widetilde{\psi}_{jh}^{\ell} \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\, d\widetilde{\Omega}_{ij}^{x}}^{\substack{\textit{Intermediate Demand}\\ \textit{Value-Added}_h}}. \tag{11}
$$

Equation (10) segments the first-order variation of the labor income share into four effects. First, *competitive income* tells us that lower profit margins in sector i will increase Λ_h in a magnitude that is proportional to the sector's size and firm-to-worker centrality, i.e., $\psi^{\ell}_{ih}\,\lambda_i$. Second, according to *final demand recomposition*, as the representative household shifts their expenditure towards sector i, Λ_h rises in a magnitude proportional to the sector's firm-to-worker centrality ψ^{ℓ}_{ih}. Third, *labor demand recomposition* tells us that Λ_h rises with $\widetilde{\Omega}^{\ell}_{ih}$ in a magnitude proportional to the sector i's aggregate cost share $\mu_i\,\lambda_i$. Finally, according to *intermediate demand recomposition*, as sector i shifts their costs towards goods from sector j, Λ_h rises in a magnitude proportional to the aggregate cost share of the consumer of intermediate inputs and the supplier's firm-to-worker centrality, i.e., $\psi^{\ell}_{jh}\,\mu_i\,\lambda_i$.

Equation (11) segments the first-order variation of the value-added share into three channels. First, according to *final demand value-added*, as the representative household shifts their expenditure towards sector i, $\widetilde{\Lambda}_i$ rises in a magnitude proportional to the sector's worker-to-firm centrality or value-added share $\widetilde{\psi}^{\ell}_{ih}$. Second, *labor demand value-added* tells us that $\widetilde{\Lambda}_h$ rises with $\widetilde{\Omega}^{\ell}_{ih}$ in a magnitude proportional to the sector i's cost-based Domar weight $\widetilde{\lambda}_i$. Finally, according to *intermediate demand value-added*, as sector i shifts their costs towards goods from sector j, $\widetilde{\Lambda}_h$ rises in a magnitude proportional to the cost-based Domar weight for the consumer of intermediate inputs and the supplier's worker-to-firm centrality, i.e., $\widetilde{\psi}^{\ell}_{jh}\,\widetilde{\lambda}_i$.

Additionally, equation (10) and equation (11) allow me to make the following conclusions. First, value-added shares are only influenced directly by demand shocks, while labor income shares respond to demand shocks and markdown variations. Second, upstream centralities and markdowns are sufficient statistics to capture the labor income share elasticities, while downstream centralities sufficiently represent the value-added variations. Third, value-added shares are more responsive in levels than labor income shares to demand shocks; remember that cost-based centralities weakly dominate revenue-based centralities.

Theorem 2 characterizes the equilibrium aggregate real output Y and the aggregate labor supply L in terms of the aggregate labor share Γ, which in equilibrium equals the aggregate labor wedge.

Theorem 2. *Assume an aggregate labor supply function* $L = F\left(\{L_h\}_{h\in\mathscr{H}}\right)$ *with elasticities* *equal to the value-added shares, i.e.,* $d\log L/d\log L_h = \widetilde{\Lambda}_h$. *In equilibrium, the aggregate output and labor supply satisfies*

$$\frac{U_L}{U_Y} + \Gamma\,\frac{Y}{L} = 0 \qquad \textit{with} \qquad \Gamma = \sum_{h\in\mathscr{H}} \Lambda_h = \delta_b^{-1} \quad \forall b \in \mathscr{H}, \tag{12}$$

and the variation for Γ *is, to a first-order,*

$$d\,\Gamma = \overbrace{\sum_{i\in\mathscr{N}} \psi_i^{\ell}\,\lambda_i\,d\log\mu_i}^{\substack{\textit{Competitive}\\ \textit{Income}}} + \overbrace{\sum_{i\in\mathscr{N}} \psi_i^{\ell}\,d\,\beta_i}^{\substack{\textit{Final Demand}\\ \textit{Recomposition}}} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i\,\lambda_i\,d\,\omega_i^{\ell}}^{\substack{\textit{Labor Demand}\\ \textit{Recomposition}}} + \overbrace{\sum_{j\in\mathscr{N}} \psi_j^{\ell} \sum_{i\in\mathscr{N}} \mu_i\,\lambda_i\,d\,\widetilde{\Omega}_{ij}^{x}}^{\substack{\textit{Intermediate Demand}\\ \textit{Recomposition}}}. \tag{13}$$

Equation (12) characterizes the aggregate labor supply and the aggregate labor wedge. The labor wedge Γ relates the aggregate marginal rate of substitution with the economy's marginal rate of transformation Y/L. The labor wedge equals the aggregate labor share (i.e., $\sum_{h\in\mathscr{H}} \Lambda_h$), and also the inverse of the distortion centralities for all types of labor. For this reason, the aggregate labor wedge captures how the whole set of economic distortions influences the aggregate labor supply and output. The symmetry in distortion centralities restricts the space of solutions that the representative household entertains as a solution for the composition of the aggregate labor supply. The intuition behind this is that when labor income faces distortions, the best that the representative household can do is to make all types of labor equally undervalued. For example, if there were two types of labor with asymmetric distortion centralities, let me say $\delta_h > \delta_b$, the representative household could obtain higher labor compensation without facing disutility costs by raising the relatively overvalued supply b while reducing the undervalued supply h.

The equivalence between Γ and the aggregate labor share, and the symmetry in distortion centralities, requires that the elasticities for the aggregate labor supply in response to each type of labor are equal to its value-added shares. This assumption comes from the function $F\,(.)$ being CRS. As it will become apparent in Theorem 3, this is the only assumption consistent with a segmentation of the first-order effects on output from TFP and the labor supply.

Equation (13) segments the first-order variation of the aggregate labor share into four channels. These channels are analogous to the mechanisms already introduced by equation (10). However, there are two differences. First, the sector-specific *payment centralities* replace the firm-to-worker upstream centralities as part of the sufficient statistics. Now, Γ increases more as the profit margins shrink in large sectors with high payment centralities or as the representative household or firms shift their demand towards sectors with high payment centralities. Second, up to a first-order, Γ is inelastic to the within-sector composition of the labor bundle (to be more precise, variations in the matrix α); what matters now is the variations in the sectoral cost intensity on labor captured by ω_ℓ.

Corollary 1. ***Bigio & La'O (2020).*** *Without distortions, the variation for Γ is, to a first-order,*

$$d\log\Gamma = \sum_{i\in\mathcal{N}} \lambda_i \, d\log\mu_i.$$

Corollary 1 tells us that Domar weights sufficiently capture the effect from markdowns on the aggregate labor share, i.e., $\frac{d\log\Gamma}{d\log\mu_i} = \lambda_i$. This is the main result that Bigio & La'O (2020) obtain for a representative household production network economy with one type of labor around the efficient equilibrium. Theorems 1 and 2 capture the extension from their findings to any inefficient equilibrium with one or multiple types of labor. The proof for this result is straightforward: around the efficient equilibrium, the payment centralities are unitary (i.e., $\psi_i^\ell = 1 \quad \forall i \in \mathcal{N}$), and due to homogenous of degree one consumption and production aggregators, demand shocks are neutral on the aggregate labor share, up to a first-order.

5.3 Aggregate TFP

Theorem 3 characterizes aggregate real output Y in equilibrium and its first-order variation around the equilibrium.

Theorem 3. *In equilibrium, real GDP satisfies*

$$Y = Q_Y\left(\{C_i\}_{i\in\mathcal{N}}\right) = TFP\ F\left(\{L_h\}_{h\in\mathcal{H}}\right). \tag{14}$$

The variations in Y and TFP are, to a first-order,

$$d\,log\,Y = d\,log\,TFP + \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h\, d\,log\,L_h, \tag{15}$$

$$d\,log\,TFP = Technology - Misallocation, \tag{16}$$

where

$$Technology = \sum_{i \in \mathscr{N}} \widetilde{\lambda}_i\, d\,log\,\mathscr{A}_i \qquad and \qquad Misallocation = \overbrace{d\,log\,\Gamma}^{\substack{Income\\Misallocation}} \; - \overbrace{\sum_{i \in \mathscr{N}} \widetilde{\lambda}_i\, d\,log\,\mu_i}^{Competitiveness}.$$

From equation (14), real GDP in equilibrium has two representations. First, as a CRS function Q_Y that aggregates consumption across sectors. Second, as the product of TFP, and the CRS function F that aggregates labor. Equation (15) segments the output response into TFP and labor supply variations. Notice that the partial equilibrium elasticity for real output on the labor supply of type h equals the value-added share $\widetilde{\Lambda}_h$. The assumption that $d\,log\,L/d\,log\,L_h = \widetilde{\Lambda}_h$ is what segments the first-order effects on output from TFP and L, which makes it consistent with the neoclassic representation $Y = TFP\,L$.

Equation (16) divides the first-order variation of TFP into three components. First, *technology* captures the direct effect of changes in productivity under a fixed allocation of resources. Second, *competitiveness* portrays the reallocation effects from distortions in the absence of variations in the aggregate labor share. These two components tell us that in the absence of Γ variations, the effects on TFP from productivity and markdown shocks in sector i are proportional to its cost-based sales Domar weight $\widetilde{\lambda}_i$. Third, *income misallocation* portrays how TFP responds to variations in the aggregate labor share, and it captures how the income sources change for the representative household, either through labor compensation or rebated profits.

Equations (15) and (16) simplify our understanding of how the equilibrium output for an economy at the production possibility frontier (PPF) changes in response to supply and demand shocks. *Technology* captures how the PPF shifts in response to productivity shocks, and *misallocation* portrays how the economy moves along the PPF boundaries due to labor and intermediate input reallocation between firms.

Corollary 2. ***Hulten (1978)***. *Without distortions, the variation for TFP is, to a first-order,*

$$d\,log\,TFP = \sum_{i \in \mathscr{N}} \lambda_i\, d\,log\,\mathscr{A}_i.$$

Corollary 2 provides the local approximation around the undistorted equilibrium for TFP in response to supply and demand shocks. The proof comes from introducing Corollary 1 in Theorem 3. The original result from Hulten (1978) considers only productivity shocks. An extended version from Bigio & La'O (2020) also accounts for shocks in distortions. The novelty from Corollary 2 is to show that around the efficient equilibrium, up to a first-order, demand shocks are neutral on TFP.

The main difference between my economy and the environment from Baqaee & Farhi (2020) is labor supply endogeneity. For this reason, they label *allocative efficiency* = −*misallocation*. Here, I refrain from using the *efficiency* tag, as with endogenous labor, an increase in real GDP driven by better allocation of resources is not necessarily welfare-improving. The difference between our TFP decompositions is given by

$$\text{Income Misallocation} = \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h\, d\,log\,\Lambda_h = \sum_{h \in \mathscr{H}} \delta_h\, d\,\Lambda_h = \Gamma^{-1} \sum_{h \in \mathscr{H}} d\,\Lambda_h = d\,log\,\Gamma.$$

Their decomposition uses the first definition, while Theorem 3 utilizes the symmetry in distortion centralities across types of labor from Theorem 2. The implication is that variations in the aggregate labor share are sufficient to capture the effects on TFP, and tracing the perturbations for all the labor income shares, as in their model, is no longer necessary. Furthermore, this result associates in a single equation TFP and the aggregate labor wedge, the two equilibrium objects that, according to Chari et al. (2007), account for the bulk of business cycle fluctuations.

Income misallocation captures how aggregate *misallocation* improves as the aggregate labor share falls. This process portrays how workers become relatively more affordable, allowing firms in heavily distorted supply chains, that operate with high marginal productivities, to increase their input demand. The argument that misallocation falls as the profit share rises might sound counterintuitive to the reader. Profits are the source of revenue dilution that generates misallocation. How is it possible that the cause of the malady can also cure it? I want to emphasize two things. First, this result has a second-best nature. More precisely, when distortions generate wedges in the marginal productivities between firms, more distance between the labor income shares and value-added shares allows firms with high input marginal productivities to exploit these variations. These relocation gains do not happen under the first-best solution simply because there are no wedges in the marginal productivities of inputs between firms. Second, *income misallocation* captures only reallocation effects driven by variations in the labor share, while *misallocation* portrays the effect on TFP.

Corollary 3. *The partial equilibrium misallocation effects are given by*

1. ***Distortions****. To a 1% increase in* μ_i*:* $\Gamma^{-1}\,\psi_i^\ell\,\lambda_i - \widetilde{\lambda}_i$.
2. ***Final Expenditure****. To a 1 percentage point (pp) shift from* β_j *to* β_i*:* $\Gamma^{-1}\left(\psi_i^\ell - \psi_j^\ell\right)$.
3. ***Labor Demand****. To a 1 pp shift from* ω_i^x *to* ω_i^ℓ*:* $\mu_i\,\lambda_i\left(1 - \sum_{j\in\mathcal{N}}\psi_j^\ell\ \omega_{ij}\right)$.
4. ***Intermediate Demand****. To a 1 pp shift from* $\widetilde{\Omega}_{im}^x$ *to* $\widetilde{\Omega}_{ij}^x$*:* $\mu_i\,\lambda_i\left(\psi_j^\ell - \psi_m^\ell\right)$.

Corollary 3 characterizes the partial equilibrium response of *misallocation* to shocks. These effects do not account for the general equilibrium endogenous substitution effects that variations in relative prices and wages might induce. The proof comes from introducing Theorem 2 in Theorem 3.

There are four conclusions to extract from these results. First, a reduction in the profit margins for sector i will increase misallocation if $\psi_i^\ell > \Gamma\frac{\widetilde{\lambda}_i}{\lambda_i}$. If the sector's payment centrality is too large, their new demand will reallocate inputs towards the production in supply chains with low marginal productivities. Hence, targeted antitrust efforts should increase sectoral competition in sectors with relatively low payment centralities. However, the threshold for when it is convenient or not to reduce distortions is sector-specific; it depends on the ratio of cost-based to revenue-based Domar weights (i.e., $\widetilde{\lambda}_i/\lambda_i$), or what Liu (2019) denominates as *sectoral distortion centrality*. Second, misallocation increases as the representative household shifts consumption expenditure from low to high payment centrality sectors. Third, misallocation rises as firms shift their cost from intermediate inputs to labor (notice that $1 > \sum_{j\in\mathcal{N}}\psi_j^\ell\,\omega_{ij}$). Fourth, misallocation increases as firms shift intermediate input demand from low to high payment centrality sectors. Therefore, the sectoral payment centralities are necessary statistics to understand the aggregate effects of supply and demand shocks.

6 The Labor Share and Payment Centralities

The empirical implementation of my model relies upon four types of money flows: (1) firm-to-firm in the supply of intermediate inputs, (2) firm-to-workers in the supply of labor, (3) consumer-to-firm in the supply of final goods, and (4) firm-to-household in the distribution of dividends. In this section, I describe the data sources used to implement the model, estimate the sectoral payment centralities, and show the importance of these centralities on the variation of the labor share in the United States.

6.1 Data

The first source is the input-output (IO) tables published by the Bureau of Economic Analysis (BEA) from 1997 to 2021. These tables measure the intermediate input transactions, labor costs, and final expenditure for 71 North American Industry Classification System (NAICS) 3-digit level industries. I exclude industries corresponding to federal, state, and local governments, resulting in a matched data set of 66 industries. The BEA provides the IO use and make tables. The use tables depict usage across multiple categories of goods and services to produce industrial-level output. The make tables characterize industrial production of multiple categories of goods and services $g \in \mathscr{G}$. The interaction between the use and make tables produces the industry-to-industry IO table. The BEA has IO use and make tables that go back to 1946, but only after 1997 did these tables start to identify the sectoral labor costs as an independent component of value-added, which is essential for identifying sectoral distortions. I use these tables to calibrate $\forall i \in \mathscr{N}$ and $\forall t \in \{1997, \cdots, 2021\}$:

$$\text{Cost}_{i,t} = \text{Labor Cost}_{i,t} + \text{Intermediate Cost}_{i,t},$$

$$\text{Value Added}_{i,t} = \text{Labor Cost}_{i,t} + \text{Rents}_{i,t},$$

$$\text{Sales}_{i,t} = \text{Value Added}_{i,t} + \text{Intermediate Cost}_{i,t},$$

$$\omega^{\ell}_{i,t} = \frac{\text{Labor Cost}_{i,t}}{\text{Cost}_{i,t}}, \qquad \mu_{i,t} = \frac{\text{Cost}_{i,t}}{\text{Sales}_{i,t}}, \qquad \omega_{ij,t} = \sum_{g \in \mathscr{G}} \omega^{use}_{ig,t}\, \omega^{make}_{gj,t},$$

$$\text{Intermediate Sales}_{i,t} = \sum_{g \in \mathscr{G}} \omega^{make}_{gi,t} \sum_{j \in \mathscr{N}} \text{Sales of } g \text{ to } j_t,$$

$$\text{Final Sales}_{i,t} = \text{Sales}_{i,t} - \text{Intermediate Sales}_{i,t}, \qquad \beta_{i,t} = \frac{\text{Final Sales}_{i,t}}{\sum_{j \in \mathscr{N}} \text{Final Sales}_{j,t}},$$

where $\omega^{use}_{ig,t}$ and $\omega^{make}_{gj,t}$ stand for the weights from the use and make matrices.

The following three are the main criticisms about this implementation. First, sectoral costs do not include capital costs, which are considered rents. This is a limitation of the BEA's IO data, which does not separately identify capital costs within value-added. Consequently, my accounting measures of markdowns are a lower bound, which could be a concern for sectors with high capital intensity. Second, I use the accounting profit approach (AP) that defines the markdown as the ratio of costs over sales. The AP approach is directly implementable with the BEA's IO data and is similar to the method used by Harberger (1954). Alternatively, for measuring wedges, one could implement a user-cost approach (UC) following Gutiérrez & Philippon (2016), or a production function approach (PF) using the methodology from Loecker & Warzynski (2012). However, the UC approach requires sectoral estimates of capital costs and stocks, riskless rates, and industry-specific depreciations and risk premiums, which could be obtained by matching the BEA's IO data with the sectoral KLEMS data, and the PF approach requires parametric estimation methods that provide the elasticity of the production function to a variable input. I only implement the AP approach, as my primary objective is to introduce the sectoral *payment centralities* concept and show its importance on the labor share variations, not to evaluate the correctness for different markdown definitions. Third, I define final sales as the residual between total and intermediate sales, which agglomerates in the representative household's bundle all forms of final use, i.e., exports, private and public expenditure, investment and changes in inventories.

Figure II portrays the observed and the model-based equilibrium labor share. The correlation between these two measures is 0.99%, and the main difference is on the level: the actual labor share is smaller than the model's. The reason for this level difference is that the model's measure for nominal GDP given by equation (5), ignores the industrial production from the federal, state, and local governments, making it smaller than actual nominal GDP. Additionally, Figure III shows the observed and the model-based sales distribution for 2021. The correlation between these two sales distributions is 1. The aggregate labor share and the sectoral sales distribution show that this simple static production network model implementation successfully captures aggregate and granular moments in equilibrium.

6.2 Markdowns

The first four columns in Tables III, IV, and V show the sectoral markdowns for 2021, the growth in markdowns between 1997 and 2021, and the mean and standard deviations for sectoral markdowns. The fifth and sixth columns show the slope parameter and standard error for the following regression

$$\Delta \mu_{i,t} = \tau_{0,i} + \tau_{1,i}\, \mu_{i,t-1} + \epsilon_{i,t},$$

which measures the stability of industrial markdowns across time.

The five industries with the lowest markdowns in 2021 were housing, pipeline transportation, rental and leasing of intangibles, credit intermediation, and oil and gas extraction, where respectively, 12.97%, 41.29%, 47.03%, 49.31%, and 54.02% of revenue is used to cover labor or intermediate input costs. The five industries with the highest markdowns in 2021 were warehousing and storage, nursing and residential care, social assistance, apparel and leather, and management of companies, where respectively, 96.72%, 96.03%, 95.77%, 95.21%, and 92.23% of revenue covers labor or intermediate input costs.

On the one hand, the five industries with the strongest reductions in markdowns between 1997 and 2021 were pipeline transportation, computers and electronics, credit intermediation, mining -except oil and gas-, and rail transportation, with reductions of, respectively, -35.71%, -19.15%, -18.67%, -17.16%, and -15.14%. On the other hand, the five industries with the most substantial increase in markdowns for the same period were recreational and gambling, air transportation, warehousing and storage, other services, and water transportation, with increases of, respectively, 15.25%, 13.97%, 11.05%, 9.87%, and 8.92%.

Sectoral markdowns fell between 1997 and 2021. For 46 industries, the autocorrelation parameter is statistically negative for 46 sectors at the 10%, for 25 sectors at the 5%, and for 10 industries at the 1%. Not a single autocorrelation parameter is statistically positive at the 10%. This shows that rents and market concentration have risen across the economy.

6.3 Payment Centralities

The first four columns in Tables VI, VII, and VIII show the sectoral payment centralities for 2021, the growth in payment centralities between 1997 and 2021, and the mean and standard deviations for sectoral payment centralities. The fifth and sixth columns show the slope parameter and standard error for the following regression

$$\Delta\, \psi^{\ell}_{i,t} = \phi_{0,i} + \phi_{1,i}\, \phi^{\ell}_{i,t-1} + \epsilon_{i,t},$$

which measures the stability of payment centralities across time. The last column shows the partial equilibrium elasticity of TFP to markdown variations $\widetilde{\lambda}_i - \Gamma^{-1}\, \psi^{\ell}_i\, \lambda_i$ from Corollary 3.

The five industries with the lowest payment centralities in 2021 were housing, oil and gas extraction, petroleum and coal, pipeline transportation, and rental and leasing of intangibles, where respectively, 6.96%, 26.88%, 27.40%, 28.95%, and 29.67% of revenue reached labor compensation. The five industries with the highest payment centralities in 2021 were computer systems design, nursing and residential care, social assistance, management of companies, and warehousing and storage, where respectively, 82.25%, 78.16%, 76.78%, 74.05%, and 74.56% of revenue reached labor compensation.

On the one hand, the five industries with the strongest reductions in payment centralities between 1997 and 2021 are internet and information services, pipeline transportation, mining -except oil and gas-, primary metals, and rail transportation, with reductions of, respectively, -17.75%, -17.47%, -15.24%, -14.45%, and -14.41%. On the other hand, the five industries with the most substantial increase in payment centralities for the same period are forestry and fishing, recreational and gambling, air transportation, water transportation, and computer systems design, with increases of, respectively, 19.87%, 12.49%, 8.99%, 8.12%, and 7.82%.

Sectoral payment centralities fell between 1997 and 2021. For 22 industries, the autocorrelation parameter is statistically negative at the 10%, for 15 sectors at the 5%, and for 7 industries at the 1%. Not a single autocorrelation parameter is statistically positive at the 10%.

Increasing sectoral markdowns will have a positive partial equilibrium effect on TFP for 52 industries. For the other 14 industries, reducing distortions by raising markdowns will harm TFP. Out of the top 16 industries with the highest payment centralities, reducing distortions on 12 will reduce TFP. In other words, as argued in Subsection 5.3, high payment centrality industries operate with low marginal productivities, and for this reason, rising distortions will shrink TFP, as labor and intermediate inputs away from highly distorted sectors that operate with marginal productivities.

Figures IV, V, and VI, and Table IX show that payment centralities are positively correlated with markdowns, negatively correlated with intermediate input cost intensity, and uncorrelated with Domar weights. The positive correlation with markdowns is understood once we notice that the upper bound for sectoral payment centralities is the markdown (i.e., $\mu_i \geq \psi_i^\ell$), and this bound is effective when firms in a sector use no intermediate inputs (i.e., $\omega_i^\ell = 1$). In other words, the higher the markdown, the higher the share of revenue that directly or indirectly reaches labor compensation. The negative correlation with sectoral intermediate input cost intensity is because as the latter increases, the longer the upstream path that firms' revenue travels to reach labor compensation, and the more likely it is to become rents in other stages of production.

Figures VII, VIII, and IX, and Table X show that the difference between markdowns and payment centralities $\mu_i - \psi_i^\ell$ are positively correlated with markdowns and intermediate input cost intensity, and uncorrelated with Domar weights. The explanation for the positive correlation with markdowns is that for a fixed intermediate input cost intensity, the higher (lower) the markdown from a sector, the more likely it is that in upstream stages of production, revenue flows face lower (higher) markdowns, and as a consequence $\mu_i - \psi_i^\ell$ will rise (fall). Now, as ω_i^x rises, the more the payment centrality from an industry depends on indirect upstream paths, which face rent extraction, and hence, $\mu_i - \psi_i^\ell$ will rise.

6.4 Aggregate Labor Share Decomposition

From Figure II, we can see that the aggregate labor share increased between 1997 and 2000, it fell between 2000 and 2010, and it raised from 2010 to 2020. For this reason, I am going to segment the exposition of the aggregate labor share variations decompositions into three periods: (i) 1998 to 2021 - long run, (ii) 2001 to 2010 - pre Great Recession, and (iii) 2011 to 2020 - post Great Recession.

Figure X portrays the labor share counterfactual dynamics and Table XI the counterfactural level and difference to the actual labor share for the three periods, if we were to ignore, one at the time, the channels from the variational decomposition from Theorem 2. Table XII shows the covariance decomposition for the variations in the aggregate labor share for the three periods. Tables XIII, XIV, XV, and XVI shows the counterfactual difference to the actual labor share for the three periods, if we were to ignore, one at the time, an industry-specific channel from Theorem 2.

In the long run, the *competitive income* channel mattered the most in terms of levels and volatility, driving down the aggregate labor share by 1.51% and explaining 82.16% of its volatility. Higher distortions in the credit intermediation, computers and electronics, and chemical products industries were the main drivers behind this channel, explaining, respectively, an aggregate labor share reduction of 0.80%, 0.40%, and 0.33%. Final, labor, and intermediate demand recomposition played a secondary role in the long run, with each one explaining, respectively, an increase of 0.49%, 0.41%, and 0.34% in the aggregate labor share, and 0.99%, 30.10%, and -13.25% of its volatility. On the one hand, the higher final expenditure share on wholesale trade and hospitals, the larger labor cost share in computers and electronics and credit intermediation, and the more robust demand for intermediate inputs from wholesale trade pushed up the aggregate labor share by, respectively, 1.35%, 1.07%, 1.08%, 0.81%, and 0.65%. On the other hand, the lower final expenditure share on computers and electronics, the smaller labor cost share from the wholesale trade sector, and the lower intermediate input cost intensity from computers and electronics drove down the aggregate labor share by, respectively, 1.34%, 1.10%, and 0.58%.

Before the Great Recession, the *competitive income* channel was also the main driver, reducing the aggregate labor share by 4.78% and explaining 153.75% of its volatility. Higher distortions in the internet and information services, computers and electronics, publishing, telecommunications, securities and investments, and wholesale trade industries were the main drivers behind this channel, explaining, respectively, an aggregate labor share reduction of 0.53%, 0.50%, 0.46%, 0.36%, 0.35%, and 0.30%. Final, labor, and intermediate demand recomposition played a secondary role before the Great Recession, with each one explaining, respectively, a reduction of 0.53%, 0.13%, and 0.29% in the aggregate labor share, and 9.30%, -61.52%, and -1.53% of its volatility. The main drivers behind this demand-driven reduction in the aggregate labor share were the lower final expenditure share on construction, computers and electronics, the smaller labor cost share from wholesale trade, and the weaker intermediate input cost intensity from computers and electronics drove down the aggregate labor share by, respectively, 1.52%, 0.91%, 0.52%, and 0.34%.

After the Great Recession, the *labor demand recomposition* channel was the main driver, increasing the aggregate labor share by 2.00%, and explaining 115.30% of its volatility. Higher labor cost intensity from the credit intermediation and the computers and electronics industries were the main drivers behind this channel, explaining, respectively, an aggregate labor share increase of 0.68% and 0.41%. The *competitive income* mechanism was also relevant, increasing the aggregate labor share by 1.78% and explaining 36.09% of its volatility. Lower distortions in the miscellaneous professional services, oil and gas extraction, administrative services, and air transportation industries were the main drivers behind this channel, explaining, respectively, an aggregate labor share increase of 0.39%, 0.27%, 0.25%, and 0.21%. Final and intermediate demand recomposition had a secondary role after the Great Recession, with each one explaining, respectively, variations of 0.29% and -0.12% on the aggregate labor share.

Notice that the variations in the aggregate labor share are also decomposed as

$$d\,\Gamma = \overbrace{\sum_{i\in\mathcal{N}} \psi_i^{\ell}\, d\,\beta_i}^{\substack{\textit{Final Demand}\\ \textit{Recomposition}}} + \overbrace{\sum_{i\in\mathcal{N}} \beta_i\, d\,\psi_i^{\ell}}^{\substack{\textit{Payment Centrality}\\ \textit{Recomposition}}} \quad ,$$

where the *payment centrality recomposition* channel combines *competitive income*, *labor* and *intermediate demand recomposition.* The covariance decomposition from Table XII shows that *payment centrality recomposition* explained 99.1% of the variation in the aggregate labor share in the long run, 161.5% before the Great Recession, and 110.8% after the Great Recession.

7 Parametric Model

In this section, I derive the parametric statistics that characterize in terms of primitives the first-order variations derived in Section 5. For this parametric environment, I identify a linear system of equations that solves the endogenous variations in wages, final expenditure, and sales.

7.1 Normalized CES environment

Following Baqaee & Farhi (2019a,b, 2020, 2023), I extend the normalized CES function introduced by de La Grandville (1989) and Klump & de La Grandville (2000) to an economy with intermediate goods. The overlined variables correspond to equilibrium values. Firm z_i in sector i uses the normalized CES composite

$$\frac{y_{z_i}}{\overline{y}_{z_i}} = A_i \left(\omega_i^\ell \left(\frac{\ell_{z_i}}{\overline{\ell}_{z_i}} \right)^{\frac{\theta_i - 1}{\theta_i}} + \omega_i^x \left(\frac{X_{z_i}}{\overline{X}_{z_i}} \right)^{\frac{\theta_i - 1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i - 1}} \quad \text{and} \quad \frac{X_{z_i}}{X_{z_i}} = \left(\sum_{j \in \mathcal{N}} \omega_{ij} \left(\frac{x_{z_i j}}{\overline{x}_{z_i j}} \right)^{\frac{\theta_i^x - 1}{\theta_i^x}} \right)^{\frac{\theta_i^x}{\theta_i^x - 1}}.$$

In this production function, productivity shocks are Hicks-neutral normalized to 1 in equilibrium, θ_i stands for the elasticity of substitution between labor and intermediate inputs, and θ_i^x stands for the elasticity of substitution between intermediates. Similarly, the consumption aggregator for the representative household is given by

$$\frac{Y}{\overline{Y}} = \left(\sum_{i \in \mathcal{N}} \beta_i \left(\frac{C_i}{\overline{C}_i} \right)^{\frac{\varrho - 1}{\varrho}} \right)^{\frac{\varrho}{\varrho - 1}},$$

where ϱ stands for the elasticity of substitution. The benefit from the normalized CES is that the parameters ω_i^ℓ, ω_i^x, ω_{ij}, and β_i have the same interpretation as in Section 4, and do not depend on deep parameters such as the elasticities of substitution (Klump et al., 2012).

The representative household operates using the following utility function

$$U(Y, L) = \frac{\left(Y\left(1 - E^{-\gamma} L\right)^{\varphi}\right)^{1-\sigma} - 1}{1 - \sigma},$$

with $\varphi > 0$. This utility function allows for greater flexibility in parametrizing the income and substitution effects on the labor supply.

Proposition 3. *The change in aggregate labor supply in response to wage and income shocks is, to a first-order,*

$$d\log L = \zeta_w \, d\log w - \zeta_e \, d\log E.$$

Where the corresponding elasticities are given by $\zeta_w = \frac{1}{1-\varphi\gamma}\frac{\varphi}{\Gamma}$ *and* $\zeta_e = \frac{1}{1-\varphi\gamma}\left(\frac{\varphi}{\Gamma} - \gamma\frac{E^{\gamma}}{L}\right)$.

Proposition 3 characterizes the endogenous first-order variation of the labor supply in terms of elasticities for the (1) substitution effect ζ_w and (2) income effect ζ_e. These elasticities depend on equilibrium values and the deep preference parameters γ and φ.

This utility function nests the following preferences. First, by assuming $\gamma = 0$, I obtain King, Plosser, & Rebelo's (1988) preferences with symmetric substitution and income effects. Second, by using $\gamma = \gamma\Gamma\frac{E^{\gamma}}{L}$, I obtain Greenwood, Hercowitz, & Huffman's (1988) preferences with no income effect. Finally, in its most general form, this utility is inspired by Jaimovich & Rebelo's (2009), and for this reason, it allows for asymmetric income and substitution effects. However, relative to the latter utility preferences, this specification allows for a direct effect from consumption expenditure in labor supply disutility through the parameter γ. The disutility effects from increasing the labor supply become weaker as this parameter increases, and as a consequence, there are stronger demographic and substitution effects.

7.2 Sufficient Endogenous Statistics

Theorem 4 characterizes a $N+2$ linear system of equations that solves for the endogenous first-order variation for final expenditure, the wage, and sales.

Theorem 4. *In a CES economy, the variation in final expenditure, the wage, and sales, in response to supply and demand shocks is, to a first-order,*

$$d\log E = \overbrace{\frac{(1+\zeta_w)\,\Gamma}{1+\zeta_e\Gamma} d\log w}^{\substack{\textit{Wage Effect}\\ \textit{on Expenditure}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\lambda_i}{(1+\zeta_e\Gamma)} \left((1-\mu_i)\, d\log S_i - \mu_i\, d\log \mu_i\right)}^{\substack{\textit{Corporate Income Effect}\\ \textit{on Expenditure}}};$$

$$\begin{aligned} d\log w = {} & \overbrace{\frac{\zeta_e}{1+\zeta_w}\, d\log E}^{\substack{\textit{Expenditure Effect}\\ \textit{on Wages}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\Omega^{\ell}_i \lambda_i}{(1+\zeta_w)\,\Gamma} \left((\theta_i-1)\, d\log A_i + \theta_i\, d\log \mu_i\right)}^{\substack{\textit{Direct Effect}\\ \textit{on Wages}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\Omega^{\ell}_i \lambda_i \theta_i}{(1+\zeta_w)\,\Gamma} d\log \omega^{\ell}_i}^{\substack{\textit{Labor Demand}\\ \textit{on Wages}}} \\ & \overbrace{-\sum_{j\in\mathcal{N}} \left(\sum_{i\in\mathcal{N}} \frac{\Omega^{\ell}_i \lambda_i}{(1+\zeta_w)\,\Gamma} (\theta_i - 1)\, \widetilde{\psi}^x_{ij} \right) (d\log A_j + d\log \mu_j)}^{\substack{\textit{Supplier Effect}\\ \textit{on Wages}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\Omega^{\ell}_i \lambda_i}{(1+\zeta_w)\,\Gamma}\, d\log S_i}^{\substack{\textit{Sales Effect}\\ \textit{on Wages}}}; \end{aligned}$$

$$\begin{aligned} d\log S_i = {} & \overbrace{\frac{\beta_i}{\lambda_i}\, d\log E}^{\substack{\textit{Expenditure Effect}\\ \textit{on Sales}}} + \overbrace{\sum_{j\in\mathcal{N}} \frac{\Omega^x_{ji}\lambda_j}{\lambda_i}\, d\log S_j}^{\substack{\textit{Sales Effect}\\ \textit{on Sales}}} + \overbrace{\sum_{j\in\mathcal{N}} \frac{\Omega^x_{ji}\lambda_j}{\lambda_i} \left((\theta_j-1)\, d\log A_j + \theta_j\, d\log \mu_j\right)}^{\substack{\textit{Direct Effect}\\ \textit{on Sales}}} \\ & + \overbrace{\sum_{j\in\mathcal{N}} \left(\frac{\beta_i}{\lambda_i} (\rho - 1) \left(\widetilde{\psi}^x_{ij} - \widetilde{\lambda}_j \right) + \sum_{q\in\mathcal{N}} \frac{\Omega^x_{qi}\lambda_q}{\lambda_i} (\theta_q - 1) \left(\widetilde{\psi}^x_{ij} - \widetilde{\psi}^x_{qj} \right) \right) (d\log A_j + d\log \mu_j)}^{\textit{Supplier Effect on Sales}} \\ & + \overbrace{\sum_{j\in\mathcal{N}} \left(\sum_{q\in\mathcal{N}} \frac{\Omega^x_{qi}\lambda_q}{\lambda_i} (\theta^x_q - \theta_q) \left(\widetilde{\psi}^x_{ij} - \sum_{m\in\mathcal{N}} \omega_{qm} \widetilde{\psi}^x_{mj} \right) \right) (d\log A_j + d\log \mu_j)}^{\textit{Indirect Supplier Effect on Sales}} \\ & + \overbrace{\frac{\beta_i}{\lambda_i} \varrho\, d\log \beta_i}^{\substack{\textit{Final Demand}\\ \textit{on Sales}}} + \overbrace{\sum_{j\in\mathcal{N}} \frac{\Omega^x_{ji}\,\lambda_j}{\lambda_i} \left(\theta_j\, d\log \omega^x_j + \theta^x_j\, d\log \omega_{ji} \right)}^{\substack{\textit{Intermediate Demand}\\ \textit{on Sales}}}. \end{aligned}$$

The first-order variation for final expenditure depends on two channels. First, in the *wage effect*, a wage increase directly raises income. Additionally, it triggers a substitution effect on the labor supply captured by $(1+\zeta_w)$, which is proportional to the labor income share Γ. Second, for the *corporate income effect*, dividends from sector i depend both on sales and their markdowns: (i) an increase in sales augments dividend income by the rent extraction share

$1-\mu_i$, and (ii) an increase in markdowns reduces profits by the cost share μ_i. These two paths for dividend income variation are proportional to the Domar weight λ_i. These two channels increase consumption expenditure and trigger an income effect that reduces the labor supply, attenuating their magnitudes by $1+\zeta_e\,\Gamma$.

The first-order variation in wages depends on five channels. These channels trigger a substitution effect that increases the labor supply and attenuates their influence on wages by $1+\zeta_w$. Additionally, the effect on w from the channels that depict variations in sector i's labor demand are proportional to the direct revenue-based centrality Ω_i^ℓ and the sales to labor income ratio λ_i/Γ. First, for the *expenditure effect*, in response to an increase in their total income, their labor supply falls by ζ_e, and wages rise. Second, the *direct effect* captures the increase in labor demand from firms that receive either productivity or markdown shocks. Firm i increases their demand for labor in response to a positive productivity shock as long as there is substitutability in their production (i.e., $\theta_i > 1$) and in response to lower distortions as long as the production function is not Leontief (i.e., $\theta_i > 0$). Third, the *labor demand effect* captures the impact of higher labor cost intensity. Firm i raises their demand for labor when ω_i^ℓ increases, and consequently wages, as long as the production function is not Leontief. Fourth, the *supplier effect* portrays the variations in firms' labor demand in response to productivity and markdown shocks to its intermediate input suppliers. Firm i decreases labor demand in response to positive productivity shocks and lower profit margins to its direct or indirect intermediate supplier j, as long as there is substitutability in their production. The magnitude of this effect is proportional to the cost-based firm-to-firm centrality $\widetilde{\psi}_{ij}^x$. Fifth, the *sales effect* characterizes how more sales expand labor demand.

For firms in sector i, the first-order variation for their sales depends on seven channels. The channels that represent variation in the demand of final goods by the representative household are proportional to expenditure intensity to Domar weight ratio β_i/λ_i, and those that illustrate changes in the demand for intermediate goods by firms in sector j are proportional to the direct flow of revenue Ω_{ji}^x and the Domar weight ratio λ_j/λ_i. First, the *expenditure effect* captures how higher final expenditure increases demand for final goods. Second, the *sales effect* portrays how higher firms' sales increase demand for intermediate goods. Third, the *direct effect* characterizes the increase in intermediate input demand from firms that receive either productivity or markdown shocks. Firm j increases their demand for good i in response to positive productivity shocks as long as there is substitutability and in response to higher markdowns as long as the production function is not Leontief. Fourth, the *supplier effect* characterizes the variations in the representative household's and firms' demand for goods in response to productivity and markdown shocks to its direct or indirect suppliers. Under substitutability, the household and firm q increase their demand for good i in response to higher productivity or markdowns to its direct or indirect supplier j if their cost-based centrality to firm j is smaller than the one that firms in sector i have. In other words, when firm j reduces its price, the demand by the household and firms from sector q for the good i rises if their cost-based exposure to the shock is weaker than the one from firms in sector i, i.e., $\widetilde{\psi}^x_{ij} > \widetilde{\lambda}_j$ and $\widetilde{\psi}^x_{ij} > \widetilde{\psi}^x_{qj}$. In other words, when these conditions are satisfied, the household and firm q demand directly or indirectly more goods from sector i than from sector j, simply because they face more substantial price reductions through good i. Fifth, the *indirect supplier effect* captures the variations in firms' demand for goods in response to productivity and markdown shocks to the direct or indirect supplier of firms' direct suppliers. When the substitutability of intermediate inputs is stronger than the substitutability between labor and intermediate inputs (i.e., $\theta^x_q > \theta_q$), firm q increases their demand for good i in response to higher productivity or markdowns to its direct or indirect supplier j if the weighted cost-based centrality of its direct suppliers to firm j is smaller than the one that firms in sector i have (i.e., $\widetilde{\psi}^x_{ij} > \sum_{m \in \mathcal{N}} \omega_{qm} \widetilde{\psi}^x_{mj}$). Lastly, the *final* and *intermediate demand effects* capture the impact of higher cost intensity on the household or firms.

The solution in Theorem 4 represents an alternative to Baqaee & Farhi's (2020) results for the following four reasons: (1) it does not require the production network covariance operator introduced by Baqaee & Farhi (2019a); (2) it captures the influence of the labor supply substitution and income elasticities; (3) it decomposes the mechanisms behind the first-order variations; and (4) using the nominal GDP as the numeraire is not necessary.

7.3 A Simple Economy

Figure XI displays the supply of labor and goods for a simple two-firm economy. The first firm utilizes only labor to produce y_1, while the second firm requires labor and intermediate inputs supplied by firm one. The second firm demands labor with a cost intensity of ω_ℓ and intermediate inputs from firm one with an intensity of ω_x. The representative household consumes goods from both firms, and the expenditure share on the good from firm one is captured by β.

The horizontal and the vertical production networks are the two boundaries of this economy. On the one hand, when $\omega_\ell = 1$, there is no demand for intermediate inputs, and we operate at the horizontal production network where both firms require only labor. On the other hand, when $\omega_\ell = \beta = 0$, we operate at the vertical production network where there is only one path for labor's value-added to reach final consumption.

Proposition 4. *For the simple economy, the aggregate labor share first-order variations in response to supply and demand shocks are given by*

$$\frac{\partial\,\Gamma}{\partial\,log\,A_1} = \omega_\ell\,(\beta\,(1-\beta)\,(\mu_1 - \mu_2\,(\omega_\ell + \mu_1\,\omega_x))\,(\varrho - 1) - \mu_2\,(1-\beta)\,\omega_x\,(1-\mu_1)\,(\theta_2 - 1))\,;$$

$$\frac{\partial\,\Gamma}{\partial\,log\,A_2} = \beta\,(\mu_2\,(1-\beta)\,(\omega_\ell + \mu_1\,\omega_x) - \mu_1\,(\omega_\ell - \beta))\,(\varrho - 1)\,;$$

$$\frac{\partial\,\Gamma}{\partial\,log\,\mu_1} = \mu_1\,\beta + (1-\beta)\,(\mu_1\,\mu_2\,\omega_x + \omega_\ell\,\beta\,(\mu_1 - \mu_2\,(\omega_\ell + \mu_1\,\omega_x))\,(\varrho - 1) - \omega_\ell\,\omega_x\,\mu_2\,(1-\mu_1)\,(\theta_2 - 1))\,;$$

$$\frac{\partial\,\Gamma}{\partial\,log\,\mu_2} = \mu_2\,(1-\beta)\,(\omega_\ell + \mu_1\,\omega_x) + \beta\,(\mu_2\,\omega_\ell\,(1-\beta) + \mu_1\,(\beta - \omega_\ell + \mu_2\,\omega_x\,(1-\beta)))\,(\varrho - 1)\,;$$

$$\frac{\partial\,\Gamma}{\partial\,\beta} = \varrho\,(\mu_1\,(1-\mu_2\,\omega_x) - \mu_2\,\omega_\ell)\,; \qquad \frac{\partial\,\Gamma}{\partial\,\omega_\ell} = \mu_2\,(1-\beta)\,(1-\mu_1\,\theta_2)\,.$$

For the horizontal economy production network, first, the variation for the aggregate labor share in response to productivity shocks is given by

$$\frac{\partial\,\Gamma}{\partial\,log\,A_1} = -\frac{\partial\,\Gamma}{\partial\,log\,A_2} = (\mu_1 - \mu_2)\,\beta\,(1-\beta)\,(\varrho - 1)\,.$$

This condition tells us that under consumption substitutability (i.e., $\varrho > 1$), the representative household shifts their expenditure towards the firm that faces lower marginal costs due to the productivity shock. If the firm receiving the productivity shock has lower profit margins, the aggregate labor share will rise. For example, the aggregate labor share will increase when firm one receives the productivity shock and $\mu_1 > \mu_2$. These effects depend exclusively on the *final demand recomposition* channel, where the productivity shocks trigger an endogenous reallocation of expenditure such that $\frac{\partial \beta}{\partial \, log \, A_1} = -\frac{\partial \beta}{\partial \, log \, A_2} = \beta \, (1 - \beta) \, (\varrho - 1)$, with the difference in payment centralities given by $\psi_1^\ell - \psi_2^\ell = \mu_1 - \mu_2$. Second, the aggregate labor share variation in response to markdown shocks is given by

$$\frac{\partial \, \Gamma}{\partial \, log \, \mu_1} = \mu_1 \, \beta + (\mu_1 - \mu_2) \, \beta \, (1 - \beta) \, (\varrho - 1) \, ;$$

$$\frac{\partial \, \Gamma}{\partial \, log \, \mu_2} = \mu_2 \, (1 - \beta) - (\mu_1 - \mu_2) \, \beta \, (1 - \beta) \, (\varrho - 1) \, .$$

These effects agglomerate two channels, the *competitive income* channel $\mu_1 \, \beta$ and $\mu_2 \, (1 - \beta)$, and the *final demand recomposition*, which has a magnitude and an explanation equivalent to the productivity shock case. Finally, in response to an exogenous expenditure recomposition

$$\frac{\partial \, \Gamma}{\partial \, \beta} = (\mu_1 - \mu_2) \, \varrho,$$

which tells us that the elasticity of substitution mediates the strength of this channel.

For the vertical economy production network, first, the variation for the aggregate labor share in response to productivity shocks is given by

$$\frac{\partial \, \Gamma}{\partial \, log \, A_1} = \frac{\partial \, \Gamma}{\partial \, log \, A_2} = 0.$$

In this case, the productivity shock cannot trigger substitutability in final expenditure as all the final consumption is on the second firm's goods. Second, the aggregate labor share elasticity in response to markdown variations is unitary because $\Gamma = \mu_1 \, \mu_2$, so

$$\frac{\partial \, log \, \Gamma}{\partial \, log \, \mu_1} = \frac{\partial \, log \, \Gamma}{\partial \, log \, \mu_2} = 1.$$

7.4 Elasticities of Substitution

I identify the elasticities of substitution, and the substitution and income effects using the following multistage grid optimization method.[3] The values for the grid will be chosen by minimizing the sum squared residual between the actual and model-based variations for the aggregate labor share derived using Theorem 4 between 1998 and 2021.

For this optimization method, I require measures for shocks. Productivity shocks come from the BEA's Integrated Industry-Level Production Account (KLEMS). Following La'O & Tahbaz-Salehi (2022), I will use the variations in sectoral TFP as a measure of productivity variation. Specifically, in my model, sectoral TFP variations differ from sectoral productivity shocks. Still, I equate these two notions, not only because it is the standard in the literature but also because the alternative requires having measures of sectoral prices that allow me to directly estimate better sectoral Solow residuals, which is outside this project's scope. Variations in markdowns and demand shocks come from the BEA's IO data.

In the first stage, I assume symmetry in the elasticities of substitution between industries, i.e., $\theta = \theta_i$ and $\theta^x = \theta_i^x \ \forall i \in \mathscr{N}$. The solution for this step is $\theta = 0.2$, $\theta^x = 0.0$, $\varrho = 1.0$, $\zeta_w = 4.0$, and $\zeta_e = 0.0$, with and R^2 of 54.28%. Interestingly, these values are close to the standard values found throughout the input-output literature (Boehm et al., 2014; Atalay, 2017; Baqaee, 2018; Baqaee & Farhi, 2020), where $\theta = 0.5$, $\theta^x = 0.2$, and $\varrho = 0.9$ are used. In the second stage, starting from the previous values, I allow for asymmetries in the elasticities of substitution between labor and intermediate inputs. Now, the R^2 increases to 94.25%. In the third stage, starting from stage two values, I allow for asymmetries in the elasticities of substitution between intermediate inputs, which raises the R^2 to 94.31%. In the fourth stage, starting from stage three values, I simultaneously solve for asymmetric parameters in θ_i and θ_i^x, and now the R^2 is 94.31%. Tables XVII and XVIII display the parameters estimated in stages 2, 3, and 4, and Table XIX show the parameters from regressing the actual variation for the aggregate labor share on the model prediction for each of the stages. Figure XII portrays the variations of the aggregate labor share and stage four prediction.

[3] For the elasticities of substitution θ_i, θ_i^x, and ϱ, I use the following grid values: 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0, 4.0, 6.0, 8.0, 10.0, and 20.0. I use 0, 1, 2, 3, 4, and 5 for the substitution and income effect.

This exercise shows that the asymmetries in the industry-level elasticities of substitution are essential for improving the model's fit. By moving from stage one to stage four estimations, the model's fit rises from 54.28% to 94.21%, and the predictive power from the model goes from insignificant to significant at the 1%. Most empirical implementations of production network models ignore the heterogeneity in industry-level elasticities of substitution.

7.5 Counterfactual Shocks

In this section, I will estimate the endogenous variation in the labor share in response to two types of industry-level counterfactual exercises. First, I assume exogenous productivity and markdown shocks with a magnitude of 1%. Table XX captures the variation of Γ in levels due to productivity shocks and Table XXI due to exogenous variations in markdowns. Tables XXII and XXIII portray regressions of the endogenous variations in the labor share on sectoral Domar weights, markdowns, and payment centralities. These regressions show that the variations in the labor share are positively correlated with the Domar weights, and these correlations are significant at 1%. These results state that shocks in larger sectors have a wider effect on the aggregate labor share.

For this reason, the second type of counterfactual exercise normalizes productivity and markdown shocks using a magnitude equal to the inverse of the sectoral cost-based Domar weights, i.e., $\widetilde{\lambda}_i^{-1}$. According to Theorem 3, the previous assumption implies that the TFP elasticities through the *technology* and *competitiveness* channels are equal to 1. The heterogeneity in the TFP response depends exclusively on the heterogeneity in the *income misallocation* channel, in other words, the heterogeneity in the labor share variations. For this second counterfactual exercise, Table XXIV captures the variation of Γ in levels due to productivity shocks and Table XXV due to exogenous variations in markdowns, and Tables XXVI and XXVII portray regressions of the endogenous variations in the labor share on sectoral Domar weights, markdowns, and payment centralities.

These regressions give us two lessons. The first lesson is that the variations in the aggregate labor share in response to productivity shocks are uncorrelated with Domar weights, markdowns, and payment centralities. This is a consequence of Theorem 2, where productivity shocks only affect the aggregate labor share indirectly through endogenous demand recomposition in response to substitution effects. The second lesson is that the variations in the aggregate labor share in response to markdowns are positively correlated with markdowns and payment centralities. However, in a multivariate regression, only payment centralities are positively correlated. Therefore, if the objective is to increase the aggregate labor share, industrial policy should push for competition in industries with high payment centralities. However, Theorem 3 tells us that this comes at a higher cost in terms of TFP reductions.

Notice that for both productivities and markdowns, positive shocks in the oil and gas extraction industry generate the strongest reduction in the labor share. The oil and gas extraction industry has the second weakest payment centrality after housing. However, the labor share increases weakly in response to productivity and markdown variations for housing, which tells us that despite their positive correlation, there is no strict monotonic relationship between payment centralities and the labor share variation. This implies that the position of an industry in the productivity network, which goes beyond the concept of payment centralities, is essential to understand how a shock generates the endogenous firm and household substitution effects described by Theorem 4 that indirectly influence the aggregate labor share variation.

8 Conclusion

This paper provided the first nonparametric decomposition for the variations in the labor share for a production network economy with distortions. This decomposition segmented the sources of variation in supply and demand shocks. I found that among the necessary sufficient statistics to understand these variations, *payment centralities* are essential to measure the strength of the labor share response to some of these supply and demand mechanisms. Variations in *payment centralities* explain most of the labor share volatility.

The simple representative household, closed-economy, production network model behind the decompositions introduced by this paper does an excellent job in empirically replicating the variations in the labor share and describing the granular sources behind these variations. However, it leaves some important questions unanswered about the sources of variation in the aggregate labor share; among those, I consider that the most important are the following four: 1) What is the role of heterogeneity in households' consumption bundles and variations in the consumption expenditure distribution?; 2) What is the role of foreign shocks?; 3) What is the role of nonlinearities?; and 4) How relevant is the introduction of alternative measures of distortions? I expect future research will tackle these questions and amplify our understanding of the granular sources behind the labor share variations.

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Online Appendix

1 Proofs for the nonparametric model

1.1 Firms

1.1.1 Aggregators' Problem

For every sector $i \in \mathscr{N}$, the perfectly competitive aggregator chooses $\left\{y_i,\, (y_{z_i})_{z_i \in [0,1]}\right\}$ to maximize

$$\bar{\pi}_i = p_i y_i - \int p_{z_i} y_{z_i}\, dz_i$$

subject to the CES technology (4) and taking prices $\left\{p_i, (p_{z_i})_{z_i \in [0,1]}\right\}$ as given.

Taking first order conditions I arrive to the usual Dixit & Stiglitz's (1977) CES demand function

$$y_{z_i} = \left(\frac{p_i}{p_{z_i}}\right)^{\frac{1}{1-\mu_i}} y_i \qquad \forall z_i \in [0,1]\,, \tag{17}$$

from here $\frac{\partial p_{z_i}}{\partial y_{z_i}} = -\left(1-\mu_i\right)\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i}\frac{p_i}{y_{z_i}}$ and $p_i = \left(\int p_{z_i}^{\frac{\mu_i}{\mu_i - 1}}\, dz_i\right)^{\frac{\mu_i - 1}{\mu_i}}$.

1.1.2 Monopolistically Competitive Firms' problem

Firm z_i in sector $i \in \mathscr{N}$ chooses $\left\{y_{z_i},\, p_{z_i},\, \{\ell_{z_i h}\}_{h \in \mathscr{H}},\, \{x_{z_i j}\}_{j \in \mathscr{N}}\right\}$ to maximize

$$\pi_{z_i} = p_{z_i}\, y_{z_i} - \underbrace{\sum_{h \in \mathscr{H}} w_h\, \ell_{z_i h}}_{= p^{\ell}_{z_i}\, L_{z_i}} - \underbrace{\sum_{j \in \mathscr{N}} p_j\, x_{z_i j}}_{= p^{x}_{z_i}\, X_{z_i}}, \tag{18}$$

subject to equation (17),

$$y_{z_i} = A_i\, Q_i\left(L_{z_i}, X_{z_i}\right), \quad L_{z_i} = A_i^{\ell}\, Q_i^{\ell}\left(\left\{A_{ih}^{\ell}\, \ell_{z_i h}\right\}_{h \in \mathscr{H}}\right), \quad X_{z_i} = A_i^{x}\, Q_i^{x}\left(\left\{A_{ij}^{x}\, x_{z_i j}\right\}_{j \in \mathscr{N}}\right), \tag{19}$$

and taking $\left\{ \{w_h\}_{h\in\mathscr{H}},\, \{p_j\}_{j\in\mathscr{N}} \right\}$ as given.

Notice that firm z_i's revenue derivative with respect to any variable q is given by

$$\begin{aligned}
\frac{\partial\, p_{z_i}\, y_{z_i}}{\partial q} &= \left(p_{z_i} + \frac{\partial p_{z_i}}{\partial y_{z_i}}\, y_{z_i}\right)\frac{\partial y_{z_i}}{\partial q} \\
&= \left(p_{z_i} - (1-\mu_i)\left(\frac{y_{z_i}}{y_i}\right)^{\mu_i - 1} p_i\right)\frac{\partial y_{z_i}}{\partial q} = \mu_i\, p_{z_i}\frac{\partial y_{z_i}}{\partial q}.
\end{aligned}$$

Firms z_i's optimality conditions are given by

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\left(L_{z_i},\, X_{z_i}\right)}{\partial\, L_{z_i}} = p_{z_i}^{\ell}, \tag{20}$$

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\left(L_{z_i},\, X_{z_i}\right)}{\partial\, X_{z_i}} = p_{z_i}^{x}, \tag{21}$$

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\left(L_{z_i},\, X_{z_i}\right)}{\partial\, L_{z_i}}\, A_i^{\ell}\, \frac{\partial\, Q_i^{\ell}\left(\left\{A_{ib}^{\ell}\, \ell_{z_i b}\right\}_{b\in\mathscr{H}}\right)}{\partial\, \ell_{z_i h}} = w_h \qquad \forall h \in \mathscr{H} : \partial\, y_{z_i}/\partial\, \ell_{z_i h} > 0, \tag{22}$$

$$\mu_i\, p_{z_i}\, A_i\, \frac{\partial\, Q_i\left(L_{z_i},\, X_{z_i}\right)}{\partial\, X_{z_i}}\, A_i^{x}\, \frac{\partial\, Q_i^{x}\left(\left\{A_{im}^{x}\, x_{z_i m}\right\}_{m\in\mathscr{N}}\right)}{\partial\, x_{z_i j}} = p_j \qquad \forall j \in \mathscr{N} : \partial\, y_{z_i}/\partial\, x_{z_i j} > 0. \tag{23}$$

Representing elasticities with $e\left(a, b\right) = \left(\partial a/\partial b\right)\left(b/a\right)$ the former first order conditions for firm z_i are also captured by

$$\omega_{z_i}^{\ell} = e\left(y_{z_i},\, L_{z_i}\right) = \frac{1}{\mu_i}\, \frac{p_{z_i}^{\ell}\, L_{z_i}}{p_{z_i}\, y_{z_i}}, \tag{24}$$

$$\omega_{z_i}^{x} = e\left(y_{z_i},\, X_{z_i}\right) = \frac{1}{\mu_i}\, \frac{p_{z_i}^{x}\, X_{z_i}}{p_{z_i}\, y_{z_i}}, \tag{25}$$

$$e\left(y_{z_i},\, \ell_{z_i h}\right) = \frac{1}{\mu_i}\, \frac{w_h\, \ell_{z_i h}}{p_{z_i}\, y_{z_i}} \qquad \forall h \in \mathscr{H} \tag{26}$$

$$e\left(y_{z_i},\, x_{z_i j}\right) = \frac{1}{\mu_i}\, \frac{p_j\, x_{z_i j}}{p_{z_i}\, y_{z_i}} \qquad \forall j \in \mathscr{N} \tag{27}$$

Combining equations (20) with (22), and (21) with (23)

$$\alpha_{z_i h} = e\left(L_{z_i},\, \ell_{z_i h}\right) = \frac{w_h\, \ell_{z_i h}}{p^{\ell}_{z_i}\, L_{z_i}}, \qquad \forall h \in \mathscr{H} \tag{28}$$

$$\omega_{z_i j} = e\left(X_{z_i},\, x_{z_i j}\right) = \frac{p_j\, x_{z_i j}}{p^{x}_{z_i}\, X_{z_i}} \qquad \forall j \in \mathscr{N} \tag{29}$$

Additionally, combining (26), (27), and using the implicit function theorem

$$e\left(\ell_{z_i h},\, \ell_{z_i b}\right) = -\frac{w_b\, \ell_{z_i b}}{w_h\, \ell_{z_i h}} \qquad \forall h, b \in \mathscr{H} \tag{30}$$

$$e\left(x_{z_i j},\, x_{z_i m}\right) = -\frac{p_m\, x_{z_i m}}{p_j\, x_{z_i j}} \qquad \forall j, m \in \mathscr{N}. \tag{31}$$

Introducing equations (26)-(27) in the cost function

$$\begin{aligned} c_{z_i}\left(\vartheta,\, \rho\right) &= p^{\ell}_{z_i}\, L_{zi} + p^{x}_{z_i}\, X_{zi} = \sum_{h \in \mathscr{H}} w_h\, \ell_{z_i h} + \sum_{j \in \mathscr{N}} p_j\, x_{z_i j} \\ &= \mu_i\, p_{z_i}\, y_{z_i} \left(\sum_{h \in \mathscr{H}} e\left(y_{z_i},\, \ell_{z_i h}\right) + \sum_{j \in \mathscr{N}} e\left(y_{z_i},\, x_{z_i j}\right) \right). \end{aligned} \tag{32}$$

From CRS in $Q_i\left(L_{z_i},\, X_{z_i}\right)$, $Q^{\ell}_i\left(\left\{A^{\ell}_{ih}\, \ell_{z_i h}\right\}_{h \in \mathscr{H}}\right)$, and $Q^{x}_i\left(\left\{A^{x}_{ij}\, x_{z_i j}\right\}_{j \in \mathscr{N}}\right)$

$$\begin{aligned} &\sum_{h \in \mathscr{H}} e\left(y_{z_i},\, \ell_{z_i h}\right) + \sum_{j \in \mathscr{N}} e\left(y_{z_i},\, x_{z_i j}\right) \\ &= e\left(y_{z_i},\, L_{z_i}\right) \sum_{h \in \mathscr{H}} e\left(L_{z_i},\, \ell_{z_i h}\right) + e\left(y_{z_i},\, X_{z_i}\right) \sum_{j \in \mathscr{N}} e\left(X_{z_i},\, x_{z_i j}\right) \\ &= e\left(y_{z_i},\, L_{z_i}\right) + e\left(y_{z_i},\, X_{z_i}\right) = 1, \end{aligned}$$

which implies that in (32) $c_{z_i}\left(\vartheta,\, \rho\right) = \mu_i\, p_{z_i}\, y_{z_i}$, and from here I obtain $\omega^{\ell}_{z_i} = e\left(y_{z_i},\, L_{z_i}\right)$, $\omega^{x}_{z_i} = e\left(y_{z_i},\, X_{z_i}\right)$, $\widetilde{\Omega}^{\ell}_{z_i h} = e\left(y_{z_i},\, \ell_{z_i h}\right)$, and $\widetilde{\Omega}^{x}_{z_i j} = e\left(y_{z_i},\, x_{z_i j}\right)$.

1.2 Household's Problem

The representative household chooses $\left\{Y, L, \{C_i\}_{i\in\mathscr{N}}, \{L_h\}_{h\in\mathscr{H}}\right\}$ to maximize $U(Y,\, L)$ subject to $Y = Q_Y\left(\{C_i\}_{i\in\mathscr{N}}\right)$, $L = F\left(\{L_h\}_{h\in\mathscr{H}}\right)$, and the budget constraint

$$GDP = p_Y\, Y = \sum_{i\in\mathscr{N}} p_i\, C_i \leq w\, L + \Pi, \tag{33}$$

$$w\, L = \sum_{h\in\mathscr{H}} w_h\, L_h, \qquad \Pi = \sum_{i\in\mathscr{N}} \left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i\right), \tag{34}$$

and taking as given

$$\left\{w_h,\, \left\{p_i,\, \bar{\pi}_i,\, \left(\pi_{z_i}\right)_{z_i\in[0,1]}\right\}_{i\in\mathscr{N}}\right\}.$$

The first order conditions for household $h \in \mathscr{H}$ are given by

$$\frac{U_Y}{p_Y} = \frac{U_Y}{p_i}\frac{\partial\, Y}{\partial\, C_i} = \gimel \qquad \forall i \in \mathscr{N} :\ \frac{\partial\, Y}{\partial\, C_i} > 0 \tag{35}$$

$$-\frac{U_L}{w} = -\frac{U_L}{w_h}\frac{\partial\, L}{\partial\, L_h} = \gimel \qquad \forall h \in \mathscr{H}, \tag{36}$$

where $\gimel$ stands for the lagrange multiplier for the budget constraint.

Combining (35) and (36), the former first order conditions can be represented by

$$\frac{w_h}{p_y} = -\frac{U_L}{U_Y}\frac{\partial\, L}{\partial\, L_h}, \tag{37}$$

$$\frac{p_i}{p_Y} = \frac{\partial\, Y}{\partial\, C_i} \qquad \forall i \in \mathscr{N} :\ \frac{\partial Y}{\partial C_i} > 0. \tag{38}$$

Using the implicit function theorem, equations (37) and (38) can be represented in terms of elasticities as

$$e(Y,\, L) = \frac{w\, L}{p_Y\, Y}, \tag{39}$$

$$\beta_i = e\,(Y,\, C_i) = \frac{p_i\, C_i}{p_Y\, Y} \qquad \forall i \in \mathscr{N}, \tag{40}$$

$$e\,(C_i,\, C_m) + \frac{p_m\, C_m}{p_i\, C_i} = 0 \qquad \forall i, m \in \mathscr{N}:\ \frac{\partial\, Y}{\partial\, C_i} > 0 \quad \text{and} \quad \frac{\partial\, Y}{\partial\, C_m}, \tag{41}$$

$$e\,(C_i,\, L_h) = \frac{w_h\, L_h}{p_i\, C_i} \qquad \forall i \in \mathscr{N}:\ \frac{\partial\, Y}{\partial\, C_i} > 0. \tag{42}$$

Now let me say a couple of things about the aggregate labor supply. First, the optimal solution that relates real GDP and the aggregate labor supply via

$$-\frac{U_L}{U_Y} = \underbrace{\frac{w\, L}{p_Y\, Y}}_{=\Gamma}\, \frac{Y}{L}. \tag{43}$$

The interpretation for this equation is that the aggregate marginal rate of substitution between real GDP and the aggregate factor supply equals the aggregate marginal rate of transformation times an aggregate wedge Γ, which in equilibrium equals the aggregate labor share. Given that $Y = TFP\, L$, the aggregate marginal rate of transformation equals TFP.

Second, the optimal allocation for the representative household relates aggregate output and labor supply of type h via

$$\begin{aligned} -\frac{U_L}{w_h}\, \frac{\partial\, L}{\partial\, L_h} &= \frac{U_Y}{p_Y} \\ -U_L\, \widetilde{\Lambda}_h\, \frac{L}{w_h\, L_h} &= U_Y\, \frac{Y}{p_Y\, Y} \\ -\frac{U_L}{U_Y} &= \frac{\Lambda_h}{\widetilde{\Lambda}_h}\, \frac{Y}{L}. \end{aligned} \tag{44}$$

The second line comes from the fact that $\widetilde{\Lambda}_h = \frac{L_h}{L}\, \frac{\partial\, L}{\partial\, L_h}$.

Equations (43) and (44) imply that the representative household requires that

$$\Gamma = \frac{\Lambda_h}{\widetilde{\Lambda}_h} = \delta_h^{-1} \quad \forall h \in \mathscr{H}. \tag{45}$$

Any deviation from this condition is an inefficient composition for the aggregate labor supply.

Adding up over all types of labor

$$\Gamma \underbrace{\sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h}_{=1} = \sum_{h\in\mathscr{H}} \Lambda_h$$

$$\Gamma = \sum_{h\in\mathscr{H}} \Lambda_h. \tag{46}$$

1.3 Proof for Proposition 1

1.3.1 Proof of Necessity

First, using equations (17), (23), and (41), I can obtain the first subset of conditions in Proposition 1

$$\begin{aligned} &\frac{\partial Y/\partial C_j}{\partial Y/\partial C_i} = \frac{p_j}{p_i} = \mu_i \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial y_{z_i}}{\partial x_{z_i j}} \quad \forall i,j \in \mathscr{N},\ \forall z_i \in [0,1]\,, \\ &\text{such that} \quad \partial Y/\partial C_i > 0, \quad \partial Y/\partial C_j > 0, \quad \text{and} \quad \partial y_{z_i}/\partial x_{z_i j} > 0. \end{aligned} \tag{47}$$

Notice that in this first subset of equilibrium conditions, the representative household consumes both from the sectors i and j, and firms z_i also has to demand intermediate inputs from sector j.

Second, using equations (17), (22), and (42), I can obtain

$$\begin{aligned} &-\frac{w_b}{w_h}\frac{U_L}{U_Y}\frac{\partial L/\partial L_h}{\partial Y/\partial C_i} = \frac{w_b}{p_i} = \mu_i \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial y_{z_i}}{\partial \ell_{z_i b}} \quad \forall i \in \mathscr{N},\ \forall z_i \in [0,1]\,,\ \forall h,b \in \mathscr{H}, \\ &\text{such that} \quad \partial Y/\partial C_i > 0, \quad U_L < 0, \quad \partial L/\partial L_h, \quad \text{and} \quad \partial y_{z_i}/\partial \ell_{z_i b} > 0. \end{aligned} \tag{48}$$

Notice that in this second subset of equilibrium conditions, the condition that links the demand from firm z_i for labor of type b and the marginal rate of substitution between the labor supply of labor of type h and the consumption of goods form sector i does not require that firm z_i hires labor of type h. What is necessary for this relationship to exist is that firm z_i hires labor $b \in \mathscr{H}$, and that the representative household consumes from sector i. Whenever $b \neq h$, the wage-rate-differential wedge w_b/w_h arises.

Finally, the resource constraints

$$y_i = C_i + \sum_{j \in \mathscr{N}} \int x_{z_j i}\, dz_j \quad \forall i \in \mathscr{N}, \qquad \text{and} \qquad L_h = \sum_{i \in \mathscr{N}} \int \ell_{z_i h}\, dz_i \quad \forall h \in \mathscr{H}, \tag{49}$$

are necessary conditions for the equilibrium allocation.

1.3.2 Proof of Sufficiency

Now, I am going to prove that there exists a strictly positive price system

$$\left\{ \left\{ \left(p_{z_i}\right)_{z_i \in [0,1]},\, p_i \right\}_{i \in \mathscr{N}},\, \{w_h\}_{h \in \mathscr{H}} \right\},$$

that implements a specific allocation for firms

$$\left\{ \left(y_{z_i},\, \{\ell_{z_i h}\}_{h \in \mathscr{H}},\, \{x_{z_i j}\}_{j \in \mathscr{N}} \right)_{z_i \in [0,1]},\, y_i \right\}_{i \in \mathscr{N}},$$

and a representative household allocation

$$\left\{ Y,\, L, \{C_i\}_{i \in \mathscr{N}}, \{L_h\}_{h \in \mathscr{H}} \right\},$$

as an equilibrium.

Let me start by using a normalized price system in which a CRS function defines the GDP deflator

$$\overline{p}_Y = Q^p\left(\{p_i\}_{i \in \mathscr{N}} \right) = 1. \tag{50}$$

Using equation (22), prices for firm z_i are given by

$$\begin{aligned} & p_{z_i} = \frac{w_h}{\mu_i}\left(\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}\right)^{-1} \quad \text{if} \quad \exists h \in \mathscr{H} : \frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}} > 0 \\ \text{otherwise} \quad & p_{z_i} = \frac{w_h}{\mu_i}\left(\frac{\partial\, y_{z_i}}{\partial\, x_{z_i \underline{j}}}\right)^{-1}\left(\frac{\partial\, y_{z_{\bar{j}}}}{\partial\, \ell_{\bar{j}h}}\right)^{-1}\prod_{j\in\mathscr{N}_{z_i}}\frac{1}{\mu_j}\left(\frac{y_{z_j}}{y_j}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\bar{j}\}}\left(\frac{\partial\, y_{z_j}}{\partial\, x_{z_j j+1}}\right)^{-1} \end{aligned} \tag{51}$$

where $\mathscr{N}_{z_i} = \left\{\underline{j}, \underline{j}+1, \cdots, \bar{j}-1, \bar{j}\right\}$ captures a sequence of sectors for which there is sequence of firms that establish a connection between the labor supply from households of type h and the intermediate input demand from firm z_i. What I strictly need for this proof is that $\forall i \in \mathscr{N}$, there $\exists h \in \mathscr{H}$, such that for every firm in sector i, there is some direct or indirect demand of the factor supplied by a worker of type h, and that for every type of worker $h \in \mathscr{H}$, there exists a sector $i \in \mathscr{N}$ that satisfies this condition.

As a consequence, prices for sector $i \in \mathscr{N}$ are given by

$$\begin{aligned} p_i = \frac{w_h}{\mu_i}\Bigg(&\int \mathbb{1}\left\{\ell_{z_i h} > 0\right\}\left(\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}\right)^{\frac{\mu_i}{1-\mu_i}} d\, z_i \\ &+ \int \mathbb{1}\left\{\ell_{z_i h} = 0\right\}\left(\frac{\partial\, y_{z_i}}{\partial\, x_{z_i \underline{j}}}\frac{\partial\, y_{z_{\bar{j}}}}{\partial\, \ell_{z_{\bar{j}} h}}\prod_{j\in\mathscr{N}_{z_i}}\mu_j\left(\frac{y_j}{y_{z_j}}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\bar{j}\}}\frac{\partial\, y_{z_j}}{\partial\, x_{z_j j+1}}\right)^{\frac{\mu_i}{1-\mu_i}} d\, z_i\Bigg)^{\frac{\mu_i-1}{\mu_i}}. \end{aligned} \tag{52}$$

From equation (50) wages are given by

$$\begin{aligned} w_h = Q^p\Bigg(\Bigg\{&\frac{1}{\mu_i}\Bigg(\int \mathbb{1}\left\{\ell_{z_i h} > 0\right\}\left(\frac{\partial\, y_{z_i}}{\partial\, \ell_{z_i h}}\right)^{\frac{\mu_i}{1-\mu_i}} d\, z_i \\ &+ \int \mathbb{1}\left\{\ell_{z_i h} = 0\right\}\left(\frac{\partial\, y_{z_i}}{\partial\, x_{z_i \underline{j}}}\frac{\partial\, y_{z_{\bar{j}}}}{\partial\, \ell_{z_{\bar{n}} h}}\prod_{j\in\mathscr{N}_{z_i}}\mu_j\left(\frac{y_j}{y_{z_j}}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\bar{j}\}}\frac{\partial\, y_{z_j}}{\partial\, x_{z_j j+1}}\right)^{\frac{\mu_i}{1-\mu_i}} d\, z_i\Bigg)^{\frac{\mu_i-1}{\mu_i}}\Bigg\}_{i\in\mathscr{N}}\Bigg)^{-1}. \end{aligned} \tag{53}$$

Notice that prices and wages are strictly positive because the marginal productivities of factors and intermediate inputs have to be strictly positive when there is some demand.

Now, I need to prove that starting from the set of equilibrium conditions represented in equations (47), (48), and (49), and under the system of prices represented in equations (52) and (53), the optimality conditions for firms and households hold.

To obtain equations (41) and (42), assume that firms in sector i directly or indirectly demand workers of type h, and firms in sector j directly or indirectly demand workers of type b. This assumption is made without loss of generality as it holds for any combination of pairs $i, j \in \mathscr{N}$ and $h, b \in \mathscr{H}$. Introducing equations (47) and (48) in (52)

$$p_i = w_h \left(\left(-\frac{w_b}{w_h} \frac{U_Y}{U_L} \frac{\partial Y / \partial C_i}{\partial L / \partial L_b} \right)^{\frac{\mu_i}{\mu_i - 1}} \int \left(\frac{y_i}{y_{z_i}} \right)^{\mu_i} d\, z_i \right)^{\frac{\mu_i - 1}{\mu_i}} = -w_b \frac{U_Y}{U_L} \frac{\partial Y / \partial C_i}{\partial L / \partial L_b},$$

$$p_j = -w_b \frac{U_Y}{U_L} \frac{\partial Y / \partial C_j}{\partial L / \partial L_b}.$$

This proofs equation (42). Dividing these two conditions, I arrive to $\frac{p_j}{p_i} = \frac{\partial Y / \partial C_j}{\partial Y / \partial C_i}$, which is equation (41).

Equation (39) comes from multiplying equation (42) by C_i, adding up over all sectors, using the assumption that $Q_Y \left(\{C_i\}_{i \in \mathscr{N}} \right)$ is CRS in conjunction with Euler's homogeneous function theorem, and the implicit function theorem

$$w_b U_Y \underbrace{\sum_{i \in \mathscr{N}} C_i \frac{\partial Y}{\partial C_i}}_{= Y} = -U_L \frac{\partial L}{\partial L_b} \underbrace{\sum_{i \in \mathscr{N}} p_i C_i}_{= GDP},$$

this implies that $\frac{w_b}{p_Y} = -\frac{U_L}{U_Y} \frac{\partial L}{\partial L_b}$, which is equation (39).

Equation (40) comes from dividing equation (39) by equation (42)

$$\frac{p_i}{p_Y} = \frac{\partial Y}{\partial C_i}.$$

Now for firms, I obtain equation (27) from equation (47), using the implicit function theorem, and introducing equations (17) and (41)

$$\frac{p_i}{p_j} \frac{\partial Y / \partial C_j}{\partial Y / \partial C_i} = \mu_i \frac{p_i}{p_j} \left(\frac{y_i}{y_{z_i}} \right)^{1 - \mu_i} \frac{\partial y_{z_i}}{\partial x_{z_i j}}$$

$$\underbrace{\frac{p_i}{p_j}\frac{\partial Y/\partial C_j}{\partial Y/\partial C_i}}_{=1} = \mu_i \frac{p_i}{p_j}\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i}\frac{\partial y_{z_i}}{\partial x_{z_ij}}$$

$$\frac{\partial y_{z_i}}{\partial x_{z_ij}} = \frac{1}{\mu_i}\frac{p_j}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i,j \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial x_{z_ij}} > 0.$$

Equation (25) comes from adding up equation (27) over all sectors, and using the assumption that $Q_i^x\left(\left\{A_{ij}^x\, x_{z_ij}\right\}_{j\in\mathscr{N}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem

$$\mu_i\, p_{z_i}\frac{\partial y_{z_i}}{\partial X_{z_i}} A_i^x \underbrace{\sum_{j\in\mathscr{N}} x_{z_ij}\frac{\partial Q_i^x\left(\left\{A_{ij}^x\, x_{z_ij}\right\}_{j\in\mathscr{N}}\right)}{\partial x_{z_ij}}}_{=Q_i^x\left(\left\{A_{ij}^x\, x_{z_ij}\right\}_{j\in\mathscr{N}}\right)} = \underbrace{\sum_{j\in\mathscr{N}} p_j\, x_{z_ij}}_{=p_{z_i}^x\, X_{z_i}}$$

$$\frac{\partial y_{z_i}}{\partial X_{z_i}} = \frac{1}{\mu_i}\frac{p_{z_i}^x}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial X_{z_i}} > 0.$$

Equation (26) comes from introducing equations (17) and (42) in equation (48)

$$\underbrace{-\frac{p_i}{w_b}\frac{U_L}{U_Y}\frac{\partial L/\partial L_b}{\partial Y/\partial C_i}}_{=1} = \mu_i \frac{p_i}{w_h}\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i}\frac{\partial y_{z_i}}{\partial \ell_{z_ih}}$$

$$\frac{\partial y_{z_i}}{\partial \ell_{z_ih}} = \frac{1}{\mu_i}\frac{w_h}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial \ell_{z_ih}} > 0.$$

Equation (24) comes from adding up equation (26) over all types of labor, and using the assumption that $Q_i^l\left(\left\{A_{ih}^\ell\, \ell_{z_ih}\right\}_{h\in\mathscr{H}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem

$$\mu_i\, p_{z_i}\frac{\partial y_{z_i}}{\partial L_{z_i}} A_i^\ell \underbrace{\sum_{h\in\mathscr{H}} \ell_{z_ih}\frac{\partial Q_i^\ell\left(\left\{A_{ih}^\ell\, \ell_{z_ib}\right\}_{b\in\mathscr{H}}\right)}{\partial \ell_{z_ih}}}_{=Q_i^\ell\left(\left\{A_{ih}^\ell\, \ell_{z_ih}\right\}_{h\in\mathscr{H}}\right)} = \underbrace{\sum_{h\in\mathscr{H}} w_h\, \ell_{z_ih}}_{=p_{z_i}^\ell\, L_{z_i}}$$

$$\frac{\partial y_{z_i}}{\partial L_{z_i}} = \frac{1}{\mu_i}\frac{p_{z_i}^\ell}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial L_{z_i}} > 0.$$

What remains to be proven is is that households' budget constraints hold. Adding up equation

(33), and introducing equation (34)

$$\sum_{i\in\mathscr{N}} p_i\, C_i = \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{i\in\mathscr{N}} \left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i\right).$$

Introducing zero-profit condition on aggregator firms ($\bar{\pi}_i = 0 \quad \forall i \in \mathscr{N}$), equation (18), and rearranging terms

$$\begin{aligned}\sum_{i\in\mathscr{N}} p_i\, C_i &= \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{i\in\mathscr{N}} \int \Big(p_{z_i}\, y_{z_i} - \sum_{j\in\mathscr{N}} p_j\, x_{z_i j} - \sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}\Big) dz_i \\ &= \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{i\in\mathscr{N}} \int \Big(p_{z_i}\, y_{z_i} - \sum_{j\in\mathscr{N}} p_j\, x_{z_i j} - \sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}\Big) dz_i.\end{aligned}$$

From zero profits for aggregators $p_i y_i = \int p_{z_i} y_{z_i}$, and using equations (49), the households' budget constraints holds

$$0 = \sum_{h\in\mathscr{H}} w_h \underbrace{\left(L_h - \sum_{i\in\mathscr{N}} \int \ell_{z_i h}\, dz_i\right)}_{=0} + \sum_{i\in\mathscr{N}} p_i \underbrace{\left(y_i - C_i - \sum_{j\in\mathscr{N}} \int x_{z_j i}\, dz_j\right)}_{=0}.$$

1.4 Equilibrium Centralities from Subsection 4.2

1.4.1 Goods Market Equilibrium Conditions

Introducing equations (25), (27), (29), and (40) in the goods market resource constraint (49) for sector $i \in \mathscr{N}$

$$S_i = p_i\, C_i + \sum_{j\in\mathscr{N}} \int p_i\, x_{z_j i}\, dz_j = \beta_i\, GDP + \sum_{j\in\mathscr{N}} \mu_j \int \omega^x_{z_j}\, \omega_{z_j i}\, p_{z_j}\, y_{z_j}\, dz_j.$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$S_i = \beta_i\, GDP + \sum_{j\in\mathscr{N}} \Omega^x_{ji}\, S_j, \tag{54}$$

where $\Omega^x_{ij} \equiv \mu_i \, \omega^\ell_i \, \omega_{ij}$.

In matrix form, this equation is represented by

$$\left(I_N - \widetilde{\Omega}'_x \, diag\,(\mu)\right) S = \beta \; GDP,$$

$$S = \left(I_N - \widetilde{\Omega}'_x \, diag\,(\mu)\right)^{-1} \beta \; GDP, \tag{55}$$

where $S \equiv [\, S_1, \cdots, S_N]'$, $\beta \equiv [\,\beta_1, \cdots, \beta_N]'$, $\mu \equiv [\,\mu_1, \cdots, \mu_N]'$, and the matrices $\Omega_x \equiv diag\,(\mu)\, \widetilde{\Omega}_x$, $\Psi_x \equiv (I_N - \Omega_x)^{-1}$, and

$$\Omega_x \equiv \begin{pmatrix} \Omega^x_{11} & \cdots & \Omega^x_{1N} \\ \vdots & \ddots & \vdots \\ \Omega^x_{N1} & \cdots & \Omega^x_{NN} \end{pmatrix}.$$

By dividing element i in equation (54) by nominal GDP, I arrive to the following equation that relates the revenue-based Domar weights and the expenditure shares

$$\lambda = \Psi'_x \, \beta = (I_N - \Omega'_x)^{-1} \, \beta, \tag{56}$$

where $\lambda \equiv [\,\lambda_1, \cdots, \lambda_N]'$. In equilibrium, λ_i captures the share of aggregate expenditure that reaches sector i's revenue.

Let me define

$$\widetilde{\lambda} \equiv \widetilde{\Psi}'_x \, \beta \equiv \left(I_N - \widetilde{\Omega}'_x\right)^{-1} \beta,$$

where

$$\widetilde{\Psi}_x \equiv \left(I_N - \widetilde{\Omega}_x\right)^{-1}$$

Then, in equation (56)

$$\lambda = \Psi_x' \left(I_N - \widetilde{\Omega}_x' \right) \widetilde{\lambda},$$

which allows me to define the cost-based Domar weights

$$\widetilde{\lambda} \equiv \widetilde{\Psi}_x' \left(I_N - \Omega_x' \right) \lambda. \tag{57}$$

To understand the cost-based Domar weights, notice that

$$\widetilde{S}_i \equiv p_i \, C_i + \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ji}^x \, \widetilde{S}_j = \widetilde{\lambda}_i \, GDP$$

where $\widetilde{S}_i = \widetilde{\lambda}_i \, GDP$. Remember that in equilibrium, $\widetilde{\Omega}_{ji}^x$ captures the cost share in sector j of intermediate goods supplied by sector i. And for this reason, $\widetilde{S}_i$ represents the value-added that passes through sector i. For this reason, $\widetilde{\lambda}_i$ captures the aggregate value-added share that passes through sector i. Notice that $\omega_\ell' \widetilde{\lambda} = \mathbb{1}_N' \left(I_N - \widetilde{\Omega}_x' \right) \widetilde{\Psi}_x' \beta = 1$, and for this reason $\omega_i^\ell \widetilde{\lambda}_i$ is the aggregate share of value-added from sector generated by workers in sector i.

Finally, I am going to prove that the value-added that passes through a sector is greater than or equal to its revenue, i.e., that $\widetilde{\lambda}_i \geq \lambda_i$ holds $\forall i \in \mathscr{N}$. Let me start with

$$\widetilde{\Psi}_x - \Psi_x = \widetilde{\Psi}_x - \Psi_x = \sum_{q=1}^{\infty} \left(\widetilde{\Omega}_x^q - \Omega_x^q \right).$$

Notice that $\widetilde{\Omega}_x - \Omega_x = \left(I_N - diag\left(\mu\right) \right) \widetilde{\Omega}_x \succcurlyeq 0_N \, 0_N'$, because $\mu_i \in (0, 1]$ and $\widetilde{\Omega}^x \succcurlyeq 0_N \, 0_N'$ ($A \succcurlyeq B$ means that matrix A is elementwise greater than or equal than matrix B). Now, from induction, for $q > 1$ assume that $\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} \succcurlyeq 0_N \, 0_N'$, then

$$\begin{aligned}
\widetilde{\Omega}_x^q - \Omega_x^q &= \left(\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} diag\left(\mu\right) \right) \widetilde{\Omega}_x \\
&= \left(\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} + \Omega_x^{q-1} \left(I_N - diag\left(\mu\right) \right) \right) \widetilde{\Omega}_x \succcurlyeq 0_N \, 0_N'.
\end{aligned}$$

Therefore $\widetilde{\Psi}_x \succcurlyeq \Psi_x$. As a consequence $\widetilde{\lambda} - \lambda = \left(\widetilde{\Psi}_x - \Psi_x \right)' \beta \succcurlyeq 0_N$.

1.4.2 Labor Market Equilibrium Conditions

Introducing equations (24), (26), and (28) in the factor market clearing condition (49) for household $h \in \mathscr{H}$

$$J_h = w_h\, L_h = \sum_{i \in \mathscr{N}} \int w_h\, \ell_{z_i h}\, dz_i = \sum_{i \in \mathscr{N}} \mu_i \int \omega^{\ell}_{z_i}\, \alpha_{z_i h}\, S_{z_i}\, dz_i.$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$J_h = \sum_{i \in \mathscr{N}} \mu_i\, \widetilde{\Omega}^{\ell}_{ih}\, S_i, \tag{58}$$

where $\widetilde{\Omega}^{\ell}_{ih} \equiv \omega^{\ell}_i\, \alpha_{ih}$.

In matrix form, these equations are represented by

$$J = \widetilde{\Omega}'_{\ell}\, diag\,(\mu)\;S = \Omega'_{\ell}\, S, \tag{59}$$

where the matrices are given by

$$\Omega_{\ell} \equiv \begin{pmatrix} \Omega^{\ell}_{11} & \cdots & \Omega^{\ell}_{1H} \\ \vdots & \ddots & \vdots \\ \Omega^{\ell}_{N1} & \cdots & \Omega^{\ell}_{NH} \end{pmatrix},$$

$$\Omega_{\ell} \equiv diag\,(\mu)\;\widetilde{\Omega}_{\ell}$$

and $J \equiv [\,J_1, \cdots, J_H]'$.

By dividing element h in equation (58) by nominal GDP, I arrive at the following equation that relates the labor income shares and the revenue-based Domar weights

$$\Lambda = \Omega'_{\ell}\, \lambda, \tag{60}$$

where $\Lambda \equiv [\Lambda_1, \cdots, \Lambda_H]'$.

Similarly, I define the cost-based factor Domar weights as

$$\widetilde{\Lambda} \equiv \widetilde{\Omega}_\ell' \widetilde{\lambda}, \tag{61}$$

where $\mathbb{1}_H' \widetilde{\Lambda} = \mathbb{1}_H' \alpha' \, diag(\omega_\ell) \, \widetilde{\lambda} = \omega_\ell' \widetilde{\lambda} = 1$.

Notice that $\widetilde{\Lambda} \succcurlyeq \Lambda$ because

$$\begin{aligned}\widetilde{\Lambda} - \Lambda &= \widetilde{\Omega}_\ell' \widetilde{\lambda} - \Omega_\ell' \lambda \\ &= \underbrace{\widetilde{\Omega}_\ell'}_{\succcurlyeq 0_H 0_N'} \underbrace{\left(\widetilde{\lambda} - \lambda\right)}_{\succcurlyeq 0_N} + \widetilde{\Omega}_\ell' \underbrace{\left(I_N - diag(\mu)\right)}_{\succcurlyeq 0_N 0_N'} \lambda.\end{aligned}$$

The firm-to-worker and worker-to-firm centrality matrices are respectively given by

$$\Psi_\ell = \Psi_x \Omega_\ell, \qquad \widetilde{\Psi}_\ell = \widetilde{\Psi}_x \widetilde{\Omega}_\ell, \tag{62}$$

where $\widetilde{\Psi}_\ell \mathbb{1}_H = \widetilde{\Psi}_x \widetilde{\Omega}_\ell \mathbb{1}_H = \widetilde{\Psi}_x \omega_\ell = \widetilde{\Psi}_x \left(I_N - \widetilde{\Omega}_x\right) \mathbb{1}_N = \mathbb{1}_N$. Additionally $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$ because

$$\widetilde{\Psi}_\ell - \Psi_\ell = \underbrace{\left(\widetilde{\Psi}_x - \Psi_x\right)}_{\succcurlyeq 0_N 0_N'} \underbrace{\widetilde{\Omega}_\ell}_{\succcurlyeq 0_N 0_H'} + \underbrace{\Psi_x}_{\succcurlyeq 0_N 0_N'} \underbrace{\left(I_N - diag(\mu)\right)}_{\succcurlyeq 0_N 0_N'} \widetilde{\Omega}_\ell.$$

1.4.3 Labor Wedges

From equations (25), (29), and (40), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$x_{ji} = \mu_j \, \omega_j^x \, \omega_{ji} \, y_j \, \frac{\beta_j}{\beta_i} \frac{C_i}{C_j} \qquad \forall i, j \in \mathcal{N}.$$

From equation (49), the goods market resource constraint for goods produced firms in sector

i in terms of household h's consumption is given by

$$y_i = C_i + \frac{C_i}{\beta_i} \sum_{j \in \mathscr{N}} \mu_j \, \omega_j^x \omega_{ji} \, y_j \, \frac{\beta_j}{C_j}.$$

In matrix representation, this equation is given by

$$y = C + diag\left(\beta^{\circ -1} \circ C\right) \Omega_x' diag\left(\beta^{\circ -1} \circ C\right)^{-1} y,$$

$$y = \left[I_N - diag\left(\beta^{\circ -1} \circ C\right) \Omega_x' diag\left(\beta^{\circ -1} \circ C\right)^{-1}\right]^{-1} C,$$

$$y = diag\left(\beta^{\circ -1} \circ C\right) \left[I_N - \Omega_x'\right]^{-1} diag\left(\beta^{\circ -1} \circ C\right)^{-1} C,$$

$$diag\left(\beta^{\circ -1} \circ C\right)^{-1} y = \Psi_x' \, \beta, \tag{63}$$

where $\circ$ stands for the Hadamard product, $^\circ$ for the Hadamard power.

Now, from equations (24), (28), (40), and (42), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$\ell_{ih} = -\frac{U_Y}{U_L} \frac{1}{\partial \, L / \partial \, L_h} \, \mu_i \, \omega_i^\ell \, \alpha_{ih} \, y_i \, \beta_i \frac{Y}{C_i} \qquad \forall h \in \mathscr{H}, \quad \text{and} \quad \forall i \in \mathscr{N}.$$

In matrix representation, these conditions are portrayed by

$$\ell_h \frac{\partial \, L}{\partial \, L_h} = -\frac{U_Y}{U_L} Y \, diag\left(\Omega_\ell \, o_H\left(h\right)\right) diag\left(\beta^{\circ -1} \circ C\right)^{-1} y,$$

where $o_H\left(h\right)$ stands for a vector of zeros with size H that has a one in position h.

Adding up, the labor market equilibrium from equation (49) in terms of first-order conditions is given by

$$L_h \frac{\partial \, L}{\partial \, L_h} = -\frac{U_Y}{U_L} Y \, \underbrace{\mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H\left(h\right)\right) diag\left(\beta^{\circ -1} \circ C\right)^{-1} y}_{=\Gamma_h}.$$

Consequently, equilibrium labor supply is characterized by

$$L_h \frac{\partial L}{\partial L_h} + \Gamma_h \frac{U_Y}{U_L} Y = 0.$$

Dividing by L and accounting for $\widetilde{\Lambda}_h = \frac{L_h}{L}\frac{\partial L}{\partial L_h}$

$$-\frac{U_L}{U_Y} = \frac{\Gamma_h}{\widetilde{\Lambda}_h}\frac{Y}{L}.$$

From equations (56) and (60), Γ_h is given by

$$\begin{aligned} \Gamma_h &= \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H\left(h\right)\right) \Psi_x' \, \beta' \, \chi = \mathbb{1}_N' \, diag\left(\Omega_\ell \, o_H\left(h\right)\right) \lambda \\ &= \sum_{i\in\mathscr{N}} \Omega_{ih}^\ell \sum_{j\in\mathscr{N}} \psi_{ji}^x \sum_{b\in\mathscr{H}} \beta_{bj} \, \chi_b = \chi_h^{-1} \sum_{i\in\mathscr{N}} \Omega_{ih}^\ell \, \lambda_i = \Lambda_h, \end{aligned} \tag{64}$$

which provides an additional form of reaching equation (44).

Taking equation (63)

$$\begin{aligned} \Lambda_h &= \mathbb{1}_N' \, diag\left(\Omega_\ell o_H\left(h\right)\right) \Psi_x' \, \beta \\ &= \mathbb{1}_N' \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) diag\left(\mu\right) \left(I_N - \widetilde{\Omega}_x' \, diag\left(\mu\right)\right)^{-1} \beta \\ &= \mathbb{1}_N' \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) \left(diag\left(\mu\right)^{-1} - \widetilde{\Omega}_x'\right)^{-1} \beta. \end{aligned} \tag{65}$$

1.4.4 Household Budget Constraint Equilibrium Conditions

Introducing equations (24) and (25) in the profit equation (18)

$$\pi_{z_i} = \left(1 - \mu_i\right) p_{z_i} \, y_{z_i}. \tag{66}$$

Introducing equations (24), (26), (28), (49), and (66) in the representative household's budget constraint (33)

$$GDP = \sum_{i \in \mathscr{N}} \int \left(\mu_i \, \omega^\ell_{z_i} \sum_{h \in \mathscr{H}} \alpha_{z_i h} + (1 - \mu_i) \right) p_{z_i} \, y_{z_i} \, dz_i. \tag{67}$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$GDP = \sum_{i \in \mathscr{N}} \left(\mu_i \sum_{h \in \mathscr{H}} \widetilde{\Omega}^\ell_{ih} + (1 - \mu_i) \right) S_i. \tag{68}$$

In matrix form, these equations are represented by

$$GDP = (\Omega_\ell \, \mathbb{1}_H + \Omega_\pi)' \, S, \tag{69}$$

where the matrices are given by $\Omega_\pi = \mathbb{1}_N - \mu$.

By dividing by nominal GDP, I arrive at the following equation that relates the expenditure shares and the revenue-based Domar weights

$$1 = (\Omega_\ell \, \mathbb{1}_H + \Omega_\pi)' \, \lambda. \tag{70}$$

Using this equilibrium condition to define nominal GDP

$$\begin{aligned} GDP &= \sum_{h \in \mathscr{H}} J_h + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, S_i \\ &= \sum_{h \in \mathscr{H}} \sum_{i \in \mathscr{N}} \Omega^\ell_{ih} \, S_i + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, S_i \\ &= \sum_{i \in \mathscr{N}} \mu_i \, \omega^\ell_i \, S_i + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, S_i = \sum_{i \in \mathscr{N}} (1 - \mu_i \, \omega^x_i) \, S_i. \end{aligned} \tag{71}$$

1.5 Proof for Propositions in Section 5

1.5.1 Proof for Proposition 2

Using the following equations, I obtain a first-order approximation around the equilibrium for prices

$$p_{z_i}^{\ell} = \frac{\sum_{h\in\mathscr{H}} w_h \, \ell_{z_i h}}{A_i^{\ell} \, Q_i^{\ell} \left(\left\{ A_{ih}^{\ell} \, \ell_{z_i h} \right\}_{h\in\mathscr{H}} \right)}, \tag{72}$$

$$p_{z_i}^{x} = \frac{\sum_{j\in\mathscr{N}} p_j \, x_{z_i j}}{A_i^{x} \, Q_i^{x} \left(\left\{ A_{ij}^{x} \, x_{z_i j} \right\}_{j\in\mathscr{N}} \right)}, \tag{73}$$

$$p_{z_i} = \frac{\left(p_{z_i}^{\ell} \, L_{z_i} + p_{z_i}^{x} \, X_{z_i} \right)}{\mu_i \, A_i \, Q_i \left(L_{z_i}, \, X_{z_i} \right)}, \tag{74}$$

$$p_Y = \frac{\sum_{i\in\mathscr{N}} p_i \, C_i}{Q_Y \left(\left\{ C_i \right\}_{i\in\mathscr{N}} \right)}. \tag{75}$$

From equation (72)

$$\widehat{p}_{z_i}^{\ell} = \frac{A_i^{\ell}}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, A_i^{\ell}}} \widehat{A}_i^{\ell} + \sum_{h\in\mathscr{H}} \left(\frac{w_h}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, w_h}} \widehat{w}_h + \frac{A_{ih}^{\ell}}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, A_{ih}^{\ell}}} \widehat{A}_{ih}^{\ell} + \frac{\ell_{z_i h}}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, \ell_{z_i h}}} \widehat{\ell}_{z_i h} \right),$$

where $\frac{A_i^{\ell}}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, A_i^{\ell}}} = -1$, $\frac{w_h}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, w_h}} = \alpha_{z_i h}$, $\frac{A_{ih}^{\ell}}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, A_{ih}^{\ell}}} = -\alpha_{z_i h}$, $\frac{\ell_{z_i h}}{p_{z_i}^{\ell}} \overline{\frac{\partial \, p_{z_i}^{\ell}}{\partial \, \ell_{z_i h}}} = \alpha_{z_i h} - e \left(L_{z_i}, \ell_{z_i h} \right) = 0$ from equation (28), and $\widehat{x} = log \left(x / \overline{x} \right)$ stands for the log deviation around the equilibrium for variable x. As a consequence

$$\widehat{p}_{z_i}^{\ell} = -\widehat{A}_i^{\ell} + \sum_{h\in\mathscr{H}} \alpha_{z_i h} \left(\widehat{w}_h - \widehat{A}_{ih}^{\ell} \right). \tag{76}$$

Similarly, from equations (73), (74), and (75)

$$\widehat{p}_{z_i}^{x} = -\widehat{A}_i^{x} + \sum_{j\in\mathscr{N}} \omega_{z_i j} \left(\widehat{p}_j - \widehat{A}_{ij}^{x} \right), \tag{77}$$

$$\widehat{p}_{z_i} = \omega_{z_i}^{\ell} \, \widehat{p}_{z_i}^{\ell} + \omega_{z_i}^{x} \, \widehat{p}_{z_i}^{x} - \widehat{A}_i - \widehat{\mu}_i, \tag{78}$$

$$\widehat{p}_Y = \sum_{i \in \mathscr{N}} \beta_i \, \widehat{p}_i. \tag{79}$$

From imposing symmetry in the decision of monopolistically competitive firms within the same sector, these equations are represented in matrix form by

$$\widehat{p}_\ell = \alpha \, \widehat{w} - \widehat{A}_\ell - \left(\alpha \circ \underline{\widehat{\mathrm{A}}}_\ell \right) \mathbb{1}_H, \tag{80}$$

$$\widehat{p}_x = \mathscr{W} \, \widehat{p} - \widehat{A}_x - \left(\mathscr{W} \circ \underline{\widehat{\mathrm{A}}}_x \right) \mathbb{1}_N, \tag{81}$$

$$\widehat{p} = diag\left(\omega_\ell\right) \widehat{p}_\ell + diag\left(\omega_x\right) \widehat{p}_x - \widehat{A} - \widehat{\mu}, \tag{82}$$

$$\widehat{p}_Y = \beta' \, \widehat{p}. \tag{83}$$

Introducing equations (80) and (81) in equation (82)

$$\widehat{p} = \widetilde{\Psi}_x \left(\widetilde{\Omega}_\ell \, \widehat{w} - \widehat{\mathscr{A}} - \widehat{\mu} \right), \tag{84}$$

and introducing equation (84) in equation (83)

$$\widehat{p}_Y = \widetilde{\lambda}' \left(\widetilde{\Omega}_\ell \, \widehat{w} - \widehat{\mathscr{A}} - \widehat{\mu} \right). \tag{85}$$

The matrices previously used are defined by

$$\alpha \equiv \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1H} \\ \vdots & \ddots & \vdots \\ \alpha_{N1} & \cdots & \alpha_{NH} \end{pmatrix}, \qquad \mathscr{W} \equiv \begin{pmatrix} \omega_{11} & \cdots & \omega_{1N} \\ \vdots & \ddots & \vdots \\ \omega_{N1} & \cdots & \omega_{NN} \end{pmatrix}, \qquad \widetilde{\Psi}_x \equiv \begin{pmatrix} \widetilde{\psi}^x_{11} & \cdots & \widetilde{\psi}^x_{1N} \\ \vdots & \ddots & \vdots \\ \widetilde{\psi}^x_{N1} & \cdots & \widetilde{\psi}^x_{NN} \end{pmatrix},$$

$\widehat{\mathscr{A}} \equiv \widehat{A} + diag\left(\omega_\ell\right) \widehat{A}_\ell + \left(\widetilde{\Omega}_\ell \circ \underline{\widehat{\mathrm{A}}}_\ell \right) \mathbb{1}_H + diag\left(\omega_x\right) \widehat{A}_x + \left(\widetilde{\Omega}_x \circ \underline{\widehat{\mathrm{A}}}_x \right) \mathbb{1}_N$, $\widehat{A} \equiv \left[\widehat{A}_1, \cdots, \widehat{A}_N \right]'$, $\widehat{A}_\ell \equiv \left[\widehat{A}^\ell_1, \cdots, \widehat{A}^\ell_N \right]'$, $\widehat{A}_x \equiv \left[\widehat{A}^x_1, \cdots, \widehat{A}^x_N \right]'$, $\underline{\widehat{\mathrm{A}}}_\ell = \left[\underline{\widehat{\mathrm{A}}}^\ell_1, \cdots, \underline{\widehat{\mathrm{A}}}^\ell_N \right]'$, $\underline{\widehat{\mathrm{A}}}^\ell_i = \left[\widehat{A}^\ell_{i1}, \cdots, \widehat{A}^\ell_{iH} \right]'$, $\underline{\widehat{\mathrm{A}}}_x = \left[\underline{\widehat{\mathrm{A}}}^x_1, \cdots, \underline{\widehat{\mathrm{A}}}^x_n \right]'$, $\underline{\widehat{\mathrm{A}}}^x_i = \left[\widehat{A}^x_{i1}, \cdots, \widehat{A}^x_{iN} \right]'$, $\widehat{p} \equiv \left[\widehat{p}_1, \cdots, \widehat{p}_N \right]'$, $\widehat{p}_\ell \equiv \left[\widehat{p}^\ell_1, \cdots, \widehat{p}^\ell_N \right]'$, $\widehat{p}_x \equiv \left[\widehat{p}^x_1, \cdots, \widehat{p}^x_N \right]'$, $\widehat{\mu} \equiv \left[\widehat{\mu}_1, \cdots, \widehat{\mu}_N \right]'$, and $\widehat{w} \equiv \left[\widehat{w}_1, \cdots, \widehat{w}_H \right]'$.

1.5.2 Proof for Theorems 1 and 2

From equations (64) and (65)

$$\Lambda_h\,\widehat{\Lambda}_h = \mathbb{1}_N'\,diag\left(\Omega_\ell\,o_H\left(h\right)\right)\Psi_x'\begin{pmatrix}\beta_1\,\widehat{\beta}_1\\ \vdots\\ \beta_N\,\widehat{\beta}_N\end{pmatrix} + \sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\,\lambda_i\,\left(\widehat{\omega}_i^{\ell}+\widehat{\alpha}_{ih}\right)$$
$$+\,\mathbb{1}_N'\,diag\left(\widetilde{\Omega}_\ell\,o_H\left(h\right)\right)\frac{d\left(diag\left(\mu\right)^{-1}-\widetilde{\Omega}_x'\right)^{-1}}{d\,log\,\widetilde{\Omega}_x}\beta$$
$$+\,\mathbb{1}_N'\,diag\left(\widetilde{\Omega}_\ell\,o_H\left(h\right)\right)\frac{d\left(diag\left(\mu\right)^{-1}-\widetilde{\Omega}_x'\right)^{-1}}{d\,log\,\mu}\beta.$$

Using equations (56), (60), and (62), and the fact that for any invertible matrix A, $\frac{d\,A^{-1}}{d\,x} = -A^{-1}\frac{d\,A}{d\,x}A^{-1}$, the previous equation becomes

$$\widehat{\Lambda}_h = \Lambda_h^{-1}\sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\sum_{j\in\mathscr{N}}\psi_{ji}^{x}\,\beta_j\,\widehat{\beta}_j + \Lambda_h^{-1}\sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\,\lambda_i\,\left(\widehat{\omega}_i^{\ell}+\widehat{\alpha}_{ih}\right)$$
$$-\,\Lambda_h^{-1}\,\mathbb{1}_N' diag\left(\Omega_\ell\,o_H\left(h\right)\right)\Psi_x'\frac{d\left(diag\left(\mu\right)^{-1}-\widetilde{\Omega}_x'\right)}{d\,log\,\widetilde{\Omega}_x}diag\left(\mu\right)\,\lambda$$
$$-\,\Lambda_h^{-1}\,\mathbb{1}_N'\,diag\left(\Omega_\ell\,o_H\left(h\right)\right)\Psi_x'\frac{d\left(diag\left(\mu\right)^{-1}-\widetilde{\Omega}_x'\right)}{d\,log\,\mu}diag\left(\mu\right)\,\lambda.$$

$$\widehat{\Lambda}_h = \frac{1}{\Lambda_h}o_H\left(h\right)'\,\Psi_\ell'\,diag\left(\widehat{\mu}\right)\,\lambda$$
$$+\frac{1}{\Lambda_h}\left(\sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\,\lambda_i\,\left(\widehat{\omega}_i^{\ell}+\widehat{\alpha}_{ih}\right)+\sum_{j\in\mathscr{N}}\psi_{jh}^{\ell}\left(\beta_j\,\widehat{\beta}_j+\sum_{i\in\mathscr{N}}\Omega_{ij}^{x}\,\lambda_i\,\left(\widehat{\omega}_i^{x}+\widehat{\omega}_{ij}\right)\right)\right). \tag{86}$$

Now, using equation (64)

$$d\,\Lambda_h = \sum_{i\in\mathscr{H}}\psi_{ih}^{\ell}\,\lambda_i\,d\,log\,\mu_i + \sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\,d\,\widetilde{\Omega}_{ih}^{\ell} + \sum_{j\in\mathscr{N}}\psi_{jh}^{\ell}\left(d\,\beta_j+\sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\,d\,\widetilde{\Omega}_{ij}^{x}\right). \tag{87}$$

Adding up across worker types

$$d\,\Gamma = \sum_{h\in\mathscr{H}}d\,\Lambda_h = \sum_{i\in\mathscr{H}}\psi_i^{\ell}\,\lambda_i\,d\,log\,\mu_i + \sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\,d\,\omega_i^{\ell} + \sum_{j\in\mathscr{N}}\psi_j^{\ell}\left(d\,\beta_j+\sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\,d\,\widetilde{\Omega}_{ij}^{x}\right).$$

Symmetry in distortion centralities was shown in equation (45).

Similarly, from equation (61) the value-added by labor of type h can be represented as

$$\widetilde{\Lambda}_h = \mathbb{1}'_N \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right)\left(I_N - \widetilde{\Omega}'_x\right)^{-1} \beta. \tag{88}$$

Hence

$$\widetilde{\Lambda}_h \, \widehat{\widetilde{\Lambda}}_h = \mathbb{1}'_N \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) \widetilde{\Psi}'_x \begin{pmatrix} \beta_1 \, \widehat{\beta}_1 \\ \vdots \\ \beta_N \, \widehat{\beta}_N \end{pmatrix}$$

$$+ \sum_{i\in\mathscr{N}} \widetilde{\Omega}^\ell_{ih} \, \widetilde{\lambda}_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) + \mathbb{1}'_N \, diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) \frac{d\left(I_N - \widetilde{\Omega}'_x\right)^{-1}}{d \, log \, \widetilde{\Omega}_x} \beta.$$

Using the fact that for any invertible matrix A, $\frac{d\,A^{-1}}{d\,x} = -A^{-1}\frac{d\,A}{d\,x}A^{-1}$, the previous equation becomes

$$\widetilde{\Lambda}_h \widehat{\widetilde{\Lambda}}_h = \sum_{i\in\mathscr{N}} \widetilde{\Omega}^\ell_{ih} \sum_{j\in\mathscr{N}} \widetilde{\psi}^x_{ji} \, \beta_j \, \widehat{\beta}_j + \sum_{i\in\mathscr{N}} \widetilde{\Omega}^\ell_{ih} \, \widetilde{\lambda}_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) - \mathbb{1}'_N diag\left(\widetilde{\Omega}_\ell \, o_H\left(h\right)\right) \widetilde{\Psi}'_x \frac{d\left(I_N - \widetilde{\Omega}'_x\right)}{d \, log \, \widetilde{\Omega}_x} \widetilde{\lambda}.$$

$$\widehat{\widetilde{\Lambda}}_h = \frac{1}{\widetilde{\Lambda}_h} \left(\sum_{i\in\mathscr{N}} \widetilde{\Omega}^\ell_{ih} \, \widetilde{\lambda}_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) + \sum_{j\in\mathscr{N}} \widetilde{\psi}^\ell_{jh} \left(\beta_j \, \widehat{\beta}_j + \sum_{i\in\mathscr{N}} \widetilde{\Omega}^x_{ij} \, \widetilde{\lambda}_i \, \left(\widehat{\omega}^x_i + \widehat{\omega}_{ij}\right)\right)\right).$$

Then

$$d\,\widetilde{\Lambda}_h = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \, d\,\widetilde{\Omega}^\ell_{ih} + \sum_{j\in\mathscr{N}} \widetilde{\psi}^\ell_{jh} \left(d\,\beta_j + \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \, d\,\widetilde{\Omega}^x_{ij}\right). \tag{89}$$

Notice that $\sum_{h\in\mathscr{H}} d\,\widetilde{\Lambda}_h = 0$, which makes sense because $\sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h = 1$.

1.5.3 Proof for Theorem 3

The first order approximation for equation (33) is given by

$$\widehat{GDP} = \sum_{h\in\mathscr{H}} \Lambda_h \left(\widehat{w}_h + \widehat{L}_h\right) + (1-\Gamma)\,\widehat{\Pi}. \tag{90}$$

The first order approximation for dividend income in equations (34) and (66) is given by

$$\widehat{\Pi} = \frac{1}{\Pi} \sum_{i\in\mathscr{N}} \int S_{z_i} \left((1-\mu_i)\,\widehat{S}_{z_i}\,dz_i - \mu_i\,\widehat{\mu}_i\right) dz_i. \tag{91}$$

Introducing equation (91) in equation (90)

$$GDP\,\widehat{GDP} = \sum_{h\in\mathscr{H}} J_h \left(\widehat{w}_h + \widehat{L}_h\right) + \sum_{i\in\mathscr{N}} \int p_{z_i}\,y_{z_i}\left((1-\mu_i)\left(\widehat{p}_{z_i} + \widehat{y}_{z_i}\right) - \mu_i\,\widehat{\mu}_i\right) dz_i. \tag{92}$$

From equations (85) and (92), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$\widehat{Y} = \widehat{GDP} - \widehat{p}_Y = \sum_{h\in\mathscr{H}} \Lambda_h \left(\widehat{w}_h + \widehat{L}_h\right) - \widetilde{\Lambda}'\,\widehat{w} + \widetilde{\lambda}'\left(\widehat{\mathscr{A}} + \widehat{\mu}\right) + \sum_{i\in\mathscr{N}} \lambda_i \left((1-\mu_i)\,\widehat{S}_i - \mu_i\,\widehat{\mu}_i\right).$$

Then

$$\widehat{Y} = \widetilde{\lambda}'\left(\widehat{\mathscr{A}} + \widehat{\mu}\right) + \Lambda'\,\widehat{J} - \widetilde{\Lambda}'\,\widehat{J} + \sum_{i\in\mathscr{N}} \lambda_i \left((1-\mu_i)\,\widehat{S}_i - \mu_i\,\widehat{\mu}_i\right) + \widetilde{\Lambda}'\,\widehat{L}$$

Therefore

$$\frac{Y}{\overline{\overline{Y}}} = \overline{\eta}\,\mathscr{D}(\mathscr{A})\;\mathscr{D}(\mu)\;\mathscr{D}(J)\;\mathscr{D}(\Pi)\;F\left(\{L_h\}_{h\in\mathscr{H}}\right) \tag{93}$$

where $L = F\left(\{L_h\}_{g\in\mathscr{H}}\right)$ is a CRS function such that $\frac{d\log F(\{L_h\}_{h\in\mathscr{H}})}{d\log L_h} = \widetilde{\Lambda}_h$, and

$$\mathscr{D}(\mathscr{A}) = exp\left\{\widetilde{\Lambda}'\,\widehat{\mathscr{A}}\right\}, \qquad \mathscr{D}(\mu) = exp\left\{\widetilde{\Lambda}'\,\widehat{\mu}\right\},$$

$$\mathcal{D}\left(\Pi\right)=exp\left\{\sum_{i\in\mathcal{N}}\lambda_i\left(\left(1-\mu_i\right)\widehat{S}_i-\mu_i\,\widehat{\mu}_i\right)\right\},$$

$$\mathcal{D}\left(J\right)=exp\left\{\Lambda'\,\widehat{J}-\widetilde{\Lambda}'\,\widehat{J}\right\},\tag{94}$$

and $\overline{\eta}$ stands for a constant.

As a consequence

$$Y=\eta\,\mathcal{D}\left(\mathcal{A}\right)\,\mathcal{D}\left(\mu\right)\,\mathcal{D}\left(J\right)\,\mathcal{D}\left(\Pi\right)\,F\left(\left\{L_h\right\}_{h\in\mathcal{H}}\right)=TFP\,F\left(\left\{L_h\right\}_{h\in\mathcal{H}}\right)\tag{95}$$

$$TFP=\eta\,\mathcal{D}\left(\mathcal{A}\right)\,\mathcal{D}\left(\mu\right)\,\mathcal{D}\left(\mathcal{J}\right)\,\mathcal{D}\left(\Pi\right)$$

with $\eta=\overline{\eta}\,\overline{Y}$.

Add and subtract $\widehat{GDP}$ to express equation (95) in terms of Domar weights and labor income shares

$$\begin{aligned}\widehat{Y}&=\widetilde{\lambda}'\,\widehat{\mathcal{A}}+\left(\widetilde{\lambda}-diag\left(\mu\right)\,\lambda\right)'\widehat{\mu}+\Lambda'\,\widehat{\Lambda}-\widetilde{\Lambda}'\,\widehat{\Lambda}+\lambda' diag\left(\mathbb{1}_N-\mu\right)\,\widehat{\lambda}+\widetilde{\Lambda}'\,\widehat{L}\\&\quad+\underbrace{\sum_{h\in\mathcal{H}}\left(\Lambda_h+\sum_{i\in\mathcal{N}}\left(1-\mu_i\right)\lambda_i-\sum_{h\in\mathcal{H}}\widetilde{\Lambda}_h\right)\widehat{GDP}}_{=0}\\&=\widetilde{\lambda}'\,\widehat{\mathcal{A}}+\left(\widetilde{\lambda}-diag\left(\mu\right)\,\lambda\right)'\widehat{\mu}+\Lambda'\,\widehat{\Lambda}-\widetilde{\Lambda}'\,\widehat{\Lambda}+\lambda' diag\left(\mathbb{1}_N-\mu\right)\,\widehat{\lambda}+\widetilde{\Lambda}'\,\widehat{L}.\end{aligned}$$

where the last equality is given by equations (70).

The $H+N$ vector $\mathcal{R}$ captures the revenue distribution for the representative household

$$\mathcal{R}'=\begin{bmatrix}\Lambda_1 & \cdots & \Lambda_H & \left(1-\mu_1\right)\lambda_1 & \cdots & \left(1-\mu_N\right)\lambda_N\end{bmatrix}.$$

The first H elements capture the share of labor income for labor, and the last N elements capture the share of profits by each sector on household h's expenditure. As the elements of this vector add up to one, its first-order approximation is given by

$$0=\sum_{h\in\mathcal{H}}\Lambda_h\,\widehat{\Lambda}_h+\sum_{i\in\mathcal{N}}\lambda_i\left(\left(1-\mu_i\right)\widehat{\lambda}_i-\mu_i\,\widehat{\mu}_i\right).\tag{96}$$

This implies that

$$\widehat{Y} = \widetilde{\lambda}' \widehat{\mathscr{A}} + \widetilde{\lambda}' \widehat{\mu} - \widetilde{\Lambda}' \widehat{\Lambda} + \widetilde{\Lambda}' \widehat{L}. \tag{97}$$

Now, using equations (64) and (87), and the definitions $\delta_h = \widetilde{\Lambda}_h / \Lambda_h$, $\delta_h = \Gamma^{-1}\ \forall h \in \mathscr{H}$, and $d\,\Gamma = \sum_{h\in\mathscr{H}} d\,\Lambda_h$

$$\begin{aligned}
\widehat{TFP} &= \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) - \sum_{h\in\mathscr{H}} \delta_h\, d\,\Lambda_h = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) - \Gamma^{-1} \sum_{h\in\mathscr{H}} d\,\Lambda_h \\
&= \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) - \widehat{\Gamma} \\
&= \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\, \widehat{\mathscr{A}}_i + \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i\, \widehat{\mu}_i - \Gamma^{-1} \sum_{i\in\mathscr{N}} \psi_i^{\ell}\, \lambda_i\, d\,log\,\mu_i \\
&\quad - \Gamma^{-1} \sum_{i\in\mathscr{N}} \mu_i\, \lambda_i\, d\,\widetilde{\omega}_i^{\ell} - \Gamma^{-1} \sum_{j\in\mathscr{N}} \psi_j^{\ell} \left(d\,\beta_j + \sum_{i\in\mathscr{N}} \mu_i\, \lambda_i\, d\,\widetilde{\Omega}_{ij}^{x} \right).
\end{aligned} \tag{98}$$

2 Proofs for the normalized nested-CES model

2.1 Firms

The competitive aggregator firm from sector $i \in \mathscr{N}$ operates under the same environment as in the section 1 of this Online Appendix.

The monopolistically competitive firm z_i chooses $\left\{ y_{z_i}, \{\ell_{z_i h}\}_{h\in\mathscr{H}}, \{x_{z_i j}\}_{j\in\mathscr{N}} \right\}$ to maximize

$$\pi_{z_i} = p_{z_i} y_{z_i} - \underbrace{\sum_{h\in\mathscr{H}} w_h \ell_{z_i h}}_{= p_{z_i}^{\ell} L_{z_i}} - \underbrace{\sum_{j\in\mathscr{N}} p_j x_{z_i j}}_{= p_{z_i}^{x} X_{z_i}},$$

subject to

$$\frac{y_{z_i}}{\overline{y}_{z_i}} = A_i \left(\omega_i^{\ell} \left(\frac{L_{z_i}}{\overline{L}_{z_i}} \right)^{\frac{\theta_i - 1}{\theta_i}} + \omega_i^{x} \left(\frac{X_{z_i}}{\overline{X}_{z_i}} \right)^{\frac{\theta_i - 1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i - 1}},$$

$$\frac{L_{z_i}}{\overline{L}_{z_i}} = \left(\sum_{h\in\mathscr{H}} \alpha_{ih} \left(\frac{\ell_{z_ih}}{\overline{\ell}_{z_ih}}\right)^{\frac{\theta_i^\ell-1}{\theta_i^\ell}}\right)^{\frac{\theta_i^\ell}{\theta_i^\ell-1}},$$

$$\frac{X_{z_i}}{\overline{X}_{z_i}} = \left(\sum_{j\in\mathscr{N}} \omega_{ij} \left(\frac{x_{z_ij}}{\overline{x}_{z_ij}}\right)^{\frac{\theta_i^x-1}{\theta_i^x}}\right)^{\frac{\theta_i^x}{\theta_i^x-1}}.$$

From here, the first order conditions are given by

$$p_{z_i}^\ell L_{z_i} = \left(\mu_i\omega_i^\ell\right)^{\theta_i} \left(A_i \frac{p_{z_i}\overline{y}_{z_i}}{p_{z_i}^\ell \overline{L}_{z_i}}\right)^{\theta_i-1} p_{z_i}y_{z_i}, \tag{99}$$

$$p_{z_i}^x X_{z_i} = \left(\mu_i\omega_i^x\right)^{\theta_i} \left(A_i \frac{p_{z_i}\overline{y}_{z_i}}{p_{z_i}^x \overline{X}_{z_i}}\right)^{\theta_i-1} p_{z_i}y_{z_i}, \tag{100}$$

$$w_h\ell_{z_ih} = \left(\mu_i\omega_i^\ell\right)^{\theta_i} \alpha_{ih}^{\theta_i^\ell} \left(A_i \frac{p_{z_i}\overline{y}_{z_i}}{p_{z_i}^\ell \overline{L}_{z_i}}\right)^{\theta_i-1} \left(\frac{p_{z_i}^\ell \overline{L}_{z_i}}{w_h\overline{\ell}_{z_ih}}\right)^{\theta_i^\ell-1} p_{z_i}y_{z_i}, \tag{101}$$

$$p_jx_{z_ij} = \left(\mu_i\omega_i^x\right)^{\theta_i} \omega_{ij}^{\theta_i^x} \left(A_i \frac{p_{z_i}\overline{y}_{z_i}}{p_{z_i}^x \overline{X}_{z_i}}\right)^{\theta_i-1} \left(\frac{p_{z_i}^x \overline{X}_{z_i}}{p_j\overline{x}_{z_ij}}\right)^{\theta_i^x-1} p_{z_i}y_{z_i}. \tag{102}$$

In the point of normalization $A_i = 1\ \forall i \in \mathscr{N}$

$$\overline{p}_{z_i}^\ell \overline{L}_{z_i} = \mu_i\omega_i^\ell \overline{p}_{z_i}\overline{y}_{z_i}, \qquad \overline{p}_{z_i}^x \overline{X}_{z_i} = \mu_i\omega_i^x \overline{p}_{z_i}\overline{y}_{z_i},$$

$$\overline{w}_h\overline{\ell}_{z_ih} = \alpha_{ih}\overline{p}_{z_i}^\ell \overline{L}_{z_i}, \qquad \overline{p}_j\overline{x}_{z_ij} = \omega_{ij}\overline{p}_{z_i}^x \overline{X}_{z_i}.$$

Finally, prices are given by

$$p_{z_i}^\ell = \frac{1}{\overline{L}_{z_i}} \left(\sum_{h\in\mathscr{H}} \alpha_{ih}^{\theta_i^\ell} \left(w_h\overline{\ell}_{z_ih}\right)^{1-\theta_i^\ell}\right)^{\frac{1}{1-\theta_i^\ell}}, \tag{103}$$

$$p_{z_i}^x = \frac{1}{\overline{X}_{z_i}} \left(\sum_{j\in\mathscr{N}} \omega_{ij}^{\theta_i^x} \left(p_j\overline{x}_{z_ij}\right)^{1-\theta_i^x}\right)^{\frac{1}{1-\theta_i^x}}, \tag{104}$$

$$p_{z_i} = \frac{1}{A_i\mu_i\overline{y}_{z_i}} \left(\omega_i^{\ell\,\theta_i} \left(p_{z_i}^\ell \overline{L}_{z_i}\right)^{1-\theta_i} + \omega_i^{x\,\theta_i} \left(p_{z_i}^x \overline{X}_{z_i}\right)^{1-\theta_i}\right)^{\frac{1}{1-\theta_i}}. \tag{105}$$

2.2 Representative Household

The representative household chooses $\left\{Y, L, \{C_i\}_{i\in\mathscr{N}}, \{L_h\}_{h\in\mathscr{H}}\right\}$ to maximize

$$U\left(\widetilde{Y},\widetilde{L}\right) = \frac{\left[\widetilde{Y}\left(1-E^{-\gamma}\widetilde{L}\right)^{\varphi}\right]^{1-\sigma}-1}{1-\sigma},$$

subject to $Y = n\,\widetilde{L}$ and $L = n\,\widetilde{L}$, where n stands for the population size, we start from a unitary normalization such that $\overline{n} = 1$, and

$$\frac{Y}{\overline{\overline{Y}}} = \left(\sum_{i\in\mathscr{N}}\beta_i\left(\frac{C_i}{\overline{C}_i}\right)^{\frac{\varrho-1}{\varrho}}\right)^{\frac{\varrho}{\varrho-1}},$$

$$E = p_Y\,Y = \sum_{i\in\mathscr{N}} p_i C_i \leq \sum_{h\in\mathscr{H}} w_h L_h + \Pi,$$

$$\Pi = \sum_{i\in\mathscr{N}}\left(\bar{\pi}_i + \int \pi_{z_i}\,dz_i\right).$$

The first order conditions are given

$$\widetilde{Y}^{-\sigma}\left(1-E^{-\gamma}\widetilde{L}\right)^{\varphi(1-\sigma)}\left(1+\varphi\,\gamma\frac{E^{-\gamma}\widetilde{L}}{1-E^{-\gamma}\widetilde{L}}\right)\frac{\partial\,Y}{\partial\,C_i} = \varkappa\,n\,p_i, \tag{106}$$

$$\widetilde{Y}^{-\sigma}\left(1-E^{-\gamma}\widetilde{L}\right)^{\varphi(1-\sigma)}\left(1+\varphi\,\gamma\frac{E^{-\gamma}\widetilde{L}}{1-E^{-\gamma}\widetilde{L}}\right) = \varkappa\,n\,p_Y, \tag{107}$$

$$\varphi\,\widetilde{Y}^{1-\sigma}\left(1-E^{-\gamma}\widetilde{L}\right)^{\varphi(1-\sigma)-1}E^{-\gamma}\frac{\partial\,L}{\partial\,L_h} = \varkappa\,n\,w_h, \tag{108}$$

$$\varphi\,\widetilde{Y}^{1-\sigma}\left(1-E^{-\gamma}\widetilde{L}\right)^{\varphi(1-\sigma)-1}E^{-\gamma} = \varkappa\,n\,w, \tag{109}$$

where $\frac{\partial Y}{\partial C_i} = \beta_i\left(\frac{\overline{Y}}{\overline{C}_i}\right)^{\frac{\varrho-1}{\varrho}}\left(\frac{Y}{C_i}\right)^{\frac{1}{\varrho}}$, and $\varkappa$ stands for the Lagrange multiplier for the budget constraint.

From equations (106) and (107)

$$p_i C_{hi} = \beta_i^{\varrho}\left(\frac{p_Y\overline{Y}}{p_i\overline{C}_i}\right)^{\varrho-1}E, \tag{110}$$

or $\overline{p}_i\,\overline{C}_i = \beta_i\,p_Y\,\overline{Y}$ in the point of normalization.

From equations (107) and (109)

$$w\,L = \frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}. \tag{111}$$

From equations (108) and (109)

$$\frac{\partial\,L}{\partial\,L_h} = \frac{w_h}{w}.$$

Using the fact that $\frac{L_h}{L}\frac{\partial\,L}{\partial\,L_h} = \widetilde{\Lambda}_h$

$$\widetilde{\Lambda}_h = \frac{w_h\,L_h}{w\,L}. \tag{112}$$

Introducing equation (112) in equation (111) gives

$$w_h\,L_h = \widetilde{\Lambda}_h \frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}. \tag{113}$$

Now, from equation (113), the first order approximation for the factor supply schedule

$$L_h\widehat{L_h} = \widetilde{\Lambda}_h \frac{\partial\,L_h}{\partial\,\widetilde{\Lambda}_h}\widehat{\widetilde{\Lambda}}_h + n\frac{\partial\,L_h}{\partial\,n}\widehat{n} + w\frac{\partial\,L_h}{\partial\,w}\widehat{w} + w_h\frac{\partial\,L_h}{\partial\,w_h}\widehat{w}_h + E\frac{\partial\,L_h}{\partial\,E}\widehat{E}.$$

$$\frac{\partial\,L_h}{\partial\,\widetilde{\Lambda}_h} = \frac{1}{w_h}\left(\frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}\right), \quad \frac{\partial\,L_h}{\partial\,n} = \widetilde{\Lambda}_h \frac{w}{w_h}\frac{E^{\gamma}}{1 - \varphi\,\gamma}, \quad \frac{\partial\,L_h}{\partial\,w} = \widetilde{\Lambda}_h \frac{n}{w_h}\frac{E^{\gamma}}{1 - \varphi\,\gamma},$$

$$\frac{\partial\,L_h}{\partial\,w_h} = -\frac{\widetilde{\Lambda}_h}{w_h^2}\left(\frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}\right), \quad \frac{\partial\,L_h}{\partial\,E} = -\frac{\widetilde{\Lambda}_h}{1 - \varphi\,\gamma}\left(\frac{\varphi}{w_h} - \gamma\,n\,\frac{w}{w_h}\,E^{\gamma-1}\right)$$

$$\begin{aligned}\widehat{L}_h = {} & \frac{\widetilde{\Lambda}_h}{w_h L_h}\left(\frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}\right)\widehat{\widetilde{\Lambda}}_h + \frac{\widetilde{\Lambda}_h}{w_h L_h}\frac{n\,w\,E^{\gamma}}{1 - \varphi\,\gamma}\widehat{n} + \frac{\widetilde{\Lambda}_h}{w_h L_h}\frac{n\,w\,E^{\gamma}}{1 - \varphi\,\gamma}\widehat{w} \\ & - \frac{\widetilde{\Lambda}_h}{w_h L_h}\left(\frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}\right)\widehat{w}_h - \frac{\widetilde{\Lambda}_h}{w_h L_h}\left(\frac{\varphi\,E - \gamma\,n\,w\,E^{\gamma}}{1 - \varphi\,\gamma}\right)\widehat{E}\end{aligned}$$

$$\begin{aligned}\widehat{L}_h = {} & \frac{1}{w\,L}\left(\frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}\right)\widehat{\widetilde{\Lambda}}_h + \frac{1}{w\,L}\frac{n\,w\,E^{\gamma}}{1 - \varphi\,\gamma}\widehat{n} + \frac{1}{w\,L}\frac{n\,w\,E^{\gamma}}{1 - \varphi\,\gamma}\widehat{w} \\ & - \frac{1}{w\,L}\left(\frac{n\,w\,E^{\gamma} - \varphi\,E}{1 - \varphi\,\gamma}\right)\widehat{w}_h - \frac{1}{w\,L}\left(\frac{\varphi\,E - \gamma\,n\,w\,E^{\gamma}}{1 - \varphi\,\gamma}\right)\widehat{E}.\end{aligned} \tag{114}$$

Now, from equation (111), the first order approximation for the factor supply schedule

$$L\widehat{L} = n\frac{\partial L}{\partial n}\widehat{n} + w\frac{\partial L}{\partial w}\widehat{w} + E\frac{\partial L}{\partial E}\widehat{E}.$$

$$\frac{\partial L}{\partial n} = \frac{E^{\gamma}}{1-\varphi\gamma}, \quad \frac{\partial L}{\partial w} = \frac{\varphi E}{(1-\varphi\gamma)\,w^2}, \quad \frac{\partial L}{\partial E} = -\frac{1}{1-\varphi\gamma}\left(\frac{\varphi}{w} - \gamma E^{\gamma-1}\right)$$

$$\widehat{L} = \frac{1}{1-\varphi\gamma}\left(E^{\gamma}\frac{n}{L}\widehat{n} + \frac{\varphi}{\Gamma}\widehat{w} - \left(\frac{\varphi}{\Gamma} - \gamma\frac{E^{\gamma}}{L}\right)\widehat{E}\right) = \zeta_n\widehat{n} + \zeta_w\widehat{w} - \zeta_e\widehat{E} \tag{115}$$

$$\zeta_n = \frac{E^{\gamma}}{1-\varphi\gamma}\frac{n}{L}, \qquad \zeta_w = \frac{1}{1-\varphi\gamma}\frac{\varphi}{\Gamma}, \qquad \zeta_e = \frac{1}{1-\varphi\gamma}\left(\frac{\varphi}{\Gamma} - \gamma\frac{E^{\gamma}}{L}\right).$$

Under KPR preferences ($\gamma = 0$)

$$\widehat{L} = \frac{n}{L}\widehat{n} + \frac{\varphi}{\Gamma}\left(\widehat{w} - \widehat{E}\right) = \zeta^n\widehat{n} + \zeta^w\widehat{w} - \zeta^e\widehat{E}$$

$$\zeta^n = \frac{n}{L}, \qquad \zeta^w = \zeta^e = \frac{\varphi}{\Gamma}.$$

Under GHH preferences ($\zeta^e = 0$) γ and φ are given by equation $\varphi = \gamma E^{\gamma}\frac{\Gamma}{L}$.

Finally, the GDP deflator is given by

$$p_Y = \frac{1}{\overline{Y}}\left(\sum_{i\in\mathscr{N}} \beta_i^{\varrho}\left(p_i\overline{C}_i\right)^{1-\varrho}\right)^{\frac{1}{1-\varrho}}. \tag{116}$$

2.3 Equilibrium conditions

From now on, I will assume that there is only one type of labor, i.e., $H = 1$.

2.3.1 Goods markets

From equations (102) and (110), the goods produced by sector $i \in \mathscr{N}$ must satisfy under symmetry for firms in the same sector

$$S_i = \sum_{j\in\mathscr{N}} p_i x_{ji} + \sum_{h\in\mathscr{H}} p_i C_{hi},$$

$$S_i = \sum_{j \in \mathscr{N}} \left(\mu_j \omega_j^x\right)^{\theta_j} \omega_{ji}^{\theta_j^x} \left(A_j \frac{p_j \overline{y}_j}{p_j^x \overline{X}_j}\right)^{\theta_j - 1} \left(\frac{p_j^x \overline{X}_j}{p_i \overline{x}_{ji}}\right)^{\theta_j^x - 1} S_j + \beta_i^{\varrho} \left(\frac{p_Y \overline{Y}}{p_i \overline{C}_i}\right)^{\varrho - 1} E. \tag{117}$$

In the steady state this relationship is simplified into

$$S_i = \sum_{j \in \mathscr{N}} \Omega_{ji}^x S_j + \beta_i E$$

which in matrix is represented by equation (55).

The first order approximation for equation (117) is given by

$$\begin{aligned}
&\lambda_i \widehat{S}_i = \beta_i \left(\varrho \widehat{\beta}_i + (\varrho - 1)(\widehat{p}_Y - \widehat{p}_i) + \widehat{E}\right) \\
&+ \sum_{j \in \mathscr{N}} \Omega_{ji}^x \lambda_j \left[\theta_j \left(\widehat{\omega}_j^x + \widehat{\mu}_j\right) + \theta_j^x \widehat{\omega}_{ji} + (\theta_i - 1)\left(\widehat{A}_j + \widehat{p}_j\right) + \left(\theta_j^x - \theta_j\right) \widehat{p}_j^x - \left(\theta_j^x - 1\right) \widehat{p}_i + \widehat{S}_j\right].
\end{aligned}$$

In matrix form this equation is given by

$$\begin{aligned}
diag(\lambda)\, \widehat{S} = & \left(\beta \circ \widehat{\beta}\right) \varrho + \beta \left((\varrho - 1)\, \widehat{p}_Y + \widehat{E}\right) - diag(\beta)(\varrho - 1)\, \widehat{p} \\
& + \left(\Omega_x \circ \widehat{\mathscr{W}}\right)' diag(\theta_x)\, \lambda - diag\left(\Omega_x' diag(\theta_x - \mathbb{1}_N)\, \lambda\right) \widehat{p} \\
& + \Omega_x' diag(\lambda) \left(diag(\theta)(\widehat{\omega}_x + \widehat{\mu}) + diag(\theta - \mathbb{1}_N)\left(\widehat{A} + \widehat{p}\right) + diag(\theta_x - \theta)\, \widehat{p}_x + \widehat{S}\right)
\end{aligned}$$

$$\begin{aligned}
diag(\lambda)\, \widehat{S} = \Psi_x' \Bigg\{ & \left(\beta \circ \widehat{\beta}\right) \varrho + \beta \left((\varrho - 1)\, \widehat{p}_Y + \widehat{E}\right) - diag(\beta)(\varrho - 1)\, \widehat{p} \\
& + \left(\Omega_x \circ \widehat{\mathscr{W}}\right)' diag(\theta_x)\, \lambda - diag\left(\Omega_x' diag(\theta_x - \mathbb{1}_N)\, \lambda\right) \widehat{p} \\
& + \Omega_x' diag(\lambda) \left(diag(\theta)(\widehat{\omega}_x + \widehat{\mu}) + diag(\theta - \mathbb{1}_N)\left(\widehat{A} + \widehat{p}\right) + diag(\theta_x - \theta)\, \widehat{p}_x\right) \Bigg\}.
\end{aligned} \tag{118}$$

2.3.2 Budget Constraint

From the representative household's budget constraint, consumption expenditure must satisfy under symmetry for firms in the same cluster

$$E = w\, L + \sum_{i \in \mathscr{N}} (1 - \mu_i)\, S_i.$$

In the steady state this relationship is represented in matrix form by equation (69).

The first order approximation for this equation is given by

$$\widehat{E} = \Lambda\,\widehat{J} + \sum_{i\in\mathcal{N}} \lambda_i \left((1-\mu_i)\,\widehat{S}_i - \mu_i \widehat{\mu}_i \right).$$

In matrix form this equation is given by

$$\widehat{E} = \Lambda\,\widehat{J} + \lambda' \left(diag\,(\mathbb{1}_N - \mu)\,\widehat{S} - diag\,(\mu)\,\widehat{\mu} \right). \tag{119}$$

2.3.3 Labor Market

From equation (101), equilibrium in the factor market for household h must satisfy

$$w\,L = \sum_{i\in\mathcal{N}} w\,\ell_i = \sum_{i\in\mathcal{N}} \left(\mu_i \omega_i^{\ell}\right)^{\theta_i} \left(A_i \frac{p_i \overline{y}_i}{w_h \overline{\ell}_{ih}} \right)^{\theta_i - 1} S_i. \tag{120}$$

In steady state this relationship is simplified into

$$J = \sum_{i\in\mathcal{N}} \mu_i\,\omega_i^{\ell}\,S_i = \sum_{i\in\mathcal{N}} \Omega_i^{\ell} S_i$$

which in matrix form is represented by equation (59).

The first order approximation for equation (120) is given by

$$\Lambda\,\widehat{J} = \sum_{i\in\mathcal{N}} \Omega_i^{\ell} \lambda_i \left[\theta_i \left(\widehat{\omega}_i^{\ell} + \widehat{\mu}_i \right) + (\theta_i - 1) \left(\widehat{A}_i + \widehat{p}_i \right) - (\theta_i - 1)\,\widehat{w}_h + \widehat{S}_i \right].$$

In matrix form this equation is given by

$$\Lambda\,\widehat{J} = \Omega_\ell' diag\,(\lambda) \left(diag\,(\theta)\,\left(\widehat{\omega}_\ell + \widehat{\mu}\right) + diag\,(\theta - \mathbb{1}_N) \left(\widehat{A} + \widehat{p} \right) + \widehat{S} \right) - \Omega_\ell'\,diag\,(\theta - \mathbb{1}_N)\,\lambda\,\widehat{w}. \tag{121}$$

2.3.4 Prices

The first-order approximation for equations (104), (105), and (116), under symmetry for firms in the same sector is given by

$$\widehat{p}_i^x = \sum_{j\in\mathcal{N}} \omega_{ij}\widehat{p}_j,$$

$$\widehat{p}_i = \omega_i^\ell \widehat{w} + \omega_i^x \widehat{p}_i^x - \widehat{A}_i - \widehat{\mu}_i,$$

$$\widehat{p}_Y = \sum_{i\in\mathcal{N}} \beta_i \widehat{p}_i.$$

In matrix form this equation is given by

$$\widehat{p}_x = \mathscr{W}\widehat{p},$$

$$\widehat{p} = \omega_\ell \widehat{w} + diag\left(\omega_x\right)\widehat{p}_x - \widehat{A} - \widehat{\mu},$$

$$\widehat{p}_Y = \beta'\,\widehat{p}.$$

The last three equations can be simplified into

$$\widehat{p} = \widetilde{\Psi}_\ell \widehat{w} - \widetilde{\Psi}_x\left(\widehat{A} + \widehat{\mu}\right), \tag{122}$$

$$\widehat{p}_x = \mathscr{W}\widetilde{\Psi}_x\left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right), \tag{123}$$

$$\widehat{p}_Y = \widetilde{\lambda}'\left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right). \tag{124}$$

2.3.5 Sufficient equations - Proof for Theorem 4

Labor Income

Introducing equations (115) and (122) in equation (121)

$$
\begin{aligned}
&\Lambda\ (1+\zeta_w)\,\widehat{w} = -\Omega_\ell'\,diag\,(\theta-\mathbb{1}_N)\,\lambda\,\widehat{w} + \zeta_e\,\Lambda\,\widehat{E} - \zeta_n\,diag\,(\Lambda)\,\widehat{n} \\
&+\Omega_\ell'\,diag\,(\lambda)\left(diag\,(\theta)\ (\widehat{\omega}_\ell+\widehat{\mu}) + diag\,(\theta-\mathbb{1}_N)\left(\widetilde{\Psi}_\ell\,\widehat{w} + \left(I_N-\widetilde{\Psi}_x\right)\widehat{A} - \widetilde{\Psi}_x\,\widehat{\mu}\right) + \widehat{S}\right) \\
&=\Omega_\ell'\,diag\,(\lambda)\,diag\,(\theta-\mathbb{1}_N)\left(I_N-\widetilde{\Psi}_x\right)\widehat{A} + \Omega_\ell'\,diag\,(\lambda)\left(diag\,(\theta) - diag\,(\theta-\mathbb{1}_N)\,\widetilde{\Psi}_x\right)\widehat{\mu} - \zeta_n\,\Lambda\,\widehat{n} \\
&+\Omega_\ell'\,diag\,(\theta\circ\lambda)\,\widehat{\omega}_\ell + \Omega_\ell'\,diag\,(\lambda)\,\widehat{S} + \zeta_e\,\Lambda\,\widehat{E} + \Omega_\ell'\,diag\,(\theta-\mathbb{1}_N)\left(diag\,(\lambda)\underbrace{\widetilde{\Psi}_\ell}_{=\mathbb{1}_N} - \lambda\right)\widehat{w}
\end{aligned}
\tag{125}
$$

This implies that

$$
\begin{aligned}
\Lambda\,(1+\zeta_w)\,\widehat{w} = &\sum_{i\in\mathcal{N}}\left(\Omega_i^\ell\,\lambda_i\,(\theta_i-1) - \sum_{j\in\mathcal{N}}\Omega_j^\ell\,\lambda_j\,(\theta_j-1)\,\widetilde{\psi}_{ji}^x\right)\widehat{A}_i \\
&+\sum_{i\in\mathcal{N}}\left(\Omega_i^\ell\,\lambda_i\,\theta_i - \sum_{j\in\mathcal{N}}\Omega_j^\ell\,\lambda_j\,(\theta_j-1)\,\widetilde{\psi}_{ji}^x\right)\widehat{\mu}_i + \sum_{i\in\mathcal{N}}\Omega_i^\ell\,\lambda_i\,\theta_i\,\widehat{\omega}_i^\ell \\
&+\sum_{i\in\mathcal{N}}\Omega_i^\ell\,\lambda_i\,\widehat{S}_i + \Lambda\left(\zeta_e\widehat{E} - \zeta_n\widehat{n}\right).
\end{aligned}
$$

Final Expenditure

Let me start by introducing equations (121) and (122) in equation (119)

$$
\begin{aligned}
\widehat{E} = &\,\Omega_\ell' diag\,(\lambda)\left(diag\,(\theta)\ (\widehat{\omega}_\ell+\widehat{\mu}) + diag\,(\theta-\mathbb{1}_N)\left(\widehat{A}+\widehat{p}\right) + \widehat{S}\right) \\
&-\Omega_\ell'\,diag\,(\theta-\mathbb{1}_N)\,\lambda\,\widehat{w} + \lambda'\left(diag\,(\mathbb{1}_N-\mu)\,\widehat{S} - diag\,(\mu)\,\widehat{\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\Omega_\ell' + \mathbb{1}_N'\,diag\,(\mathbb{1}_N-\mu)\right)diag\,(\lambda)\,\widehat{S} + \Omega_\ell'\,diag\,(\lambda)\,diag\,(\theta-\mathbb{1}_N)\left(I_N-\widetilde{\Psi}_x\right)\widehat{A} \\
&+\Omega_\ell'\,diag\,(\theta-\mathbb{1}_N)\underbrace{\left(diag\,(\lambda)\,\widetilde{\Psi}_\ell - \lambda\right)}_{=0_N}\widehat{w} + \Omega_\ell'\,diag\,(\lambda)\ diag\,(\theta)\ \widehat{\omega}_\ell \\
&+\left(\Omega_\ell' diag\,(\lambda)\left(diag\,(\theta) - diag\,(\theta-\mathbb{1}_N)\,\widetilde{\Psi}_x\right) - \lambda'\,diag\,(\mu)\right)\widehat{\mu}.
\end{aligned}
$$

After taking equation (125) into account

$$
(1+\zeta_e\,\Lambda)\,\widehat{E} = \Lambda\left((1+\zeta_w)\,\widehat{w} + \zeta_n\,\widehat{n}\right) - \lambda'\,diag\,(\mu)\,\widehat{\mu} + \mathbb{1}_N'\,diag\,(\mathbb{1}_N-\mu)\,diag\,(\lambda)\,\widehat{S}. \tag{126}
$$

This implies that

$$\widehat{E} = \frac{1}{1+\zeta_e \Lambda} \left(\Lambda \left((1+\zeta_w)\, \widehat{w} + \zeta_n \widehat{n} \right) + \sum_{i \in \mathscr{N}} \left((1-\mu_i)\, \lambda_i \, \widehat{S}_i - \mu_i \, \lambda_i \, \widehat{\mu}_i \right) \right).$$

Sales

Now, introducing equations (122), (123), and (124) in equation (118)

$$\begin{aligned}
diag\,(\lambda)\,\widehat{S} = {} & \lambda\,\widehat{E} + \lambda\,(\varrho - 1)\,\widehat{p}_Y + \Psi_x'\,\Omega_x'\,diag\,(\lambda)\,diag\,(\theta_x - \theta)\,\widehat{p}_x \\
& + \Psi_x' \left(\Omega_x' diag\,(\lambda)\,diag\,(\theta - \mathbb{1}_N) - diag\,(\beta)\,(\varrho-1) - diag\left(\Omega_x' diag\,(\theta_x - \mathbb{1}_N)\,\lambda\right)\right)\widehat{p} \\
& + \Psi_x' \left(\Omega_x' diag\,(\lambda) \left(diag\,(\theta)\,(\widehat{\omega}_x + \widehat{\mu}) + diag\,(\theta - \mathbb{1}_N)\,\widehat{A}\right)\right) \\
& + \Psi_x' \left(\left(\beta \circ \widehat{\beta}\right)\rho + \left(\Omega_x \circ \widehat{\mathscr{W}}\right)' diag\,(\theta_x)\,\lambda\right) \\
= {} & \lambda\,\widehat{E} + \Psi_x' \left(\beta\,(\varrho-1)\,\beta' - diag\,(\beta)\,(\varrho-1)\right)\widetilde{\Psi}_x \left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right) \\
& + \Psi_x' \left(\Omega_x' diag\,(\lambda)\,diag\,(\theta_x - \mathbb{1}_N)\,\mathscr{W} - diag\left(\Omega_x' diag\,(\theta_x - \mathbb{1}_N)\,\lambda\right)\right)\widetilde{\Psi}_x \left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right) \\
& + \Psi_x' \Omega_x' diag\,(\lambda)\,diag\,(\theta - \mathbb{1}_N)\,(I_N - \mathscr{W})\,\widetilde{\Psi}_x \left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right) \\
& + \Psi_x' \left(\Omega_x' diag\,(\lambda) \left(diag\,(\theta)\,(\widehat{\omega}_x + \widehat{\mu}) + diag\,(\theta - \mathbb{1}_N)\,\widehat{A}\right)\right) \\
& + \Psi_x' \left(\left(\beta \circ \widehat{\beta}\right)\rho + \left(\Omega_x \circ \widehat{\mathscr{W}}\right)' diag\,(\theta_x)\,\lambda\right) \\
= {} & \lambda\,\widehat{E} + \Psi_x' \Omega_x' diag\,(\lambda) \left(diag\,(\theta - \mathbb{1}_N)\,\widehat{A} + diag\,(\theta)\,\widehat{\mu}\right) \\
& - \Psi_x' \left(\beta\,(\varrho-1)\,\widetilde{\lambda}' - diag\,(\beta)\,(\varrho-1)\,\widetilde{\Psi}_x\right)\left(\widehat{A} + \widehat{\mu}\right) \\
& - \Psi_x' \left(\Omega_x' diag\,(\lambda)\,diag\,(\theta - \mathbb{1}_N) - diag\left(\Omega_x' diag\,(\theta - \mathbb{1}_N)\,\lambda\right)\right)\widetilde{\Psi}_x \left(\widehat{A} + \widehat{\mu}\right) \\
& - \Psi_x' \left(\Omega_x' diag\,(\lambda)\,diag\,(\theta_x - \theta)\,\mathscr{W} - diag\left(\Omega_x' diag\,(\theta_x - \theta)\,\lambda\right)\right)\widetilde{\Psi}_x \left(\widehat{A} + \widehat{\mu}\right) \\
& + \Psi_x' \underbrace{\left(\Omega_x' diag\,(\lambda)\,diag\,(\theta - \mathbb{1}_N) - diag\left(\Omega_x' diag\,(\theta - \mathbb{1}_N)\,\lambda\right)\right)\mathbb{1}_N}_{=0_N} \widehat{w} \\
& + \Psi_x' \underbrace{\left(\Omega_x' diag\,(\lambda)\,diag\,(\theta_x - \theta)\,\mathscr{W} - diag\left(\Omega_x' diag\,(\theta_x - \theta)\,\lambda\right)\right)\mathbb{1}_N}_{=0_N} \widehat{w} \\
& + \Psi_x' \left(\left(\beta \circ \widehat{\beta}\right)\rho + \Omega_x'\,diag\,(\lambda)\,diag\,(\theta)\,\widehat{\omega}_x + \left(\Omega_x \circ \widehat{\mathscr{W}}\right)' diag\,(\theta_x)\,\lambda\right)
\end{aligned} \tag{127}$$

This implies that

$$\begin{aligned}
\lambda_i \widehat{S}_i = \lambda_i \widehat{E} &+ \sum_{j \in \mathcal{N}} \psi^x_{ji} \sum_{m \in \mathcal{N}} \Omega^x_{mj} \lambda_m \left((\theta_m - 1) \widehat{A}_m + \theta_m \widehat{\mu}_m \right) \\
&- \sum_{j \in \mathcal{N}} \left(\sum_{m \in \mathcal{N}} \psi^x_{mi} \, \beta_m \, (\varrho - 1) \left(\widetilde{\lambda}_j - \widetilde{\psi}^x_{mj} \right) \right) \left(\widehat{A}_j + \widehat{\mu}_j \right) \\
&- \sum_{j \in \mathcal{N}} \left(\sum_{m \in \mathcal{N}} \psi^x_{mi} \sum_{n \in \mathcal{N}} \Omega^x_{nm} \, \lambda_n \, (\theta_n - 1) \left(\widetilde{\psi}^x_{nj} - \widetilde{\psi}^x_{mj} \right) \right) \left(\widehat{A}_j + \widehat{\mu}_j \right) \\
&- \sum_{j \in \mathcal{N}} \left(\sum_{m \in \mathcal{N}} \psi^x_{mi} \sum_{n \in \mathcal{N}} \Omega^x_{nm} \lambda_n \, (\theta^x_n - \theta_n) \left(\sum_{q \in \mathcal{N}} \omega_{nq} \widetilde{\psi}^x_{qj} - \widetilde{\psi}^x_{mj} \right) \right) \left(\widehat{A}_j + \widehat{\mu}_j \right) \\
&+ \sum_{j \in \mathcal{N}} \psi^x_{ji} \left(\rho \, \beta_j \, \widehat{\beta}_j + \sum_{f \in \mathcal{N}} \Omega^x_{fj} \, \lambda_f \left(\theta_f \, \widehat{\omega}^x_f + \theta^x_f \, \widehat{\omega}_{fj} \right) \right).
\end{aligned}$$

Summary of Sufficient Equations

Equations (125), (126), and (127) represent a system of $N + 2$ equations on $N + 2$ unknowns that captures the elasticities of wages, consumption expenditure and sales in response to exogenous productivity, markdown, labor supply, preferences technology, and equity allocation shocks. This solution can be used to capture the variation of prices from equations (122), (123), and (124). From here using equations (121) it is possible to obtain the variations of factor income.

3 Simple Economy

$$\widetilde{\Omega}_\ell = \begin{pmatrix} 1 \\ \omega_\ell \end{pmatrix}, \qquad \widetilde{\Omega}_x = \begin{pmatrix} 0 & 0 \\ \omega_x & 0 \end{pmatrix}, \qquad \Omega_\ell = \begin{pmatrix} \mu_1 \\ \mu_2 \, \omega_\ell \end{pmatrix}, \qquad \Omega_x = \begin{pmatrix} 0 & 0 \\ \mu_2 \, \omega_x & 0 \end{pmatrix},$$

$$\widetilde{\Psi}_x = \begin{pmatrix} 1 & 0 \\ -\omega_x & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \omega_x & 1 \end{pmatrix}, \quad \Psi_x = \begin{pmatrix} 1 & 0 \\ \mu_2 \, \omega_x & 1 \end{pmatrix}, \quad \widetilde{\Psi}_\ell = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Psi_\ell = \begin{pmatrix} \mu_1 \\ \mu_2 \left(\omega_\ell + \mu_1 \omega_x \right) \end{pmatrix},$$

$$\widetilde{\lambda} = \begin{pmatrix} 1 & \omega_x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ 1 - \beta \end{pmatrix} = \begin{pmatrix} \omega_x + \beta \, \omega_\ell \\ 1 - \beta \end{pmatrix}, \qquad \lambda = \begin{pmatrix} 1 & \mu_2 \, \omega_x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ 1 - \beta \end{pmatrix} = \begin{pmatrix} \beta + (1 - \beta) \, \mu_2 \, \omega_x \\ 1 - \beta \end{pmatrix},$$

$$\Gamma = \mu_1 \, \beta + (1 - \beta) \, \mu_2 \left(\omega_\ell + \mu_1 \, \omega_x \right).$$

From Theorem 4

$$
\begin{aligned}
(1+\zeta_w)\,\Gamma\, d\,log\, w = {} & \zeta_e\,\Gamma\, d\,log\, E + \mu_1\,\lambda_1\, d\,log\, S_1 + \mu_2\,\omega_\ell\,\lambda_2\, d\,log\, S_2 - \mu_2\,\omega_2\,\lambda_2\,(\theta_2-1)\,\omega_x\, d\,log\, A_1 \\
& + (\mu_1\,\lambda_1 - \mu_2\,\omega_\ell\,\omega_x\,\lambda_2\,(\theta_2-1))\, d\;log\;\mu_1 + \mu_2\,\omega_\ell\,\lambda_2\, d\,log\,\mu_2 + \mu_2\,\lambda_2\, d\,\omega_\ell;
\end{aligned}
\tag{128}
$$

$$
\begin{aligned}
\lambda_1\, d\,log\, S_1 = {} & \beta\, d\,log\, E + \mu_2\,\omega_x\,\lambda_2\, d\,log\, S_2 \\
& + \omega_\ell\,(1-\beta)\,(\beta\,(\varrho-1) + \mu_2\,\omega_x\,(\theta_2-1))\,(d\,log\, A_1 + d\,log\,\mu_1) + \beta\,(\beta-\omega_\ell)\,(\varrho-1)\, d\,log\, A_2 \\
& + (\beta\,(\beta-\omega_\ell)\,(\varrho-1) + \mu_2\,\omega_x\;(1-\beta))\, d\;log\;\mu_2 + \varrho\, d\,\beta + \mu_2\,\lambda_2\,\theta_2\, d\,\omega_x;
\end{aligned}
\tag{129}
$$

$$
\begin{aligned}
\lambda_2\, d\,log\, S_2 = {} & (1-\beta)\, d\,log\, E - \beta\,(1-\beta)\,\omega_\ell\,(\varrho-1)\,(d\,log\, A_1 + d\,log\,\mu_1) \\
& + \beta\,(1-\beta)\,(\varrho-1)\,(d\,log\, A_2 + d\,log\,\mu_2) - \varrho\, d\,\beta.
\end{aligned}
\tag{130}
$$

From normalizing with the GDP deflator

$$
d\,log\, w = \widetilde{\lambda}_1\,(d\,log\, A_1 + d\,log\,\mu_1) + \widetilde{\lambda}_2\,(d\,log\, A_2 + d\,log\,\mu_2)\,. \tag{131}
$$

Introducing (131) in (128),

$$
\begin{aligned}
& \zeta_e\,\Gamma\, d\,log\, E + \mu_1\,\lambda_1\, d\,log\, S_1 + \mu_2\,\omega_\ell\,\lambda_2\, d\,log\, S_2 \\
& = \Big((1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1 + \mu_2\,\omega_\ell\,\omega_w\,\lambda_2\,(\theta_2-1)\Big)\, d\,log\, A_1 + (1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2\, d\,log\, A_2 \\
& + \Big((1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1 + \mu_2\,\omega_\ell\,\omega_w\,\lambda_2\,(\theta_2-1) - \mu_1\,\lambda_1\Big)\, d\,log\,\mu_1 \\
& + \Big((1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2 - \mu_2\,\omega_\ell\,\lambda_2\Big)\, d\,log\,\mu_2 + \mu_2\,\lambda_2\, d\,\omega_x.
\end{aligned}
\tag{132}
$$

Introducing (130) in (129)

$$
\begin{aligned}
\lambda_1\, d\,log\, S_1 = {} & (\beta + \mu_2\,\omega_x\,(1-\beta))\, d\,log\, E \\
& + \omega_\ell\,(1-\beta)\,((1-\mu_2\,\omega_x)\,\beta\,(\varrho-1) + \mu_2\,\omega_x\,(\theta_2-1))\,(d\,log\, A_1 + d\,log\,\mu_1) \\
& + \beta\,(\beta - \omega_\ell + \mu_2\,\omega_x\;(1-\beta))\,(\varrho-1)\, d\,log\, A_2 \\
& + (\beta\,(\beta - \omega_\ell + \mu_2\,\omega_x\,(1-\beta))\,(\varrho-1) + \mu_2\,\omega_x\;(1-\beta))\, d\;log\;\mu_2 \\
& + \varrho\,(1-\mu_2\,\omega_x)\, d\,\beta + \mu_2\,\lambda_2\,\theta_2\, d\,\omega_x.
\end{aligned}
\tag{133}
$$

Introducing (130) and (133) in (132)

$$
\begin{aligned}
&(1+\zeta_e)\,\Gamma\, d\,log\, E \\
&= \Big(\omega_\ell\, \beta\,(1-\beta)\,(\mu_2\,(\omega_\ell+\mu_1\,\omega_x)-\mu_1)\,(\varrho-1)+\mu_2\,(1-\beta)\,\omega_\ell\,\omega_x\,(1-\mu_1)\,(\theta_2-1)+(1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1\Big)\, d\,log\, A_1 \\
&+ \Big(\beta\,(\varrho-1)\,(\mu_1\,(\omega_\ell-\beta)-\mu_2\,(1-\beta)\,(\omega_\ell+\mu_1\,\omega_x))+(1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2\Big)\, d\,log\, A_2 \\
&+ \Big(\omega_\ell\,\beta\,(1-\beta)\,(\mu_2\,(\omega_\ell+\mu_1\,\omega_x)-\mu_1)\,(\varrho-1)+\omega_\ell\,\omega_x\,\mu_2\,(1-\beta)\,(1-\mu_1)\,(\theta_2-1)+(1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1-\mu_1\,\lambda_1\Big)\, d\,log\,\mu_1 \\
&+ \Big((1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2-\mu_2\,(1-\beta)\,(\omega_\ell+\mu_1\,\omega_x)-\beta\,(\mu_2\,\omega_\ell\,(1-\beta)+\mu_1\,(\beta-\omega_\ell+\mu_2\,\omega_x\,(1-\beta)))\,(\varrho-1)\Big)\, d\,log\,\mu_2 \\
&+ \varrho\,(\mu_2\,\omega_\ell-\mu_1\,(1-\mu_2\,\omega_x))\, d\,\beta+\mu_2\,(1-\beta)\,(1-\mu_1\,\theta_2)\, d\,\omega_x.
\end{aligned}
\tag{134}
$$

Using equations (131) and (134)

$$
d\,log\,\Gamma = (1+\zeta_w)\, d\,log\, w-(1+\zeta_e)\, d\,log\, E.
$$

3.1 Productivity A_1

$$
\begin{aligned}
&\frac{\partial\,\Gamma}{\partial\,log\,A_1} = (1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1 \\
&- \Big(\omega_\ell\,\beta\,(1-\beta)\,(\mu_2\,(\omega_\ell+\mu_1\,\omega_x)-\mu_1)\,(\varrho-1)+\mu_2\,(1-\beta)\,\omega_\ell\,\omega_x\,(1-\mu_1)\,(\theta_2-1)+(1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1\Big) \\
&= \omega_\ell\,(\beta\,(1-\beta)\,(\mu_1-\mu_2\,(\omega_\ell+\mu_1\,\omega_x))\,(\varrho-1)-\mu_2\,(1-\beta)\,\omega_x\,(1-\mu_1)\,(\theta_2-1))\,.
\end{aligned}
$$

Horizontal Economy - $\boldsymbol{\omega_\ell=1}$: $\frac{\partial\,\Gamma}{\partial\,log\,A_1} = \beta\,(1-\beta)\,(\mu_1-\mu_2)\,(\varrho-1)$.

Vertical Economy - $\boldsymbol{\omega_\ell=0}$ and $\boldsymbol{\beta=0}$: $\frac{\partial\,\Gamma}{\partial\,log\,A_1} = 0$.

3.2 Productivity A_2

$$
\begin{aligned}
&\frac{\partial\,\Gamma}{\partial\,log\,A_2} = (1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2-\Big(\beta\,(\mu_1\,(\omega_\ell-\beta)-\mu_2\,(1-\beta)\,(\omega_\ell+\mu_1\,\omega_x))\,(\varrho-1)+(1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2\Big) \\
&= \beta\,(\mu_2\,(1-\beta)\,(\omega_\ell+\mu_1\,\omega_x)-\mu_1\,(\omega_\ell-\beta))\,(\varrho-1)\,.
\end{aligned}
$$

Horizontal Economy - $\boldsymbol{\omega_\ell=1}$: $\frac{\partial\,\Gamma}{\partial\,log\,A_2} = -\beta\,(1-\beta)\,(\mu_1-\mu_2)\,(\varrho-1)$.

Vertical Economy - $\boldsymbol{\omega_\ell=0}$ and $\boldsymbol{\beta=0}$: $\frac{\partial\,\Gamma}{\partial\,log\,A_2} = 0$.

3.3 Markdown μ_1

$$\frac{\partial \Gamma}{\partial \, log \, \mu_1} = (1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1$$
$$-\Big(\omega_\ell\,\beta\,(1-\beta)\,(\mu_2\,(\omega_\ell+\mu_1\,\omega_x)-\mu_1)\,(\varrho-1)+\omega_\ell\,\omega_x\,\mu_2\,(1-\beta)\,(1-\mu_1)\,(\theta_2-1)+(1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_1-\mu_1\,\lambda_1\Big)$$
$$=\mu_1\,\beta+\mu_1\,\mu_2\,\omega_x\,(1-\beta)+\omega_\ell\,\beta\,(1-\beta)\,(\mu_1-\mu_2\,(\omega_\ell+\mu_1\,\omega_x))\,(\varrho-1)-\omega_\ell\,\omega_x\,\mu_2\,(1-\beta)\,(1-\mu_1)\,(\theta_2-1)\,.$$

Horizontal Economy - $\boldsymbol{\omega_\ell = 1}$: $\frac{\partial \Gamma}{\partial \, log \, \mu_1} = \beta\,(\mu_1+(1-\beta)\,(\mu_1-\mu_2)\,(\varrho-1))$.

Vertical Economy - $\boldsymbol{\omega_\ell = 0}$ and $\boldsymbol{\beta = 0}$: $\frac{\partial \Gamma}{\partial \, log \, \mu_1} = \mu_1\,\mu_2$.

3.4 Markdown μ_2

$$\frac{\partial \Gamma}{\partial \, log \, \mu_2} = (1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2$$
$$-\Big((1+\zeta_w)\,\Gamma\,\widetilde{\lambda}_2-\mu_2\,(1-\beta)\,(\omega_\ell+\mu_1\,\omega_x)-\beta\,(\mu_2\,\omega_\ell\,(1-\beta)+\mu_1\,(\beta-\omega_\ell+\mu_2\,\omega_x\,(1-\beta)))\,(\varrho-1)\Big)$$
$$=\mu_2\,(1-\beta)\,(\omega_\ell+\mu_1\,\omega_x)+\beta\,(\mu_2\,\omega_\ell\,(1-\beta)+\mu_1\,(\beta-\omega_\ell+\mu_2\,\omega_x\,(1-\beta)))\,(\varrho-1)\,.$$

Horizontal Economy - $\boldsymbol{\omega_\ell = 1}$: $\frac{\partial \Gamma}{\partial \, log \, \mu_2} = (1-\beta)\,(\mu_2-\beta\,(\mu_1-\mu_2)\,(\varrho-1))$.

Vertical Economy - $\boldsymbol{\omega_\ell = 0}$ and $\boldsymbol{\beta = 0}$: $\frac{\partial \Gamma}{\partial \, log \, \mu_1} = \mu_1\,\mu_2$.

3.5 Consumption Patterns β

$$\frac{\partial \Gamma}{\partial \, \beta} = \varrho\,(\mu_1\,(1-\mu_2\,\omega_x)-\mu_2\,\omega_\ell)$$

Horizontal Economy - $\boldsymbol{\omega_\ell = 1}$: $\frac{\partial \Gamma}{\partial \, \beta} = \varrho\,(\mu_1-\mu_2)$.

Vertical Economy - $\boldsymbol{\omega_\ell = 0}$ and $\boldsymbol{\beta = 0}$: $\frac{\partial \Gamma}{\partial \, \beta} = \varrho\,\mu_1\,(1-\mu_2)$.

3.6 Labor Intensity ω_ℓ

$$\frac{\partial \Gamma}{\partial \, \omega_\ell} = \mu_2\,(1-\beta)\,(1-\mu_1\,\theta_2)$$

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Tables

TABLE I

Direct Centralities

Matrix	***Definition***	***In Equilibrium***	***Properties***
ω_ℓ	$\omega_i^\ell \equiv \frac{\partial \, log\, c_i(\vartheta,\rho)}{\partial \, log\, p_i^\ell}$	Cost share of L_i	$\omega_i^\ell + \omega_i^x = 1$
ω_x	$\omega_i^x \equiv \frac{\partial \, log\, c_i(\vartheta,\rho)}{\partial \, log\, p_i^x}$	Cost share of X_i	
$\widetilde{\Omega}_\ell$	$\widetilde{\Omega}_{ih}^\ell \equiv \frac{\partial \, log\, c_i(\vartheta,\rho)}{\partial \, log\, w_h}$	Cost share of ℓ_{ih}	$\sum_{h\in\mathscr{H}} \widetilde{\Omega}_{ih}^\ell + \sum_{j\in\mathscr{N}} \widetilde{\Omega}_{ij}^x = 1$
$\widetilde{\Omega}_x$	$\widetilde{\Omega}_{ij}^x \equiv \frac{\partial \, log\, c_i(\vartheta,\rho)}{\partial \, log\, p_j}$	Cost share of x_{ij}	
$diag\,(\omega_\ell)\,\alpha = \widetilde{\Omega}_\ell$	$\alpha_{ih} \equiv \frac{\partial \, log\, p_i^\ell\, L_i}{\partial \, log\, w_h}$	Cost share of ℓ_{ih} in L_i	$\sum_{h\in\mathscr{H}} \alpha_{ih} = 1$
$diag\,(\omega_x)\,\mathscr{W} = \widetilde{\Omega}_x$	$\omega_{ij} \equiv \frac{\partial \, log\, p_i^x\, X_i}{\partial \, log\, p_j}$	Cost share of x_{ij} in X_i	$\sum_{j\in\mathscr{N}} \omega_{ij} = 1$
β	$\beta_i \equiv \frac{\partial \, log\, E}{\partial \, log\, p_i}$	Cost share of C_i	$\sum_{i\in\mathscr{N}} \beta_i = 1$
$\Omega_\ell \equiv diag\,(\mu)\,\widetilde{\Omega}_\ell$	$\Omega_{ih}^\ell \equiv \frac{\partial \, log\, S_i}{\partial \, log\, w_h}$	Share of S_i for ℓ_{ih}	$\sum_{h\in\mathscr{H}} \Omega_{ih}^\ell + \sum_{j\in\mathscr{N}} \Omega_{ij}^x + \Omega_i^\pi = 1$
$\Omega_x \equiv diag\,(\mu)\,\widetilde{\Omega}_x$	$\Omega_{ij}^x \equiv \frac{\partial \, log\, S_i}{\partial \, log\, p_j}$	Share of S_i for x_{ij}	
$\Omega_\pi = \mathbb{1}_N - \mu$	$\Omega_i^\pi = \frac{\pi_i}{S_i}$	Share of S_i for Π	

TABLE II

Network Adjusted Centralities

Matrix	***Definition in Equilibrium***	***Properties***
Downstream or Cost-Based Centralities		
$\widetilde{\Psi}_x = \left(I - \widetilde{\Omega}_x\right)^{-1}$	$\widetilde{\psi}^x_{ij}$ *firm-to-firm* Centrality of j in the costs of i	
$\widetilde{\Psi}_\ell = \widetilde{\Psi}_x \, \widetilde{\Omega}_\ell$	$\widetilde{\psi}^\ell_{ih}$ *worker-to-firm* Value-added share by h in the production of i	$\sum_{h \in \mathscr{H}} \widetilde{\psi}^\ell_{ih} = 1$
$\widetilde{\lambda} = \widetilde{\Psi}'_x \, \beta$	$\widetilde{\lambda}_i$ *cost-based Domar weight* Share of aggregate value-added that passes through i	$\sum_{i \in \mathscr{N}} \omega^\ell_i \, \widetilde{\lambda}_i = 1$
$\widetilde{\Lambda} = \widetilde{\Psi}'_\ell \, \beta$	$\widetilde{\Lambda}_h$ *cost-based labor share* Share of aggregate value-added generated by h	$\sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h = 1$
Upstream or Revenue-Based Centralities		
$\Psi_x = (I - \Omega_x)^{-1}$	ψ^x_{ij} *firm-to-firm* Share of S_i that reaches S_j	
$\Psi_\ell = \Psi_x \, \Omega_\ell$	ψ^ℓ_{ih} *firm-to-worker* Share of S_i that reaches J_h	$\psi^\ell_i = \sum_{h \in \mathscr{H}} \psi^\ell_{ih}$
$\lambda = \Psi'_x \, \beta$	λ_i *revenue-based Domar weight* Aggregate sales share S_i/GDP	$\sum_{i \in \mathscr{N}} \lambda_i \geq 1$
$\Lambda = \Psi'_\ell \, \beta$	Λ_h *revenue-based labor share* Labor income share J_h/GDP	$\Gamma = \sum_{h \in \mathscr{H}} \Lambda_h \leq 1$
Other Definitions		
$\delta = diag\,(\Lambda)^{-1} \, \widetilde{\Lambda}$	δ_h *distortion centrality* Measure for how undervalue is L_h	$\delta_h = \widetilde{\Lambda}_h / \Lambda_h$

TABLE III

Sectoral Markdowns

Rank	***Sector***	$\boldsymbol{\mu_{i,2021}}$	$\boldsymbol{\mu_{i,2021} - \mu_{i,1997}}$	***Mean***	***St Dev***	***Autocorr***	***St Error***
1	Housing	12.97%	1.88	11.34%	0.0077	-0.111	0.1423
2	Pipeline transportation	41.29%	-35.71	54.46%	0.1573	-0.193	0.1146
3	Rental & leasing intangibles	47.03%	4.43	47.90%	0.0403	-0.406^{**}	0.1615
4	Credit intermediation	49.31%	-18.67	62.43%	0.0562	-0.030	0.1191
5	Oil & gas extraction	54.02%	-4.47	52.63%	0.0891	-0.211	0.1279
6	Utilities	55.88%	2.46	60.62%	0.0520	-0.3431^{**}	0.1536
7	Computers & electronics	61.81%	-19.45	72.95%	0.0992	-0.030	0.0623
8	Chemical products	62.14%	-11.46	70.66%	0.0389	-0.139	0.1494
9	Mining, except oil & gas	62.41%	-17.16	67.21%	0.0635	-0.209^{*}	0.1028
10	Telecommunications	62.60%	-2.03	65.08%	0.0375	-0.154	0.1174
11	Publishing industries	65.12%	-10.34	68.62%	0.0604	-0.289^{*}	0.1435
12	Arts, sports & museums	65.35%	-7.49	68.51%	0.0382	-0.258^{*}	0.1386
13	Internet & information services	66.17%	-11.92	68.70%	0.1618	-0.270^{*}	0.1437
14	Farms	68.70%	3.27	67.52%	0.0309	-0.459^{**}	0.1776
15	Legal services	68.95%	-3.80	68.70%	0.0314	-0.332^{**}	0.1480
16	Primary metals	69.08%	-9.56	75.34%	0.0382	-0.090	0.1100
17	Rail transportation	70.23%	-15.14	76.94%	0.0419	-0.143	0.0936
18	Other real estate	71.72%	3.84	69.84%	0.0392	-0.244^{*}	0.1394
19	Wholesale trade	73.31%	-1.60	74.30%	0.0153	-0.259^{*}	0.1450
20	Motion, pictures & sound	73.40%	-2.36	64.61%	0.0609	-0.410^{**}	0.1639
21	Accommodation	74.17%	-1.65	75.97%	0.0156	-0.964^{***}	0.2194
22	Nonmetallic minerals	74.45%	-4.42	79.84%	0.0338	-0.0408	0.0952

Notes: Columns $\mu_{i,2021}$ displays industry-level markdowns for 2021. Column $\mu_{i,2021} - \mu_{i,1997}$ shows the level difference in percentage points between markdowns in 2021 and 1997. Column *Mean* and *St Dev* display the temporal mean and standard deviation for industry-level markdowns between 1997 and 2021. Columns *Autocorr* and *St Error* portray the slope parameter and its standard error for the following industry-level regression $\Delta\, \mu_{i,t} = \tau_{0,i} + \tau_{1,i}\, \mu_{i,t-1} + \epsilon_{i,t}$.

TABLE IV

Sectoral Markdowns

Rank	***Sector***	$\boldsymbol{\mu_{i,2021}}$	$\boldsymbol{\mu_{i,2021} - \mu_{i,1997}}$	***Mean***	***St Dev***	***Autocorr***	***St Error***
23	Transit & ground transportation	74.84%	2.52	69.20%	0.0357	-0.296^{*}	0.1635
24	Other transportation equipment	76.39%	-13.17	82.02%	0.0468	-0.101	0.0793
25	Wood products	77.53%	-13.87	89.75%	0.0403	0.126	0.1314
26	Miscellaneous manufacturing	78.21%	-2.90	79.69%	0.0272	-0.315^{*}	0.1554
27	Waste & remediation services	79.00%	0.89	79.29%	0.0231	-0.437^{**}	0.1767
28	Motor vehicles & parts dealers	79.12%	0.76	81.04%	0.0160	-0.304^{**}	0.1434
29	Insurance carriers	79.27%	6.99	75.15%	0.0348	-0.677^{***}	0.2055
30	Other retail	79.37%	-0.70	78.81%	0.0121	-0.629^{***}	0.1927
31	Water transportation	79.70%	8.92	73.94%	0.0358	-0.317^{*}	0.1680
32	Forestry & fishing	80.28%	8.45	76.18%	0.0297	-0.299^{*}	0.1503
33	Electrical equipment	80.83%	-0.70	82.10%	0.0335	-0.492^{**}	0.1843
34	Petroleum & coal	81.42%	2.87	78.55%	0.0451	-0.718^{***}	0.2065
35	Paper products	81.47%	0.90	82.30%	0.0191	-0.806^{***}	0.2061
36	Misc. professional services	81.82%	4.96	78.53%	0.0165	-0.243	0.1623
37	Support activities for mining	81.85%	3.64	77.34%	0.0524	-0.307^{*}	0.1584
38	Machinery	82.22%	-6.22	85.02%	0.0216	-0.156	0.1064
39	Truck transportation	82.25%	1.43	80.75%	0.0130	-0.413^{**}	0.1806
40	Securities & investment	82.27%	1.11	89.53%	0.0814	-0.702^{***}	0.2022
41	Food, beverage & tobacco	83.68%	-4.46	85.73%	0.0156	-0.636^{***}	0.1949
42	Construction	84.04%	0.37	82.13%	0.0125	-0.300^{*}	0.1579
43	Printing services	84.18%	-10.11	86.40%	0.0385	-0.114^{**}	0.0455
44	Food & beverage stores	84.20%	3.45	81.46%	0.0250	-0.100	0.1047

Notes: Columns $\mu_{i,2021}$ displays industry-level markdowns for 2021. Column $\mu_{i,2021} - \mu_{i,1997}$ shows the level difference in percentage points between markdowns in 2021 and 1997. Column *Mean* and *St Dev* display the temporal mean and standard deviation for industry-level markdowns between 1997 and 2021. Columns *Autocorr* and *St Error* portray the slope parameter and its standard error for the following industry-level regression $\Delta\, \mu_{i,t} = \tau_{0,i} + \tau_{1,i}\, \mu_{i,t-1} + \epsilon_{i,t}$.

TABLE V

Sectoral Markdowns

Rank	***Sector***	$\boldsymbol{\mu_{i,2021}}$	$\boldsymbol{\mu_{i,2021} - \mu_{i,1997}}$	***Mean***	***St Dev***	***Autocorr***	***St Error***
45	Other services	85.00%	9.87	81.54%	0.0327	-0.172^{*}	0.0932
46	Ambulatory healthcare	85.77%	1.82	84.64%	0.0088	-0.297^{*}	0.1587
47	Funds, trusts & fin. vehicles	85.78%	-2.81	83.51%	0.0494	-0.251^{*}	0.1353
48	Fabricated metal products	86.00%	2.26	85.63%	0.0087	-0.791^{***}	0.1859
49	Recreational & gambling	86.05%	15.25	81.33%	0.0381	-0.456^{***}	0.1378
50	Administrative services	86.11%	1.43	84.36%	0.0207	-0.098	0.0995
51	Other transportation activities	86.13%	6.19	82.65%	0.0239	-0.325^{*}	0.1630
52	Food & drinking services	86.70%	0.63	86.29%	0.0102	-0.273^{*}	0.1471
53	Air transportation	87.25%	13.97	77.75%	0.0874	-0.311^{*}	0.1607
54	Plastics & rubber products	87.54%	2.07	86.62%	0.0176	-0.717^{***}	0.2038
55	Educational services	88.72%	5.31	84.79%	0.0202	-0.334^{*}	0.1819
56	Motor vehicles bodies	89.11%	0.52	89.03%	0.0258	-0.477^{**}	0.1815
57	Textile mills and textiles	89.40%	-2.42	90.95%	0.0190	-0.487^{**}	0.1856
58	Furniture	89.81%	2.89	89.00%	0.0164	-0.362^{**}	0.1555
59	Hospitals	90.92%	0.01	90.75%	0.0072	-0.277^{*}	0.1474
60	Computer systems design	91.92%	0.99	89.51%	0.0424	-0.117	0.1023
61	General merchandise stores	91.96%	4.76	89.61%	0.0227	-0.080	0.0828
62	Management of companies	92.23%	2.58	91.17%	0.0073	-0.244^{*}	0.1215
63	Apparel & leather	95.21%	3.38	92.48%	0.0204	-0.427^{**}	0.1847
64	Social assistance	95.77%	8.92	90.20%	0.0315	0.023	0.0488
65	Nursing & residential care	96.03%	0.63	95.17%	0.0075	-0.529^{**}	0.1947
66	Warehousing & storage	96.72%	11.05	86.44%	0.0579	-0.039	0.1035

Notes: Columns $\mu_{i,2021}$ displays industry-level markdowns for 2021. Column $\mu_{i,2021} - \mu_{i,1997}$ shows the level difference in percentage points between markdowns in 2021 and 1997. Column *Mean* and *St Dev* display the temporal mean and standard deviation for industry-level markdowns between 1997 and 2021. Columns *Autocorr* and *St Error* portray the slope parameter and its standard error for the following industry-level regression $\Delta\, \mu_{i,t} = \tau_{0,i} + \tau_{1,i}\, \mu_{i,t-1} + \epsilon_{i,t}$.

TABLE VI

Sectoral Payment Centralities

Rank	*Sector*	$\psi^{\ell}_{i,2021}$	$\psi^{\ell}_{i,2021} - \psi^{\ell}_{i,1997}$	*Mean*	*St Dev*	*Autocorr*	*St Error*	*Par Equ*
1	Housing	6.96%	0.83	6.29%	0.0036	-0.215	0..1573	0.1060
2	Oil & gas extraction	26.88%	-6.00	29.04%	0.0659	-0.252^{*}	0.1398	0.0559
3	Petroleum & coal	27.40%	-3.82	27.00%	0.0627	-0.293^{*}	0.1478	0.0385
4	Pipeline transportation	28.95%	-17.47	36.99%	0.0881	-0.152	0.1103	0.0015
5	Rental & leasing intangibles	29.67%	0.27	30.97%	0.0180	-0.533^{***}	0.1872	0.0192
6	Farms	30.96%	-1.05	32.00%	0.0225	-0.428^{**}	0.1761	0.0280
7	Utilities	32.18%	-0.32	33.95%	0.0268	-0.213	0.1328	0.0417
8	Chemical products	33.39%	-10.31	38.19%	0.040	-0.061	0.0667	0.0762
9	Primary metals	35.52%	-14.45	42.76%	0.0667	-0.084	0.0858	0.0502
10	Telecommunications	38.03%	-4.69	40.71%	0.0356	-0.087	0.0900	0.0353
11	Mining, except oil & gas	38.68%	-15.24	42.53%	0.0684	-0.105	0.0649	0.0138
12	Other real estate	40.28%	1.18	39.99%	0.0332	-0.173	0.1195	0.0768
13	Credit intermediation	41.60%	-8.11	47.01%	0.0335	-0.116	0.1197	0.0796
14	Food, beverage & tobacco	41.76%	-3.90	43.17%	0.0240	-0.142	0.1027	0.0243
15	Rail transportation	44.82%	-14.41	53.03%	0.0600	-0.065	0.0685	0.0008
16	Internet & information services	45.66%	-17.75	5.51%	0.1510	-0.164	0.1131	0.0224
17	Nonmetallic minerals	47.60%	-6.38	51.63%	0.0353	-0.033	0.0689	0.0127
18	Paper products	49.55%	-2.78	50.27%	0.0204	-0.194	0.1212	0.0108
19	Arts, sports & museums	50.09%	-5.77	52.10%	0.0351	-0.347^{**}	0.1564	0.0069
20	Motor vehicles bodies	50.11%	-5.96	53.32%	0.0389	-0.266^{*}	0.1461	0.0147
21	Motion, pictures & sound	50.94%	0.20	43.35%	0.0423	-0.338^{**}	0.1610	0.0064
22	Wood products	51.13%	-8.50	59.22%	0.0325	-0.138	0.1614	0.0124

Notes: Columns $\psi^{\ell}_{i,2021}$ displays industry-level payment centralities for 2021. Column $\psi^{\ell}_{i,2021} - \psi^{\ell}_{i,1997}$ shows the level difference in percentage points between payment centralities in 2021 and 1997. Column *Mean* and *St Dev* display the temporal mean and standard deviation for industry-level payment centralities between 1997 and 2021. Columns *Autocorr* and *St Error* portray the slope parameter and its standard error for the following industry-level regression $\Delta\,\psi^{\ell}_{i,t} = \phi_{0,i} + \phi_{1,i}\,\phi^{\ell}_{i,t-1} + \epsilon_{i,t}$. Column *Par Equ* shows the industry level estimate for $\widetilde{\lambda}_{i,t} - \Gamma_t^{-1}\,\lambda_{i,t}\,\psi^{\ell}_{i,t}$ which is the partial equilibrium effect from distortions in Corollary 3.

TABLE VII

Sectoral Payment Centralities

Rank	***Sector***	$\boldsymbol{\psi^{\ell}_{i,2021}}$	$\boldsymbol{\psi^{\ell}_{i,2021} - \psi^{\ell}_{i,1997}}$	***Mean***	***St Dev***	***Autocorr***	***St Error***	***Par Equ***
23	Plastics & rubber products	51.63%	-2.39	51.13%	0.0290	-0.103	0.0839	0.0137
24	Funds, trusts & fin. vehicles	51.73%	-1.44	53.00%	0.0351	-0.619^{***}	0.1978	0.0012
25	Insurance carriers	52.63%	-2.07	53.60%	0.0292	-0.623^{***}	0.1975	0.0514
26	Water transportation	52.82%	8.12	46.20%	0.0399	-0.101	0.1208	$5.089e^{-5}$
27	Electrical equipment	53.07%	-2.02	54.91%	0.0355	-0.171	0.1215	0.0102
28	Other transportation equipment	53.39%	-12.13	57.60%	0.0539	-0.093	0.0711	0.0031
29	Wholesale trade	53.77%	-6.96	57.72%	0.0361	-0.023	0.0565	0.0038
30	Machinery	54.22%	-7.37	57.49%	0.0391	-0.082	0.0798	0.0162
31	Textile mills and textiles	54.55%	-3.86	55.84%	0.0292	-0.156	0.1094	0.0023
32	Fabricated metal products	54.61%	-3.81	56.84%	0.0295	-0.118	0.1035	0.0274
33	Computers & electronics	54.83%	-1.13	56.43%	0.0462	-0.151	0.1136	0.0167
34	Legal services	54.99%	-2.60	55.46%	0.023	-0.198	0.1208	0.0123
35	Transit & ground transportation	55.33%	2.83	51.38%	0.0301	-0.061	0.0936	0.0014
36	Truck transportation	55.88%	-2.01	54.42%	0.0323	-0.172	0.1116	-0.0003
37	Other retail	56.12%	-7.25	59.98%	0.0274	-0.044	0.0702	-0.0034
38	Miscellaneous manufacturing	56.15%	-1.27	56.13%	0.0270	-0.143	0.1079	0.0016
39	Accommodation	56.18%	-1.84	58.18%	0.0215	-0.904^{***}	0.2163	0.0021
40	Publishing industries	56.49%	0.99	52.67%	0.034	-0.280^{*}	0.1505	0.0026
41	Waste & remediation services	57.16%	0.56	56.16%	0.0125	-0.495^{**}	0.1869	0.0038
42	Construction	58.05%	-2.07	58.38%	0.0193	-0.152	0.1060	0.0250
43	Printing services	58.18%	-7.62	60.06%	0.0381	-0.078	0.0535	0.0023
44	Furniture	59.99%	-2.21	61.38%	0.0200	-0.143	0.1126	0.0039

Notes: Columns $\psi^{\ell}_{i,2021}$ displays industry-level payment centralities for 2021. Column $\psi^{\ell}_{i,2021} - \psi^{\ell}_{i,1997}$ shows the level difference in percentage points between payment centralities in 2021 and 1997. Column *Mean* and *St Dev* display the temporal mean and standard deviation for industry-level payment centralities between 1997 and 2021. Columns *Autocorr* and *St Error* portray the slope parameter and its standard error for the following industry-level regression $\Delta\,\psi^{\ell}_{i,t} = \phi_{0,i} + \phi_{1,i}\,\phi^{\ell}_{i,t-1} + \epsilon_{i,t}$. Column *Par Equ* shows the industry level estimate for $\widetilde{\lambda}_{i,t} - \Gamma_t^{-1}\,\lambda_{i,t}\,\psi^{\ell}_{i,t}$ which is the partial equilibrium effect from distortions in Corollary 3.

TABLE VIII

Sectoral Payment Centralities

Rank	*Sector*	$\psi^{\ell}_{i,2021}$	$\psi^{\ell}_{i,2021} - \psi^{\ell}_{i,1997}$	*Mean*	*St Dev*	*Autocorr*	*St Error*	*Par Equ*
45	Support activities for mining	60.36%	-1.22	57.06%	0.058	-0.240^{*}	0.1365	0.0011
46	Air transportation	60.77%	8.99	52.26%	0.0944	-0.324^{*}	0.1631	0.0012
47	Food & drinking services	61.75%	0.98	62.49%	0.0102	-0.401^{**}	0.1565	0.0030
48	Securities & investment	63.23%	-1.98	69.08%	0.0605	-0.658^{***}	0.2030	0.0080
49	Misc. professional services	63.52%	3.02	61.88%	0.0139	-0.355^{**}	0.1654	0.0364
50	Other transportation activities	63.64%	-1.35	61.69%	0.0304	-0.207	0.1238	0.0067
51	Food & beverage stores	64.14%	-1.26	64.08%	0.0139	-0.233^{*}	0.1304	-0.0026
52	Motor vehicles & parts dealers	64.93%	2.66	66.11%	0.0209	-0.540^{***}	0.1723	-0.0024
53	Recreational & gambling	65.15%	12.49	60.01%	0.0351	-0.455^{**}	0.1645	-0.0010
54	Forestry & fishing	66.86%	19.87	56.39%	0.0672	-0.023	0.0564	0.0059
55	Administrative services	67.12%	-1.52	68.40%	0.0189	-0.186	0.1276	0.0414
56	Other services	67.36%	7.77	63.82%	0.0210	-0.182	0.1107	-0.0015
57	Hospitals	67.90%	5.21	68.18%	0.0322	-0.324^{**}	0.1377	-0.0002
58	Apparel & leather	70.20%	-2.83	71.03%	0.0173	-0.122	0.0908	-0.0178
59	Ambulatory healthcare	71.42%	3.37	68.62%	0.0127	-0.149	0.1555	-0.0225
60	General merchandise stores	72.20%	1.09	72.02%	0.0083	-0.628^{***}	0.1922	-0.0045
61	Educational services	72.84%	5.01	68.58%	0.0237	-0.150	0.1405	-0.0052
62	Warehousing & storage	73.56%	-2.90	65.78%	0.0534	-0.177^{*}	0.1030	0.0020
63	Management of companies	74.05%	0.55	74.68%	0.0122	-0.185	0.1243	0.0175
64	Social assistance	76.78%	6.56	71.48%	0.0276	-0.011	0.0959	-0.0062
65	Nursing & residential care	78.16%	1.85	75.97%	0.0162	-0.286^{*}	0.1617	-0.0066
66	Computer systems design	82.25%	7.82	75.47%	0.0338	-0.137	0.1449	-0.0055

Notes: Columns $\psi^{\ell}_{i,2021}$ displays industry-level payment centralities for 2021. Column $\psi^{\ell}_{i,2021} - \psi^{\ell}_{i,1997}$ shows the level difference in percentage points between payment centralities in 2021 and 1997. Column *Mean* and *St Dev* display the temporal mean and standard deviation for industry-level payment centralities between 1997 and 2021. Columns *Autocorr* and *St Error* portray the slope parameter and its standard error for the following industry-level regression $\Delta\,\psi^{\ell}_{i,t} = \phi_{0,i} + \phi_{1,i}\,\phi^{\ell}_{i,t-1} + \epsilon_{i,t}$. Column *Par Equ* shows the industry level estimate for $\widetilde{\lambda}_{i,t} - \Gamma_t^{-1}\,\lambda_{i,t}\,\psi^{\ell}_{i,t}$ which is the partial equilibrium effect from distortions in Corollary 3.

TABLE IX

Payment Centrality Regressions

	(1)	**(2)**	**(3)**	**(4)**
μ_i	0.812***			0.666***
	(0.0700)			(0.0306)
ω_i^x		-0.596***		-0.439***
		(0.0686)		(0.0244)
λ_i			-0.962	0.176
			(0.6408)	(0.1557)
Intercept	-0.087**	0.908***	0.566***	0.292***
	(0.0549)	(0.0441)	(0.0248)	(0.0327)
N	66			
R^2	67.78%	54.06%	3.40%	94.84%

Notes: Parameters and standard errors for: (1) $\psi^{\ell}_{i,2021} = \tau_0 + \tau_1\, \mu_{i,2021} + e_i$, (2) $\psi^{\ell}_{i,2021} = \tau_0 + \tau_1\, \omega^x_{i,2021} + e_i$, and (3) $\psi^{\ell}_{i,2021} = \tau_0 + \tau_1\, \lambda_{i,2021} + e_i$. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

TABLE X

Markdown minus Payment Centrality Regressions

	(1)	(2)	(3)	(4)
μ_i	0.187***			0.333***
	(0.0700)			(0.0306)
ω_i^x		0.358***		0.439***
		(0.0408)		(0.0244)
λ_i			-0.405	-0.176
			(0.3869)	(0.1557)
Intercept	0.0875	0.010	0.243***	-0.292***
	(0.0549)	(0.0262)	(0.0149)	(0.0327)
N	66			
R^2	10.05%	54.53%	1.68%	85.30%

Notes: Parameters and standard errors for: (1) $\psi^{\ell}_{i,2021} - \mu_{i,2021} = \tau_0 + \tau_1\, \mu_{i,2021} + e_i$, (2) $\psi^{\ell}_{i,2021} - \mu_{i,2021} = \tau_0 + \tau_1\, \omega^x_{i,2021} + e_i$, and (3) $\psi^{\ell}_{i,2021} - \mu_{i,2021} = \tau_0 + \tau_1\, \lambda_{i,2021} + e_i$. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

TABLE XI

Differential in Γ leaving aside one channel at the time

	Competitive Income	***Final Demand***	***Labor Demand***	***Intermediate Demand***
A. Between 1998 and 2021: Γ in 2021 was 52.55%				
Level	54.06%	52.06%	52.14%	52.20%
Difference	-1.51%	0.49%	0.41%	0.34%
B. Between 2001 and 2010: Γ in 2010 was 49.36%				
Level	54.14%	49.89%	49.49%	49.65%
Difference	-4.78%	-0.53%	-0.13%	-0.29%
C. Between 2011 and 2020: Γ in 2020 was 53.31%				
Level	51.53%	53.02%	51.31%	53.43%
Difference	1.78%	0.29%	2.00%	-0.12%

Notes: In table A. I take Γ_{1997} and estimate counterfactual $d\Gamma_t$ between 1998 and 2021 by leaving aside one at the time one of the channels in equation (13). In table B, I start with Γ_{2000} and do the same with $d\Gamma_t$ between 2001 and 2010. In table C, I start with Γ_{2010} and do the same with $d\Gamma_t$ between 2011 and 2020. *Level* reports the counterfactual aggregate labor share level in the final year, and *Difference* the distance relative to the actual labor share in the final year.

TABLE XII

Γ Covariance Decomposition

Competitive Income	***Final Recomp.***	***Labor Recomp.***	***Intermediate Recomp.***
A. Between 1998 and 2021			
82.16%	0.99%	30.10%	-13.25%
B. Between 2001 and 2010			
153.75%	9.30%	-61.52%	-1.53%
C. Between 2011 and 2020			
36.09%	-10.82%	115.30%	-40.57%

Notes: From equation (13), the covariance decomposition is given by $Var(d\Gamma_t) = Cov(Competitive\ Income, d\Gamma_t) + Cov(Final\ Demand\ Recomposition\ , d\Gamma_t) + Cov(Labor\ Demand\ Recomposition\ , d\Gamma_t) + Cov(Intermediate\ Demand\ Recomposition\ , d\Gamma_t)$.

TABLE XIII

Differential in Γ leaving aside *Competitive Income* for one industry at the time

A. Between 1997 and 2021

1	Other services	0.34%
2	Misc. professional services	0.28%
	⋮	
62	Publishing industries	-0.26%
63	Securities & investment	-0.27%
64	Chemical products	-0.33%
65	Computers & electronics	-0.40%
66	Credit intermediation	-0.80%

B. Between 2001 and 2010

1	Other services	0.28%
	⋮	
56	Misc. professional services	-0.22%
57	Utilities	-0.23%
58	Administrative services	-0.25%
59	Other real estate	-0.27%
60	Computers systems design	-0.29%
61	Wholesale trade	-0.30%
62	Securities & investment	-0.35%
63	Telecommunications	-0.36%
64	Publishing industries	-0.46%
65	Computers & electronics	-0.50%
66	Internet & inf. services	-0.53%

C. Between 2011 and 2020

1	Misc. professional services	0.39%
2	Oil & gas extraction	0.27%
3	Administrative services	0.25%
4	Air transportation	0.21%
	⋮	
65	Securities & investment	-0.31%
66	Credit intermediation	-0.58%

TABLE XIV

Differential in Γ leaving aside *Final Demand Recomposition* for one industry at the time

A. Between 1997 and 2021

1	Wholesale trade	1.35%
2	Hospitals	1.07%
3	Internet & inf. services	0.62%
4	Misc. professional services	0.61%
5	Securities & investment	0.60%
6	Ambulatory healthcare	0.53%
7	Other retail	0.52%
8	Computer systems design	0.51%
	⋮	
62	Apparel & leather	-0.47%
63	Food & beverage stores	-0.53%
64	Machinery	-0.71%
65	Motor vehicles bodies	-0.77%
66	Computers & electronics	-1.34%

B. Between 2001 and 2010

1	Hospitals	1.03%
2	Ambulatory healthcare	0.79%
3	Wholesale trade	0.68%
	⋮	
64	Motor vehicles bodies	-0.69%
65	Computers & electronics	-0.91%
66	Construction	-1.52%

C. Between 2011 and 2020

1	Construction	1.19%
2	Other retail	0.56%
3	Oil & gas extraction	0.52%
	⋮	
66	Petroleum & coal	-0.41%

Notes: In Table XIII only sectors with more than 0.2% in absolute value are included. In Table XIV only sectors with more than 0.4% in absolute value are included. For each estimation, using Theorem 2 a counterfactual sequence for Γ is constructued. This sequence excludes the effects from one industry in one specific channel at the time.

TABLE XV

Differential in Γ leaving aside *Labor Demand Recomposition* for one industry at the time

A. Between 1998 and 2021

1	Computers & electronics	1.08%
2	Credit intermediation	0.81%
3	Publishing industries	0.51%
4	Computer systems design	0.38%
5	Ambulatory healthcare	0.33%
	⋮	
61	Hospitals	-0.33%
62	Telecommunications	-0.34%
63	Administrative services	-0.39%
64	Insurance carriers	-0.61%
65	Other retail	-0.74%
66	Wholesale trade	-1.10%

B. Between 2001 and 2010

1	Computers & electronics	0.60%
2	Securities & investment	0.35%
3	Utilities	0.26%
	⋮	
64	Insurance carriers	-0.23%
65	Motor vehicles bodies	-0.24%
66	Wholesale trade	-0.52%

C. Between 2011 and 2020

1	Credit intermediation	0.68%
2	Computers & electronics	0.41%
3	Publishing industries	0.35%
	⋮	
63	Insurance carriers	-0.38%
64	Administrative services	-0.39%
65	Other retail	-0.52%
66	Wholesale trade	-0.89%

TABLE XVI

Differential in Γ leaving aside *Intermediate Demand Recomposition* for one industry at the time

A. Between 1998 and 2021

1	Wholesale trade	0.65%
2	Other retail	0.41%
3	Insurance carriers	0.31%
4	Administrative services	0.27%
5	Hospitals	0.23%
6	Telecommunications	0.20%
	⋮	
63	Computer system design	-0.20%
64	Publishing industries	-0.27%
65	Credit intermediation	-0.42%
66	Computers & electronics	-0.58%

B. Between 2001 and 2010

1	Wholesale trade	0.31%
	⋮	
65	Securities & investment	-0.21%
66	Computers & electronics	-0.34%

C. Between 2011 and 2020

1	Administrative services	0.24%
2	Other retail	0.22%
3	Insurance carriers	0.21%
	⋮	
66	Credit intermediation	-0.32%

Notes: In Tables XV and XVI only sectors with more than 0.2% in absolute value are included. For each estimation, using Theorem 2 a counterfactual sequence for Γ is constructed. This sequence excludes the effects from one industry in one specific channel at the time.

TABLE XVII

Elasticities of Substitution

Sector	Second Stage	Third Stage	Fourth Stage	
	θ_i	θ_i^x	θ_i	θ_i^x
Farms	4.0	0.0	4.0	0.0
Forestry & fishing	8.0	20.0	8.0	20.0
Oil & gas extraction	4.0	0.0	4.0	0.0
Mining, except oil & gas	4.0	0.0	4.0	0.0
Support activities for mining	1.5	0.0	1.5	0.0
Utilities	4.0	0.0	4.0	0.0
Construction	0.0	0.0	0.0	0.0
Wood products	0.0	20.0	0.0	20.0
Nonmetallic minerals	0.0	20.0	0.0	20.0
Primary metals	0.0	20.0	0.0	20.0
Fabricated metal products	0.0	20.0	0.0	0.0
Machinery	0.6	0.0	0.6	0.0
Computers & electronics	0.0	0.0	0.0	0.0
Electrical equipment	0.0	20.0	0.0	20.0
Motor vehicles bodies	0.6	0.0	0.6	0.0
Other transportation equipment	0.0	0.0	0.0	0.0
Furniture	8.0	0.0	8.0	0.0
Miscellaneous manufacturing	4.0	0.0	4.0	0.0
Food, beverage & tobacco	0.8	0.0	1.0	0.0
Textile mills and textiles	0.0	0.0	0.0	0.0
Apparel & leather	6.0	0.0	6.0	0.0
Paper products	0.6	0.0	0.6	0.0
Printing services	0.0	0.0	0.0	0.0
Petroleum & coal	2.0	0.0	2.0	0.0
Chemical products	0.8	0.0	0.8	0.0
Plastics & rubber products	0.4	0.0	0.2	0.0
Wholesale trade	0.0	0.0	0.0	0.0
Motor vehicles & parts dealers	0.6	0.0	0.6	0.0
Food & beverage stores	0.8	0.0	0.8	0.0
General merchandise stores	2.0	0.0	2.0	0.0
Other retail	0.0	0.0	0.0	0.0
Air transportation	0.0	0.0	0.0	0.0
Rail transportation	1.5	0.0	1.5	0.0

Notes: Elasticities of substitution for the second, third, and fourth estimation stage described in Section 7.4.

TABLE XVIII

Elasticities of Substitution

Sector	Second Stage θ_i	Third Stage θ_i^x	Fourth Stage θ_i	Fourth Stage θ_i^x
Water transportation	1.0	0.0	1.0	0.0
Truck transportation	0.8	0.0	0.8	0.0
Transit & ground transportation	0.0	0.0	0.0	0.0
Pipeline transportation	6.0	20.0	6.0	20.0
Other transportation activities	6.0	0.0	6.0	0.0
Warehousing & storage	6.0	0.0	6.0	0.0
Publishing industries	0.0	0.0	0.0	0.0
Motion, pictures & sound	1.0	0.0	1.0	0.0
Telecommunications	0.0	0.0	0.0	0.0
Internet & inf. services	2.0	20.0	2.0	0.0
Credit intermediation	0.0	0.0	0.0	0.0
Securities & investment	0.8	0.0	0.8	0.0
Insurance carriers	0.0	0.0	0.0	0.0
Funds, trusts & fin. vehicles	4.0	0.0	4.0	0.0
Housing	0.0	0.0	0.0	0.0
Other real estate	0.0	0.0	0.0	20.0
Rental & leasing intangibles	1.0	0.0	1.0	0.0
Legal services	4.0	0.0	4.0	0.0
Computer systems design	0.0	0.0	0.0	0.0
Misc. professional services	0.0	0.0	0.0	0.0
Management of companies	0.0	0.0	0.0	0.0
Administrative services	0.0	0.0	0.0	0.0
Waste & remediation services	0.0	0.0	0.0	0.0
Educational services	0.0	0.0	0.0	20.0
Ambulatory healthcare	0.0	0.0	0.0	0.0
Hospitals	0.0	0.0	0.0	0.0
Nursing & residential care	0.0	0.0	0.0	0.0
Social assistance	0.2	20.0	0.0	20.0
Arts, sports & museums	0.0	0.0	0.0	0.0
Recreational & gambling	0.0	0.0	0.0	0.0
Accommodation	0.2	0.0	0.2	0.0
Food & beverage stores	0.4	0.0	0.2	0.0
Other services	0.0	0.0	0.0	0.0

Notes: Elasticities of substitution for the second, third, and fourth estimation stage described in Section 7.4.

TABLE XIX

$d\,\Gamma$ on $d\,\widehat{\Gamma}$

	(1)	(2)	(3)	(4)
$d\,\widehat{\Gamma}$	0.8389***	1.0066***	1.0032***	1.0032***
	(0.1641)	(0.0529)	(0.0525)	(0.0525)
Intercept	-0.0011	-0.0001	-0.0001	-0.0001
	(0.0009)	(0.0003)	(0.0003)	(0.0003)
N	24			
R^2	54.28%	94.25%	94.31%	94.31%

Notes: Each column captures the slope and intercept parameter, and their standard errors, for the regression $d\,\Gamma_t = \tau_0 + \tau_1\, d\,\widehat{\Gamma}_t + e_i$, where $d\,\Gamma_t$ stands for the observed aggregate labor share variation from the BEA's IO tables, and $d\,\widehat{\Gamma}_t$ from column n corresponds to the stage $n \in \{1, 2, 3, 4\}$ parametric estimate using the method from Section 7.4. Column 5 uses the parameters from stage 4 excluding the years 2007 to 2010. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

TABLE XX

$d\Gamma$ in response to $d\,log\,A_i = 1\%$

	Sector	$d\Gamma$		Sector	$d\Gamma$
1	Other real estate	$1.76e^{-4}$	34	Other transportation equipment	$9.66e^{-6}$
2	Insurance carriers	$1.30e^{-4}$	35	Waste & remediation services	$8.72e^{-6}$
3	Chemical products	$1.11e^{-4}$	36	Forestry & fishing	$8.40e^{-6}$
4	Misc. professional services	$1.07e^{-4}$	37	Miscellaneous manufacturing	$7.74e^{-6}$
5	Primary metals	$9.80e^{-5}$	38	Printing services	$7.71e^{-6}$
6	Administrative services	$9.01e^{-5}$	39	Accommodation	$6.06e^{-6}$
7	Credit intermediation	$7.52e^{-5}$	40	Other transportation activities	$5.21e^{-6}$
8	Telecommunications	$6.77e^{-5}$	41	Air transportation	$4.93e^{-6}$
9	Internet & information services	$4.93e^{-5}$	42	Warehousing & storage	$4.83e^{-6}$
10	Fabricated metal products	$4.69e^{-5}$	43	Transit & ground transportation	$4.09e^{-6}$
11	Utilities	$4.50e^{-5}$	44	Educational services	$4.05e^{-6}$
12	Computers & electronics	$4.34e^{-5}$	45	Other retail	$3.12e^{-6}$
13	Management of companies	$3.76e^{-5}$	46	Petroleum & coal	$2.34e^{-6}$
14	Nonmetallic minerals	$3.22e^{-5}$	47	Motor vehicles & parts dealers	$2.27e^{-6}$
15	Rental & leasing intangibles	$3.16e^{-5}$	48	Ambulatory healthcare	$2.15e^{-6}$
16	Motor vehicles bodies	$2.87e^{-5}$	49	Recreational & gambling	$1.06e^{-6}$
17	Securities & investment	$2.78e^{-5}$	50	Funds, trusts & fin. vehicles	$1.05e^{-6}$
18	Legal services	$2.77e^{-5}$	51	Furniture	$1.05e^{-6}$
19	Food, beverage & tobacco	$2.57e^{-5}$	52	Truck transportation	$8.85e^{-7}$
20	Plastics & rubber products	$2.50e^{-5}$	53	Food & beverage stores	$8.41e^{-7}$
21	Electrical equipment	$2.41e^{-5}$	54	Hospitals	$6.84e^{-7}$
22	Food & drinking service	$2.21e^{-5}$	55	Textile mills and textiles	$6.80e^{-7}$
23	Mining, except oil & gas	$2.06e^{-5}$	56	Housing	$4.71e^{-7}$
24	Arts, sports & museums	$1.97e^{-5}$	57	General merchandise stores	$3.74e^{-7}$
25	Wood products	$1.97e^{-5}$	58	Water transportation	$1.18e^{-7}$
26	Machinery	$1.84e^{-5}$	59	Apparel & leather	$7.03e^{-8}$
27	Paper products	$1.84e^{-5}$	60	Nursing & residential care	$1.11e^{-8}$
28	Wholesale trade	$1.83e^{-5}$	61	Social assistance	$4.36e^{-9}$
29	Construction	$1.66e^{-5}$	62	Rail transportation	$3.45e^{-9}$
30	Motion, pictures & sound	$1.60e^{-5}$	63	Support activities for mining	$-1.75e^{-7}$
31	Other services	$1.52e^{-5}$	64	Pipeline transportation	$-2.09e^{-7}$
32	Publishing industries	$1.24e^{-5}$	65	Farms	$-5.52e^{-6}$
33	Computer systems design	$1.05e^{-5}$	66	Oil & gas extraction	$-6.80e^{-5}$

Notes: I assume an industry-level productivity shock of 1% and solve the system of equations from Theorem 4. $d\Gamma = \Gamma\, d\,log\,\Gamma$ with $d\,log\,\Gamma = d\,log\,w\,L - d\,log\,E$.

TABLE XXI

$d\Gamma$ in response to $d\,log\,\mu_i = 1\%$

	Sector	$d\Gamma$		Sector	$d\Gamma$
1	Misc. professional services	$7.05e^{-4}$	34	Other transportation equipment	$9.22e^{-5}$
2	Wholesale trade	$5.96e^{-4}$	35	General merchandise stores	$8.92e^{-5}$
3	Construction	$5.88e^{-4}$	36	Rental & leasing intangibles	$8.90e^{-5}$
4	Administrative services	$5.07e^{-4}$	37	Food & beverage stores	$8.89e^{-5}$
5	Other real estate	$4.93e^{-4}$	38	Plastics & rubber products	$8.76e^{-5}$
6	Insurance carriers	$4.93e^{-4}$	39	Petroleum & coal	$8.59e^{-5}$
7	Ambulatory healthcare	$4.40e^{-4}$	40	Housing	$8.55e^{-5}$
8	Hospitals	$3.65e^{-4}$	41	Warehousing & storage	$7.31e^{-5}$
9	Other retail	$3.58e^{-4}$	42	Accommodation	$7.24e^{-5}$
10	Food & drinking service	$3.24e^{-4}$	43	Farms	$6.98e^{-5}$
11	Management of companies	$3.09e^{-4}$	44	Nonmetallic minerals	$6.71e^{-5}$
12	Credit intermediation	$2.98e^{-4}$	45	Arts, sports & museums	$6.67e^{-5}$
13	Securities & investment	$2.77e^{-4}$	46	Paper products	$6.61e^{-5}$
14	Other services	$2.65e^{-4}$	47	Electrical equipment	$6.26e^{-5}$
15	Chemical products	$2.53e^{-4}$	48	Miscellaneous manufacturing	$5.92e^{-5}$
16	Food, beverage & tobacco	$2.42e^{-4}$	49	Motion, pictures & sound	$5.87e^{-5}$
17	Telecommunications	$2.38e^{-4}$	50	Wood products	$5.82e^{-5}$
18	Computer systems design	$2.36e^{-4}$	51	Air transportation	$5.72e^{-5}$
19	Motor vehicles bodies	$2.12e^{-4}$	52	Funds, trusts & fin. vehicles	$5.10e^{-5}$
20	Internet & information services	$1.98e^{-4}$	53	Waste & remediation services	$4.46e^{-5}$
21	Fabricated metal products	$1.54e^{-4}$	54	Recreational & gambling	$4.33e^{-5}$
22	Computers & electronics	$1.54e^{-4}$	55	Mining, except oil & gas	$4.12e^{-5}$
23	Publishing industries	$1.53e^{-4}$	56	Printing services	$3.07e^{-5}$
24	Educational services	$1.46e^{-4}$	57	Forestry & fishing	$2.81e^{-5}$
25	Primary metals	$1.44e^{-4}$	58	Furniture	$2.50e^{-5}$
26	Utilities	$1.44e^{-4}$	59	Transit & ground transportation	$2.31e^{-5}$
27	Legal services	$1.38e^{-4}$	60	Rail transportation	$1.99e^{-5}$
28	Truck transportation	$1.34e^{-4}$	61	Support activities for mining	$1.91e^{-5}$
29	Machinery	$1.31e^{-4}$	62	Textile mills and textiles	$1.49e^{-5}$
30	Other transportation activities	$1.15e^{-4}$	63	Water transportation	$1.19e^{-5}$
31	Motor vehicles & parts dealers	$1.07e^{-4}$	64	Pipeline transportation	$8.96e^{-6}$
32	Nursing & residential care	$1.02e^{-4}$	65	Apparel & leather	$7.75e^{-6}$
33	Social assistance	$9.98e^{-5}$	66	Oil & gas extraction	$-7.96e^{-6}$

Notes: I assume an industry-level markdown shock of 1% and solve the system of equations from Theorem 4. $d\Gamma = \Gamma\,d\,log\,\Gamma$ with $d\,log\,\Gamma = d\,log\,w\,L - d\,log\,E$..

TABLE XXII

$\frac{d\Gamma}{d\,log\,A_i}$ **with** $d\,log\,A_i = 1\%$

	(1)	(2)	(3)	(4)
λ_i	$5.674e^{-4***}$			$5.698e^{-4***}$
	$(1.531e^{-4})$			$(1.598e^{-4})$
μ_i		$-3.221e^{-5}$		$4.102e^{-5}$
		$(3.168e^{-5})$		$(5.237e^{-5})$
ψ_i^{ℓ}			$-4.173e^{-5}$	$-5.580e^{-5}$
			$(3.193e^{-5})$	$(5.214e^{-5})$
Intercept	$7.543e^{-6}$	$4.798e^{-5*}$	$4.564e^{-5**}$	$5.928e^{-6}$
	$(5.934e^{-6})$	$(2.488e^{-5})$	$(1.782e^{-5})$	$(2.567e^{-6})$
N	66			
R^2	17.65%	1.58%	2.59%	19.17%

Notes: Each column captures the slope and intercept parameter, and their standard errors, for the regression for the outcomes in Table XX on Domar weights, markdowns, and sectoral centralities. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

TABLE XXIII

$\frac{d\Gamma}{d\log\mu_i}$ **with** $d\log\mu_i = 1\%$

	(1)	(2)	(3)	(4)
λ_i	0.0049^{***}			$5.484e^{-3***}$
	(0.00039)			$(3.233e^{-4})$
μ_i		$1.081e^{-4}$		$2.166e^{-4**}$
		$(1.394e^{-4})$		$(1.059e^{-4})$
ψ_i^ℓ			$1.836e^{-4}$	$1.968e^{-4*}$
			$(1.400e^{-4})$	$(1.054e^{-4})$
Intercept	$2.687e^{-5*}$	$8.061e^{-5}$	-0.0001	$-2.599e^{-4***}$
	$(1.538e^{-5})$	$(1.095e^{-4})$	(0.0003)	$(5.194e^{-5})$
N		66		
R^2	71.22%	0.93%	2.61%	82.80%

Notes: Each column captures the slope and intercept parameter, and their standard errors, for the regression for the outcomes in Table XXI on Domar weights, markdowns, and sectoral centralities. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

TABLE XXIV

$d\Gamma$ in response to $d\,log\,A_i$ such that $Technology = 1$

	Sector	$d\Gamma$		Sector	$d\Gamma$
1	Primary metals	16.54	34	Publishing industries	4.17
2	Nonmetallic minerals	16.52	35	Accommodation	4.05
3	Electrical equipment	13.71	36	Food, beverage & tobacco	3.90
4	Other real estate	12.75	37	Food & drinking services	3.60
5	Arts, sports & museums	12.37	38	Other services	3.27
6	Mining, except oil & gas	11.59	39	Warehousing & storage	3.17
7	Computers & electronics	11.43	40	Computer systems design	2.79
8	Printing services	11.31	41	Other transportation activities	1.86
9	Motion, pictures & sound	10.92	42	Educational services	1.82
10	Insurance carriers	10.79	43	Wholesale trade	1.59
11	Chemical products	10.78	44	Recreational & gambling	1.51
12	Rental & leasing intangibles	10.44	45	Textile mills and textiles	1.32
13	Wood products	9.93	46	Motor vehicles & parts dealers	1.28
14	Telecommunications	9.93	47	Furniture	1.22
15	Fabricated metal products	9.72	48	Construction	1.22
16	Internet & information services	9.63	49	Funds, trusts & fin. vehicles	0.97
17	Plastics & rubber products	9.62	50	Food & beverage stores	0.59
18	Paper products	9.16	51	Apparel & leather	0.55
19	Forestry & fishing	8.66	52	Water transportation	0.50
20	Legal services	8.23	53	Other retail	0.48
21	Waste & remediation services	8.08	54	Petroleum & coal	0.42
22	Transit & ground transportation	7.97	55	Truck transportation	0.34
23	Utilities	7.41	56	Ambulatory healthcare	0.34
24	Administrative services	7.39	57	General merchandise stores	0.29
25	Misc. professional services	7.06	58	Hospitals	0.13
26	Miscellaneous manufacturing	6.69	59	Housing	0.03
27	Credit intermediation	6.13	60	Nursing & residential care	$8.47e^{-5}$
28	Motor vehicles bodies	5.71	61	Rail transportation	$7.44e^{-5}$
29	Management of companies	5.38	62	Social assistance	$3.35e^{-5}$
30	Other transportation equipment	5.06	63	Support activities for mining	-0.35
31	Securities & investment	4.96	64	Pipeline transportation	-0.63
32	Machinery	4.85	65	Farms	-1.30
33	Air transportation	4.35	66	Oil & gas extraction	-10.06

Notes: I assume an industry-level productivity shock of $\widetilde{\lambda}_i^{-1}$ and solve the system of equations from Theorem 4. $d\Gamma = \Gamma\ d\,log\,\Gamma$ with $d\,log\,\Gamma = d\,log\ w\,L - d\,log\,E$. A x increase captures a labor share that is x percentage points higher.

TABLE XXV

$d\Gamma$ in response to $d\,log\,\mu_i$ such that $Competitiveness = 1$

	Sector	$d\Gamma$		Sector	$d\Gamma$
1	Nursing & residential care	78.10	34	Administrative services	41.67
2	Social assistance	76.75	35	Waste & remediation services	41.37
3	Ambulatory healthcare	71.21	36	Other transportation services	41.20
4	General merchandise stores	70.88	37	Legal services	41.13
5	Hospitals	69.77	38	Insurance carriers	40.68
6	Educational services	66.11	39	Computers & electronics	40.54
7	Food & beverage stores	62.42	40	Motion, pictures & sound	39.97
8	Computer systems design	62.39	41	Support activities for mining	39.04
9	Recreational & gambling	61.40	42	Internet & information services	38.84
10	Apparel & leather	60.85	43	Food, beverage & tobacco	36.70
11	Motor vehicles & parts dealers	60.31	44	Other real estate	35.75
12	Other services	59.96	45	Electrical equipment	35.54
13	Other retail	55.12	46	Telecommunications	34.94
14	Truck transportation	53.00	47	Machinery	34.56
15	Food & drinking services	52.93	48	Nonmetallic minerals	34.45
16	Wholesale trade	51.77	49	Plastics & rubber products	34.18
17	Publishing industries	51.50	50	Paper products	32.92
18	Water transportation	51.30	51	Fabricated metal products	32.08
19	Miscellaneous manufacturing	51.26	52	Rental & leasing intangibles	29.35
20	Air transportation	50.57	53	Wood products	29.29
21	Securities & investment	49.49	54	Textile mills and textiles	29.26
22	Accommodation	48.43	55	Furniture	29.22
23	Other transportation equipment	48.37	56	Forestry & fishing	29.00
24	Warehousing & storage	48.10	57	Pipeline transportation	27.20
25	Funds, trusts & fin. vehicles	47.10	58	Chemical products	24.46
26	Misc. professional services	46.51	59	Primary metals	24.43
27	Transit & ground transportation	45.17	60	Credit intermediation	24.35
28	Printing services	45.04	61	Utilities	23.69
29	Management of companies	44.30	62	Mining, except oil & gas	23.17
30	Construction	43.52	63	Farms	16.42
31	Rail transportation	42.98	64	Petroleum & coa	15.72
32	Motor vehicles bodies	42.40	65	Housing	6.98
33	Arts, sports & museums	41.80	66	Oil & gas extraction	-1.11

Notes: I assume an industry-level markdown shock of $\widetilde{\lambda}_i^{-1}$ and solve the system of equations from Theorem 4. $d\Gamma = \Gamma\, d\,log\,\Gamma$ with $d\,log\,\Gamma = d\,log\; w\,L - d\,log\,E$. A x% increase captures a labor share that is x percentage points higher.

TABLE XXVI

$\frac{d\Gamma}{d\log A_i}$ **with** $d\log A_i$ **such that** $Technology = 1\%$

	(1)	(2)	(3)	(4)
λ_i	-0.1440			-0.2158
	(0.2282)			(0.2365)
μ_i		-0.0469		-0.0197
		(0.0429)		(0.0775)
ψ_i^{ℓ}			-0.0553	-0.0465
			(0.0433)	(0.0771)
Intercept	0.0546***	0.0869**	0.0805***	0.0969**
	(0.0088)	(0.0337)	(0.0241))	(0.0380)
N	66			
R^2	0.61%	1.83%	2.48%	3.78%

Notes: Each column captures the slope and intercept parameter, and their standard errors, for the regression for the outcomes in Table XXIV on Domar weights, markdowns, and sectoral centralities. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

TABLE XXVII

$\frac{d\Gamma}{d\log\mu_i}$ with $d\log\mu_i$ such that $Competitiveness = 1\%$

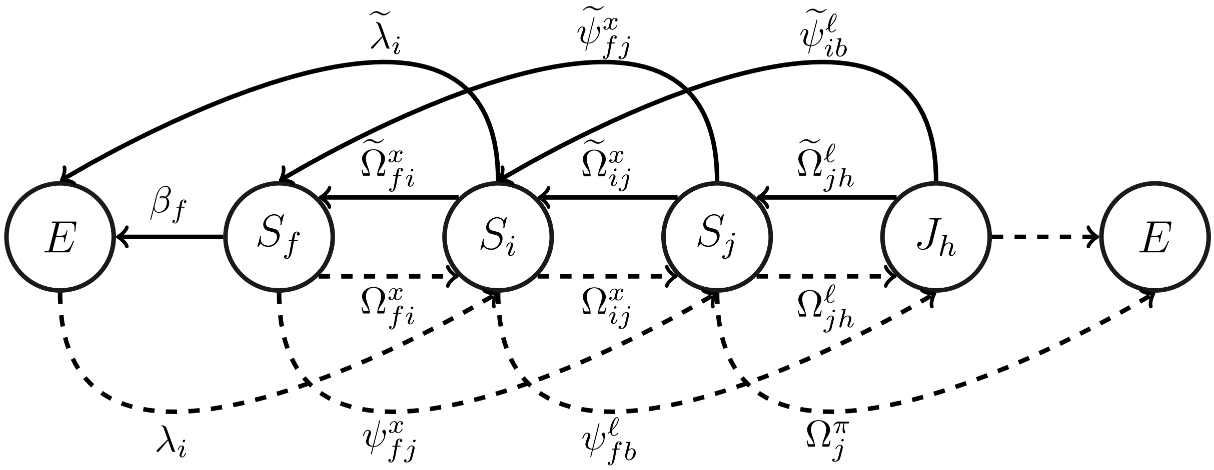
	(1)	(2)	(3)	(4)
λ_i	-0.3573			0.4667
	(0.7186)			(0.4136)
μ_i		0.6933***		-0.1315
		(0.1051)		(0.1355)
ψ_i^ℓ			0.9168***	1.0430***
			(0.0429)	(0.1349)
Intercept	0.4363***	-0.1086	-0.0683	-0.0477
	(0.0278)	(0.0825)	(0.0429)	(0.0664)
N	66			
R^2	0.38%	40.46%	68.94%	70.25%

Notes: Each column captures the slope and intercept parameter, and their standard errors, for the regression for the outcomes in Table XXV on Domar weights, markdowns, and sectoral centralities. $*$ means significant at the 10%, $**$ at the 5%, and $***$ at the 1%.

Figures and Illustrations

FIGURE I

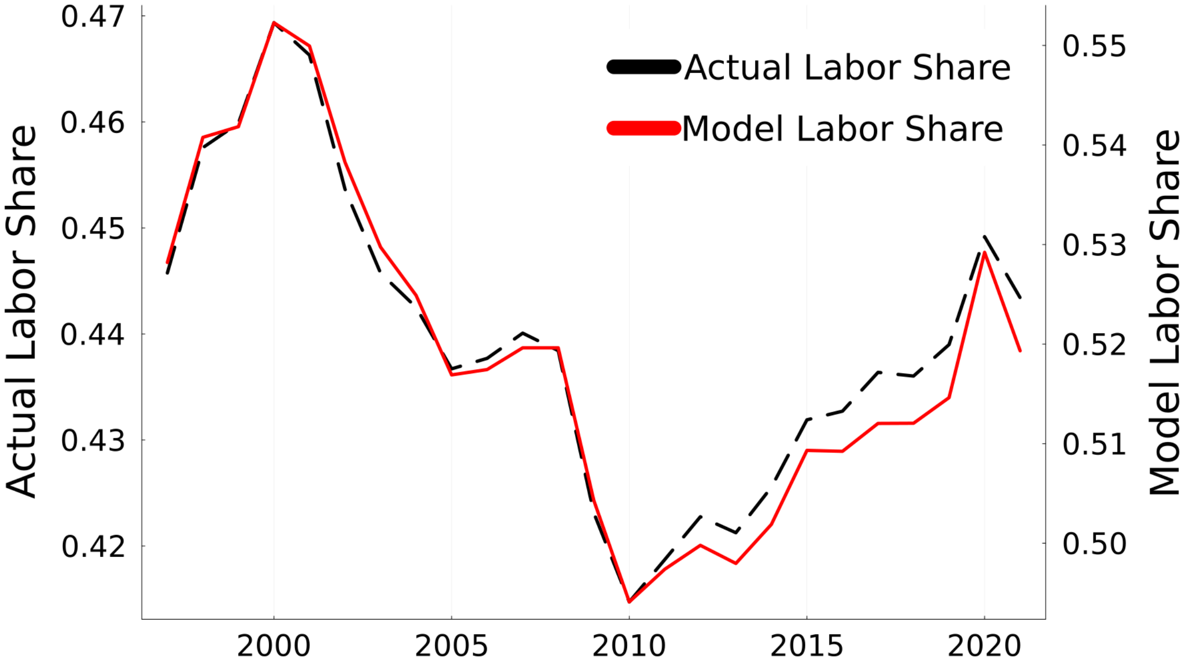
Measures of Centrality



Notes: Continuous and dashed arrows represent the cost-based and revenue-based centrality measures, respectively.

FIGURE II

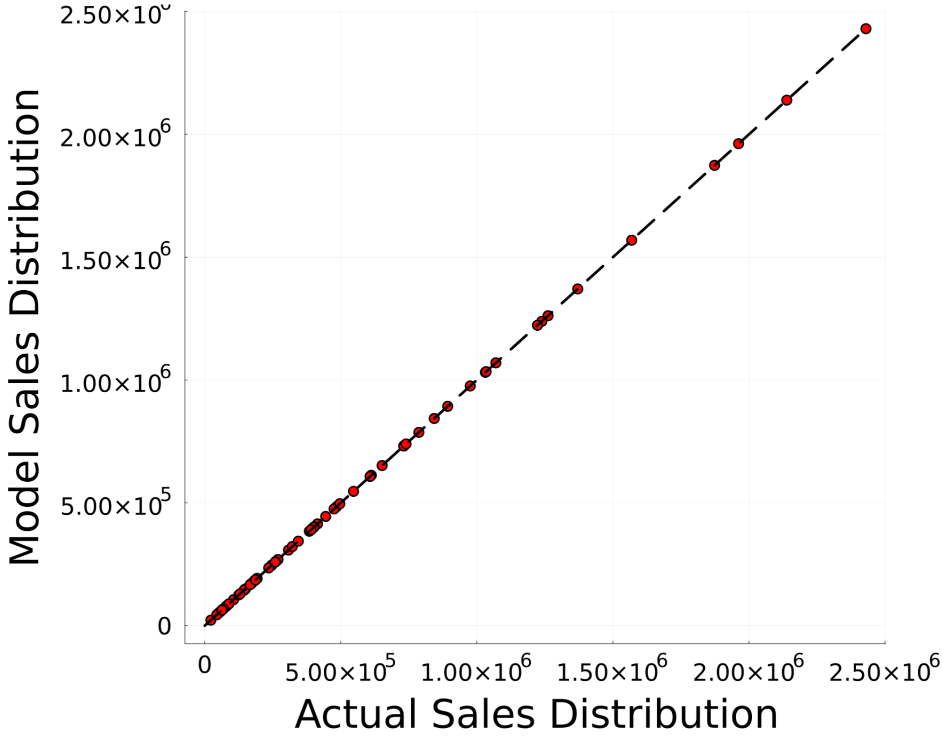
The Aggregate Labor Share



Notes: The actual labor share comes from dividing total labor costs from the BEA's IO matrix with nominal GDP. The model's labor share comes from solving the followin system of equations $\lambda = \Psi_x' \, \beta$ and $\Lambda = \Omega_\ell' \, \lambda$.

FIGURE III

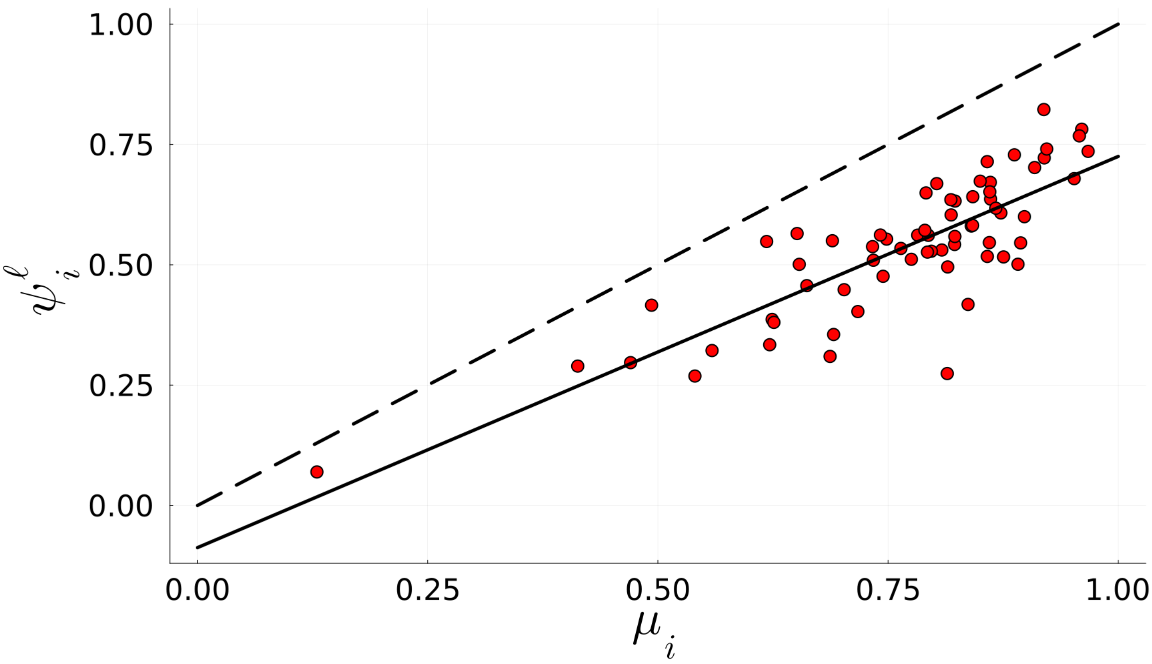
The Sales Distribution



Notes: The actual sales distribution is given by the BEA's IO matrix. The model's sales distribution comes from solving $\lambda = \Psi_x' \, \beta$ and dividing by $GDP = \sum_{i \in \mathcal{N}} (1 - \mu_i \, \omega_i^x) \, S_i$.

FIGURE IV

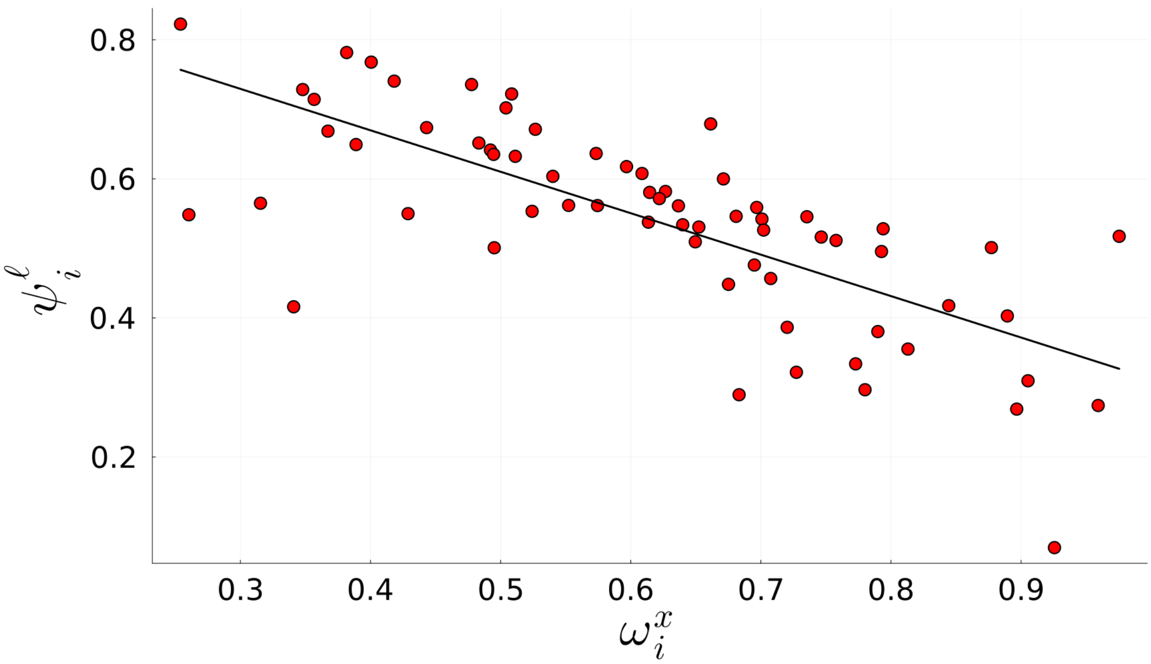
Markdowns and Payment Centralities



Notes: Scatterplot captures combination of industry-level markdowns and payment centralities in 2021. Dashed line is the 45 degre line, and continuous line is the line of best fit given by the linear regression $\psi^{\ell}_{i,2021} = \tau_0 + \tau_1\, \mu_{i,2021} + e_i$.

FIGURE V

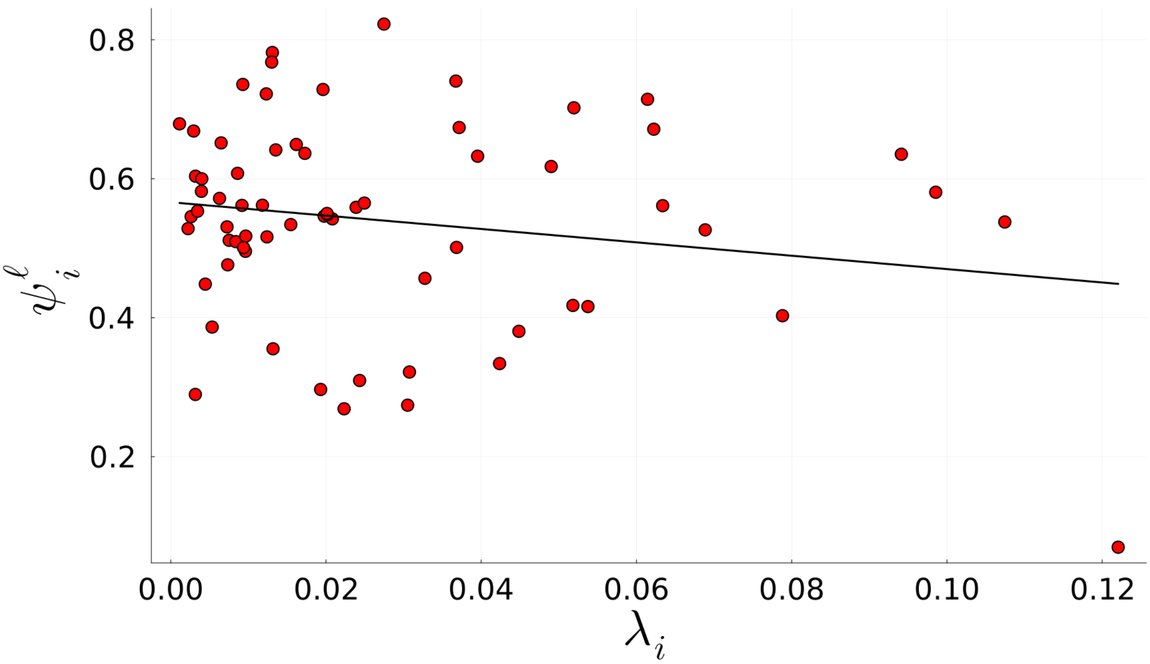
Intermediate Input Intensities and Payment Centralities



Notes: Scatterplot captures combination of industry-level intermediate input cost intensities and payment centralities in 2021. Continuous line is the line of best fit given by the linear regression $\psi^{\ell}_{i,2021} = \tau_0 + \tau_1\, \omega^{x}_{i,2021} + e_i$.

FIGURE VI

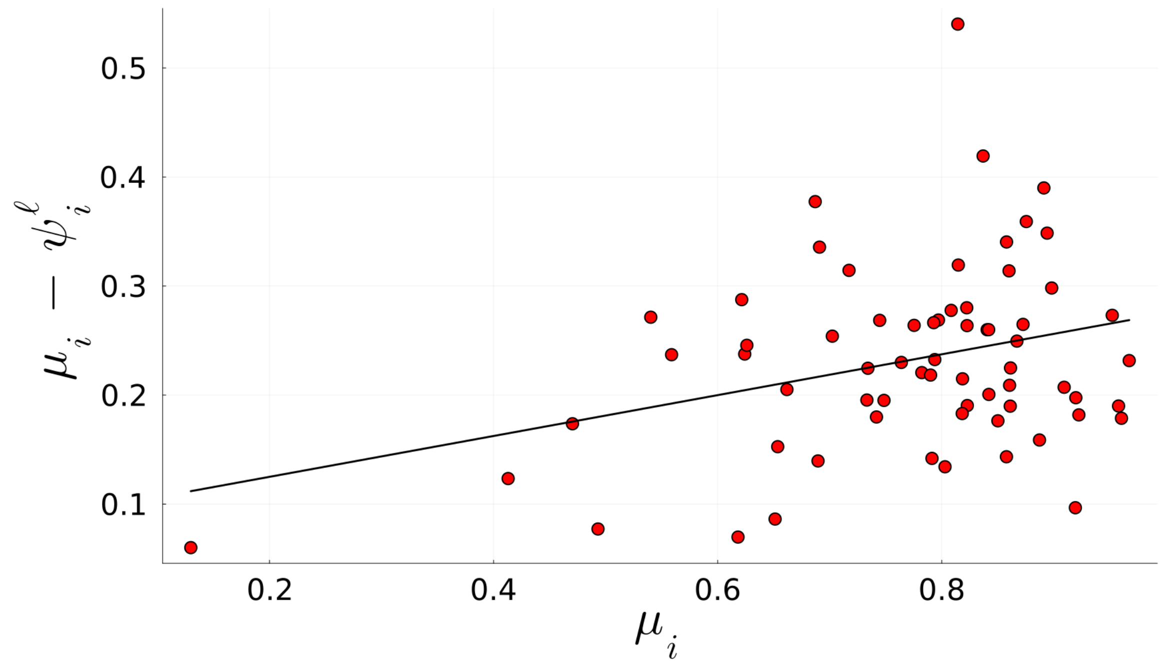
Domar weights and Payment Centralities



Notes: Scatterplot captures combination of industry-level Domar weights and payment centralities in 2021. Continuous line is the line of best fit given by the linear regression $\psi^{\ell}_{i,2021} = \tau_0 + \tau_1\,\lambda_{i,2021} + e_i$.

FIGURE VII

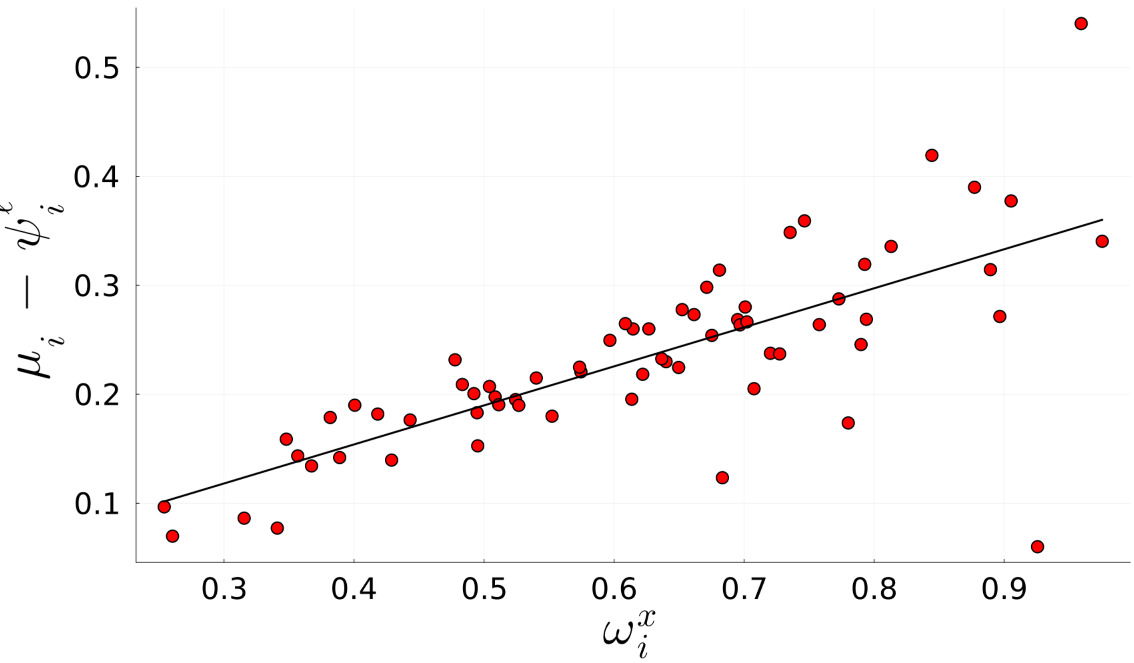
Markdowns and differences between Markdowns and Payment Centralities



Notes: Scatterplot captures combination of industry-level markdowns and the difference between markdowns and payment centralities in 2021. Continuous line is the line of best fit given by the linear regression $\psi^{\ell}_{i,2021} - \mu_{i,2021} = \tau_0 + \tau_1\, \mu_{i,2021} + e_i$.

FIGURE VIII

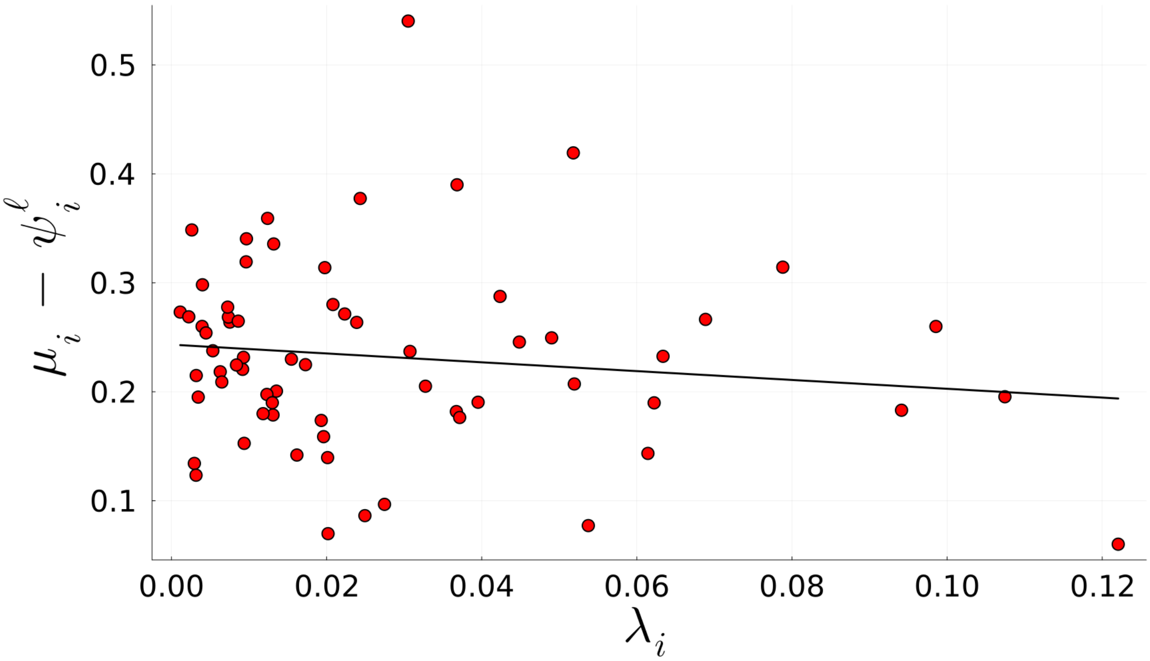
Intermediate Input Intensities and differences between Markdowns and Payment Centralities



Notes: Scatterplot captures combination of industry-level intermediate input intensities and the difference between markdowns and payment centralities in 2021. Continuous line is the line of best fit given by the linear regression $\psi^{\ell}_{i,2021} - \mu_{i,2021} = \tau_0 + \tau_1\,\omega^{x}_{i,2021} + e_i$.

FIGURE IX

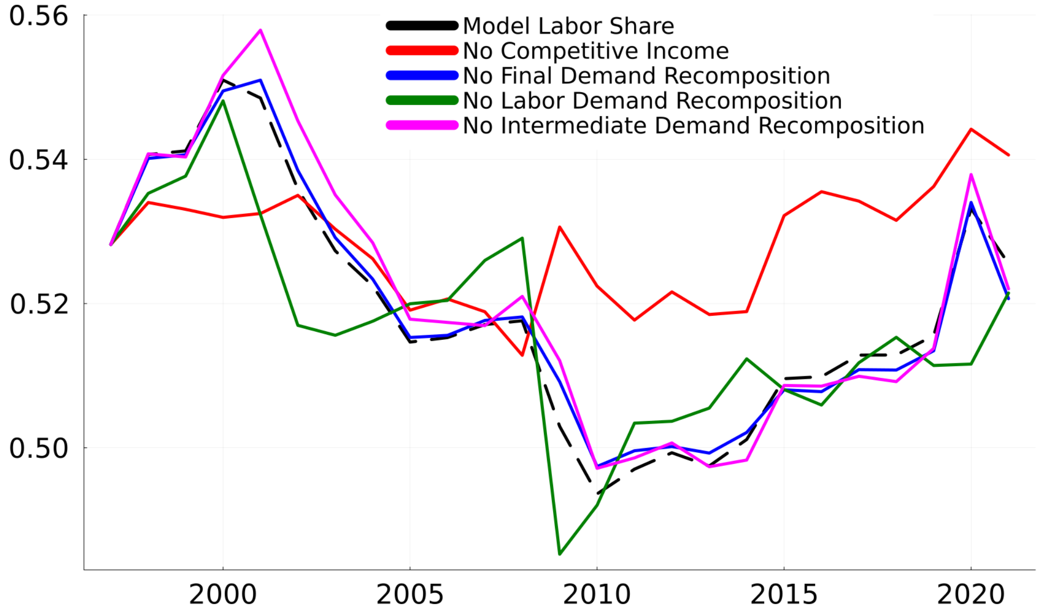
Domar weights and differences between Markdowns and Payment Centralities



Notes: Scatterplot captures combination of industry-level Domar weights and the difference between markdowns and payment centralities in 2021. Continuous line is the line of best fit given by the linear regression $\psi^{\ell}_{i,2021} - \mu_{i,2021} = \tau_0 + \tau_1\,\lambda_{i,2021} + e_i$.

FIGURE X

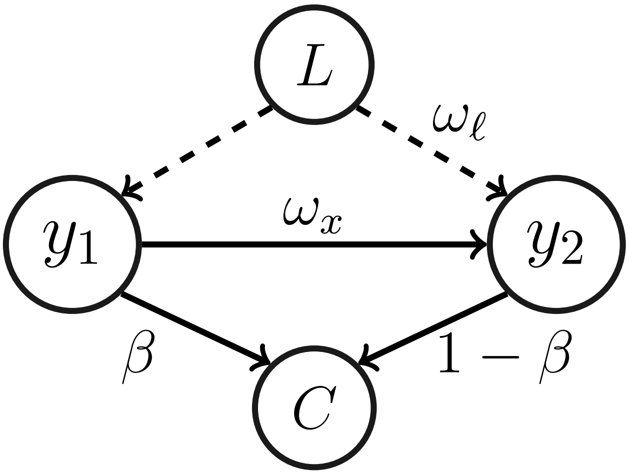
Counterfactual Aggregate Labor Share



Notes: The dashed line comes from using $\hat{\Gamma}_t$ given by equation (13) using $d\,log\,\mu_i$, $d\,\beta_i$, $d\,\omega_i^\ell$, and $d\,\widetilde{\Omega}_{ij}^x$ obtained from temporal differences in the BEA's IO matrices. The red line leaves aside the *competitive income* channel by assuming $d\,log\,\mu_i = 0$. The blue lines comes from leaving aside *final demand recomposition* channel by assuming $d\,\beta_i = 0$. The green line comes from leaving aside the *labor demand recomposition* channel by assuming $d\,\omega_i^\ell = 0$. The purple line comes from leaving aside the *intermediate demand recomposition* channel by assuming $d\,\widetilde{\Omega}_{ij} = 0$.

FIGURE XI

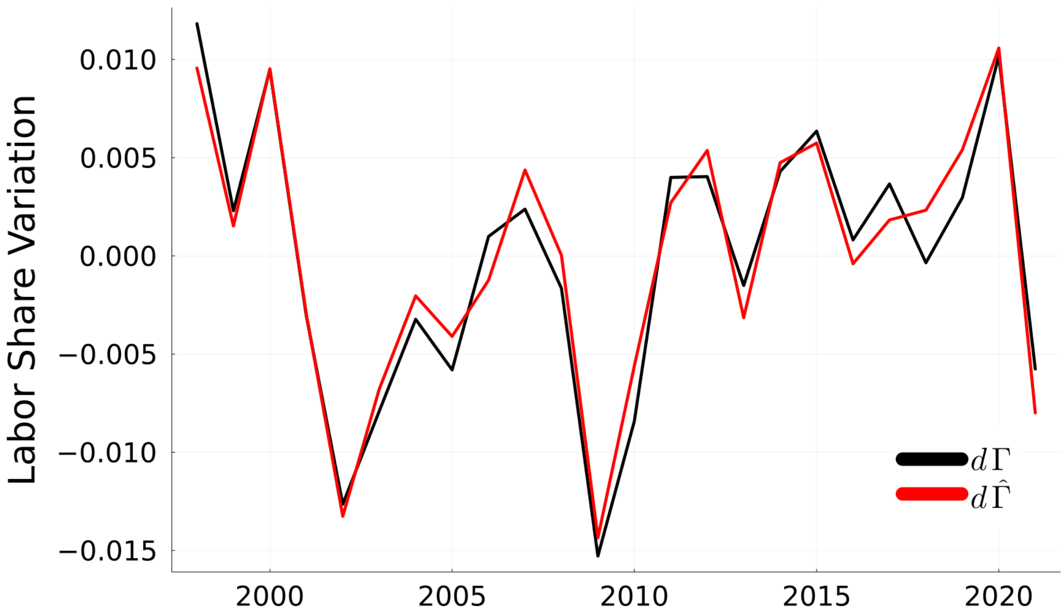
Simple Economy



Note: Continuous arrows represent the flow of goods and dashed arrows the supply of labor.

FIGURE XII

Observed and Model Labor Share Variation



Notes: $d\Gamma_t$ stands for the observed aggregate labor share variation from the BEA's IO tables, and $d\widehat{\Gamma}_t$ for the stage four estimation for the method described in Section 7.4.