Inequality and Misallocation under Production Networks

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Abstract

In this paper, I develop an aggregation theory for distorted production network economies with heterogeneous households. I provide general decompositions for how the aggregate and distributional effects of shocks are sensitive to underlying consumer and firm heterogeneity. The workers' value-added over labor income ratios (distortion centralities) gauge the importance of workers in the production of heavily distorted firms and are sufficient statistics for the effect of income distribution variations on TFP. The average distortion centrality faced by a household's expenditure (expenditure centrality) and a firm's revenue (revenue centrality) are sufficient statistics for the effect of expenditure variations on TFP. Labor misallocation rises and TFP falls as labor income shifts toward high distortion centrality workers, consumption shifts toward high expenditure centrality households, or demand shifts toward high revenue centrality firms. The reason is that when aggregate expenditure on relatively undistorted firms rises, their labor demand increases, reallocating workers from distorted firms with high marginal productivity to relatively undistorted firms with low marginal productivity. These second-best results show how distributional variations affect aggregate output by changing the aggregate allocation efficiency of workers. I estimate the first production network model with household heterogeneity for the United States. I show that variations in the income distribution have been responsible for 20% of the TFP volatility. Additionally, income distribution variations reduced misallocation between 2001 and 2009, and accentuated misallocation after the Great Recession. Heterogeneities in the production network are essential in explaining income and real consumption inequalities.

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1 Introduction

> *"While we often must focus on aggregates for macroeconomic policy, it is impossible to think coherently about national well-being while ignoring inequality and poverty, neither of which is visible in aggregate data. Indeed, and except in exceptional cases, macroeconomic aggregates themselves depend on distribution."*
>
> – Deaton (2016)

Modern economies are a complex web of market interactions shaped by the decisions of billions of agents. Agents determine their production strategies, consumption patterns, and work levels based on income, prevailing prices, and market access. These economic interactions rely on multilayered networks through which disaggregated flows of goods, services, and payments circulate. Within these complex economies, understanding the aggregate and distributional effects of microeconomic shocks is foundational for a theory of macroeconomic aggregation. Developing this theory is challenging because it requires an explanation for how the propagation of shocks depends on the positions of firms and households within the network and the network structure.

In this paper, I contribute to the theory of macroeconomic aggregation by building a neoclassic environment for production network economies with heterogeneous households and distortions. In this environment, labor supply can be endogenous or exogenous. My main objective is to provide decompositions that capture the effect of distributional variations on aggregate measures of real output and idiosyncratic measures of real consumption. The main theoretical contribution of this paper shows that variations in the distributions of labor income and consumption expenditure influence total factor productivity (TFP), and these effects are only neutral under highly restrictive conditions. I prove this by identifying sufficient statistics that capture, under a general setting, the aggregate and idiosyncratic effects of distributional variations. The common intuition behind the mechanisms introduced in this paper is that tracing the variations in how expenditure flows through the economy is crucial for understanding the aggregate and distributional effects of the reallocation of workers among firms. The TFP decomposition I introduce is the first for a distorted general production network economy with household heterogeneity in preferences and income.

Using my model, I estimate the first empirical implementation of a production network environment with heterogeneous households for the United States. The model indicates that distributional variations increased TFP by 8.2% before the Great Recession (2001 to 2009) and reduced TFP by 7.5% after the Great Recession (2010 to 2020). These results contribute to the secular stagnation literature by introducing evidence for a new distributional channel based on factoral misallocation.

Theoretical Contribution

The theory of aggregation for multisector input-output economies has relied on the Domar aggregation developed by Hulten (1978), which builds on the growth accounting work from Solow (1957) and Domar (1961). Hulten's theorem is a macroeconomic envelope condition for a perfectly competitive representative household economy. This theorem states that the first-order variation for the aggregate efficiency wedge of the production possibility frontier depends exclusively on productivity shocks, the sales distribution is a sufficient statistic for these aggregate effects, the microeconomic structure of the network is irrelevant, and the reallocation of factors is neutral.

Hulten's theorem relies on an undistorted allocation of workers and intermediate inputs. In this allocation, firms operate at their competitive margin, so the value-added that passes through a firm coincides with its revenue. The symmetry between the propagation of costs (from labor costs to final expenditure) and the propagation of revenue (from final expenditure to labor income) is essential for this theorem. Introducing rebated distortions (e.g., taxation, tariffs, financial constraints, nominal rigidities, and market power) breaks this symmetry, changes the system of prices, alters the choices of firms and households, and keeps factors and intermediate inputs away from their undistorted allocation. For this reason, Baqaee & Farhi (2020) find that in a distorted input-output representative-household economy with exogenous factoral supply, the first-order variation in TFP depends on a technological component and the endogenous reallocation of factors and intermediate inputs across firms. Technological shocks shift the aggregate production possibility frontier, and reallocation moves the equilibrium along the edges of the production possibility frontier. Now, the firms', workers', and households' location in the network and the microeconomic structure of the network structure are necessary to understand the aggregate effects of microeconomic shocks.

I segment the influence of the reallocation of resources on TFP into effects from exogenous variations in distortions and endogenous changes in income distribution. I show that for each worker, the value-added to labor income ratio, which I call *distortion centrality*, indicates the extent of their income exposure to heavily distorted supply chains. The distribution of distortion centralities is a ranking for the negative effect on the aggregate marginal labor productivity of one additional percentage point of labor income share for a type of worker. Aggregate misallocation worsens as labor income shifts towards workers with large distortion centralities because resources reallocate towards relatively undistorted firms that operate with low marginal productivities. Consequently, for an economy with distortions, the allocation of resources improves as the income distribution becomes more distant from the value-added distribution. This decomposition aligns with the findings from Baqaee & Farhi (2020). However, the workers' distortion centralities and the corresponding second-best results offer a novel approach for measuring distributional effects on the aggregate efficiency wedge.

My main contribution comes from decomposing the distributional sources of variation for the aggregate efficiency wedge. I do this by deriving the first-order approximation for the labor income shares. I show that variations in a worker's labor income share depend on two sources. First, changes in the households' consumption expenditure distribution holding fixed the economy's demand structure. Second, changes in the economy's demand structure holding fixed the households' consumption expenditure distribution. The economy's demand structure has as a sufficient statistic a matrix of bilateral centralities that represent the share of expenditure from each household that reaches the labor income for every worker. These bilateral centralities depend on the whole set of distortions that the economy faces, the households' consumption patterns, and the firms' demand for labor and intermediate inputs. Hence, understanding the changes in the economy's demand structure requires decomposing the bilateral centrality variations into four channels: (i) distortions, (ii) endogenous changes in households' demand structure, (iii) endogenous changes in firms' labor demand structure, and (iv) endogenous changes in firms' intermediate input demand structure. Endogenous shifts in the demand structure reflect how expenditure shares adjust based on relative price variations. These decompositions are related to Bigio & La'O (2020), who obtain the first-order variation for the aggregate labor wedge around the efficient equilibrium in a production network representative household economy with one type of endogenous labor. The aggregate labor wedge measures the effect of distortions on the labor supply decision, and in equilibrium, it equals the aggregate labor share. Their paper shows that around the efficient equilibrium, the first-order variation for the aggregate labor share depends exclusively on distortions, and the sales distribution is a sufficient statistic for these aggregate effects. My decompositions for the labor income shares extend these results to a production network economy with heterogeneous households and distortions on the equilibrium.

Using the variations of the labor income shares, I decompose the effect on TFP from endogenous perturbations in the income distribution into (i) changes in the households' consumption expenditure distribution holding fixed the economy's demand structure, and (ii) changes in the economy's demand structure holding fixed the households' consumption expenditure distribution. I define each household and firm's exposure to distortion centralities through expenditure or revenue as *expenditure centrality* and *revenue centrality*, respectively. These metrics serve as sufficient statistics for the expenditure reallocation effects on TFP. The distribution of expenditure centralities is a ranking for the negative effect on the aggregate marginal labor productivity of one additional percentage point of expenditure share for a type of consumer. The distribution of revenue centralities is a ranking for the negative effect on the aggregate marginal labor productivity of one additional percentage point of expenditure on a firm. High expenditure or revenue centralities indicate that a significant share of households' consumption expenditure and firms' revenue will reach workers essential for production in distorted supply chains through labor compensation from firms operating in relatively efficient supply chains. When expenditure leans towards consumers or firms with high centrality measures, aggregate misallocation worsens because resources reallocate from distorted firms with high marginal productivities to relatively undistorted firms with low marginal productivities. The TFP decomposition allows me to establish two neutrality results. First, expenditure redistribution and demand recomposition have no first-order effect on TFP around the undistorted equilibrium. Second, expenditure redistribution is neutral on TFP if the expenditure centralities are symmetric across households. One case in which the symmetry in expenditure centralities is satisfied is when consumption preferences are homogenous. This latter result indicates that aggregate demand non-homotheticity is necessary but not sufficient for the redistribution of expenditure to influence TFP.

By comparing the decentralized market solution with the allocation from a constrained social planner that centralizes households' decisions, I show that the decentralized solution faces additional externalities from the individual labor supply decision. The planner's solution requires symmetric distortion centralities, which equalizes the effects of labor income redistribution on the aggregate marginal labor productivity. In other words, symmetry in distortion centralities represents the optimal composition of the aggregate labor supply. If this condition is not satisfied, there is still space to augment aggregate welfare by shifting the production possibility frontier through the aggregate efficiency wedge. Workers do not internalize this condition when choosing their labor supply, creating an externality on aggregate welfare. The constrained social planner resembles a representative household with the additional problem of choosing distributional allocations. Consequently, the representative household economy coincides with the heterogeneous household model only under the highly restrictive condition of symmetry in distortion centralities, and only then are the allocations efficient from the perspective of the constrained planner. This result allows me to prove that for a representative household or a constrained planner environment, the distributional effects on aggregate misallocation are proportional to the growth in the aggregate labor share. Hence, changes in the aggregate labor share are sufficient to represent the effects on TFP, and tracing the variations for the whole income distribution is no longer necessary.

Household heterogeneity also allows me to consider the distributional effects of microeconomic shocks. For this reason, I introduce the positional terms of trade (PTT) as an object that captures idiosyncratic efficiency wedges. I use the term "positional" because they depend on the location of households across multiple networks. The changes in PTTs are a distributional decomposition of the variations in TFP.

Empirical Contribution

My model requires four types of money flows: (1) business-to-business in the supply of intermediate inputs, (2) business-to-workers in the supply of labor, (3) consumer-to-business in the supply of final goods, and (4) business-to-households in the distribution of dividends. I use the following data sources from 1997 to 2021 to capture these bilateral linkages. First, I use the sectoral input-output tables from the Bureau of Economic Analysis for business-to-business transactions. Second, I combine the county business patterns from the Census Bureau with the occupational employment and wage statistics from the Bureau of Labor Statistics to measure geographic and occupational intensity for business-to-worker transactions. Third, I employ the state-level personal consumption by product type from the Bureau of Economic Analysis and create a product-to-sector crosswalk that measures consumer-to-business transactions. Finally, I use the integrated industry-level production accounts from the Bureau of Economic Analysis to identify sectoral productivity shocks.

The technological component was the primary source of growth in TFP; without it, TFP would have grown 24% less. Without the productivity shocks in oil and gas extraction and the computer and electronic products industries, my model predicts that TFP would have grown 11.1% and 6.6% less, respectively. The reallocation of resources had a secondary role; TFP would have grown 2.5% more without variations in profit margins and 2.8% more without distributional-driven misallocation. Nevertheless, almost 60% of the volatility in TFP was attributable to the reallocation of resources. Out of this, 40% corresponds to changes in profit margins and the remaining 20% to variations in the income distribution.

For specific business cycles, the distributional-driven misallocation of resources had a significant role. For the cycle before the Great Recession (2002 to 2009), TFP's growth would have been 8.2% lower without the variations in the income distribution. The primary drivers of this growth-enhancing environment were the higher sectoral profit margins, particularly in the oil and gas extraction and the computer and electronic products industries. After the Great Recession (2010 to 2020), without the variations in the income distribution, TFP growth would have been 7.5% higher. The main culprits behind this stagnated growth environment were the higher labor demand from the credit intermediation industry and the higher final and intermediate demand for wholesale trade goods.

These aggregate variations hide a rich story of distributional effects. According to the PTTs, the last two decades, on the one hand, have been unfavorable for low-skill industrial workers with occupations heavily exposed to the printing, shoe, leather, and textile industries. On the other hand, the same shocks have benefited high-skill workers in computer science and mathematics occupations.

Related Literature

This article relates to the literature on disaggregated national accounts, production networks, heterogeneous agents, growth accounting, and misallocation. The most foundational is the literature on disaggregated national accounts with heterogeneous consumers and producers. The roots of this literature trace back to the work from Cantillon (1756) and Quesnay (1758), who considered that a successful system of macroeconomic accounts should build up from bilateral flows that add up to the national aggregates. These principles inspired the diagrams of circular flow developed by Lahn (1903), Foster (1922), Knight (1933), Meade & Stone (1941), and Kuznets (1946), and the measures of inter-industrial trade from Leontief (1928, 1986). These studies are the foundation for modern national accounts (Stone, 1961). However, the disaggregated transactions these accounts collect still need to be completed. For example, they capture no information about the flows between firms and households. For this reason, Andersen, Hansen, Huber, Johannesen, & Straub (2022) take a step forward in measuring these flows in Denmark, where accessible administrative data and credit card transaction data from the largest retail bank allow them to estimate direct bilateral flows. My model defines new measures of bilateral centrality that utilize these disaggregated flows to capture the importance of the direct and indirect channels that connect any two households or firms throughout the economy.

The production network literature builds on the canonical multisector models from Hulten (1978) and Long & Plosser (1983). The main emphasis of this literature has been on the propagation of sectoral productivity shocks (Foerster et al., 2011; Horvath, 1998, 2000; Dupor, 1999; Acemoglu et al., 2012, 2016; Carvalho et al., 2021). However, the same models have been used to study the propagation of sectoral distortions under specific (Basu, 1995; Ciccone, 2002; Yi, 2003; Jones, 2011; Asker et al., 2014) and generic (Jones, 2013; Baqaee, 2018; Liu, 2019; Baqaee & Farhi, 2020; Bigio & La'O, 2020) input-output structures. The literature on production networks belongs to the broader attempt to map the aggregate effects from "granular" microeconomic shocks that follow the seminal work from Gabaix (2011). My model nests all of these environments and shocks as specific cases.

Within the extensive work on heterogenous agents, my article is related to the literature on asymmetries in marginal propensities to demand goods and labor. These publications show that static marginal propensities to consume can be heterogenous across regions, countries, sectors, or categories of goods (Clayton et al., 2018; Jaravel, 2019; Cravino et al., 2020; Argente & Lee, 2021; Huneeus et al., 2021). This argument is captured in production network environments with heterogeneous households by the models from Baqaee & Farhi (2019b, 2022) and Devereux et al. (2023). My model differs from Baqaee & Farhi (2019b) in taking distortions into account and from Baqaee & Farhi (2019b, 2022) in the use of the first-order decomposition for the labor income shares and in the inclusion of a microfounded labor-leisure tradeoff; however, relative to these papers, the most crucial difference is that I represent the production network as separated substochastic matrices, which allows me to introduce new measures of bilateral centrality. Relative to the open economy environment with production networks and an endogenous labor supply from Devereux et al. (2023), my model generalizes its distributional implications from a Cobb-Douglas environment to a generic nonparametric specification. The treatment of the elastic labor supply borrows from the representative household environment in Bigio & La'O (2020).

Finally, in the growth accounting literature opened by Solow (1957), and developed by Domar (1961); Hulten (1978); Jorgenson et al. (1987); Hall & Diamond (1990); Basu & Fernald (2002); Petrin & Levinsohn (2012); Osotimehin (2019); Baqaee & Farhi (2020), I develop a segmentation of the allocative component from the aggregate TFP that depends on the variations in distortions, the demand structure, and the consumption distribution. The aggregate and distributional decomposition of the effects from the reallocation of resources relates my model with the misallocation literature (Restuccia & Rogerson, 2008; Hsieh & Klenow, 2009).

Layout

The structure of the paper is as follows. Section 2 introduces the multisector input-output model with heterogeneous households and distortions. Section 3 characterizes the equilibrium and the centrality measures. Section 4 presents sufficient statistics for aggregate TFP and household-level PTTs under a nonparametric environment. Section 5 characterizes how the equilibrium and the sufficient statistics would change if a constrained social planner centralized households' decisions. Section 6 describes the data and the quantitative implementation for aggregate TFP and distributional PTTs. Section 7 introduces a parametric setting that disciplines endogenous variations. Section 8 presents the most simple economy for which the distributional effects on TFP will show up. Section 9 evaluates the aggregate and distributional effects from a manifold of sectoral shocks in productivities and markdowns. Section 10 identifies four general classes of economies for which there are zero first-order distributional reallocation gains on TFP, which allows me to understand the economic structure and primitives necessary for variations in the income and consumption distributions that allow for non-technological growth. Section 11 concludes.

2 General Framework

In this section, I set up a static nonparametric general equilibrium model with constant-returns-to-scale (CRS) for economies with N sectors and H types of households. Sector $i \in \mathscr{N} = \{1, \cdots, N\}$ consists of two types of firms: (i) a unit mass of monopolistic competitive firms indexed by $z_i \in [0, 1]$ producing differentiated goods, and (ii) a perfectly competitive producer that aggregates the industry's differentiated goods into a uniform sectoral good that can be consumed by households or used by other firms as intermediate inputs. Firms differ along three dimensions; first, monopolistic firms across sectors operate under different technologies; second, monopolistic firms within sectors have heterogeneous input demand; and third, sectoral aggregators face different distortions. Households of type $h \in \mathscr{H} = \{1, \cdots, H\}$ consume sectoral goods using the income received from their endogenous labor supply and rebated profits. Households differ along three dimensions; first, their preferences; second, a type-specific horizontally differentiated labor supply; and third, the composition of their equity portfolio. Financial markets are incomplete, and households cannot cross-insure their idiosyncratic income shocks.

2.1 Production

Monopolistic firms within sectors produce differentiated goods using the same technology. The production for firm z_i in sector i follows

$$y_{z_i} = A_i \, Q_i \left(L_{z_i}, X_{z_i} \right), \quad L_{z_i} = A_i^\ell \, Q_i^\ell \left(\left\{ A_{ih}^\ell \, \ell_{z_i h} \right\}_{h \in \mathscr{H}} \right), \quad X_{z_i} = A_i^x \, Q_i^x \left(\left\{ A_{ij}^x \, x_{z_i j} \right\}_{j \in \mathscr{N}} \right), \tag{1}$$

where y_{z_i} stands for output, A_i is the sector-specific Hicks-neutral productivity term. L_{z_i} is the labor composite that depends on the productivity A_i^ℓ. $\ell_{z_i h}$ is the amount of labor hired from household h and is influenced by the productivity A_{ih}^ℓ. X_{z_i} is the intermediate input composite that depends on the productivity A_i^x. $x_{z_i j}$ is the amount of intermediate input goods purchased from sector j and is influenced by the productivity A_{ij}^x.

The technologies $Q_i : \mathbb{R}_+^2 \to \mathbb{R}_+$, $Q_i^\ell : \mathbb{R}_+^H \to \mathbb{R}_+$, and $Q_i^x : \mathbb{R}_+^N \to \mathbb{R}_+$ are neoclassical and satisfy the following regularity conditions: they are positive, finite, and for the set of labor types and intermediate inputs for which there is effective demand, they are monotonically increasing, twice continuously differentiable, strictly concave, and the Inada conditions hold.

The profits for firms z_i are given by

$$\pi_{z_i} = p_{z_i} y_{z_i} - \underbrace{\sum_{h \in \mathscr{H}} w_h \, \ell_{z_i h}}_{= p_{z_i}^\ell L_{z_i}} - \underbrace{\sum_{j \in \mathscr{N}} p_j \, x_{z_i j}}_{= p_{z_i}^x X_{z_i}}, \tag{2}$$

where p_{z_i} is the price of its output, $p_{z_i}^\ell$ is the price for the labor composite, $p_{z_i}^x$ is the price for the intermediate input composite, w_h is the wage received by households of type h, and p_j is the market price for the good produced by the competitive aggregator in sector j.

The competitive firm in sector i guarantees a homogeneous good by aggregating sectoral production using the following CES production function

$$y_i = \left(\int {y_{z_i}}^{\mu_i} \, dz_i \right)^{\frac{1}{\mu_i}}, \tag{3}$$

where $\mu_i \leq 1$ stands for the sector-specific markdown, and y_{z_i} represents the demand of goods produced by firm z_i. The aggregator takes prices as given and maximizes profits given by $\bar{\pi}_i = p_i y_i - \int p_{z_i} y_{z_i}\, dz_i$.

2.2 Households

Households of type h share the preference utility function $U_h(C_h, L_h)$, where C_h stands for real consumption, and L_h for the labor supply. The utility $U_h : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfies the following regularity conditions: $U_{C_h} > 0$, $U_{L_h} \leq 0$, twice continuously differentiable, strictly concave, and the Inada conditions hold. The composite real consumption $C_h = Q^c_h\left(\{C_{hi}\}_{i \in \mathscr{N}}\right)$ depends on the final consumption C_{hi} of goods from sector i. The consumption aggregation technology $Q^c_h : \mathbb{R}^N_+ \to \mathbb{R}_+$ is neoclassical: positive, finite, homogeneous of degree one, and for the set of goods for which there is effective final demand, it is monotonically increasing, twice continuously differentiable, strictly concave, and the Inada conditions hold.

Each household is infinitesimal, and for this reason, they take prices and wages as given. Consequently, for any two households with type h, their choices are equivalent, and the notation of the model becomes simpler by assuming a type-specific representative household with a budget constraint given by

$$E_h = p^c_h C_h = \sum_{i \in \mathscr{N}} p_i C_{hi} \leq J_h + \Pi_h, \qquad \text{and} \qquad \Pi_h = \sum_{i \in \mathscr{N}} \kappa_{ih} \left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i \right). \tag{4}$$

Expenditure E_h must not be greater than income; the latter includes labor income $J_h = w_h L_h$, and dividend income Π_h. Households of type h own a fraction κ_{ih} of the firms in sector i.

2.3 Market Clearing

For this economy, the technologies, productivities, markdowns, and ownership distributions are primitives. Monopolistic competition is the only source of market imperfections. These distortions reallocate resources and imply no wasted resources. Hence, the goods market clearing is given by

$$y_i = \sum_{h \in \mathscr{H}} C_{hi} + \sum_{j \in \mathscr{N}} x_{ji} \quad \forall i \in \mathscr{N}, \tag{5}$$

where $x_{ji} \equiv \int x_{z_j i}\, dz_j$ is the total amount of intermediate inputs from sector i bought by all monopolistic firms in sector j. Labor market clearing requires $L_h = \ell_h \ \forall h \in \mathscr{H}$, with $\ell_h = \sum_{i \in \mathscr{N}} \int \ell_{z_i h}\, dz_i$.

2.4 Remarks

This environment also applies to the following three generalizations. First, following McKenzie (1959), economies with variable (increasing or decreasing) return to scale can be handled by appropriately introducing producer-specific fixed entrepreneurial factors in a constant return model. Second, without loss of generality, the model and the following results apply to any production factor with endogenous or exogenous supply, not only labor. Finally, the effects of markdowns are isomorphic to other sector-specific distortions that deviate the system of prices from its first-best solution, such as taxes and financial constraints.

A potential limitation of my model is that I assume segmentation of the labor supply across types of households. The parsimony from this premise allows me to bypass three problems. First, I do not need to consider an ownership matrix that specifies the factor share supplied by each household type. Second, I do not need to consider the cross-elasticities in preferences that arise from the supply of multiple factors by the same household. Third, I can abstract from strategic complementarities between multiple types of households in the supply of the same factor.

3 Equilibrium, Centralities, and Information Theory

In this section, first, I characterize the equilibrium for this economy. Second, I introduce measures of bilateral centrality across firms and households, and measures of aggregate centrality that portray each firm or household's role in the economy. Third, I explain how the concept of relative entropy borrowed from information theory serves as a measure of statistical distance between distributions. This section is essential to understand the first-order approximations that make up the main contribution of this paper.

3.1 Equilibrium Characterization

Let $e \equiv (\mathscr{A}, \mu, \kappa)$ represent the aggregate state, and $\mathscr{E}$ denote the measurable collection of all possible realizations for this state. The matrix $\mathscr{A} \equiv (A, A_\ell, A_x, \underline{A}_\ell, \underline{A}_x)$ collects all productivity measures,[1] and sectoral markdowns are captured by $\mu \equiv (\mu_1, \cdots, \mu_N)'$. The equity matrix $\kappa \equiv (\kappa_1, \cdots, \kappa_N)'$ of size $N \times H$ contains the ownership distribution of firms in sector i represented by the vector $\kappa_i \equiv (\kappa_{i1}, \cdots, \kappa_{iH})'$, with $\kappa_i' \mathbb{1}_H = 1$, and where $\mathbb{1}_H$ is an H sized vector of ones.

For this economy, a mapping of the realization of the aggregate state to an allocation $\vartheta = (\vartheta(e))_{e \in \mathscr{E}}$ and the price system $\rho = (\rho(e))_{e \in \mathscr{E}}$ is represented by the set of functions

$$\vartheta(e) \equiv \left\{ \left\{ \left(y_{z_i}(e), \{\ell_{z_i h}(e)\}_{h \in \mathscr{H}}, \{x_{z_i j}(e)\}_{j \in \mathscr{N}} \right)_{z_i \in [0,1]}, y_i(e), \{C_{hi}(e)\}_{h \in \mathscr{H}} \right\}_{i \in \mathscr{N}}, \{C_h(e), L_h(e)\}_{h \in \mathscr{H}} \right\},$$

$$\rho(e) \equiv \left\{ \left\{ \left(p_{z_i}(e), p^\ell_{z_i}(e), p^x_{z_i}(e) \right)_{z_i \in [0,1]}, p_i(e) \right\}_{i \in \mathscr{N}}, \{w_h(e), p^c_h(e)\}_{h \in \mathscr{H}} \right\}.$$

Definition 1. For any realization of the aggregate state e in the state space $\mathscr{E}$, an equilibrium is the combination of an allocation and a price system (ϑ, ρ) such that:

(i) given wages $\{w_h(e)\}_{h \in \mathscr{H}}$ and prices $\{p_j(e)\}_{j \in \mathscr{N}}$, monopolistically competitive firms' labor $\{\ell_{z_i h}(e)\}_{h \in \mathscr{H}}$ and intermediate input demand $\{x_{z_i j}(e)\}_{j \in \mathscr{N}}$, output $y_{z_i}(e)$, and price $p_{z_i}(e)$

[1] $A \equiv (A_1, \cdots, A_N)'$, $A_\ell = (A^\ell_1, \cdots, A^\ell_N)'$, $A_x \equiv (A^x_1, \cdots, A^x_N)'$, $\underline{A}_\ell = (\underline{A}^\ell_1, \cdots, \underline{A}^\ell_N)'$, $\underline{A}_x = (\underline{A}^x_1, \cdots, \underline{A}^x_N)'$, $\underline{A}^\ell_i = (A^\ell_{i1}, \cdots, A^\ell_{iH})'$, and $\underline{A}^x_i = (A^x_{i1}, \cdots, A^x_{iN})'$.

maximize their profits;

(ii) given prices $[p_{z_i}(e)]_{z_i\in[0,1]}$, aggregator firms' good demand $[y_{z_i}(e)]_{z_i\in[0,1]}$, and output $y_i(e)$ maximize their profits;

(iii) given prices $\{p_i(e)\}_{i\in\mathscr{N}}$ and wages $\{w_h(e)\}_{h\in\mathscr{H}}$, households' consumption $\{C_{hi}(e)\}_{i\in\mathscr{N}}$ and labor supply $L_h(e)$ maximize utility while satisfying their budget constraints;

(iv) goods and labor markets clear.

Proposition 1. The set of functions (ϑ, ρ) are an equilibrium if and only if the following set of conditions are jointly satisfied

$$\begin{aligned} &\frac{\partial C_h(e)/\partial C_{hj}(e)}{\partial C_h(e)/\partial C_{hi}(e)} = \mu_i(e)\left(\frac{y_i(e)}{y_{z_i}(e)}\right)^{1-\mu_i(e)}\frac{\partial y_{z_i}(e)}{\partial x_{z_ij}(e)} \quad \forall i,j\in\mathscr{N},\ \forall z_i\in[0,1], \\ &\forall h\in\mathscr{H},\ \text{and } \forall e\in\mathscr{E} \text{ such that } C_{hi}(e)>0,\ C_{hj}(e)>0,\ \text{and } x_{z_ij}(e)>0, \end{aligned} \tag{6}$$

$$\begin{aligned} &-\frac{w_b(e)}{w_h(e)}\frac{U_{L_h}}{U_{C_{hi}}} = \mu_i(e)\left(\frac{y_i(e)}{y_{z_i}(e)}\right)^{1-\mu_i(e)}\frac{\partial y_i(e)}{\partial \ell_{ib}(e)} \quad \forall i\in\mathscr{N},\ \forall z_i\in[0,1], \\ &\forall h,b\in\mathscr{H},\ \text{and } \forall e\in\mathscr{E} \text{ such that } C_{hi}(e)>0,\ \text{and } \ell_{ib}(e)>0, \end{aligned} \tag{7}$$

and resource constraints

$$\begin{aligned} y_i(e) &= \sum_{h\in\mathscr{H}} C_{hi}(e) + \sum_{j\in\mathscr{N}}\int x_{z_ji}(e)\,dz_j \quad \forall i\in\mathscr{N}, \\ \text{and} \qquad L_h(e) &= \sum_{i\in\mathscr{N}}\int \ell_{z_ih}(e)\,dz_i \quad \forall h\in\mathscr{H}. \end{aligned} \tag{8}$$

Proposition 1 identifies the set of equilibrium allocations. In equation (6), for a firm z_i, the markdown-adjusted marginal productivity from using the good from sector j as an intermediate input has to equate for every household the marginal rate of substitution between goods i and j.[2] In equation (7), for a firm z_i, the markdown-adjusted marginal productivity from using the labor supplied by households of type b, has the equate for every household a wage-adjusted marginal rate of substitution between the consumption of the good from sector i and their labor supply.

Notice that in the set of conditions captured by equation (7), the only thing that is necessary for the existence of an equilibrium relationship between the labor demand from firm z_i and the labor supply from households of type h, is the consumption from the latter of the goods supplied by sector i. Whenever firm z_i hires households of type b, and $b\neq h$, the differential wage adjustment w_b/w_h arises in these equilibrium conditions. This wage ratio is a point of difference with Bigio & La'O's (2020) representative-household economy, where they only consider the endogenous supply of one factor. For households of type h, a higher w_b/w_h is isomorphic to an increase in the marginal rate of substitution between consumption and labor supply, and in equilibrium, it requires a higher marginal productivity

[2] In the right-hand side of equation (6), notice that for fixed marginal rates of substitution, and under no variation in the relative production of firms within a sector (i.e., $y_i(e)/y_{z_i}(e)$ fixed $\forall z_i\in[0,1]$), an increase in $\mu_i(e)$ has a heterogeneous effect across firms in sector i. On the one hand, for firms with relatively low levels of production (more precisely $1<\mu_i\,log\,(y_i/y_{z_i})$) the markdown increase forces a reduction in the demand for intermediate inputs. On the other hand, it increases the demand for intermediate inputs for the rest of the firms. Furthermore, notice that for fixed marginal rates of substitution and markdown $\mu_i(e)$, an increase in $y_i(e)/y_{z_i}(e)$ requires a reduction in the demand for intermediate inputs. The same analysis holds for equation (7).

in firm z_i of the labor supplied by households of type b. Additionally, there is an isomorphism between distortionary markdown increases and positive productivity shocks in equations (6) and (7): both will increase the markdown-adjusted marginal productivities from labor and intermediate goods.

Furthermore, a relevant technicality is that Proposition 1 does not require final consumption in each sector. The usual assumption for this type of proof in the production network literature is that $\forall i \in \mathscr{N}$, the representative household's consumption technology satisfies $\partial C / \partial C_i > 0$ (see Bigio & La'O (2020) and La'O & Tahbaz-Salehi (2022)). The equivalent assumption under heterogeneous households is that $\forall i \in \mathscr{N}$, there $\exists h \in \mathscr{H}$ such that $\partial C_h / \partial C_{hi} > 0$, but this assumption does not match the empirical input-output tables, where it is not uncommon to find sectors for which there is no direct registered final consumption, e.g., oil and gas extraction. The less stringent assumption that I make instead is that $\forall h \in \mathscr{H}$, there $\exists i \in \mathscr{N}$ such that for all the firms in this sector, it is possible to establish a direct or indirect demand of labor supplied by workers of type h.

To make the notation cleaner, the definitions and implementation of the model in the following sections are conditional in a specific aggregate state $e \in \mathscr{E}$, e.g., $\mu(e)$ is portrayed by μ. Finally, I will abstract from within sector firm heterogeneity by imposing the assumption of symmetry, i.e., $\ell_{ih} = \ell_{z_i h}$, and $x_{ij} = x_{z_i j}$ $\forall z_i \in [0,1]$, $\forall i, j \in \mathscr{N}$ and $\forall h \in \mathscr{H}$.[3] For this reason, I will refer indistinguishably to firm z_i as firm i.

3.2 Measures of Centrality

For the following measures, downstream or cost centrality refers to the propagation of costs from the supply of labor or intermediate inputs through supply chains, and upstream or revenue centrality refers to the propagation of money flows from the demand for labor and goods through payment chains. Table 1 summarizes the direct centralities and Table 2 the network centralities.

3.2.1 Direct Centralities

The vectors $\omega_\ell \equiv \left(\omega_1^\ell, \cdots, \omega_N^\ell\right)'$ and $\omega_x \equiv \left(\omega_1^x, \cdots, \omega_N^x\right)'$ portray the direct cost centralities from composites. Its elements $\omega_i^\ell \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_i^\ell} = \frac{p_i^\ell L_i}{c_i(\vartheta,\rho)}$ and $\omega_i^x \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_i^x} = \frac{p_i^x X_i}{c_i(\vartheta,\rho)}$ capture respectively firm i's cost elasticities to p_i^ℓ and p_i^x, and in equilibrium they equal the cost share of the labor and intermediate input composites. For this reason, $\omega_i^\ell + \omega_i^x = 1$.

The matrices $\widetilde{\Omega}_\ell$ and $\widetilde{\Omega}_x$ depict direct labor and intermediate input downstream centralities. Its elements $\widetilde{\Omega}_{ih}^\ell \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log w_h} = \frac{w_h \ell_{ih}}{c_i(\vartheta,\rho)}$ and $\widetilde{\Omega}_{ij}^x \equiv \frac{\partial \log c_i(\vartheta,\rho)}{\partial \log p_j} = \frac{p_j x_{ij}}{c_i(\vartheta,\rho)}$ capture respectively firm i's cost elasticities to w_h and p_j, and in equilibriym they equal the cost share of the labor supplied by households of type h and the good from firm j. The fact that $\sum_{h \in \mathscr{H}} \widetilde{\Omega}_{ih}^\ell + \sum_{j \in \mathscr{N}} \widetilde{\Omega}_{ij}^x = 1$ indicate that all costs come from labor or intermediate inputs.

Using these definitions, I obtain the labor network $\alpha \equiv diag\left(\omega_\ell\right)^{-1} \widetilde{\Omega}_\ell$ and the input-output network

[3] As a consequence $y_i = y_{z_i}$, $p_i = p_{z_i}$, $L_i = L_{z_i}$, and $X_i = X_{z_i}$.

$\mathscr{W} \equiv diag\left(\omega_x\right)^{-1} \widetilde{\Omega}_x$, where $diag$ stands for the diagonal operator. Its elements $\alpha_{ih} \equiv \frac{\partial \, log \, p_i^\ell \, L_i}{\partial \, log \, w_h} = \frac{w_h \, \ell_{ih}}{p_i^\ell \, L_i}$ and $\omega_{ij} \equiv \frac{\partial \, log \, p_i^x \, X_i}{\partial \, log \, p_j} = \frac{p_j \, x_{ij}}{p_i^x \, X_i}$ capture respectively firm i's composite cost elasticities to w_h and p_j, and in equilibrium they equal the corresponding composites' cost share of the labor supplied by households of type h and the good from firm j. Notice that $\sum_{h \in \mathscr{H}} \alpha_{ih} = 1$ and $\sum_{j \in \mathscr{N}} \omega_{ij} = 1$.

From here, I can define the revenue-based upstream centrality matrices $\Omega_\ell \equiv diag\left(\mu\right) \widetilde{\Omega}_\ell$ and $\Omega_x \equiv diag\left(\mu\right) \widetilde{\Omega}_x$. Since $\mu_i \in (0, 1] \; \forall i \in \mathscr{N}$, $\widetilde{\Omega}_\ell \succcurlyeq \Omega_\ell$ and $\widetilde{\Omega}_x \succcurlyeq \Omega_x$, where $\succcurlyeq$ stands for elementwise greater than or equal to. Its elements $\Omega_{ih}^\ell \equiv \frac{\partial \, log \, S_i}{\partial \, log \, w_h} = \frac{w_h \, \ell_{ih}}{S_i}$ and $\Omega_{ij}^x \equiv \frac{\partial \, log \, S_i}{\partial \, log \, p_j} = \frac{p_j \, x_{ij}}{S_i}$ capture respectively the elasticities of firm i's sales to w_h and p_j, and in equilibrium they equal the sales share of payments for labor supplied by workers of type h and goods from firm j. Additionally, $\Omega_{ih}^\pi = \frac{\kappa_{ih} \, \pi_i}{S_i}$ portrays the equilibrium sales share of firm i's profits rebated back to households of type h. The fact that $\sum_{h \in \mathscr{H}} \Omega_{ih}^\ell + \sum_{j \in \mathscr{N}} \Omega_{ij}^x + \sum_{b \in \mathscr{H}} \Omega_{ib}^\pi = 1$ indicate that all revenue generated by firm i ends as payments for labor, intermediate inputs, or dividends.

Finally, for households, the consumption network $\beta = \left(\beta_1, \cdots, \beta_H\right)'$ contains the vectors $\beta_h \equiv \left(\beta_{h1}, \cdots, \beta_{hN}\right)'$. Its element $\beta_{hi} \equiv \frac{\partial \, log \, E_h}{\partial \, log \, p_i} = \frac{p_i \, C_{hi}}{E_h}$ captures the expenditure elasticity for households of type h to p_i, and in equilibrium they equal the expenditure share on the good supplied by firm i. For this reason $\sum_{i \in \mathscr{N}} \beta_{hi} = 1$.

Table 1: Direct Centralities

Matrix	***Definition***	***In Equilibrium***	***Properties***
ω_ℓ	$\omega_i^\ell \equiv \frac{\partial \, log \, c_i(\vartheta, \rho)}{\partial \, log \, p_i^\ell}$	Cost share of L_i	$\omega_i^\ell + \omega_i^x = 1$
ω_x	$\omega_i^x \equiv \frac{\partial \, log \, c_i(\vartheta, \rho)}{\partial \, log \, p_i^x}$	Cost share of X_i	
$\widetilde{\Omega}_\ell$	$\widetilde{\Omega}_{ih}^\ell \equiv \frac{\partial \, log \, c_i(\vartheta, \rho)}{\partial \, log \, w_h}$	Cost share of ℓ_{ih}	$\sum\limits_{h \in \mathscr{H}} \widetilde{\Omega}_{ih}^\ell + \sum\limits_{j \in \mathscr{N}} \widetilde{\Omega}_{ij}^x = 1$
$\widetilde{\Omega}_x$	$\widetilde{\Omega}_{ij}^x \equiv \frac{\partial \, log \, c_i(\vartheta, \rho)}{\partial \, log \, p_j}$	Cost share of x_{ij}	
$diag\left(\omega_\ell\right) \alpha = \widetilde{\Omega}_\ell$	$\alpha_{ih} \equiv \frac{\partial \, log \, p_i^\ell \, L_i}{\partial \, log \, w_h}$	Cost share of ℓ_{ih} in L_i	$\sum\limits_{h \in \mathscr{H}} \alpha_{ih} = 1$
$diag\left(\omega_x\right) \mathscr{W} = \widetilde{\Omega}_x$	$\omega_{ij} \equiv \frac{\partial \, log \, p_i^x \, X_i}{\partial \, log \, p_j}$	Cost share of x_{ij} in X_i	$\sum\limits_{j \in \mathscr{N}} \omega_{ij} = 1$
β	$\beta_{hi} \equiv \frac{\partial \, log \, E_h}{\partial \, log \, p_i}$	Cost share of C_{hi}	$\sum\limits_{i \in \mathscr{N}} \beta_{hi} = 1$
κ	$\kappa_{ih} \equiv \frac{d \, \Pi_h}{d \, \pi_i}$	Equity share of h in i	$\sum\limits_{h \in \mathscr{H}} \kappa_{ih} = 1$
$\Omega_\ell \equiv diag\left(\mu\right) \widetilde{\Omega}_\ell$	$\Omega_{ih}^\ell \equiv \frac{\partial \, log \, S_i}{\partial \, log \, w_h}$	Share of S_i for ℓ_{ih}	
$\Omega_x \equiv diag\left(\mu\right) \widetilde{\Omega}_x$	$\Omega_{ij}^x \equiv \frac{\partial \, log \, S_i}{\partial \, log \, p_j}$	Share of S_i for x_{ij}	$\sum\limits_{h \in \mathscr{H}} \left(\Omega_{ih}^\ell + \Omega_{ih}^\pi\right) + \sum\limits_{j \in \mathscr{N}} \Omega_{ij}^x = 1$
$\Omega_\pi = diag\left(\mathbb{1}_N - \mu\right) \kappa$	$\Omega_{ih}^\pi = \frac{\kappa_{ih} \, \pi_i}{S_i}$	Share of S_i for Π_h	

3.2.2 Network Adjusted Centralities

The firm-to-firm downstream centrality matrix or cost-based Leontief inverse matrix is given by $\widetilde{\Psi}_x \equiv \left(I - \widetilde{\Omega}_x\right)^{-1} \equiv \sum_{q=0}^{\infty} \widetilde{\Omega}_x^q$. Its element $\widetilde{\psi}_{ij}^x$ captures the centrality of intermediate inputs supplied by firm j on the costs of firm i. Similarly, I define the firm-to-firm upstream centrality matrix or revenue-based Leontief inverse matrix $\Psi_x \equiv (I - \Omega_x)^{-1} \equiv \sum_{q=0}^{\infty} \Omega_x^q$, where its element ψ_{ij}^x represents the revenue share from firm i that through the payment of intermediate input reaches sales of firm j.

The firm-to-consumer downstream centrality matrix is given by $\widetilde{\mathscr{B}} \equiv \beta \, \widetilde{\Psi}_x$. Its element $\widetilde{\mathscr{B}}_{hi} = \sum_{j \in \mathscr{N}} \beta_j \, \widetilde{\psi}_{ji}^x$ captures all direct or indirect paths through which the costs of firm i can reach the expenditure for households of type h. The cost-based sales Domar weight $\widetilde{\lambda}_i = \sum_{h \in \mathscr{H}} \chi_h \, \widetilde{\mathscr{B}}_{hi}$ stands for the average firm-to-consumer centrality from sector i, where $\chi_h = E_h/GDP$ represents the expenditure share for households of type h. Likewise, I define the consumer-to-firm upstream centrality matrix $\mathscr{B} \equiv \beta \, \Psi_x$, where its element $\mathscr{B}_{hi} = \sum_{j \in \mathscr{N}} \beta_j \, \psi_{ji}^x$ represents the share of expenditure from households of type h that through the payment chain reaches the revenue of firm i. The revenue-based sales Domar weight $\lambda_i = \sum_{h \in \mathscr{H}} \chi_h \, \mathscr{B}_{hi} = S_i/GDP$ stands for the average consumer-to-firm centrality towards sector i, and in equilibrium it coincides with the ratio of sales to GDP. These definitions generalize the supplier centrality vector from Baqaee (2018), or the influence vector from Acemoglu et al. (2012), to an environment with heterogeneous households and distortions.

The worker-to-firm downstream centrality matrix is given by $\widetilde{\Psi}_\ell \equiv \widetilde{\Psi}_x \, \widetilde{\Omega}_\ell$. Given that $\sum_{h \in \mathscr{H}} \widetilde{\psi}_{ih}^\ell = 1$, all costs for a firm can be traced back through the production network to some original labor cost. As a consequence, $\widetilde{\psi}_{ih}^\ell$ is the value-added share by workers of type h on the production process of firm i. In the same way, I define the firm-to-worker upstream centrality matrix $\Psi_\ell \equiv \Psi_x \, \Omega_\ell$, where the element ψ_{ih}^ℓ represents the revenue share from firm i that reaches labor income for workers of type h.

The worker-to-consumer downstream centrality matrix is given by $\widetilde{\mathscr{C}} \equiv \beta \, \widetilde{\Psi}_\ell$. Given that $\sum_{b \in \mathscr{H}} \widetilde{\mathscr{C}}_{hb} = 1$, its element $\widetilde{\mathscr{C}}_{hb}$ represents the value-added share for households of type h attributed to workers of type b. The cost-based factor Domar weight $\widetilde{\Lambda}_h = \sum_{b \in \mathscr{H}} \chi_b \widetilde{\mathscr{C}}_{bh}$ stands for the average worker-to-consumer centrality from workers of type h. Consequently, $\widetilde{\Lambda}_h$ is the share of aggregate value-added by their labor. All the costs from this economy originate in labor costs, and for this reason, $\sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h = 1$. Similarly, the consumer-to-worker upstream centrality matrix is given by $\mathscr{C} \equiv \beta \, \Psi_\ell$, where its element $\mathscr{C}_{hb}$ portrays the share of consumption expenditure from households of type h that reaches labor income for workers of type b. The revenue-based factor Domar weight $\Lambda_h = \sum_{b \in \mathscr{H}} \chi_b \, \mathscr{C}_{bh} = J_h/GDP$ stands for the average consumer-to-worker centrality towards workers of type h. In equilibrium Λ_h coincides with the ratio of labor income to GDP.

Cost-based centralities are greater than or equal to revenue-based centralities, i.e., $\widetilde{\Psi}_x \succcurlyeq \Psi_x$, $\widetilde{\mathscr{B}} \succcurlyeq \mathscr{B}$, $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$, $\widetilde{\mathscr{C}} \succcurlyeq \mathscr{C}$, $\widetilde{\lambda} \succcurlyeq \lambda$, and $\widetilde{\Lambda} \succcurlyeq \Lambda$. For this reason, for workers of type h, $\delta_h = \widetilde{\Lambda}_h/\Lambda_h \geq 1$ is a measure of *distortion centrality* that captures how undervalued a worker is in the market. When workers supply their labor to sectors that operate in heavily distorted supply chains, their distortion centrality will be high, and a higher share of their value-added will reach households' income via rebated distortions. For this reason, $M_h = \sum_{b \in \mathscr{H}} \mathscr{C}_{hb} \, \delta_b$ and $F_i = \sum_{h \in \mathscr{H}} \psi_{ih}^\ell \, \delta_h$ capture the average distortion centrality faced by the consumption expenditure from households of type h and the revenue from firms in sector i. For M_h and F_i to be relatively high, it is necessary that the consumer-to-worker $\{\mathscr{C}_{hb}\}_{b\in\mathscr{H}}$ and the firm-to-worker $\{\psi^{\ell}_{ih}\}_{h\in\mathscr{H}}$ centralities are high, and this requires that the demand for goods and inputs is relatively undistorted. For this reason, M_h and F_i will be respectively called household h's *expenditure centrality* and firm i's *revenue centrality.*

Table 2: Network Adjusted Centralities

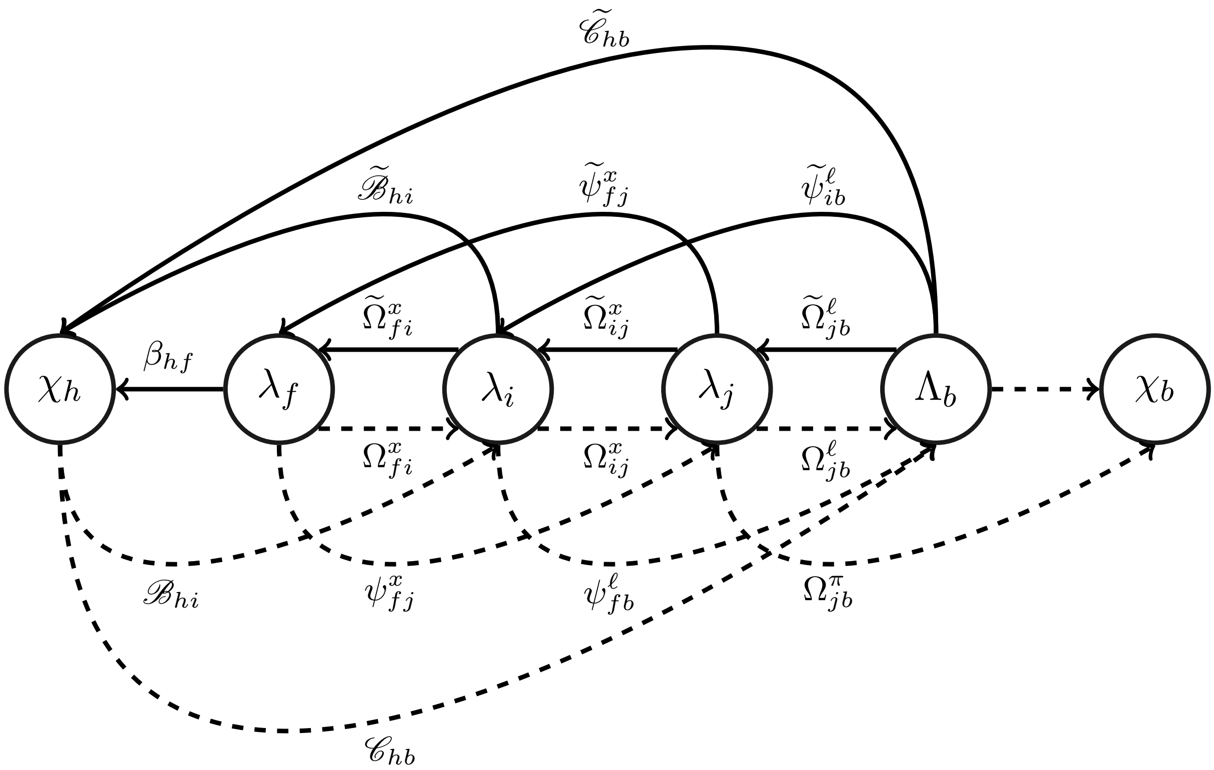
Matrix	***Definition in Equilibrium***	***Properties***
	Downstream or Cost-Based Centralities	
$\widetilde{\Psi}_x = \left(I - \widetilde{\Omega}_x\right)^{-1}$	$\widetilde{\psi}^x_{ij}$ *firm-to-firm* Centrality of j in the costs of i	
$\widetilde{\mathscr{B}} = \beta\,\widetilde{\Psi}_x$	$\widetilde{\mathscr{B}}_{hi}$ *firm-to-consumer* Centrality of i in the costs of h	
$\widetilde{\Psi}_\ell = \widetilde{\Psi}_x\,\widetilde{\Omega}_\ell$	$\widetilde{\psi}^\ell_{ih}$ *worker-to-firm* Value-added share by h in the production of i	$\sum_{h\in\mathscr{H}} \widetilde{\psi}^\ell_{ih} = 1$
$\widetilde{\mathscr{C}} = \beta\,\widetilde{\Psi}_\ell$	$\widetilde{\mathscr{C}}_{hb}$ *worker-to-consumer* Value-added share by b in the consumption of h	$\sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} = 1$
$\widetilde{\lambda} = \widetilde{\mathscr{B}}'\,\chi$	$\widetilde{\lambda}_i$ *cost-based Domar weight* Share of aggregate value-added that passes through i	$\sum_{i\in\mathscr{N}} \omega^\ell_i\,\widetilde{\lambda}_i = 1$
$\widetilde{\Lambda} = \widetilde{\mathscr{C}}'\,\chi$	$\widetilde{\Lambda}_h$ *cost-based labor share* Share of aggregate value-added generated by h	$\sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h = 1$
	Upstream or Revenue-Based Centralities	
$\Psi_x = (I - \Omega_x)^{-1}$	ψ^x_{ij} *firm-to-firm* Share of S_i that reaches S_j	
$\mathscr{B} = \beta\,\Psi x$	$\mathscr{B}_{hi}$ *consumer-to-firm* Share of E_h that reaches S_i	
$\Psi_\ell = \Psi_x\,\Omega_\ell$	ψ^ℓ_{ih} *firm-to-worker* Share of S_i that reaches J_h	$\psi^\ell_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih}$
$\mathscr{C} = \beta\,\Psi_\ell$	$\mathscr{C}_{hb}$ *consumer-to-worker* Share of E_h that reaches J_h	$\mathscr{C}_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}$
$\lambda = \mathscr{B}'\,\chi$	λ_i *revenue-based Domar weight* Aggregate sales share S_i/GDP	$\sum_{i\in\mathscr{N}} \lambda_i \geq 1$
$\Lambda = \mathscr{C}'\,\chi$	Λ_h *revenue-based labor share* Labor income share J_h/GDP	$\Gamma = \sum_{h\in\mathscr{H}} \Lambda_h \leq 1$
$\chi = (\Omega_\ell + \Omega_\pi)'\,\lambda$	χ_h *expenditure share* Consumption expenditure share χ_h/GDP	$\sum_{h\in\mathscr{H}} \chi_h = 1$
	Other Definitions	
$\delta = diag\,(\Lambda)^{-1}\,\Lambda$	δ_h *distortion centrality* Measure for how undervalue is L_h	$\delta_h = \widetilde{\Lambda}_h/\Lambda_h$
$M = \mathscr{C}\,\delta$	M_h *expenditure centrality* Average distortion centrality faced by E_h	$M_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}\,\delta_b$
$F = \Psi_\ell\,\delta$	F_i *revenue centrality* Average distortion centrality faced by S_i	$F_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih}\,\delta_h$

Additionally, for households of type h and firm i, I will respectively use $\mathscr{C}_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}$ and $\psi_i^\ell = \sum_{h\in\mathscr{H}} \psi_{ih}^\ell$ to capture their payment centrality, i.e., the share of their expenditure that reaches households' income via labor income. Notice that the cost-based equivalent for $\mathscr{C}_h$ and ψ_i^ℓ are equal to one, which implies that these measures will shrink as the influence from distortions rises.

Finally, in equilibrium, the expenditure shares are connected to the revenue-based Domar weights via the following relationship $\chi_h = \Lambda_h + \sum_{i\in\mathscr{N}} \Omega_{ih}^\pi \lambda_i$, and by definition $\sum_{h\in\mathscr{H}} \chi_h = 1$.

3.2.3 A Diagrammatic Recap

Figure 1: Measures of Centrality



Notes: Continuous and dashed arrows represent the cost-based and revenue-based centrality measures, respectively.

Figure 1 illustrates the centrality measures for firms $f, i, j \in \mathscr{N}$ and households $h, b \in \mathscr{H}$. Household b supplies labor to firm j, firm j supplies intermediate inputs to firm i, firm i supplies intermediate inputs to firm f, and firm f supplies goods to household h. Firm f does not demand intermediate inputs from firm j, but it is exposed to its costs through the demand and supply of intermediate inputs from firm i; the indirect linkages between f and j are captured by $\widetilde{\psi}_{fj}^x$ and ψ_{fj}^x. Firm i does not demand labor from worker b, but it is exposed to its costs through the demand of labor and supply of intermediate inputs from firm j; the indirect linkages between i and b are captured by $\widetilde{\psi}_{ib}^\ell$ and ψ_{ib}^ℓ. Household h does not demand goods from firm i, but it is exposed to its costs through the demand of intermediate inputs and supply of final goods from firm f; the indirect linkages between h and i are captured by $\widetilde{\mathscr{B}}_{hi}$ and $\mathscr{B}_{hi}$. Household h is exposed to the labor costs from worker b through the demand and supply from firms f, i, and j; the indirect linkages between h and b are captured by $\widetilde{\mathscr{C}}_{hb}$ and $\mathscr{C}_{hb}$. Finally, income for household b comes from the labor compensation and profits from firm j which are captured by Ω^{ℓ}_{jb} and Ω^{π}_{jb}. This is not an exhaustive list for all the relationships that characterize this economy; for example, it ignores $\widetilde{\mathscr{C}}_{bh}$ and $\mathscr{C}_{bh}$.

3.2.4 Networks as a Markov Chain

Alternatively, the transition matrix that represents the downstream probabilities of cost propagation between agents (firms and households) in a Markov chain, and its corresponding generalized downstream Leontief inverse matrix that gathers the effects from all of the previous cost-based centrality matrices are given by

$$\widetilde{\Omega} = \begin{pmatrix} \widetilde{\Omega}_x & \widetilde{\Omega}_\ell \\ \beta & 0 \end{pmatrix}, \qquad \left(I - \widetilde{\Omega}\right)^{-1} = \begin{pmatrix} \widetilde{\Psi}_x + \widetilde{\Psi}_\ell \left(I - \widetilde{\mathscr{C}}\right)^{-1} \widetilde{\mathscr{B}} & \widetilde{\Psi}_\ell \left(I - \widetilde{\mathscr{C}}\right)^{-1} \\ \left(I - \widetilde{\mathscr{C}}\right)^{-1} \widetilde{\mathscr{B}} & \left(I - \widetilde{\mathscr{C}}\right)^{-1} \end{pmatrix}.$$

Instead of using the absorbing Markov chain, I work independently with the substochastic matrices.[4] In this sense, my notation is closer to the models from Hulten (1978), Long & Plosser (1983), Acemoglu et al. (2012, 2016), and Bigio & La'O (2020), with the added complexity of accounting for consumption and income heterogeneity at the household level. My decision to operate with substochastic matrices differs from Baqaee & Farhi (2019a,b, 2020, 2022), where the Markov transition matrix is the production network, and its Leontief inverse lumps together all of the measures of centrality previously introduced. The segmentation of the production network in its different components allows me to analytically separate the different channels for the propagation of shocks through the economic network and introduce bilateral measures for each firm or household's centrality on every other firm or household across the economy.

3.3 Information Theory

This subsection introduces the variation of the relative entropy as a measure of distance between distributions. I will use it in Section 4 to characterize the aggregate and distributional effects from variations in the income distribution. Skipping this section will not affect the reader's understanding of the model's central mechanism.

A discrete random variable $\mathcal{Q}$ with G mutually exclusive events is distributed according to the probability vector $q = [q_1, \cdots, q_G]'$. The natural units of information carried by an event g are given by $I\left(f|\mathcal{Q}\right) = -\,log\, q_g$.[5] Shannon's (1948) entropy captures the average amount of information conveyed by a random draw, or similarly the expected surprise from observing an event,[6] and is given

[4] The upstream probabilities of money flow between agents are portrayed by the Markov chain $\Omega = \left(\begin{smallmatrix} \Omega_x & \Omega_\ell + \Omega_\pi \\ \beta & 0 \end{smallmatrix}\right)$.

[5] This function satisfied the two properties. First, it is decreasing, i.e., $q_a < q_b$ implies that $I\left(a|\mathcal{Q}\right) > I\left(b|\mathcal{Q}\right)$. Second, it is additive, i.e., $I\left(a\, b|\mathcal{Q}\right) = I\left(a|\mathcal{Q}\right) + I\left(b|\mathcal{Q}\right)$. Monotonicity captures the idea that less probable events convey more information, and additivity means that combined information is the sum of separate information.

[6] In information theory, maximum entropy is equivalent to maximal surprise under current knowledge. For the case

by $H(q) = \sum_{g=1}^{G} q_g \, I(g|\mathcal{Q}) = -\sum_{g=1}^{G} q_g \log q_g$. The excess surprise from using the distribution $\widetilde{q}$ instead of the true distribution q is given by the Kullback-Leibler (KL) divergence or relative entropy $\mathcal{K}(q|\widetilde{q}) = -\sum_{g=1}^{G} q_g \log(\widetilde{q}_g/q_g)$. From Gibbs's inequality $\mathcal{K}(q|\widetilde{q}) \geq 0$, which captures the idea that using an incorrect probability model $\widetilde{q}$ will introduce a positive bias in the measure of average expected information conveyed by a random draw. This excess surprise measures the statistical distance between the two distributions q and $\widetilde{q}$. However, the KL divergence is not a metric, as it does not satisfy the properties of symmetry and triangle inequality.

The first-order variation on the relative entropy when the distribution $\widetilde{q}$ changes are given by

$$d\,\mathcal{K}(q|\widetilde{q}) = -\sum_{g=1}^{G} q_g \, d\log \widetilde{q}_g.$$

When $q = \widetilde{q}$ this implies that $d\,\mathcal{K}(q|\widetilde{q}) = 0$, which reflects that the average expected excess information from changing the model distribution $\widetilde{q}$ around the true distribution does not add any excess surprise up to a first-order. In other words, the information conveyed by $\widetilde{q}$ satisfies an envelope condition around q.[7]

4 Aggregate and Distributional Accounting

In this section, I derive the nonparametric ex-post sufficient statistics necessary to characterize the first-order variations in prices, labor income shares, labor wedges, aggregate TFP, and household-level terms of trade. I call these measures ex-post because they assume that the necessary variations are observable and do not depend on underlying model primitives. First, I present the price variation in response to exogenous shocks and show that these effects propagate downstream through the cost of intermediate and final goods. Second, I characterize the first-order variation for the labor income shares. Third, I decompose the first-order variation for aggregate TFP and the household-level positional terms of trade (PTT) and establish a connection with the labor income shares that allow me to decompose the aggregate and distributional effects from the endogenous reallocation of labor across firms into variations of (i) exogenous distortions, (ii) endogenous variations in the expenditure distribution keeping the demand structure fixed, and (iii) endogenous recomposition in the demand structure from firms and households in response to relative price variations while keeping the expenditure distribution fixed.

4.1 Price Variation

Proposition 2 captures the network-adjusted response of prices to supply shocks. These shocks propagate downstream through the costs of intermediate inputs and final goods, and the cost-based firm-to-firm and firm-to-consumer centrality measures capture their magnitude.

of distributions with no prior information, the uniform distribution maximizes the entropy.

[7]For this envelope condition it is required that when $\widetilde{q}$ changes to $\widetilde{q}^*$, the new distribution satisfies $\mathbb{1}_G' \widetilde{q}^* = 1$.

Proposition 2. The change in sector i's prices and household h's price index in response to productivity, markdown, and factor cost shocks are, to a first-order,

$$\begin{aligned}
d\log p_i^\ell &= -d\log A_i^\ell - \sum_{h\in\mathscr{H}} \alpha_{ih}\left(d\log A_{ih}^\ell - d\log w_h\right), \\
d\log p_i^x &= -d\log A_i^x - \sum_{j\in\mathscr{N}} \omega_{ij}\left(d\log A_{ij}^x - d\log p_j\right), \\
d\log p_i &= -\sum_{j\in\mathscr{N}} \widetilde{\psi}_{ij}^x \left(d\log \mathscr{A}_j + d\log \mu_j\right) + \sum_{h\in\mathscr{H}} \widetilde{\psi}_{ih}^\ell \, d\log w_h, \\
d\log p_h^c &= -\sum_{i\in\mathscr{N}} \widetilde{\mathscr{B}}_{hi} \left(d\log \mathscr{A}_i + d\log \mu_i\right) + \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \, d\log w_b,
\end{aligned}$$

where $d\log \mathscr{A}_i = d\log A_i + \omega_i^\ell \, d\log A_i^\ell + \omega_i^x \, d\log A_i^x + \sum_{h\in\mathscr{H}} \widetilde{\Omega}_{ih}^\ell \, d\log A_{ih}^\ell + \sum_{j\in\mathscr{N}} \widetilde{\Omega}_{ij}^x \, d\log A_{ij}^x$.

First, non-Hicks neutral productivity shocks directly influence firms' composite bundle prices. Second, firm i's compound measure of productivity $d\log \mathscr{A}_i$ incorporates Hicks-neutral, labor-specific, and input-specific augmenting productivity shocks, and its effect on prices across all firms and households is isomorphic to an increase in the markdown for firm i. Third, labor costs have a direct effect on the labor bundle price that propagates through the supply of intermediate inputs to other firms and finally reaches consumption bundle prices.

4.2 Labor Wedges and the Income Distribution

Theorem 1 portrays the equilibrium characterization of the households' labor supply and the endogenous variation of the labor income distribution. This theorem represents an extension of the labor wedge decompositions from Bigio & La'O (2020) to an environment with heterogeneous households and a distorted equilibrium. For workers of type h, the labor wedge Γ_h gauges how the whole set of economic distortions influences their labor supply decision.

Theorem 1. In equilibrium, the labor supply from households of type h satisfies

$$\frac{U_{L_h}}{U_{C_h}} + \Gamma_h \frac{C_h}{L_h} = 0 \qquad \text{with} \qquad \Gamma_h = \frac{\Lambda_h}{\chi_h}. \tag{9}$$

The change of Λ_h in response to variations in the consumption distribution and consumer-to-worker centralities is, to a first-order,

$$d\Lambda_h = \overbrace{\sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \, d\chi_b}^{\substack{\textit{Distributive} \\ \textit{Income}_h}} + \overbrace{\sum_{b\in\mathscr{H}} \chi_b \, d\mathscr{C}_{bh}}^{\substack{\textit{Income} \\ \textit{Centrality}_h}}, \tag{10}$$

$$\substack{\textit{Income} \\ \textit{Centrality}_h} = \overbrace{\sum_{i\in\mathscr{N}} \psi_{ih}^\ell \sum_{b\in\mathscr{H}} \chi_b \, d\beta_{bi}}^{\substack{\textit{Final Demand} \\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \psi_{ih}^\ell \sum_{j\in\mathscr{N}} \mu_j \lambda_j \, d\widetilde{\Omega}_{ji}^x}^{\substack{\textit{Intermediate Demand} \\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i \lambda_i \, d\widetilde{\Omega}_{ih}^\ell}^{\substack{\textit{Labor Demand} \\ \textit{Recomposition}_h}} + \overbrace{\sum_{i\in\mathscr{N}} \psi_{ih}^\ell \lambda_i \, d\log \mu_i}^{\substack{\textit{Competitive} \\ \textit{Income}_h}}. \tag{11}$$

The decentralized labor wedge Γ_h from equation (9) relates the marginal rate of substitution between consumption and the labor supply with the household's average labor rate of transformation on consumption C_h/L_h. In equilibrium, the decentralized labor wedge equals the share of labor income to consumption expenditure, i.e., J_h/E_h. This wedge is decentralized because each household independently chooses it, and it differs from the centralized labor wedge that in Section 5 is chosen by the constrained social planner. For an economy without distortions, labor compensation is the only source of income and $\Gamma_h = 1$.

Equation (10) divides the first-order variation of the labor income share into changes in the consumption distribution and changes in the consumer-to-worker centralities. First, *distributive income* captures how the revenue share for workers of type h increases as the expenditure share grows for households whose expenditure has a relatively high upstream centrality on their labor income. For example, Λ_h will increase in response to an endogenous redistribution of expenditure from type q to type b households if $\mathscr{C}_{bh} > \mathscr{C}_{qh}$. Second, *income centrality* portrays how the revenue share for workers of type h increases as the consumer-to-worker centralities on their labor income rise.[8]

The income centrality variation collects four different effects. The *final* and *intermediate demand recomposition* characterize the income distribution effects from households' and firms' expenditure reallocation, respectively. These two channels convey that the labor revenue share for workers of type h will increase as the households' consumption patterns or the firms' cost structure shifts towards sectors with a high firm-to-worker centrality on their labor income. For example, Λ_h rises in response to a cost reallocation from sector j to sector i, by any firm or household, if $\psi^{\ell}_{ih} > \psi^{\ell}_{jh}$. The *labor demand recomposition* portrays the influence on the labor income share from higher labor demand; the magnitude of this effect is more prominent for big and relatively undistorted sectors. Finally, the *competitive income* tells us that lower profit margins in a sector will increase the labor income share for workers of type h in a magnitude proportional to the sector's size and the sector's centrality on the labor income of these workers.

4.3 Aggregate Accounting

Theorem 2 characterizes aggregate real output Y in equilibrium and its first-order variation around the equilibrium.

Theorem 2. In equilibrium, real GDP satisfies

$$Y = Q_Y\left(\{C_h\}_{h\in\mathscr{H}}\right) = TFP\ F\left(\{L_h\}_{h\in\mathscr{H}}\right), \tag{12}$$

where TFP captures total factor productivity, and Q_Y and F satisfy $d\log Q_Y/d\log C_h = \chi_h$ and $d\log F/d\log L_h = \widetilde{\Lambda}_h$.

[8] The following variations in relative entropies capture these two insights: $\Lambda_h\, d\mathscr{K}\left(\Lambda_h^{-1}\mathscr{C}'\chi|\chi\right) + \textit{Distributive Income}_h = 0$ and $\Lambda_h\, d\mathscr{K}\left(\Lambda_h^{-1}\mathscr{C}'\chi|\mathscr{C}_{\uparrow h}\right) + \textit{Income Centrality}_h = 0$ with $\mathscr{C}_{\uparrow h} = (\mathscr{C}_{1h}, \cdots, \mathscr{C}_{Hh})'$. This underscores how Λ_h increases as the statistical distances of $\Lambda_h^{-1}\mathscr{C}'\chi$ relative to the distributions χ and $\mathscr{C}_{\uparrow h}$ fall.

The change in Y and TFP are, to a first-order

$$d\log Y = d\log TFP + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \, d\log L_h, \tag{13}$$

$$d\log TFP = Technology + Competitiveness - Misallocation, \tag{14}$$

where

$$Technology = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \, d\log \mathscr{A}_i, \qquad Competitiveness = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \, d\log \mu_i,$$

and *Misallocation* has the following four equivalent definitions

$$1. \quad \overbrace{\sum_{h\in\mathscr{H}} \delta_h \, d\Lambda_h}^{Entropic\ TT}, \quad 2. \quad \overbrace{\sum_{h\in\mathscr{H}} (\delta_h - 1)\, \Lambda_h \, d\log J_h}^{Labor\ Terms\ of\ Trade\ (TT)} - \overbrace{\sum_{i\in\mathscr{N}} \lambda_i \left((1-\mu_i)\, d\log S_i - d\,\mu_i\right)}^{Corporate\ Income},$$

$$3. \quad \overbrace{\sum_{h\in\mathscr{H}} M_h \, d\chi_h}^{Distributive\ TT} + \overbrace{\sum_{h\in\mathscr{H}} \chi_h \sum_{b\in\mathscr{H}} \delta_b \, d\mathscr{C}_{hb}}^{Centrality\ TT},$$

$$\begin{aligned} 4. \quad & \sum_{h\in\mathscr{H}} M_h \, d\chi_h + \overbrace{\sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} F_i \, d\beta_{hi}}^{Final\ Demand\ TT} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i \, \lambda_i \sum_{j\in\mathscr{N}} F_j \, d\widetilde{\Omega}^x_{ij}}^{Intermediate\ Demand\ TT} \\ & + \overbrace{\sum_{i\in\mathscr{N}} \mu_i \, \lambda_i \sum_{h\in\mathscr{H}} \delta_h \, d\widetilde{\Omega}^\ell_{ih}}^{Labor\ Demand\ TT} + \overbrace{\sum_{i\in\mathscr{N}} \lambda_i \, F_i \, d\log \mu_i}^{Competitive\ TT}, \end{aligned}$$

with $\delta_h = \widetilde{\Lambda}_h/\Lambda_h$, $M_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb}\, \delta_b$ and $F_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih}\, \delta_h$.

From equation (12), real GDP in equilibrium has two representations. First, as a CRS function Q_Y that aggregates consumption across households, with elasticities equal to the expenditure shares χ. Second, as the product of TFP, and a CRS function F that aggregates labor with elasticities equal to the value-added weights $\widetilde{\Lambda}$.

Equation (13) segments the output response into a TFP and a factoral component. Equation (14) divides the first-order variation of TFP into three components. First, *technology* captures the direct effect of changes in productivity under a fixed allocation of resources. Second, *competitiveness* portrays the reallocation effects from distortions in the absence of income distribution variations. These two components tell us that in the absence of distributional reallocation, the effects on TFP of productivity and markdown changes in sector i are proportional to its cost-based sales Domar weight $\widetilde{\lambda}_i$. Third, *misallocation* portrays the aggregate efficiency losses from reallocating labor across firms in response to changes in the income distribution. The last two components capture the effects on TFP from the reallocation of labor across firms arising from exogenous variations in distortions and endogenous changes in the labor income shares. For this reason, Baqaee & Farhi (2020) label *competitiveness* – *misallocation* as the variation in *allocative efficiency*. Here, I refrain from using the *efficiency* tag as with endogenous labor, an increase in real GDP is not necessarily welfare-improving. Finally, Hulten's

(1978) theorem holds in the absence of distortions (i.e., $d\log TFP = \lambda' \, d\log \mathcal{A}$), which implies that variations in the labor income distribution and distortions around the first-best equilibrium generate reallocation of resources that are allocative-neutral on output (i.e., $\textit{competitiveness} = \textit{misallocation}$).

Theorem 2 also contains four equivalent definitions for the *misallocation* component, and each one gives us a different intuition about the effects on TFP from changes in the income distribution. All four definitions capture the idea that aggregate labor misallocation rises as workers with high distortion centralities move away from firms that operate in heavily distorted supply chains. Markdowns generate profits that dilute revenue as consumption expenditure flows upstream in a production network. For this reason, upstream firms operating in heavily distorted supply chains receive less revenue and demand less labor than in equivalent economies without distortions. Workers with high distortion centralities are essential for heavily distorted sectors. As the labor demand from firms in heavily distorted supply chains falls, high δ workers move into more efficient forms of production, aggravating aggregate labor misallocation.

In the first definition, the *entropic terms of trade* capture a reduction in the statistical distance between the value-added and the labor income distributions as measured by $-d\,\mathcal{K}\left(\widetilde{\Lambda}|\Lambda\right)$. This result coincides with the main theorem from Baqaee & Farhi (2020). Labor misallocation improves as labor income shifts from high to low distortion centrality workers, or what is equivalent when the distance between the value-added and the labor income distribution rises. The increase in the statistical distance between the value-added and the labor income distributions portrays how high-distortion centrality workers become relatively more affordable, which allows them to reallocate in response to higher labor demand from sectors in heavily distorted supply chains. The vector of *distortion centralities* δ is a sufficient statistic for the effect of labor income distributional variations on TFP.

The second definition segments *misallocation* into changes in the *labor terms of trade* and *corporate income*, which are in terms of nominal variations. The *labor terms of trade* tell us that under fixed aggregate nominal dividends, misallocation worsens when labor income increases, more so for highly undervalued workers. As labor income rises for workers with high distortion centralities, they are becoming relatively more expensive, and the labor demand from firms in heavily distorted supply chains that require them falls, aggravating labor misallocation. The aggregate misallocation increase arising in response to a higher income share for workers of type h is proportional to their income share and distortion centrality.

Corporate income tells us that misallocation falls as dividends increase under a fixed nominal labor income distribution. Aggregate nominal dividends can increase in response to higher sales or lower markdowns. For a fixed nominal labor income distribution, more aggregate dividends generate a higher nominal GDP and increase the distance between the labor compensation and the value added for each worker. All workers are suddenly becoming relatively more affordable, which allows labor demand to increase. The increase in the labor demand is more substantial for firms that operate in relatively inefficient supply chains that force them to produce with high marginal productivities. Hence, workers will reallocate towards inefficient supply chains, ameliorating aggregate labor misallocation.

The argument that aggregate labor misallocation falls as the aggregate corporate income share rises might sound counterintuitive to the reader. Profits are the source of revenue dilution that generates labor misallocation. How is it possible that the cause of the malady can also cure it? For this reason, I want to emphasize two things. First, this argument holds under a fixed nominal labor income distribution. Second, *misallocation* captures only distributional sources of misallocation, while *misallocation* $-$ *competitiveness* represents the total increase in aggregate misallocation. For example, assume a markdown reduction in sector i of 1% such that the nominal labor income and sales distribution are inelastic. In response to this shock, distributional *misallocation* will fall by $\lambda_i\,\mu_i$, and total misallocation will increase by $\widetilde{\lambda}_i - \lambda_i\,\mu_i$, which is strictly positive if $\widetilde{\lambda}_i > \lambda_i$ or $\mu_i < 1$.

Additionally, all the definitions in Theorem 2 come from accounting identities, and normalization relative to the price of a numeraire is unnecessary. A normalization only becomes necessary to discipline the variations of the endogenous variables, but not to characterize the ex-post sufficient statistics, which assume that these variations are readily observable, e.g., in Section 7, we will require a numeraire to solve for the changes in Λ that characterize the entropic terms of trade in response to a shock. This lack of normalization is a point of difference with the comparable theorem from Baqaee & Farhi (2020), who instead assume a fixed nominal GDP. Using nominal GDP as the numeraire creates uncertainty about the fundamental real unit of account, as real GDP will no longer be neutral to pure nominal variations, e.g., Y has to increase as P_Y falls. Their assumption implies that the first and second definitions of *misallocation* are equivalent. Consequently, under their assumption, there would be no effect from measuring misallocation just in terms of nominal labor income changes and ignoring *corporate income*, i.e., *misallocation* $= \sum_{h\in\mathscr{H}} \delta_h \, d\,J_h$. Furthermore, as I will show in Section 7, with an endogenous labor supply, the nominal GDP normalization used by Baqaee & Farhi (2020, 2022) is non-neutral on TFP whenever the substitution and income effects on the labor supply are asymmetric.

The last two definitions require the labor income share variations from Theorem 1. The third definition splits *misallocation* into variations in the consumption distribution and consumer-to-worker centralities. First, the *distributive terms of trade* imply that labor misallocation worsens as expenditure shifts towards households with high *expenditure efficiency*. Consumers of type h have a high *expenditure efficiency* M_h when the dot product of their vector of consumer-to-worker centralities $\mathscr{C}_{\uparrow h} = (\mathscr{C}_{h1}, \ldots, \mathscr{C}_{hH})'$ and the vector of distortion centralities δ is high. High consumer-to-worker centralities imply that the consumption bundle from a household relies heavily on goods produced by relatively undistorted supply chains. Hence, a high M_h implies that households of type h demand goods produced by firms within efficient supply chains that rely heavily on workers essential for firms within inefficient supply chains. Aggregate misallocation increases with χ_h when M_h is high because aggregate expenditure flows towards efficient firms that demand labor from high distortion centrality workers, increasing the labor demand from these firms and reallocating workers from inefficient to efficient supply chains. The vector of *expenditure efficiencies* M is a sufficient statistic for the effect of expenditure distributional variations on TFP. For example, TFP will improve in response to an endogenous redistribution of expenditure from type h to type b households if $M_h > M_b$. Notice that for a representative household economy, the *distributive terms of trade* are always null. Second, the *centrality terms of trade* indicate that misallocation worsens as the consumer-to-worker centralities from a household increase, and the magnitude of this effect is more prominent when it takes place on workers with high distortion centralities. This channel captures the distributional effects on TFP from the endogenous recomposition in the demand structure from firms and households while keeping the expenditure distribution fixed.

Corollary 1. Distributive Neutrality. Endogenous changes in the distribution of consumption expenditure are neutral on TFP if the expenditure efficiency is symmetric across all households, i.e., $M_h = M \ \forall h \in \mathscr{H}$. This condition nests the following economic structures: (i) undistorted economy, i.e., $\mu_i = 1 \ \forall i \in \mathscr{N}$; (ii) symmetric consumption bundles, i.e., $\beta_{hf} = \beta_f \ \forall h \in \mathscr{H}$; (iii) no intermediate inputs and symmetric distortions across firms, i.e., $\omega_i^x = 0$ and $\mu_i = \mu \ \forall i \in \mathscr{N}$; and (iv) no intermediate inputs and sectoral specific labor supply, i.e., $\omega_i^\ell = \alpha_{ii} = 1 \ \forall i \in \mathscr{N}$.

Corollary 1 establishes the condition under which an aggregate production function can disregard changes in the consumption expenditure distribution without introducing first-order biases on TFP. The symmetry in M_h's across households nests a manifold of common economic structures, including environments where the first welfare theorem holds, economies where all households share the same consumption bundle, and models without intermediate inputs, in which either, there is a common distortion or labor supply is sector specific. First, in an efficient economy, all distortion centralities equal 1, and the households' expenditure reaches income only through labor compensation ($\mathscr{C}_h = 1$ $\forall h \in \mathscr{H}$). For any form of heterogeneity in consumption bundles, expenditure efficiency equals 1 for all households. Hence, perturbations in the expenditure distribution might change how money flows through the economy and reallocate workers across sectors, but these effects are neutral on aggregate. Second, when consumption bundles are symmetric, the consumer-to-worker centralities from all households toward the same worker are the same, i.e., $\mathscr{C}_{hb} = d_b \ \forall h \in \mathscr{H}$. Hence, independently of the distortion centrality vector δ, expenditure efficiency is the same for all households. Third, in an economy without intermediate inputs and symmetric distortions, the consumer's marginal rates of substitution across goods are unaffected by markdowns, and there is no labor misallocation. Just as in the first scenario, perturbations in the expenditure distribution might reallocate workers, but these effects are neutral on the aggregate. Finally, there is no labor misallocation for an economy without intermediate inputs and with sector-specific labor supply. These cases prove that consumption bundle heterogeneity and, as a consequence, aggregate non-homotheticity are necessary but not sufficient for the variations in the consumption expenditure distribution to influence TFP.

The last definition separates the *centrality terms of trade* into four different effects that capture endogenous demand recomposition. The *final demand* and *intermediate demand terms of trade* represent how misallocation worsens with an increase in the demand for goods produced by firms with high *revenue efficiency*. Firms in sector i have a high *revenue efficiency* F_i when the dot product of their firm-to-worker centralities and the vector of distortion centralities δ is high. High firm-to-worker centralities imply that the firm faces high markdowns or the intermediate input bundle relies heavily on goods produced by relatively undistorted supply chains. Hence, a high F_i implies that firms of type i produce within relatively efficient supply chains and require, directly or indirectly, workers that are essential for firms within inefficient supply chains. The *labor demand terms of trade* portray how misallocation increases as the demand from high distortion centrality workers from big and relatively undistorted sectors rises. Finally, the *competitive terms of trade* capture the effects on TFP from the reallocation of workers in response to variations in labor demand driven by markdowns. For example, assume a markdown reduction in sector i of 1% such that the expenditure distribution and the final, intermediate, and labor demand terms of trade are inelastic. In response to this shock, distributional *misallocation* will fall by $\lambda_i\, F_i$, and total misallocation will increase by $\widetilde{\lambda}_i - \lambda_i\, F_i$. Contrary to the analogous case in the *corporate income* channel, total misallocation does not necessarily increase. When markdown reduction occurs in sectors with high *revenue efficiency*, the corresponding reduction in labor demand allows workers to move toward firms that operate in distorted supply chains and total misallocation falls.

The revenue efficiencies, distortion centralities, markdowns, and revenue-based Domar weights are sufficient statistics for the four channels captured by the *centrality terms of trade*. Corollary 2 establishes the conditions under which endogenous recompositions in the demand structure from firms and households are neutral on TFP.

Corollary 2. Demand Neutrality.

1. Demand structure variations are neutral on TFP around the first-best equilibrium.
2. Household h's demand variations are neutral on TFP if all of the firms from which it demands final goods share the same expenditure efficiency, i.e., $F_i = F\ \forall i \in \mathscr{N} : \beta_{hi} > 0$.
3. Firm i's demand variations are neutral on TFP if: (a) the firm demands no intermediate inputs and all of its workers have a symmetric distortion centrality, i.e., $\delta_h = \delta\ \forall h \in \mathscr{H} : \widetilde{\Omega}^{\ell}_{ih} > 0$; (b) the firm demands no labor and all of its intermediate input suppliers share the expenditure efficiency, i.e., $F_j = F\ \forall j \in \mathscr{N} : \widetilde{\Omega}^{x}_{ij} > 0$; or (c) the distortion centrality from all of its workers and the expenditure efficiency from all its suppliers are symmetric, i.e., $\delta_h = F_j\ \forall h \in \mathscr{H} : \widetilde{\Omega}^{\ell}_{ih} > 0$ and $\forall j \in \mathscr{N} : \widetilde{\Omega}^{x}_{ij} > 0$.

Corollary 3 segments the *labor terms of trade* into three effects: (i) real income, (ii) consumer price index (CPI), and (iii) real exchange rate (RER). First, the *real labor income effect* captures the aggregate net exposure to real labor income variations. As in its nominal counterpart, the change in household h's real income is proportional to their distortion centrality and labor income share. Consequently, variations on the real labor income for households with unitary distortion centralities are neutral on TFP. Second, the *CPI effect* shows how misallocation rises more in response to increases in the bundle price for households whose expenditure is heavily dependent on corporate income than for households whose expenditure depends mainly on labor income. Finally, the *RER effect* illustrates that for households of type h, a depreciation in their average bilateral real exchange rate ε_h increases misallocation in a magnitude proportional to their expenditure share.[9]

Corollary 3. The variation in the labor terms of trade is given by

$$Labor\ TT = \overbrace{\sum_{h\in\mathscr{H}} (\delta_h - 1)\,\Lambda_h\, d\log \mathcal{G}_h}^{Real\ Income\ effect} + \overbrace{\sum_{h\in\mathscr{H}} (1-\Gamma_h)\,\chi_h\, d\log p^c_h}^{CPI\ effect} + \overbrace{\sum_{h\in\mathscr{H}} \chi_h\, d\log \varepsilon_h}^{RER\ effect},$$

[9]An increase in ε_{hb} captures a depreciation of the bundle of type h relative to type b households.

where $\mathcal{J}_h = J_h/p^c_h$ stands for real income and $d\log\varepsilon_h = \sum_{b\in\mathcal{H}}\widetilde{\mathscr{C}}_{hb}\, d\log\varepsilon_{hb}$ is the average change in the bilateral real exchange rate for households of type h, with $\varepsilon_{hb} = p^c_b/p^c_h$.

4.4 Distributional Accounting

Theorem 3 characterizes household-level real consumption in equilibrium and its first-order variation. For this, I need to introduce the *positional terms of trade* (PTT) as an equilibrium object that captures the efficiency of the labor supply from all workers on the idiosyncratic real consumption bundle for a specific type of household. I use the term *positional* because it depends on the households' location in the production network, and it serves an analogous function to the TFP from Theorem 2. $d\log TFP = \sum_{h\in\mathcal{H}}\chi_h\, d\log PTT_h$ captures the relationship between TFP and PTTs, and shows that TFP growth is the aggregation of idiosyncratic efficiency growth.

Theorem 3. In equilibrium, real consumption for households of type h satisfies

$$C_h = Q^c_h\left(\{C_{hi}\}_{i\in\mathcal{N}}\right) = PTT_h\; f_h\left(\{L_b\}_{b\in\mathcal{H}}\right), \tag{15}$$

where PTT_h captures the positional terms of trade, and f_h satisfies $d\log f_h/d\log L_b = \widetilde{\mathscr{C}}_{hb}$.

The change in C_h and PTT_h are, to a first-order

$$d\log C_h = d\log PTT_h + \sum_{b\in\mathcal{H}}\widetilde{\mathscr{C}}_{hb}\, d\log L_b, \tag{16}$$

$$d\log PTT_h = \mathit{Technology}_h + \mathit{Competitiveness}_h - \mathit{Misallocation}_h, \tag{17}$$

where

$$\mathit{Technology}_h = \sum_{i\in\mathcal{N}}\widetilde{\mathscr{B}}_{hi}\, d\log\mathscr{A}_i, \quad \mathit{Competitiveness}_h = \sum_{i\in\mathcal{N}}\widetilde{\mathscr{B}}_{hi}\, d\log\mu_i,$$

and $\mathit{Misallocation}_h$ has the following four equivalent definitions

$$1.\ \overbrace{\sum_{b\in\mathcal{H}}\delta_{b|h}\, d\Lambda_b}^{\mathit{Entropic}\ TT_h} - d\log\chi_h, \quad 2.\ \overbrace{\sum_{b\in\mathcal{H}}\widetilde{\mathscr{C}}_{hb}\, d\log J_b - \Gamma_h\, d\log J_h}^{\mathit{Labor}\ TT_h} - \overbrace{\sum_{i\in\mathcal{N}}\kappa_{ih}\frac{\lambda_i}{\chi_h}\left((1-\mu_i)\, d\log S_i - d\mu_i\right)}^{\mathit{Corporate\ Income}_h},$$

$$3.\ \overbrace{\sum_{b\in\mathcal{H}}M_{b|h}\, d\chi_b}^{\mathit{Distributive}\ TT_h} + \overbrace{\sum_{b\in\mathcal{H}}\chi_b\sum_{q\in\mathcal{H}}\delta_{q|h}\, d\log\mathscr{C}_{bq}}^{\mathit{Centrality}\ TT_h} - d\log\chi_h,$$

$$4.\ \sum_{b\in\mathcal{H}}M_{b|h}\, d\chi_b + \overbrace{\sum_{b\in\mathcal{H}}\chi_b\sum_{i\in\mathcal{N}}F_{i|h}\, d\beta_{bi}}^{\mathit{Final\ Demand}\ TT_h} + \overbrace{\sum_{i\in\mathcal{N}}\mu_i\,\lambda_i\sum_{j\in\mathcal{N}}F_{j|h}\, d\widetilde{\Omega}^x_{ij}}^{\mathit{Intermediate\ Demand}\ TT_h}$$
$$+\overbrace{\sum_{i\in\mathcal{N}}\mu_i\,\lambda_i\sum_{b\in\mathcal{H}}\delta_{b|h}\, d\widetilde{\Omega}^\ell_{ib}}^{\mathit{Labor\ Demand}\ TT_h} + \overbrace{\sum_{i\in\mathcal{N}}\lambda_i\, F_{i|h}\, d\log\mu_i}^{\mathit{Competitive}\ TT_h} - d\log\chi_h,$$

with $\delta_{b|h} = \widetilde{\mathscr{C}}_{hb}/\Lambda_b$, $M_{b|h} = \sum_{q\in\mathscr{H}} \mathscr{C}_{bq}\, \delta_{q|h}$, and $F_{i|h} = \sum_{q\in\mathscr{H}} \psi^{\ell}_{iq}\, \delta_{q|h}$.

In equation (15), real consumption for households of type h has two representations. First, as a CRS function Q^c_h that aggregates final goods. Second, as the product of PTT_h, and a CRS function f_h that aggregates labor with elasticities equal to the idiosyncratic value-added or worker-to-consumer centralities that reach consumers of type h, i.e., $\widetilde{\mathscr{C}}_{\downarrow h} = \left(\widetilde{\mathscr{C}}_{h1}, \cdots, \widetilde{\mathscr{C}}_{hH}\right)'$. Equation (16) segments the household-level real consumption response into PTT and a factoral component. Equation (17) divides the first-order variation of the PTT for households of type h into three components. Just as in its aggregate counterpart from Theorem 2, for households of type h, $technology_h$ captures the direct effect of changes in productivity under a fixed allocation of resources, and $competitiveness_h$ portrays the direct effect from distortions. These two components tell us that in the absence of distributional reallocation, the effects on PTT_h of productivity and markdown changes in sector i are proportional to the firm-to-consumer cost-based centrality $\widetilde{\mathscr{B}}_{hi}$. Third, $misallocation_h$ represents the endogenous distributional losses in response to changes in the income distribution. The relationship between aggregate *misallocation* and idiosyncratic *misallocation* is represented by $misallocation = \sum_{h\in\mathscr{H}} \chi_h\, misallocation_h$. From here, we can see that the allocative-neutrality from Hulten's (1978) theorem implies that idiosyncratic *misallocation* effects from productivity shocks are zero-sum around the equilibrium without distortions.

Theorem 3 also contains four equivalent definitions for the idiosyncratic *misallocation* component. All four definitions capture the idea that *misallocation* is favorable for households when their expenditure share increases or when workers' new allocation is more favorable for their consumption bundle. Equation (18) represents the total income share variation for households of type h, which captures that the expenditure share increases with labor or corporate income.

$$d\,\chi_h = d\,\Lambda_h + \sum_{i\in\mathscr{N}} \left((1-\mu_i)\left(\lambda_i\, d\,\kappa_{ih} + \kappa_{ih}\, d\,\lambda_i\right) - \kappa_{ih}\,\lambda_i\, d\,\mu_i\right). \tag{18}$$

First, the idiosyncratic *entropic terms of trade* capture a reduction in the statistical distance between the individual value-added and the labor income distributions as measured by $-d\,\mathscr{K}\left(\widetilde{\mathscr{C}}_{\downarrow h}|\Lambda\right)$. The reallocation of labor makes consumers of type h better off as labor income shifts toward workers who, from h's perspective, are relatively undervalued. $\delta_{b|h}$ represents the distortion centrality for workers of type b conditional on the value-added distribution from households of type h. From the perspective of h, a worker is overvalued when $0 \leq \delta_{b|h} < 1$. Consumers of type h are better off when the labor income share shifts toward workers they perceive as relatively overvalued. For example, they are better off as labor income shifts from type b to type q workers as long as $\delta_{b|h} > \delta_{q|h}$. This effect portrays how workers who are essential for producing the consumption bundle of h are becoming relatively more affordable, which allows them to reallocate in response to higher labor demand from firms in supply chains that are relevant for h.

The second definition segments idiosyncratic *misallocation* into nominal labor and corporate income changes. The individual *labor terms of trade* show that distributional reallocations are favorable for households as their labor income rises or as the indirect labor costs from their consumption bundle fall.

The idiosyncratic *corporate income* tells us that a household is better off as their dividends increase.

The third definition separates the *entropic terms of trade* in terms of changes in the consumption distribution and the consumer-to-worker centralities. First, for consumers of type h, the *distributive terms of trade* imply that the new allocation of labor makes them worse-off as expenditure shifts toward households that from h's perspective have a high *expenditure efficiency.* $M_{b|h}$ represents the *expenditure efficiency* for consumers of type b from the perspective of households of type h. A high $M_{b|h}$ implies that households of type b demand goods produced by firms within efficient supply chains that rely heavily on workers essential for producing the consumption bundle for households of type h. In other words, a high $M_{b|h}$ implies that, on average, the expenditure from type b households reaches workers that h considers undervalued. The reallocation of labor worsens households of type h as χ_b rises when $M_{b|h}$ is high because aggregate expenditure flows towards efficient firms that demand labor from workers that are essential for consumers of type h, reallocating these workers from inefficient to efficient firms. The vector of idiosyncratic *expenditure efficiencies* $M_{|h} = \left(M_{1|h}, \ldots, M_{H|h}\right)'$ is a sufficient statistic for the effect of expenditure distributional variations on PTT_h. Second, the idiosyncratic *centrality terms of trade* indicate that misallocation worsens as households of type h are worse off as the consumer-to-worker centralities from any other households increase, mainly when this increase benefits workers that from the perspective of h are undervalued.

The last definition segments the idiosyncratic *centrality terms of trade* into four different effects that capture endogenous demand recomposition. The idiosyncratic *final demand* and *intermediate demand terms of trade* capture households of type h are worse off with an increase in the demand for goods produced by firms that from h's perspective have a high *revenue efficiency.* $F_{i|h}$ represents the *revenue efficiency* for firm i from the perspective of households of type h. A high $F_{i|h}$ implies that firms in sector i produce within relatively efficient supply chains and require, directly or indirectly, workers essential for producing the consumption bundle for households of type h. In other words, a high $F_{i|h}$ implies that, on average, the revenue from type i firms reaches workers that h considers undervalued. The idiosyncratic *labor demand terms of trade* portray how consumers of type h are worse off as big and relatively undistorted sectors demand more labor from workers they consider undervalued. Finally, the idiosyncratic *competitive terms of trade* represent the effects on PTT_h from the reallocation of workers in response to variations in labor demand driven by markdowns. For example, assume a markdown reduction in sector i of 1% such that the expenditure distribution and the final, intermediate, and labor demand terms of trade are inelastic. In response to this shock, PTT_h falls by $\widetilde{\mathscr{B}}_{hi} - \mathscr{B}_{hi}\, F_{i|h}$, i.e., a reduction in markdowns can be favorable for consumers of type h when it takes place in a high $F_{i|h}$ sector, as it allows workers that are essential for h toward relatively more inefficient firms.

Corollary 4. The variation in the labor terms of trade for consumers of type h is given by

$$Labor\ TT_h = \overbrace{\sum_{b \in \mathscr{H}} \widetilde{\mathscr{C}}_{hb}\, d\log \mathcal{J}_b - \Gamma_h\, d\log \mathcal{J}_h}^{Real\ Income\ Effect_h} + \overbrace{(1-\Gamma_h)\, d\log p^c_h}^{CPI\ effect_h} + \overbrace{\sum_{b \in \mathscr{H}} \widetilde{\mathscr{C}}_{hb}\, d\log \varepsilon_{hb}}^{RER\ effect_h}.$$

Corollary 4 is the distributional equivalent of Corollary 3, and segments the idiosyncratic *labor terms* *of trade* for households of type h into three effects: (i) real income, (ii) CPI, and (iii) RER. First, as in its nominal counterpart, the change in household h's real income is proportional to the labor wedge Γ_h, and the variation in the real income for workers of type b is proportional to their value-added contribution $\widetilde{\mathscr{C}}_{hb}$. Second, the *CPI effect* shows how idiosyncratic reallocation worsens as the price of the consumption bundle rises, and the magnitude from this effect is proportional to the household h's share of corporate income. Finally, the *RER effect* illustrates that for households of type h, a depreciation of their consumption bundle relative to households of type b has an effect on the *labor terms of trade* that is proportional to $\widetilde{\mathscr{C}}_{hb}$.

5 Constrained Social Planner Economy

Assume the existence of an aggregate welfare function $W(Y, L)$ where Y and $L = F\left(\{L_h\}_{h\in\mathscr{H}}\right)$ are the same functions as in equation (12). A constrained social planner maximizes $W(Y, L)$ by choosing $Y, L, \left\{C_h, L_h, \{C_{hi}\}_{i\in\mathscr{N}}\right\}_{h\in\mathscr{H}}$, subject to

$$p_Y Y = \sum_{h\in\mathscr{H}} p_h^c C_h = \sum_{h\in\mathscr{H}}\sum_{i\in\mathscr{N}} p_i C_{hi} \leq wL + \Pi = \sum_{h\in\mathscr{H}} \left(w_h L_h + \Pi_h\right),$$

taking prices, wages, and profits as given.

This social planner tells each household how much to work, collects all labor and corporate income, buys final goods, and distributes them across households in a manner that respects preferences on consumption. This planner is not concerned by the stability of its regime, as it does not account for households' compatibility incentives. The constraints on this social planner are cognitive and instrumental. First, the planner is unaware of the general equilibrium effects of its demand on prices. Second, the planner cannot develop policies that tackle distortions directly, e.g., flexible Pigouvian taxes that subsidize heavily distorted firms by taxing relatively undistorted sectors.

This primitive social planner is akin to a representative household that maximizes welfare by choosing its consumption bundle and labor supply. The only difference between a representative household and a constrained social planner is that the latter has to decide how to distribute goods across households.

Theorem 4 is the analogous of Theorems 1 and 2 for the constrained social planner problem. It characterizes the equilibrium aggregate real output Y and the aggregate labor supply L in terms of the aggregate labor wedge Γ, which in equilibrium equals the aggregate labor share. It also presents the local variation for the aggregate distributional *misallocation* under the planner's problem.

Theorem 4. In equilibrium, the aggregate output and labor supply satisfies

$$\frac{W_L}{W_Y} + \Gamma\frac{Y}{L} = 0 \qquad \text{with} \qquad \Gamma = \sum_{h\in\mathscr{H}} \Lambda_h = \delta_b^{-1} \quad \forall b \in \mathscr{H}, \tag{19}$$

and the change in the *misallocation* component of TFP and Γ are, to a first order

$$Misallocation = d\,log\,\Gamma, \tag{20}$$

$$d\,\Gamma = \overbrace{\sum_{h\in\mathscr{H}} \mathscr{C}_h\, d\,\chi_h}^{Distributive\ TT} + \overbrace{\sum_{h\in\mathscr{H}} \chi_h\, \mathscr{C}_h}^{Centrality\ TT}, \tag{21}$$

$$Centrality\ TT = \overbrace{\sum_{h\in\mathscr{H}} \chi_h \sum_{i\in\mathscr{N}} \psi_i^{\ell}\, d\,\beta_{hi}}^{Final\ Demand\ TT} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i\, \lambda_i \sum_{j\in\mathscr{N}} \psi_j^{\ell}\, d\,\widetilde{\Omega}_{ij}^{x}}^{Intermediate\ Demand\ TT} + \overbrace{\sum_{i\in\mathscr{N}} \mu_i\, \lambda_i\, d\,\omega_i^{\ell}}^{Labor\ TT} + \overbrace{\sum_{i\in\mathscr{N}} \psi_i^{\ell}\, \lambda_i\, d\,log\,\mu_i}^{Competitive\ TT}. \tag{22}$$

Equation (19) characterizes the aggregate labor supply and the aggregate labor wedge. The centralized labor wedge Γ relates the aggregate marginal rate of substitution with the economy's marginal rate of transformation Y/L. The centralized labor wedge equals the aggregate labor share (i.e., $\sum_{h\in\mathscr{H}} \Lambda_h$), and also the inverse of the distortion centralities for all workers. The symmetry in distortion centralities restricts the space of labor income and expenditure distributions that the planner entertains as a solution. Consequently, decentralized solutions that violate the symmetry in distortion centralities will be inefficient from the social planner's perspective. In other words, the decentralized economy withstands externalities on aggregate welfare.

Equation (20) shows that for the social planner, there is a negative monotonic relationship between aggregate labor *misallocation* and the aggregate labor wedge. Just as in the *corporate income* component from Theorem 2, for a specific level of aggregate labor income, more aggregate dividends entail an increase in the distance between the labor compensation and the value added by each worker. Some workers are suddenly becoming more affordable, which allows labor demand to increase, mainly for firms that operate in relatively inefficient supply chains that produce with high marginal productivities. Hence, workers will reallocate towards inefficient supply chains, ameliorating aggregate labor misallocation.

Additionally, this result shows that under endogenous labor supply, there is a tight connection between the TFP decomposition from Baqaee & Farhi (2020) and the first-order variation of the labor wedge from Bigio & La'O (2020). Consequently, for a representative household economy with endogenous labor supply, the TFP decomposition from Baqaee & Farhi (2020) is simplified as the endogenous effect from labor income corresponds to the variation of the aggregate labor wedge. Furthermore, this result associates in a single equation the two equilibrium objects that, according to Chari et al. (2007), account for the bulk of business cycle fluctuations. In other words, under distortions and endogenous labor supply, if the social planner wanted to increase TFP, they could symmetrically amplify the distortion centralities for all workers and reduce the decentralized labor wedge. This increase in the corporate income share represents a linear drift towards a second-best equilibrium.

This result does not imply that total labor misallocation increases with the labor share. *Misallocation* both in Theorems 2 and 4 captures only the effect from endogenous changes in the income distribution. The definition of total aggregate labor misallocation is *competitiveness − misallocation*. As a consequence, antitrust policies that aim to reduce firms' profit margins and the aggregate profit share can be successful in increasing TFP, as long as *competitiveness* $>$ *misallocation*.

Equation (21) describes two mechanisms through which the social planner can increase the aggregate labor wedge. First, the *distributive terms of trade* imply that the aggregate labor income share rises as income shifts towards households with a high *payment centrality*. For consumers of type h, their payment centrality $\mathscr{C}_h$ represents the share from their expenditure that reaches labor compensation. Aggregate misallocation increases with χ_h when $\mathscr{C}_h$ is high because the aggregate labor share rises. The vector of payment centralities is a sufficient statistic for the effect of expenditure distributional variations on TFP. For example, TFP improves in response to an endogenous redistribution of expenditure from type h to type b households if $\mathscr{C}_h > \mathscr{C}_b$. Second, the *centrality terms of trade* indicate that the aggregate labor income share increases with the payment centrality from any household, more so for households with a large expenditure share. This channel captures the distributional effects on TFP from the endogenous recomposition in the demand structure from firms and households while keeping the expenditure distribution fixed. The social planner chooses the recomposition of demand for households.

Equation (22) further divides the *centrality terms of trade* into four sources of endogenous demand recomposition analogous to the channels in Theorem 2. The difference now is that the vector of sectoral payment centralities $\psi_\ell = \left(\psi_1^\ell, \ldots, \psi_N^\ell\right)'$ replaces the vector F of *revenue efficiencies* as a sufficient statistic.

Furthermore, in the absence of distortions, the effect from markdowns on the centralized labor wedge is sufficiently captured by the Domar weights, i.e., $\frac{d\log\Gamma}{d\log\mu_i} = \lambda_i$. This local variation is the main result from Bigio & La'O (2020), and Theorem 4 captures the extension from their findings to any inefficient equilibrium in which a constrained social planner makes the distributional decisions on behalf of heterogenous households.

6 Growth Accounting for the United States in the XXIst Century

My model builds upon four types of money flows: (1) firm-to-firm in the supply of intermediate inputs, (2) firm-to-workers in the supply of labor, (3) consumer-to-firm in the supply of final goods, and (4) firm-to-households in the distribution of dividends. In this section I describe the date sources that I use to implement the model, and I estimate the TFP and PTT decompositions from Section 4.

6.1 Data

The first source is the input-output (IO) tables constructed by the Bureau of Economic Analysis (BEA) from 1997 to 2021. These tables measure the intermediate input transactions, labor costs, and final expenditure for 71 North American Industry Classification System (NAICS) 3-digit level industries. As usual, I exclude industries corresponding to federal, state, and local governments, resulting in a matched data set of 66 industries. The IO tables are not readily available, as the BEA provides only

IO use and make tables. The use tables depict industrial consumption across multiple categories of goods and services, and the make tables characterize industrial production of multiple categories of goods and services. The interaction between the use and make tables produces the IO network. The BEA has IO use and make tables that go back to 1946, but only after 1997 did these tables start to identify the sectoral labor costs as an independent component of value-added, which is essential for my identification of sectoral distortions. I use this tables to calibrate $\forall i \in \mathcal{N}$

$$\omega_i^\ell = \frac{\text{Labor Cost}_i}{\text{Total Cost}_i}, \qquad \mu_i = \frac{\text{Total Cost}_i}{\text{Sales}_i}, \qquad \omega_{ij} = \frac{\text{Sales from } j \text{ to } i}{\text{Intermediate Cost}_i},$$

$$\text{Total Cost}_i = \text{Labor Cost}_i + \text{Intermediate Cost}_i, \qquad \text{Value Added}_i = \text{Labor Cost}_i + \text{Rents}_i,$$

$$\text{Sales}_i = \text{Value Added}_i + \text{Intermediate Cost}_i.$$

The second source is the 1997 to 2021 county business patterns (CBP) from the Census Bureau. The CBP is an annual series that, for each industry, provides economic data at the county, metropolitan statistical area, state, and national levels. For each subnational level, the CBP includes the number of workers and their income in each NAICS industry up to the 6-digit level. The employment statistics count full- and part-time workers with an active payroll in the pay period that includes March 12 and their average annual income. The CBP draws its information from administrative records of the Internal Revenue Agency, the Bureau of Labor Statistics (BLS), and the Social Security Administration, which gives it a higher degree of trustworthiness than voluntary census responses. There are two widely known issues with the CBP.

The first issue is that the Census Bureau suppresses a significant proportion of the data to protect individual employers' confidentiality.[10] To make matters worse, since 2007, the non-suppressed observations have included a random noise infusion multiplier that further complicates its implementation. A whole research agenda on antisuppression algorithms tries to fill the gaps in the CBP. The data mining techniques developed by this literature utilize the additional information available due to the industrial and geographical hierarchical nature, which justifies a manifold of bounds and aggregation constraints across hierarchies. Two current gold standards solve this problem: first, the two-staged algorithm from Isserman & Westervelt (2006), and second, the linear programming solution from Eckert et al. (2020).[11] For my calibration, the problem with both identification methods is their emphasis on the number of workers rather than their compensation. For this reason, I develop a three-staged estimation for the average annual payroll. The first and second stages consist of the Isserman & Westervelt (2006) algorithm for the number of workers with an initial guess given by the Eckert et al. (2020) solution. The third stage utilizes the two-staged employment estimates and analogous hierarchical bounds and aggregation constraints for income.

The second issue is that the CBP only covers some forms of private employment. The CBP does

[10] For example, Isserman & Westervelt (2006) document that for 2002, the suppression rate was two-thirds - almost 1.5 million out of the 2.2 million records.

[11] An alternative and more straightforward solution is the employment estimate using midpoints of establishment size groups as in Clapp et al. (1992), Glaeser et al. (1992), Porter (2003). The issue is that this method does not use all the hierarchical information embedded within the CBP.

not include workers in agriculture production, railroads, government, and private households. I use the BEA's Regional Economic Information System (REIS) to fill this gap and obtain state-level employment and income measures for agricultural production and railroad workers. The REIS uses the Quarterly Census of Employment and Statistics from the Bureau of Labor Statistics (BLS). Its main limitation relative to the CBP is that it only provides 2-digit NAICS statistics.

The third source is the BLS Occupational Employment and Wage Statistics (OEWS). This dataset contains employment and wage estimates for approximately 830 occupations under the Standard Occupational Classification System (SOC). These estimations are available at the level of the country, industries, states, and metropolitan and nonmetropolitan areas. The BLS also suppresses data on the OEWS to protect the confidentiality of employers and workers, although this problem is less pervasive than in the CBP. For this reason, I implement a two-stage antisuppression algorithm that depends on hierarchical aggregation constraints, and it is analogous to the one implemented for the CBP.

I use the CBP and OEWS to obtain labor network estimates at (a) geographic, (b) occupation, and (c) geographic and occupation levels. For example, to capture, for hospitals (i), the labor share of dentists in Maine (h), I define

$$\alpha_{ih} \propto \overbrace{\begin{array}{c}\text{Hospital's share of}\\ \text{labor expenditure}\\ \text{in Maine}\end{array}}^{\text{Spatial Demand (CBP)}} \times \overbrace{\begin{array}{c}\text{Hospital's share of}\\ \text{labor expenditure}\\ \text{in dentists}\end{array}}^{\text{Occupational Demand (OEWS)}} \times \overbrace{\begin{array}{c}\text{Maine's share of}\\ \text{labor expenditure}\\ \text{in dentists}\end{array}}^{\text{Occupational Supply (OEWS)}}. \tag{23}$$

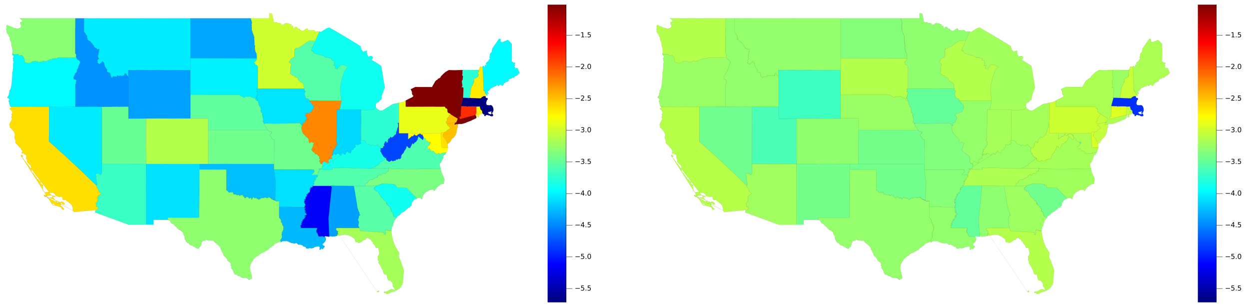
Each of these three factors portrays a different feature of the labor market. First, *spatial demand* captures sectoral heterogeneity in labor demand at the subnational level. Without this factor, sectors with the same occupational demand would have symmetric labor bundles. Second, *occupational demand* represents sectoral heterogeneity in labor demand across occupations. Without this factor, sectors with the same spatial demand would have symmetric labor bundles. Third, the *occupational supply* illustrates geographic heterogeneity in the availability of occupations. Without this factor, sectors would have the same occupational demand across states. The *demand* and *supply* labels are misnomers, as these three components are all equilibrium objects. However, these labels illustrate how the first two factors capture heterogeneity from the firms, while the last factor represents differences in the availability of occupations across space. For the geographic specifications, I use only the *spatial demand* factor, while for the occupational specifications, I employ the *occupational demand* factor exclusively.

Figure 2 illustrates the implementation of Equation (23) with heatmaps for the estimation of the cost intensity from the financial sector in economists, and from the ambulatory health industry in dentists. On the one hand, there is geographical concentration in the intensity of the financial sector in hiring economists in New York, Connecticut, Illinois, New Jersey, and California. On the other hand, there is no geographic concentration in the intensity of the ambulatory health industry in hiring dentists.

The fourth source is the BEA state-level personal consumption expenditure by product type (PCE). The PCE classifies consumption expenditure into 113 types of products. Of these, only 71 categories

Figure 2: Sectoral Labor Intensity

A. Financial Sector in Economists **B. Ambulatory Health in Dentists**



Note: The corresponding α_{ih} are divided by the labor force from each state to make them comparable.

are non-redundant or refer to new goods. Using the IO make matrix from the categories left, I build a product-to-sector crosswalk that specifies the state-level final consumption share for each of the 66 sectors in the IO tables. From here, households within the same state will share the same consumption bundle.

The fifth and final data source is the sectoral TFP measure from the BEA's Integrated Industry-Level Production Account (KLEMS). Following La'O & Tahbaz-Salehi (2022), I will use the variations in sectoral TFP as an exogenous measure of productivity variation. Specifically, in my model, sectoral TFP variations differ from sectoral productivity shocks. Still, I equate these two notions in the exogenous variations, not only because it is the standard in the literature but also because the alternative requires having measures of sectoral prices that allow me to directly estimate the sectoral Solow residuals, which I expect could only improve the model's fit with the aggregate data. In this sense, my decision to measure exogenous productivity shocks from sectoral KLEMS's TFP variations imposes the most stringent benchmark for testing the model's implications.

To capture the variations between periods t and $t+1$, I estimate the equilibrium in period t, and introduce the variations captured by the data between period t and $t+1$. For example, the technological component of TFP between period t and $t+1$ is given by

$$Technology_{t+1} = \sum_{i \in \mathcal{N}} \widetilde{\lambda}_{i,t} \, d\log \mathcal{A}_{i,t+1}.$$

6.2 Quantitative Implementation

6.2.1 Aggregate Accounting

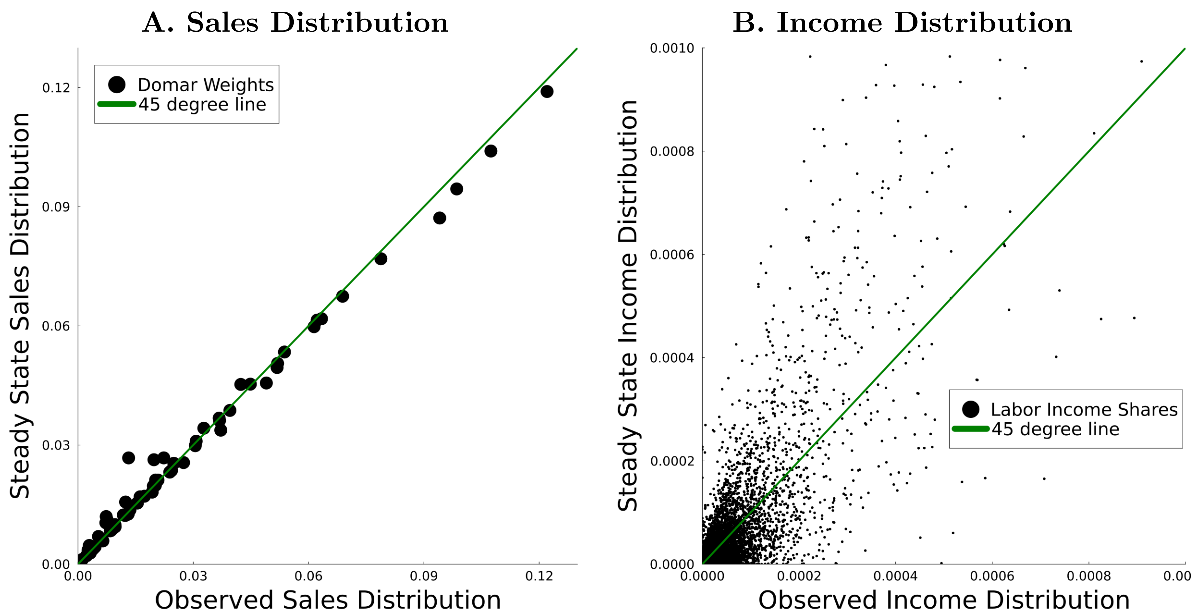
Figure 3 shows two scatterplots that compare the observed sales and income distributions in 2021 with their equivalent equilibrium distributions. The match is almost perfect for the sales distribution, and the R^2 of 0.982 for the OLS regression of the observed λ on its equilibrium equivalent confirms this. The imprecision of the model estimation for the income distribution comes from the uncertainty about the expenditure from each sector on each type of worker. For example, the CBP provides information about the compensation from the financial sector to workers in Illinois, and the OEWS captures the nationwide compensation from the financial sector to economists. However, there is no accessible data that provides compensation from the financial sector to economists in Illinois. For this reason, Equation (23) provides the proxy required for the model implementation. Nevertheless, despite this uncertainty, the R^2 for the OLS regression of the observed Λ on its equilibrium equivalent for 2021 is 0.682, and the t-value for its slope coefficient is 286.

Tables 3 and 4 report the results from two types of OLS regressions for observed TFP growth, first on the model prediction of TFP growth, and second on the three components from the decomposition in equation (14).

$$d\,log\,TFP_t = a_0 + a_1\; d\,\widehat{log\,TFP}_t + \epsilon_t,$$

$$d\,log\,TFP_t = b_0 + b_1\; Technology_t + b_2\; Competitiveness_t + b_3\; Misallocation_t + u_t.$$

Figure 3: Sales and Income Distribution



Observed* λ *on	
Equilibrium λ	1.022***
	(3.4e-3)
Intercept	-5.1e-4***
	(1.2e-4)
R^2	0.982
Observations	1,650

Observed* Λ *on	
Equilibrium Λ	0.438***
	(1.5e-3)
Intercept	7.6e-6***
	(1.6e-7)
R^2	0.682
Observations	38,189

Notes: Table A plots the observed revenue-based Domar weights λ_i^{obs} and its equilibrium counterpart λ_i. Table B plots the observed labor income shares Λ_h^{obs} and its equilibrium counterpart Λ_h. The equilibrium values are estimated using the system of equations in Table 2. The first regressions is $\lambda_i^{obs} = a_0 + a_1\;\lambda_i + \epsilon_i$, and the sample is given by the 66 NAICS industries in the years from 1997 to 2021. The second regression is $\Lambda_h^{obs} = b_0 + b_1\;\Lambda_h + u_h$, and the sample is given by the 38,189 types of workers that come from the state and occupation interaction in the year 2021.

Table 3 uses the regressors from an estimation without input-output networks, while Table 4 allows for intermediate input markets. Each table contains four estimations with different assumptions about the number of households: (i) representative household ($H = 1$), (ii) heterogeneity by occupation ($H = 750$), (iii) geographical heterogeneity by county ($H = 3,136$), and (iv) heterogeneity by the interaction of states and occupations ($H = 38,190$). The latter is my preferred specification because it simultaneously accounts for skill and geographic heterogeneity. Under this specification, accounting for intermediate input linkages boosts the R^2 of the model prediction for TFP growth from 5% to 50%. Additionally, in the regression of observed TFP growth on the three components from equation (14) with intermediate input markets, R^2 increases to 75%, and the *technology* and *competitiveness* components are significant at the 1% level.

Table 3: Explanatory Power of the Model Without IO Networks

	Rep. Household		***Occupation***		***County***		***State & Ocupation***	
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**
$d\log TFP$	0.523		0.503		0.388		-0.265	
	(0.366)		(0.350)		(0.316)		(0.264)	
Technology		1.341***		0.789***		0.796***		0.847***
		(0.308)		(0.267)		(0.266)		(0.289)
Competitiveness		0.212		0.320		0.454		0.986
		(0.423)		(0.489)		(0.373)		(0.695)
Misallocation		0.573*		0.450		0.335		-0.105
		(0.329)		(0.437)		(0.315)		(0.360)
Intercept	0.012***	0.011***	0.012***	0.012***	0.013***	0.012***	0.015***	0.012***
	(3.2e-3)	(2.0e-3)	(3.2e-3)	(2.2e-3)	(3.2e-3)	(2.1e-3)	(3.0e-3)	(2.2e-3)
Observations	22							
N	66							
H	1		750		3,136		38,190	
R^2	9.2%	71.4%	9.35%	62.4%	7.00%	62.5%	4.8%	60.4%
Adj. R^2	9.2%	68.4%	9.35%	58.4%	7.00%	58.6%	4.8%	56.2%

Notes: Columns 1, 3, 5, and 7 report the results for the regression $d\log TFP_t = a_0 + a_1\, d\widehat{\log TFP}_t + \epsilon_t$. Columns 2, 4, 6, and 8 report the results for the regression $d\log TFP_t = b_0 + b_1\ Technology_t + b_2\ Competitiveness_t + b_3\ Misallocation_t + u_t$. Equation (14) is estimated solving for the system of equations in Table 2 without accounting for intermediate input linkages, i.e. $\omega_i^x = 0\ \forall i \in \mathcal{N}$. Estimations have 66 sectors and use variations from 1997 to 2019, hence $N = 66$ and $obs = 22$. Columns 1 and 2 use a representative household estimation. Columns 3 and 4 use heterogeneity by occupation. Columns 5 and 6 use heterogeney by county. Columns 7 and 8 use heterogeneity by interaction of states and occupations.

Figure 4 shows the dynamics for TFP, technology, competitiveness, and misallocation normalizing their initial 1997 level at 100. Table 5 captures the counterfactual growth on TFP relative to the model prediction leaving aside the technology, competitiveness, or misallocation channels. Table 6 and Table 7 portray the counterfactual growth on TFP relative to the model prediction leaving aside the technology or competitiveness effect from a specific industry. Table 8 depicts the variance decomposition for TFP growth, Table 9 the variance decomposition for technology across industries, and Table 10 for competitiveness.

The model prediction for TFP follows observed TFP in terms of levels and variations until 2014 (Figure 4A). After 2014, the model predicts no growth in TFP and a strong reduction in response to the 2020 COVID shock. The static and closed-economy nature of the model is, in my opinion, the reason why the model fails to capture TFP variations after 2014. From 2014 to 2021, the net international investment position as a percentage of GDP almost doubled from -40% to -77.8%. It is reasonable to expect that this increase in external liabilities could be behind an intertemporal demand-driven growth in TFP that this model completely misses. This model emphasizes capturing multiple sources of supply-driven growth and their dependence on the heterogeneity of firms and households. For this reason, in the absence of measurement errors, this result brings some evidence about the lack of domestic supply-driven sources of growth for TFP after 2014.[12]

Table 4: Explanatory Power of the Model With IO Networks

	Rep. Household		***Occupation***		***County***		***State & Ocupation***	
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**
$d\log TFP$	0.370***		0.311***		0.316***		0.311***	
	(0.072)		(0.069)		(0.065)		(0.069)	
Technology		0.478***		0.414***		0.416***		0.413***
		(0.097)		(0.081)		(0.083)		(0.082)
Competitiveness		0.398***		0.341***		0.350***		0.342***
		(0.062)		(0.054)		(0.053)		(0.054)
Misallocation		0.074		0.172		0.164		0.168
		(0.138)		(0.125)		(0.135)		(0.125)
Intercept	0.010***	0.009	0.011***	0.010***	0.011***	0.010***	0.011***	0.010***
	(2.1e-3)	(2.0e-3)	(2.2e-3)	(1.8e-3)	(2.1e-3)	(1.9e-3)	(2.3e-3)	(1.9e-3)
Observations	22							
N	66							
H	1		750		3,136		38,190	
R^2	56.9%	75.2%	49.9%	75.8%	54.0%	75.4%	49.9%	75.5%
Adj. R^2	56.9%	72.6%	49.9%	73.3%	54.0%	72.8%	49.9%	73.2%

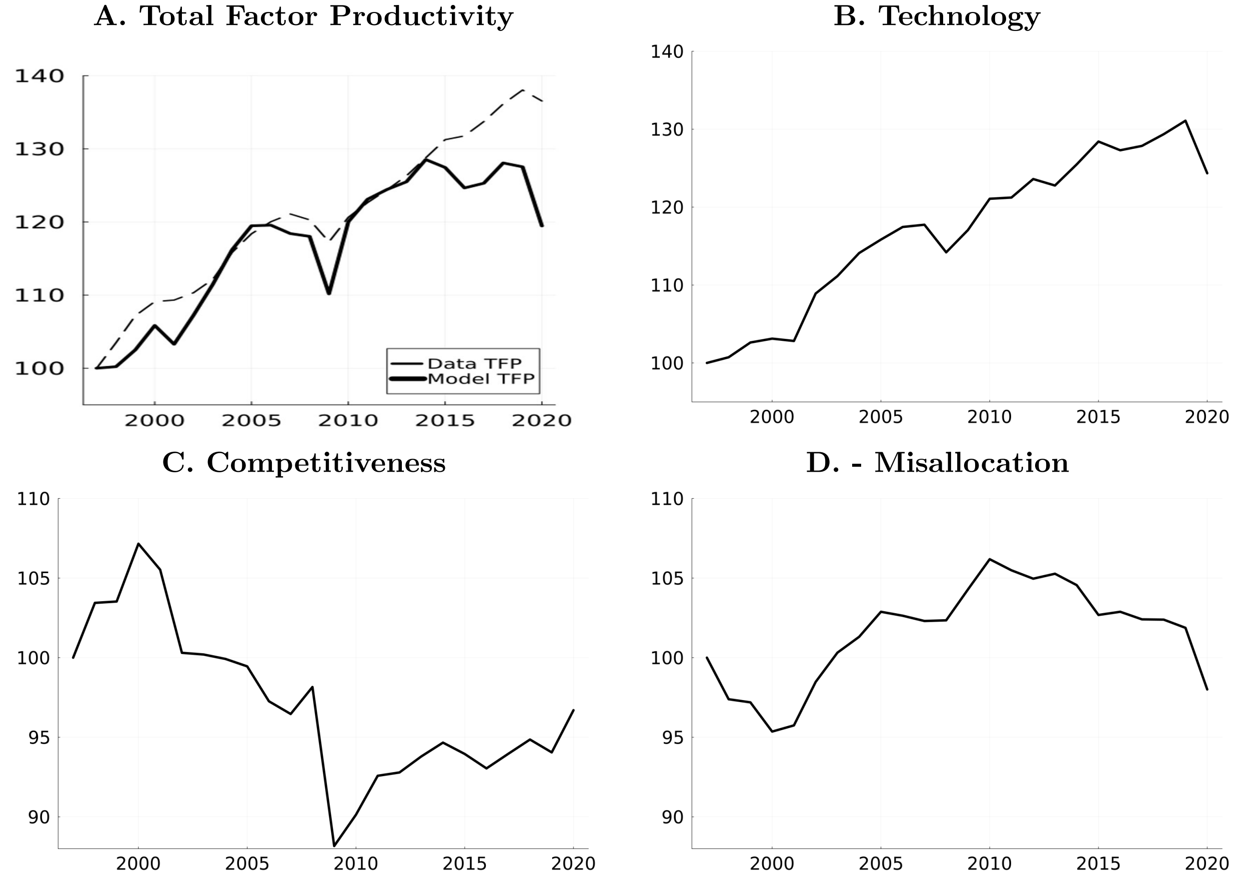
Notes: Columns 1, 3, 5, and 7 report the results for the regression $d\log TFP_t = a_0 + a_1\, d\widehat{\log TFP}_t + \epsilon_t$. Columns 2, 4, 6, and 8 report the results for the regression $d\log TFP_t = b_0 + b_1\ Technology_t + b_2\ Competitiveness_t + b_3\ Misallocation_t + u_t$. Equation (14) is estimated solving for the system of equations in Table 2 accounting for intermediate input linkages. Estimations have 66 sectors and use variations from 1997 to 2019, hence $N = 66$ and $obs = 22$. Columns 1 and 2 use a representative household estimation. Columns 3 and 4 use heterogeneity by occupation. Columns 5 and 6 use heterogeney by county. Columns 7 and 8 use heterogeneity by interaction of states and occupations.

From 1998 to 2020, the growth of TFP was mainly attributable to technological shocks, while competitiveness and misallocation had a negative secondary role (Figure 4 and Table 5A). On the one hand, without productivity shocks, TFP in 2020 would have been 23.4% lower. On the other hand, leaving aside the effects of competitiveness or misallocation, TFP would have grown 2.5% and 2.8% more. The productivity shocks in the oil and gas extraction, computer and electronics, telecommunications, and computer system design industries were the main drivers of technologically driven growth. Without them, TFP would have been respectively 11.1%, 6.6%, 2.8%, and 2.3% lower. The productivity shocks in the construction, chemical products, and credit intermediation industries stood in the way of growth. Without them, TFP would have been respectively, 2.9%, 2.8%, and 1.8% higher (Table 6A). Despite the secondary role of aggregate competitiveness, the higher profit margins of the credit

[12] Two potential sources of measurement error are of my concern. First, observed growth in TFP is the difference between growth in real GDP and the labor force. However, from equation (13), the variations in the labor force participation from heterogeneous workers are not symmetric and depend on their aggregate value-added contribution given by the distribution $\bar{\Lambda}$. Second, the nature of productivity growth might have changed after 2014 in a way not captured by the BEA's sectoral KLEMS Solow's residual estimation.

intermediation, chemical products, and computer and electronics sectors hindered TFP growth, while the lower profit margins from the housing and insurance sectors boosted TFP. Without them, TFP would have been respectively, 4.1%, 2.6%, and 1.3% higher, and 1.6% and 1.5% lower (Table 7A).

Figure 4: TFP Decomposition



Notes: The observed growth in TFP comes from the difference between growth of real GDP and variations in the labor force participation. Theorem 2 provides the decompositions for technology, competitiveness, and misallocation. Using equation (14), I estimate the model prediction for TFP growth. The three channels and TFP are normalized to 100 in the year 1997.

Furthermore, during the same period, 55.6% of the volatility was attributable to the reallocation of resources. Out of this, 34.6% was due to variations in aggregate competitiveness and 21% due to changes in the income distribution. Productivity shocks explained the remaining 44.4% of the volatility (Table 8A). Productivity shocks in the oil and gas extraction, insurance, air transportation, utilities, and financial sector were the main sources of technological-driven volatility (Table 9A). Variations in the profit margins for the oil and gas extraction, financial, utilities, and chemical product sectors were the main drivers of competitive-driven volatility (Table 10A).

The secondary role of competitiveness and misallocation from 1997 to 2020 reflects a structural change of direction during the Great Recession (GR). Competitiveness and misallocation fell from 2002 to 2009 and increased from 2010 to 2020 (Figures 4C and 4D). The self-compensating nature of competitiveness and misallocation is not surprising, as increases in profit margins are correlated negatively with competitiveness and positively with misallocation.

Table 5: Counterfactual TFP Growth Differential

A. Between 1998 and 2020

Heterogeneity	***Model***	***Technology***	***Competitiveness***	***Misallocation***
Rep. Household	20.3%	-25.0%	2.3%	2.6%
Occupation	19.0%	-26.8%	4.7%	3.3%
County	18.3%	-26.7%	4.9%	4.0%
State & Occupation	18.2%	-23.4%	2.5%	2.8%

B. Between 2002 and 2009

Heterogeneity	***Model***	***Technology***	***Competitiveness***	***Misallocation***
Rep. Household	5.2%	-13.9%	19.7%	-8.8%
Occupation	6.5%	-17.4%	23.3%	-9.7%
County	5.0%	-17.1%	23.1%	-8.1%
State & Occupation	4.2%	-13.0%	19.3%	-8.2%

C. Between 2010 and 2020

Heterogeneity	***Model***	***Technology***	***Competitiveness***	***Misallocation***
Rep. Household	9.0%	-6.2%	-10.2%	8.0%
Occupation	8.3%	-5.9%	-10.5%	8.8%
County	8.7%	-5.9%	-10.4%	8.3%
State & Occupation	9.0%	-6.3%	-9.8%	7.5%

Notes: The model estimation comes from compounding variations on TFP from equation (14). To be more precise, $TFP_t = TFP_0 \times \Pi_{q=1}^{t} exp\,(d\,log\,TFP_q)$. The results in columns technology, competitiveness and misallocation come from obtaining sequences for $d\,log\,TFP_q$ that leave out one of the channels from equation (14) at the time.

For the cycle before the Great Recession (2001 to 2009), on the one hand, growth in TFP was driven by technology and reductions in misallocation. Without the growth in productivity or reductions in misallocation, TFP would have been 13% and 8.2% lower, respectively. On the other hand, the reductions in aggregate competitiveness hindered growth, and in their absence, TFP would have been 19.3% higher. The productivity shocks in the oil and gas extraction, computer and electronics, and telecommunication sector were the main drivers of technologically driven growth. Without them, TFP would have been respectively 5.35%, 2.84%, and 2.27% lower (Table 6B). The higher profit margins in oil and gas extraction stood in the way of growth, and in their absence, TFP would have been 6.59% higher (Table 7B).

For the cycle after the GR (2010 to 2020), on the one hand, growth in TFP was driven by increases in technology and competitiveness. Without the growth in productivity or competitiveness, TFP would have been 6.3% and 9.8% lower, respectively. On the other hand, the increases in misallocation hindered growth, and in their absence, TFP would have been 7.5% higher. The main growth drivers were the productivity shocks and the reduction in the profit margins from the oil and gas extraction sector. Without them, TFP would have been respectively 5.41% and 6.34% lower. Additionally, the reductions in the profit margins for the housing sector enabled growth, and in their absence, TFP would have been 3.09% lower. Furthermore, reductions in productivity and higher profits margins for the credit intermediation and the chemical products industries hindered growth (Table 6C and Table 7C). Most of the TFP volatility after the GR was attributable to technology and misallocation (Table 8C). Productivity shocks in the air transportation and insurance sectors were the primary technological sources of volatility (Tables 9C).

Table 6: Counterfactual TFP Growth Without Sectoral Technology

A. Between 1998 and 2020

1	Oil & gas extraction	-11.11%
2	Computer & electronics	-6.64%
3	Telecommunications	-2.85%
4	Computer systems design	-2.30%
5	Administrative services	-1.74%
6	Insurance carriers	-1.45%
7	Farms	-1.34%
8	Primary metals	-1.28%
	⋮	
63	Rental & leasing	1.41%
64	Credit intermediation	1.77%
65	Chemical Products	2.84%
66	Construction	2.87%

B. Between 2002 and 2009

1	Oil & gas extraction	-5.35%
2	Computer & electronics	-2.84%
3	Telecommunications	-2.27%
4	Utilities	-1.92%
5	Administrative services	-1.06%
	⋮	
66	Construction	1.76%

C. Between 2010 and 2020

1	Oil & gas extraction	-5.41%
2	Computer systems design	-1.29%
3	Management of companies	-1.26%
4	Housing	-1.14%
5	Other real estate	-1.01%
	⋮	
64	Air transportation	1.03%
65	Chemical products	1.90%
66	Credit intermediation	2.73%

Table 7: Counterfactual TFP Growth Without Sectoral Competitiveness

A. Between 1998 and 2020

1	Housing	-1.65%
2	Insurance carriers	-1.53%
3	Misc. professional services	-1.10%
4	Other services	-0.89%
	⋮	
63	Publishing industries	0.80%
64	Computer and electronics	1.34%
65	Chemical products	2.57%
66	Credit intermediation	4.10%

B. Between 2002 and 2009

1	Securities & investment	-0.86%
	⋮	
58	Wholesale trade	0.92%
59	Publishing industries	0.93%
60	Internet, & inf. services	0.99%
61	Chemical products	1.35%
62	Telecommunications	1.43%
63	Computer and electronics	1.48%
64	Housing	1.57%
65	Utilities	1.87%
66	Oil & gas extraction	6.59%

C. Between 2010 and 2020

1	Oil & gas extraction	-6.34%
2	Housing	-3.09%
3	Insurance carriers	-0.98%
4	Misc. professional services	-0.87%
5	Administrative services	-0.82%
	⋮	
64	Primary metals	0.80%
65	Chemical products	0.84%
66	Credit intermediation	3.86%

Notes: For Table 6, observations larger than 1.2% in absolute value are included in table A and 1% for tables B and C. For For Table 7, only observations larger than 0.8% in absolute value are included. For each estimation, using Theorem 2 a counterfactual sequence for $\left\{Technology_q\right\}_{q=1}^{t}$ or $\left\{Competitiveness_q\right\}_{q=1}^{t}$ is constructed. This sequence excludes the productivity or markdown shocks from one industry at the time. The counterfactural sequence is used to estimate $\{d\,log\,TFP_q\}_{q=1}^{t}$ using equation (14) and $TFP_t = TFP_0 \times \Pi_{q=1}^{t} exp\,(d\,log\,TFP_q)$.

Figure 5 shows the dynamics for misallocation and its components normalizing their initial 1997 level at 100. Table 11 captures the counterfactual growth on TFP relative to the model prediction leaving aside each one of the components from misallocation. Tables 12-15 portray the counterfactual growth on TFP relative to the model prediction leaving aside the competitive, labor demand, final demand, and intermediate demand terms of trade from a specific industry. Table 16 depicts the variance decomposition for variations in misallocation and Tables 17-20 the variance decomposition for each one of the misallocation components across industries.

Table 8: TFP Covariance Decomposition

A. Between 1998 and 2020

Heterogeneity	*Technology*	*Competitiveness*	*-Misallocation*
Rep. Household	41.3%	39.3%	19.4%
Occupation	40.1%	41.5%	18.4%
County	37.2%	46.8%	16.1%
State & Occupation	44.4%	34.6%	21.0%

B. Between 2002 and 2009

Heterogeneity	*Technology*	*Competitiveness*	*-Misallocation*
Rep. Household	21.3%	68.6%	10.1%
Occupation	12.7%	85.2%	2.1%
County	10.6%	85.0%	4.4%
State & Occupation	28.3%	61.2%	10.5%

C. Between 2010 and 2020

Heterogeneity	*Technology*	*Competitiveness*	*-Misallocation*
Rep. Household	56.7%	8.2%	35.1%
Occupation	55.3%	13.1%	31.6%
County	55.4%	15.6%	29.0%
State & Occupation	58.1%	4.9%	37.0%

Notes: From equation (14), the covariance decomposition is given by $Var\left(d\log TFP_t\right) = Cov\left(d\log TFP_t,\ Technology_t\right) + Cov\left(d\log TFP_t,\ Competitiveness_t\right) - Cov\left(d\log TFP_t,\ Misallocation_t\right)$. The estimates for each one of these components are provided by Theorem 2.

From 1998 to 2020, misallocation barely increased, and without its effect on growth, TFP would have grown 2.8% more (Table 5A). However, this apparent lack of variation was due to a worsening in the labor demand terms of trade, partially compensated by the improvement in the competitive, final, and intermediate demand terms of trade. Without the increase in the labor demand terms of trade, TFP would have grown 15.6%, and in the absence of the reduction in the competitive, intermediate, and final demand terms of trade, TFP would have been 5.9%, 4.5%, and 2.6% higher (Table 11A). The worsening in the labor demand terms of trade has its main culprits in the higher labor demand from the credit intermediation, computer and electronics, oil and gas extraction, and publishing sectors. Without them, TFP would have been 2.40%, 2.25%, 1.79%, and 1.34% higher, respectively. Labor demand by the wholesale trade and insurance sectors acted as a buffer, and in their absence, TFP would have been 1.62% and 1.61% lower, respectively. The higher profit margins for the credit intermediation and the chemical products sectors explain the improvement in the competitive terms of trade. Without them, TFP would have been 2.16% and 1.06% lower. The shift of final and intermediate demand toward computers and electronics fostered the improvement in the final and intermediate demand terms of trade. In their absence, TFP would have been 1.50% and 1.24% lower, respectively. The shift of final and intermediate demand toward the wholesale trade sector worsened the final and intermediate demand terms of trade. In their absence, TFP would have been 1.18% and 1.21% higher, respectively (Tables 12A-15A). During this period, the primary sources of variation for misallocation were the profit margins from the financial, chemical products, and utilities sectors, and the labor demand from oil and gas extraction (Tables 16A-20A).

Table 9: Technology Covariance Decomposition by Industry

A. Between 1998 and 2020

1	Oil & gas extraction	12.09%
2	Insurance carriers	9.39%
3	Air transportation	9.32%
4	Utilities	8.83%
5	Securities & investment	5.90%
6	Chemical products	4.84%
7	Motor vehicles	4.33%

B. Between 2002 and 2009

1	Oil & gas extraction	20.71%
2	Securities & investment	20.12%
3	Utilities	17.40%
4	Chemical products	7.64%
5	Insurance carriers	7.31%
6	Motor vehicles	5.18%
7	Internet & inf. services	5.08%
8	Credit intermediation	4.78%
	⋮	
66	Petroleum & coal	-8.64%

C. Between 2010 and 2020

1	Air transportation	14.42%
2	Insurance carriers	12.46%
3	Arts, sports & museums	6.55%
4	Management of companies	5.61%
5	Oil & gas extraction	4.15%
6	Housing	4.63 %
7	Motor vehicles	4.62%
8	Petroleum & coal	4.61%
9	Other real estate	4.60%
10	Food Services	4.26%
	⋮	
66	Farms	-4.58%

Table 10: Competitiveness Covariance Decomposition by Industry

A. Between 1998 and 2020

1	Oil & gas extraction	21.16%
2	Securities & investment	12.00%
3	Utilities	11.03%
4	Chemical products	10.82%
5	Rental & leasing	4.39%

B. Between 2002 and 2009

1	Securities & investment	18.24%
2	Chemical products	14.79%
3	Utilities	14.16%
4	Oil & gas extraction	12.79%
5	Insurance carriers	8.36%
6	Credit intermediation	6.52%
5	Rental & leasing	4.19%
	⋮	
64	Legal services	-4.90%
65	Telecommunications	-5.05%
66	Housing	-9.14%

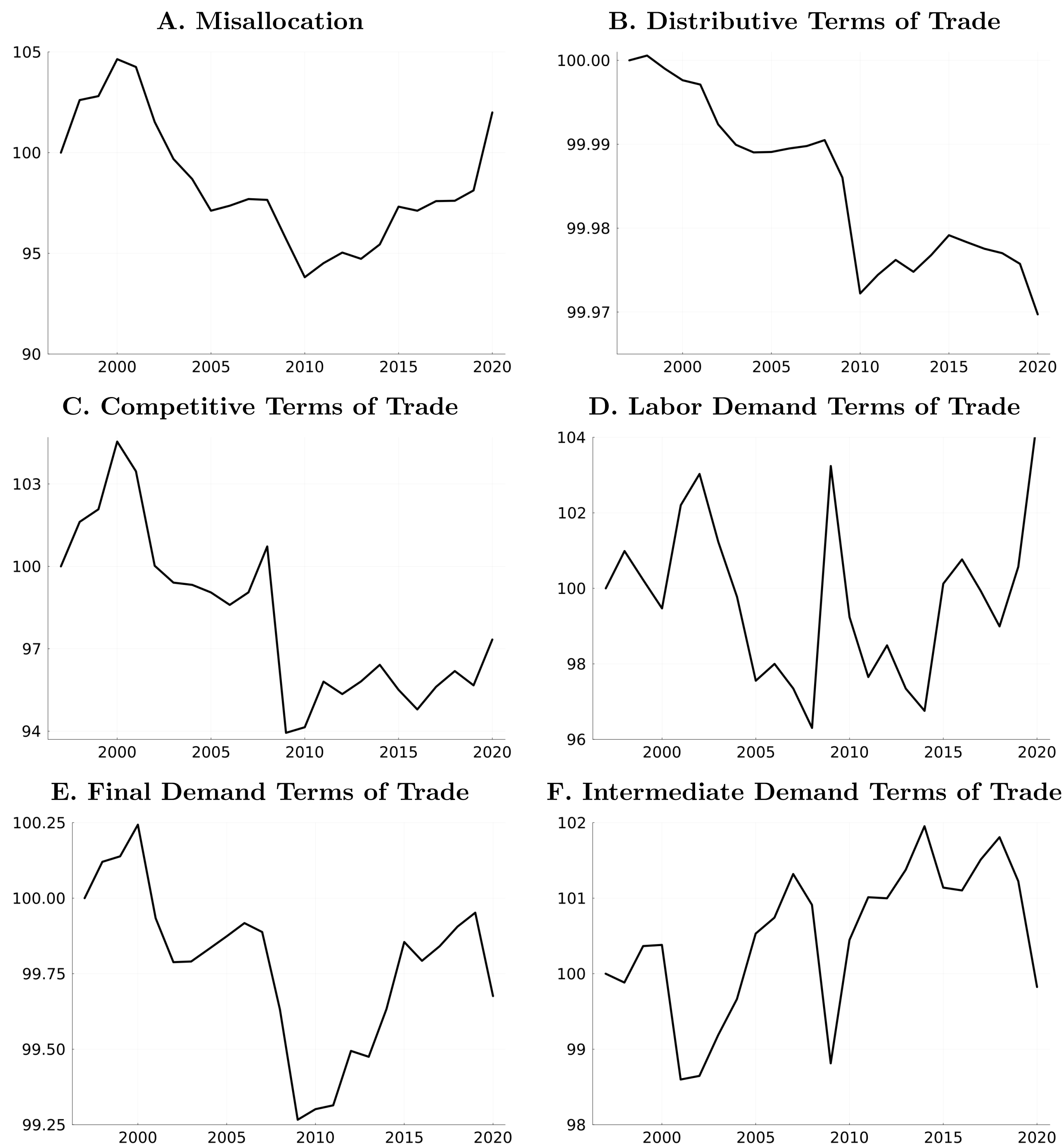
C. Between 2010 and 2020

1	Oil & gas extraction	28.82%
2	Chemical products	16.08%
3	Securities & investment	14.94%
4	Rental & leasing	12.91%
5	Insurance carriers	12.57%
6	Telecommunications	9.59%
7	Air transportation	9.04%
8	Food, beverages & tobacco	6.07%
9	Wholesale trade	4.99%
10	Petroleum & coal	4.20%
	⋮	
64	Farms	-9.15%
65	Credit intermediation	-12.62%
66	Other real estate	-20.84%

Notes: Only sectors with more than 4% in absolute value are included. From Theorem 2, the covariance decomposition for $Technology_t$ and $Competitiveness_t$ are respectively given by $Var\,(Technology_t) = \sum_{i\in\mathcal{N}} Cov\left(\widetilde{\lambda}_{i,t}\, d\log A_{i,t},\ Technology_t\right)$ and $Var\,(Competitiveness_t) = \sum_{i\in\mathcal{N}} Cov\left(\widetilde{\lambda}_{i,t}\, d\log \mu_{i,t},\ Competitiveness_t\right)$.

Before the GR misallocation improved, and without its effect on growth, TFP would have grown 8.2% less (Table 5B). The improvement in the competitive terms of trade explained the reduction in misallocation. Without it, TFP would have grown 11.1% less (Table 11B). The improvement in the competitive terms of trade mainly originates in the higher profit margins from the oil and gas extraction, computer and electronic, and internet and information services sectors. Withouth them, TFP would have been 1.46%, 1.11%, and 1.01% lower (Table 12B). The main sources of volatility in misallocation were the profit margins for the financial, chemical product, and utility sectors (Tables

Figure 5: Misallocation Decomposition



Notes: Theorem 2 provides the decomposition for *Misallocation*. Each channels is normalized to 100 in the year 1997.

16B-17B).

Table 11: Counterfactual TFP Growth Differential in the Absence of Misallocation Components

A. Between 1998 and 2020

Heterogeneity	***Distributive TT***	***Competitive TT***	***Labor DTT***	***Final DTT***	***Intermediate DTT***
Rep. Household	0%	-3.4%	6.3%	0.4%	-1.3%
Occupation	0%	-5.9%	15.1%	-2.0%	-4.2%
County	0.1%	-5.2%	14.2%	-0.9%	-4.4%
State & Occupation	0.1%	-5.9%	15.6%	-2.6%	-4.5%

B. Between 2002 and 2009

Heterogeneity	***Distributive TT***	***Competitive TT***	***Labor DTT***	***Final DTT***	***Intermediate DTT***
Rep. Household	0%	-9.3%	1.1%	-0.9%	-0.2%
Occupation	0%	-11.0%	3.4%	-1.9%	-0.8%
County	0.1%	-10.4%	3.4%	-0.7%	-1.0%
State & Occupation	0.1%	-11.1%	3.4%	-2.0%	-0.9%

C. Between 2010 and 2020

Heterogeneity	***Distributive TT***	***Competitive TT***	***Labor DTT***	***Final DTT***	***Intermediate DTT***
Rep. Household	0%	3.9%	1.2%	1.7%	0.9%
Occupation	0%	2.9%	7.2%	0.2%	-1.8%
County	0.1%	3.0%	3.5%	2.1%	-1.5%
State & Occupation	0.1%	2.8%	7.4%	-0.1%	-1.7%

Notes: The model estimation comes from compounding variations on TFP from equation (14). To be more precise, $TFP_t = TFP_0 \times \Pi_{q=1}^{t} exp\,(d\,log\,TFP_q)$. The results in each column come from obtaining sequences for $d\,log\,TFP_q$ that leave out one of the *misallocation* channels from Theorem 2 at the time.

Table 12: Counterfactual TFP Growth Without Sectoral Competitive TT

A. Between 1998 and 2020

1	Credit intermediation	-2.16%
2	Chemical products	-1.06%
3	Computer & electronics	-0.98%
4	Publishing industries	-0.80%
5	Internet & inf. services	-0.69%
	⋮	
64	Insurance carriers	0.77%
65	Other services	0.81%
66	Misc. professional services	0.87%

B. Between 2002 and 2009

1	Oil & gas extraction	-1.46%
2	Computer & electronics	-1.11%
3	Internet & inf. services	-1.01%
4	Wholesale trade	-0.92%
5	Telecommunications	-0.86%
6	Utilities	-0.84%
7	Publishing industries	-0.82%

C. Between 2010 and 2020

1	Credit intermediation	-2.0%
2	Securities & investment	-0.52%
	⋮	
64	Administrative services	0.62%
65	Misc. professional services	0.70%
66	Oil & gas extraction	1.91%

Table 13: Counterfactual TFP Growth Without Sectoral Labor Demand TT

A. Between 1998 and 2020

1	Wholesale trade	-1.62%
2	Insurance carriers	-1.61%
3	Other retail	-1.07%
	⋮	
61	Utilities	0.69%
62	Computer systems design	0.82%
63	Publishing industries	1.34%
64	Oil & gas extraction	1.79%
65	Computer & electronics	2.28%
66	Credit intermediation	2.40%

B. Between 2002 and 2009

1	Securities & investment	-0.96%
	⋮	
64	Computer & electronicss	0.85%
65	Utilities	1.02%
66	Oil & gas extraction	2.20%

C. Between 2010 and 2020

1	Wholesale trade	-1.70%
2	Insurance carriers	-1.03%
3	Administrative services	-0.93%
4	Other retail	-0.83%
	⋮	
64	Publishing industries	0.89%
65	Computer & electronics	0.98%
66	Credit intermediation	2.44%

Table 14: Counterfactual TFP Growth Without Sectoral Final Demand TT

A. Between 1998 and 2020

1	Computer & electronics	-1.50%
2	Motor vehicles	-0.91%
3	Machinery	-0.88%
4	Apparel & leather	-0.51%
	⋮	
62	Securities & investment	0.87%
63	Misc. professional services	0.94%
64	Hospitals	0.95%
65	Internet & inf. services	1.01%
66	Wholesale trade	1.18%

B. Between 2002 and 2009

1	Construction	-1.22%
2	Motor vehicles	-0.82%
	⋮	
66	Hospitals	0.58%

C. Between 2010 and 2020

1	Computer & electronis	-0.52%
	⋮	
63	Other retail	0.59%
64	Internet & inf. services	0.60%
65	Construction	0.89%
66	Wholesale trade	1.08%

Table 15: Counterfactual TFP Growth Without Sectoral Intermediate Demand TT

A. Between 1998 and 2020

1	Computer & electronics	-1.24%
2	Credit intermediation	-0.90%
3	Publishing industries	-0.76%
4	Computer systems design	-0.45%
5	Ambulatory health	-0.42%
	⋮	
61	Telecommunications	0.52%
62	Administrative services	0.54%
63	Hospitals	0.56%
64	Insurance carriers	0.74%
65	Other retail	0.90%
66	Wholesale trade	1.21%

B. Between 2002 and 2009

1	Computer & electronics	-0.48%
	⋮	
66	Securities & investment	0.49%

C. Between 2010 and 2020

1	Credit intermediation	-0.97%
2	Publishing industries	-0.51%
3	Computer & electronics	-0.49%
	⋮	
63	Insurance carriers	0.52%
64	Administrative services	0.63%
65	Other retail	0.66%
66	Wholesale trade	1.12%

Notes: In Tables 12 and 13 only sectors with more than 0.6% in absolute value are included. In Table 14 only sectors with more than 0.5% in absolute value are included. In Table 15 only sectors with more than 4% in absolute value are included. For each estimation, using Theorem 2 a counterfactual sequence for is constructed. This sequence excludes the effects from one industry in one specific channel at the time.

Table 16: Misallocation Covariance Decomposition

A. Between 1998 and 2020

Heterogeneity	*Distributive TT*	*Competitive TT*	*Labor DTT*	*Final DTT*	*Intermediate DTT*
Rep. Household	0%	61.9%	48.9%	3.5%	-14.3%
Occupation	0%	73.1%	37.0%	0.9%	-11.0%
County	0.1%	68.4%	49.6%	-4.8%	-13.3%
State & Occupation	0.1%	73.6%	38.6%	2.3%	-14.6%

B. Between 2002 and 2009

Heterogeneity	*Distributive TT*	*Competitive TT*	*Labor DTT*	*Final DTT*	*Intermediate DTT*
Rep. Household	0%	136.8%	-30.9%	-7.9%	2.0%
Occupation	0%	234.1%	-160.7%	-2.7%	29.3%
County	0.1%	197.1%	-95.2%	-15.1%	13.1%
State & Occupation	0.1%	155.9%	-78.4%	3.5%	18.9%

C. Between 2010 and 2020

Heterogeneity	*Distributive TT*	*Competitive TT*	*Labor DTT*	*Final DTT*	*Intermediate DTT*
Rep. Household	0%	18.0%	125.8%	-3.2%	-40.6%
Occupation	0%	24.0%	129.5%	-12.8%	-40.7%
County	0.1%	18.9%	140.5%	-17.0%	-42.5%
State & Occupation	0.1%	19.2%	131.4%	-3.1%	-47.6%

Notes: The fourth definition for *Misallocation* in Theorem 2 is used for its covariance decomposition.

After the GR misallocation worsened, and without its effect on growth, TFP would have grown 7.5% more (Table 5C). The increase in labor demand and competitive terms of trade explained the rise in misallocation. Without them, TFP would have grown 7.4% and 2.8% more, respectively (Table 11C). The worsening in the labor demand terms of trade has its main culprits in the higher labor demand from the credit intermediation sector and the increase for competitive terms of trade in the lower profit margins for the oil and gas extraction industries. Without them, TFP would have been 2.44% and 1.91% higher, respectively (Tables 12C-13C). The main sources of volatility were the labor demand from the oil and gas extraction and the chemical product sectors (Tables 16C-18B).

The distributive terms of trade had a minuscule role in the misallocation variation and the volatility (Tables 11 and 16). My explanation is the low heterogeneity at the state level in consumption bundles and, consequently, in the average expenditures' average distortion centralities M_h.

Table 17: Competitive TT Covariance Decomposition by Industry

A. Between 1998 and 2020

1	Securities & investment	22.92%
2	Chemical products	12.37%
3	Utilities	10.31%
4	Food, beverages & tobacco	6.87%
5	Oil & gas extraction	6.09%
6	Insurance carriers	5.77%
7	Computer & electronics	4.56%
8	Misc. manufacturing	4.08%

B. Between 2002 and 2009

1	Securities & investment	27.94%
2	Chemical products	14.51%
3	Utilities	11.17%
4	Insurance carriers	7.81%
5	Food, beverages & tobacco	7.62%
6	Misc manufacturing	4.00%

C. Between 2010 and 2020

1	Securities & investment	19.28%
2	Insurance carriers	11.17%
3	Air transportation	9.71%
4	Chemical products	9.57%
5	Food, beverages & tobacco	7.71%
6	Apparel & leather	7.32%
7	Oil & gas	5.07%
8	Misc. manufacturing	4.30%
	⋮	
64	Other real estate	-6.05%
65	Credit intermediation	-6.95%
66	Farms	-8.40%

Table 18: Labor Demand TT Covariance Decomposition by Industry

A. Between 1998 and 2020

1	Oil & gas extraction	18.49%
2	Chemical products	12.03%
3	Utilities	10.18%
4	Securities & investment	6.49%
5	Insurance carriers	4.77%
6	Petroleum & coal	4.16%
7	Computer & electronics	4.11%

B. Between 2002 and 2009

1	Oil & gas extraction	19.61%
2	Utilities	14.01%
3	Chemical products	12.48%
4	Insurance carriers	7.56%
5	Computer & electronicSs	6.21%
6	Securities & investment	5.67%
7	Food, beverages & tobacco	4.03%

C. Between 2010 and 2020

1	Oil & gas extraction	17.45%
2	Chemical production	12.09%
3	Utilities	6.61%
4	Petroleum & coal	4.52%
5	Food, beverages & tobacco	4.30%
6	Primary metals	4.09%

Table 19: Final Demand TT Covariance Decomposition by Industry

A. Between 1998 and 2020

1	Accommodation	31.06%
2	Apparel & leathers	29.34%
3	Arts, sports & museums	25.19%
4	Food services	23.58%
5	Air transportation	21.21%
	⋮	
66	Food, beverages & tobacco	-36.91%

B. Between 2002 and 2009

1	Motor vehicles	51.73%
2	Furniture	39.99%
3	Computer & electronics	25.94%
	⋮	
66	Food, beverages & tobacco	-16.00%

C. Between 2010 and 2020

1	Accommodation	54.88%
2	Arts, sports & museums	50.58%
3	Food services	46.49%
4	Apparel & leather	39.87%
5	Air transportation	33.81%
6	Hospitals	25.26%
7	Misc. professional services	21.43%
8	Recreational & gambling	19.01%
9	Ambulatory health care	18.04%
10	Petroleum & coal	16.49%
	⋮	
64	Misc. manufacturing	-26.47%
65	Computer & electronics	-31.90%
66	Food, beverages & tobacco	-59.61%

Table 20: Intermediate Demand TT Covariance Decomposition by Industry

A. Between 1998 and 2020

1	Securities & investment	12.24%
2	Chemical products	11.36%
3	Oil & gas extraction	9.91%
4	Hospitals	6.31%
5	Apparel & leather	5.61%
6	Utilities	5.25%
7	Misc. manufacturing	4.10%

B. Between 2002 and 2009

1	Chemical products	14.36%
2	Computer & electronics	14.09%
3	Oil & gas extraction	9.13%
4	Apparel & leather	8.96%
5	Movies & music	4.49%
6	Management of companies	4.41%
7	Misc. manufacturing	4.34%

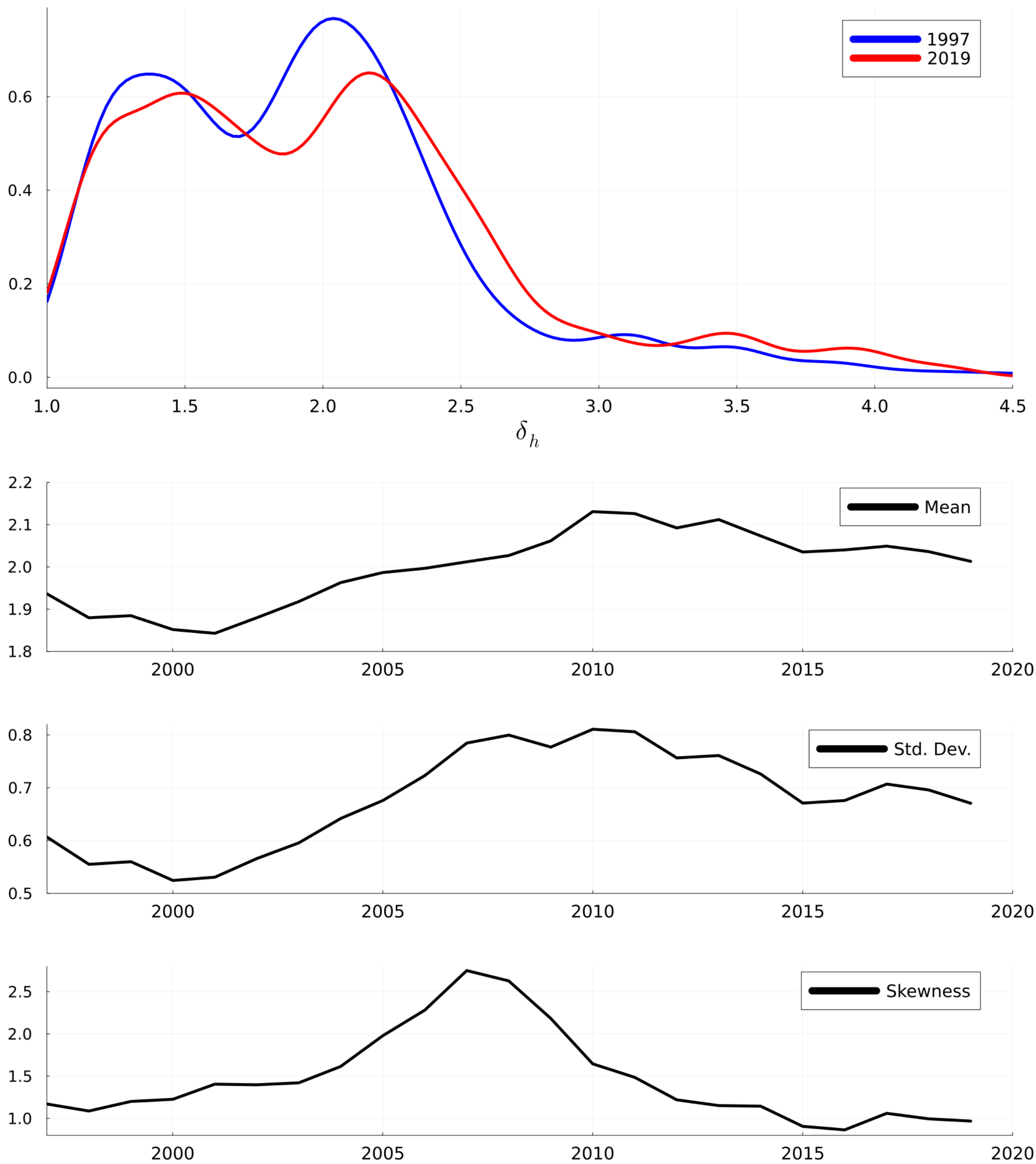
C. Between 2010 and 2020

1	Chemical products	13.71%
2	Oil & gas extraction	11.97%
3	Hospitals	9.52%
4	Accommodation	7.16%
5	Credit intermediation	7.13%
6	Air transportation	5.74%
7	Utilities	5.55%
8	Primary metals	4.50%
9	Ambulatory health care	4.36%
	⋮	
66	Food services	-6.00%

Notes: In Tables 17, 18, and 20 only sectors with more than 0.4% in absolute value are included. In Table 19 only sectors with more than 16% in absolute value are included. The fourth definition of *misallocation* in Theorem 2 is used for the different covariance decompositions.

6.2.2 Distributional Accounting

Figure 6: Distortion Centrality Density and Moments

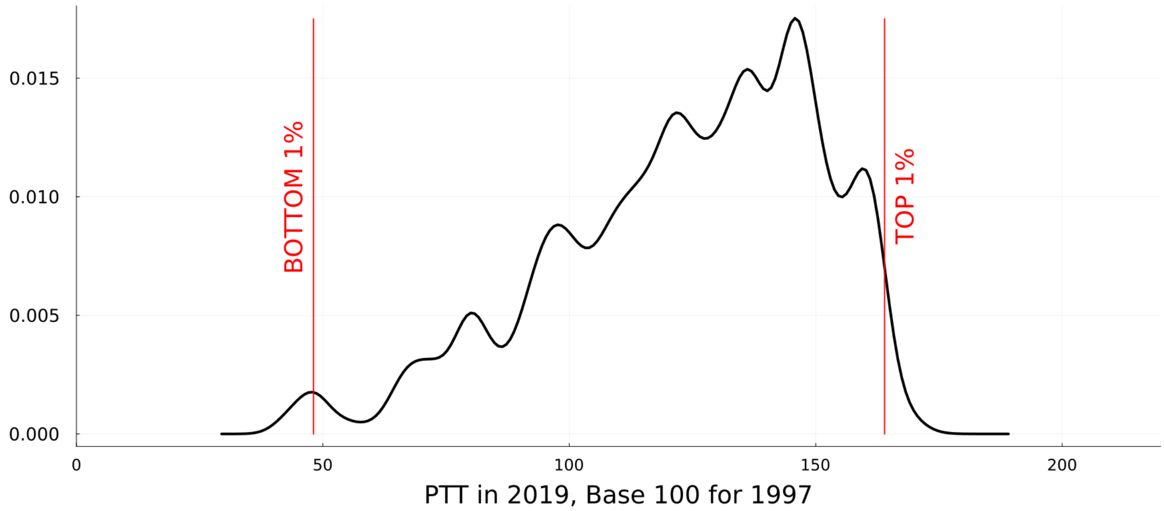


Notes: Distortion centralities are given by the ratios of cost- over revenue-based labor income rates, i.e. $\delta_h = \widetilde{\Lambda}_h/\Lambda_h$. These values are estimated using the system of equations in Table 2.

Figure 6 portrays the density of the distortion centralities for 1997 and 2019, and the variations across time of its first three moments. These distributions had an average distortion centrality of 2, and their skewness was positive. Before the GR, the three moments increased, and after GR, there was a partial reversal in mean and variance, while the skewness had a full reversal to its original level from its 2007 peak.

Figure 7 portrays the density of PTTs for the state and occupation interaction under the assumption that the 1997 level of PTTs is 100 for all workers. On average, there has been growth in PTTs, and the density has a negative skewness, which captures a heavy left tail of workers for which the shocks from the last two decades have not been favorable. The tails from this distribution tell us that the last two decades of shocks, on the one hand, have favored logging workers, workers with mathematical and computational occupations, and compensation managers, and on the other hand, the same set of shocks have been unfavorable for industrial workers with occupation exposed to the printing, shoe and leather, and textile industries.

Figure 7: Positional Terms of Trade in 2019



Top 1%	
Occupation	
Logging Workers	37%
Computer Occ.	13%
Mathematical Sciences Occ.	10%
Compensation Managers	7%

Bottom 1%	
Occupation	
Printing Workers	40%
Shoe & Leather Operator	26%
Textile Machine Operator	15%
Miscellaneous Textile	12%

Notes: The PTT estimations come from compounding variations using the decompositions from Theorem 3. To be more precise, $PTT_{h,t} = PTT_{h,0} \times \Pi_{q=1}^{t} exp\left(d\,log\,PTT_{h,q}\right)$ where $PTT_{h,1997} = 100$. The top and bottom 1% tables shows the occupational classifications that capture most of the households within the tails. For example, 37% of the households on the top 1% of the PTT distribution have occupations that are classified as logging workers.

7 Parametric Accounting

In this section, I derive the parametric ex-ante statistics necessary to characterize the first-order variations derived in Sections 4 and 5. These ex-ante measures depend on the model primitives. In this parametric environment, I identify a linear system of equations that solves the endogenous first-order variations in wages, household expenditure, and sales. This section finishes with a discussion about how the numeraire selection is non-neutral when the labor supply substitution and income effects are asymmetric.

7.1 Normalized CES Environment

Following Baqaee & Farhi (2019a,b, 2020, 2022), I extend the normalized CES function introduced by de La Grandville (1989) and Klump & de La Grandville (2000) to an economy with intermediate goods. The overlined variables correspond to the equilibrium values. Firm z_i in sector i uses the normalized CES composite

$$\frac{y_{z_i}}{\overline{y}_{z_i}} = A_i \left(\omega_i^{\ell} \sum_{h \in \mathscr{H}} \alpha_{ih} \left(\frac{\ell_{z_i h}}{\overline{\ell}_{z_i h}} \right)^{\frac{\theta_i - 1}{\theta_i}} + \omega_i^{x} \sum_{j \in \mathscr{N}} \omega_{ij} \left(\frac{x_{z_i j}}{\overline{x}_{z_i j}} \right)^{\frac{\theta_i - 1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i - 1}}.$$

In this production function, productivity shocks are Hicks-neutral normalized to 1 in equilibrium, and θ_i stands for the elasticity of substitution. Similarly, the consumption aggregator for the representative household of type h is given by

$$\frac{C_h}{\overline{C}_h} = \left(\sum_{i \in \mathscr{N}} \beta_{hi} \left(\frac{C_{hi}}{\overline{C}_{hi}} \right)^{\frac{\varrho_h - 1}{\varrho_h}} \right)^{\frac{\varrho_h}{\varrho_h - 1}},$$

where ϱ_h stands for the elasticity of substitution. The benefit from the normalized CES is that the parameters ω_i^{ℓ}, α_{ih}, ω_i^{x}, ω_{ij}, and β_{hi} have the same interpretation as in Section 3, and do not depend on deep parameters such as the elasticities of substitution (Klump et al., 2012).

Household h, which has an initial population size of n_h, operates under the following utility function

$$U_h\left(c_h, \widetilde{L}_h\right) = \frac{\left(c_h \left(1 - E_h^{-\gamma_h} \widetilde{L}_h\right)^{\varphi_h}\right)^{1-\sigma} - 1}{1 - \sigma},$$

with $C_h = n_h\, c_h$, $L_h = n_h\, \widetilde{L}_h$, and $\varphi_h > 0$. c_h and $\widetilde{L}_h$ represent the normalized real consumption and labor supply, which makes preferences independent from the population size. This utility function allows for greater flexibility in parametrizing the income and substitution effects on the labor supply.

Proposition 3. The change in labor supply from type h workers in response to demographic, wage, and income shocks is, to a first-order,

$$d\log L_h = \zeta_h^n\, d\log n_h + \zeta_h^w\, d\log w_h - \zeta_h^e\, d\log E_h.$$

Where the corresponding elasticities are given by

$$\zeta_h^n = \frac{E_h^{\gamma_h}}{1 - \varphi_h \gamma_h} \frac{n_h}{L_h}, \qquad \zeta_h^w = \frac{1}{1 - \varphi_h \gamma_h} \frac{\varphi_h}{\Gamma_h}, \qquad \zeta_h^e = \zeta_h^w - \gamma_h \zeta_h^n.$$

Proposition 3 characterizes the endogenous first-order variation of the labor supply in terms of elasticities for the: (1) demographic effect ζ_h^n; (2) substitution effect ζ_h^w; and (3) income effect ζ_h^e. These elasticities depend on equilibrium values and the deep preference parameters γ_h and φ_h.

This utility function nests the following preferences. First, by assuming $\gamma_h = 0$, I obtain King, Plosser, & Rebelo's (1988) preferences with symmetric substitution and income effects, more precisely, $\zeta_h^n = n_h/L_h$ and $\zeta_h^w = \zeta_h^e = \varphi_h/\Gamma_h$. Second, by using the preference parameters that solve $\gamma_h = \frac{1}{2\varphi_h}\left(1+\Gamma_h^{-1/2}\sqrt{\Gamma_h - 2\varphi_h^2}\right)$, $\varphi_h = \frac{1}{\gamma_h}\left(1 - E_h^{\gamma_h}\frac{n_h}{L_h}\right)$ and $\zeta_h^n = 1$, I obtain Greenwood, Hercowitz, & Huffman's (1988) preferences with no income effect, i.e., $\zeta_h^e = 0$. Finally, in its most general form, this utility is inspired by Jaimovich & Rebelo's (2009), and for this reason, it allows for asymmetric income and substitution effects. However, relative to the latter utility preferences, this specification allows for a direct effect from consumption expenditure in labor supply disutility through the parameter γ_h. The disutility effects from increasing the labor supply become weaker as this parameter increases, and as a consequence, there are stronger demographic and substitution effects.

7.2 Sufficient Endogenous Statistics

Theorem 5 characterizes a $2H+N$ linear system of equations that solves for the endogenous first-order variation of consumption expenditure, wages, and sales. These equations capture partial (PE) and general (GE) equilibrium effects.

Theorem 5. In a CES economy, the variation in consumption expenditure, wages, and sales, in response to productivity, distortion, and demographic shocks are, to a first-order,

$$d\log E_h = \overbrace{\frac{\zeta_h^n \Gamma_h}{1+\zeta_h^e\Gamma_h}\, d\log n_h}^{\substack{\textit{Demographic Effect}\\ \textit{on Expenditure (PE)}}} + \overbrace{\frac{(1+\zeta_h^w)\,\Gamma_h}{1+\zeta_h^e\Gamma_h} d\log w_h}^{\substack{\textit{Wage Effect}\\ \textit{on Expenditure (GE)}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\kappa_{ih}\lambda_i}{(1+\zeta_h^e\Gamma_h)\,\chi_h}\left((1-\mu_i)\, d\log S_i - \mu_i\, d\log\mu_i\right)}^{\substack{\textit{Corporate Income Effect}\\ \textit{on Expenditure (PE + GE)}}};$$

$$\begin{aligned} d\log w_h = {} & \overbrace{\frac{\zeta_h^e}{1+\zeta_h^w}\, d\log E_h}^{\substack{\textit{Expenditure Effect}\\ \textit{on Wages (GE)}}} - \overbrace{\frac{\zeta_h^n}{1+\zeta_h^w} d\log n_h}^{\substack{\textit{Demographic Effect}\\ \textit{on Wages (PE)}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\Omega_{ih}^{\ell}\lambda_i}{(1+\zeta_h^w)\,\Lambda_h}\left((\theta_i-1)\, d\log A_i + \theta_i\, d\log\mu_i\right)}^{\substack{\textit{Direct Effect}\\ \textit{on Wages (PE)}}} \\ & \overbrace{-\sum_{j\in\mathcal{N}}\left(\sum_{i\in\mathcal{N}} \frac{\Omega_{ih}^{\ell}\lambda_i}{(1+\zeta_h^w)\,\Lambda_h}(\theta_i-1)\,\widetilde{\psi}_{ij}^{x}\right)(d\log A_j + d\log\mu_j)}^{\substack{\textit{Supplier Effect}\\ \textit{on Wages (PE)}}} + \overbrace{\sum_{i\in\mathcal{N}} \frac{\Omega_{ih}^{\ell}\lambda_i}{(1+\zeta_h^w)\,\Lambda_h}\, d\log S_i}^{\substack{\textit{Sales Effect}\\ \textit{on Wages (GE)}}} \\ & \overbrace{-\left(\sum_{i\in\mathcal{N}} \frac{\Omega_{ih}^{\ell}\lambda_i}{(1+\zeta_h^w)\,\Lambda_h}(\theta_i-1)\right) d\log w_h}^{\substack{\textit{Direct Substitution Effect}\\ \textit{on Wages (GE)}}} + \overbrace{\sum_{b\in\mathcal{H}}\left(\sum_{i\in\mathcal{N}} \frac{\Omega_{ih}^{\ell}\lambda_i}{(1+\zeta_h^w)\,\Lambda_h}(\theta_i-1)\,\widetilde{\psi}_{ib}^{\ell}\right) d\log w_b}^{\substack{\textit{Supplier Substitution Effect}\\ \textit{on Wages (GE)}}}; \end{aligned}$$

$$
\begin{aligned}
d\log S_i =& \overbrace{\sum_{h\in\mathscr{H}} \frac{\beta_{hi}\chi_h}{\lambda_i}\, d\log E_h}^{\substack{\textit{Expenditure Effect}\\ \textit{on Sales (GE)}}} + \overbrace{\sum_{j\in\mathscr{N}} \frac{\Omega^x_{ji}\lambda_j}{\lambda_i}\, d\log S_j}^{\substack{\textit{Sales Effect}\\ \textit{on Sales (GE)}}} + \overbrace{\sum_{j\in\mathscr{N}} \frac{\Omega^x_{ji}\lambda_j}{\lambda_i}\left((\theta_j-1)\, d\log A_j + \theta_j\, d\log \mu_j\right)}^{\substack{\textit{Direct Effect}\\ \textit{on Sales (PE)}}} \\
&+ \overbrace{\sum_{j\in\mathscr{N}} \left(\sum_{h\in\mathscr{H}} \frac{\beta_{hi}\chi_h}{\lambda_i}(\rho_h-1)\left(\widetilde{\psi}^x_{ij} - \widetilde{\mathscr{B}}_{hj}\right) + \sum_{q\in\mathscr{N}} \frac{\Omega^x_{qi}\lambda_q}{\lambda_i}(\theta_q-1)\left(\widetilde{\psi}^x_{ij} - \widetilde{\psi}^x_{qj}\right)\right)(d\log A_j + d\log\mu_j)}^{\substack{\textit{Supplier Effect}\\ \textit{on Sales (PE)}}} \\
&+ \overbrace{\sum_{h\in\mathscr{H}} \left(\sum_{b\in\mathscr{H}} \frac{\beta_{bi}\chi_b}{\lambda_i}(\varrho_b-1)\left(\widetilde{\mathscr{C}}_{bh} - \widetilde{\psi}^\ell_{ih}\right) + \sum_{j\in\mathscr{N}} \frac{\Omega^x_{ji}\lambda_j}{\lambda_i}(\theta_j-1)\left(\widetilde{\psi}^\ell_{jh} - \widetilde{\psi}^\ell_{ih}\right)\right) d\log w_h}^{\substack{\textit{Supplier Substitution Effect}\\ \textit{on Sales (GE)}}}.
\end{aligned}
$$

For households of type h, the first-order variation for their consumption expenditure depends on three channels. First, for the *demographic effect*, in response to an increase in their labor force, the factoral supply will rise by ζ^n_h, and its effect on consumption expenditure is proportional to the labor income share Γ_h. Second, for the *wage effect*, a wage increase directly impacts income. However, it additionally triggers a substitution effect on the labor supply captured by ζ^w_h. The magnitude of this substitution effect on consumption expenditure is proportional to the labor income share Γ_h. Finally, for the *corporate income effect*, dividends from sector i depend both on sales and their markdowns: (i) an increase in sales augments dividend income by the rent extraction share $1-\mu_i$, and (ii) an increase in markdowns reduces profits by the cost share μ_i. These two paths for dividend income variation are proportional to the equity participation share κ_{ih} and the sales-to-expenditure ratio λ_i/χ_h. These three channels increase consumption expenditure and trigger an income effect that reduces the labor supply attenuating their magnitudes by $1+\zeta^e_h\Gamma_h$.

For workers of type h, the first-order variation for their wages depends on seven channels. These channels trigger a substitution effect that increases the labor supply and attenuates their influence on wages by $1+\zeta^w_h$. Additionally, the effect on w_h from the channels that depict variations in sector i's labor demand are proportional to the direct revenue-based centrality Ω^ℓ_{ih} and the sales to labor income ratio λ_i/Λ_h. First, for the *expenditure effect*, in response to an increase in their total income, their labor supply falls by ζ^e_h, and wages rise. Second, for the *demographic effect*, in response to an increase in their labor force, their labor supply rises by ζ^n_h, and wages fall. Third, the *direct effect* captures the increase in labor demand for these workers from the firms that receive either productivity or markdown shocks. Firm i increases their demand for workers of type h in response to a positive productivity shock as long as there is substitutability in their production (i.e., $\theta_i > 1$) and in response to lower distortions as long as the production function is not Leontief (i.e., $\theta_i > 0$). Fourth, the *supplier effect* portrays the variations in firms' labor demand in response to productivity and markdown shocks to its intermediate input suppliers. Firm i decreases its demand for workers of type h, as long as there is substitutability in their production, in response to positive productivity shocks and markdown reductions to its direct or indirect intermediate supplier j. The magnitude of this effect is proportional to the cost-based firm-to-firm centrality $\widetilde{\psi}^x_{ij}$. Fifth, the *sales effect* characterizes how sales increases expand labor demand. Sixth, the *direct substitution effect* portrays the variation in firms' labor demand for workers of type h in response to variations in w_h. Firm i increases their demand for workers of type h when w_h falls and there is substitutability in production. Finally, the *supplier substitution effect* captures the variations in firms' labor demand for workers of type h in response to wage changes for all other workers. Firm i increases their demand for workers of type h when the wage from workers of type b rises and there is substitutability in production. The magnitude of this effect is proportional to the cost-based worker-to-firm centrality $\widetilde{\psi}^{\ell}_{ib}$.

For firms in sector i, the first-order variation for their sales depends on five channels. The channels that represent variation in the demand of final goods by households of type h are proportional to their consumption share β_{hi} and the Domar weight ratio χ_h/λ_i, and those that illustrate changes in the demand for intermediate goods by firms in sector j are proportional to the direct revenue exposure Ω^{x}_{ji} and the Domar weight ratio λ_j/λ_i. First, the *expenditure effect* captures how higher household expenditure increases demand for final goods. Second, the *sales effect* portrays how higher firms' sales increase demand for intermediate goods. Third, the *direct effect* characterizes the increase in intermediate input demand from firms that receive either productivity or markdown shocks. Firm j increases their demand for good i in response to positive productivity shocks as long as there is substitutability and in response to lower distortions as long as the production function is not Leontief. Fourth, the *supplier effect* characterizes the variations in households' and firms' demand for goods in response to productivity and markdown shocks to its direct or indirect suppliers. Under substitutability, household h and firm q increase their demand for good i in response to increases in productivity or markdowns to its direct or indirect supplier j if their cost-based centrality to firm j is smaller than the one that firms in sector i have. In other words, when firm j reduces its price, the demand by households of type h and firms from sector q for the good i rises if their cost-based exposure to the shock is weaker than the one from firms in sector i, i.e., $\widetilde{\psi}^{x}_{ij} > \widetilde{\mathscr{B}}_{hj}$ and $\widetilde{\psi}^{x}_{ij} > \widetilde{\psi}^{x}_{qj}$. Finally, the *supplier substitution effect* portrays the increase in households' and firms' demand for goods in response to wage variations. Household b and firm j increase their demand for good i in response to the increase in prices from higher wages for workers of type h if there is substitutability and their cost-based centralities to firm j are larger than the one that firms in sector i have, i.e., $\widetilde{\mathscr{C}}_{bh} > \widetilde{\psi}^{\ell}_{ih}$ and $\widetilde{\psi}^{\ell}_{jh} > \widetilde{\psi}^{\ell}_{ih}$.

The solution in Theorem 5 represents an alternative to Baqaee & Farhi's (2022) results for the following five reasons: (1) it does not require the production network covariance operator introduced by Baqaee & Farhi (2019a); (2) it utilizes the measures of centrality from the substochastic Markov chain; (3) it captures the influence of the labor supply demographic, substitution, and income elasticities; (4) it decomposes the channels from productivity, markdown, and wage variations in direct effects, and effects through intermediate input suppliers; and (5) the variations are expressed in nominal terms and not in Domar weights because using the nominal GDP as the numeraire is not required.

7.3 Numeraire Non-Neutrality

From Walras' Law, to solve the model, take $2H + N - 1$ of the equations in Theorem 5, and normalize the variation in this system by using Y as the numeraire, which implies that there are no variations in the global GDP deflator, i.e., $d\,log\,p_Y = 0$. This follows Hulten (1978), Baqaee & Farhi (2019a), and Bigio & La'O (2020) who also use Y as the real unit of account. Now, as mentioned in Section 4, this is not the only normalization that the literature has used, as Baqaee & Farhi (2020, 2022) use nominal GDP as the numeraire.

Proposition 4 portrays the differences in GDP growth and household-level real consumption between using Y or nominal GDP as the numeraire.

Proposition 4. The differences between normalizing with $d\log p_Y = 0$ and $d\log GDP = 0$ are, to a first-order:

$$\frac{d\log Y|d\log GDP = 0}{d\log Y|d\log p_Y = 0} = \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{1+\zeta_h^e}{1+\zeta_h^w};$$

$$(d\log C_h|d\log p_Y = 0) - (d\log C_h|d\log GDP = 0)$$
$$= \frac{\sum_{q\in\mathscr{H}} \widetilde{\mathscr{C}}_{hq} \frac{\zeta_q^w - \zeta_q^e}{1+\zeta_q^w}}{\sum_{q\in\mathscr{H}} \widetilde{\Lambda}_q \frac{1+\zeta_q^e}{1+\zeta_q^w}} \left(\sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(d\log \mathscr{A}_i + d\log \mu_i\right) + \sum_{b\in\mathscr{H}} \widetilde{\Lambda}_b \frac{\zeta_b^n\, d\log n_b - \zeta_b^e\, d\log \chi_b - d\log \Lambda_b}{1+\zeta_b^w} \right).$$

Proposition 4 characterizes the biases in growth between these two numeraire assumptions. The biases exist if there is an endogenous factor supply with asymmetric substitution and income effects. At the aggregate level, the bias is multiplicative, while at the household level, it is additive. The biases from assuming nominal GDP as the unit of account are positive if the income effect strictly dominates the substitution effect, i.e., $\zeta_h^e > \zeta_h^w \ \forall h \in \mathscr{H}$. There are no biases from normalization if $\zeta_h^e = \zeta_h^w \ \forall h \in \mathscr{H}$. One advantage of Theorems 1, 2, 3, and 4 over the comparable results in Baqaee & Farhi (2020, 2022), is that my derivations for these sufficient ex-post statistics do not require a normalization assumption. For this reason, they are independent of these biases.

8 Simple Horizontal Economy

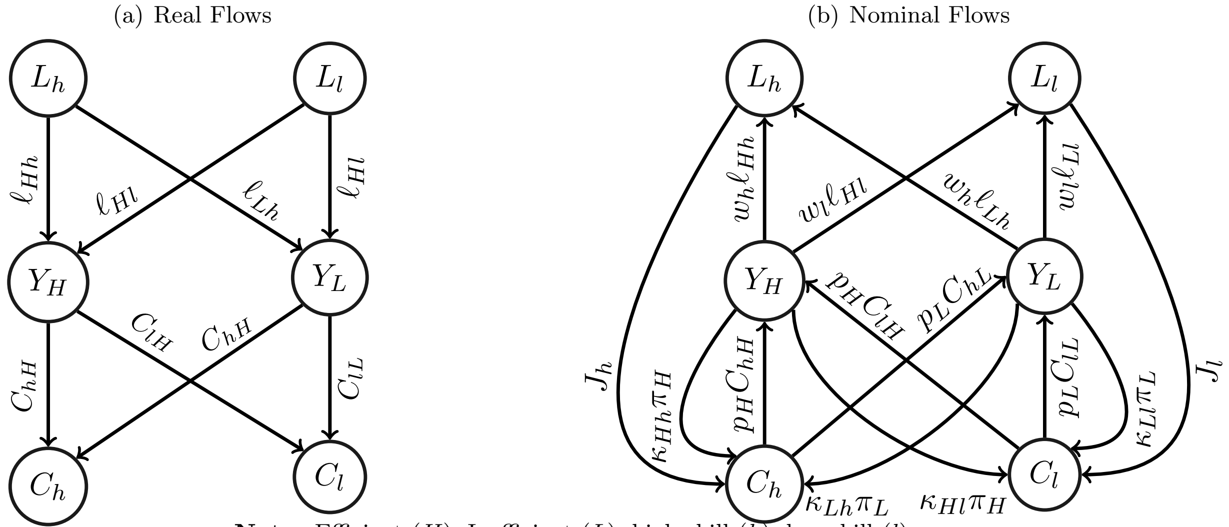
In order to simplify the understanding of the main mechanisms behind the effects of the income and consumption expenditure distributions on aggregate TFP, I will use the following horizontal economy with two types of workers and two firms. This example distills the model from this paper to the most basic structure for which there are still distributional effects on TFP.

Firms are efficient (H) or inefficient (L), and workers can be high-skill (h) or low-skill (l). The production from firms follows a Cobb-Douglas production function $y_i = A_i\, \ell_{ih}^{\alpha_i}\, \ell_{il}^{1-\alpha_i}$, where α_i is the firm i's eintensity in high-skill workers. The consumption aggregator for each household $r \in \{h, l\}$ follows a normalized CES

$$\frac{C_r}{\overline{C}_r} = \left(\beta_r \left(\frac{C_{rH}}{\overline{C}_{rH}}\right)^{\frac{\varrho-1}{\varrho}} + (1-\beta_r)\left(\frac{C_{rL}}{\overline{C}_{rL}}\right)^{\frac{\varrho-1}{\varrho}} \right).$$

Where β_r is the preference parameter on the efficient good and ϱ is the households' elasticity of substitution. Figure 8 represents the real and nominal flows for this economy.

Figure 8: Horizontal Economy



Note: Efficient (H), Inefficient (L), high-skill (h), low-skill (l).

Let me assume that the efficient firm requires more high skill, the high-skill prefers to consume efficient goods, and profits are symmetrically distributed: (i) $\mu_H \geq \mu_L$, (ii) $\alpha_H \geq \alpha_L$, (iii) $\beta_h \geq \beta_l$, and (iv) $\kappa_{H\,h} = \kappa_{L\,h} = 0.5$. Additionally, $\mu_H + \mu_L = \alpha_H + \alpha_L = \beta_h + \beta_l = 1$. These restrictions on the parameter space simplify the solution.

First, in the absence of distortions ($\mu_H = \mu_L = 1$), the equilibrium for this economy implies symmetry in expenditure across firms and householdsl, i.e., $\lambda_H = \chi_h = \Lambda_h = \frac{1}{2}$. Additionally, $\frac{\ell_{Hh}}{L_h} = \alpha_H$ of the high-skill's work is supplied to the efficient firm. Furthermore, the labor supply equilibrium condition for each household is given by $-\frac{U_{L_r}}{U_{C_r}} = \frac{C_r}{L_r}$.

With distortions, the equilibrium is characterized by the following solution in terms of sales, expenditure, labor income, and value-added shares

$$\lambda_H = \frac{1 - \mu_L \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)}{2 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)}, \qquad \chi_h = \frac{1 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \mu_H\right)}{2 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)},$$

$$\Lambda_h = \frac{\alpha_L + \mu_H \left(\alpha_H - \alpha_L\right)\left(1 - \mu_L \left(\beta_h - \beta_l\right)\right)}{2 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)}, \qquad \Lambda_l = \frac{\alpha_H - \mu_H \left(\alpha_H - \alpha_L\right)\left(1 + \mu_L \left(\beta_h - \beta_l\right)\right)}{2 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)},$$

$$\widetilde{\Lambda}_h = \frac{1 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)\left(\alpha_H - \mu_H \left(\alpha_H - \alpha_L\right)\right)}{2 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)}.$$

Additionally, the aggregate labor income share is given

$$\Lambda = \Lambda_h + \Lambda_l = \frac{1 - 2\,\mu_H\,\mu_L \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)}{2 - \left(\alpha_H - \alpha_L\right)\left(\beta_h - \beta_l\right)}.$$

The Domar weights, the value-added shares, and the aggregate labor share are symmetric in the absence of α and β heterogeneity. Absorption and labor income shares are symmetric without α and μ heterogeneity. Furthermore, the labor supply equilibrium condition now is given by $-\frac{U_{L_r}}{U_{C_r}} = \Gamma_r \, \frac{C_r}{L_r}$.

The labor wedges Γ_r imply that the feasible production set might differ from the production possibility frontier in the undistorted economy.

I will compare a benchmark with μ, α, and β heterogeneity with three alternative scenarios in which I shut down one of the types of heterogeneity at the time.

Symmetric μ: $\lambda_H = \widetilde{\Lambda}_h = \chi_h = \frac{1}{2}$, $\lambda_h = \lambda_l = \frac{1}{4}$, and $\frac{\ell_{Hh}}{L_h} = \alpha_H$. This allocation depends on the relationship between the marginal rate of substitution between goods and the markdown-adjusted marginal productivities of labor

$$\frac{U_{C_{rH}}}{U_{C_{rL}}} = \frac{\mu_L}{\mu_H} \frac{d\,y_L/d\,\ell_{Lr}}{d\,y_H/d\,\ell_{Hr}}.$$

Under symmetric markdowns, the composition of the consumption bundle and the expenditure distribution are the same as in the undistorted equilibrium. Consequently, the allocation of workers is the same, and α_H of the high-skill labor is supplied to the efficient firm.

If we consider the possibility of heterogeneity in the distribution of profits, the allocation of workers will no longer be the same. For example, if the high-skill households received all profits, more labor would be allocated to the efficient firm, i.e., $\frac{\ell_{Hh}}{L_h} = \alpha_H + \alpha_H\,\alpha_L\,(\beta_h - \beta_l)$. The difference in the allocation of workers relative to the undistorted economy is not a misallocation, as it does not originate from the effects of distortions on the marginal rates of substitution between goods. The solution is still on the Pareto set. The composition of the consumption bundle for each household is still the same; the difference now is that the high-skill receive a higher share of aggregate income and use it to consume more goods from the efficient sector.

Symmetric α: The distributions are the same as in the symmetric μ scenario. However, the distortions on the marginal rates of substitution incline consumption bundles towards the efficient good. Consequently, there is an excess of labor supplied to the efficient sector $\frac{\ell_{Hh}}{L_h} = 2\alpha_H\,\mu_H > \alpha_H$. This case indicates that the four distributions λ, χ, Λ, and $\widetilde{\Lambda}$ are insufficient to identify if there is misallocation.

Symmetric β: $\lambda_H = \widetilde{\Lambda}_h = \frac{1}{2}$, $\chi_h = \frac{1}{2} + \frac{1}{4}\,(\alpha_H - \alpha_L)\,(\mu_H - \mu_L)$, $\Lambda_h = \frac{1}{2}\,(\alpha_H\,\mu_H + \alpha_L\,\mu_L)$, and $\Lambda_l = \frac{1}{2}\,((1-\alpha_H)\,\mu_H + (1-\alpha_L)\,\mu_L)$. Additionally, distortions on the marginal rates of substitution incline consumption bundles towards the efficient good, and there is an excess of labor supplied to the efficient sector $\frac{\ell_{Hh}}{L_h} = \frac{\alpha_H\,\mu_H}{\alpha_H\,\mu_H + \alpha_L\,\mu_L} > \alpha_H$.

Bilateral centralities are given by

$$\psi_{ih} = \alpha_i\,\mu_i, \qquad \psi_{il} = (1-\alpha_i)\,\mu_i, \qquad \mathscr{C}_{rh} = \beta_r\,\psi_{Hh} + (1-\beta_r)\,\psi_{Lh}, \qquad \mathscr{C}_{rl} = \beta_r\,\psi_{Hl} + (1-\beta_r)\,\psi_{Ll}.$$

To simplify the exposition, let me consider only a shock to the productivity from the inefficient sector

$d\log A_L = 1\%$. Theorem 1 tells us that the labor income share variations are given by

$$d\Lambda_h = \underbrace{(\beta_h - \beta_l)(\mu_H - \alpha_L)\, d\chi_h}_{\text{Distributive Income}_h} + \underbrace{(\alpha_H - \mu_L)\sum \chi_r\, d\beta_r}_{\text{Income Centrality}_h},$$

$$d\Lambda_l = \underbrace{(\beta_h - \beta_l)(\mu_H - \alpha_H)\, d\chi_h}_{\text{Distributive Income}_l} + \underbrace{(\mu_H - \alpha_H)\sum \chi_r\, d\beta_r}_{\text{Income Centrality}_l},$$

$d\beta_r = (\varrho - 1)\,\beta_r\,(1-\beta_r)\; d\log\frac{p_L}{p_H}$, $d\log\frac{p_L}{p_H} = -1 + (\alpha_H - \alpha_L)\, d\log\frac{w_l}{w_h}$, and $d\log\frac{w_h}{w_l} = \frac{1}{1+\zeta^w} d\log\frac{\Lambda_h}{\Lambda_l} + \frac{\zeta^e}{1+\zeta^w} d\log\frac{\chi_h}{\chi_l}$.

From Theorem 2, TFP growth is given by[13]

$$d\log TFP = \lambda_L - \text{Misallocation}$$

$$\begin{aligned}\text{Misallocation} &= \overbrace{\boldsymbol{a}\,(\mu_H - \mu_L)\,(\alpha_H - \alpha_L)\;\, d\Lambda_\ell}^{\textit{Labor Income Reallocation}} + \delta_h d\Lambda \\ &= \underbrace{\boldsymbol{b}\,(\mu_H - \mu_L)\,(\beta_h - \beta_l)\, d\chi_h}_{\textit{Distributive Terms of Trade}} + \underbrace{\boldsymbol{b}\,(\mu_H - \mu_L)\sum \chi_r\, d\beta_r}_{\textit{Final Demand Terms of Trade}}\;.\end{aligned}$$

Misallocation increases with the labor income share for the low-skill, the aggregate labor share, the income share for the high-skill, and the expenditure intensity on the efficient good. These channels reallocate labor from the inefficient to the efficient firm, accentuating misallocation.

From Theorem 5, distributional variations are given by

$$d\lambda_H = -\frac{2\,(\varrho-1)\,\beta_h\beta_l}{2-(\alpha_H-\alpha_L)\,(\beta_h-\beta_l)+2\,(\varrho-1)\,(\alpha_H-\alpha_L)\,\frac{\beta_h\beta_l}{1+\zeta^w}\left(\frac{\alpha_H\Lambda-\mu_H\Lambda_h-\mu_L\Lambda_l}{\Lambda_h\,\Lambda_l}+\zeta^e\frac{\alpha_H-\alpha_L}{2\,\chi_h\,\chi_l}\right)},$$

$$d\Lambda_h = (\alpha_H - \mu_L)\, d\lambda_H, \qquad d\Lambda_l = (\mu_H - \alpha_H)\, d\lambda_H, \qquad d\chi_h = \frac{1}{2}(\alpha_H - \alpha_L)\, d\lambda_H.$$

Table 21 summarizes the requirements in heterogeneity for the four misallocation channels. The distributive TT requires heterogeneity in μ, α, and β. The labor income reallocation needs heterogeneity in μ, α. The final demand TT and the aggregate labor share channels require heterogeneity in μ.

Table 21: Heterogeneity Requirements

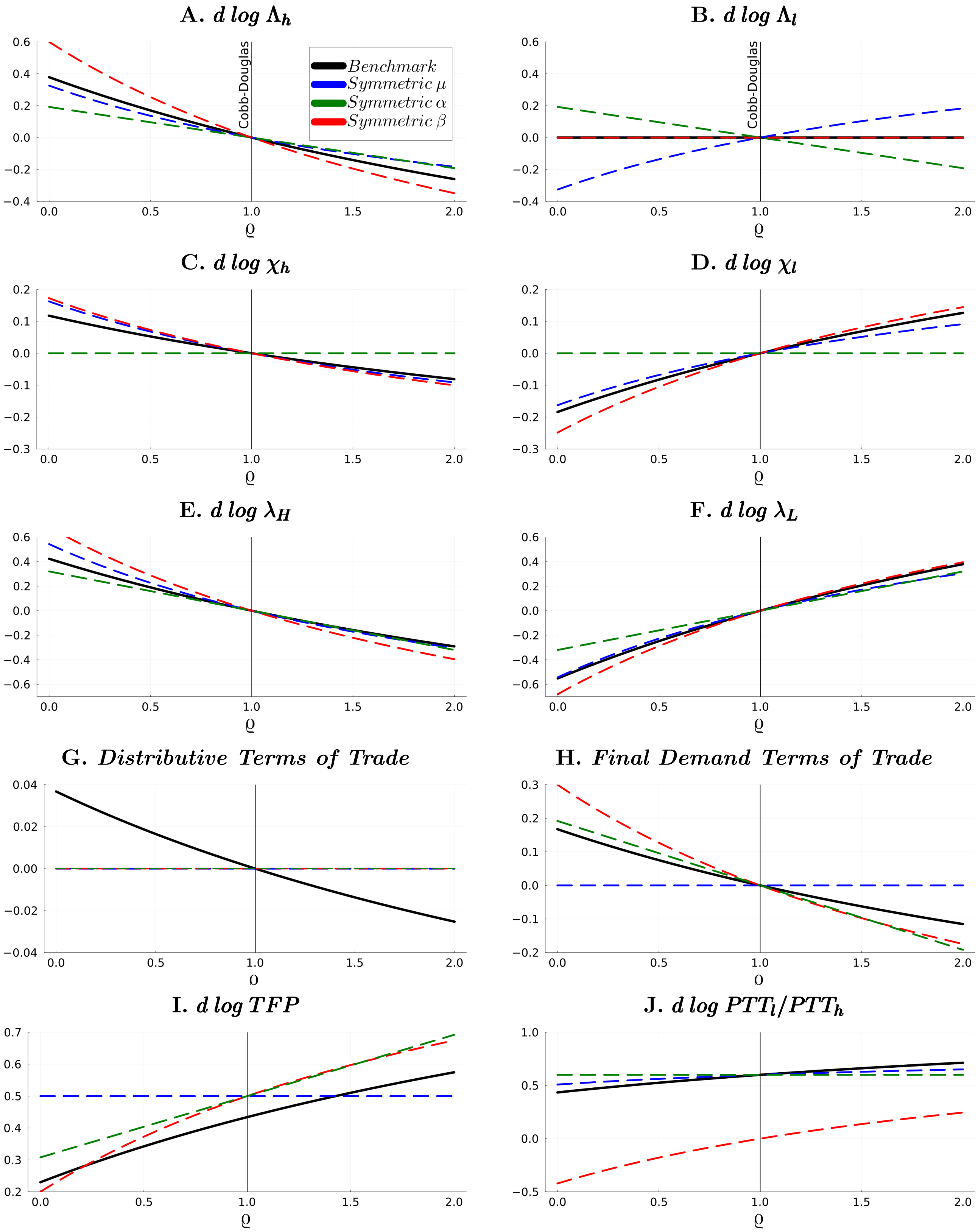
Channel	$\boldsymbol{\mu_H = \mu_L}$	$\boldsymbol{\alpha_H = \alpha_L}$	$\boldsymbol{\beta_h = \beta_l}$
Distributive Terms of Trade	*No*	*No*	*No*
Labor Income Reallocation	*No*	*No*	*Yes*
Final Demand Terms of Trade	*No*	*Yes*	*Yes*
Aggregate Labor Share	*No*	*Yes*	*Yes*

Notes: *No* means that the channel is null, *Yes* that the channel is non-zero.

Figure 9 portrays for the benchmark and three alternative specifications, under different values for ϱ, the elasticities in response to the productivity shock in the inefficient firm for the labor income shares,

[13] $\boldsymbol{a} = \frac{1+(\alpha_H-\alpha_L)(\beta_h-\beta_l)(1+\boldsymbol{c})}{(\alpha_H\,\mu_H+\alpha_L\,\mu_L-\boldsymbol{c})(\alpha_H\,\mu_L+\alpha_L\,\mu_H-\boldsymbol{c})}$, $\boldsymbol{b} = \delta_l + (\alpha_H - \alpha_L)\,(\mu_H - \alpha_L)\,\boldsymbol{a}$, and $\boldsymbol{c} = \mu_H\,\mu_L\,(\alpha_H - \alpha_L)\,(\beta_h - \beta_l)$.

Figure 9: Simple Horizontal Economy



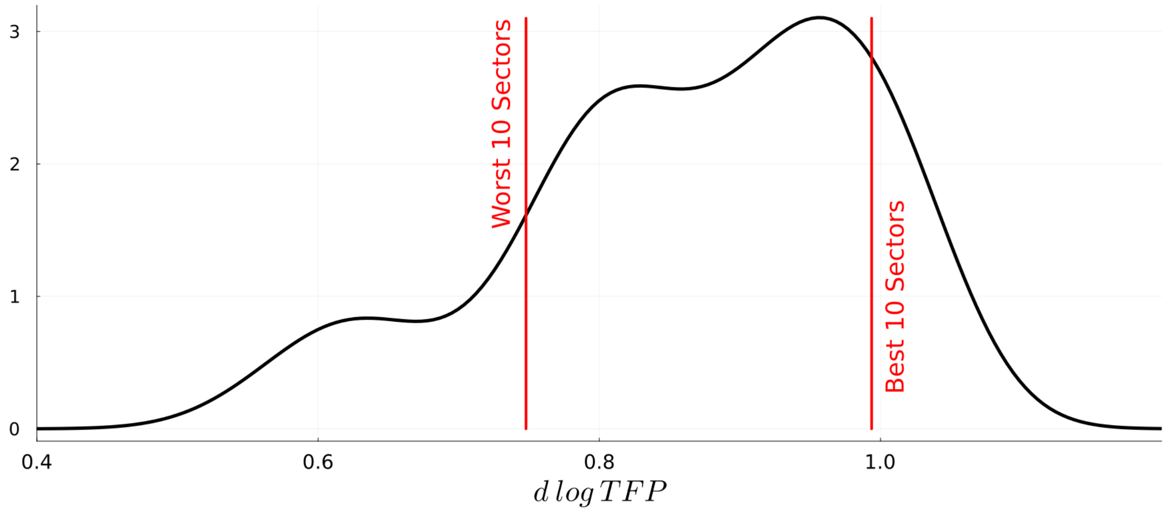
Notes: The horizontal axis captures the households' elasticity of substitution. $\varrho = 1$ represents the Cobb-Douglas economy. The benchmark is estimated using $\mu_H = \alpha_H = \beta_h = 0.8$ and $\mu_L = \alpha_L = \beta_l = 0.2$. The symmetric μ case with $\alpha_H = \beta_h = 0.8$, $\alpha_L = \beta_l = 0.2$, and $\mu_H = \mu_L = 0.5$. The symmetric α case with $\mu_H = \beta_h = 0.8$, $\mu_L = \beta_l = 0.2$, and $\alpha_H = \alpha_L = 0.5$. The symmetric β case with $\mu_H = \alpha_H = 0.8$, $\mu_L = \alpha_L = 0.2$, and $\beta_h = \beta_l = 0.5$. These elasticities are in responde to a productivity shock from the inefficient sector such that $d\log A_L = 1\%$.

expenditure shares, Domar weights, the distributive terms of trade, the final demand terms of trade, TFP, and the difference in PTTs.

For the benchmark specification, under substitutability, consumers can shift towards the now more abundant inefficient good. This effect increases the inefficient firm's Domar weight and the expenditure share for the low-skill. Consequently, the efficient firm and the expenditure share from the high-skill falls. The lower expenditure share from the high-skill and the expenditure shift towards the inefficient firm reduce misallocation. For this reason, misallocation falls, and the TFP elasticity is higher than the inefficient firm's equilibrium Domar weight. The low-skill households face a more favorable increase in their PTT due to the higher exposure from their consumption bundle to the inefficient good.

9 Counterfactual Industrial Policy

Figure 10: $d\log TFP$ density in response to sectoral productivity shock



Best 10 Sectors

1	Nursing & residential care	1.035%
2	Social assistance	1.033%
3	Merchandise stores	1.027%
4	Hospital	1.022%
5	Ambulatory health care	1.021%
6	Computer systems design	1.013%
7	Apparel & leather	1.008%
8	Food & beverage stores	1.005%
9	Educational services	0.998%
10	Other retail	0.993%

Worst 10 Sectors

1	Oil & gas extraction	0.558%
2	Primary metals	0.595%
3	Chemical products	0.601%
4	Mining, except oil & gas	0.616%
5	Utilities	0.628%
6	Petroleum & coal	0.639%
7	Farms	0.658%
8	Rental & leasing	0.680%
9	Other real estate	0.715%
10	Paper products	0.747%

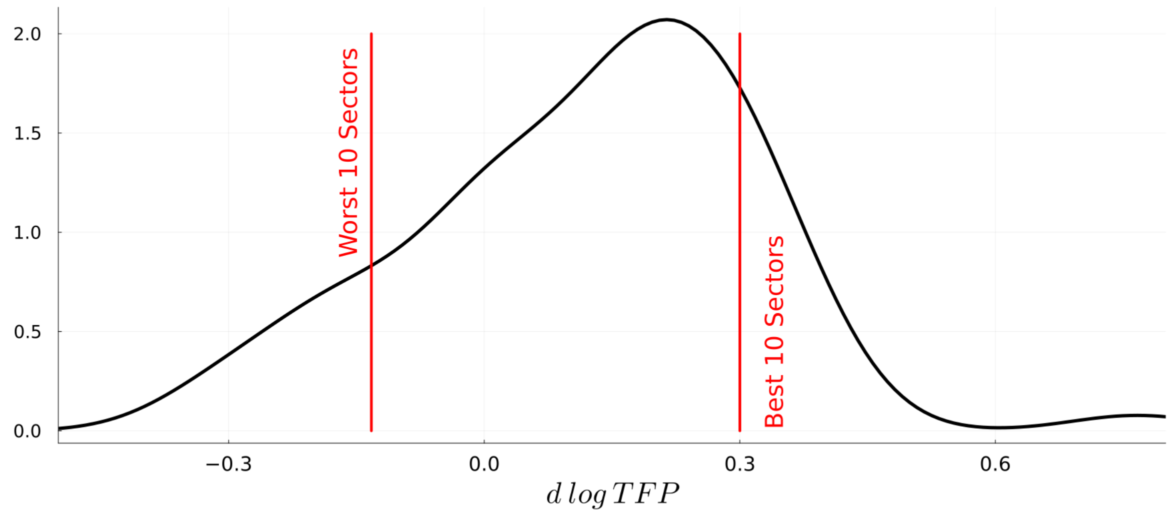
Notes: Density of $d\log TFP$ in response to independent sectoral productivity shocks of 1% for each of the 66 NAICS industries considered. The lower tables show the best and worst 10 sectors and the magnitude for the TFP elasticity.

Section 2 in the Online Appendix provides a nested CES extension to the model from Section 7. This section parameterizes such a model using the following elasticities of substitution, which are consistent with the values estimated and used throught the input-output literature (Boehm et al., 2014;

Atalay, 2017; Baqaee, 2018; Baqaee & Farhi, 2020). I assume for all sectors an unitary elasticity of substitution between types of labor, an elasticity of substitution of 0.2 between intermediate inputs, an elasticity of substitution of 0.5 between the labor and intermediate input aggregates, and an elasticity of substitution of 0.9 for the consumption aggregators.

This parametric setting allows me to discipline the endogenous variations in the model and estimate the aggregate and distributional effects from a manifold of sectoral shocks. Here, I evaluate the effects of two shocks: a sectoral productivity shock such that aggregate technology equals 1% and a sectoral increase in markdowns such that aggregate competitiveness equals 1%. The 1% assumption allows me to make their effect comparable, as the differences in TFP will depend exclusively on the asymmetric response from misallocation to the endogenous variations in the income distribution.

Figure 11: $d\log TFP$ density in response to sectoral markdow shock



	Best 10 Sectors	
1	Housing	0.766%
2	Credit intermediation	0.409%
3	Furniture	0.376%
4	Pipeline transportation	0.360%
5	Oil & gas extraction	0.355%
6	Mining, except oil & gas	0.349%
7	Primary metals	0.342%
8	Petroleum & coal	0.328%
9	Chemical products	0.316%
10	Rental & leasing	0.300%

	Worst 10 Sectors	
1	Nursing & residential care	-0.329%
2	Social assistance	-0.303%
3	Merchandise stores	-0.274%
4	Hospital	-0.219%
5	Ambulatory health care	-0.201%
6	Educational services	-0.191%
7	Apparel & leather	-0.163%
8	Computer systems design	-0.154%
9	Recreational & gambling	-0.135%
10	Food & beverage stores	-0.132%

Notes: Density of $d\log TFP$ in response to independent sectoral markdown shocks of 1% for each of the 66 NAICS industries considered. The lower tables show the best and worst 10 sectors and the magnitude for the TFP elasticity.

Figure 10 displays the density for TFP elasticities in response to sectoral productivity shocks of a magnitude such that aggregate technology equals 1%. If the costs from stimulating a productivity shock of this magnitude were symmetric across sectors, on the one hand, the best technological shocks would be to the healthcare, social assistance, retail, computer design, and education industries. On the other hand, the worst technological shocks would be to extractive, chemical, utilities, farms, real estate, and paper industries. In particular, almost 45% of the initial technological stimulus from a productivity shock to the oil and gas extraction industry is lost due to higher labor misallocation.

Figure 11 displays the density for TFP elasticities in response to sectoral markdown shocks of a magnitude such that aggregate competitiveness equals 1%. If the costs from stimulating competition by this magnitude were symmetric across sectors, on the one hand, the best antitrust interventions would be in the housing, extractive, chemical, and rental industries. On the other hand, it would be an awful idea to push for antitrust interventions in healthcare, social assistance, retail, computer design, recreation, and education sectors. For the last set of industries, the corresponding increase in misallocation more than washes off the gains in aggregate competitiveness.

Table 22: Aggregate TFP on sectoral characteristics

	$\frac{d\log TFP}{d\log A_i}$ *from Figure 10*				$\frac{d\log TFP}{d\log \mu_i}$ *from Figure 11*			
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**
λ_i	0.149			0.905*	1.331			-0.084
	(0.591)			(0.523)	(0.941)			(0.736)
μ_i		0.391***		0.198		-0.943***		-0.925***
		(0.094)		(0.134)		(0.125)		(0.189)
F_i			0.241***	0.181**			-0.401***	-0.016
			(0.052)	(0.076)			(0.083)	(0.107)
Intercept	0.860***	0.562***	0.599***	0.486***	0.084**	0.848***	0.563***	0.855***
	(0.022)	(0.074)	(0.059)	(0.079)	(0.035)	(0.098)	(0.094)	(0.111)
N	66				66			
R^2	0.001	0.212	0.245	0.301	0.030	0.471	0.266	0.471
Adj. R^2	0.001	0.245	0.217	0.279	0.030	0.471	0.266	0.454

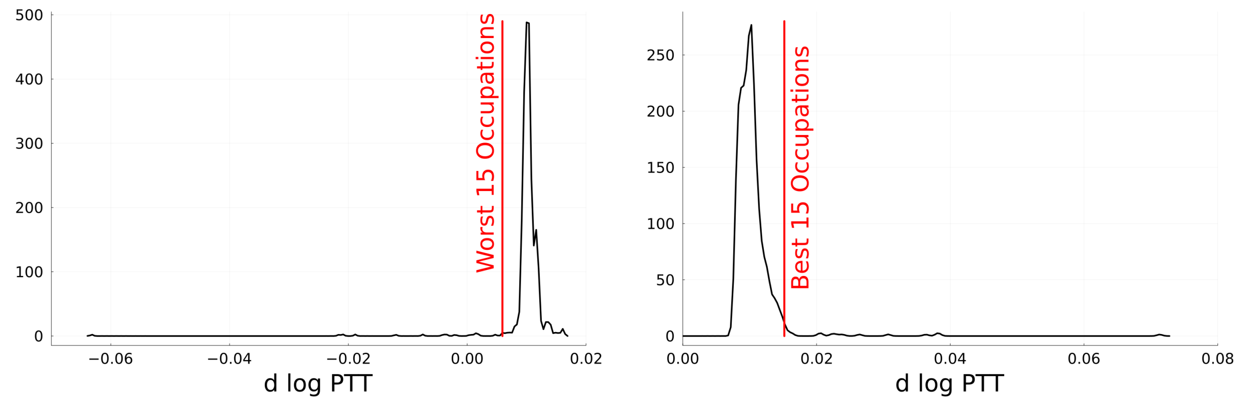
Notes: Columnns 1 to 4 report regressions of $\frac{d\log A_i}{d\log A_i}$. Columnns 5 to 8 report regressions of $\frac{d\log A_i}{d\log \mu_i}$. Columns 1 and 5 report univariate regressions on the sectoral Domar weights λ_i. Columns 2 and 6 report univariate regressions on the sectoral markdown μ_i. Columns 3 and 7 report univariate regressions on the sectoral *revenue efficiency* F_i. Columns 4 and 8 report multivariate regressions on λ_i, μ_i, and F_i.

Table 22 report the results from OLS regressions for the TFP elasticities estimated in Figures 10 and 11 on sectoral Domar weights, markdowns, and revenue efficiency. Not surprisingly, the 1% normalization makes the Domar weights insignificant. Markdowns and the revenue efficiencies have a positive correlation with the TFP response to productivity shocks and a negative correlation with the TFP response to higher competition. However, in the multivariate regressions, the revenue efficiency captures the positive correlation with the effect of productivity shocks, and the markdown captures the negative correlation with the TFP response to higher competition. The latter results show that industrial policies that incentivize productivity should aim for high F_i sectors. In contrast, antitrust policies that increase competition should target low μ_i industries.

Figure 12 displays the density for the PTT elasticities in response to the productivity and markdown shocks for the oil and gas industry estimated in Figures 10 and 11. In response to the productivity shock, the distribution of PTTs has a negative skewness and a positive skewness in response to the competition shock. The long tails from these distributions capture the effect on fifteen occupations heavily exposed to the oil and gas extraction sector. The complementarity in households' preferences (i.e., $\varrho < 1$) explains the difference in terms of skewness. On the one hand, the productivity shock introduces a supply shock that increases the quantity of goods. However, preference complementarity forces households to substitute their expenditure towards relatively inefficient sectors. Consequently, the final demand for oil and gas extraction falls, and correspondingly, the labor income share for occupations with heavy exposure to this sector shrink. On the other hand, an increase in competition introduces a labor demand shock that raises the labor income share for occupations with heavy exposure to the oil and gas extraction industry. Notice that the occupations that face the worst PTT elasticities in response to the productivity shock are almost the same occupations that face the best PTT elasticities in response to the increase in competition. The bilateral centrality from the revenue of the oil and gas extraction sector on the labor income from these occupations is high.

Figure 12: Density of $d\,log\,PTT$ to Oil & Gas Extraction Shocks

A. Productivity Shock **B. Markdown Shock**



Worst 15 Occupations	
1. Wellhead Pumpers	-6.32%
2. Service Unit Operators, Oil & Gas	
3. Petroleum Engineers	-2.15%
4. Rotary Drill Operators, Oil & Gas	-2.08%
5. Roustabouts, Oil & Gas	-1.88%
6. Geoscientists	-1.20%
7. Hydrologic Technicians	-0.74%
8. Geological Technicians	-0.40%
9. Mining & Geological Engineers	-0.35%
10. Gas Compressor & Pumping Station Operators	-0.20%
11. Extraction Workers, All Other	0.04%
12. Rentier	0.14%
13. Gas Plant Operators	0.19%
14. Petroleum Pump System Operators Refinery Operators, and Gaugers	0.47%
15. Pump Operators, Except Wellhead Pumpers	0.59%

Best 15 Occupations	
1. Wellhead Pumpers	7.13%
2. Service Unit Operators, Oil & Gas	3.82%
3. Petroleum Engineers	3.81%
4. Rotary Drill Operators, Oil & Gas	3.63%
5. Roustabouts, Oil & Gas	3.07%
6. Geoscientists	2.64%
7. Hydrologic Technicians	2.38%
8. Geological Technicians	2.28%
9. Mining & Geological Engineers	2.22%
10. Extraction Workers, All Others	2.06%
11. Petroleum Pump System Operators Refinery Operators, and Gaugers	2.04%
12. Gas Plant Operators	1.63%
13. Gas Compressor & Pumping Station Operators	1.60%
14. Pump Operators, Except Wellhead Pumpers	1.55%
15. Pourers and Casters, Metal	1.51%

Notes: Table A reports the $d\,log\,PTT_h$ density in response to a 1% productivity shock in the oil and gas extraction industry. The lower table reports the worst 15 occupations in terms of their $d\,log\,PTT$ in response to the productivity shock. Table B reports the $d\,log\,PTT_h$ density in response to a 1% markdown shock in the oil and gas extraction industry. The lower table reports the best 15 occupations in terms of their $d\,log\,PTT$ in response to the markdown shock.

10 Allocative Neutrality

Theorem 6 identifies four general classes of economies for which there are zero first-order aggregate gains from the reallocation of resources. These cases allow me to characterize the primitives necessary to obtain changes in the income and consumption distributions that allow for non-technological growth. By allocative neutrality I mean that $\mathit{Competitiveness} = \mathit{Misallocation}$, and consequently $d\log TFP = \mathit{Technology}$.

Theorem 6. For the following economies and shocks, allocative neutrality is satisfied:

1. For a Cobb-Douglas economy ($\theta_i = \varrho_h = 1 \ \forall i \in \mathscr{N}$ and $\forall h \in \mathscr{H}$) in response to productivity or demographic shocks.

2. For a Leontief economy ($\theta_i = \varrho_h = 0 \ \forall i \in \mathscr{N}$ and $\forall h \in \mathscr{H}$) with inelastic labor supply in response to a markdown shock if: (i) payment centrality is symmetric across households, i.e., $\mathscr{C}_h = \tau \in (0, 1] \ \forall h \in \mathscr{H}$; or (ii) $\mathscr{C}$ is nonsingular and $\mathscr{B}\,\Omega_\pi$ has its eigenvalues within the unit circle.

3. For a horizontal economy with symmetric distortions ($\mu_i = \mu \ \forall i \in \mathscr{N}$) in response to productivity, markdown, and demographic shocks.

4. In a vertical economy in response to productivity, markdown, and demographic shocks.

1. Cobb-Douglas Neutrality

For the class of Cobb-Douglas economies, there is no first-order variation in aggregate misallocation in response to exogenous supply shocks. Technological shocks that change the productivity from a sector or demographic shocks are allocative-neutral on aggregate TFP because the sales, labor income, and expenditure shares are inelastic. Consequently, there is also distributional allocative-neutrality and $d\log PTT_h = \mathit{Technology}_h \ \forall h \in \mathscr{H}$. This result extends the Cobb-Douglas neutrality benchmark from Baqaee & Farhi (2020) to an environment with heterogeneous households and endogenous labor supply.

2. Leontief Neutrality

For the class of Leontief economies with inelastic labor supplies, shocks in markdowns are allocative neutral if one of the two conditions introduced by Theorem 6 are satisfied. However, before discussing the merits and implications of these conditions, let me build up some base intuition. Baqaee & Farhi (2020) establish that markdown shocks are allocative neutral for a representative household Leontief economy with inelastic labor. The reason is that regardless of prices, the household will consume fixed ratios of goods, and the firms will demand fixed ratios of labor and intermediate inputs. As a result, variations in distortions influence prices but not the demand for final goods or intermediate inputs. Consequently, the allocation of workers across firms does not change in response to markdown variations.

Extending this result to an environment with heterogeneous households is more complex. The reason is that any endogenous shift in consumption expenditure between households with heterogenous consumption bundles will modify aggregate final demand and the allocation of workers across firms. However, we know that in response to the markdown shocks, up to a first-order, the endogenous reallocation of real consumption for this class of economies will satisfy

$$0 = \sum_{h \in \mathscr{H}} \chi_h \, \mathscr{C}_h \, d\,log\, C_h. \tag{24}$$

Hence, from Theorem 2, allocative neutrality will be satisfied if the payment centrality from households is symmetric and real GDP will be inelastic, i.e., $d\,log\,Y = \sum_{h \in \mathscr{C}} \chi_h \, d\,log\,C_h = 0$. Payment centralities are symmetric if consumption bundles are homogenous.

Now, the problem is that symmetry between households in their payment centralities is an extremely restrictive condition. In general, Equation (24) implies that

$$d\,log\,Y = \sum_{h \in \mathscr{H}} \chi_h \, (1 - \mathscr{C}_h) \, d\,log\,C_h.$$

This last equation tells us that for a Leontief economy with inelastic labor that faces shocks in markdowns, there is space for GDP growth through the reallocation of workers, when real consumption is endogenously shifted towards households with relatively high consumption expenditure and small payment centralities. The households with small payment centralities are the ones who have a smaller share of their expediture reaching households through labor income. Consequently, increasing consumption expenditure to these households has the largest negative effect on the aggregate labor wedge.

Furthermore, aggregate distributive neutrality for this class of models is also guaranteed whenever the consumer-to-worker upstream centrality matrix $\mathscr{C}$ is nonsingular and $\mathscr{B}\,\Omega_\pi$ has its eigenvalues within the unit circle. Ω_π is a $N \times H$ matrix, where its ih element is given by $(1 - \mu_i)\,\kappa_{ih}$, i.e., the share of revenue from sector i that reaches income for households of type h through corporate profits. First, the invertibility of $\mathscr{C}$ portrays that there needs to be a sufficiently high level of heterogeneity in consumption bundles. Second, having all of the eigenvalues from $\mathscr{B}\,\Omega_\pi$ within the unit circle implies that its determinant is less than one, and consequently $\sum_{q=0}^{\infty} (\mathscr{B}\,\Omega_\pi)^q = (I - \mathscr{B}\,\Omega_\pi)^{-1}$. Notice that the hb element from $\mathscr{B}\,\Omega_\pi$ is given by $\sum_{i \in \mathscr{N}} \mathscr{B}_{hi}\,(1 - \mu_i)\,\kappa_{ib}$ and captures the share of expenditure from households of type h that reaches the income from households of type b through corporate profits. In other words, the second condition imposes an equilibrium convergence criteria according to which there can be no explosive paths from consumption expenditure to dividend income. Here the Gershgorin circle theorem is useful, as it tells us that all eigenvalues for a matrix are within the unit circle if the off-diagonal element summation for each of its rows is less than one (Gershgorin, 1931). For row h, the off-diagonal elements add up to $\sum_{i \in \mathscr{N}} \mathscr{B}_{hi}\,(1 - \mu_i)\,(1 - \kappa_{ih})$. For an economy without intermediate inputs, this condition holds as $\sum_{i \in \mathscr{N}} \mathscr{B}_{hi} \leq 1$. However, the proof escapes me for an economy with intermediate inputs, mainly because consumer-to-firm upstream centralities can be larger than one.

3. Horizontal Economy

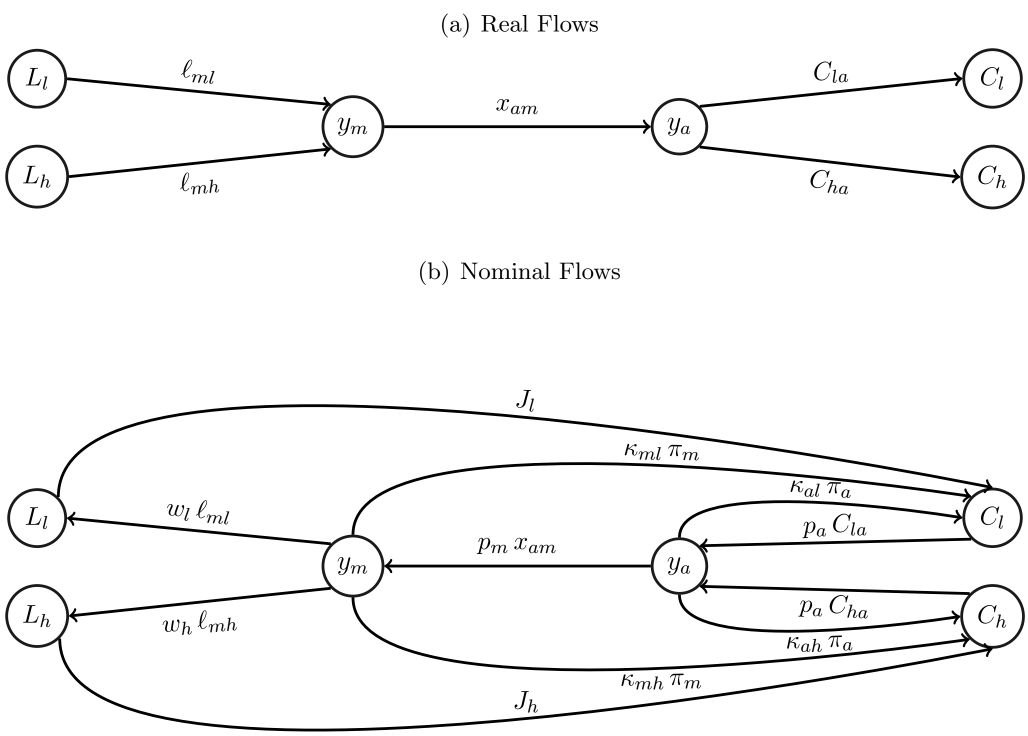
For the class of general horizontal economies with N sectors and H households, allocative neutrality in response to productivity, markdown, and demographic shocks is satisfied if distortions are symmetric across sectors. The reason is that under symmetric markdowns, the final demand from households and the labor demand from firms are undistorted. This is because distortions cancel out in the households' marginal rates of substitution. For this reason, the allocation of workers across firms is already efficient, and up to a first-order, the endogenous reallocation from workers in response to any of these three types of shocks is neutral on TFP.

Bigio & La'O (2020) prove that for a horizontal representative household economy with symmetric distortions, one type of labor, and endogenous labor supply, shocks in sectoral distortions are neutral on TFP. Theorem 6 extend this result to productivity and demographic shocks and a heterogenous household economy with multiple types of labor.

To understand how the presence of intermediate inputs would alter this result, let me get back to the simple horizontal economy introduced in Section 8, with the additional assumption that the inefficient firm demands meals from the efficient firm. Under symmetric distortions, household final demand is the same as in the first-best equilibrium. However, the distortions alter the inefficient firm's marginal rates of substitution between labor demand and intermediate inputs, and the allocation of workers is longer efficient.

4. Vertical Economy

Figure 13: Vertical Economy



Note: Low-skill (l), high-skill (h), manufacturing (m), and agriculture (a).

For the class of vertical economies, productivity, markdown, and demographic shocks are allocative neutral. Notice that this result is independent of the markdowns that sectors face. Figure 13 represents a vertical economy with two firms and two households. The manufacturing firm demands labor from high- and low-skill workers and supplies intermediate inputs to the agricultural firm. Households only consume agricultural goods.

Bigio & La'O (2020) prove that for a vertical representative household economy with one type of endogenous labor, shocks in sectoral distortions are neutral on TFP. Theorem 6 extend this allocative neutrality result to productivity and demographic shocks and to a heterogenous household economy with multiple types of labor. The reason is that in a vertical economy, workers are hired only by the most upstream firm, they have nowhere else to go, and their allocation coincides with the first-best equilibrium.

11 Conclusion

In this paper, I build an aggregation theory for a general production network economy with heterogeneous households and endogenous labor supply. I provide nonparametric characterizations of the local effects that endogenous variations in the income distribution, the consumption expenditure distribution, and the demand structure from firms and households have on aggregate TFP and the households' positional terms of trade. These results show that the channels via which expenditure enters and flows through the economy matter as they influence the allocation of workers across firms. Furthermore, under distortions, the decentralized decision from households about the level of their labor supply introduces externalities on aggregate welfare. A constrained social planner that centralizes household decisions could solve these externalities by making all workers symmetrically undervalued.

The first empirical implementation of a production network environment with heterogenous households for the United States allows me to quantitatively implement the sufficient statistics that decompose the growth of TFP. Not surprisingly, the aggregate increase of TFP during the first two decades of the XXIst century has been technologically driven. However, the distributional effects on TFP have been relevant during specific business cycles. Distributionally driven TFP fostered growth and increased TFP by 8.2% before the Great Recession, while it hindered growth and reduced TFP by 7.5% after the Great Recession. The latter result serves as evidence in favor of a distributional explanation behind the lackluster growth that the US economy experienced over the last decade.

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Online Appendix

1 Proofs for the nonparametric model

1.1 Firms

1.1.1 Aggregators' Problem

For every sector $i \in \mathscr{N}$, the perfectly competitive aggregator chooses $\left\{y_i,\, (y_{z_i})_{z_i \in [0,1]}\right\}$ to maximize

$$\bar{\pi}_i = p_i y_i - \int p_{z_i} y_{z_i}\, dz_i$$

subject to the CES technology (3) and taking prices $\left\{p_i, (p_{z_i})_{z_i \in [0,1]}\right\}$ as given.

Taking first order conditions I arrive to the usual Dixit & Stiglitz's (1977) CES demand function

$$y_{z_i} = \left(\frac{p_i}{p_{z_i}}\right)^{\frac{1}{1-\mu_i}} y_i \qquad \forall z_i \in [0,1]\,, \tag{25}$$

from here $\frac{\partial p_{z_i}}{\partial y_{z_i}} = -\,(1-\mu_i)\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{p_i}{y_{z_i}}$ and $p_i = \left(\int p_{z_i}^{\frac{\mu_i}{\mu_i - 1}}\, dz_i\right)^{\frac{\mu_i - 1}{\mu_i}}$.

1.1.2 Monopolistically Competitive Firms' problem

Firm z_i in sector $i \in \mathscr{N}$ chooses $\left\{y_{z_i},\, p_{z_i},\, \{\ell_{z_i h}\}_{h\in\mathscr{H}}\,,\, \{x_{z_i j}\}_{j\in\mathscr{N}}\right\}$ to maximize

$$\pi_{z_i} = p_{z_i}\, y_{z_i} - \underbrace{\sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}}_{=\, p^{\ell}_{z_i} L_{z_i}} - \underbrace{\sum_{j\in\mathscr{N}} p_j\, x_{z_i j}}_{=\, p^{x}_{z_i} X_{z_i}}, \tag{26}$$

subject to equation (25),

$$y_{z_i} = A_i\, Q_i\,(L_{z_i}, X_{z_i})), \quad L_{z_i} = A^{\ell}_i\, Q^{\ell}_i \left(\left\{A^{\ell}_{ih}\, \ell_{z_i h}\right\}_{h\in\mathscr{H}}\right), \quad X_{z_i} = A^{x}_i\, Q^{x}_i \left(\left\{A^{x}_{ij}\, x_{z_i j}\right\}_{j\in\mathscr{N}}\right), \tag{27}$$

and taking $\left\{\{w_h\}_{h\in\mathscr{H}}\,,\, \{p_j\}_{j\in\mathscr{N}}\right\}$ as given.

Notice that firm z_i's revenue derivative with respect to any variable q is given by

$$\begin{aligned}\frac{\partial\, p_{z_i}\, y_{z_i}}{\partial q} &= \left(p_{z_i} + \frac{\partial p_{z_i}}{\partial y_{z_i}}\, y_{z_i}\right) \frac{\partial y_{z_i}}{\partial q} \\ &= \left(p_{z_i} - (1-\mu_i)\left(\frac{y_{z_i}}{y_i}\right)^{\mu_i - 1} p_i\right) \frac{\partial y_{z_i}}{\partial q} = \mu_i\, p_{z_i} \frac{\partial y_{z_i}}{\partial q}.\end{aligned}$$

Firms z_i's optimality conditions are given by

$$\mu_i \, p_{z_i} \, A_i \, \frac{\partial \, Q_i \, (L_{z_i}, \, X_{z_i})}{\partial \, L_{z_i}} = p^{\ell}_{z_i}, \tag{28}$$

$$\mu_i \, p_{z_i} \, A_i \, \frac{\partial \, Q_i \, (L_{z_i}, \, X_{z_i})}{\partial \, X_{z_i}} = p^{x}_{z_i}, \tag{29}$$

$$\mu_i \, p_{z_i} \, A_i \, \frac{\partial \, Q_i \, (L_{z_i}, \, X_{z_i})}{\partial \, L_{z_i}} \, A^{\ell}_i \, \frac{\partial \, Q^{\ell}_i \left(\left\{ A^{\ell}_{ib} \, \ell_{z_i b} \right\}_{b \in \mathscr{H}} \right)}{\partial \, \ell_{z_i h}} = w_h \qquad \forall h \in \mathscr{H} : \partial \, y_{z_i} / \partial \, \ell_{z_i h} > 0, \tag{30}$$

$$\mu_i \, p_{z_i} \, A_i \, \frac{\partial \, Q_i \, (L_{z_i}, \, X_{z_i})}{\partial \, X_{z_i}} \, A^{x}_i \, \frac{\partial \, Q^{x}_i \left(\{ A^{x}_{im} \, x_{z_i m} \}_{m \in \mathscr{N}} \right)}{\partial \, x_{z_i j}} = p_j \qquad \forall j \in \mathscr{N} : \partial \, y_{z_i} / \partial \, x_{z_i j} > 0. \tag{31}$$

Representing elasticities with $e\,(a, b) = (\partial a / \partial b)\,(b/a)$ the former first order conditions for firm z_i are also represented by

$$\omega^{\ell}_{z_i} = e\,(y_{z_i}, \, L_{z_i}) = \frac{1}{\mu_i} \, \frac{p^{\ell}_{z_i} \, L_{z_i}}{p_{z_i} \, y_{z_i}}, \tag{32}$$

$$\omega^{x}_{z_i} = e\,(y_{z_i}, \, X_{z_i}) = \frac{1}{\mu_i} \, \frac{p^{x}_{z_i} \, X_{z_i}}{p_{z_i} \, y_{z_i}}, \tag{33}$$

$$e\,(y_{z_i}, \, \ell_{z_i h}) = \frac{1}{\mu_i} \, \frac{w_h \, \ell_{z_i h}}{p_{z_i} \, y_{z_i}} \qquad \forall h \in \mathscr{H} \tag{34}$$

$$e\,(y_{z_i}, \, x_{z_i j}) = \frac{1}{\mu_i} \, \frac{p_j \, x_{z_i j}}{p_{z_i} \, y_{z_i}} \qquad \forall j \in \mathscr{N} \tag{35}$$

Combining equations (28) with (30), and (29) with (31)

$$\alpha_{z_i h} = e\,(L_{z_i}, \, \ell_{z_i h}) = \frac{w_h \, \ell_{z_i h}}{p^{\ell}_{z_i} \, L_{z_i}}, \qquad \forall h \in \mathscr{H} \tag{36}$$

$$\omega_{z_i j} = e\,(X_{z_i}, \, x_{z_i j}) = \frac{p_j \, x_{z_i j}}{p^{x}_{z_i} \, X_{z_i}} \qquad \forall j \in \mathscr{N} \tag{37}$$

Additionally, combining (34), (35), and using the implicit function theorem

$$e\,(\ell_{z_i h}, \, \ell_{z_i b}) = - \frac{w_b \, \ell_{z_i b}}{w_h \, \ell_{z_i h}} \qquad \forall h, b \in \mathscr{H} \tag{38}$$

$$e\,(x_{z_i j}, \, x_{z_i m}) = - \frac{p_m \, x_{z_i m}}{p_j \, x_{z_i j}} \qquad \forall j, m \in \mathscr{N}. \tag{39}$$

Introducing equations (34)-(35) in the cost function

$$\begin{aligned} c_{z_i}\,(\vartheta, \, \rho) &= p^{\ell}_{z_i} \, L_{zi} + p^{x}_{z_i} \, x_{zi} = \sum_{h \in \mathscr{H}} w_h \, \ell_{z_i h} + \sum_{j \in \mathscr{N}} p_j \, x_{z_i j} \\ &= \mu_i \, p_{z_i} \, y_{z_i} \left(\sum_{h \in \mathscr{H}} e\,(y_{z_i}, \, \ell_{z_i h}) + \sum_{j \in \mathscr{N}} e\,(y_{z_i}, \, x_{z_i j}) \right). \end{aligned} \tag{40}$$

From CRS in $Q_i(L_{z_i}, X_{z_i})$, $Q_i^\ell\left(\{A_{ih}^\ell\,\ell_{z_ih}\}_{h\in\mathscr{H}}\right)$, and $Q_i^x\left(\left\{A_{ij}^x\,x_{z_ij}\right\}_{j\in\mathscr{N}}\right)$

$$\begin{aligned}
&\sum_{h\in\mathscr{H}} e(y_{z_i}, \ell_{z_ih}) + \sum_{j\in\mathscr{N}} e(y_{z_i}, x_{z_ij}) \\
&= e(y_{z_i}, L_{z_i}) \sum_{h\in\mathscr{H}} e(L_{z_i}, \ell_{z_ih}) + e(y_{z_i}, X_{z_i}) \sum_{j\in\mathscr{N}} e(X_{z_i}, x_{z_ij}) \\
&= e(y_{z_i}, L_{z_i}) + e(y_{z_i}, X_{z_i}) = 1,
\end{aligned}$$

which implies that in (40) $c_{z_i}(\vartheta, \rho) = \mu_i\, p_{z_i}\, y_{z_i}$, and from here I obtain $\omega_{z_i}^\ell = e(y_{z_i}, L_{z_i})$, $\omega_{z_i}^x = e(y_{z_i}, X_{z_i})$, $\widetilde{\Omega}_{z_ih}^\ell = e(y_{z_i}, \ell_{z_ih})$, and $\widetilde{\Omega}_{z_ij}^x = e(y_{z_i}, x_{z_ij})$.

1.2 Households' Problem

Household $h \in \mathscr{H}$ chooses $\left\{\{C_{hi}\}_{i\in\mathscr{N}}, L_h\right\}$ to maximize $U_h(C_h, L_h)$ subject to $C_h = Q_h^c\left(\{C_{hi}\}_{i\in\mathscr{N}}\right)$, the budget constraint

$$E_h = p_h^c\, C_h = \sum_{i\in\mathscr{N}} p_i\, C_{hi} \leq w_h\, L_h + \Pi_h, \tag{41}$$

$$\Pi_h = \sum_{i\in\mathscr{N}} \kappa_{ih} \left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i\right), \tag{42}$$

and taking as given

$$\left\{w_h, \left\{p_i, \kappa_{ih}, \bar{\pi}_i, (\pi_{z_i})_{z_i\in[0,1]}\right\}_{i\in\mathscr{N}}\right\}.$$

The first order conditions for household $h \in \mathscr{H}$ are given by

$$U_{C_h} = \daleth_h\, p_h^c, \tag{43}$$

$$U_{L_h} = -\daleth_h\, w_h, \tag{44}$$

$$U_{C_h} \frac{\partial C_h}{\partial C_{hi}} = \daleth_h\, p_i \qquad \forall i \in \mathscr{N}:\ \partial C_h/\partial C_{hi} > 0 \tag{45}$$

where $\daleth_h$ stands for the lagrange multiplier for household h's budget constraint.

Combining (43) with (44), and (43) with (45), the former first order conditions for household h can be represented by

$$\frac{w_h}{p_h^c} U_{C_h} = -U_{L_h}, \tag{46}$$

$$\frac{p_i}{p_h^c} = \frac{\partial C_h}{\partial C_{hi}} \qquad \forall i \in \mathscr{N}:\ \partial C_h/\partial C_{hi} > 0. \tag{47}$$

Using the implicit function theorem, equations (46) and (47) can be represented in terms of elasticities as

$$e\left(C_h,\, L_h\right) = \frac{w_h\, L_h}{p_h^c\, C_h}, \tag{48}$$

$$\beta_{hi} = e\left(C_h,\, C_{hi}\right) = \frac{p_i\, C_{hi}}{p_h^c\, C_h} \qquad \forall i \in \mathscr{N}, \tag{49}$$

$$e\left(C_{hi},\, C_{hm}\right) + \frac{p_m\, C_{hm}}{p_i\, C_{hi}} = 0 \qquad \forall i, m \in \mathscr{N} :\ \partial C_h/\partial C_{hi} > 0, \tag{50}$$

$$e\left(C_{hi},\, L_h\right) = \frac{w_h\, L_h}{p_i\, C_{hi}} \qquad \forall i \in \mathscr{N} :\ \partial C_h/\partial C_{hi} > 0. \tag{51}$$

1.3 Proof for Proposition 1

1.3.1 Proof of Necessity

First, using equations (25), (31), and (50), I can obtain the first subset of conditions in Proposition 1

$$\begin{gathered}\frac{\partial C_h/\partial C_{hj}}{\partial C_h/\partial C_{hi}} = \frac{p_j}{p_i} = \mu_i \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial y_{z_i}}{\partial x_{z_i j}} \quad \forall i, j \in \mathscr{N},\ \forall z_i \in [0,1]\,,\ \forall h \in \mathscr{H}, \\ \text{such that} \quad \partial C_h/\partial C_{hi} > 0, \quad \partial C_h/\partial C_{hj} > 0, \quad \text{and} \quad \partial y_{z_i}/\partial x_{z_i j} > 0.\end{gathered} \tag{52}$$

Notice that in this first subset of equilibrium conditions, household h has to consume both from the sectors i and j, and firms z_i also has to demand intermediate inputs from sector j.

Second, using equations (25), (30), and (51), I can obtain

$$\begin{gathered}-\frac{w_b}{w_h}\frac{U_{L_h}}{U_{C_{hi}}} = \frac{w_b}{p_i} = \mu_i \left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i} \frac{\partial y_{z_i}}{\partial \ell_{z_i b}} \quad \forall i \in \mathscr{N},\ \forall z_i \in [0,1]\,,\ \forall h, b \in \mathscr{H}, \\ \text{such that} \quad \partial C_h/\partial C_{hi} > 0, \quad U_{L_h} \neq 0, \quad \text{and} \quad \partial y_{z_i}/\partial \ell_{z_i b} > 0.\end{gathered} \tag{53}$$

Notice that in this second subset of equilibrium conditions, the condition that links the demand from firm z_i for workers of type b and the marginal rate of substitution between the labor supply from households of type h and their consumption of goods form sector i does not require that firm z_i hires workers of type h. What is necessary for this relationship to exist is that firm z_i hires labor from any worker $b \in \mathscr{H}$, and that household h consumes from sector i. Whenever $b \neq h$, the distributional factor-rate-differential wedge w_b/w_h arises.

Finally, the resource constraints

$$y_i = \sum_{h \in \mathscr{H}} C_{hi} + \sum_{j \in \mathscr{N}} \int x_{z_j i}\, dz_j \quad \forall i \in \mathscr{N}, \qquad \text{and} \qquad L_h = \sum_{i \in \mathscr{N}} \int \ell_{z_i h}\, dz_i \quad \forall h \in \mathscr{H}, \tag{54}$$

are necessary conditions for the equilibrium allocation.

1.3.2 Proof of Sufficiency

Now, I am going to prove that for any exogenous equity distributions $\left\{\{\kappa_{ih}\}_{i\in\mathscr{N}}\right\}_{h\in\mathscr{H}}$, there exists a strictly positive price system

$$\left\{\left\{(p_{z_i})_{z_i\in[0,1]}\,,\,p_i\right\}_{i\in\mathscr{N}}\,,\,\{w_h\}_{h\in\mathscr{H}}\right\},$$

that implements a specific allocation for firms

$$\left\{\left(y_{z_i},\,\{\ell_{z_ih}\}_{h\in\mathscr{H}}\,,\,\{x_{z_ij}\}_{j\in\mathscr{N}}\right)_{z_i\in[0,1]}\,,\,y_i\right\}_{i\in\mathscr{N}},$$

and a household allocation

$$\left\{\{C_{hi}\}_{i\in\mathscr{N}}\,,\,C_h,\,L_h\right\}_{h\in\mathscr{H}},$$

as an equilibrium.

Let me start by using a normalized price system in which a CRS function defines the GDP deflator

$$\overline{p}_Y = Q^p\left(\{p_i\}_{i\in\mathscr{N}}\right) = 1. \tag{55}$$

Using equation (30), prices for firm z_i are given by

$$\begin{aligned}
&p_{z_i} = \frac{w_h}{\mu_i}\left(\frac{\partial\,y_{z_i}}{\partial\,\ell_{z_ih}}\right)^{-1} \quad \text{if} \quad \exists h\in\mathscr{H}:\frac{\partial\,y_{z_i}}{\partial\,\ell_{z_ih}}>0\\
\text{otherwise}\quad &p_{z_i} = \frac{w_h}{\mu_i}\left(\frac{\partial\,y_{z_i}}{\partial\,x_{z_i\underline{j}}}\right)^{-1}\left(\frac{\partial\,y_{z_{\overline{j}}}}{\partial\,\ell_{\overline{j}h}}\right)^{-1}\prod_{j\in\mathscr{N}_{z_i}}\frac{1}{\mu_j}\left(\frac{y_{z_j}}{y_j}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\overline{j}\}}\left(\frac{\partial\,y_{z_j}}{\partial\,x_{z_jj+1}}\right)^{-1}
\end{aligned} \tag{56}$$

where $\mathscr{N}_{z_i} = \left\{\underline{j},\underline{j}+1,\cdots,\overline{j}-1,\overline{j}\right\}$ captures a sequence of sectors for which there is sequence of firms that establish a connection between the labor supply from households of type h and the intermediate input demand from firm z_i. What I strictly need for this proof is that $\forall i\in\mathscr{N}$, there $\exists h\in\mathscr{H}$, such that for every firm in sector i, there is some direct or indirect demand of the factor supplied by a worker of type h, and that for every type of worker $h\in\mathscr{H}$, there exists a sector $i\in\mathscr{N}$ that satisfies this condition.

As a consequence, prices for sector $i\in\mathscr{N}$ are given by

$$\begin{aligned}
p_i = \frac{w_h}{\mu_i}&\left(\int \mathbb{1}\left\{\ell_{z_ih}>0\right\}\left(\frac{\partial\,y_{z_i}}{\partial\,\ell_{z_ih}}\right)^{\frac{\mu_i}{\mu_i-1}}d\,z_i\right.\\
&\left.+\int \mathbb{1}\left\{\ell_{z_ih}=0\right\}\left(\frac{\partial\,y_{z_i}}{\partial\,x_{z_i\underline{j}}}\frac{\partial\,y_{z_{\overline{j}}}}{\partial\,\ell_{z_{\overline{j}}h}}\prod_{j\in\mathscr{N}_{z_i}}\mu_j\left(\frac{y_j}{y_{z_j}}\right)^{1-\mu_j}\prod_{j\in\mathscr{N}_{z_i}\setminus\{\overline{j}\}}\frac{\partial\,y_{z_j}}{\partial\,x_{z_jj+1}}\right)^{\frac{\mu_i}{\mu_i-1}}d\,z_i\right)^{\frac{1-\mu_i}{\mu_i}}.
\end{aligned} \tag{57}$$

From equation (55) wages are given by

$$
\begin{aligned}
w_h = Q^p \Bigg(\Bigg\{ \frac{1}{\mu_i} \Bigg(\int \mathbb{1}\{\ell_{z_i h} > 0\} \left(\frac{\partial y_{z_i}}{\partial \ell_{z_i h}} \right)^{\frac{\mu_i}{\mu_i - 1}} d z_i \\
+ \int \mathbb{1}\{\ell_{z_i h} = 0\} \left(\frac{\partial y_{z_i}}{\partial x_{z_i \underline{j}}} \frac{\partial y_{z_{\overline{j}}}}{\partial \ell_{z_{\overline{n}} h}} \prod_{j \in \mathscr{N}_{z_i}} \mu_j \left(\frac{y_j}{y_{z_j}} \right)^{1-\mu_j} \prod_{j \in \mathscr{N}_{z_i} \setminus \{\overline{j}\}} \frac{\partial y_{z_j}}{\partial x_{z_j j+1}} \right)^{\frac{\mu_i}{\mu_i - 1}} d z_i \Bigg)^{\frac{1-\mu_i}{\mu_i}} \Bigg\}_{i \in \mathscr{N}} \Bigg)^{-1} .
\end{aligned} \tag{58}
$$

Notice that prices and wages are strictly positive because the marginal productivities of factors and intermediate inputs have to be strictly positive when there is some demand.

Now, I need to prove that starting from the set of equilibrium conditions represented in equations (52), (53), and (54), and under the system of prices represented in equations (57) and (58), the optimality conditions for firms and households hold.

To obtain equations (50) and (51), assume that firms in sector i directly or indirectly demand workers of type h, and firms in sector j directly or indirectly demand workers of type b. This assumption is made without loss of generality as it holds for any combination of pairs $i, j \in \mathscr{N}$ and $h, b \in \mathscr{H}$. Introducing equations (52) and (53) in (57)

$$
p_i = w_h \left(\left(- \frac{w_b}{w_h} \frac{U_{C_{bi}}}{U_{L_b}} \right)^{\frac{\mu_i}{1-\mu_i}} \int \left(\frac{y_i}{y_{z_i}} \right)^{\mu_i} d z_i \right)^{\frac{1-\mu_i}{\mu_i}} = - w_b \frac{U_{C_{bi}}}{U_{L_b}},
$$

$$
p_j = - w_b \frac{U_{C_{bj}}}{U_{L_b}}.
$$

This proofs equation (51). Dividing these two conditions, I arrive to $\frac{p_j}{p_i} = \frac{U_{C_{bj}}}{U_{C_{bi}}}$, which is equation (50).

Equation (48) comes from multiplying equation (51) by C_{bi}, adding up over all sectors, using the assumption that $Q^c \left(\{C_{bi}\}_{i \in \mathscr{N}} \right)$ is CRS in conjunction with Euler's homogeneous function theorem, and the implicit function theorem

$$
w_b U_{C_b} \underbrace{\sum_{i \in \mathscr{N}} C_{bi} \frac{\partial C_b}{\partial C_{bi}}}_{= C_b} = - U_{L_b} \underbrace{\sum_{i \in \mathscr{N}} p_i C_{bi}}_{= p_b^c C_b},
$$

this implies that $\frac{w_b}{p_b^c} = - \frac{U_{L_b}}{U_{C_b}}$, which is equation (48).

Equation (49) comes from dividing equation (48) by equation (51)

$$
\frac{p_i}{p_b^c} = \frac{\partial C_b}{\partial C_{bi}}.
$$

Now for firms, I obtain equation (35) from equation (52), using the implicit function theorem, and introducing equations (25) and (50)

$$\frac{p_i}{p_j}\,\frac{\partial C_b/\partial C_{bj}}{\partial C_b/\partial C_{bi}} = \mu_i\,\frac{p_i}{p_j}\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i}\frac{\partial y_{z_i}}{\partial x_{z_i j}}$$

$$\underbrace{\frac{p_i}{p_j}\,\frac{\partial C_b/\partial C_{bj}}{\partial C_b/\partial C_{bi}}}_{=1} = \mu_i\,\frac{p_i}{p_j}\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i}\frac{\partial y_{z_i}}{\partial x_{z_i j}}$$

$$\frac{\partial y_{z_i}}{\partial x_{z_i j}} = \frac{1}{\mu_i}\,\frac{p_j}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i,j \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial x_{z_i j}} > 0.$$

Equation (33) comes from adding up equation (35) over all sectors, and using the assumption that $Q_i^x\left(\left\{A_{ij}^x\,x_{z_i j}\right\}_{j\in\mathscr{N}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem

$$\mu_i\,p_{z_i}\,\frac{\partial y_{z_i}}{\partial X_{z_i}}\,A_i^x\underbrace{\sum_{j\in\mathscr{N}} x_{z_i j}\,\frac{\partial Q_i^x\left(\left\{A_{ij}^x\,x_{z_i j}\right\}_{j\in\mathscr{N}}\right)}{\partial x_{z_i j}}}_{=Q_i^x\left(\left\{A_{ij}^x\,x_{z_i j}\right\}_{j\in\mathscr{N}}\right)} = \underbrace{\sum_{j\in\mathscr{N}} p_j\,x_{z_i j}}_{=p_{z_i}^x\,X_{z_i}}$$

$$\frac{\partial y_{z_i}}{\partial X_{z_i}} = \frac{1}{\mu_i}\,\frac{p_{z_i}^x}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial X_{z_i}} > 0.$$

Equation (34) comes from introducing equations (25) and (51) in equation (53)

$$\underbrace{-\frac{p_i}{w_b}\,\frac{U_{L_b}}{U_{C_{bi}}}}_{=1} = \mu_i\,\frac{p_i}{w_h}\left(\frac{y_i}{y_{z_i}}\right)^{1-\mu_i}\frac{\partial y_{z_i}}{\partial \ell_{z_i h}}$$

$$\frac{\partial y_{z_i}}{\partial \ell_{z_i h}} = \frac{1}{\mu_i}\,\frac{w_h}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial \ell_{z_i h}} > 0.$$

Equation (32) comes from adding up equations (34) over all households, and using the assumption that $Q_i^l\left(\left\{A_{ih}^\ell\,\ell_{z_i h}\right\}_{h\in\mathscr{H}}\right)$ is CRS in conjunction with Euler's homogeneous function theorem

$$\mu_i\,p_{z_i}\,\frac{\partial y_{z_i}}{\partial L_{z_i}}\,A_i^\ell\underbrace{\sum_{h\in\mathscr{H}} \ell_{z_i h}\,\frac{\partial Q_i^\ell\left(\left\{A_{ih}^\ell\,\ell_{z_i b}\right\}_{b\in\mathscr{H}}\right)}{\partial \ell_{z_i h}}}_{=Q_i^\ell\left(\left\{A_{ih}^\ell\,\ell_{z_i h}\right\}_{h\in\mathscr{H}}\right)} = \underbrace{\sum_{h\in\mathscr{H}} w_h\,\ell_{z_i h}}_{=p_{z_i}^\ell\,L_{z_i}}$$

$$\frac{\partial y_{z_i}}{\partial L_{z_i}} = \frac{1}{\mu_i}\,\frac{p_{z_i}^\ell}{p_{z_i}} \qquad \forall z_i \in [0,1] \quad \text{and} \quad \forall i \in \mathscr{N} : \frac{\partial y_{z_i}}{\partial L_{z_i}} > 0.$$

What remains to be proven is is that households' budget constraints hold. Adding up equation (41), and introducing equation (42)

$$\sum_{h\in\mathscr{H}}\sum_{i\in\mathscr{N}} p_i\, C_{hi} = \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{h\in\mathscr{H}}\sum_{i\in\mathscr{N}} \kappa_{ih}\left(\bar{\pi}_i + \int \pi_{z_i}\, dz_i\right).$$

Introducing zero-profit condition on aggregator firms ($\bar{\pi}_i = 0 \quad \forall i \in \mathscr{N}$), equation (26), and rearranging terms

$$\begin{aligned}
\sum_{h\in\mathscr{H}}\sum_{i\in\mathscr{N}} p_i\, C_{hi} &= \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{b\in\mathscr{H}}\sum_{i\in\mathscr{N}} \kappa_{ib} \int \Big(p_{z_i}\, y_{z_i} - \sum_{j\in\mathscr{N}} p_j\, x_{z_i j} - \sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}\Big) dz_i \\
&= \sum_{h\in\mathscr{H}} w_h\, L_h + \sum_{i\in\mathscr{N}} \int \Big(p_{z_i}\, y_{z_i} - \sum_{j\in\mathscr{N}} p_j\, x_{z_i j} - \sum_{h\in\mathscr{H}} w_h\, \ell_{z_i h}\Big) dz_i \underbrace{\sum_{b\in\mathscr{H}} \kappa_{ib}}_{=1}.
\end{aligned}$$

From zero profits for aggregators $p_i y_i = \int p_{z_i} y_{z_i}$, and using equations (54), the households' budget constraints holds

$$0 = \sum_{h\in\mathscr{H}} w_h \underbrace{\left(L_h - \sum_{i\in\mathscr{N}} \int \ell_{z_i h}\, dz_i\right)}_{=0} + \sum_{i\in\mathscr{N}} p_i \underbrace{\left(y_i - \sum_{h\in\mathscr{H}} C_{hi} - \sum_{j\in\mathscr{N}} \int x_{z_j i}\, dz_j\right)}_{=0}.$$

1.4 Equilibrium Centralities from Subsection 3.2

1.4.1 Goods Market Equilibrium Conditions

Introducing equations (33), (35), (37), and (49) in the goods market resource constraint (54) for sector $i \in \mathscr{N}$

$$S_i = \sum_{h\in\mathscr{H}} p_i\, C_{hi} + \sum_{j\in\mathscr{N}} \int p_i\, x_{z_j i}\, dz_j = \sum_{h\in\mathscr{H}} \beta_{hi}\, E_h + \sum_{j\in\mathscr{N}} \mu_j \int \omega^x_{z_j}\, \omega_{z_j i}\, p_{z_j}\, y_{z_j}\, dz_j.$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$S_i = \sum_{h\in\mathscr{H}} \beta_{hi}\, E_h + \sum_{j\in\mathscr{N}} \Omega^x_{ji}\, S_j, \tag{59}$$

where $\Omega^x_{ij} \equiv \mu_i\, \omega^\ell_i\, \omega_{ij}$.

In matrix form, this equation is represented by

$$\left(I_N - \widetilde{\Omega}'_x\, diag\,(\mu)\right) S = \beta'\, E,$$

$$S = \mathscr{B}' E, \tag{60}$$

where $S \equiv [S_1, \cdots, S_N]'$, $E \equiv [E_1, \cdots, E_H]'$, $\mu \equiv [\mu_1, \cdots, \mu_N]'$, and the matrices

$$\beta \equiv \begin{pmatrix} \beta_{11} & \cdots & \beta_{1N} \\ \vdots & \ddots & \vdots \\ \beta_{H1} & \cdots & \beta_{HN} \end{pmatrix},$$

$$\Omega_x \equiv diag\,(\mu)\,\widetilde{\Omega}_x, \qquad \Psi_x \equiv (I_N - \Omega_x)^{-1}, \qquad \mathscr{B} \equiv \beta\,\Psi_x.$$

By dividing element i in equation (59) by nominal GDP, I arrive to the following equation that relates the revenue-based Domar weights and the expenditure shares

$$\lambda = \mathscr{B}'\,\chi, \tag{61}$$

where $\lambda \equiv [\lambda_1, \cdots, \lambda_N]'$, and $\chi \equiv [\chi_1, \cdots, \chi_H]'$. In equilibrium, λ_i captures the share of aggregate expenditure that reaches sector i's revenue.

Let me define

$$\widetilde{\mathscr{B}} \equiv \beta\,\widetilde{\Psi}_x \equiv \beta\,\Psi_x\,(I_N - \Omega_x)\,\widetilde{\Psi}_x \equiv \mathscr{B}\,(I_N - \Omega_x)\,\widetilde{\Psi}_x,$$

where

$$\widetilde{\Psi}_x \equiv \left(I_N - \widetilde{\Omega}_x\right)^{-1}$$

$$\Omega_x \equiv \begin{pmatrix} \Omega^x_{11} & \cdots & \Omega^x_{1N} \\ \vdots & \ddots & \vdots \\ \Omega^x_{N1} & \cdots & \Omega^x_{NN}. \end{pmatrix},$$

Then, in equation (61)

$$\lambda = \Psi'_x\left(I_N - \widetilde{\Omega}'_x\right)\widetilde{\mathscr{B}}'\chi,$$

which allows me to define the cost-based Domar weights

$$\widetilde{\lambda} \equiv \widetilde{\Psi}'_x\,(I_N - \Omega'_x)\,\lambda \equiv \widetilde{\mathscr{B}}'\,\chi. \tag{62}$$

To understand the cost-based Domar weights, notice that

$$\widetilde{S}_i \equiv \sum_{h\in\mathscr{H}} p_i\,C_{hi} + \sum_{j\in\mathscr{N}} \widetilde{\Omega}^x_{ji}\,\widetilde{S}_j = \sum_{h\in\mathscr{H}} \widetilde{\mathscr{B}}_{hi}\,E_h$$

where $\widetilde{S}_i = \widetilde{\lambda}_i\,GDP$. Remember that in equilibrium, $\widetilde{\Omega}^x_{ji}$ captures the cost share in sector j of intermediate goods supplied by sector i. And for this reason, $\widetilde{S}_i$ represents the value-added that passes through sector i. For this reason, for a specific consumption expenditure distribution χ, $\widetilde{\lambda}_i$ captures the aggregate value-added share that passes through sector i. Notice that $\omega_\ell' \widetilde{\lambda} = \mathbb{1}_N' \left(I_N - \widetilde{\Omega}_x'\right) \widetilde{\Psi}_x' \beta' \chi = 1$, and for this reason $\omega_i^\ell \widetilde{\lambda}_i$ is the aggregate share of value-added from sector generated by workers in sector i.

Finally, I am going to prove that the value-added that passes through a sector is greater than or equal to its revenue, i.e., that $\widetilde{\lambda}_i \geq \lambda_i$ holds $\forall i \in \mathscr{N}$. Let me start with

$$\widetilde{\Psi}_x - \Psi_x = \widetilde{\Psi}_x - \Psi_x = \sum_{q=1}^{\infty} \left(\widetilde{\Omega}_x^q - \Omega_x^q\right).$$

Notice that $\widetilde{\Omega}_x - \Omega_x = (I_N - diag\,(\mu))\,\widetilde{\Omega}_x \succcurlyeq 0_N\,0_N'$, because $\mu_i \in (0,1]$ and $\widetilde{\Omega}^x \succcurlyeq 0_N\,0_N'$ ($A \succcurlyeq B$ means that matrix A is elementwise greater than or equal than matrix B). Now, from induction, for $q > 1$ assume that $\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} \succcurlyeq 0_N\,0_N'$, then

$$\begin{aligned}\widetilde{\Omega}_x^q - \Omega_x^q &= \left(\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} diag\,(\mu)\right)\widetilde{\Omega}_x \\ &= \left(\widetilde{\Omega}_x^{q-1} - \Omega_x^{q-1} + \Omega_x^{q-1}\,(I_N - diag\,(\mu))\right)\widetilde{\Omega}_x \succcurlyeq 0_N\,0_N'.\end{aligned}$$

Therefore $\widetilde{\Psi}_x \succcurlyeq \Psi_x$. As a consequence $\widetilde{\mathscr{B}} - \mathscr{B} = \beta\left(\widetilde{\Psi}_x - \Psi_x\right) \succcurlyeq 0_H\,0_N'$, and $\widetilde{\lambda} - \lambda = \left(\widetilde{\mathscr{B}} - \mathscr{B}\right)' \chi \succcurlyeq 0_N$.

1.4.2 Labor Market Equilibrium Conditions

Introducing equations (32), (34), and (36) in the factor market clearing condition (54) for household $h \in \mathscr{H}$

$$J_h = w_h\,L_h = \sum_{i \in \mathscr{N}} \int w_h\,\ell_{z_i h}\,dz_i = \sum_{i \in \mathscr{N}} \mu_i \int \omega_{z_i}^\ell\,\alpha_{z_i h}\,S_{z_i}\,dz_i.$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$J_h = \sum_{i \in \mathscr{N}} \mu_i\,\widetilde{\Omega}_{ih}^\ell\,S_i, \tag{63}$$

where $\widetilde{\Omega}_{ih}^\ell \equiv \omega_i^\ell\,\alpha_{ih}$.

In matrix form, these equations are represented by

$$J = \widetilde{\Omega}_\ell'\,diag\,(\mu)\;S = \Omega_\ell'\,S, \tag{64}$$

where the matrices are given by

$$\Omega_\ell \equiv \begin{pmatrix} \Omega^\ell_{11} & \cdots & \Omega^\ell_{1H} \\ \vdots & \ddots & \vdots \\ \Omega^\ell_{N1} & \cdots & \Omega^\ell_{NH} \end{pmatrix},$$

$$\Omega_\ell \equiv diag\,(\mu)\;\widetilde{\Omega}_\ell$$

and $J \equiv [\,J_1, \cdots, J_H]'$.

By dividing element h in equation (63) by nominal GDP, I arrive at the following equation that relates the labor income shares and the revenue-based Domar weights

$$\Lambda = \Omega'_\ell\,\lambda, \tag{65}$$

where $\Lambda \equiv [\,\Lambda_1, \cdots, \Lambda_H]'$.

Similarly, I define the cost-based factor Domar weights as

$$\widetilde{\Lambda} \equiv \widetilde{\Omega}'_\ell\,\widetilde{\lambda}, \tag{66}$$

where $\mathbb{1}'_H\,\widetilde{\Lambda} = \mathbb{1}'_H\,\alpha'\,diag\,(\omega_\ell)\;\widetilde{\lambda} = \omega'_\ell\,\widetilde{\lambda} = 1$.

Notice that $\widetilde{\Lambda} \succcurlyeq \Lambda$ because

$$\begin{aligned}\widetilde{\Lambda} - \Lambda &= \widetilde{\Omega}'_\ell\,\widetilde{\lambda} - \Omega'_\ell\,\lambda \\ &= \underbrace{\widetilde{\Omega}'_\ell}_{\succcurlyeq 0_H 0'_N}\;\underbrace{\left(\widetilde{\lambda} - \lambda\right)}_{\succcurlyeq 0_N} + \widetilde{\Omega}'_\ell\,\underbrace{(I_N - diag\,(\mu))}_{\succcurlyeq 0_N 0'_N}\,\lambda.\end{aligned}$$

The firm-to-worker and worker-to-firm centrality matrices are respectively given by

$$\Psi_\ell = \Psi_x\,\Omega_\ell, \qquad \widetilde{\Psi}_\ell = \widetilde{\Psi}_x\,\widetilde{\Omega}_\ell, \tag{67}$$

where $\widetilde{\Psi}_\ell\,\mathbb{1}_H = \widetilde{\Psi}_x\,\widetilde{\Omega}_\ell\,\mathbb{1}_H = \widetilde{\Psi}_x\,\omega_\ell = \widetilde{\Psi}_x\left(I_N - \widetilde{\Omega}_x\right)\mathbb{1}_N = \mathbb{1}_N$. Additionally $\widetilde{\Psi}_\ell \succcurlyeq \Psi_\ell$ because

$$\widetilde{\Psi}_\ell - \Psi_\ell = \underbrace{\left(\widetilde{\Psi}_x - \Psi_x\right)}_{\succcurlyeq 0_N 0'_N}\;\underbrace{\widetilde{\Omega}_\ell}_{\succcurlyeq 0_N 0'_H} + \underbrace{\Psi_x}_{\succcurlyeq 0_N 0'_N}\;\underbrace{(I_N - diag\,(\mu))}_{\succcurlyeq 0_N 0'_N}\,\widetilde{\Omega}_\ell.$$

Similarly, the consumer-to-worker and worker-to-consumer centrality matrices are respectively given by

$$\mathscr{C} = \mathscr{B}\,\Omega_\ell, \qquad \widetilde{\mathscr{C}} = \widetilde{\mathscr{B}}\,\widetilde{\Omega}_\ell, \tag{68}$$

where $\widetilde{\mathscr{C}}\,\mathbb{1}_H = \widetilde{\mathscr{B}}\,\widetilde{\Omega}_\ell\,\mathbb{1}_H = \beta\,\widetilde{\Psi}_x\,\omega_\ell = \beta\,\widetilde{\Psi}_x\left(I_N - \widetilde{\Omega}_x\right)\mathbb{1}_N = \mathbb{1}_H$, $\widetilde{\mathscr{C}}'\,\chi = \widetilde{\Omega}'_\ell\,\widetilde{\mathscr{B}}'\,\chi = \widetilde{\Omega}'_\ell\,\widetilde{\lambda} = \widetilde{\Lambda}$,

$\mathscr{C}'\chi = \Omega_\ell' \mathscr{B}'\chi = \Omega_\ell' \lambda = \Lambda$, and $\widetilde{\mathscr{C}} \succcurlyeq \mathscr{C}$ because

$$\widetilde{\mathscr{C}} - \mathscr{C} = \underbrace{\left(\widetilde{\mathscr{B}} - \mathscr{B}\right)}_{\succcurlyeq 0_H 0_N'} \underbrace{\widetilde{\Omega}_\ell}_{\succcurlyeq 0_N 0_H'} + \underbrace{\mathscr{B}}_{\succcurlyeq 0_H 0_N'} \underbrace{\left(I_N - diag\left(\mu\right)\right)}_{\succcurlyeq 0_N 0_N'} \widetilde{\Omega}_\ell.$$

1.4.3 Labor Wedges

From equations (33), (37), and (49), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$x_{ji} = \mu_j \, \omega_j^x \, \omega_{ji} \, y_j \, \frac{\beta_{hj}}{\beta_{hi}} \frac{C_{hi}}{C_{hj}} \qquad \forall h \in \mathscr{H}, \quad \text{and} \quad \forall i, j \in \mathscr{N}.$$

From equation (54), the goods market resource constraint for goods produced firms in sector i in terms of household h's consumption is given by

$$y_i = \sum_{b \in \mathscr{H}} C_{bi} + \frac{C_{hi}}{\beta_{hi}} \sum_{j \in \mathscr{N}} \mu_j \, \omega_j^x \omega_{ji} \, y_j \, \frac{\beta_{hj}}{C_{hj}}.$$

In matrix representation, this equation is given by

$$y = C' \mathbb{1}_H + diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right) \Omega_x' diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1} y,$$

$$y = \left[I_N - diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right) \Omega_x' diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1}\right]^{-1} C' \mathbb{1}_H,$$

$$y = diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right) \left[I_N - \Omega_x'\right]^{-1} diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1} C' \mathbb{1}_H,$$

$$diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1} y = \Psi_x' diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1} C' \mathbb{1}_H,$$

where $\circ$ stands for the Hadamard product, $^\circ$ for the Hadamard power, and $o_H\left(h\right)$ for a vector of zeros with size H that has a one in position h.

Notice from equation (49) that $\beta_{hi} \frac{E_h}{C_{hi}} = p_i = \beta_{bi} \frac{E_b}{C_{bi}}$, and as a consequence

$$diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1} C' \mathbb{1}_H = \begin{pmatrix} \sum_{b \in \mathscr{H}} \beta_{h1} \frac{C_{b1}}{C_{h1}} \\ \vdots \\ \sum_{b \in \mathscr{H}} \beta_{hN} \frac{C_{bF}}{C_{hF}} \end{pmatrix} = E_h^{-1} \beta' E.$$

Then

$$diag\left(\left(\beta^{\circ -1} \circ C\right)' o_H\left(h\right)\right)^{-1} y = E_h^{-1} \Psi_x' \beta' E. \tag{69}$$

Now, from equations (32), (36), (49), and (51), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$\ell_{ih} = -\frac{U_{C_h}}{U_{L_h}}\,\mu_i\,\omega_i^{\ell}\,\alpha_{ih}\,y_i\,\beta_{hi}\frac{C_h}{C_{hi}} \qquad \forall h \in \mathscr{H}, \quad \text{and} \quad \forall i \in \mathscr{N}.$$

In matrix representation, these conditions are portrayed by

$$\ell_h = -\frac{U_{C_h}}{U_{L_h}}\,C_h\,diag\left(\Omega_\ell\,o_H\left(h\right)\right)diag\left(\left(\beta^{\circ -1}\circ C\right)'\,o_H\left(h\right)\right)^{-1}y.$$

Adding up, the labor market equilibrium from equation (54) in terms of first-order conditions is given by

$$L_h = -\frac{U_{C_h}}{U_{L_h}}C_h\,\underbrace{\mathbb{1}_N'\,diag\left(\Omega_\ell\,o_H\left(h\right)\right)diag\left(\left(\beta^{\circ -1}\circ C\right)'\,o_H\left(h\right)\right)^{-1}y}_{=\Gamma_h}.$$

Consequently, equilibrium labor supply is characterized by

$$L_h + \Gamma_h\frac{U_{C_h}}{U_{L_h}}C_h = 0. \tag{70}$$

Taking equation (69)

$$\begin{aligned}
\Gamma_h &= E_h^{-1}o_H\left(h\right)'\,\mathscr{C}'\,E \\
&= E_h^{-1}\,\mathbb{1}_N'\,diag\left(\Omega_\ell o_H\left(h\right)\right)\Psi_x'\,\beta'\,E \\
&= E_h^{-1}\,\mathbb{1}_N'\,diag\left(\widetilde{\Omega}_\ell\,o_H\left(h\right)\right)diag\left(\mu\right)\left(I_N-\widetilde{\Omega}_x'\,diag\left(\mu\right)\right)^{-1}\beta'\,E \\
&= E_h^{-1}\,\mathbb{1}_N'\,diag\left(\widetilde{\Omega}_\ell\,o_H\left(h\right)\right)\left(diag\left(\mu\right)^{-1}-\widetilde{\Omega}_x'\right)^{-1}\beta'\,E.
\end{aligned} \tag{71}$$

Finally, using equations (61) and (65), in the steady state is given by

$$\begin{aligned}
\Gamma_h &= \chi_h^{-1}\,\mathbb{1}_N'\,diag\left(\Omega_\ell\,o_H\left(h\right)\right)\Psi_x'\,\beta'\,\chi = \chi_h^{-1}\,\mathbb{1}_N'\,diag\left(\Omega_\ell\,o_H\left(h\right)\right)\lambda \\
&= \chi_h^{-1}\sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\sum_{j\in\mathscr{N}}\psi_{ji}^{x}\sum_{b\in\mathscr{H}}\beta_{bj}\,\chi_b = \chi_h^{-1}\sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\,\lambda_i = \frac{\Lambda_h}{\chi_h}\leq 1.
\end{aligned} \tag{72}$$

1.4.4 Household Budget Constraint Equilibrium Conditions

Introducing equations (32) and (33) in the profit equation (26)

$$\pi_{z_i} = \left(1-\mu_i\right)p_{z_i}\,y_{z_i}. \tag{73}$$

Introducing equations (32), (34), (36), (54), and (73) in the household budget constraint for household

$h \in \mathscr{H}$ (41)

$$E_h = \sum_{i \in \mathscr{N}} \int \Big(\mu_i \, \omega^{\ell}_{z_i} \, \alpha_{z_i h} + \kappa_{ih} \, (1 - \mu_i) \Big) \, p_{z_i} \, y_{z_i} \, dz_i. \tag{74}$$

Imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$E_h = \sum_{i \in \mathscr{N}} \Big(\mu_i \, \widetilde{\Omega}^{\ell}_{ih} + \kappa_{ih} \, (1 - \mu_i) \Big) \, S_i. \tag{75}$$

In matrix form, these equations are represented by

$$E = (\Omega_\ell + \Omega_\pi)' \, S, \tag{76}$$

where the matrices are given by $\Omega_\pi = diag \, (\mathbb{1}_N - \mu) \, \kappa$, and

$$\kappa \equiv \begin{pmatrix} \kappa_{11} & \cdots & \kappa_{1H} \\ \vdots & \ddots & \vdots \\ \kappa_{N1} & \cdots & \kappa_{NH} \end{pmatrix}.$$

By dividing element h in equation (75) by nominal GDP, I arrive at the following equation that relates the expenditure shares and the revenue-based Domar weights

$$\chi = (\Omega_\ell + \Omega_\pi)' \, \lambda. \tag{77}$$

Thus $\mathbb{1}'_H \, (\Omega_\ell + \Omega_\pi)' \, \lambda = \mathbb{1}'_H \, \Lambda + \mathbb{1}'_N \, diag \, (\mathbb{1}_N - \mu) \;\; \lambda = \sum_{h \in \mathscr{H}} \Lambda_h + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, \lambda_i = 1$. Using this equilibrium condition to define nominal GDP

$$\begin{aligned} GDP &= \sum_{h \in \mathscr{H}} J_h + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, S_i \\ &= \sum_{h \in \mathscr{H}} \sum_{i \in \mathscr{N}} \Omega^{\ell}_{ih} \, S_i + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, S_i \\ &= \sum_{i \in \mathscr{N}} \mu_i \, \omega^{\ell}_i \, S_i + \sum_{i \in \mathscr{N}} (1 - \mu_i) \, S_i = \sum_{i \in \mathscr{N}} (1 - \mu_i \, \omega^{x}_i) \, S_i. \end{aligned} \tag{78}$$

1.4.5 Nominal GDP

To define nominal GDP, I start by aggregating the good market clearing condition from equation (54) for all sectors

$$\sum_{i \in \mathscr{N}} S_i = \sum_{i \in \mathscr{N}} \left(\sum_{h \in \mathscr{H}} p_i C_{hi} + \sum_{j \in \mathscr{N}} p_i \int x_{z_j i} \, dz_j \right).$$

Then

$$
\begin{aligned}
GDP &\equiv \sum_{h \in \mathscr{H}} E_h \\
&= \sum_{i \in \mathscr{N}} \left(S_i - \sum_{j \in \mathscr{N}} p_j \int x_{z_i j} \, dz_i \right),
\end{aligned}
$$

using equations (33), (35), (37), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$
GDP = \sum_{i \in \mathscr{N}} \left(1 - \mu_i \sum_{j \in \mathscr{N}} \widetilde{\Omega}^x_{ij} \right) S_i = \sum_{i \in \mathscr{N}} \left(1 - \omega^x_i \, \mu_i \right) S_i. \tag{79}
$$

The last definition coincides with the total value-added generated by firms

$$
\begin{aligned}
GDP &= \sum_{i \in \mathscr{N}} \left(\left(\omega^\ell_i + \omega^x_i \right) S_i - \omega^x_i \, \mu_i \, S_i \right) \\
&= \sum_{i \in \mathscr{N}} \left(\sum_{h \in \mathscr{H}} w_h \ell_{ih} - \mu_i \frac{\sum_{h \in \mathscr{H}} w_h \ell_{ih}}{\mu_i S_i} S_i + \omega^\ell_i S_i + (1 - \mu_i) \, \omega^x_i S_i \right) \\
&= \sum_{i \in \mathscr{N}} \left(\sum_{h \in \mathscr{H}} w_h \ell_{ih} + (1 - \mu_i) \, S_i \right).
\end{aligned} \tag{80}
$$

1.5 Proof for Propositions in Section 4

1.5.1 Proof for Proposition 2

Using the following equations, I obtain a first-order approximation around the equilibrium for prices

$$
p^\ell_{z_i} = \frac{\sum_{h \in \mathscr{H}} w_h \, \ell_{z_i h}}{A^\ell_i \, Q^\ell_i \left(\left\{ A^\ell_{ih} \, \ell_{z_i h} \right\}_{h \in \mathscr{H}} \right)}, \tag{81}
$$

$$
p^x_{z_i} = \frac{\sum_{j \in \mathscr{N}} p_j \, x_{z_i j}}{A^x_i \, Q^x_i \left(\left\{ A^x_{ij} \, x_{z_i j} \right\}_{j \in \mathscr{N}} \right)}, \tag{82}
$$

$$
p_{z_i} = \frac{\left(p^\ell_{z_i} \, L_{z_i} + p^x_{z_i} \, X_{z_i} \right)}{\mu_i \, A_i \, Q_i \left(L_{z_i}, \, X_{z_i} \right)}, \tag{83}
$$

$$
p^c_h = \frac{\sum_{i \in \mathscr{N}} p_i \, C_{hi}}{Q^c_h \left(\{ C_{hi} \}_{i \in \mathscr{N}} \right)}. \tag{84}
$$

From equation (81)

$$
\widehat{p}^\ell_{z_i} = \frac{A^\ell_i}{p^\ell_{z_i}} \overline{\frac{\partial \, p^\ell_{z_i}}{\partial \, A^\ell_i}} \widehat{A}^\ell_i + \sum_{h \in \mathscr{H}} \left(\frac{w_h}{p^\ell_{z_i}} \overline{\frac{\partial \, p^\ell_{z_i}}{\partial \, w_h}} \widehat{w}_h + \frac{A^\ell_{ih}}{p^\ell_{z_i}} \overline{\frac{\partial \, p^\ell_{z_i}}{\partial \, A^\ell_{ih}}} \widehat{A}^\ell_{ih} + \frac{\ell_{z_i h}}{p^\ell_{z_i}} \overline{\frac{\partial \, p^\ell_{z_i}}{\partial \, \ell_{z_i h}}} \widehat{\ell}_{z_i h} \right),
$$

where $\frac{A_i^\ell}{p_{z_i}^\ell}\overline{\frac{\partial p_{z_i}^\ell}{\partial A_i^\ell}} = -1$, $\frac{w_h}{p_{z_i}^\ell}\overline{\frac{\partial p_{z_i}^\ell}{\partial w_h}} = \alpha_{z_i h}$, $\frac{A_{ih}^\ell}{p_{z_i}^\ell}\overline{\frac{\partial p_{z_i}^\ell}{\partial A_{ih}^\ell}} = -\alpha_{z_i h}$, $\frac{\ell_{z_i h}}{p_{z_i}^\ell}\overline{\frac{\partial p_{z_i}^\ell}{\partial \ell_{z_i h}}} = \alpha_{z_i h} - e\left(L_{z_i}, \ell_{z_i h}\right) = 0$ from equation (36), and $\widehat{x} = log\left(x/\overline{x}\right)$ stands for the log deviation around the equilibrium for variable x. As a consequence

$$\widehat{p}_{z_i}^\ell = -\widehat{A}_i^\ell + \sum_{h\in\mathscr{H}} \alpha_{z_i h}\left(\widehat{w}_h - \widehat{A}_{ih}^\ell\right). \tag{85}$$

Similarly, from equations (82), (83), and (84)

$$\widehat{p}_{z_i}^x = -\widehat{A}_i^x + \sum_{j\in\mathscr{N}} \omega_{z_i j}\left(\widehat{p}_j - \widehat{A}_{ij}^x\right), \tag{86}$$

$$\widehat{p}_{z_i} = \omega_{z_i}^\ell\,\widehat{p}_{z_i}^\ell + \omega_{z_i}^x\,\widehat{p}_{z_i}^x - \widehat{A}_i - \widehat{\mu}_i, \tag{87}$$

$$\widehat{p}_h^c = \sum_{i\in\mathscr{N}} \beta_{hi}\,\widehat{p}_i. \tag{88}$$

From imposing symmetry in the decision of monopolistically competitive firms within the same sector, these equations are represented in matrix form by

$$\widehat{p}_\ell = \alpha\,\widehat{w} - \widehat{A}_\ell - \left(\alpha\circ\widehat{\underline{A}}_\ell\right)\mathbb{1}_H, \tag{89}$$

$$\widehat{p}_x = \mathscr{W}\,\widehat{p} - \widehat{A}_x - \left(\mathscr{W}\circ\widehat{\underline{A}}_x\right)\mathbb{1}_N, \tag{90}$$

$$\widehat{p} = diag\left(\omega_\ell\right)\widehat{p}_\ell + diag\left(\omega_x\right)\widehat{p}_x - \widehat{A} - \widehat{\mu}, \tag{91}$$

$$\widehat{p}_c = \beta\,\widehat{p}. \tag{92}$$

Introducing equations (89) and (90) in equation (91)

$$\widehat{p} = \widetilde{\Psi}_x\left(\widetilde{\Omega}_\ell\,\widehat{w} - \widehat{\mathscr{A}} - \widehat{\mu}\right), \tag{93}$$

and introducing equation (93) in equation (92)

$$\widehat{p}_c = \widetilde{\mathscr{B}}\left(\widetilde{\Omega}_\ell\,\widehat{w} - \widehat{\mathscr{A}} - \widehat{\mu}\right). \tag{94}$$

The matrices previously used are defined by

$$\alpha \equiv \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1H} \\ \vdots & \ddots & \vdots \\ \alpha_{N1} & \cdots & \alpha_{NH} \end{pmatrix}, \qquad \mathscr{W} \equiv \begin{pmatrix} \omega_{11} & \cdots & \omega_{1N} \\ \vdots & \ddots & \vdots \\ \omega_{N1} & \cdots & \omega_{NN} \end{pmatrix},$$

$$\widetilde{\Psi}_x \equiv \begin{pmatrix} \widetilde{\psi}^x_{11} & \cdots & \widetilde{\psi}^x_{1N} \\ \vdots & \ddots & \vdots \\ \widetilde{\psi}^x_{N1} & \cdots & \widetilde{\psi}^x_{NN} \end{pmatrix}, \qquad \widetilde{\mathscr{B}} \equiv \beta\, \widetilde{\Psi}_x \equiv \begin{pmatrix} \widetilde{\mathscr{B}}_{11} & \cdots & \widetilde{\mathscr{B}}_{1N} \\ \vdots & \ddots & \vdots \\ \widetilde{\mathscr{B}}_{H1} & \cdots & \widetilde{\mathscr{B}}_{HN} \end{pmatrix},$$

$\widehat{\mathscr{A}} \equiv \widehat{A} + diag\,(\omega_\ell)\,\widehat{A}_\ell + \left(\widetilde{\Omega}_\ell \circ \underline{\widehat{\mathrm{A}}}_\ell\right) \mathbb{1}_H + diag\,(\omega_x)\,\widehat{A}_x + \left(\widetilde{\Omega}_x \circ \underline{\widehat{\mathrm{A}}}_x\right) \mathbb{1}_N$, $\widehat{A} \equiv \left[\widehat{A}_1, \cdots, \widehat{A}_N\right]'$, $\widehat{A}_\ell \equiv \left[\widehat{A}^\ell_1, \cdots, \widehat{A}^\ell_N\right]'$, $\widehat{A}_x \equiv \left[\widehat{A}^x_1, \cdots, \widehat{A}^x_N\right]'$, $\underline{\widehat{\mathrm{A}}}_\ell = \left[\underline{\widehat{\mathrm{A}}}^\ell_1, \cdots, \underline{\widehat{\mathrm{A}}}^\ell_N\right]'$, $\underline{\widehat{\mathrm{A}}}^\ell_i = \left[\widehat{A}^\ell_{i1}, \cdots, \widehat{A}^\ell_{iH}\right]'$, $\underline{\widehat{\mathrm{A}}}_x = \left[\underline{\widehat{\mathrm{A}}}^x_1, \cdots, \underline{\widehat{\mathrm{A}}}^x_n\right]'$, $\underline{\widehat{\mathrm{A}}}^x_i = \left[\widehat{A}^x_{i1}, \cdots, \widehat{A}^x_{iN}\right]'$, $\widehat{p} \equiv [\widehat{p}_1, \cdots, \widehat{p}_N]'$, $\widehat{p}_\ell \equiv \left[\widehat{p}^\ell_1, \cdots, \widehat{p}^\ell_N\right]'$, $\widehat{p}_x \equiv [\widehat{p}^x_1, \cdots, \widehat{p}^x_N]'$, $\widehat{\mu} \equiv [\widehat{\mu}_1, \cdots, \widehat{\mu}_N]'$, and $\widehat{w} \equiv [\widehat{w}_1, \cdots, \widehat{w}_H]'$.

1.5.2 Proof for Theorem 1

From equations (71) and (72)

$$\begin{aligned}
\Lambda_h \widehat{\Gamma}_h =& \mathbb{1}'_N\, diag\,(\Omega_\ell\, o_H\,(h))\, \mathscr{B}' \left(\begin{pmatrix} \chi_1\, \widehat{E}_1 \\ \vdots \\ \chi_H\, \widehat{E}_H \end{pmatrix} - \chi\, \widehat{E}_h \right) + \mathbb{1}'_N\, diag\,(\Omega_\ell\, o_H\,(h))\, \Psi'_x \begin{pmatrix} \sum_{b\in\mathscr{H}} \beta_{b1}\, \chi_b\, \widehat{\beta}_{b1} \\ \vdots \\ \sum_{b\in\mathscr{H}} \beta_{bN}\, \chi_b\, \widehat{\beta}_{bN} \end{pmatrix} \\
&+ \sum_{i\in\mathscr{N}} \Omega^\ell_{ih}\, \lambda_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) + \mathbb{1}'_N\, diag\left(\widetilde{\Omega}_\ell\, o_H\,(h)\right) \frac{d\left(diag\,(\mu)^{-1} - \widetilde{\Omega}'_x\right)^{-1}}{d\,log\,\widetilde{\Omega}_x} \beta'\,\chi \\
&+ \mathbb{1}'_N\, diag\left(\widetilde{\Omega}_\ell\, o_H\,(h)\right) \frac{d\left(diag\,(\mu)^{-1} - \widetilde{\Omega}'_x\right)^{-1}}{d\,log\,\mu} \beta'\,\chi.
\end{aligned}$$

Using equations (61), (65), (68), and (67), and the fact that for any invertible matrix A, $\frac{d\,A^{-1}}{d\,x} = -A^{-1} \frac{d\,A}{d\,x} A^{-1}$, the previous equation becomes

$$\begin{aligned}
\widehat{\Gamma}_h =& \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{E}_b - \widehat{E}_h + \Lambda_h^{-1} \sum_{i\in\mathscr{N}} \Omega^\ell_{ih} \sum_{j\in\mathscr{N}} \Psi^x_{ji} \sum_{b\in\mathscr{H}} \beta_{bj}\, \chi_b\, \widehat{\beta}_{bj} + \Lambda_h^{-1} \sum_{i\in\mathscr{N}} \Omega^\ell_{ih}\, \lambda_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) \\
&- \Lambda_h^{-1}\, \mathbb{1}'_N diag\,(\Omega_\ell\, o_H\,(h))\, \Psi'_x \frac{d\left(diag\,(\mu)^{-1} - \widetilde{\Omega}'_x\right)}{d\,log\,\widetilde{\Omega}_x} diag\,(\mu)\;\lambda \\
&- \Lambda_h^{-1}\, \mathbb{1}'_N\, diag\,(\Omega_\ell\, o_H\,(h))\, \Psi'_x \frac{d\left(diag\,(\mu)^{-1} - \widetilde{\Omega}'_x\right)}{d\,log\,\mu} diag\,(\mu)\;\lambda.
\end{aligned}$$

$$\begin{aligned}
\widehat{\Gamma}_h &= \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{E}_b - \widehat{E}_h + \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{\mathscr{C}}_{bh} \\
&= \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\Lambda_h} \widehat{E}_b - \widehat{E}_h + \frac{1}{\Lambda_h} o_H\,(h)'\, \Psi'_\ell\, diag\,(\widehat{\mu})\;\lambda \\
&+ \frac{1}{\Lambda_h} \left(\sum_{i\in\mathscr{N}} \Omega^\ell_{ih}\, \lambda_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) + \sum_{j\in\mathscr{N}} \psi^\ell_{jh} \left(\sum_{b\in\mathscr{H}} \beta_{bj}\, \chi_b\, \widehat{\beta}_{bj} + \sum_{i\in\mathscr{N}} \Omega^x_{ij}\, \lambda_i\, \left(\widehat{\omega}^x_i + \widehat{\omega}_{ij}\right) \right) \right).
\end{aligned} \tag{95}$$

Add and subtract $\widehat{GDP}$ to express equation (95) in terms of sales and factor Domar weights

$$\begin{aligned}\Gamma_h \widehat{\Gamma}_h &= \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\chi_h} \widehat{\chi}_b - \Gamma_h \widehat{\chi}_h + \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\chi_h} \widehat{\mathscr{C}}_{bh} \\ &= \sum_{b\in\mathscr{H}} \mathscr{C}_{bh} \frac{\chi_b}{\chi_h} \widehat{\chi}_b - \Gamma_h \widehat{\chi}_h + \frac{1}{\chi_h} o_H (h)' \Psi_\ell' \, diag\left(\widehat{\mu}\right) \lambda \\ + \frac{1}{\chi_h} &\left(\sum_{i\in\mathscr{N}} \Omega^\ell_{ih} \lambda_i \left(\widehat{\omega}^\ell_i + \widehat{\alpha}_{ih}\right) + \sum_{j\in\mathscr{N}} \psi^\ell_{jh} \left(\sum_{b\in\mathscr{H}} \beta_{bj} \chi_b \widehat{\beta}_{bj} + \sum_{i\in\mathscr{N}} \Omega^x_{ij} \lambda_i \left(\widehat{\omega}^x_i + \widehat{\omega}_{ij}\right)\right)\right),\end{aligned}$$

where $\Lambda_h = \sum_{b\in\mathscr{H}} \chi_b \mathscr{C}_{bh}$ is used. Now, using equation (72)

$$\begin{aligned} d\,\Lambda_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{bh}\, d\,\chi_b + \sum_{b\in\mathscr{H}} \chi_b\, d\,\mathscr{C}_{bh} = \sum_{b\in\mathscr{H}} \mathscr{C}_{bh}\, d\,\chi_b + \sum_{i\in\mathscr{H}} \psi^\ell_{ih}\, \lambda_i\, d\,log\, \mu_i \\ + \sum_{i\in\mathscr{N}} \mu_i\, \lambda_i\, d\,\widetilde{\Omega}^\ell_{ih} + \sum_{j\in\mathscr{N}} \psi^\ell_{jh} \left(\sum_{b\in\mathscr{H}} \chi_b\, d\,\beta_{bj} + \sum_{i\in\mathscr{N}} \mu_i\, \lambda_i\, d\,\widetilde{\Omega}^x_{ij}\right). \end{aligned} \tag{96}$$

1.5.3 Proof for Theorem 3

The first order approximation for equation (41) is given by

$$\widehat{E}_h = \Gamma_h \left(\widehat{w}_h + \widehat{L}_h\right) + \left(1 - \Gamma_h\right) \widehat{\Pi}_h. \tag{97}$$

The first order approximation for dividend income in equations (42) and (73) is given by

$$\widehat{\Pi}_h = \frac{1}{\Pi_h} \sum_{i\in\mathscr{N}} \kappa_{ih} \int S_{z_i} \left(\left(1 - \mu_i\right) \left(\widehat{\kappa}_{ih} + \widehat{S}_{z_i}\right) dz_i - \mu_i \,\widehat{\mu}_i \right) dz_i. \tag{98}$$

Introducing equation (98) in equation (97)

$$E_h \widehat{E}_h = J_h \left(\widehat{w}_h + \widehat{L}_h\right) + \sum_{i\in\mathscr{N}} \kappa_{ih} \int p_{z_i}\, y_{z_i} \left(\left(1 - \mu_i\right) \left(\widehat{\kappa}_{ih} + \widehat{p}_{z_i} + \widehat{y}_{z_i}\right) - \mu_i \,\widehat{\mu}_i \right) dz_i. \tag{99}$$

From equations (94) and (99), and imposing symmetry in the decision of monopolistically competitive firms within the same sector

$$\begin{aligned}\widehat{C}_h &= \widehat{E}_h - \widehat{p}^c_h \\ &= \Gamma_h \left(\widehat{w}_h + \widehat{L}_h\right) - \widetilde{\mathscr{C}}_h' \,\widehat{w} + \widetilde{\mathscr{B}}_h' \left(\widehat{\mathscr{A}} + \widehat{\mu}\right) + \sum_{i\in\mathscr{N}} \kappa_{ih} \frac{\lambda_i}{\chi_h} \left(\left(1 - \mu_i\right) \left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i \,\widehat{\mu}_i \right)\end{aligned}$$

where $\widetilde{\mathscr{B}}_h = \left[\widetilde{\mathscr{B}}_{h1}, \cdots, \widetilde{\mathscr{B}}_{hN}\right]'$, and $\widetilde{\mathscr{C}}_h = \left[\widetilde{\mathscr{C}}_{h1}, \cdots, \widetilde{\mathscr{C}}_{hH}\right]'$. Then

$$\widehat{C}_h = \widetilde{\mathscr{B}}_h' \left(\widehat{\mathscr{A}} + \widehat{\mu}\right) + \Gamma_h \widehat{J}_h - \widetilde{\mathscr{C}}_h' \widehat{J} + \sum_{i\in\mathscr{N}} \kappa_{ih} \frac{\lambda_i}{\chi_h} \left(\left(1 - \mu_i\right) \left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i \,\widehat{\mu}_i \right) + \widetilde{\mathscr{C}}_h' \widehat{L}$$

Therefore

$$\frac{C_h}{\overline{C}_h} = \overline{\eta}_h \, \mathscr{D}_h\left(\mathscr{A}\right) \, \mathscr{D}_h\left(\mu\right) \, \mathscr{D}_h\left(J\right) \, \mathscr{D}_h\left(\Pi\right) \, f_h\left(\{L_b\}_{b\in\mathscr{H}}\right) \tag{100}$$

where $f_h\left(\{L_b\}_{b\in\mathscr{H}}\right)$ is a CRS function such that $\frac{d\log f_h\left(\{L_b\}_{b\in\mathscr{H}}\right)}{d\log L_b} = \widetilde{\mathscr{C}}_{hb}$, and

$$\mathscr{D}_h\left(\mathscr{A}\right) = exp\left\{\widetilde{\mathscr{B}}_h' \, \widehat{\mathscr{A}}\right\}, \qquad \mathscr{D}_h\left(\mu\right) = exp\left\{\widetilde{\mathscr{B}}_h' \, \widehat{\mu}\right\},$$

$$\mathscr{D}_h\left(\Pi\right) = exp\left\{\sum_{i\in\mathscr{N}} \kappa_{ih} \, \frac{\lambda_i}{\chi_h} \left(\left(1-\mu_i\right)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i \, \widehat{\mu}_i\right)\right\},$$

$$\mathscr{D}_h\left(J\right) = exp\left\{\Gamma_h \, \widehat{J}_h - \widetilde{\mathscr{C}}_h' \, \widehat{J}\right\}, \tag{101}$$

and $\overline{\eta}_h$ stands for a constant.

As a consequence

$$C_h = \eta_h \, \mathscr{D}_h\left(\mathscr{A}\right) \, \mathscr{D}_h\left(\mu\right) \, \mathscr{D}_h\left(J\right) \, \mathscr{D}_h\left(\Pi\right) \, f_h\left(\{L_b\}_{b\in\mathscr{H}}\right) = PTT_h \, f_h\left(\{L_b\}_{b\in\mathscr{H}}\right) \tag{102}$$

$$PTT_h = \eta_h \, \mathscr{D}_h\left(\mathscr{A}\right) \, \mathscr{D}_h\left(\mu\right) \, \mathscr{D}_h\left(\mathscr{J}\right) \, \mathscr{D}_h\left(\Pi\right)$$

with $\eta_h = \overline{\eta}_h \, \overline{C}_h$.

Add and subtract $\widehat{GDP}$ to express equation (102) in terms of Domar weights and labor income shares

$$\begin{aligned}
\widehat{C}_h &= \widetilde{\mathscr{B}}_h' \, \widehat{\mathscr{A}} + \left(\widetilde{\mathscr{B}}_h - \chi_h^{-1} \, diag\left(\lambda\right) \, diag\left(\mu\right) \, \kappa \, o_H\left(h\right)\right)' \widehat{\mu} + \Gamma_h \, \widehat{\Lambda}_h - \widetilde{\mathscr{C}}_h' \, \widehat{\Lambda} \\
&\quad + \chi_h^{-1} \, o_H\left(h\right)' \, \kappa' \, diag(\lambda) \, diag\left(\mathbb{1}_N - \mu\right) \left(\widehat{\kappa} \, o_H\left(h\right) + \widehat{\lambda}\right) + \widetilde{\mathscr{C}}_h' \, \widehat{L} \\
&\quad + \underbrace{\left(\Gamma_h + \chi_h^{-1} \sum_{i\in\mathscr{N}} \kappa_{ih}\left(1-\mu_i\right)\lambda_i - \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb}\right)}_{=0} \widehat{GDP} \\
&= \widetilde{\mathscr{B}}_h' \, \widehat{\mathscr{A}} + \left(\widetilde{\mathscr{B}}_h - \chi_h^{-1} \, diag\left(\lambda\right) \, diag\left(\mu\right) \, \kappa \, o_H\left(h\right)\right)' \widehat{\mu} + \Gamma_h \, \widehat{\Lambda}_h - \widetilde{\mathscr{C}}_h' \, \widehat{\Lambda} \\
&\quad + \chi_h^{-1} \, o_H\left(h\right)' \, \kappa' \, diag(\lambda) \, diag\left(\mathbb{1}_N - \mu\right) \left(\widehat{\kappa} \, o_H\left(h\right) + \widehat{\lambda}\right) + \widetilde{\mathscr{C}}_h' \, \widehat{L}.
\end{aligned}$$

where the last equality is given by equations (68) and (77).

The $N+1$ vector $\mathscr{R}_h$ captures the revenue distribution for household h

$$\mathscr{R}_h' = \begin{bmatrix}\Gamma_h & \chi_h^{-1}\lambda' diag\left(\Omega_\pi \, o_H\left(h\right)\right)\end{bmatrix} = \frac{1}{\chi_h}\begin{bmatrix}\Lambda_h & \kappa_{1h}\left(1-\mu_1\right)\lambda_1 & \cdots & \kappa_{Nh}\left(1-\mu_N\right)\lambda_N\end{bmatrix}.$$

The first element captures the share of labor income in household h's expenditure, and the last N elements capture the share of profits by each sector on household h's expenditure. As the elements of this vector add up to one, its first-order approximation is given by

$$\widehat{\chi}_h = \Gamma_h \widehat{\Lambda}_h + \chi_h^{-1} \sum_{i\in\mathscr{N}} \kappa_{ih} \lambda_i \left((1-\mu_i) \left(\widehat{\kappa}_{ih} + \widehat{\lambda}_i \right) - \mu_i \widehat{\mu}_i \right). \tag{103}$$

This implies that

$$\widehat{C}_h = \widetilde{\mathscr{B}}_h' \widehat{\mathscr{A}} + \widetilde{\mathscr{B}}_h' \widehat{\mu} + \widehat{\chi}_h - \widetilde{\mathscr{C}}_h' \widehat{\Lambda} + \widetilde{\mathscr{C}}_h' \widehat{L}. \tag{104}$$

Now, using equations (72) and (96), and the definitions $\delta_{b|h} = \widetilde{\mathscr{C}}_{hb}/\widetilde{\Lambda}_b$, $M_{q|h} = \sum_{b\in\mathscr{H}} \mathscr{C}_{qb}\, \delta_{b|h}$, and $F_{i|h} = \sum_{q\in\mathscr{H}} \psi_{iq}^{\ell}\, \delta_{q|h}$

$$\begin{aligned}
\widehat{PTT}_h &= \sum_{i\in\mathscr{N}} \widetilde{\mathscr{B}}_{hi} \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i \right) + \widehat{\chi}_h - \sum_{b\in\mathscr{H}} F_{b|h}\, d\chi_b - \sum_{q\in\mathscr{H}} \chi_q \sum_{b\in\mathscr{H}} \delta_{b|h}\, d\mathscr{C}_{qb} \\
&= \sum_{i\in\mathscr{N}} \widetilde{\mathscr{B}}_{hi} \widehat{\mathscr{A}}_i + \sum_{i\in\mathscr{N}} \widetilde{\mathscr{B}}_{hi} \widehat{\mu}_i + \widehat{\chi}_h - \sum_{b\in\mathscr{H}} M_{b|h}\, d\chi_b - \sum_{i\in\mathscr{N}} \lambda_i F_{i|h} \widehat{\mu}_i \\
&- \sum_{i\in\mathscr{N}} \mu_i \lambda_i \sum_{b\in\mathscr{H}} \delta_{b|h}\, d\widetilde{\Omega}_{ib}^{\ell} - \sum_{b\in\mathscr{H}} \chi_b \sum_{i\in\mathscr{N}} F_{i|h}\, d\beta_{bi} - \sum_{i\in\mathscr{N}} \mu_i \lambda_i \sum_{j\in\mathscr{N}} F_{j|h}\, d\widetilde{\Omega}_{ij}^{x}.
\end{aligned} \tag{105}$$

1.5.4 Proof for Theorem 2

The first-order approximation for nominal GDP is given by

$$\widehat{GDP} = \sum_{h\in\mathscr{H}} \chi_h \widehat{E}_h = \sum_{h\in\mathscr{H}} \chi_h \left(\widehat{p}_h^c + \widehat{C}_h \right).$$

From here, I define the GDP deflator as the Divisia weighted variation of idiosyncratic price bundles

$$\widehat{p}_Y = \sum_{h\in\mathscr{H}} \chi_h \widehat{p}_h^c = \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \widehat{w}_h - \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{A}_i + \widehat{\mu}_i \right). \tag{106}$$

Hence, the first-order approximation for real GDP is given by

$$\widehat{Y} = \widehat{GDP} - \widehat{p}_Y = \sum_{h\in\mathscr{H}} \chi_h \widehat{C}_h, \tag{107}$$

and this equation represented as deviation from the equilibrium is given by

$$Y = Q_Y \left(\{C_h\}_{h\in\mathscr{H}} \right) = \eta_Y\, \mathscr{D}(\mathscr{A})\, \mathscr{D}(\mu)\, \mathscr{D}(J)\, \mathscr{D}(\Pi)\, F\left(\{L_h\}_{h\in\mathscr{H}} \right),$$

where η_Y is a constant, $Q_Y \left(\{C_h\}_{h\in\mathscr{H}} \right)$ is a CRS function such that $\frac{d\log Q_Y\left(\{C_h\}_{h\in\mathscr{H}}\right)}{d\log C_h} = \chi_h$, and $F\left(\{L_h\}_{h\in\mathscr{H}} \right)$ is a CRS function such that $\frac{d\log F\left(\{L_h\}_{h\in\mathscr{H}}\right)}{d\log L_h} = \widetilde{\Lambda}_h$.

Additionally,

$$\mathcal{D}\left(q\right)=Q_{\mathcal{D}(q)}\left(\left\{\mathcal{D}_{h}\left(q\right)\right\}_{h\in\mathcal{H}}\right),\qquad\qquad L=F\left(\left\{L_{h}\right\}_{h\in\mathcal{H}}\right).$$

where $Q_{\mathcal{D}(q)}\left(\left\{\mathcal{D}_{h}\left(q\right)\right\}_{h\in\mathcal{H}}\right)$ is a CRS function such that $\frac{d\,log\,Q_{\mathcal{D}(q)}\left(\left\{\mathcal{D}_{h}(q)\right\}_{h\in\mathcal{H}}\right)}{d\,log\,\mathcal{D}_{h}(q)}=\chi_{h}$.

As a consequence

$$Y=\eta_{Y}\;\mathcal{D}\left(\mathcal{A}\right)\;\mathcal{D}\left(\mu\right)\;\mathcal{D}\left(\Pi\right)\;\mathcal{D}\left(J\right)\;L=TFP\;L, \tag{108}$$

$$TFP=\eta_{Y}\;\mathcal{D}\left(\mathcal{A}\right)\;\mathcal{D}\left(\mu\right)\;\mathcal{D}\left(\Pi\right)\;\mathcal{D}\left(J\right).$$

Notice that so far, not a single parametric assumption has been made to obtain these first-order decompositions. The elasticity χ_h for the functions $Q_Y\left(\left\{C_h\right\}_{h\in\mathcal{H}}\right)$ and $Q_{\mathcal{D}(q)}\left(\left\{\mathcal{D}_{h}\left(q\right)\right\}_{h\in\mathcal{H}}\right)$, and $\widetilde{\Lambda}_h$ for the function $F\left(\left\{L_{h}\right\}_{h\in\mathcal{H}}\right)$, come respectively from the first-order approximation of nominal GDP in its different components, and the fact that $\widetilde{\Lambda}_h=\sum_{b\in\mathcal{H}}\chi_b\,\widetilde{\mathcal{C}}_{bh}$.

The aggregate labor terms of trade are equal to

$$log\,\mathcal{D}\left(J\right)=\sum_{h\in\mathcal{H}}\left(\Lambda_h\,\widehat{J}_h-\chi_h\,\widetilde{\mathcal{C}}_h'\,\widehat{J}\right)=\sum_{h\in\mathcal{H}}\Lambda_h\,\widehat{J}_h-\sum_{b\in\mathcal{H}}\widehat{J}_b\sum_{i\in\mathcal{N}}\widetilde{\Omega}^{\ell}_{ib}\sum_{h\in\mathcal{H}}\chi_h\,\widetilde{\mathcal{B}}_{hi}.$$

Using equations (62) and (66)

$$log\,\mathcal{D}\left(J\right)=\sum_{h\in\mathcal{H}}\Lambda_h\widehat{J}_h-\sum_{b\in\mathcal{H}}\widehat{J}_b\sum_{i\in\mathcal{N}}\widetilde{\Omega}^{\ell}_{ib}\,\widetilde{\lambda}_i=\left(\Lambda-\widetilde{\Lambda}\right)'\,\widehat{J}. \tag{109}$$

Therefore, starting from equation (107), and using (62), the first-order approximation for real GDP is given by

$$\begin{aligned}\widehat{Y}=&\sum_{h\in\mathcal{H}}\left(\chi_h\,\widetilde{\mathcal{B}}_h'\left(\widehat{\mathcal{A}}+\widehat{\mu}\right)+\left(\Lambda_h-\widetilde{\Lambda}_h\right)\widehat{J}_h+\widetilde{\Lambda}_h\,\widehat{L}_h\right)\\&+\sum_{i\in\mathcal{N}}\lambda_i\left(\left(1-\mu_i\right)\widehat{S}_i-\mu_i\,\widehat{\mu}_i\right)\underbrace{\sum_{h\in\mathcal{H}}\kappa_{ih}}_{=1}+\sum_{i\in\mathcal{N}}\left(1-\mu_i\right)\lambda_i\underbrace{\sum_{h\in\mathcal{H}}\kappa_{ih}\,\widehat{\kappa}_{ih}}_{=0}\end{aligned}$$

$$\widehat{Y}=\widetilde{\lambda}'\,\widehat{\mathcal{A}}+\left(\widetilde{\lambda}-diag\left(\mu\right)\lambda\right)'\widehat{\mu}+\left(\Lambda-\widetilde{\Lambda}\right)'\widehat{J}+\lambda'\,diag\left(\mathbb{1}_N-\mu\right)\widehat{S}+\widetilde{\Lambda}'\,\widehat{L}. \tag{110}$$

Add and subtract $\widehat{GDP}$ to express equation (110) in terms of Domar weights and labor income shares

$$\begin{aligned}
\widehat{Y} &= \widetilde{\lambda}'\,\widehat{\mathscr{A}} + \left(\widetilde{\lambda} - diag\,(\mu)\;\lambda\right)'\widehat{\mu} + \left(\Lambda - \widetilde{\Lambda}\right)'\widehat{\Lambda} + \lambda'\,diag\,(\mathbb{1}_N - \mu)\;\widehat{\lambda} + \widetilde{\Lambda}'\,\widehat{L} \\
&\quad + \underbrace{\left(\left(\Lambda - \widetilde{\Lambda}\right)'\mathbb{1}_H + \lambda'\,diag\,(\mathbb{1}_N - \mu)\,\mathbb{1}_N\right)}_{=\sum_{h\in\mathscr{H}}\Lambda_h + \sum_{i\in\mathscr{N}}(1-\mu_i)\lambda_i - \sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h = 0}\widehat{GDP} \\
&= \widetilde{\lambda}'\,\widehat{\mathscr{A}} + \left(\widetilde{\lambda} - diag\,(\mu)\;\lambda\right)'\widehat{\mu} + \left(\Lambda - \widetilde{\Lambda}\right)'\,\widehat{\Lambda} + \lambda'\,diag\,(\mathbb{1}_N - \mu)\;\widehat{\lambda} + \widetilde{\Lambda}'\,\widehat{L}.
\end{aligned}$$

where the last equality is given by equations (66) and (77).

The $2H$ vector $\mathscr{R}$ captures the revenue distribution across households

$$\mathscr{R}' = \begin{bmatrix}\lambda'\,\Omega_\ell & \lambda'\,\Omega_\pi\end{bmatrix} = \begin{bmatrix}\Lambda_1 & \cdots & \Lambda_H & \sum_{i\in\mathscr{N}}\kappa_{i1}\,(1-\mu_i)\,\lambda_i & \cdots & \sum_{i\in\mathscr{N}}\kappa_{iH}\,(1-\mu_i)\,\lambda_i\end{bmatrix}.$$

The first H elements capture for each households its labor income share, and the last H elements portray for each household the share of its profits in total expenditure. As the elements of this vector add up to one, its first order approximation is given by

$$\begin{aligned}
0 &= \sum_{h\in\mathscr{H}}\Lambda_h\,\widehat{\Lambda}_h + \sum_{h\in\mathscr{H}}\sum_{i\in\mathscr{N}}\kappa_{ih}\,\lambda_i\left((1-\mu_i)\left(\widehat{\kappa}_{ih} + \widehat{\lambda}_i\right) - \mu_i\,\widehat{\mu}_i\right) \\
&= \sum_{h\in\mathscr{H}}\Lambda_h\,\widehat{\Lambda}_h + \sum_{i\in\mathscr{N}}\lambda_i\left((1-\mu_i)\left(\underbrace{\sum_{h\in\mathscr{H}}\kappa_{ih}\,\widehat{\kappa}_{ih}}_{=0} + \widehat{\lambda}_i\underbrace{\sum_{h\in\mathscr{H}}\kappa_{ih}}_{=1}\right) - \mu_i\,\widehat{\mu}_i\underbrace{\sum_{h\in\mathscr{H}}\kappa_{ih}}_{=1}\right) \\
&= \sum_{h\in\mathscr{H}}\Lambda_h\,\widehat{\Lambda}_h + \sum_{i\in\mathscr{N}}\lambda_i\left((1-\mu_i)\;\widehat{\lambda}_i - \mu_i\,\widehat{\mu}_i\right).
\end{aligned}$$

This implies that

$$\widehat{Y} = \widetilde{\lambda}'\,\widehat{\mathscr{A}} + \widetilde{\lambda}'\,\widehat{\mu} - \widetilde{\Lambda}'\,\widehat{\Lambda} + \widetilde{\Lambda}'\,\widehat{L} \tag{111}$$

which under the additional assumption that the labor supply is inelastic coincides with Theorem 1 in Baqaee & Farhi (2020).

Now, using equations (72) and (96)

$$\begin{aligned}
\widehat{TFP} &= \sum_{i\in\mathscr{N}}\widetilde{\lambda}_i\left(\widehat{\mathscr{A}}_i + \hat{\mu}_i\right) - \sum_{h\in\mathscr{H}}M_h\,d\,\chi_h - \sum_{h\in\mathscr{H}}\delta_h\sum_{b\in\mathscr{H}}\chi_b\,d\,\mathscr{C}_{bh} \\
&= \sum_{i\in\mathscr{N}}\widetilde{\lambda}_i\widehat{\mathscr{A}}_i + \sum_{i\in\mathscr{N}}\left(\widetilde{\lambda}_i - \lambda_i\,F_i\right)\widehat{\mu}_i - \sum_{h\in\mathscr{H}}M_h\,\widehat{\chi}_h - \sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\sum_{h\in\mathscr{H}}\delta_h\,d\,\widetilde{\Omega}^\ell_{ih} \\
&\quad - \sum_{h\in\mathscr{H}}\chi_h\sum_{i\in\mathscr{N}}F_i\,d\,\beta_{hi} - \sum_{i\in\mathscr{N}}\mu_i\,\lambda_i\sum_{j\in\mathscr{N}}F_j\,d\,\widetilde{\Omega}^x_{ij}.
\end{aligned} \tag{112}$$

1.5.5 Proof for Corollary 1

If $M_h = M \ \forall h \in \mathscr{H}$, then *Distributive TT* $= M \sum_{h\in\mathscr{H}} d\,\chi_h = 0$ because $\sum_{h\in\mathscr{H}} d\,\chi_h = 0$. **(i) Undistorted economy.** $\delta_h = 1$ and $M_h = 1 \ \forall h \in \mathscr{H}$. **(ii) Symmetric consumption bundles.** $\beta = \mathbb{1}_H\,\overline{\beta}'$ where $\overline{\beta}$ stands for the common vector of consumption. Hence $\mathscr{C} = \beta\,\Psi_x\,\Omega_\ell = \mathbb{1}_H\,\overline{\beta}'\,\Psi_x\,\Omega_\ell = \mathbb{1}_H\,\overline{\mathscr{C}}'$. This implies that $M_h = \overline{\mathscr{C}}'\,\delta \ \forall h \in \mathscr{H}$ where δ stands for the vector of distortion centralities. **(iii) No intermediate imputs and symmetric distortions.** $\mathscr{C} = \beta\,\Omega_\ell = \mu\,\beta\,\alpha$. Hence, $\mathscr{C}\,\mathbb{1}_H = \mu\,\beta\,\alpha\,\mathbb{1}_H = \mu\,\beta\,\mathbb{1}_H = \mu\,\mathbb{1}$. This implies that $M_h = \mu \sum_{b\in\mathscr{H}} \delta_b \ \forall h \in \mathscr{H}$. **(iv) No intermediate imputs and sectoral specific labor supply.** $\widetilde{\Omega}_\ell = I_N$. Hence, $\mathscr{C} = \beta\,diag\,(\mu)$, and $\delta_i = \mu_i^{-1}$. Consequently, $\mathscr{C}\,\delta = \beta\,\mathbb{1}_N = \mathbb{1}_N$.

1.5.6 Proof for Corollary 2

Without distortions $\delta_h = 1 \ \forall h \in \mathscr{H}$. Hence $M_h = \sum_{b\in\mathscr{H}} \mathscr{C}_{hb} = 1 \ \forall h \in \mathscr{H}$ and $F_i = \sum_{h\in\mathscr{H}} \psi^\ell_{ih} = 1$ $\forall i \in \mathscr{N}$. As a result, the final demand terms of trade are given by

$$\sum_{h\in\mathscr{H}} \chi_h \underbrace{\sum_{i\in\mathscr{N}} \beta_{hi}}_{=0} = 0,$$

and the *labor TT*+ *intermediate TT* are given by

$$\sum_{i\in\mathscr{N}} \lambda_i \underbrace{\left(\sum_{h\in\mathscr{H}} d\,\widetilde{\Omega}^\ell_{ih} + \sum_{j\in\mathscr{N}} d\,\widetilde{\Omega}^x_{ij} \right)}_{=0} = 0.$$

Cases 2 and 3 should be obvious from the previous proof.

1.5.7 Proof for Corollaries 3 and 4

To proof Corollary 4, notice that

$$\widetilde{\mathscr{B}}'_h\,diag\,(\omega_\ell)\,\mathbb{1}_N = \beta'_h\,\widetilde{\Psi}_x\,\omega_\ell = \beta'_h\,\widetilde{\Psi}_x \left(I_N - \widetilde{\Omega}_x \right) \mathbb{1}_N = 1.$$

Therefore, equation (101) can be represented as

$$\begin{aligned}
\mathscr{D}_h\,(J) &= exp\left\{ \widetilde{\mathscr{B}}'_h\,diag\,(\omega_\ell) \left(\mathbb{1}_N\,\Gamma_h\,\widehat{J}_h - \alpha\,\widehat{J} \right) \right\} = exp\left\{ \widetilde{\mathscr{B}}'_h\,diag\,(\omega_\ell) \left(\mathbb{1}_N\,\Gamma_h\,o_H\,(h)' - \alpha \right) \left(\widehat{J} \pm \widehat{p}_c \right) \right\} \\
&= exp\left\{ \Gamma_h\,\widehat{p}^c_h \pm \widehat{p}^c_h + \widetilde{\mathscr{B}}'_h\,diag\,(\omega_\ell) \left[\left(\mathbb{1}_N\,\Gamma_h\,o_H\,(h)' - \alpha \right) \left(\widehat{J} - \widehat{p}_c \right) - \alpha\,\widehat{p}_c \right] \right\} \\
&= exp\left\{ -\left(1 - \Gamma_h\right) \widehat{p}^c_h + \widetilde{\mathscr{B}}'_h\,diag\,(\omega_\ell) \left(\mathbb{1}_N\,\Gamma_h\,o_H\,(h)' - \alpha \right) \left(\widehat{J} - \widehat{p}_c \right) - \widetilde{\mathscr{C}}'_{\downarrow h} \left(\widehat{p}_c - \mathbb{1}_H\,\widehat{p}^c_h \right) \right\} \\
&= exp\left\{ -\left(1 - \Gamma_h\right) \widehat{p}^c_h + \widetilde{\mathscr{B}}'_h\,diag\,(\omega_\ell) \left(\mathbb{1}_N\,\Gamma_h\,o_H\,(h)' - \alpha \right) \left(\widehat{J} - \widehat{p}_c \right) - \widetilde{\mathscr{C}}'_{\downarrow h}\,\widehat{\varepsilon}_h \right\},
\end{aligned}$$

where $\varepsilon_{hb} = p_b^c / p_h^c$, and $\widehat{\varepsilon}_h = [\widehat{\varepsilon}_{h1}, \cdots, \widehat{\varepsilon}_{hH}]'$.

Then $\mathscr{D}_h(J)$ can be decomposed in nominal wedges as follows

$$\mathscr{D}_h(J) = \mathscr{D}_h(\mathcal{J})\, \mathscr{D}_h(p_c)\, \mathscr{D}_h(\varepsilon), \tag{113}$$

where

$$\mathscr{D}_h(\mathcal{J}) = exp\left\{\Gamma_h\left(\widehat{J}_h - \widehat{p}_h^c\right) - \widetilde{\mathscr{C}}_{\downarrow h}'\left(\widehat{J} - \widehat{p}_c\right)\right\},$$

$$\mathscr{D}_h(p_c) = exp\left\{(\Gamma_h - 1)\, \widehat{p}_h^c\right\},$$

$$\mathscr{D}_h(\varepsilon) = exp\left\{-\widetilde{\mathscr{C}}_{\downarrow h}'\, \widehat{\varepsilon}_h\right\}.$$

Corollary 3 comes from the expenditure weighted summation of the idiosyncratice nominal wedges, e.g.,

$$\mathscr{D}(\varepsilon) = \sum_{h \in \mathscr{H}} \chi_h\, \mathscr{D}_h(\varepsilon).$$

1.6 Proof for Theorem 4

Now, in order to obtain the first-order decomposition for the aggregate labor wedge, let me assume the existence of an aggregate welfare function and a constraint social planner that centralizes the decision for all households by solving

$$\underset{\left\{Y, L, \left\{C_h, L_h, \{C_{hi}\}_{i \in \mathscr{N}}\right\}_{h \in \mathscr{H}}\right\}}{Max} \quad W(Y, L)$$

subject to

$$p_Y\, Y = \sum_{h \in \mathscr{H}} p_h^c C_h = \sum_{h \in \mathscr{H}} \sum_{i \in \mathscr{N}} p_i C_{hi} \leq wL + \Pi = \sum_{h \in \mathscr{H}} (w_h L_h + \Pi_h)$$

and taking prices, wages, and profits as given.

The first order conditions for the constrained social planner satisfies

$$\frac{W_Y}{p_Y} = -\frac{W_L}{w} = \frac{W_Y}{p_h^c} Y_{C_h} = -\frac{W_L}{w_h} L_{L_h} = \frac{W_Y}{p_i} Y_{C_h} \frac{d\,C_h}{d\,C_{hi}} = \daleth \qquad \forall h \in \mathscr{H},$$

where $\daleth$ stands for the lagrange multiplier, $W_q = \frac{d\,W(Y,L)}{d\,q}$, and $Y_{C_q} = \frac{d\,Q_Y\left(\{C_h\}_{h \in \mathscr{H}}\right)}{d\,C_q}$, and $L_{L_q} = \frac{d\,F\left(\{L_h\}_{h \in \mathscr{H}}\right)}{d\,L_q}$.

First, the optimal solution for the constrained social planner relates real GDP and aggregate labor supply via

$$\frac{W_L}{W_Y} + \underbrace{\frac{w\, L}{p_Y\, Y}}_{=\Gamma} \frac{Y}{L} = 0. \tag{114}$$

The interpretation for this equation is that the aggregate marginal rate of substitution between real GDP and the aggregate factor supply equals the aggregate marginal rate of transformation times an aggregate wedge Γ, which in equilibrium equals the aggregate labor share.

Second, the optimal allocation for the constrained social planner relates idiosyncratic real consumption and labor supply for household h via

$$\begin{aligned}
\frac{W_L}{w_h}\, L_{L_h} + \frac{W_Y}{p_h^c}\, Y_{C_h} &= 0 \\
W_L\, \widetilde{\Lambda}_h\, \frac{L}{w_h\, L_h} + W_Y\, \chi_h \frac{Y}{p_h^c\, C_h} &= 0 \\
\frac{W_L}{W_Y} + \frac{\Lambda_h}{\widetilde{\Lambda}_h} \frac{Y}{L} &= 0.
\end{aligned} \tag{115}$$

Equations (114) and (115) imply that the representative household requires that

$$\Gamma = \frac{\Lambda_h}{\widetilde{\Lambda}_h} \quad \forall h \in \mathscr{H}. \tag{116}$$

Any deviation from this condition under the decentralized solution implies an inefficient allocation from the perspective of the constrained social planner. Using equation (72), the relationship between the aggregate and the idiosyncratic labor wedges is given by

$$\Gamma = \frac{\chi_h}{\widetilde{\Lambda}_h}\, \Gamma_h \quad \forall h \in \mathscr{H}. \tag{117}$$

Adding up over all households

$$\begin{aligned}
\Gamma \underbrace{\sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h}_{=1} &= \sum_{h \in \mathscr{H}} \chi_h\, \Gamma_h = \sum_{h \in \mathscr{H}} \Lambda_h \\
\Gamma &= \sum_{h \in \mathscr{H}} \Lambda_h.
\end{aligned} \tag{118}$$

Taking the first-order approximation and using equation (72)

$$\widehat{\Gamma} = \sum_{h \in \mathscr{H}} \frac{\Lambda_h}{\Gamma}\, \widehat{\Lambda}_h = \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h\, \widehat{\Lambda}_h = \sum_{h \in \mathscr{H}} \frac{\Lambda_h}{\Gamma}\, \widehat{\Lambda}_h = \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h \left(\widehat{\Gamma}_h + \widehat{\chi}_h \right). \tag{119}$$

As a consequence, from equation (96)

$$\begin{aligned}\Gamma\,\widehat{\Gamma} &= \sum_{h\in\mathscr{H}}\chi_h\left(\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\right)\widehat{\chi}_h+\sum_{h\in\mathscr{H}}\chi_h\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\widehat{\mathscr{C}}_{hb}\\ &=\sum_{h\in\mathscr{H}}\chi_h\left(\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\right)\widehat{\chi}_h+\sum_{h\in\mathscr{H}}o_H\left(h\right)'\,\Psi_\ell'\,diag\left(\widehat{\mu}\right)\,\lambda\\ &\quad+\sum_{h\in\mathscr{H}}\left(\sum_{i\in\mathscr{N}}\Omega_{ih}^{\ell}\,\lambda_i\left(\widehat{\omega}_i^{\ell}+\widehat{\alpha}_{ih}\right)+\sum_{j\in\mathscr{N}}\psi_{jh}^{\ell}\left(\sum_{b\in\mathscr{H}}\beta_{bj}\,\chi_b\widehat{\beta}_{bj}+\sum_{i\in\mathscr{N}}\Omega_{ij}^{x}\,\lambda_i\left(\widehat{\omega}_i^{x}+\widehat{\omega}_{ij}\right)\right)\right).\end{aligned} \tag{120}$$

Using the fact that $\widehat{\Gamma}=\widetilde{\Lambda}'\widehat{\Lambda}$

$$\widehat{Y}=\widetilde{\lambda}'\,\widehat{\mathscr{A}}+\widetilde{\lambda}'\,\widehat{\mu}-\widehat{\Gamma}+\widetilde{\Lambda}'\,\widehat{L}.$$

1.7 Information Theory and Aggregate Efficiency

1.7.1 Shannon's Entropy, Cross Entropy, and Kullback-Leibler Divergence

A discrete random variable $\mathscr{Q}$ with G mutually exclusive events is distributed according to the probability given by the vector $q'=[q_1,\cdots,q_G]'$ such that $\mathbb{1}_G'q=1$. Denominate the information carried by an event g as[14]

$$I\left(g|\mathscr{Q}\right)=-\,log\,q_g.$$

Notice that this function satisfies two properties

1. **Decreasing:** $q_a<q_b$ implies $I\left(a|\mathscr{Q}\right)>I\left(b|\mathscr{Q}\right)$. Less probable events convey more information.
2. **Additive:** $I\left(ab|\mathscr{Q}\right)=I\left(a|Q\right)+I\left(b|Q\right)$. Combined information is the sum of separate information.

Shannon entropy (Shannon, 1948) captures the average amount of information conveyed by a random draw, or similarly the expected surprise from observing an event, and is given by

$$H\left(q\right)=\sum_{g=1}^{G}q_g\,I\left(g|\mathscr{Q}\right)=-\sum_{g=1}^{G}q_g\,log\,q_g.$$

Maximum entropy is equivalent to maximal surprise. For the case of a distribution for which we have no previous knowledge that imposes constraints, maximal surprise takes place with the uniform distribution.

[14] This definition implicitly uses the natural logarithm, and for this reason, information is measured in "natural units" nats.

When instead of using the true probability distribution q, an estimated probability distribution $\widetilde{q}$ is used, the expected measured surprise is given by the cross entropy

$$CE\left(q,\widetilde{q}\right)=-\sum_{g=1}^{G}q_{g}\;log\,\widetilde{q}_{g}.$$

The excess surprise from using the distribution $\widetilde{q}$ instead of the true distribution q is given by the Kullback-Leibler (KL) divergence or relative entropy $\mathscr{K}\left(q|\widetilde{q}\right)$. The KL divergence is related to the Shannon and Cross entropy via the following equation

$$CE\left(q,\widetilde{q}\right)=H\left(q\right)+\mathscr{K}\left(q|\widetilde{q}\right).$$

This implies that KL divergence is given by

$$\mathscr{K}\left(q|\widetilde{q}\right)=-\sum_{f=1}^{F}q_{f}\;log\,\left(\frac{\widetilde{q}_{f}}{q_{f}}\right).$$

From Gibbs's inequality $\mathscr{K}\left(q|\widetilde{q}\right)\geq 0$, which captures the idea that using an incorrect probability distribution $\widetilde{q}$ will introduce a positive bias in the measured average expected information that is conveyed by a random draw. $\mathscr{K}\left(q|\widetilde{q}\right)$ is a measure of the statistical distance between the two distributions q and $\widetilde{q}$. However, unfortunately, this is not a metric, as it does not satisfy the properties of symmetry and triangle inequality.

Now, the first-order variation of the KL divergence in response to changes to the estimated probability distribution $\widetilde{q}$ is given by

$$d\,\mathscr{K}\left(q|\widetilde{q}\right)=-\sum_{g=1}^{G}q_{g}\;d\,log\,\widetilde{q}_{g}.$$

When $q=\widetilde{q}$, the property $\sum_{g=1}^{G}q_{g}=1$ implies that $d\,\mathscr{K}\left(q|\widetilde{q}\right)=0$.[15] The latter results reflects that to a first-order, the average expected excess information from changing the measured distribution $\widetilde{q}$ around the true distribution q does not add any excess surprise. In other words, the information conveyed by the measured distribution $\widetilde{q}$ satisfies an envelope condition around q.[16]

1.7.2 Implementation in the model

Aggregate KL divergence

Take the distribution of revenue given by the vector $\mathscr{R}'=\begin{bmatrix}\lambda'\Omega_{\ell} & \lambda'\Omega_{\pi}\end{bmatrix}$, and the value-added distri-

[15]This requires that when $\widetilde{q}$ changes from $\widetilde{q}_0$ to $\widetilde{q}_1$ $\mathbb{1}'_J\widetilde{q}_0=\mathbb{1}'_J\widetilde{q}_1=1$.

[16]Notice that the first-order approximation that was used to derived equation (111) comes from

$$d\,\mathscr{K}\left(\mathscr{R}|\widetilde{\mathscr{R}}\right)|_{\mathscr{R}=\widetilde{\mathscr{R}}}=0.$$

bution given by $\widetilde{\Lambda}$. Their respective Shannon entropies are given by

$$H\left(\mathscr{R}\right)=-\sum_{h\in\mathscr{H}}\left(\Lambda_h\,log\,\Lambda_h+\left(\sum_{i\in\mathscr{N}}\kappa_{ih}\left(1-\mu_i\right)\lambda_i\right)\,log\,\left(\sum_{i\in\mathscr{N}}\kappa_{ih}\left(1-\mu_i\right)\lambda_i\right)\right),$$

$$H\left(\widetilde{\Lambda}\right)=-\sum_{h=1}^{H}\widetilde{\Lambda}_h\,log\,\widetilde{\Lambda}_h.$$

The former measures the expected information about the income distribution from a random draw of one "unit" of household revenue. The latter measures the expected information about the value-added distribution by a random draw of one "unit" of value added.

If the revenue distribution $\mathscr{R}$ is used instead of $\widetilde{\Lambda}$ to infer the share of value added by each worker, the cross entropy is given by

$$CE\left(\widetilde{\Lambda},\mathscr{R}\right)=-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,log\,\Lambda_h.$$

The relative entropy gives the excess surprise carried by using the revenue distribution

$$\mathscr{K}\left(\widetilde{\Lambda}|\mathscr{R}\right)=-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,log\,\left(\Lambda_h/\widetilde{\Lambda}_h\right),$$

and the first-order effect from variation in the revenue distribution $\mathscr{R}$ on this measure of excess surprise is given by

$$d\,\mathscr{K}\left(\widetilde{\Lambda}|\mathscr{R}\right)=-\sum_{h\in\mathscr{H}}\widetilde{\Lambda}_h\,d\,log\,\Lambda_h=-Misallocation.$$

This implies that the aggregate misallocation and $\mathscr{K}\left(\widetilde{\Lambda}|\mathscr{R}\right)$ are negatively correlated. Consequently, an increase in the statistical distance between the distributions $\mathscr{R}$ and $\widetilde{\Lambda}$ captures a reduction in labor misallocation.

Households' KL divergence

The value-added distribution for households of type h is given by the vector $\widetilde{\mathscr{C}}'_{\downarrow h}=\begin{pmatrix}\widetilde{\mathscr{C}}_{h1} & \cdots & \widetilde{\mathscr{C}}_{hH}\end{pmatrix}$, with $\widetilde{\mathscr{C}}'_{\downarrow h}\,\mathbb{1}_H=1$.

If instead of using $\widetilde{\mathscr{C}}_{\downarrow h}$ to infer the value added by each worker to the consumption household h, the revenue distribution $\mathscr{R}$ was used, the cross entropy would be given by

$$CE\left(\widetilde{\mathscr{C}}_{\downarrow h},\mathscr{R}\right)=-\sum_{b\in\mathscr{H}}\widetilde{\mathscr{C}}_{hb}\,log\,\Lambda_b,$$

the excess surprise carried by using this distribution is given by the relative entropy

$$\mathscr{K}\left(\widetilde{\mathscr{C}}_{\downarrow h}|\mathscr{R}\right)=-\sum_{b\in\mathscr{H}}\widetilde{\mathscr{C}}_{hb}\,log\,\left(\frac{\Lambda_b}{\widetilde{\mathscr{C}}_{hb}}\right),$$

and the first-order effect from variations in $\mathscr{R}$ is given by

$$d\mathscr{K}\left(\widetilde{\mathscr{C}}_{\downarrow h}|\mathscr{R}\right) = -\sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb}\, d\log\Lambda_b,$$

which is equivalent $-Entropy\ TT_h$ in Theorem 3. This implies that $Entropy\ TT_h$ and $\mathscr{K}\left(\widetilde{\mathscr{C}}_{\downarrow h}|\mathscr{R}\right)$ are negatively correlated. Consequently, an increase in the statistical distance between $\mathscr{R}$ and $\widetilde{\mathscr{C}}_{\downarrow h}$ captures a favourable distributional variation for households of type h.

Finally, notice that

$$d\mathscr{K}\left(\widetilde{\Lambda}|\mathscr{R}\right) = \sum_{h\in\mathscr{H}} \chi_h\, d\mathscr{K}\left(\widetilde{\mathscr{C}}_{\downarrow h}|\mathscr{R}\right).$$

1.8 Benchmarks

1.8.1 Productivity shock in sector k around the efficient equilibrium

From equation (104)

$$\frac{\partial\log PTT_h}{\partial\log\mathscr{A}_k} = \mathscr{B}_{hk} + \frac{\partial\log\chi_h}{\partial\log\mathscr{A}_k} - \sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\frac{\partial\log\Lambda_b}{\partial\log\mathscr{A}_k}, \qquad \frac{\partial\log C_h}{\partial\log\mathscr{A}_k} = \frac{\partial\log PTT_h}{\partial\log\mathscr{A}_k} + \sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\frac{\partial\log L_b}{\partial\log\mathscr{A}_k}.$$

From equation (112)

$$\frac{\partial\log TFP}{\partial\log\mathscr{A}_k} = \lambda_k, \qquad \frac{\partial\log Y}{\partial\log\mathscr{A}_k} = \frac{\partial\log TFP}{\partial\log\mathscr{A}_k} + \sum_{h\in\mathscr{H}}\Lambda_h\frac{\partial\log L_h}{\partial\log\mathscr{A}_k}.$$

Notice that $\frac{\partial\log TFP}{\partial\log\mathscr{A}_k} = \lambda_k$ is Hulten's (1978) theorem, i.e., in an efficient economy, sectoral productivity shocks have first-order effects on TFP equal to the Domar weights.

From equation (96)

$$\frac{\partial\Gamma_h}{\partial\log\mathscr{A}_k} = \sum_{b\in\mathscr{H}}\chi_b\frac{\mathscr{C}_{bh}}{\chi_h}\left(\frac{\partial\log\chi_b}{\partial\log\mathscr{A}_k} + \frac{\partial\log\mathscr{C}_{bh}}{\partial\log\mathscr{A}_k}\right) - \frac{\partial\log\chi_h}{\partial\log\mathscr{A}_k} = 0. \tag{121}$$

Because under efficiency $\sum_{b\in\mathscr{H}}\chi_b\mathscr{C}_{bh} = \Lambda_h$, and $\sum_{b\in\mathscr{H}}\chi_b\mathscr{C}_{bh}\left(\widehat{\chi}_b + \widehat{\mathscr{C}}_{bh}\right) = \widehat{\Lambda}_h$, and from equation (103) we have that $\frac{\partial\log\chi_h}{\partial\log\mathscr{A}_k} = \frac{\partial\log\Lambda_h}{\partial\log\mathscr{A}_k}$.

Finally, from equation (120)

$$\frac{\partial\Gamma}{\partial\log\mathscr{A}_k} = \underbrace{\sum_{h\in\mathscr{H}}\chi_h\overbrace{\left(\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\right)}^{=1}\frac{\partial\log\chi_h}{\partial\log\mathscr{A}_k}}_{=0} + \sum_{h\in\mathscr{H}}\chi_h\underbrace{\sum_{b\in\mathscr{H}}\mathscr{C}_{hb}\frac{\partial\log\mathscr{C}_{hb}}{\partial\log\mathscr{A}_k}}_{=0} = 0. \tag{122}$$

Equations (121) and (122) proof that in an efficient economy, sectoral productivity shocks have zero first-order effects on the aggregate and idiosyncratic factoral wedges.

1.8.2 Markdown shock in sector k around the efficient equilibrium

From equation (104)

$$\frac{\partial \log PTT_h}{\partial \log \mu_k} = \mathscr{B}_{hk} + \frac{\partial \log \chi_h}{\partial \log \mu_k} - \sum_{b\in\mathscr{H}} \mathscr{C}_{hb} \frac{\partial \log \Lambda_b}{\partial \log \mu_k}, \qquad \frac{\partial \log C_h}{\partial \log \mu_k} = \frac{\partial \log TFP_h}{\partial \log \mu_k} + \sum_{b\in\mathscr{H}} \mathscr{C}_{hb} \frac{\partial \log L_b}{\partial \log \mu_k}. \tag{123}$$

From equation (112)

$$\frac{\partial \log TFP}{\partial \log \mu_k} = 0, \qquad \frac{\partial \log Y}{\partial \log \mu_k} = \sum_{h\in\mathscr{H}} \Lambda_h \frac{\partial \log L_h}{\partial \log \mu_k}. \tag{124}$$

From equation (96)

$$\frac{\partial \Gamma_h}{\partial \log \mu_k} = \sum_{b\in\mathscr{H}} \chi_b \frac{\mathscr{C}_{bh}}{\chi_h} \left(\frac{\partial \log \chi_b}{\partial \log \mu_k} + \frac{\partial \log \mathscr{C}_{bh}}{\partial \log \mu_k} \right) - \frac{\partial \log \chi_h}{\partial \log \mu_k} = \frac{\partial \log \Lambda_h}{\partial \log \mu_k} - \frac{\partial \log \chi_h}{\partial \log \mu_k} = \kappa_{kh} \frac{\lambda_k}{\chi_h}. \tag{125}$$

Because from equation (103) $\frac{\partial \log \chi_h}{\partial \log \mathscr{A}_k} = \frac{\partial \log \Lambda_h}{\partial \log \mathscr{A}_k} - \frac{\kappa_{kh}}{\chi_h} \lambda_k$.

Finally, from equation (120)

$$\frac{\partial \Gamma}{\partial \log \mu_k} = \sum_{h\in\mathscr{H}} \chi \frac{\partial \Gamma_h}{\partial \log \mu_k} + \underbrace{\sum_{h\in\mathscr{H}} \chi_h \frac{\partial \log \chi_h}{\partial \log \mu_k}}_{=0} = \lambda_k. \tag{126}$$

Notice that equations (124) and (126) are the main result (Theorem 2) from Bigio & La'O (2020). Starting from the efficient equilibrium, firm level distortions have zero first-order aggregate effects on TFP, and nonzero first-order effects on the aggregate labor wedge equal to the Domar weights.

Equations (123) and (125) bring out the distributional story that is absent in the aggregate variables. First, starting from the efficient equilibrium, firm level distortions have non-zero first-order effects on idiosyncratic PTT that are zero sum, and second, nonzero first-order effects on the idiosyncratic factor wedge that depend on the equity distribution and the expenditure share.

1.8.3 Factor supply shock for household k around the efficient equilibrium

From equation (104)

$$\frac{\partial \log PTT_h}{\partial \log L_k} = \frac{\partial \log \chi_h}{\partial \log L_k} - \sum_{b\in\mathscr{H}} \mathscr{C}_{hb} \frac{\partial \log \Lambda_b}{\partial \log L_k} \qquad \frac{\partial \log C_h}{\partial \log L_k} = \frac{\partial \log PTT_h}{\partial \log L_k} + \mathscr{C}_{hk} + \sum_{\substack{b\in\mathscr{H} \\ b\neq k}} \mathscr{C}_{hb} \frac{\partial \log L_b}{\partial \log L_k}.$$

From equation (112)

$$\frac{\partial \log TFP}{\partial \log L_k} = 0, \qquad \frac{\partial \log Y}{\partial \log L_k} = \Lambda_k + \sum_{\substack{h\in\mathscr{H} \\ h\neq k}} \Lambda_h \frac{\partial \log L_h}{\partial \log L_k}.$$

From equation (96)

$$\frac{\partial \Gamma_h}{\partial \log L_k} = \sum_{b\in\mathscr{H}} \chi_b \frac{\mathscr{C}_{bh}}{\chi_h} \left(\frac{\partial \log \chi_b}{\partial \log L_k} + \frac{\partial \log \mathscr{C}_{bh}}{\partial \log L_k} \right) - \frac{\partial \log \chi_h}{\partial \log L_k} = 0.$$

Finally, from equation (120)

$$\frac{\partial \Gamma}{\partial \log L_k} = \underbrace{\sum_{h\in\mathscr{H}} \chi_h \overbrace{\left(\sum_{b\in\mathscr{H}} \mathscr{C}_{hb}\right)}^{=1} \frac{\partial \log \chi_h}{\partial \log L_k}}_{=0} + \sum_{h\in\mathscr{H}} \chi_h \underbrace{\sum_{b\in\mathscr{H}} \mathscr{C}_{hb} \frac{\partial \log \mathscr{C}_{hb}}{\partial \log L_k}}_{=0} = 0.$$

1.8.4 Productivity shock in sector k around an inefficient equilibrium

From equations (100) and (104)

$$\begin{aligned}\frac{\partial \log PTT_h}{\partial \log \mathcal{A}_k} &= \widetilde{\mathscr{B}}_{hk} + \Gamma_h \frac{\partial \log J_h}{\partial \log \mathcal{A}_k} - \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log J_b}{\partial \log \mathcal{A}_k} + \sum_{i\in\mathscr{N}} \kappa_{ih} (1-\mu_i) \frac{\lambda_i}{\chi_h} \frac{\partial \log S_i}{\partial \log \mathcal{A}_k} \\ &= \widetilde{\mathscr{B}}_{hk} + \frac{\partial \log \chi_h}{\partial \log \mathcal{A}_k} - \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log \Lambda_b}{\partial \log \mathcal{A}_k},\end{aligned}$$

$$\frac{\partial \log C_h}{\partial \log \mathcal{A}_k} = \frac{\partial \log PTT_h}{\partial \log \mathcal{A}_k} + \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log L_b}{\partial \log \mathcal{A}_k}.$$

From equations (110) and (111)

$$\frac{\partial \log TFP}{\partial \log \mathcal{A}_k} = \widetilde{\lambda}_k + \sum_{h\in\mathscr{H}} \left(\Lambda_h - \widetilde{\Lambda}_h\right) \frac{\partial \log J_h}{\partial \log \mathcal{A}_k} + \sum_{i\in\mathscr{N}} (1-\mu_i) \lambda_i \frac{\partial \log S_i}{\partial \log \mathcal{A}_k} = \widetilde{\lambda}_i - \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log \Lambda_h}{\partial \log \mathcal{A}_k},$$

$$\frac{\partial \log Y}{\partial \log \mathcal{A}_k} = \frac{\partial \log TFP}{\partial \log \mathcal{A}_k} + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log L_h}{\partial \log \mathcal{A}_k}.$$

From equation (96) and $\mathscr{C}'\chi = \Lambda$

$$\frac{\partial \log \Gamma_h}{\partial \log \mathcal{A}_k} = \frac{\partial \log \Lambda_h}{\partial \log \mathcal{A}_k} - \frac{\partial \log \chi_h}{\partial \log \mathcal{A}_k}.$$

Finally, from equation (120)

$$\frac{\partial \log \Gamma}{\partial \log \mathcal{A}_k} = \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log \Gamma_h}{\partial \log \mathcal{A}_k} = \frac{1}{\Gamma} \left(\sum_{h\in\mathscr{H}} \mathscr{C}_h \frac{\partial \chi_h}{\partial \log \mathcal{A}_k} + \sum_{h\in\mathscr{H}} \chi_h \sum_{b\in\mathscr{H}} \frac{\partial \mathscr{C}_{hb}}{\partial \log \mathcal{A}_k} \right).$$

1.8.5 Markdown shock in sector k around an inefficient equilibrium

From equations (100) and (104)

$$\begin{aligned}\frac{\partial \log PTT_h}{\partial \log \mu_k} &= \widetilde{\mathscr{B}}_{hk} - \kappa_{kh}\mu_k \frac{\lambda_k}{\chi_h} + \Gamma_h \frac{\partial \log J_h}{\partial \log \mu_k} - \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log J_b}{\partial \log \mu_k} + \sum_{i\in\mathscr{N}} \kappa_{ih} (1-\mu_i) \frac{\lambda_i}{\chi_h} \frac{\partial \log S_i}{\partial \log \mu_k} \\ &= \widetilde{\mathscr{B}}_{hk} + \frac{\partial \log \chi_h}{\partial \log \mu_k} - \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log \Lambda_b}{\partial \log \mu_k} + \sum_{i\in\mathscr{N}} \kappa_{ih} (1-\mu_i) \frac{\lambda_i}{\chi_h} \frac{\partial \log S_i}{\partial \log \mu_k},\end{aligned}$$

$$\frac{\partial \log C_h}{\partial \log \mu_k} = \frac{\partial \log PTT_h}{\partial \log \mu_k} + \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log L_b}{\partial \log \mu_k}.$$

From equations (110) and (111)

$$\frac{\partial \log TFP}{\partial \log \mu_k} = \widetilde{\lambda}_k - \mu_k\lambda_k + \sum_{h\in\mathscr{H}} \left(\Lambda_h - \widetilde{\Lambda}_h\right) \frac{\partial \log J_h}{\partial \log \mu_k} + \sum_{i\in\mathscr{N}} (1-\mu_i) \lambda_i \frac{\partial \log S_i}{\partial \log \mu_k} = \widetilde{\lambda}_k - \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log \Lambda_h}{\partial \log \mu_k},$$

$$\frac{\partial \log Y}{\partial \log \mu_k} = \frac{\partial \log TFP}{\partial \log \mu_k} + \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log L_h}{\partial \log \mu_k}.$$

From equation (96) and $\mathscr{C}'\chi = \Lambda$

$$\frac{\partial \log \Gamma_h}{\partial \log \mu_k} = \frac{\partial \log \Lambda_h}{\partial \log \mu_k} - \frac{\partial \log \chi_h}{\partial \log \mu_k}.$$

Finally, from equation (120)

$$\frac{\partial \log \Gamma}{\partial \log \mu_k} = \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log \Gamma_h}{\partial \log \mu_k} = \frac{1}{\Gamma} \left(\sum_{h \in \mathscr{H}} \mathscr{C}_h \frac{\partial \chi_h}{\partial \log \mu_k} + \sum_{h \in \mathscr{H}} \chi_h \sum_{b \in \mathscr{H}} \frac{\partial \mathscr{C}_{hb}}{\partial \log \mu_k} \right).$$

1.8.6 Factor supply shock for household k around an inefficient equilibrium

From equations (100) and (104)

$$\frac{\partial \log PTT_h}{\partial \log L_k} = \Gamma_h \frac{\partial \log J_h}{\partial \log L_k} - \sum_{b \in \mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log J_b}{\partial \log L_k} + \sum_{i \in \mathscr{N}} \kappa_{ih} (1 - \mu_i) \frac{\lambda_i}{\chi_h} \frac{\partial \log S_i}{\partial \log L_k} = \frac{\partial \log \chi_h}{\partial \log L_k} - \sum_{b \in \mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log \Lambda_b}{\partial \log L_k},$$

$$\frac{\partial \log C_h}{\partial \log L_k} = \frac{\partial \log PTT_h}{\partial \log L_k} + \widetilde{\mathscr{C}}_{hk} + \sum_{\substack{b \in \mathscr{H} \\ b \neq k}} \widetilde{\mathscr{C}}_{hb} \frac{\partial \log L_b}{\partial \log L_k}.$$

From equations (110) and (111)

$$\frac{\partial \log TFP}{\partial \log L_k} = \sum_{h \in \mathscr{H}} \left(\Lambda_h - \widetilde{\Lambda}_h \right) \frac{\partial \log J_h}{\partial \log L_k} + \sum_{i \in \mathscr{N}} (1 - \mu_i) \lambda_i \frac{\partial \log S_i}{\partial \log L_k} = - \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log \Lambda_h}{\partial \log L_k},$$

$$\frac{\partial \log Y}{\partial \log L_k} = \frac{\partial \log TFP}{\partial \log L_k} + \widetilde{\Lambda}_k + \sum_{\substack{h \in \mathscr{H} \\ h \neq k}} \widetilde{\Lambda}_h \frac{\partial \log L_h}{\partial \log L_k}.$$

From equation (96) and $\mathscr{C}'\chi = \Lambda$

$$\frac{\partial \log \Gamma_h}{\partial \log L_k} = \frac{\partial \log \Lambda_h}{\partial \log L_k} - \frac{\partial \log \chi_h}{\partial \log L_k}.$$

Finally, from equation (120)

$$\frac{\partial \log \Gamma}{\partial \log L_k} = \sum_{h \in \mathscr{H}} \widetilde{\Lambda}_h \frac{\partial \log \Gamma_h}{\partial \log L_k} = \frac{1}{\Gamma} \left(\sum_{h \in \mathscr{H}} \mathscr{C}_h \frac{\partial \chi_h}{\partial \log L_k} + \sum_{h \in \mathscr{H}} \chi_h \sum_{b \in \mathscr{H}} \frac{\partial \mathscr{C}_{hb}}{\partial \log L_k} \right).$$

2 Proofs for the normalized nested-CES model

2.1 Firms

The competitive aggregator firm from sector $i \in \mathscr{N}$ operates under the same environment as in the section 1 of this Online Appendix.

The monopolistically competitive firm z_i chooses $\left\{y_{z_i}, \{\ell_{z_ih}\}_{h\in\mathscr{H}}, \{x_{z_ij}\}_{j\in\mathscr{N}}\right\}$ to maximize

$$\pi_{z_i} = p_{z_i}y_{z_i} - \underbrace{\sum_{h\in\mathscr{H}} w_h\ell_{z_ih}}_{=p^\ell_{z_i}L_{z_i}} - \underbrace{\sum_{j\in\mathscr{N}} p_j x_{z_ij}}_{=p^x_{z_i}X_{z_i}},$$

subject to

$$\frac{y_{z_i}}{\overline{y}_{z_i}} = A_i\left(\omega_i^\ell\left(\frac{L_{z_i}}{\overline{L}_{z_i}}\right)^{\frac{\theta_i-1}{\theta_i}} + \omega_i^x\left(\frac{X_{z_i}}{\overline{X}_{z_i}}\right)^{\frac{\theta_i-1}{\theta_i}}\right)^{\frac{\theta_i}{\theta_i-1}},$$

$$\frac{L_{z_i}}{\overline{L}_{z_i}} = \left(\sum_{h\in\mathscr{H}} \alpha_{ih}\left(\frac{\ell_{z_ih}}{\overline{\ell}_{z_ih}}\right)^{\frac{\theta_i^\ell-1}{\theta_i^\ell}}\right)^{\frac{\theta_i^\ell}{\theta_i^\ell-1}},$$

$$\frac{X_{z_i}}{\overline{X}_{z_i}} = \left(\sum_{j\in\mathscr{N}} \omega_{ij}\left(\frac{x_{z_ij}}{\overline{x}_{z_ij}}\right)^{\frac{\theta_i^x-1}{\theta_i^x}}\right)^{\frac{\theta_i^x}{\theta_i^x-1}}.$$

From here, the first order conditions are given by

$$p^\ell_{z_i}L_{z_i} = \left(\mu_i\omega_i^\ell\right)^{\theta_i}\left(A_i\frac{p_{z_i}\overline{y}_{z_i}}{p^\ell_{z_i}\overline{L}_{z_i}}\right)^{\theta_i-1} p_{z_i}y_{z_i}, \tag{127}$$

$$p^x_{z_i}X_{z_i} = \left(\mu_i\omega_i^x\right)^{\theta_i}\left(A_i\frac{p_{z_i}\overline{y}_{z_i}}{p^x_{z_i}\overline{X}_{z_i}}\right)^{\theta_i-1} p_{z_i}y_{z_i}, \tag{128}$$

$$w_h\ell_{z_ih} = \left(\mu_i\omega_i^\ell\right)^{\theta_i}\alpha_{ih}^{\theta_i^\ell}\left(A_i\frac{p_{z_i}\overline{y}_{z_i}}{p^\ell_{z_i}\overline{L}_{z_i}}\right)^{\theta_i-1}\left(\frac{p^\ell_{z_i}\overline{L}_{z_i}}{w_h\overline{\ell}_{z_ih}}\right)^{\theta_i^\ell-1} p_{z_i}y_{z_i}, \tag{129}$$

$$p_jx_{z_ij} = \left(\mu_i\omega_i^x\right)^{\theta_i}\omega_{ij}^{\theta_i^x}\left(A_i\frac{p_{z_i}\overline{y}_{z_i}}{p^x_{z_i}\overline{X}_{z_i}}\right)^{\theta_i-1}\left(\frac{p^x_{z_i}\overline{X}_{z_i}}{p_j\overline{x}_{z_ij}}\right)^{\theta_i^x-1} p_{z_i}y_{z_i}. \tag{130}$$

In the point of normalization $A_i = 1\ \forall i \in \mathscr{N}$

$$\overline{p}^\ell_{z_i}\overline{L}_{z_i} = \mu_i\omega_i^\ell\overline{p}_{z_i}\overline{y}_{z_i}, \qquad \overline{p}^x_{z_i}\overline{X}_{z_i} = \mu_i\omega_i^x\overline{p}_{z_i}\overline{y}_{z_i},$$

$$\overline{w}_h\overline{\ell}_{z_ih} = \alpha_{ih}\overline{p}^\ell_{z_i}\overline{L}_{z_i}, \qquad \overline{p}_j\overline{x}_{z_ij} = \omega_{ij}\overline{p}^x_{z_i}\overline{X}_{z_i}.$$

Finally, prices are given by

$$p^\ell_{z_i} = \frac{1}{\overline{L}_{z_i}}\left(\sum_{h\in\mathscr{H}} \alpha_{ih}^{\theta_i^\ell}\left(w_h\overline{\ell}_{z_ih}\right)^{1-\theta_i^\ell}\right)^{\frac{1}{1-\theta_i^\ell}}, \tag{131}$$

$$p^x_{z_i} = \frac{1}{\overline{X}_{z_i}} \left(\sum_{j \in \mathcal{N}} \omega_{ij}^{\theta^x_i} \left(p_j \overline{x}_{z_i j} \right)^{1-\theta^x_i} \right)^{\frac{1}{1-\theta^x_i}}, \tag{132}$$

$$p_{z_i} = \frac{1}{A_i \mu_i \overline{y}_{z_i}} \left(\omega_i^{\ell\,\theta_i} \left(p^\ell_{z_i} \overline{L}_{z_i} \right)^{1-\theta_i} + \omega_i^{x\,\theta_i} \left(p^x_{z_i} \overline{X}_{z_i} \right)^{1-\theta_i} \right)^{\frac{1}{1-\theta_i}}. \tag{133}$$

2.2 Households

Household h chooses $\left\{ \{C_{hi}\}_{i \in \mathcal{N}}, L_h \right\}$ to maximize

$$U_h \left(c_h, \widetilde{L}_h \right) = \frac{\left[c_h \left(1 - E_h^{-\gamma_h} \widetilde{L}_h \right)^{\varphi_h} \right]^{1-\sigma} - 1}{1-\sigma},$$

subject to $C_h = n_h c_h$, $L_h = n_h \widetilde{L}_h$,

$$\frac{C_h}{\overline{C}_h} = \left(\sum_{i \in \mathcal{N}} \beta_{hi} \left(\frac{C_{hi}}{\overline{C}_{hi}} \right)^{\frac{\varrho_h - 1}{\varrho_h}} \right)^{\frac{\varrho_h}{\varrho_h - 1}},$$

$$E_h = p^c_h C_h = \sum_{i \in \mathcal{N}} p_i C_{hi} \leq w_h L_h + \Pi_h,$$

$$\Pi_h = \sum_{i \in \mathcal{N}} \kappa_{ih} \left(\bar{\pi}_i + \int \pi_{z_i} \, dz_i \right).$$

The first order conditions are given

$$c_h^{-\sigma} \left(1 - E_h^{-\gamma_h} \widetilde{L}_h \right)^{\varphi_h (1-\sigma)} \left(1 + \varphi_h \gamma_h \frac{E_h^{-\gamma_h} \widetilde{L}_h}{1 - E_h^{-\gamma_h} \widetilde{L}_h} \right) \frac{\partial C_h}{\partial C_{hi}} = \varkappa_h n_h p_i, \tag{134}$$

$$c_h^{-\sigma} \left(1 - E_h^{-\gamma_h} \widetilde{L}_h \right)^{\varphi_h (1-\sigma)} \left(1 + \varphi_h \gamma_h \frac{E_h^{-\gamma_h} \widetilde{L}_h}{1 - E_h^{-\gamma_h} \widetilde{L}_h} \right) = \varkappa_h n_h p^c_h, \tag{135}$$

$$\varphi_h c_h^{1-\sigma} \left(1 - E_h^{-\gamma_h} \widetilde{L}_h \right)^{\varphi_h (1-\sigma) - 1} E_h^{-\gamma_h} = \varkappa_h n_h w_h, \tag{136}$$

where $\frac{\partial C_h}{\partial C_{hi}} = \beta_{hi} \left(\frac{\overline{C}_h}{\overline{C}_{hi}} \right)^{\frac{\varrho_h - 1}{\varrho_h}} \left(\frac{C_h}{C_{hi}} \right)^{\frac{1}{\varrho_h}}$, and $\varkappa_h$ stands for the Lagrange multiplier for the budget constraint.

From equations (134) and (135)

$$p_i C_{hI} = \beta_{hi}^{\varrho_h} \left(\frac{p^c_h \overline{C}_h}{p_i \overline{C}_{hi}} \right)^{\varrho_h - 1} E_h, \tag{137}$$

or $\overline{p}_i \overline{c}_{hi} = \beta_{hi} \overline{C}_h$ in the point of normalization.

From equations (135) and (136)

$$w_h L_h = \frac{n_h w_h E_h^{\gamma_h} - \varphi_h E_h}{1 - \varphi_h \gamma_h}. \tag{138}$$

Now, from equation (138), the first order approximation for the factor supply schedule

$$L_h\widehat{L}_h = n_h\frac{\partial L_h}{\partial n_h}\widehat{n}_h + w_h\frac{\partial L_h}{\partial w_h}\widehat{w}_h + E_h\frac{\partial L_h}{\partial E_h}\widehat{E}_h.$$

$$\frac{\partial L_h}{\partial n_h} = \frac{E_h^{\gamma_h}}{1-\varphi_h\gamma_h}, \quad \frac{\partial L_h}{\partial w_h} = \frac{\varphi_h E_h}{(1-\varphi_h\gamma_h)\, w_h^2}, \quad \frac{\partial L_h}{\partial E_h} = -\frac{1}{1-\varphi_h\gamma_h}\left(\frac{\varphi_h}{w_h} - \gamma_h n_h E_h^{\gamma_h-1}\right)$$

$$\widehat{L}_h = \frac{1}{1-\varphi_h\gamma_h}\left(E_h^{\gamma_h}\frac{n_h}{L_h}\widehat{n}_h + \frac{\varphi_h}{\Gamma_h}\widehat{w}_h - \left(\frac{\varphi_h}{\Gamma_h} - \gamma_h E_h^{\gamma_h}\frac{n_h}{L_h}\right)\widehat{E}_h\right) = \zeta_h^n\widehat{n}_h + \zeta_h^w\widehat{w}_h - \zeta_h^e\widehat{E}_h \tag{139}$$

$$\zeta_h^n = \frac{E_h^{\gamma_h}}{1-\varphi_h\gamma_h}\frac{n_h}{L_h}, \qquad \zeta_h^w = \frac{1}{1-\varphi_h\gamma_h}\frac{\varphi_h}{\Gamma_h}, \qquad \zeta_h^e = \zeta_h^w - \gamma_h\zeta_h^n.$$

Under KPR preferences ($\gamma_h = 0$)

$$\widehat{L}_h = \frac{n_h}{L_h}\widehat{n}_h + \frac{\varphi_h}{\Gamma_h}\left(\widehat{w}_h - \widehat{E}_h\right) = \zeta_h^n\widehat{n}_h + \zeta_h^w\widehat{w}_h - \zeta_h^e\widehat{E}_h$$

$$\zeta_h^n = \frac{n_h}{L_h}, \qquad \zeta_h^w = \zeta_h^e = \frac{\varphi_h}{\Gamma_h}.$$

Under GHH preferences ($\zeta_h^n = 1$ and $\zeta_h^e = 0$) γ_h and φ_h are given by the system of equations

$$\gamma_h = \frac{1}{2\varphi_h}\left(1 + \Gamma_h^{-1/2}\sqrt{\Gamma_h - 4\varphi_h^2}\right), \qquad \varphi_h = \frac{1}{\gamma_h}\left(1 - E_h^{\gamma_h}\frac{n_h}{L_h}\right).$$

this implies that

$$\widehat{L}_h = \widehat{n}_h + \zeta_h^n\widehat{w}_h \quad \text{with} \quad \zeta_h^n = \frac{\varphi_h}{n_h}\frac{E_h^{1-\gamma_h}}{w_h}.$$

Finally, prices for the consumption bundle of each household are given by

$$p_h^c = \frac{1}{\overline{C}_h}\left(\sum_{i\in\mathscr{N}}\beta_{hi}^{\varrho_h}\left(p_i\overline{C}_{hi}\right)^{1-\varrho_h}\right)^{\frac{1}{1-\varrho_h}}. \tag{140}$$

2.3 Equilibrium conditions

2.3.1 Goods markets

From equations (130) and (137), the goods produced by sector $i \in \mathscr{N}$ must satisfy under symmetry for firms in the same sector

$$S_i = \sum_{j\in\mathscr{N}} p_i x_{ji} + \sum_{h\in\mathscr{H}} p_i C_{hi},$$

$$S_i = \sum_{j\in\mathscr{N}}\left(\mu_j\omega_j^x\right)^{\theta_j}\omega_{ji}^{\theta_j^x}\left(A_j\frac{p_j\overline{y}_j}{p_j^x\overline{X}_j}\right)^{\theta_j-1}\left(\frac{p_j^x\overline{X}_j}{p_i\overline{x}_{ji}}\right)^{\theta_j^x-1} S_j + \sum_{h\in\mathscr{H}}\beta_{hi}^{\varrho_h}\left(\frac{p_h^c\overline{C}_h}{p_i\overline{C}_{hi}}\right)^{\varrho_h-1} E_h. \tag{141}$$

In the steady state this relationship is simplified into

$$S_i = \sum_{j\in\mathscr{N}} \Omega^x_{ji} S_j + \sum_{h\in\mathscr{H}} \beta_{hi} E_h$$

which in matrix is represented by equation (60).

The first order approximation for equation (141) is given by

$$\begin{aligned}\lambda_i \widehat{S}_i = &\sum_{h\in\mathscr{H}} \beta_{hi}\chi_h \left((\varrho_h - 1)\left(\widehat{p}^c_h - \widehat{p}_i\right) + \widehat{E}_h \right) \\ &+ \sum_{j\in\mathscr{N}} \Omega^x_{ji}\lambda_j \left[\theta_j \widehat{\mu}_j + (\theta_i - 1)\left(\widehat{A}_j + \widehat{p}_j\right) + \left(\theta^x_j - \theta_j\right)\widehat{p}^x_j - \left(\theta^x_j - 1\right)\widehat{p}_i + \widehat{S}_j \right].\end{aligned}$$

In matrix form this equation is given by

$$\begin{aligned}diag\,(\lambda)\,\widehat{S} = &\beta' diag\,(\chi)\left(diag\,(\varrho - \mathbb{1}_H)\,\widehat{p}_c + \widehat{E}\right) - diag\,(\beta' diag\,(\varrho - \mathbb{1}_H)\,\chi)\,\widehat{p} - diag\,(\Omega'_x diag\,(\theta_x - \mathbb{1}_N)\,\lambda)\,\widehat{p} \\ &+ \Omega'_x diag\,(\lambda)\left(diag\,(\theta)\,\widehat{\mu} + diag\,(\theta - \mathbb{1}_N)\left(\widehat{A} + \widehat{p}\right) + diag\,(\theta_x - \theta)\,\widehat{p}_x + \widehat{S}\right)\end{aligned}$$

$$\begin{aligned}diag\,(\lambda)\,\widehat{S} = \Psi'_x \Big\{ &\beta' diag\,(\chi)\left(diag\,(\varrho - \mathbb{1}_H)\,\widehat{p}_c + \widehat{E}\right) - diag\,(\beta' diag\,(\varrho - \mathbb{1}_H)\,\chi)\,\widehat{p} - diag\,(\Omega'_x diag\,(\theta_x - \mathbb{1}_N)\,\lambda)\,\widehat{p} \\ &+ \Omega'_x diag\,(\lambda)\left(diag\,(\theta)\,\widehat{\mu} + diag\,(\theta - \mathbb{1}_N)\left(\widehat{A} + \widehat{p}\right) + diag\,(\theta_x - \theta)\,\widehat{p}_x \right) \Big\}.\end{aligned} \tag{142}$$

2.3.2 Household budget constraint

From the household h's budget constraint, consumption expenditure must satisfy under symmetry for firms in the same cluster

$$C_h = w_h L_h + \sum_{i\in\mathscr{N}} \kappa_{ih}\,(1 - \mu_i)\,S_i.$$

In the steady state this relationship is represented in matrix form by equation (76).

The first order approximation for this equation is given by

$$\chi_h \widehat{E}_h = \Lambda_h \widehat{J}_h + \sum_{i\in\mathscr{N}} \kappa_{ih}\lambda_i \left((1 - \mu_i)\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i \widehat{\mu}_i \right).$$

In matrix form this equation is given by

$$diag\,(\chi)\,\widehat{E} = diag\,(\Lambda)\,\widehat{J} + \kappa'\, diag\,(\lambda)\left(diag\,(\mathbb{1}_N - \mu)\,\widehat{S} - diag\,(\mu)\,\widehat{\mu}\right) + (\kappa \circ \widehat{\kappa})'\, diag\,(\mathbb{1}_N - \mu)\,\lambda. \tag{143}$$

2.3.3 Factor Markets

From equation (129), equilibrium in the factor market for household h must satisfy

$$w_h L_h = \sum_{i\in\mathscr{N}} w_h \ell_{ih} = \sum_{i\in\mathscr{N}} \left(\mu_i \omega_i^\ell\right)^{\theta_i} \alpha_{ih}^{\theta_i^\ell} \left(A_i \frac{p_i \overline{y}_i}{p_i^\ell \overline{L}_i}\right)^{\theta_i - 1} \left(\frac{p_i^\ell \overline{L}_i}{w_h \overline{\ell}_{ih}}\right)^{\theta_i^\ell - 1} S_i. \tag{144}$$

In steady state this relationship is simplified into

$$J_h = \sum_{i\in\mathscr{N}} \mu_i \omega_i^\ell \alpha_{ih} S_i = \sum_{i\in\mathscr{N}} \Omega_{ih}^\ell S_i$$

which in matrix form is represented by equation (64).

The first order approximation for equation (144) is given by

$$\Lambda_h \widehat{J}_h = \sum_{i\in\mathscr{N}} \Omega_{ih}^\ell \lambda_i \left[\theta_i \widehat{\mu}_i + (\theta_i - 1)\left(\widehat{A}_i + \widehat{p}_i\right) + \left(\theta_i^\ell - \theta_i\right)\widehat{p}_i^\ell - \left(\theta_i^\ell - 1\right)\widehat{w}_h + \widehat{S}_i\right].$$

In matrix form this equation is given by

$$\begin{aligned} &diag\left(\Lambda\right)\widehat{J} = -diag\left(\Omega_\ell' diag\left(\theta_\ell - \mathbb{1}_N\right)\lambda\right)\widehat{w} \\ &+ \Omega_\ell' diag\left(\lambda\right)\left(diag\left(\theta\right)\widehat{\mu} + diag\left(\theta - \mathbb{1}_N\right)\left(\widehat{A} + \widehat{p}\right) + diag\left(\theta_\ell - \theta\right)\widehat{p}_\ell + \widehat{S}\right). \end{aligned} \tag{145}$$

2.3.4 Prices

The first-order approximation for equations (131), (132), (133), and (140), under symmetry for firms in the same sector is given by

$$\widehat{p}_i^\ell = \sum_{h\in\mathscr{H}} \alpha_{ih}\widehat{w}_h,$$

$$\widehat{p}_i^x = \sum_{j\in\mathscr{N}} \omega_{ij}\widehat{p}_j,$$

$$\widehat{p}_i = \omega_i^\ell \widehat{p}_i^\ell + \omega_i^x \widehat{p}_i^x - \widehat{A}_i - \widehat{\mu}_i,$$

$$\widehat{p}_h^c = \sum_{i\in\mathscr{N}} \beta_{hi}\widehat{p}_i.$$

In matrix form this equation is given by

$$\widehat{p}_\ell = \alpha\widehat{w}, \tag{146}$$

$$\widehat{p}_x = \mathscr{W}\widehat{p},$$

$$\widehat{p} = diag\left(\omega_\ell\right)\widehat{p}_\ell + diag\left(\omega_x\right)\widehat{p}_x - \widehat{A} - \widehat{\mu},$$

$$\widehat{p}_c = \beta\,\widehat{p}.$$

Using equation (146), the last three equations can be simplified into

$$\widehat{p} = \widetilde{\Psi}_\ell \widehat{w} - \widetilde{\Psi}_x \left(\widehat{A} + \widehat{\mu}\right), \tag{147}$$

$$\widehat{p}_x = \mathscr{W}\widetilde{\Psi}_x \left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right), \tag{148}$$

$$\widehat{p}_c = \widetilde{\mathscr{B}} \left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right). \tag{149}$$

2.3.5 Sufficient equations

Labor Income

Introducing equations (139), (146), and (147) in equation (145)

$$\begin{aligned}
&diag\left(\Lambda\right)\left(I_H + diag\left(\zeta_w\right)\right)\widehat{w} = -diag\left(\Omega_\ell' diag\left(\theta_\ell - \mathbb{1}_N\right)\lambda\right)\widehat{w} + diag\left(\zeta_e\right) diag\left(\Lambda\right)\widehat{E} - diag\left(\zeta_n\right) diag\left(\Lambda\right)\widehat{n} \\
&+ \Omega_\ell' diag\left(\lambda\right)\left(diag\left(\theta\right)\,\widehat{\mu} + diag\left(\theta - \mathbb{1}_N\right)\left(\widetilde{\Psi}_\ell \widehat{w} + \left(I_N - \widetilde{\Psi}_x\right)\widehat{A} - \widetilde{\Psi}_x \widehat{\mu}\right) + diag\left(\theta_\ell - \theta\right)\alpha\widehat{w} + \widehat{S}\right) \\
&= \Omega_\ell' diag\left(\lambda\right) diag\left(\theta - \mathbb{1}_N\right)\left(I_N - \widetilde{\Psi}_x\right)\widehat{A} + \Omega_\ell' diag\left(\lambda\right)\left(diag\left(\theta\right) - diag\left(\theta - \mathbb{1}_N\right)\widetilde{\Psi}_x\right)\widehat{\mu} - diag\left(\zeta_n\right) diag\left(\Lambda\right)\widehat{n} \\
&+ \Omega_\ell' diag\left(\lambda\right)\widehat{S} + diag\left(\zeta_e\right) diag\left(\Lambda\right)\widehat{E} + diag\left(\Omega_\ell' diag\left(\theta_\ell - \mathbb{1}_N\right)\lambda\right)\left(\widetilde{\mathscr{C}} - I_H\right)\widehat{w} \\
&+ \left(\Omega_\ell' diag\left(\lambda\right) diag\left(\theta - \mathbb{1}_N\right)\widetilde{\Psi}_\ell - diag\left(\Omega_\ell' diag\left(\theta - \mathbb{1}_N\right)\lambda\right)\widetilde{\mathscr{C}}\right)\widehat{w} \\
&+ \left(\Omega_\ell' diag\left(\lambda\right) diag\left(\theta_\ell - \theta\right)\alpha - diag\left(\Omega_\ell' diag\left(\theta_\ell - \theta\right)\lambda\right)\widetilde{\mathscr{C}}\right)\widehat{w}
\end{aligned} \tag{150}$$

This implies that

$$\begin{aligned}
\Lambda_h\left(1 + \zeta_h^w\right)\widehat{w}_h = &\sum_{i\in\mathscr{N}}\left(\Omega_{ih}^\ell \lambda_i\left(\theta_i - 1\right) - \sum_{j\in\mathscr{N}}\Omega_{jh}^\ell \lambda_j\left(\theta_j - 1\right)\widetilde{\Psi}_{ji}^x\right)\widehat{A}_i \\
&+ \sum_{i\in\mathscr{N}}\left(\Omega_{ih}^\ell \lambda_i \theta_i - \sum_{j\in\mathscr{N}}\Omega_{jh}^\ell \lambda_j\left(\theta_j - 1\right)\widetilde{\Psi}_{ji}^x\right)\widehat{\mu}_i \\
&+ \sum_{i\in\mathscr{N}}\Omega_{ih}^\ell \lambda_i \widehat{S}_i + \Lambda_h\left(\zeta_h^e \widehat{E}_h - \zeta_h^n \widehat{n}_h\right) - \left(\sum_{i\in\mathscr{N}}\Omega_{ih}^\ell \lambda_i\left(\theta_i^\ell - 1\right)\right)\widehat{w}_h \\
&+ \sum_{b\in\mathscr{H}}\left(\sum_{i\in\mathscr{N}}\Omega_{ih}^\ell \lambda_i\left(\left(\theta_i^\ell - 1\right)\alpha_{ib} + \left(\theta_i - 1\right)\left(\widetilde{\Psi}_{ib}^\ell - \alpha_{ib}\right)\right)\right)\widehat{w}_b.
\end{aligned}$$

Final Expenditure

Let me start by introducing equations (145), (146), and (147) in equation (143)

$$
\begin{aligned}
diag(\chi)\widehat{E} = &-diag\left(\Omega_\ell' diag(\theta_\ell - \mathbb{1}_N)\lambda\right)\widehat{w} \\
&+ \Omega_\ell' diag(\lambda)\left(diag(\theta)\,\widehat{\mu} + diag(\theta - \mathbb{1})\left(\widehat{A} + \widehat{p}\right) + diag(\theta_\ell - \theta)\,\widehat{p}_\ell + \widehat{S}\right) \\
&+ \kappa'\, diag(\lambda)\left(diag(\mathbb{1}_N - \mu)\,\widehat{S} - diag(\mu)\,\widehat{\mu}\right) + (\kappa \circ \widehat{\kappa})'\, diag(\mathbb{1}_N - \mu)\,\lambda
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\Omega_\ell' + \kappa'\, diag(\mathbb{1}_N - \mu)\right) diag(\lambda)\,\widehat{S} + \Omega_\ell' diag(\lambda)\, diag(\theta - \mathbb{1}_N)\left(I_N - \widetilde{\Psi}_x\right)\widehat{A} \\
&+ \left(\Omega_\ell' diag(\lambda)\left(diag(\theta - \mathbb{1}_N)\,\widetilde{\Psi}_\ell + diag(\theta_\ell - \theta)\,\alpha\right) - diag\left(\Omega_\ell' diag(\theta_\ell - \mathbb{1}_N)\lambda\right)\right)\widehat{w} \\
&+ \left(\Omega_\ell' diag(\lambda)\left(diag(\theta) - diag(\theta - \mathbb{1}_N)\,\widetilde{\Psi}_x\right) - \kappa'\, diag(\mu)\, diag(\lambda)\right)\widehat{\mu} + (\kappa \circ \widehat{\kappa})'\, diag(\mathbb{1}_N - \mu)\,\lambda.
\end{aligned}
$$

After taking equation (150) into account

$$
\begin{aligned}
diag(\chi)\left(I_H + diag(\Gamma)\, diag(\zeta_e)\right)\widehat{E} = &\, diag(\Lambda)\left(\left(I_H + diag(\zeta_w)\right)\widehat{w} + diag(\zeta_n)\,\widehat{n}\right) \\
&+ (\kappa \circ \widehat{\kappa})'\, diag(\mathbb{1}_N - \mu)\,\lambda - \kappa'\, diag(\mu)\, diag(\lambda)\,\widehat{\mu} + \Omega_\pi' diag(\lambda)\,\widehat{S}.
\end{aligned}
\tag{151}
$$

This implies that

$$
\chi_h \widehat{E}_h = \frac{1}{1 + \zeta_h^e \Gamma_h}\left(\Lambda_h\left(\left(1 + \zeta_h^w\right)\widehat{w}_h + \zeta_h^n \widehat{n}_h\right) + \sum_{i \in \mathscr{N}} \kappa_{ih}\left(\left(1 - \mu_i\right)\lambda_i\left(\widehat{\kappa}_{ih} + \widehat{S}_i\right) - \mu_i \lambda_i \widehat{\mu}_i\right)\right).
$$

Sales

Now, introducing equations (147), (148), and (149) in equation (142)

$$
\begin{aligned}
diag(\lambda)\,\widehat{S} = &\,\mathscr{B}' diag(\chi)\,\widehat{E} + \mathscr{B}' diag(\chi)\, diag(\varrho - \mathbb{1}_H)\,\widehat{p}_c + \Psi_x' \Omega_x' diag(\lambda)\, diag(\theta_x - \theta)\,\widehat{p}_x \\
&+ \Psi_x'\left(\Omega_x' diag(\lambda)\, diag(\theta - \mathbb{1}_N) - diag\left(\beta' diag(\varrho - \mathbb{1}_H)\,\chi\right) - diag\left(\Omega_x' diag(\theta_x - \mathbb{1}_N)\,\lambda\right)\right)\widehat{p} \\
&+ \Psi_x'\left(\Omega_x' diag(\lambda)\left(diag(\theta)\,\widehat{\mu} + diag(\theta - \mathbb{1}_N)\,\widehat{A}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathscr{B}' diag(\chi)\,\widehat{E} + \Psi_x'\left(\beta' diag(\chi)\, diag(\varrho - \mathbb{1}_H)\,\beta - diag\left(\beta' diag(\varrho - \mathbb{1}_H)\,\chi\right)\right)\widetilde{\Psi}_x\left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right) \\
&+ \Psi_x'\left(\Omega_x' diag(\lambda)\, diag(\theta_x - \mathbb{1}_N)\,\mathscr{W} - diag\left(\Omega_x' diag(\theta_x - \mathbb{1}_N)\,\lambda\right)\right)\widetilde{\Psi}_x\left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right) \\
&+ \Psi_x' \Omega_x' diag(\lambda)\, diag(\theta - \mathbb{1}_N)\left(I_N - \mathscr{W}\right)\widetilde{\Psi}_x\left(\widetilde{\Omega}_\ell \widehat{w} - \widehat{A} - \widehat{\mu}\right) + \Psi_x'\left(\Omega_x' diag(\lambda)\left(diag(\theta)\,\widehat{\mu} + diag(\theta - \mathbb{1}_N)\,\widehat{A}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathscr{B}' diag(\chi)\,\widehat{E} + \Psi_x' \Omega_x' diag(\lambda)\left(diag(\theta - \mathbb{1}_N)\,\widehat{A} + diag(\theta)\,\widehat{\mu}\right) \\
&- \Psi_x'\left(\beta' diag(\chi)\, diag(\varrho - \mathbb{1}_H)\,\widetilde{\mathscr{B}} - diag\left(\beta' diag(\varrho - \mathbb{1}_H)\,\chi\right)\widetilde{\Psi}_x\right)\left(\widehat{A} + \widehat{\mu}\right) \\
&- \Psi_x'\left(\Omega_x' diag(\lambda)\, diag(\theta - \mathbb{1}_N) - diag\left(\Omega_x' diag(\theta - \mathbb{1}_N)\,\lambda\right)\right)\widetilde{\Psi}_x\left(\widehat{A} + \widehat{\mu}\right) \\
&- \Psi_x'\left(\Omega_x' diag(\lambda)\, diag(\theta_x - \theta)\,\mathscr{W} - diag\left(\Omega_x' diag(\theta_x - \theta)\,\lambda\right)\right)\widetilde{\Psi}_x\left(\widehat{A} + \widehat{\mu}\right) \\
&+ \Psi_x'\left(\beta' diag(\chi)\, diag(\varrho - \mathbb{1}_H)\,\widetilde{\mathscr{C}} - diag\left(\beta' diag(\varrho - \mathbb{1}_H)\,\chi\right)\widetilde{\Psi}_\ell\right)\widehat{w} \\
&+ \Psi_x'\left(\Omega_x' diag(\lambda)\, diag(\theta - \mathbb{1}_N) - diag\left(\Omega_x' diag(\theta - \mathbb{1}_N)\,\lambda\right)\right)\widetilde{\Psi}_\ell \widehat{w} \\
&+ \Psi_x'\left(\Omega_x' diag(\lambda)\, diag(\theta_x - \theta)\,\mathscr{W} - diag\left(\Omega_x' diag(\theta_x - \theta)\,\lambda\right)\right)\widetilde{\Psi}_\ell \widehat{w}.
\end{aligned}
\tag{152}
$$

This implies that

$$
\begin{aligned}
\lambda_i \widehat{S}_i = & \sum_{h\in\mathscr{H}} \mathscr{B}_{hi}\chi_h \widehat{E}_h + \sum_{j\in\mathscr{N}} \Psi^x_{ji} \sum_{m\in\mathscr{N}} \Omega^x_{mj}\lambda_m \left((\theta_m - 1)\,\widehat{A}_m + \theta_m \widehat{\mu}_m \right) \\
& - \sum_{j\in\mathscr{N}} \left(\sum_{m\in\mathscr{N}} \Psi^x_{mi} \sum_{h\in\mathscr{H}} \beta_{hm}\chi_h \left(\varrho_h - 1\right) \left(\widetilde{\mathscr{B}}_{hj} - \widetilde{\Psi}^x_{mj} \right) \right) \left(\widehat{A}_j + \widehat{\mu}_j \right) \\
& - \sum_{j\in\mathscr{N}} \left(\sum_{m\in\mathscr{N}} \Psi^x_{mi} \sum_{n\in\mathscr{N}} \Omega^x_{nm}\lambda_n \left(\theta_n - 1\right) \left(\widetilde{\Psi}^x_{nj} - \widetilde{\Psi}^x_{mj} \right) \right) \left(\widehat{A}_j + \widehat{\mu}_j \right) \\
& - \sum_{j\in\mathscr{N}} \left(\sum_{m\in\mathscr{N}} \Psi^x_{mi} \sum_{n\in\mathscr{N}} \Omega^x_{nm}\lambda_n \left(\theta^x_n - \theta_n\right) \left(\sum_{q\in\mathscr{N}} \omega_{nq} \widetilde{\Psi}^x_{qj} - \widetilde{\Psi}^x_{mj} \right) \right) \left(\widehat{A}_j + \widehat{\mu}_j \right) \\
& + \sum_{h\in\mathscr{H}} \left(\sum_{j\in\mathscr{N}} \Psi^x_{ji} \sum_{b\in\mathscr{H}} \beta_{bj}\chi_b \left(\varrho_b - 1\right) \left(\widetilde{\mathscr{C}}_{bh} - \widetilde{\Psi}^\ell_{jh} \right) \right) \widehat{w}_h \\
& + \sum_{h\in\mathscr{H}} \left(\sum_{j\in\mathscr{N}} \Psi^x_{ji} \sum_{m\in\mathscr{N}} \Omega^x_{mj}\lambda_m \left(\theta_m - 1\right) \left(\widetilde{\Psi}^\ell_{mh} - \widetilde{\Psi}^\ell_{jh} \right) \right) \widehat{w}_h \\
& + \sum_{h\in\mathscr{H}} \left(\sum_{j\in\mathscr{N}} \Psi^x_{ji} \sum_{m\in\mathscr{N}} \Omega^x_{mj}\lambda_m \left(\theta^x_m - \theta_m\right) \left(\sum_{q\in\mathscr{N}} \omega_{mq} \widetilde{\Psi}^\ell_{qh} - \widetilde{\Psi}^\ell_{jh} \right) \right) \widehat{w}_h .
\end{aligned}
$$

Summary of Sufficient Equations

Equations (150), (151), and (152) represent a system of $2H+N$ equations on $2H+N$ unknowns that captures the elasticities of factor rates, consumption expenditure and sales in response to exogenous productivity, markdown, labor supply, preferences technology, and equity allocation shocks. This solution can be used to capture the variation of prices from equations (146), (148), (147), and (149). From here using equations (145) it is possible to obtain the variations of factor income.

2.3.6 Proof of Theorem 6

Cobb Douglas and Productivity Shocks - Part 1 of Theorem 6

In response to a general productivity shock captured by the vector $\widehat{A}$, under the assumption that $\theta_i = \theta^\ell_i = \theta^x_i = 1\ \forall i \in \mathscr{N}$, and $\varrho_h = 1\ \forall h \in \mathscr{H}$, equations (150), (151), and (152) are given by

$$
\begin{aligned}
diag\left(\Lambda\right)\left(I_H + diag\left(\zeta_w\right)\right)\widehat{w} &= \Omega'_\ell diag\left(\lambda\right)\widehat{S} + diag\left(\zeta_e\right) diag\left(\Lambda\right)\widehat{E}, \\
diag\left(\chi\right)\left(I_H + diag\left(\Gamma\right) diag\left(\zeta_e\right)\right)\widehat{E} &= diag\left(\Lambda\right)\left(I_H + diag\left(\zeta_w\right)\right)\widehat{w} + \Omega'_\pi diag\left(\lambda\right)\widehat{S}, \\
diag\left(\lambda\right)\widehat{S} &= \mathscr{B}' diag\left(\chi\right)\widehat{C}.
\end{aligned}
$$

Add and subtract $\widehat{GDP}$ and use Y as the numeraire to obtain

$$
diag\left(\Lambda\right)\widehat{\Lambda} = \Omega'_\ell diag\left(\lambda\right)\widehat{\lambda} + \underbrace{\left(\Omega'_\ell \lambda - \Lambda\right)}_{=0}\widehat{Y},
$$

$$diag\left(\chi\right)\widehat{\chi}=diag\left(\Lambda\right)\widehat{\Lambda}+\Omega_{\pi}^{\prime}diag\left(\lambda\right)\widehat{\lambda}+\underbrace{\left(\Omega_{\ell}^{\prime}\lambda+\Omega_{\pi}^{\prime}\lambda-\chi\right)}_{=0}\widehat{Y},$$

$$diag\left(\lambda\right)\widehat{\lambda}=\mathscr{B}^{\prime}diag\left(\chi\right)\widehat{\chi}+\underbrace{\left(\mathscr{B}^{\prime}\chi-\lambda\right)}_{=0}\widehat{Y}.$$

From here it is clear that the solution to this system of equations is $\widehat{\Lambda}=\widehat{\chi}=0_H$ and $\widehat{\lambda}=0_N$. From equations (104), (111), (96), and (120)

$$d\log PTT_h=\sum_{i\in\mathscr{N}}\widetilde{\mathscr{B}}_{hi}\,d\log A_i\qquad\text{and}\qquad d\log\Gamma_h=0\qquad\forall h\in\mathscr{H},$$

$$d\log TFP=\sum_{i\in\mathscr{N}}\widetilde{\lambda}_i\,d\log A_i\qquad\text{and}\qquad d\log\Gamma=0.$$

Leontief and Markdown Shocks - Part 2 of Theorem 6

In response to a general productivity shock captured by the vector $\widehat{\mu}$, under the assumptions that

1. $\theta_i=\theta_i^{\ell}=\theta_i^{x}=0\ \forall i\in\mathscr{N}$,
2. $\varrho_h=0\ \forall h\in\mathscr{H}$,
3. $\zeta_w=\zeta_e=0_H$,

equations (150), (151), and (152) are given by

$$\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widetilde{\Psi}_{\ell}\widehat{w}=\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widetilde{\Psi}_{x}\widehat{\mu}+\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widehat{S},$$

$$diag\left(\chi\right)\widehat{E}=diag\left(\Lambda\right)\widehat{w}-\kappa^{\prime}\,diag\left(\mu\right)diag\left(\lambda\right)\widehat{\mu}+\Omega_{\pi}^{\prime}diag\left(\lambda\right)\widehat{S},$$

$$\begin{aligned}diag\left(\lambda\right)\widehat{S}=\mathscr{B}^{\prime}diag\left(\chi\right)\widehat{E}&+\Psi_{x}^{\prime}\left(\beta^{\prime}diag\left(\chi\right)\beta-diag\left(\beta^{\prime}\chi\right)+\Omega_{x}^{\prime}diag\left(\lambda\right)-diag\left(\Omega_{x}^{\prime}\lambda\right)\right)\widetilde{\Psi}_{x}\widehat{\mu}\\&-\Psi_{x}^{\prime}\left(\beta^{\prime}diag\left(\chi\right)\beta-diag\left(\beta^{\prime}\chi\right)+\Omega_{x}^{\prime}diag\left(\lambda\right)-diag\left(\Omega_{x}^{\prime}\lambda\right)\right)\widetilde{\Psi}_{\ell}\widehat{w}.\end{aligned}$$

First, let me start by setting up this system of equations in terms of Domar weights by adding and subtracting $\widehat{GDP}$

$$\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widetilde{\Psi}_{\ell}\widehat{\Lambda}=\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widetilde{\Psi}_{x}\widehat{\mu}+\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widehat{\lambda}+\Omega_{\ell}^{\prime}\left(\lambda-diag\left(\lambda\right)\widetilde{\Psi}_{\ell}\mathbb{1}_H\right)\widehat{GDP},$$

$$diag\left(\chi\right)\widehat{\chi}=diag\left(\Lambda\right)\widehat{\Lambda}-\kappa^{\prime}\,diag\left(\mu\right)diag\left(\lambda\right)\widehat{\mu}+\Omega_{\pi}^{\prime}diag\left(\lambda\right)\widehat{\lambda}+\left(\Lambda+\Omega_{\pi}^{\prime}\lambda-\chi\right)\widehat{GDP},$$

$$\begin{aligned}diag\left(\lambda\right)\widehat{\lambda}=\mathscr{B}^{\prime}diag\left(\chi\right)\widehat{\chi}&+\Psi_{x}^{\prime}\left(\beta^{\prime}diag\left(\chi\right)\beta-diag\left(\beta^{\prime}\chi\right)+\Omega_{x}^{\prime}diag\left(\lambda\right)-diag\left(\Omega_{x}^{\prime}\lambda\right)\right)\widetilde{\Psi}_{x}\widehat{\mu}\\&-\Psi_{x}^{\prime}\left(\beta^{\prime}diag\left(\chi\right)\beta-diag\left(\beta^{\prime}\chi\right)+\Omega_{x}^{\prime}diag\left(\lambda\right)-diag\left(\Omega_{x}^{\prime}\lambda\right)\right)\widetilde{\Psi}_{\ell}\widehat{\Lambda}\\&+\Psi_{x}^{\prime}\left(diag\left(\beta^{\prime}\chi\right)-\beta^{\prime}diag\left(\chi\right)\beta-\Omega_{x}^{\prime}diag\left(\lambda\right)+diag\left(\Omega_{x}^{\prime}\lambda\right)\right)\widetilde{\Psi}_{\ell}\mathbb{1}_H\widehat{GDP}.\end{aligned}$$

From $\widetilde{\Psi}_{\ell}\mathbb{1}_H=\mathbb{1}_N$, using Y as the numeraire, and equations (59) and (77)

$$\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widetilde{\Psi}_{\ell}\widehat{\Lambda}=\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widetilde{\Psi}_{x}\widehat{\mu}+\Omega_{\ell}^{\prime}diag\left(\lambda\right)\widehat{\lambda}+\Omega_{\ell}^{\prime}\underbrace{\left(\lambda-\lambda\right)}_{=0_N}\widehat{Y},$$

$$diag\,(\chi)\widehat{\chi} = diag\,(\Lambda)\,\widehat{\Lambda} - \kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu} + \Omega_\pi' diag\,(\lambda)\,\widehat{\lambda} + \underbrace{(\Lambda + \Omega_\pi'\lambda - \chi)}_{=0_N}\widehat{Y},$$

$$\begin{aligned} diag\,(\lambda)\,\widehat{\lambda} = {} & \mathscr{B}' diag\,(\chi)\,\widehat{\chi} + \Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\widehat{\mu} \\ & - \Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_\ell\widehat{\Lambda} \\ & + \Psi_x'\left(\underbrace{diag\,(\beta'\chi + \Omega_x'\lambda)\,\mathbb{1}_N}_{=\lambda} - \underbrace{(\beta'\chi + \Omega_x'\lambda)}_{=\lambda}\right)Y. \end{aligned}$$

Therefore, the system is represented by

$$\Omega_\ell' diag\,(\lambda)\,\widetilde{\Psi}_\ell\widehat{\Lambda} = \Omega_\ell' diag\,(\lambda)\,\widetilde{\Psi}_x\widehat{\mu} + \Omega_\ell' diag\,(\lambda)\,\widehat{\lambda}, \tag{153}$$

$$diag\,(\chi)\widehat{\chi} = diag\,(\Lambda)\,\widehat{\Lambda} - \kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu} + \Omega_\pi' diag\,(\lambda)\,\widehat{\lambda}, \tag{154}$$

$$diag\,(\lambda)\,\widehat{\lambda} = \mathscr{B}' diag\,(\chi)\,\widehat{\chi} + \Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right). \tag{155}$$

To illustrate the logic behind this proof let me start with an environment in which there is a representative household. For this economy we know that $\chi = 1$, $\widetilde{\Psi}_\ell = \mathbb{1}_N$, $\kappa = \mathbb{1}_N$ $\Omega_\pi = \mathbb{1}_N - \mu$, $\widetilde{\mathscr{C}} = 1$, $\alpha = \mathbb{1}_N$, $\Omega_\ell = diag\,(\mu)\,\omega_\ell$, $\lambda = \mathscr{B}'$, $\widetilde{\lambda} = \widetilde{\mathscr{B}}'$, and $\widehat{\chi} = 0$. Notice that this implies the system of equations (153), (154), and (155) is given by

$$\Lambda\widehat{\Lambda} = \Omega_\ell' diag\,(\lambda)\,\widetilde{\Psi}_x\widehat{\mu} + \Omega_\ell' diag\,(\lambda)\,\widehat{\lambda},$$

$$0 = \Lambda\widehat{\Lambda} - \mu' diag\,(\lambda)\,\widehat{\mu} + (\mathbb{1}_N - \mu)'\,diag\,(\lambda)\,\widehat{\lambda},$$

$$diag\,(\lambda)\,\widehat{\lambda} = \Psi_x'\left(\beta'\beta - diag\,(\beta) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\widehat{\mu} - \Psi_x'\left(\underbrace{\beta' + \Omega_x'\lambda}_{=\lambda} - diag\left(\underbrace{\beta + \Omega_x'\lambda}_{=\lambda}\right)\mathbb{1}_N\right)\widehat{\Lambda}.$$

Introducing the third equation in the first equation and using (59)

$$\Lambda\widehat{\Lambda} = \Omega_\ell'\left(\Psi_x'\beta'\beta + \Psi_x'\left(diag\left(\underbrace{\lambda - \beta - \Omega_x'\lambda}_{=0_N}\right)\right)\right)\widetilde{\Psi}_x\widehat{\mu},$$

$$\Lambda\widehat{\Lambda} = \mathscr{C}\widetilde{\lambda}'\widehat{\mu},$$

$$\widehat{\Lambda} = \widetilde{\lambda}'\widehat{\mu},$$

where the last line comes from $\mathscr{C} = \Lambda$. In equation (111) this implies that $\widehat{Y} = 0$.

Now, for an environment with heterogeneous households, the problem is that I cannot guarantee that $\widehat{\chi} = 0_H$. Let me start by introducing equation (155) in (154)

$$\begin{aligned} diag\,(\chi)\,\widehat{\chi} = {} & (I_H - \Omega_\pi'\mathscr{B}')^{-1}\,diag\,(\Lambda)\,\widehat{\Lambda} - (I_H - \Omega_\pi'\mathscr{B}')^{-1}\,\kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu} \\ & + (I_H - \Omega_\pi'\mathscr{B}')^{-1}\,\Omega_\pi'\Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right). \end{aligned}$$

Now introducing this last result in (155) and assuming that all of the eigenvalues for $\mathscr{B}\Omega_\pi$ are within the unit circle

$$\begin{aligned}
&diag\,(\lambda)\,\widehat{\lambda} = \mathscr{B}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1} diag\,(\Lambda)\,\widehat{\Lambda} - \mathscr{B}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu}\\
&+ \mathscr{B}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi'\Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\\
&+ \Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right).
\end{aligned}$$

Introducing this last result in equation (153)

$$\begin{aligned}
0_H &= \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1} diag\,(\Lambda)\,\widehat{\Lambda} - \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu}\\
&+ \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi'\Psi_x'\left(\beta' diag\,(\chi)\,\beta - diag\,(\beta'\chi) + \Omega_x' diag\,(\lambda) - diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\\
&+ \Omega_\ell'\left(\Psi_x'\beta' diag\,(\chi)\,\beta - \Psi_x' diag\,(\beta'\chi) + \left(I_N + \Psi_x'\Omega_x'\right) diag\,(\lambda) - \Psi_x' diag\,(\Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right).
\end{aligned}$$

Now

- Use $I_N + \Omega_x\Psi_x = I_N + \Omega_x\sum_{q=0}^{\infty}\Omega_x^q = \Psi_x$,
- Add and subtract $\mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi' diag\,(\lambda)\,\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)$,
- Use equation (59),

to obtain

$$\begin{aligned}
0_H &= \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1} diag\,(\Lambda)\,\widehat{\Lambda} - \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu}\\
&- \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi' diag\,(\lambda)\,\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\\
&+ \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi'\Psi_x'\left(\beta' diag\,(\chi)\,\beta + diag\,(\lambda - \beta'\chi - \Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\\
&+ \Omega_\ell'\Psi_x'\left(\beta' diag\,(\chi)\,\beta + diag\,(\lambda - \beta'\chi - \Omega_x'\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right),
\end{aligned}$$

$$\begin{aligned}
0_H &= \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1} diag\,(\Lambda)\,\widehat{\Lambda} - \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu}\\
&- \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi' diag\,(\lambda)\,\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\\
&+ \mathscr{C}'\left(I_H + \left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\Omega_\pi'\mathscr{B}'\right) diag\,(\chi)\,\beta\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right).
\end{aligned}$$

Notice that

- $I_H + \left(I_H - \mathscr{B}\Omega_\pi\right)^{-1}\mathscr{B}\Omega_\pi = I_N + \left(\sum_{q=0}^{\infty}\left(\mathscr{B}\Omega_\pi\right)^q\right)\mathscr{B}\Omega_\pi = I_N + \sum_{q=1}^{\infty}\left(\mathscr{B}\Omega_\pi\right)^q = \left(I_H - \mathscr{B}\Omega_\pi\right)^{-1}$,

therefore

$$0_H = \mathscr{C}'\left(I_H - \Omega_\pi'\mathscr{B}'\right)^{-1}\left(diag\,(\Lambda)\,\widehat{\Lambda} - \kappa'\,diag\,(\mu)\,diag\,(\lambda)\,\widehat{\mu} + \left(diag\,(\chi)\,\beta - \Omega_\pi' diag\,(\lambda)\right)\widetilde{\Psi}_x\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\right).$$

Now, assuming that $\mathscr{C}$ is invertible and adding up the previous vector gives me

$$0 = \widetilde{\lambda}'\widehat{\mu} - \widetilde{\Lambda}'\widehat{\Lambda} + \lambda'\left(diag\,(\mu) + diag\,(\mathbb{1}_N - \mu)\,\widetilde{\Psi}_x\right)\left(\widetilde{\Omega}_\ell\widehat{\Lambda} - \widehat{\mu}\right)$$

$$0 = \widetilde{\lambda}'\widehat{\mu} - \widetilde{\Lambda}'\widehat{\Lambda} + \lambda'\left(\widetilde{\Psi}_x - diag\left(\mu\right)\left(\widetilde{\Psi}_x - I_N\right)\right)\left(\widetilde{\Omega}_\ell\widehat{\Lambda} - \widehat{\mu}\right)$$
$$0 = \widetilde{\lambda}'\widehat{\mu} - \widetilde{\Lambda}'\widehat{\Lambda} + \lambda'\left(\widetilde{\Psi}_x - diag\left(\mu\right)\widetilde{\Omega}_x\widetilde{\Psi}_x\right)\left(\widetilde{\Omega}_\ell\widehat{\Lambda} - \widehat{\mu}\right)$$
$$0 = \widetilde{\lambda}'\widehat{\mu} - \widetilde{\Lambda}'\widehat{\Lambda} + \lambda'\left(I_N - \Omega_x\right)\left(\widetilde{\Psi}_\ell\widehat{\Lambda} - \widetilde{\Psi}_x\widehat{\mu}\right).$$

For $\widetilde{\Lambda}'\widehat{\Lambda} = \widetilde{\lambda}'\widehat{\mu}$ to be a solution I require that $\lambda'\left(I_N - \Omega_x\right) = \chi'\beta$ which implies that

1. $\lambda'\left(I_N - \Omega_x\right)\widetilde{\Psi}_x = \widetilde{\lambda}'$,
2. $\lambda'\left(I_N - \Omega_x\right)\widetilde{\Psi}_\ell = \widetilde{\Lambda}'$.

From equation (61) $\lambda = \mathscr{B}'\chi$. This implies that

$$\lambda'\left(I_N - \Omega_x\right) = \chi'\mathscr{B}\left(I_N - \Omega_x\right) = \chi'\beta\Psi_x\left(I_N - \Omega_x\right) = \chi'\beta,$$

which completes the proof.

Summing up, there are two additional conditions for an environment with heterogeneous households.

1. $\mathscr{B}\Omega_\pi$ is an $H \times H$ matrix with all of its eigenvalues within the unit circle. One way to guarantee this is by assuming that the sum of its rows is always less than 1. From the Gershgorin circle theorem then all of the eigenvalues are less than one, which implies that the determinant for $\mathscr{B}\Omega_\pi$ is less than one, and this allows from the Geometric series applied to matrices to express $\sum_{q=0}^{\infty}\left(\mathscr{B}\Omega_\pi\right)^q = \left(I_N - \mathscr{B}\Omega_\pi\right)^{-1}$. Notice that the of the sum rows always less than 1 is a sufficient but not a necessary condition.
2. $\mathscr{C}$ has to be nonsingular.

Outside the space of economies for which these two conditions are satisfied, when neutrality of shocks in wedges for Leontief economies does not hold, I can characterize the distributional conditions that are necessary for allocative growth to arise. First, introduce equation (155) in equation (153), and use equation (61) to obtain

$$0_H = \mathscr{C}'diag\left(\chi\right)\left(\widehat{\chi} + \widetilde{\mathscr{B}}\left(\widehat{\mu} - \widetilde{\Omega}_\ell\widehat{\Lambda}\right)\right).$$

Now, add and subtract $\widehat{GDP}$, substitute $\widehat{E} = \widehat{p}_c + \widehat{C}$, $\widehat{p}_c = \widetilde{\mathscr{B}}\left(\widetilde{\Omega}_\ell\widehat{w} - \widehat{\mu}\right)$, and using Y as the numeraire set $\widehat{GDP} = \widehat{Y}$, then

$$0_H = \mathscr{C}'diag\left(\chi\right)\left(\widehat{C} - \left(I_H - \widetilde{\mathscr{C}}\right)\mathbb{1}_H\widehat{Y}\right) = \mathscr{C}'diag\left(\chi\right)\widehat{C}.$$

Finally, adding up over all components in this vector and using equation (107)

$$d\,log\,Y = \sum_{h\in\mathscr{H}}\chi_h\,\left(1 - \mathscr{C}_h\right)d\,log\,c_h. \tag{156}$$

From the last equation, I can obtain an additional condition that guarantees neutrality. Assume that all consumption bundles have the same payment centrality, i.e. $\sum_{b\in\mathscr{H}} \mathscr{C}_{hb} = \tau \ \forall h \in \mathscr{H}$. Notice that this is equivalent to $\mathscr{C}\mathbb{1}_H = \mathbb{1}_H \tau$, which implies in equation (156) that $\chi'\widehat{C} = 0$, and from equation (107) that $\widehat{Y} = 0$.

Therefore, there are two sufficient conditions that guarantee neutrality for shocks in distortions for a Leontief economy with heterogeneous households, either

1. $\mathscr{B}\Omega_\pi$ has its eigenvalues within the unit circle and $\mathscr{C}$ is nonsingular, or

2. all consumers have the same payment centrality.

Notice that the second sufficient condition is not encompassed by the first one, for example take an economy with two households that satisfies the second but not the first condition

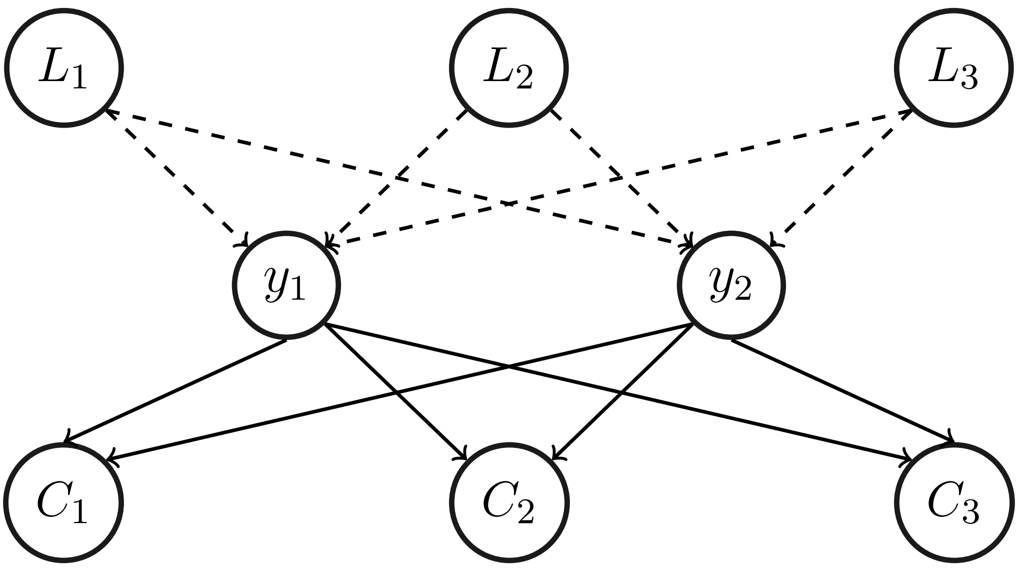
$$\mathscr{C} = \begin{pmatrix} 0.2 & 0.4 \\ 0.2 & 0.4 \end{pmatrix}.$$

Furthermore, notice that the second condition does not require homogeneous consumption bundles, for example take an economy with two households that have heterogeneous bundles but the same payment centrality

$$\mathscr{C} = \begin{pmatrix} 0.5 & 0.3 \\ 0.2 & 0.6 \end{pmatrix}.$$

3 Horizontal Economy

Figure 14: Horizontal Economy with $N = 2$ and $H = 3$



Note: Continuous arrows represent the flow of goods and dashed arrows the supply of labor.

In the horizontal economy with N firms represented in Figure 14, we have that $\omega_x = 0_N$, $\omega_\ell = \mathbb{1}_N$,

$$\alpha = \widetilde{\Omega}_\ell = \widetilde{\Psi}_\ell = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1H} \\ \vdots & \ddots & \vdots \\ \alpha_{N1} & \cdots & \alpha_{NH} \end{pmatrix}, \qquad \Omega_\ell = \Psi_\ell = \begin{pmatrix} \mu_1\alpha_{11} & \cdots & \mu_1\alpha_{1H} \\ \vdots & \ddots & \vdots \\ \mu_N\alpha_{N1} & \cdots & \mu_N\alpha_{NH} \end{pmatrix},$$

$$\widetilde{\Omega}_x = \Omega_x = 0_N 0_N', \qquad \widetilde{\Psi}_x = \Psi_x = I_N,$$

$$\widetilde{\mathscr{B}} = \mathscr{B} = \beta = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1N} \\ \vdots & \ddots & \vdots \\ \beta_{H1} & \cdots & \beta_{HN} \end{pmatrix}, \qquad \widetilde{\lambda} = \lambda = \begin{pmatrix} \sum_{h\in\mathscr{H}} \chi_h\beta_{h1} \\ \vdots \\ \sum_{h\in\mathscr{H}} \chi_h\beta_{hN} \end{pmatrix},$$

$$\widetilde{\Lambda} = \begin{pmatrix} \sum_{i\in\mathscr{N}} \alpha_{i1}\lambda_i \\ \vdots \\ \sum_{i\in\mathscr{N}} \alpha_{iH}\lambda_i \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \sum_{i\in\mathscr{N}} \alpha_{i1}\mu_i\lambda_i \\ \vdots \\ \sum_{i\in\mathscr{N}} \alpha_{iH}\mu_i\lambda_i \end{pmatrix},$$

$$\widetilde{\mathscr{C}} = \begin{pmatrix} \sum_{i\in\mathscr{N}} \beta_{1i}\alpha_{i1} & \cdots & \sum_{i\in\mathscr{N}} \beta_{1i}\alpha_{iH} \\ \vdots & \ddots & \vdots \\ \sum_{i\in\mathscr{N}} \beta_{Hi}\alpha_{i1} & \cdots & \sum_{i\in\mathscr{N}} \beta_{Hi}\alpha_{iH} \end{pmatrix}, \mathscr{C} = \begin{pmatrix} \sum_{i\in\mathscr{N}} \beta_{1i}\mu_i\alpha_{i1} & \cdots & \sum_{i\in\mathscr{N}} \beta_{1i}\mu_i\alpha_{iH} \\ \vdots & \ddots & \vdots \\ \sum_{i\in\mathscr{N}} \beta_{Hi}\mu_i\alpha_{i1} & \cdots & \sum_{i\in\mathscr{N}} \beta_{Hi}\mu_i\alpha_{iH} \end{pmatrix}.$$

In this economy, $GDP = \sum_{i\in\mathscr{N}} S_i = \sum_{h\in\mathscr{H}} J_h + \sum_{i\in\mathscr{N}} \pi_i$. Equations (150), (151), and (152) are respectively given by

$$\begin{aligned}\Lambda_h \left(1+\zeta_h^w\right) \widehat{w}_h = & \sum_{i\in\mathscr{N}} \Omega_{ih}^\ell \lambda_i \left(\widehat{\mu}_i + \widehat{S}_i\right) + \Lambda_h \left(\zeta_h^e \widehat{E}_h - \zeta_h^n \widehat{n}_h\right) \\ & + \sum_{b\in\mathscr{H}} \left(\sum_{i\in\mathscr{N}} \Omega_{ih}^\ell \lambda_i \left(\theta_i - 1\right) \alpha_{ib}\right) \widehat{w}_b - \left(\sum_{i\in\mathscr{N}} \Omega_{ih}^\ell \lambda_i \left(\theta_i - 1\right)\right) \widehat{w}_h,\end{aligned}$$

$$\chi_h \widehat{E}_h = \frac{1}{1+\zeta_h^e \Gamma_h} \left(\Lambda_h \left(\left(1+\zeta_h^w\right) \widehat{w}_h + \zeta_h^n \widehat{n}_h\right) + \sum_{i\in\mathscr{N}} \kappa_{ih} \left(\left(1-\mu_i\right) \lambda_i \widehat{S}_i - \mu_i \lambda_i \widehat{\mu}_i\right)\right),$$

$$\begin{aligned}\lambda_i \widehat{S}_i = & \sum_{h\in\mathscr{H}} \beta_{hi} \chi_h \left(\varrho_h - 1\right) \left(\widehat{A}_i + \widehat{\mu}_i - \sum_{j\in\mathscr{N}} \beta_{hj} \left(\widehat{A}_j + \widehat{\mu}_j\right)\right) \\ & + \sum_{h\in\mathscr{H}} \beta_{hi} \chi_h \widehat{E}_h + \sum_{h\in\mathscr{H}} \left(\sum_{b\in\mathscr{H}} \beta_{bi} \chi_b \left(\varrho_b - 1\right) \left(\widetilde{\mathscr{C}}_{bh} - \alpha_{ih}\right)\right) \widehat{w}_h.\end{aligned}$$

I use $2H+N-1$ of these equations, and normalize this system by taking Y as the numeraire, and for this reason p_Y is normalized to 1, which implies that

$$\sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \widehat{w}_h = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{A}_i + \widehat{\mu}_i\right).$$

3.1 Representative Household Economy

From the previous system of equations

$$\widehat{w} = \sum_{i\in\mathscr{N}} \beta_i \left(\widehat{A}_i + \widehat{\mu}_i\right),$$

$$\widehat{E} = \frac{1}{1+\zeta^e\Gamma}\left(\zeta^n\Gamma\widehat{n} + (1+\zeta^w)\,\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right) + \sum_{i\in\mathscr{N}}\beta_i\left((1-\mu_i)\,\widehat{S}_i - \mu_i\widehat{\mu}_i\right)\right),$$

$$\begin{aligned}\widehat{S}_i = {} & (\varrho-1)\left(\widehat{A}_i+\widehat{\mu}_i - \sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\widehat{\mu}_j\right)\right)\\ & + \frac{1}{1+\zeta^e\Gamma}\left(\zeta^n\Gamma\widehat{n} + (1+\zeta^w)\,\Gamma\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\widehat{\mu}_j\right) + \sum_{j\in\mathscr{N}}\beta_j\left((1-\mu_j)\,\widehat{S}_j - \mu_j\widehat{\mu}_j\right)\right).\end{aligned}$$

Notice that sales elasticities can be represented with the following matrix equation

$$\begin{pmatrix} 1+\zeta^e\Gamma-\beta_1(1-\mu_1) & \cdots & -\beta_N(1-\mu_N) \\ \vdots & \ddots & \vdots \\ -\beta_1(1-\mu_1) & \cdots & 1+\zeta^e\Gamma-\beta_N(1-\mu_N)\end{pmatrix}\begin{pmatrix}\widehat{S}_1\\ \vdots\\ \widehat{S}_N\end{pmatrix} = \begin{pmatrix} (1+\zeta^e\Gamma)\left((\varrho-1)\left(\widehat{A}_1+\widehat{\mu}_1-\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\widehat{\mu}_j\right)\right)\right) \\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\left(1-\mu_j\right)\widehat{\mu}_j\right) \\ \vdots \\ (1+\zeta^e\Gamma)\left((\varrho-1)\left(\widehat{A}_N+\widehat{\mu}_N-\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\widehat{\mu}_j\right)\right)\right) \\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\left(1-\mu_j\right)\widehat{\mu}_j\right)\end{pmatrix}.$$

To use Cramer's rule, first let me start by finding the determinant for the matrix

$$\begin{aligned}&\begin{vmatrix} 1+\zeta^e\Gamma-\beta_1(1-\mu_1) & \cdots & -\beta_j(1-\mu_j) & \cdots & -\beta_N(1-\mu_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -\beta_1(1-\mu_1) & \cdots & 1+\zeta^e\Gamma-\beta_j(1-\mu_j) & \cdots & -\beta_N(1-\mu_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -\beta_1(1-\mu_1) & \cdots & -\beta_j(1-\mu_j) & \cdots & 1+\zeta^e\Gamma-\beta_N(1-\mu_N)\end{vmatrix}\\ &= \begin{vmatrix} 1+\zeta^e\Gamma & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_1(\mu_1-1) & \cdots & \sum_{i\in\mathscr{N}}\beta_i\mu_i+\zeta^e\Gamma & \cdots & \beta_N(\mu_N-1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1+\zeta^e\Gamma\end{vmatrix}\\ &= (1+\zeta^e\Gamma)^{N-1}\left(\sum_{i\in\mathscr{N}}\beta_i\mu_i+\zeta^e\Gamma\right) = (1+\zeta^e\Gamma)^{N-1}(\Lambda+\zeta^e\Gamma).\end{aligned}$$

The first equality comes from adding all other columns to the column j , and subtracting row j from all other rows. The second equality from solving the determinant.

Now, the determinant for the matrix in which the j-th column is replaced by the N sized vector of exogenous shocks is given by

$$\begin{vmatrix} 1+\zeta^{c}\Gamma-\beta_1(1-\mu_1) & \cdots & \substack{(1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_1+\widehat{\mu}_1-\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right)\right)\\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+(1-\mu_i)\widehat{\mu}_i\right)} & \cdots & -\beta_N(1-\mu_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -\beta_1(1-\mu_1) & \cdots & \substack{(1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_j+\widehat{\mu}_j-\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right)\right)\\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+(1-\mu_i)\widehat{\mu}_i\right)} & \cdots & -\beta_N(1-\mu_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -\beta_1(1-\mu_1) & \cdots & \substack{(1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_N+\widehat{\mu}_N-\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right)\right)\\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+(1-\mu_i)\widehat{\mu}_i\right)} & \cdots & 1+\zeta^e\Gamma-\beta_N(1-\mu_N) \end{vmatrix}$$

$$= \begin{vmatrix} 1+\zeta^e\Gamma & \cdots & (1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_1-\widehat{A}_j+\widehat{\mu}_1-\widehat{\mu}_j\right) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -\beta_1(1-\mu_1) & \cdots & \substack{(1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_j+\widehat{\mu}_j-\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right)\right)\\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+(1-\mu_i)\widehat{\mu}_i\right)} & \cdots & -\beta_N(1-\mu_N) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & (1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_N-\widehat{A}_j+\widehat{\mu}_N-\widehat{\mu}_j\right) & \cdots & 1+\zeta^e\Gamma \end{vmatrix}$$

$$= \begin{vmatrix} \substack{(1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_j+\widehat{\mu}_j-\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right)\right)\\ +\zeta^n\Gamma\widehat{n}+(1+\zeta^w)\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+(1-\mu_i)\widehat{\mu}_i\right)} & \cdots & \beta_1(\mu_1-1) & \cdots & \beta_N(\mu_N-1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ (1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_1-\widehat{A}_j+\widehat{\mu}_1-\widehat{\mu}_j\right) & \cdots & 1+\zeta^e\Gamma & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ (1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_N-\widehat{A}_j+\widehat{\mu}_N-\widehat{\mu}_j\right) & \cdots & 0 & \cdots & 1+\zeta^e\Gamma \end{vmatrix}$$

$$= (1+\zeta^e\Gamma)^{N-1}\left((1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_j+\widehat{\mu}_j-\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+\widehat{\mu}_i\right)\right)+\zeta^n\Gamma\widehat{n}\right.$$
$$\left.+(1+\zeta^w)\Gamma\sum_{i\in\mathscr{N}}\beta_i\left(\widehat{A}_i+(1-\mu_i)\widehat{\mu}_j\right)+\sum_{\substack{i\in\mathscr{N}\\ i\neq j}}\beta_i(1-\mu_i)(\varrho-1)\left(\widehat{A}_i-\widehat{A}_j+\widehat{\mu}_i-\widehat{\mu}_j\right)\right).$$

The first equality comes from subtracting row j from all other rows, the second equality from substituting columns 1 for column j, and then row 1 for row j, and the third equality from Schur's complement.

Therefore[17]

$$\widehat{S}_i = \frac{1}{(1+\zeta^e)\Gamma}\left((1+\zeta^e\Gamma)(\varrho-1)\left(\widehat{A}_i+\widehat{\mu}_i-\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+\widehat{\mu}_j\right)\right)+\zeta^n\Gamma\widehat{n}\right.$$
$$\left.+(1+\zeta^w)\Gamma\sum_{j\in\mathscr{N}}\beta_j\left(\widehat{A}_j+(1-\mu_j)\widehat{\mu}_j\right)+\sum_{j\in\mathscr{N}}\beta_j(1-\mu_j)(\varrho-1)\left(\widehat{A}_j-\widehat{A}_i+\widehat{\mu}_j-\widehat{\mu}_i\right)\right).$$

3.1.1 Productivity Shock

Firm $k \in \mathscr{N}$ receives the productivity shock. This means that

- $\dfrac{\partial \log w}{\partial \log A_k} = \beta_k,$
- $\dfrac{\partial \log S_i}{\partial \log A_k} = \dfrac{1+\zeta^w}{1+\zeta^e}\beta_k + (\varrho-1)\left(\mathbb{1}\{k=i\} - \dfrac{\mu_k/\Gamma+\zeta^e}{1+\zeta^e}\beta_k\right),$

[17] Notice that $\Gamma = \Lambda$.

- $$\frac{\partial \log p_i}{\partial \log A_k} = \beta_k - \mathbb{1}\{k=i\},$$

- $$\frac{\partial \log y_i}{\partial \log A_k} = \frac{\partial \log S_i}{\partial \log A_k} - \frac{\partial \log p_i}{\partial \log A_k} = \mathbb{1}\{k=i\} + \frac{\zeta^w - \zeta^e}{1+\zeta^e}\beta_k + (\varrho - 1)\left(\mathbb{1}\{k=i\} - \frac{\mu_k/\Gamma + \zeta^e}{1+\zeta^e}\beta_k\right),$$

- $$\frac{\partial \log E}{\partial \log A_k} = \frac{1}{1+\zeta^e\Gamma}\left((1+\zeta^w)\Gamma\beta_k + \sum_{i\in\mathscr{N}}\beta_i(1-\mu_i)\frac{\partial \log S_i}{\partial \log A_k}\right) = \frac{1+\zeta^w}{1+\zeta^e}\beta_k + \frac{\varrho-1}{1+\zeta^e}\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\frac{\partial \log L}{\partial \log A_k} = \frac{\zeta^w - \zeta^e}{1+\zeta^e}\beta_k - \frac{\zeta^e}{1+\zeta^e}(\varrho-1)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\frac{\partial \log J}{\partial \log A_k} = \frac{1+\zeta^w}{1+\zeta^e}\beta_k - \frac{\zeta^e}{1+\zeta^e}(\varrho-1)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\frac{\partial \log TFP}{\partial \log A_k} = \beta_k - (1-\Gamma)\frac{\partial \log J}{\partial \log A_k} + \sum_{i\in\mathscr{N}}(1-\mu_i)\lambda_i\frac{\partial \log S_i}{\partial \log A_k} = \beta_k + (\varrho-1)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\frac{\partial \log Y}{\partial \log A_k} = \frac{\partial \log TFP}{\partial \log A_k} + \frac{\partial \log L}{\partial \log A_k} = \frac{1+\zeta^w}{1+\zeta^e}\beta_k + \frac{\varrho-1}{1+\zeta^e}\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\frac{\partial \log \Gamma}{\partial \log A_k} = \frac{\partial \log J}{\partial \log A_k} - \frac{\partial \log E}{\partial \log A_k} = (\rho-1)\left(\frac{\mu_k}{\Gamma}-1\right)\beta_k.$$

3.1.2 Markdown Shock

Firm $k \in \mathscr{N}$ receives the markdown shock. This means that

- $$\frac{\partial \log w}{\partial \log \mu_k} = \beta_k,$$

$$\frac{\partial \log S_i}{\partial \log \mu_k} = \frac{1+\zeta^w}{1+\zeta^e}(1-\mu_k)\beta_k + (\varrho-1)\left(\mathbb{1}\{k=i\} - \frac{\mu_k/\Gamma+\zeta^e}{1+\zeta^e}\beta_k\right),$$

- $$\frac{\partial \log p_i}{\partial \log \mu_k} = \beta_k - \mathbb{1}\{k=i\},$$

- $$\begin{aligned}\frac{\partial \log y_i}{\partial \log \mu_k} &= \frac{\partial \log S_i}{\partial \log \mu_k} - \frac{\partial \log p_i}{\partial \log \mu_k} \\ &= \mathbb{1}\{k=i\} + \left(\frac{\zeta^w-\zeta^e}{1+\zeta^e} - \frac{1+\zeta^w}{1+\zeta^e}\mu_k\right)\beta_k + (\varrho-1)\left(\mathbb{1}\{k=i\} - \frac{\mu_k/\Gamma+\zeta^e}{1+\zeta^e}\beta_k\right),\end{aligned}$$

- $$\begin{aligned}\frac{\partial \log E}{\partial \log \mu_k} &= \frac{1}{1+\zeta^e\Gamma}\left((1+\zeta^w)\Gamma\beta_k + \sum_{i\in\mathscr{N}}\beta_i(1-\mu_i)\frac{\partial \log S_i}{\partial \log \mu_k} - \beta_k\mu_k\right) \\ &= \frac{1+\zeta^w}{1+\zeta^e}\beta_k - \frac{\mu_k\beta_k}{1+\zeta^e\Gamma}\left(1+\frac{1+\zeta^w}{1+\zeta^e}(1-\Gamma)\right) + \frac{\varrho-1}{1+\zeta^e}\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,\end{aligned}$$

- $$\frac{\partial \log L}{\partial \log \mu_k} = \frac{\zeta^w-\zeta^e}{1+\zeta^e}\beta_k + \frac{\zeta^e\mu_k\beta_k}{1+\zeta^e\Gamma}\left(1+\frac{1+\zeta^w}{1+\zeta^e}(1-\Gamma)\right) - \frac{\zeta^e}{1+\zeta^e}(\varrho-1)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\frac{\partial \log J}{\partial \log \mu_k} = \frac{1+\zeta^w}{1+\zeta^e}\beta_k + \frac{\zeta^e\mu_k\beta_k}{1+\zeta^e\Gamma}\left(1+\frac{1+\zeta^w}{1+\zeta^e}(1-\Gamma)\right) - \frac{\zeta^e}{1+\zeta^e}(\varrho-1)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$

- $$\begin{aligned}\frac{\partial \log TFP}{\partial \log \mu_k} &= \beta_k - \mu_k\beta_k - (1-\Gamma)\frac{\partial \log J}{\partial \log A_k} + \sum_{i\in\mathscr{N}}(1-\mu_i)\lambda_i\frac{\partial \log S_i}{\partial \log A_k} \\ &= \frac{1-\mu_k+\zeta^e(\Gamma-\mu_k)}{1+\zeta^e\Gamma}\beta_k - \frac{1+\zeta^e}{1+\zeta^e\Gamma}(1-\Gamma)\mu_k\beta_k + (\varrho-1)\left(1+\zeta^e\frac{1+\zeta^e\Gamma}{1+\zeta^e}\right)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,\end{aligned}$$

- $$\frac{\partial \log Y}{\partial \log \mu_k} = \frac{\partial \log TFP}{\partial \log \mu_k} + \frac{\partial \log L}{\partial \log \mu_k} = \frac{1+\zeta^w}{1+\zeta^e}\beta_k - \frac{1}{1+\zeta^e\Gamma}\left(1+\left(1+\frac{\zeta^e}{1+\zeta^e}\left(\zeta^e-\zeta^w\right)\right)(1-\Gamma)\right)\mu_k\beta_k$$
$$+(\varrho-1)\frac{\left(1+\zeta^e\left(1+\zeta^e\Gamma\right)\right)}{1+\zeta^e}\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k,$$
$$\frac{\partial \log \Gamma}{\partial \log \mu_k} = \frac{1+\zeta^e}{1+\zeta^e\Gamma}\mu_k\beta_k + \frac{1+\zeta^w}{1+\zeta^e\Gamma}(1-\Gamma)\mu_k\beta_k - (\rho-1)\left(1-\frac{\mu_k}{\Gamma}\right)\beta_k.$$

3.1.3 Population Growth Shock

Assume a population growth of 1%, i.e. $d\log n = 1$. This means that

- $\frac{\partial \log w}{\partial \log n} = 0$,
- $\frac{\partial \log S_i}{\partial \log n} = \frac{\zeta^n}{1+\zeta^e}$,
- $\frac{\partial \log p_i}{\partial \log n} = 0$,
- $\frac{\partial \log E}{\partial \log n} = \frac{\zeta^n}{1+\zeta^e}$
- $\frac{\partial \log y_i}{\partial \log n} = \frac{\partial \log S_i}{\partial \log n} - \frac{\partial \log p_i}{\partial \log n} = \frac{\zeta^n}{1+\zeta^e}$,
- $\frac{\partial \log L}{\partial \log n} = \frac{\partial \log J}{\partial \log n} = \frac{\zeta^n}{1+\zeta^e}$
- $$\frac{\partial \log TFP}{\partial \log n} = -(1-\Gamma)\frac{\partial \log J}{\partial \log n} + \sum_{i\in\mathcal{N}}(1-\mu_i)\lambda_i\frac{\partial \log S_i}{\partial \log n} = 0,$$
- $\frac{\partial \log Y}{\partial \log n} = \frac{\partial \log TFP}{\partial \log n} + \frac{\partial \log L}{\partial \log n} = \frac{\zeta^n}{1+\zeta^e}$,
- $\frac{\partial \log \Gamma}{\partial \log n} = 0.$

3.2 Allocative Neutrality in a Horizontal Economy - Part 3 of Theorem 6

Let me assume that all firms are subject to the same distortions, i.e. $\mu_i = \mu\ \forall i \in \mathcal{N}$. Then equations (150), and (152) are given by

$$diag\left(\widetilde{\Lambda}\right)\left(I_H + diag\left(\zeta_w\right)\right)\widehat{w} = \alpha' diag\left(\lambda\right)\widehat{\mu} + \alpha' diag\left(\lambda\right)\widehat{S} + diag\left(\zeta_e\right)diag\left(\widetilde{\Lambda}\right)\widehat{E}$$
$$+\alpha' diag\left(\lambda\right)diag\left(\theta - \mathbb{1}_N\right)\alpha\widehat{w} - diag\left(\alpha' diag\left(\theta-\mathbb{1}_N\right)\lambda\right)\widehat{w},$$

$$diag\left(\lambda\right)\widehat{S} = \beta' diag\left(\chi\right)\widehat{E} + \left(\beta' diag\left(\chi\right)diag\left(\varrho - \mathbb{1}_H\right)\beta - diag\left(\beta' diag\left(\varrho-\mathbb{1}_H\right)\chi\right)\right)\left(\alpha\,\widehat{w} - \widehat{A} - \widehat{\mu}\right).$$

Now, instead of having wage elasticities, represent the system of equation in terms of factor income elasticities by replacing $\widehat{L} = \zeta_w\widehat{w} - \zeta_e\widehat{E}$

$$diag\left(\widetilde{\Lambda}\right)\widehat{J} = \alpha' diag\left(\lambda\right)\widehat{\mu} + \alpha' diag\left(\lambda\right)\widehat{S} + \left(\alpha' diag\left(\lambda\right)diag\left(\theta-\mathbb{1}_N\right)\alpha - diag\left(\alpha' diag\left(\theta-\mathbb{1}_N\right)\lambda\right)\right)\left(\widehat{J}-\widehat{L}\right),$$

$$diag\left(\lambda\right)\widehat{S} = \beta' diag\left(\chi\right)\widehat{E} + \left(\beta' diag\left(\chi\right)diag\left(\varrho-\mathbb{1}_H\right)\beta - diag\left(\beta' diag\left(\varrho-\mathbb{1}_H\right)\chi\right)\right)\left(\alpha\left(\widehat{J}-\widehat{L}\right) - \widehat{A} - \widehat{\mu}\right).$$

Now, express the system of equations in terms of add and subtract $\widehat{GDP}$ and using Y as the numeraire replace $\widehat{GDP} = \widehat{Y}$

$$diag\left(\widetilde{\Lambda}\right)\widehat{\Lambda} = \alpha' diag\left(\lambda\right)\widehat{\mu} + \alpha' diag\left(\lambda\right)\widehat{\lambda} + \left(\alpha' diag\left(\lambda\right)diag\left(\theta-\mathbb{1}_N\right)\alpha - diag\left(\alpha' diag\left(\theta-\mathbb{1}_N\right)\lambda\right)\right)\left(\widehat{\Lambda}-\widehat{L}\right)$$
$$+\underbrace{\left(\alpha'\lambda - \widetilde{\Lambda}\right)}_{=0_N}\widehat{Y} + \left(\alpha' diag\left(\lambda\right)diag\left(\theta-\mathbb{1}_N\right)\alpha - diag\left(\alpha' diag\left(\theta-\mathbb{1}_N\right)\lambda\right)\right)\mathbb{1}_H\widehat{Y},$$

$$diag\left(\lambda\right)\widehat{\lambda} = \beta' diag\left(\chi\right)\widehat{\chi} + \left(\beta' diag\left(\chi\right) diag\left(\varrho - \mathbb{1}_H\right)\beta - diag\left(\beta' diag\left(\varrho - \mathbb{1}_H\right)\chi\right)\right)\left(\alpha\left(\widehat{\Lambda} - \widehat{L}\right) - \widehat{A} - \widehat{\mu}\right)$$
$$+ \underbrace{\left(\beta'\chi - \lambda\right)}_{=0_N}\widehat{Y} + \left(\beta' diag\left(\chi\right) diag\left(\varrho - \mathbb{1}_H\right)\beta - diag\left(\beta' diag\left(\varrho - \mathbb{1}_H\right)\chi\right)\right)\alpha\mathbb{1}_H\widehat{Y}.$$

Now add up the elements for each of the vectors, notice that $\mathbb{1}' diag\left(x\right) = x'$

$$\widetilde{\Lambda}'\widehat{\Lambda} = \lambda'\widehat{\mu} + \lambda'\widehat{\lambda} + \left(\lambda' diag\left(\theta - \mathbb{1}_N\right)\alpha - \lambda' diag\left(\theta - \mathbb{1}_N\right)\alpha\right)\widehat{\Lambda} - \left(\lambda' diag\left(\theta - \mathbb{1}_N\right)\alpha - \lambda' diag\left(\theta - \mathbb{1}_N\right)\alpha\right)\widehat{L},$$

$$\lambda'\widehat{\lambda} = \chi'\widehat{\chi} + \left(\chi' diag\left(\varrho - \mathbb{1}_H\right)\beta - \chi' diag\left(\varrho - \mathbb{1}_H\right)\beta\right)\left(\alpha\left(\widehat{\Lambda} - \widehat{L}\right) - \widehat{A} - \widehat{\mu}\right).$$

Now, taking advantage of the property that $\chi'\widehat{\chi} = 0$

$$\widetilde{\Lambda}'\widehat{\Lambda} = \lambda'\widehat{\mu} + \lambda'\widehat{\lambda},$$

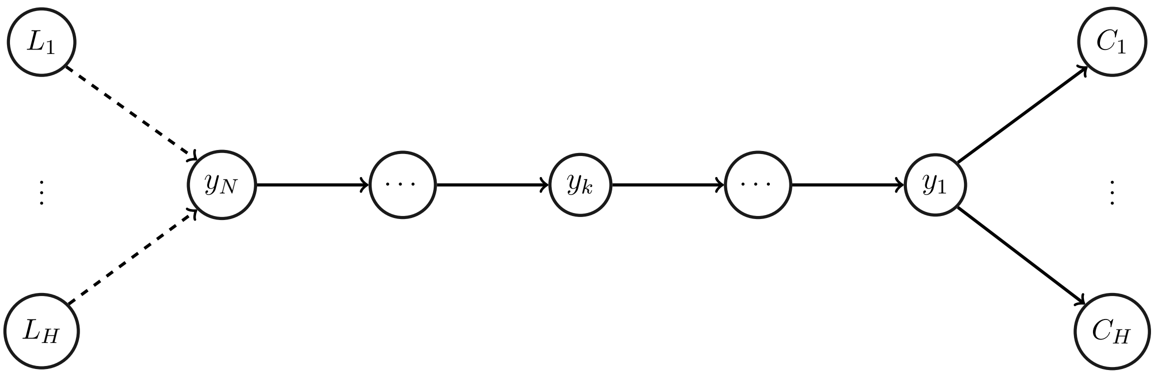
$$\lambda'\widehat{\lambda} = 0.$$

In a horizontal economy $\lambda = \widetilde{\lambda}$ and from equation (111)

$$\widehat{Y} = \widetilde{\lambda}'\widehat{A} + \widetilde{\Lambda}'\widehat{L}.$$

4 Vertical Economy

Figure 15: Vertical Economy



Note: Continuous arrows represent the flow of goods and dashed arrows the supply of labor.

In the vertical economy with F firms represented in Figure 15 we have that $\omega'_\ell = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$, $\omega'_x = \begin{pmatrix} 1 & \cdots & 1 & 0 \end{pmatrix}$, $a' = \begin{pmatrix} \alpha_{N1} & \cdots & \alpha_{NH} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \cdots & \alpha_H \end{pmatrix}$

$$\mathscr{W} = \widetilde{\Omega}_x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \alpha = \widetilde{\Omega}_\ell = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \alpha_1 & \cdots & \alpha_H \end{pmatrix} = o_N\left(N\right) a',$$

$$\Omega_x = \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ 0 & 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \Omega_\ell = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \alpha_1\mu_N & \cdots & \alpha_H\mu_N \end{pmatrix} = \mu_N\, o_N(N)\, a',$$

$$\widetilde{\Psi}_x = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad \Psi_x = \begin{pmatrix} 1 & \mu_1 & \mu_1\mu_2 & \cdots & \prod_{i=1}^{N-2}\mu_i & \prod_{i=1}^{N-1}\mu_i \\ 0 & 1 & \mu_2 & \cdots & \prod_{i=2}^{N-2}\mu_i & \prod_{i=2}^{N-1}\mu_i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \mu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$\beta = \mathbb{1}_H o_N(1)', \qquad \widetilde{\mathscr{B}} = \mathbb{1}_H \mathbb{1}_N', \qquad \mathscr{B} = \begin{pmatrix} 1 & \mu_1 & \cdots & \prod_{i=1}^{N-2}\mu_i & \prod_{i=1}^{N-1}\mu_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \mu_1 & \cdots & \prod_{i=1}^{N-2}\mu_i & \prod_{i=1}^{N-1}\mu_i \end{pmatrix},$$

$$\widetilde{\mathscr{C}} = \begin{pmatrix} \alpha_1 & \cdots & \alpha_H \\ \vdots & \ddots & \vdots \\ \alpha_1 & \cdots & \alpha_H \end{pmatrix} = \mathbb{1}_H\, a', \qquad \mathscr{C} = \begin{pmatrix} \alpha_1\prod_{i\in\mathscr{N}}\mu_i & \cdots & \alpha_H\prod_{i\in\mathscr{N}}\mu_i \\ \vdots & \ddots & \vdots \\ \alpha_1\prod_{i\in\mathscr{N}}\mu_i & \cdots & \alpha_H\prod_{i\in\mathscr{N}}\mu_i \end{pmatrix} = \prod_{i\in\mathscr{N}}\mu_i \mathbb{1}_H\, a',$$

$$\widetilde{\Psi}_\ell = \begin{pmatrix} \alpha_1 & \cdots & \alpha_H \\ \alpha_1 & \cdots & \alpha_H \\ \vdots & \ddots & \vdots \\ \alpha_1 & \cdots & \alpha_H \\ \alpha_1 & \cdots & \alpha_H \end{pmatrix} = \mathbb{1}_N\, a', \quad \Psi_\ell = \begin{pmatrix} \alpha_1\prod_{i=1}^{N}\mu_i & \cdots & \alpha_H\prod_{i=1}^{N}\mu_i \\ \alpha_1\prod_{i=2}^{N}\mu_i & \cdots & \alpha_H\prod_{i=2}^{N}\mu_i \\ \vdots & \ddots & \vdots \\ \alpha_1\prod_{i=N-1}^{N}\mu_i & \cdots & \alpha_H\prod_{i=N-1}^{N}\mu_i \\ \alpha_1\mu_N & \cdots & \alpha_H\mu_N \end{pmatrix} = \begin{pmatrix} \prod_{i\in\mathscr{N}}\mu_i \\ \prod_{i=2}^{N}\mu_i \\ \vdots \\ \mu_{N-1}\mu_N \\ \mu_N \end{pmatrix} a',$$

$$\lambda = \begin{pmatrix} 1 \\ \vdots \\ \prod_{i=1}^{k-1}\mu_i \\ \vdots \\ \prod_{i=1}^{N-1}\mu_i \end{pmatrix}, \quad \widetilde{\lambda} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \Lambda = \Omega_\ell'\lambda = \begin{pmatrix} \alpha_1\mu_N\lambda_N \\ \vdots \\ \alpha_H\mu_N\lambda_N \end{pmatrix} = \mu_N\lambda_N a, \quad \widetilde{\Lambda} = \widetilde{\Omega}_\ell'\widetilde{\lambda} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_H \end{pmatrix} = a.$$

Notice that $\mu_N\lambda_N = \sum_{h\in\mathscr{H}}\Lambda_h$, $\lambda_k = \left(\prod_{j=i}^{k-1}\mu_j\right)\lambda_i$, and $\delta_h = \mu_N\lambda_N = \prod_{i\in\mathscr{N}}\mu_i\ \forall h\in\mathscr{H}$. Additionally nominal GDP is given by

$$GDP = \mathbb{1}_N'\left(I_N - diag(\mu)\, diag(\omega_x)\right) S = \sum_{i=1}^{N-1}(1-\mu_i)\, S_i + S_N = S_1 = \sum_{h\in\mathscr{H}} E_h.$$

4.1 Allocative Neutrality in a Vertical Economy - Part 4 of Theorem 6

Distortion centralities are symmetric for all households $\delta_h = \sum_{h\in\mathscr{H}}\Lambda_h = \Lambda$. From equation (111)

$$d\log TFP = \sum_{i\in\mathscr{H}} d\log A_i + \sum_{i\in\mathscr{H}} d\log\mu_i - d\log\Lambda.$$

Because $\Lambda = \prod_{i\in\mathscr{N}} \mu_i$, $d\,log\,\Lambda = \sum_{i\in\mathscr{H}} d\,log\,\mu_i$, and

$$d\,log\,TFP = \sum_{i\in\mathscr{H}} d\,log\,A_i.$$

There is allocative neutrality in response productivity, markdown, and demographic shocks.

5 Proof for Proposition 4 - Numeraire non-neutrality

To illustrate how the numeraire choice is non-neutral in an environment with elastic labor supply, let me assume as in section (2) that labor supply has a substitution and an income effect, i.e

$$\widehat{L}_h = \zeta_h^n \widehat{n}_h + \zeta_h^w \widehat{w}_h - \zeta_h^e \widehat{E}_h.$$

After adding and subtracting $\zeta_h^w \widehat{L}_H$ and $\frac{\zeta_h^w - \zeta_h^e}{1+\zeta_h^w}\widehat{GDP}$ this can be represented by

$$\widehat{L}_h = \frac{\zeta_h^n}{1+\zeta_h^w}\widehat{n}_h + \frac{\zeta_h^w}{1+\zeta_h^w}\widehat{\Lambda}_h - \frac{\zeta_h^e}{1+\zeta_h^w}\widehat{\chi}_h + \frac{\zeta_h^w - \zeta_h^e}{1+\zeta_h^w}\widehat{GDP}.$$

Introducting this expression in equations (104) and (111):

1. With nominal GDP as the numeraire

$$\widehat{L}_h = \frac{\zeta_h^n}{1+\zeta_h^w}\widehat{n}_h + \frac{\zeta_h^w}{1+\zeta_h^w}\widehat{\Lambda}_h - \frac{\zeta_h^e}{1+\zeta_h^w}\widehat{\chi}_h.$$

   This implies that

$$\bullet\ \widehat{Y}|_{\widehat{GDP}=0} = \sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\zeta_h^n \widehat{n}_h - \widehat{\Lambda}_h - \zeta_h^e \widehat{\chi}_h}{1+\zeta_h^w};$$

$$\bullet\ \widehat{C}_h|_{\widehat{GDP}=0} = \sum_{i\in\mathscr{N}} \widetilde{\mathscr{B}}_{hi} \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \widehat{\chi}_h + \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\zeta_b^n \widehat{n}_b - \widehat{\Lambda}_b - \zeta_b^e \widehat{\chi}_b}{1+\zeta_b^w}.$$

2. With real GDP as the numeraire

$$\widehat{L}_h = \frac{\zeta_h^n}{1+\zeta_h^w}\widehat{n}_h + \frac{\zeta_h^w}{1+\zeta_h^w}\widehat{\Lambda}_h - \frac{\zeta_h^e}{1+\zeta_h^w}\widehat{\chi}_h + \frac{\zeta_h^w - \zeta_h^e}{1+\zeta_h^w}\widehat{Y}.$$

   This implies that

$$\bullet\ \widehat{Y}|_{\widehat{P_Y}=0} = \left(\sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{1+\zeta_h^e}{1+\zeta_h^w}\right)^{-1} \left(\sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\zeta_h^n \widehat{n}_h - \widehat{\Lambda}_h - \zeta_h^e \widehat{\chi}_h}{1+\zeta_h^w}\right);$$

$$\bullet\ \widehat{C}_h|_{\widehat{P_Y}=0} = \sum_{i\in\mathscr{N}} \left(\widetilde{\mathscr{B}}_{hi} + \widetilde{\lambda}_i \frac{\sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\zeta_b^w - \zeta_b^e}{1+\zeta_b^w}}{\sum_{b\in\mathscr{H}} \widetilde{\Lambda}_b \frac{1+\zeta_b^e}{1+\zeta_b^w}}\right) \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \widehat{\chi}_h + \sum_{b\in\mathscr{H}} \left(\widetilde{\mathscr{C}}_{hb} + \widetilde{\Lambda}_b \frac{\sum_{i\in\mathscr{H}} \widetilde{\mathscr{C}}_{hi} \frac{\zeta_i^w - \zeta_i^c}{1+\zeta_i^w}}{\sum_{i\in\mathscr{H}} \widetilde{\Lambda}_i \frac{1+\zeta_i^c}{1+\zeta_i^w}}\right) \frac{\zeta_b^n \widehat{n}_b - \widehat{\Lambda}_b - \zeta_b^e \widehat{\chi}_b}{1+\zeta_b^w}.$$

Under the assumption that $\zeta_h^w = \zeta_h^e = \zeta_h\ \forall h \in \mathscr{H}$

$$\bullet\ \widehat{Y}|_{\widehat{P_Y}=0} = \left(\sum_{i\in\mathscr{N}} \widetilde{\lambda}_i \left(\widehat{\mathscr{A}}_i + \widehat{\mu}_i\right) + \sum_{h\in\mathscr{H}} \widetilde{\Lambda}_h \frac{\zeta_h^n \widehat{n}_h - \widehat{\Lambda}_h - \zeta_h^e \widehat{\chi}_h}{1+\zeta_h^w}\right) = \widehat{Y}|_{\widehat{GDP}=0};$$

- $\widehat{C}_h|_{\widehat{P_Y}=0} = \sum_{i\in\mathcal{N}} \widetilde{\mathscr{B}}_{hi}\left(\widehat{\mathcal{A}}_i + \widehat{\mu}_i\right) + \widehat{\chi}_h + \sum_{b\in\mathscr{H}} \widetilde{\mathscr{C}}_{hb} \frac{\zeta_b^n \widehat{n}_b - \widehat{\Lambda}_b - \zeta_b^e \widehat{\chi}_b}{1+\zeta_b^w} = \widehat{C}_h|_{\widehat{GDP}=0}.$