Geometric Foundations of Gravity 2021

28 June 2021
Tartu, Estonia

Gravitational wave solutions in quadratic metric-affine gravity

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Structure of this presentation

A. Jimenez-Cano, Yu. N. Obukhov [AJC, Obukhov 2021]
Gravitational waves in metric-affine gravity theory.
Physical Review D **103**, 024018 (2021)
DOI: 10.1103/PhysRevD.103.024018 arXiv: 2010.14528 [gr-qc]

1. Basic objects in metric-affine theories

Geometric structures on the spacetime

- *Metric tensor:* $g_{\mu\nu}$
  - Measuring (length, volume...) // time vs space, light cones, causality // notion of scale
- *Connection:* $\Gamma_{\mu\nu}{}^{\rho}$
  - Notion of parallel in $\mathcal{M}$ // Covariant derivative ∇_μ

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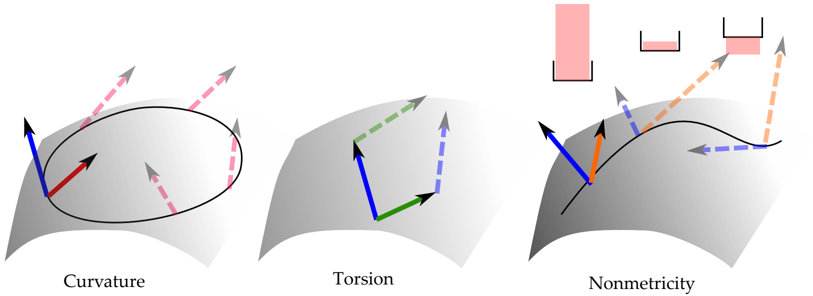
Associated geometric objects

Curvature: $$R_{\mu\nu\lambda}{}^{\rho} := \partial_\mu \Gamma_{\nu\lambda}{}^{\rho} - \partial_\nu \Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\lambda}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\lambda}{}^{\sigma}\,, \tag{1.1}$$

Torsion: $$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}\,, \tag{1.2}$$

Nonmetricity: $$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}\,. \tag{1.3}$$



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$$\text{Torsion:} \quad T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho} \,, \tag{1.5}$$

$$\text{Nonmetricity:} \quad Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho} \,. \tag{1.6}$$

Metric-affine: beyond Levi-Civita

- *Levi-Civita connection.* The only one with $T_{\mu\nu}{}^{\rho} = 0 = Q_{\mu\nu\rho}$ for a given metric:

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) . \tag{1.7}$$

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> **Metric-affine theories**
> Instead of choosing $\mathring{\Gamma}$, they consider the metric and the (general) connection as independent fields.

Metric-affine geometry can be constructed via a gauge procedure with the affine group ($\mathrm{Aff}(4,\mathbb{R}) = \mathrm{Tr}_4 \rtimes \mathrm{GL}(4,\mathbb{R})$) as structure group. [Hehl, McCrea, Mielke, Ne'eman 1995]

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Three fundamental objects:

- **Coframe**. Arbitrary basis of the cotangent space pointwise smooth:

$$\boldsymbol{e}_a = e^{\mu}{}_a \boldsymbol{\partial}_\mu \,, \qquad \boxed{\boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu} \qquad [\boldsymbol{\vartheta}^a\left(\boldsymbol{e}_b\right) = \delta^a_b \quad \Leftrightarrow \quad e_\mu{}^a e^\mu{}_b = \delta^a_b] \,. \tag{1.8}$$

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- **(Anholonomic) metric**. Components of the metric in the arbitrary basis:

$$\boxed{g_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}} \,. \tag{1.9}$$

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- **Connection 1-form**

$$\boxed{\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu}\,. \tag{1.10}$$

where $\omega_{\mu a}{}^b$ are the components of the affine connection in the anholonomic basis:

$$\omega_{\mu a}{}^b = e^\nu{}_a e_\lambda{}^b \Gamma_{\mu\nu}{}^\lambda + e_\sigma{}^b \partial_\mu e^\sigma{}_a\,. \tag{1.11}$$

N.B. $\Gamma_{\mu\nu}{}^\lambda$ and $\omega_{\mu a}{}^b$ contain the same information (for a given frame/coframe).

Connection 1-form

$$\boxed{\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu}\,. \tag{1.12}$$

- Exterior covariant derivative (of algebra-valued forms)

$$\mathbf{D}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} := \mathrm{d}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} + \quad \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\alpha}_{a\ldots}{}^{c\ldots} + \ldots \quad - \boldsymbol{\omega}_a{}^c \wedge \boldsymbol{\alpha}_{c\ldots}{}^{b\ldots} - \ldots \quad , \tag{1.13}$$

- Curvature, torsion and non-metricity forms:

$$\boldsymbol{R}_a{}^b := \mathrm{d}\boldsymbol{\omega}_a{}^b + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\omega}_a{}^c = \frac{1}{2} R_{\mu\nu a}{}^b \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.14}$$

$$\boldsymbol{T}^a := \mathbf{D}\boldsymbol{\vartheta}^a = \frac{1}{2} T_{\mu\nu}{}^a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.15}$$

$$\boldsymbol{Q}_{ab} := -\mathbf{D} g_{ab} = Q_{\mu ab} \mathrm{d}x^\mu\,. \tag{1.16}$$

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They can be decomposed according to irreps of the pseudo-orthogonal group:

$$\boldsymbol{T}^a = \underbrace{{}^{(1)}\boldsymbol{T}^a}_{\text{tensor}} + \underbrace{{}^{(2)}\boldsymbol{T}^a}_{\text{trace}} + \underbrace{{}^{(3)}\boldsymbol{T}^a}_{\text{axial}}, \qquad \boldsymbol{Q}_{ab} = \underbrace{{}^{(1)}\boldsymbol{Q}_{ab}}_{\text{tot. symm}} + \underbrace{{}^{(2)}\boldsymbol{Q}_{ab}}_{\text{tens.}} + \underbrace{{}^{(3)}\boldsymbol{Q}_{ab} + {}^{(4)}\boldsymbol{Q}_{ab}}_{\text{traces}}$$

$$\boldsymbol{R}^{ab} = \boldsymbol{W}^{[ab]} + \boldsymbol{Z}^{(ab)} \quad \Rightarrow \left\{ \begin{array}{ll} \boldsymbol{W}^{ab} & = {}^{(1)}\boldsymbol{W}^{ab} + {}^{(2)}\boldsymbol{W}^{ab} + \underbrace{{}^{(3)}\boldsymbol{W}^{ab}}_{\text{tot. antis.}} + {}^{(4)}\boldsymbol{W}^{ab} + {}^{(5)}\boldsymbol{W}^{ab} + \underbrace{{}^{(6)}\boldsymbol{W}^{ab}}_{\text{Ric. scalar}} \\ \boldsymbol{Z}^{ab} & = {}^{(1)}\boldsymbol{Z}^{ab} + {}^{(2)}\boldsymbol{Z}^{ab} + {}^{(3)}\boldsymbol{Z}^{ab} + {}^{(4)}\boldsymbol{Z}^{ab} + \underbrace{{}^{(5)}\boldsymbol{Z}^{ab}}_{\text{trace}} \end{array} \right.$$

2. Gravitational waves in (quadratic) MAG

(Quadratic) MAG Lagrangian

The most general one containing linear and quadratic invariants of $\boldsymbol{Q}_{ab}$, $\boldsymbol{T}^a$ and $\boldsymbol{R}_a{}^b$:

$$\begin{aligned}
\boldsymbol{L} = \frac{1}{2\kappa}\Big\{ & a_0 \star(\boldsymbol{\vartheta}_a \wedge \boldsymbol{\vartheta}_b) \wedge \boldsymbol{R}^{ab} - \boldsymbol{T}^a \wedge \sum_{I=1}^{3} a_I \star({}^{(I)}\boldsymbol{T}_a) && \sim R + TT \\
& - \boldsymbol{Q}_{ab} \wedge \sum_{I=1}^{4} b_I \star({}^{(I)}\boldsymbol{Q}^{ab}) - 2b_5({}^{(3)}\boldsymbol{Q}_{ac} \wedge \boldsymbol{\vartheta}^a) \wedge \star({}^{(4)}\boldsymbol{Q}^{bc} \wedge \boldsymbol{\vartheta}_b) && \sim QQ \\
& - 2\boldsymbol{\vartheta}^a \wedge \star\boldsymbol{T}^b \wedge \sum_{I=1}^{3} c_I\, {}^{(I+1)}\boldsymbol{Q}_{ab}\Big\} && \sim QT \\
- \frac{\ell_\rho^2}{2\kappa}\boldsymbol{R}^{ab} \wedge \star\Big[& \sum_{I=1}^{6} w_I\, {}^{(I)}\boldsymbol{W}_{ab} + v_1\, \boldsymbol{\vartheta}_a \wedge (\boldsymbol{e}_c \lrcorner {}^{(5)}\boldsymbol{W}^c{}_b) && \sim RR \\
& + \sum_{I=1}^{5} z_I\, {}^{(I)}\boldsymbol{Z}_{ab} + v_2\, \boldsymbol{\vartheta}_c \wedge (\boldsymbol{e}_a \lrcorner {}^{(2)}\boldsymbol{Z}^c{}_b) + \sum_{I=3}^{5} v_I\, \boldsymbol{\vartheta}_a \wedge (\boldsymbol{e}_c \lrcorner {}^{(I)}\boldsymbol{Z}^c{}_b)\Big]. && \sim RR \qquad (2.1)
\end{aligned}$$

(neither the cosmological constant term nor the odd parity invariants)

- κ and ℓ_ρ are the gravitational couplings.
- Term with a_0 is the metric-affine version of the Einstein term.
- This Lagrangian has in total a_I (3) $+b_I$ (5) $+c_I$ (3) $+w_I$ (6) $+z_I$ (5) $+v_I$ (5) $= 27$ parameters.

Ansatz for the geometry: basic fields

- We consider a line element (metric $g_{\mu\nu}$) of the Brinkmann type:

$$\mathrm{d}s^2 = \mathrm{d}\sigma\mathrm{d}\rho + U\mathrm{d}\sigma^2 - \underbrace{\delta_{AB}\mathrm{d}x^A\mathrm{d}x^B}_{\text{transversal 2D space}} \,. \tag{2.2}$$

where $U = U(\sigma, x^A)$. We introduce

$$\boxed{\boldsymbol{k} := \mathrm{d}\sigma = \boldsymbol{\vartheta}^{\hat{0}} - \boldsymbol{\vartheta}^{\hat{1}}} \qquad \text{(wave 1-form)} \qquad \rightarrow \qquad \text{dual to } \partial_\rho = k^\mu\partial_\mu \quad \text{(Killing v.).} \tag{2.3}$$

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We fix the orthonormal gauge:

- **(Anholonomic) metric**: $g_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ (Minkowski metric).
- **Coframe**

$$\boldsymbol{\vartheta}^{\widehat{0}} = \frac{1}{2}(U+1)\mathrm{d}\sigma + \frac{1}{2}\mathrm{d}\rho, \tag{2.4}$$

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$$\boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c, \tag{2.7}$$

where W_a, V_a and u_a depend on σ and x^A and are transversal:

$$W^a = \delta^a_A W^A(\sigma, x^B), \quad V^a = \delta^a_A V^A(\sigma, x^B), \quad u_a = \delta^A_a u_A(\sigma, x^B), \qquad A = 2, 3. \tag{2.8}$$

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$$ds^2 = \mathrm{d}\sigma\mathrm{d}\rho + U\mathrm{d}\sigma^2 - \underbrace{\delta_{AB}\mathrm{d}x^A\mathrm{d}x^B}_{\text{transversal 2D space}} . \tag{2.2}$$

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Unknowns: Wave's profile determined by 7 variables: U, W^A, V^A, and u_A.

Ansatz :

$$\{\boldsymbol{\vartheta}^{\widehat{0}},\boldsymbol{\vartheta}^{\widehat{1}},\boldsymbol{\vartheta}^{\widehat{A}}\} = \{\tfrac{1}{2}(U+1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \tfrac{1}{2}(U-1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \mathrm{d}x^A\},$$

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1), \qquad \boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c,$$

- Torsion

$$\boldsymbol{T}^a = -\boldsymbol{k}\wedge k^a\left[\tfrac{1}{2}\partial_A U - \delta_{AB}W^B + u_A\right]\boldsymbol{\vartheta}^A \qquad = {}^{(1)}\boldsymbol{T}^a \qquad \left(\underbrace{{}^{(2)}\boldsymbol{T}^a}_{\text{trace}} = \underbrace{{}^{(3)}\boldsymbol{T}^a}_{\text{axial}} = 0\right). \tag{2.9}$$

- Nonmetricity

$$\boldsymbol{Q}_{ab} = -2\boldsymbol{k}\, k_{(a}(W_{b)} + V_{b)}) + 2k_a k_b u_A \boldsymbol{\vartheta}^A \qquad = {}^{(1)}\boldsymbol{Q}_{ab} + {}^{(2)}\boldsymbol{Q}_{ab} \qquad \left(\underbrace{{}^{(3)}\boldsymbol{Q}_{ab} = {}^{(4)}\boldsymbol{Q}_{ab}}_{\text{traces}} = 0\right). \tag{2.10}$$

- Curvature

$$\boldsymbol{R}_a{}^b = \boldsymbol{k}\wedge(k_a \underline{\mathrm{d}} V^b + k^b \underline{\mathrm{d}} W_a) + k_a k^b \mathrm{d}(u_A \boldsymbol{\vartheta}^A) \qquad (\underline{\mathrm{d}} := \boldsymbol{\vartheta}^A \boldsymbol{e}_A \lrcorner \mathrm{d} = \mathrm{d}x^A \partial_A) \tag{2.11}$$

$$\Leftrightarrow \begin{cases} \boldsymbol{W}^{ab} = \boldsymbol{k}\wedge k^{[b}\underline{\mathrm{d}}(W^{a]} - V^{a]}) & = \ {}^{(1)}\boldsymbol{W}^{ab} + {}^{(2)}\boldsymbol{W}^{ab} + {}^{(4)}\boldsymbol{W}^{ab} \\ & \quad \left({}^{(3)}\boldsymbol{W}^{ab} = {}^{(5)}\boldsymbol{W}^{ab} = {}^{(6)}\boldsymbol{W}^{ab} = 0\right) \\ \boldsymbol{Z}^{ab} = \boldsymbol{k}\wedge k^{(b}\underline{\mathrm{d}}(W^{a)} + V^{a)}) + k^a k^b \mathrm{d}(u_A\boldsymbol{\vartheta}^A) & = \ {}^{(1)}\boldsymbol{Z}^{ab} + {}^{(2)}\boldsymbol{Z}^{ab} + {}^{(4)}\boldsymbol{Z}^{ab} \\ & \quad \left({}^{(3)}\boldsymbol{Z}^{ab} = {}^{(5)}\boldsymbol{Z}^{ab} = 0\right) \end{cases} \tag{2.12}$$

3. Field equations and solutions

$$[\text{EoM } \boldsymbol{\vartheta}^a] \qquad 0 = -\frac{a_1}{2}\Delta U + \left[\frac{a_0}{2} + a_1 - c_1\right]\partial_A W^A - \left[\frac{a_0}{2} + c_1\right]\partial_A V^A - (a_1 - 2c_1)\partial_A \underline{u}^A\,, \tag{3.1}$$

$$\begin{aligned}
[\text{EoM } \boldsymbol{\omega}_{[ab]}] \qquad 0 =& \frac{a_0 + a_1}{2}\partial_A U + \left[-\frac{a_0}{2} - a_1 + c_1\right]\underline{W}_A + \left[\frac{a_0}{2} + c_1\right]\underline{V}_A + (a_1 - 2c_1)u_A \\
&- \frac{\ell_\rho^2}{4}\left[2w_1\Delta \underline{W}_A - 2w_1\Delta \underline{V}_A + (v_4 + 2w_4)\partial_A\partial_B W^B + (v_4 - 2w_4)\partial_A\partial_B V^B\right] \\
&- \frac{\ell_\rho^2}{4}\epsilon_{AB}\epsilon^{CD}\underline{\partial}^B\left[(v_2 - 2w_2)\partial_C \underline{W}_D + (v_2 + 2w_2)\partial_C \underline{V}_D - 2v_2\partial_C u_D\right].
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
[\text{EoM } \boldsymbol{\omega}_{(ab)}] \qquad 0 =& \frac{a_1 - 2c_1}{2}\partial_A U + \left[\frac{a_0}{2} - a_1 - \frac{4(2b_1 + b_2)}{3} + 3c_1\right]\underline{W}_A \\
&+ \left[\frac{a_0}{2} - \frac{4(2b_1 + b_2)}{3} + c_1\right]\underline{V}_A + \left[a_0 + a_1 - 4c_1 - \frac{8(b_1 - b_2)}{3}\right]u_A \\
&- \frac{\ell_\rho^2}{4}\left[2z_1\Delta \underline{W}_A + 2z_1\Delta \underline{V}_A + (z_1 + z_4 + 3v_4)\partial_A\partial_B W^B + (z_1 + z_4 + v_4)\partial_A\partial_B V^B\right] \\
&- \frac{\ell_\rho^2}{4}\epsilon_{AB}\epsilon^{CD}\underline{\partial}^B\left[(2v_2 - z_1 - z_2)\partial_C \underline{W}_D - (z_1 + z_2)\partial_C \underline{V}_D - 2(z_1 - z_2 + v_2)\partial_C u_D\right].
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
0 =& \frac{2c_1 - a_1}{2}\partial_A U + \left[\frac{a_0}{2} + a_1 - \frac{4(b_1 - b_2)}{3} - 3c_1\right]\underline{W}_A \\
&+ \left[\frac{a_0}{2} - \frac{4(b_1 - b_2)}{3} - c_1\right]\underline{V}_A + \left[4c_1 - a_1 - \frac{4(b_1 + 2b_2)}{3}\right]u_A \\
&+ \frac{\ell_\rho^2}{4}\epsilon_{AB}\epsilon^{CD}\underline{\partial}^B\left[(z_1 - z_2 + 2v_2)\partial_C \underline{W}_D + (z_1 - z_2)\partial_C \underline{V}_D + 2(z_1 + z_2 - v_2)\partial_C u_D\right].
\end{aligned} \tag{3.4}$$

$$0 = \partial_\sigma\left[(z_4 - z_1 + 3v_4)\partial_A W^A + (z_4 - z_1 + v_4)\partial_A V^A - 4z_1\partial_A \underline{u}^A\right]. \tag{3.5}$$

$$[\text{EoM } \boldsymbol{\vartheta}^a] \qquad 0 = (...)\Delta U + (...)\partial_A W^A + (...)\partial_A V^A + (...)\partial_A \underline{u}^A\,, \tag{3.6}$$

$$\begin{aligned}[\text{EoM } \boldsymbol{\omega}_{[ab]}] \qquad 0 =& (...)\partial_A U + (...)\underline{W}_A + (...)\underline{V}_A + (...)u_A \\ &- \frac{\ell_\rho^2}{4}\Big[(...)\Delta \underline{W}_A + (...)\Delta \underline{V}_A + (...)\partial_A\partial_B W^B + (...)\partial_A\partial_B V^B\Big] \\ &- \frac{\ell_\rho^2}{4}\epsilon_{AB}\underline{\partial}^B\left\{\epsilon^{CD}\left[(...)\partial_{[C}\underline{W}_{D]} + (...)\partial_{[C}\underline{V}_{D]} + (...)\partial_{[C}u_{D]}\right]\right\}\,.\end{aligned} \tag{3.7}$$

$$[\text{EoM } \boldsymbol{\omega}_{(ab)}] \qquad 0 = (\text{same structure as the previous one}), \tag{3.8}$$

$$0 = (\text{same structure as the previous one without the second line}), \tag{3.9}$$

$$0 = \partial_\sigma\left[(...)\partial_A W^A + (...)\partial_A V^A + (...)\partial_A \underline{u}^A\right]\,. \tag{3.10}$$

where

$$\underline{W}_A := \delta_{AB}W^B, \qquad \underline{V}_A := \delta_{AB}V^B, \qquad \underline{u}^A := \delta^{AB}u_B, \qquad \underline{\partial}^A := \delta^{AB}\partial_B, \qquad \Delta := \delta^{AB}\partial_A\partial_B \tag{3.11}$$

and ϵ_{AB}, ϵ^{CD} correspond to the 2-dimensional Levi-Civita symbol (convention: $\epsilon_{23} := 1$, $\epsilon^{23} := 1$).

- **Step 1. Potential + copotential decomposition**

$$W^A =: \frac{1}{2}\Big(\delta^{AB}\partial_B \mathcal{W} + \epsilon^{AB}\partial_B \overline{\mathcal{W}}\Big), \tag{3.12}$$

$$V^A =: \frac{1}{2}\Big(\delta^{AB}\partial_B \mathcal{V} + \epsilon^{AB}\partial_B \overline{\mathcal{V}}\Big), \tag{3.13}$$

$$u_A =: \frac{1}{2}\Big(\partial_A \mathcal{U} + \epsilon_{AB}\delta^{BC}\partial_C \overline{\mathcal{U}}\Big). \tag{3.14}$$

- The seven variables are then $\{U, \mathcal{W}, \mathcal{V}, \mathcal{U}, \overline{\mathcal{W}}, \overline{\mathcal{V}}, \overline{\mathcal{U}}\}$.
- Useful, because

$$F^A = \frac{1}{2}\Big(\delta^{AB}\partial_B \mathcal{F} + \epsilon^{AB}\partial_B \overline{\mathcal{F}}\Big) \qquad \Rightarrow \qquad \boxed{\partial_A F^A = \frac{1}{2}\Delta\mathcal{F} \quad \text{and} \quad \epsilon_{AB}\delta^{BC}\partial_C F^A = \frac{1}{2}\Delta\overline{\mathcal{F}}}. \tag{3.15}$$

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❒ Step 2. Potential + copotential splitting of the equations

[Blagojević, Cvetković, Obukhov 2017]

$$0 = \mathbf{E}^A \equiv \delta^{AB}\partial_B\mathbf{E} + \epsilon^{AB}\partial_B\overline{\mathbf{E}} \quad \Leftrightarrow \quad \begin{cases} \mathbf{E} = f & (\Delta f = 0) \\ \overline{\mathbf{E}} = \overline{f} & (\Delta\overline{f} = 0) \end{cases} \quad \overset{\text{redef.}}{\Rightarrow} \quad \begin{cases} \mathbf{E} = 0 \\ \overline{\mathbf{E}} = 0 \end{cases} \tag{3.16}$$

❒ Step 1. Potential + copotential decomposition

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- $\{U, \mathcal{W}, \mathcal{V}, \mathcal{U}\}$ and $\{\overline{\mathcal{W}}, \overline{\mathcal{V}}, \overline{\mathcal{U}}\}$ decoupled (!!)

Step 3. Convenient change of variables

$$\boxed{\begin{aligned} \mathcal{X}_0 &= \mathcal{W} - \mathcal{V} \\ \mathcal{X}_1 &= U - \mathcal{W} + \mathcal{U}, \\ \mathcal{X}_2 &= \mathcal{W} + \mathcal{V} + \mathcal{U}, \\ \mathcal{X}_3 &= \mathcal{W} + \mathcal{V} - 2\mathcal{U}, \end{aligned}} \quad \Leftrightarrow \quad \begin{aligned} U &= \tfrac{1}{2}\mathcal{X}_0 + \mathcal{X}_1 + \tfrac{1}{2}\mathcal{X}_3, \\ \mathcal{W} &= \tfrac{1}{2}\mathcal{X}_0 + \tfrac{1}{3}\mathcal{X}_2 + \tfrac{1}{6}\mathcal{X}_3, \\ \mathcal{V} &= -\tfrac{1}{2}\mathcal{X}_0 + \tfrac{1}{3}\mathcal{X}_2 + \tfrac{1}{6}\mathcal{X}_3, \\ \mathcal{U} &= \tfrac{1}{3}\mathcal{X}_2 - \tfrac{1}{3}\mathcal{X}_3, \end{aligned} \tag{3.17}$$

$$\boxed{\begin{aligned} \overline{\mathcal{X}}_1 &= -\overline{\mathcal{W}} + \overline{\mathcal{U}}, \\ \overline{\mathcal{X}}_2 &= \overline{\mathcal{W}} + \overline{\mathcal{V}} + \overline{\mathcal{U}}, \\ \overline{\mathcal{X}}_3 &= \overline{\mathcal{W}} + \overline{\mathcal{V}} - 2\overline{\mathcal{U}}. \end{aligned}} \quad \Leftrightarrow \quad \begin{aligned} \overline{\mathcal{W}} &= -\overline{\mathcal{X}}_1 + \tfrac{1}{3}\overline{\mathcal{X}}_2 - \tfrac{1}{3}\overline{\mathcal{X}}_3, \\ \overline{\mathcal{V}} &= \overline{\mathcal{X}}_1 + \tfrac{1}{3}\overline{\mathcal{X}}_2 + \tfrac{2}{3}\overline{\mathcal{X}}_3, \\ \overline{\mathcal{U}} &= \tfrac{1}{3}\overline{\mathcal{X}}_2 - \tfrac{1}{3}\overline{\mathcal{X}}_3. \end{aligned} \tag{3.18}$$

ODD SECTOR

- By combining the equations: $\overline{\mathcal{X}}_2$ is decoupled:

$$(a_0 - 4b_1)\,\overline{\mathcal{X}}_2 - \ell_\rho^2\, z_1\, \Delta\,\overline{\mathcal{X}}_2 = 0, \tag{3.19}$$

whereas $\overline{\mathcal{X}}_1$ and $\overline{\mathcal{X}}_3$ verify

$$\begin{aligned}(a_0 + 2c_1)\,\overline{\mathcal{X}}_1 + \frac{2}{3}(a_0 + 2b_2)\,\overline{\mathcal{X}}_3 - \frac{\ell_\rho^2}{4}\Big\{&-2\,[2w_1 + 2w_2 + v_2]\,\Delta\,\overline{\mathcal{X}}_1 \\ &-\left[2w_1 + 2w_2 + v_2 + \frac{1}{3}\,(z_1 + 3z_2)\right]\Delta\,\overline{\mathcal{X}}_3\Big\} = 0,\end{aligned} \tag{3.20}$$

$$(a_0 + a_1)\,\overline{\mathcal{X}}_1 + \left(\frac{a_0}{2} + c_1\right)\overline{\mathcal{X}}_3 - \frac{\ell_\rho^2}{4}\left[-4(w_1 + w_2)\,\Delta\,\overline{\mathcal{X}}_1 - (2w_1 + 2w_2 + v_2)\Delta\overline{\mathcal{X}}_3\right] = 0 \tag{3.21}$$

ODD SECTOR

- If we take $\overline{\mathcal{X}}_I = \overline{\mathcal{X}}_I^{(0)}(\sigma)\mathrm{e}^{\mathrm{i}\overline{q}_A x^A}$:

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- If we take $\overline{\mathcal{X}}_I = \overline{\mathcal{X}}_I^{(0)}(\sigma)\mathrm{e}^{\mathrm{i}\overline{q}_A x^A}$:

$$\begin{pmatrix} 0 & a_0 - 4b_1 + 4z_1\overline{\mathcal{Q}}^2 & 0 \\ a_0 + 2c_1 - 2\overline{\mathcal{Q}}^2\Lambda_2 & 0 & \frac{2}{3}(a_0 + 2b_2) - \overline{\mathcal{Q}}^2(\Lambda_2 + \Lambda_3) \\ a_0 + a_1 - \overline{\mathcal{Q}}^2\Lambda_1 & 0 & \frac{a_0}{2} + c_1 - \overline{\mathcal{Q}}^2\Lambda_2 \end{pmatrix} \begin{pmatrix} \overline{\mathcal{X}}_1^{(0)} \\ \overline{\mathcal{X}}_2^{(0)} \\ \overline{\mathcal{X}}_3^{(0)} \end{pmatrix} = 0. \qquad (3.22)$$

Abbreviations:

$$\overline{\mathcal{Q}}^2 := \frac{\ell_\rho^2}{4}\overline{q}_A\overline{q}_B\delta^{AB}, \qquad \Lambda_1 := 4(w_1+w_2), \qquad \Lambda_2 := 2(w_1+w_2)+v_2, \qquad \Lambda_3 := \frac{1}{3}(z_1+3z_2)\,. \quad (3.23)$$

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Abbreviations:

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- The three modes propagate if

$$a_0 - 4b_1 + 4z_1\overline{\mathcal{Q}}^2 = 0, \qquad \begin{vmatrix} a_0 + 2c_1 - 2\overline{\mathcal{Q}}^2\Lambda_2 & \frac{2}{3}(a_0+2b_2) - \overline{\mathcal{Q}}^2(\Lambda_2+\Lambda_3) \\ a_0 + a_1 - \overline{\mathcal{Q}}^2\Lambda_1 & \frac{a_0}{2} + c_1 - \overline{\mathcal{Q}}^2\Lambda_2 \end{vmatrix} = 0, \quad (3.24)$$

- The amplitudes $\overline{\mathcal{X}}_I^{(0)}(\sigma)$ are arbitrary functions of σ.

EVEN SECTOR

- The equations are

$$
\begin{aligned}
(2c_1 - a_1)\,\mathcal{X}_1 + \frac{1}{3}(a_0 - 4b_1)\,\mathcal{X}_2 + \left[-\frac{a_0}{2} - c_1 + \frac{2}{3}(a_0 + 2b_2)\right]\mathcal{X}_3 &= 0, && (3.25)\\
(a_0 + a_1)\,\mathcal{X}_1 + \left(\frac{a_0}{2} + c_1\right)\mathcal{X}_3 - \frac{\ell_\rho^2}{4}\left[2(w_1 + w_4)\,\Delta\,\mathcal{X}_0 + \frac{2}{3}\,v_4\,\Delta\,\mathcal{X}_2 + \frac{1}{3}\,v_4\,\Delta\,\mathcal{X}_3\right] &= 0, && (3.26)\\
(a_1 - 2c_1)\,\mathcal{X}_1 + \frac{2}{3}(a_0 - 4b_1)\,\mathcal{X}_2 + \left[\frac{a_0}{2} + c_1 - \frac{2}{3}(a_0 + 2b_2)\right]\mathcal{X}_3 & \\
-\frac{\ell_\rho^2}{4}\left[v_4\,\Delta\,\mathcal{X}_0 + \frac{2}{3}\,(3z_1 + z_4 + 2v_4)\,\Delta\,\mathcal{X}_2 + \frac{1}{3}\,(3z_1 + z_4 + 2v_4)\,\Delta\,\mathcal{X}_3\right] &= 0, && (3.27)\\
\frac{a_0}{2}\,\Delta\,\mathcal{X}_0 - a_1\,\Delta\,\mathcal{X}_1 - c_1\,\Delta\,\mathcal{X}_3 &= 0, && (3.28)\\
\partial_\sigma\left\{v_4\,\Delta\,\mathcal{X}_0 + \frac{2}{3}(z_4 - 3z_1 + 2v_4)\,\Delta\,\mathcal{X}_2 + \frac{1}{3}(z_4 + 3z_1 + 2v_4)\,\Delta\,\mathcal{X}_3\right\} &= 0. && (3.29)
\end{aligned}
$$

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- **First four equations**

$$\begin{pmatrix} -\frac{a_0}{2} & a_1 & 0 & c_1 \\ 0 & 2c_1 - a_1 & \frac{1}{3}(a_0 - 4b_1) & -\frac{a_0}{2} - c_1 + \frac{2}{3}(a_0 + 2b_2) \\ 2(w_1 + w_4)\mathcal{Q}^2 & a_0 + a_1 & \frac{2}{3}v_4\mathcal{Q}^2 & \frac{a_0}{2} + c_1 + \frac{1}{3}v_4\mathcal{Q}^2 \\ \mathcal{Q}^2 v_4 & 0 & a_0 - 4b_1 + \frac{2}{3}\mathcal{Q}^2\Lambda_0 & \frac{1}{3}\mathcal{Q}^2\Lambda_0 \end{pmatrix} \begin{pmatrix} \mathcal{X}_0^{(0)} \\ \mathcal{X}_1^{(0)} \\ \mathcal{X}_2^{(0)} \\ \mathcal{X}_3^{(0)} \end{pmatrix} = 0. \tag{3.30}$$

Abbreviations:

$$\mathcal{Q}^2 := \frac{\ell_\rho^2}{4} q_A q_B \delta^{AB}, \qquad \Lambda_0 := 3z_1 + z_4 + 2v_4\,. \tag{3.31}$$

- This is a 4×4 matrix but its determinant is a 2nd degree polynomial in $\mathcal{Q}^2$.
- Solutions for $\mathcal{Q}^2$ in PG $\rightsquigarrow$ 2 propagating massive modes. [Blagojević, Cvetković, Obukhov 2017]

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- **Last equation**. It constrains the σ dependence:

$$v_4\,\partial_\sigma\mathcal{X}_0^{(0)} + \frac{2}{3}(\Lambda_0 - 6z_1)\,\partial_\sigma\mathcal{X}_2^{(0)} + \frac{1}{3}\Lambda_0\,\partial_\sigma\mathcal{X}_3^{(0)} = 0. \tag{3.32}$$

Conditions: Nullity of torsion and nonmetricity is equivalent to

$$u_A = 0, \qquad W^A = -V^A = \frac{1}{2}\delta^{AB}\partial_B U. \tag{3.33}$$

Non-trivial equations

$$a_0 \Delta U = 0, \tag{3.34}$$
$$v_4\, \partial_\sigma \Delta U = 0, \tag{3.35}$$
$$v_4 \ell_\rho^2 \partial_A \Delta U = 0, \tag{3.36}$$
$$(w_1 + w_4)\ell_\rho^2 \partial_A \Delta\, U = 0. \tag{3.37}$$

Remarks

- The solution of GR, $\Delta U = 0$ (and Levi-Civita), is a solution of all MAG models.

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Remarks

- The solution of GR, $\Delta U = 0$ (and Levi-Civita), is a solution of all MAG models.
- For $\boldsymbol{L}$ without the RR sector ($w_I = z_I = v_I = 0$):

$$u_A = 0, \qquad W^A = -V^A = \frac{1}{2}\delta^{AB}\partial_B U, \qquad \frac{a_0}{2}\Delta U = 0 \tag{3.38}$$

is the general solution, except for very specific parameters: [Obukhov, Vlachynsky, Esser, Hehl 1997]

$$-a_1 = \frac{a_2}{2} = 2a_3 = 2c_1 = -c_2 = -c_3 = a_0, \tag{3.39}$$

$$4b_1 = 2b_2 = -8b_3 = \frac{8b_4}{3} = 2b_5 = a_0. \tag{3.40}$$

4. Summary and conclusions

Hypothesis

- Quadratic (even) metric-affine action in vacuum
- Ansatz with 7 independent functions

Results

- Method to find the general solutions $\rightsquigarrow$ potential + copotential decomp.
- Solutions for large families of MAG theories.
- Particular solutions: Riemannian (also teleparallel and pseudo-instantons. Check the paper!)

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Limitations of this work / future work

- Solutions with matter
- Non-trivial cosmological constant
- Odd parity invariants:

$$\boldsymbol{R}^{ab} \wedge \boldsymbol{R}_{ab}, \qquad {}^{(I)}\boldsymbol{T}^{a} \wedge {}^{(J)}\boldsymbol{T}_{a} \qquad \ldots$$

- Different Ansatz (Kundt metric, other non-trivial irreds for $\boldsymbol{T}^a$ and $\boldsymbol{Q}_{ab}$).

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Thanks for your attention!

Aitäh!

References used in this presentation

- F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, [Hehl, McCrea, Mielke, Ne'eman 1995]
  Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors and breaking of dilation invariance,
  Phys. Rep. **258**, 1-177 (1995).
- A. Jimenez-Cano and Yu. N. Obukhov, [AJC, Obukhov 2021]
  Gravitational waves in metric-affine gravity theory,
  Phys. Rev. D **103**, 024018 (2021).
- Yu. N. Obukhov, E. J. Vlachynsky, W. Esser, and F. W. Hehl, [Obukhov, Vlachynsky, Esser, Hehl 1997]
  Effective Einstein theory from metric-affine gravity models via irreducible decompositions,
  Phys. Rev. D **56**, 7769-7778 (1997).
- D. Vassiliev, [Vassiliev 2002]
  Pseudoinstantons in metric-affine theory,
  Gen. Rel. Grav. **34**, 1239-1265 (2002).
- M. Blagojević, B. Cvetković, and Yu. N. Obukhov, [Blagojević, Cvetković, Obukhov 2017]
  Generalized plane waves in Poincaré gauge theory of gravity,
  Phys. Rev. D **96**, 064031 (2017).

[Hehl, McCrea, Mielke, Ne'eman 1995]

- Fundamental gravitational fields in MAG: g_{ab}, $\boldsymbol{\vartheta}^a$ and $\boldsymbol{\omega}_a{}^b$.

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- Gauge symmetry $\Rightarrow$ the most general metric-affine Lagrangian D-form is:

$$\boldsymbol{L} = \boldsymbol{L}(g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{Q}_{ab}, \boldsymbol{T}^a, \boldsymbol{R}_a{}^b). \tag{5.1}$$

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- Equations of motion:

$$\begin{aligned}
\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ab}} + \frac{\partial \boldsymbol{L}}{\partial g_{ab}} &= 0 && \equiv [\mathrm{EoM}\ g_{ab}]\,, \\
\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^a} + \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a} &= 0 && \equiv [\mathrm{EoM}\ \boldsymbol{\vartheta}^a]\,, && (5.2) \\
\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b} + \boldsymbol{\vartheta}^a \wedge \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^b} + 2 g_{bc}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ac}} &= 0 && \equiv [\mathrm{EoM}\ \boldsymbol{\omega}_a{}^b]\,, && (5.3)
\end{aligned}$$

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$$\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^a} + \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a} = 0 \qquad \equiv [\text{EoM } \boldsymbol{\vartheta}^a]\,, \tag{5.2}$$

$$\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b} + \boldsymbol{\vartheta}^a \wedge \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^b} + 2g_{bc}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ac}} = 0 \qquad \equiv [\text{EoM } \boldsymbol{\omega}_a{}^b]\,, \tag{5.3}$$

- Noether identity under $\mathrm{GL}(4,\mathbb{R})_{\text{local}}$ $\Rightarrow$ EoM of g_{ab} is redundant.

[Hehl, McCrea, Mielke, Ne'eman 1995]

- Fundamental gravitational fields in MAG: g_{ab}, $\boldsymbol{\vartheta}^a$ and $\boldsymbol{\omega}_a{}^b$.
- Gauge symmetry $\Rightarrow$ the most general metric-affine Lagrangian D-form is:

$$\boldsymbol{L} = \boldsymbol{L}(g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{Q}_{ab}, \boldsymbol{T}^a, \boldsymbol{R}_a{}^b). \tag{5.1}$$

- Equations of motion:

$$\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ab}} + \frac{\partial \boldsymbol{L}}{\partial g_{ab}} = 0 \qquad \equiv [\text{EoM } g_{ab}]\,,$$

$$\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^a} + \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a} = 0 \qquad \equiv [\text{EoM } \boldsymbol{\vartheta}^a]\,, \tag{5.2}$$

$$\mathrm{D}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b} + \boldsymbol{\vartheta}^a \wedge \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^b} + 2g_{bc}\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ac}} = 0 \qquad \equiv [\text{EoM } \boldsymbol{\omega}_a{}^b]\,. \tag{5.3}$$

- Noether identity under $\mathrm{GL}(4,\mathbb{R})_{\text{local}}$ $\Rightarrow$ EoM of g_{ab} is redundant.
- Noether identity under $\mathrm{Diff}(\mathcal{M})$ $\Rightarrow$ $\dfrac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a}$ is determined by $\boldsymbol{L}$ and the momenta.

We only need to compute the three momenta:

$$\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ab}}\,, \qquad \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^a}\,, \qquad \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b}. \tag{5.4}$$

Ansatz :

$$\{\boldsymbol{\vartheta}^{\hat{0}},\boldsymbol{\vartheta}^{\hat{1}},\boldsymbol{\vartheta}^{\hat{A}}\}=\{\tfrac{1}{2}(U+1)\mathrm{d}\sigma+\tfrac{1}{2}\mathrm{d}\rho\ ,\ \ \tfrac{1}{2}(U-1)\mathrm{d}\sigma+\tfrac{1}{2}\mathrm{d}\rho\ ,\ \ \mathrm{d}x^A\},$$

$$g_{ab}=\mathrm{diag}(1,-1,-1,-1),\qquad\qquad \boldsymbol{\omega}_a{}^b=-\boldsymbol{k}\left(k_aV^b+k^bW_a\right)+k_ak^bu_c\boldsymbol{\vartheta}^c,$$

- Torsion

$$\boldsymbol{T}^a=-\boldsymbol{k}\wedge k^a\left[\tfrac{1}{2}\partial_AU-\delta_{AB}W^B+u_A\right]\boldsymbol{\vartheta}^A\qquad={}^{(1)}\boldsymbol{T}^a \tag{5.5}$$

$$\Big(\underbrace{{}^{(2)}\boldsymbol{T}^a}_{\text{trace}}=\underbrace{{}^{(3)}\boldsymbol{T}^a}_{\text{axial}}=0\Big)\,.$$

Purely irreducible.

- Nonmetricity

$$\boldsymbol{Q}_{ab}=-2\boldsymbol{k}\,k_{(a}(W_{b)}+V_{b)})+2k_ak_bu_A\boldsymbol{\vartheta}^A\qquad={}^{(1)}\boldsymbol{Q}_{ab}+{}^{(2)}\boldsymbol{Q}_{ab} \tag{5.6}$$

$$\Big(\underbrace{{}^{(3)}\boldsymbol{Q}_{ab}={}^{(4)}\boldsymbol{Q}_{ab}}_{\text{traces}}=0\Big)\,.$$

where

$${}^{(1)}\boldsymbol{Q}_{ab}=-\frac{4}{3}\boldsymbol{k}k_{(a}(W_{b)}+V_{b)})-\frac{2}{3}k_ak_b(W_c+V_c)\boldsymbol{\vartheta}^c+\frac{4}{3}\boldsymbol{k}k_{(a}u_{b)}+\frac{2}{3}k_ak_bu_A\boldsymbol{\vartheta}^A, \tag{5.7}$$

$${}^{(2)}\boldsymbol{Q}_{ab}=-\frac{2}{3}\boldsymbol{k}k_{(a}(W_{b)}+V_{b)})+\frac{2}{3}k_ak_b(W_c+V_c)\boldsymbol{\vartheta}^c-\frac{4}{3}\boldsymbol{k}k_{(a}u_{b)}+\frac{4}{3}k_ak_bu_A\boldsymbol{\vartheta}^A. \tag{5.8}$$

Ansatz :

$$\{\boldsymbol{\vartheta}^{\widehat{0}}, \boldsymbol{\vartheta}^{\widehat{1}}, \boldsymbol{\vartheta}^{\widehat{A}}\} = \{\tfrac{1}{2}(U+1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \tfrac{1}{2}(U-1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \mathrm{d}x^A\},$$

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1), \qquad \boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c,$$

- Curvature $\quad (\underline{\mathrm{d}} := \boldsymbol{\vartheta}^A \boldsymbol{e}_A \lrcorner \mathrm{d} = \mathrm{d}x^A \partial_A)$

$$\boldsymbol{R}_a{}^b = \boldsymbol{k} \wedge (k_a \underline{\mathrm{d}} V^b + k^b \underline{\mathrm{d}} W_a) + k_a k^b \mathrm{d}(u_A \boldsymbol{\vartheta}^A)\,. \tag{5.9}$$

If we introduce

$$\overset{(\pm)}{\Omega}{}^a := \underline{\mathrm{d}}(W^a \pm V^a) = \sum_{I=1,2,4} {}^{(I)}\overset{(\pm)}{\Omega}{}^a \quad , \begin{cases} {}^{(1)}\overset{(\pm)}{\Omega}{}^a & := \frac{1}{2}\left(\overset{(\pm)}{\Omega}{}^a + \boldsymbol{\vartheta}^b \boldsymbol{e}^a \lrcorner \overset{(\pm)}{\Omega}{}_b - \boldsymbol{\vartheta}^a \boldsymbol{e}_b \lrcorner \overset{(\pm)}{\Omega}{}^b\right), \\ {}^{(2)}\overset{(\pm)}{\Omega}{}^a & := \frac{1}{2}\left(\overset{(\pm)}{\Omega}{}^a - \boldsymbol{\vartheta}^b \boldsymbol{e}^a \lrcorner \overset{(\pm)}{\Omega}{}_b\right), \\ {}^{(4)}\overset{(\pm)}{\Omega}{}^a & := \frac{1}{2}\, \boldsymbol{\vartheta}^a \boldsymbol{e}_b \lrcorner \overset{(\pm)}{\Omega}{}^b. \end{cases} \tag{5.10}$$

The transversal components of these objects are, if ${}^{(I)}\overset{(\pm)}{\Omega}{}^A = {}^{(I)}\overset{(\pm)}{\Omega}{}^A{}_B\, \boldsymbol{\vartheta}^B$,

$${}^{(1)}\overset{(\pm)}{\Omega}{}^A{}_B = \frac{1}{2}\left[\partial_B(W^A \pm V^A) + \partial^A(W_B \pm V_B) - \delta^A_B\, \partial_C(W^C \pm V^C)\right], \tag{5.11}$$

$${}^{(2)}\overset{(\pm)}{\Omega}{}^A{}_B = \frac{1}{2}\left[\partial_B(W^A \pm V^A) - \partial^A(W_B \pm V_B)\right], \tag{5.12}$$

$${}^{(4)}\overset{(\pm)}{\Omega}{}^A{}_B = \frac{1}{2}\, \delta^A_B\, \partial_C(W^C \pm V^C). \tag{5.13}$$

Ansatz :

$$\{\boldsymbol{\vartheta}^{\widehat{0}},\boldsymbol{\vartheta}^{\widehat{1}},\boldsymbol{\vartheta}^{\widehat{A}}\}=\{\tfrac{1}{2}(U+1)\mathrm{d}\sigma+\tfrac{1}{2}\mathrm{d}\rho\ ,\ \tfrac{1}{2}(U-1)\mathrm{d}\sigma+\tfrac{1}{2}\mathrm{d}\rho\ ,\ \mathrm{d}x^A\},$$

$$g_{ab}=\mathrm{diag}(1,-1,-1,-1),\qquad \boldsymbol{\omega}_a{}^b=-\boldsymbol{k}\left(k_aV^b+k^bW_a\right)+k_ak^bu_c\boldsymbol{\vartheta}^c,$$

❒ Curvature

$$\boldsymbol{R}_a{}^b=\underbrace{\boldsymbol{k}\wedge k^{[b}\overset{(-)}{\Omega}{}^{a]}}_{\boldsymbol{W}^{ab}}+\underbrace{\boldsymbol{k}\wedge k^{(b}\overset{(+)}{\Omega}{}^{a)}+k_ak^b\mathrm{d}(u_A\boldsymbol{\vartheta}^A)}_{\boldsymbol{Z}^{ab}}, \tag{5.14}$$

Non-trivial irreds in terms of $\overset{(\pm)}{\Omega}{}^a$ and its irreds:

$$\begin{aligned}
{}^{(1)}\boldsymbol{W}^{ab}&=\boldsymbol{k}\wedge{}^{(1)}\overset{(-)}{\Omega}{}^{[a}k^{b]},\\
{}^{(2)}\boldsymbol{W}^{ab}&=\boldsymbol{k}\wedge{}^{(2)}\overset{(-)}{\Omega}{}^{[a}k^{b]},\\
{}^{(4)}\boldsymbol{W}^{ab}&=\boldsymbol{k}\wedge{}^{(4)}\overset{(-)}{\Omega}{}^{[a}k^{b]},\\
{}^{(1)}\boldsymbol{Z}^{ab}&=\frac{1}{2}\boldsymbol{k}\wedge{}^{(1)}\overset{(+)}{\Omega}{}^{(a}k^{b)}+\frac{1}{4}k^ak^b\,\boldsymbol{\vartheta}_c\wedge\overset{(+)}{\Omega}{}^c+\frac{1}{2}\boldsymbol{k}\wedge k^{(a}e^{b)}\lrcorner\mathrm{d}(u_A\boldsymbol{\vartheta}^A)+\frac{1}{2}k^ak^b\mathrm{d}(u_A\boldsymbol{\vartheta}^A)+\frac{1}{2}\boldsymbol{k}\wedge\overset{(+)}{\Omega}{}^{(a}k^{b)},\\
{}^{(2)}\boldsymbol{Z}^{ab}&=\frac{1}{2}\boldsymbol{k}\wedge{}^{(2)}\overset{(+)}{\Omega}{}^{(a}k^{b)}-\frac{1}{4}k^ak^b\boldsymbol{\vartheta}_c\wedge\overset{(+)}{\Omega}{}^c-\frac{1}{2}\boldsymbol{k}\wedge k^{(a}e^{b)}\lrcorner\mathrm{d}(u_A\boldsymbol{\vartheta}^A)+\frac{1}{2}k^ak^b\mathrm{d}(u_A\boldsymbol{\vartheta}^A),\\
{}^{(4)}\boldsymbol{Z}^{ab}&=\frac{1}{2}\boldsymbol{k}\wedge{}^{(4)}\overset{(+)}{\Omega}{}^{(a}k^{b)}.
\end{aligned} \tag{5.15}$$

Trivial irreds:

$${}^{(3)}\boldsymbol{W}^{ab}={}^{(5)}\boldsymbol{W}^{ab}={}^{(6)}\boldsymbol{W}^{ab}=0\qquad {}^{(3)}\boldsymbol{Z}^{ab}={}^{(5)}\boldsymbol{Z}^{ab}=0 \tag{5.16}$$

Conditions: Nullity of curvature is equivalent to

$$W^A = W^A(\sigma), \qquad V^A = V^A(\sigma), \qquad u_A = \frac{1}{2}\partial_A \mathcal{U} \quad (\mathcal{U} = \mathcal{U}(x^B)). \tag{5.17}$$

Coframe equation

$$a_1 \Delta U + (a_1 - 2c_1)\, \Delta \mathcal{U} = 0\,. \tag{5.18}$$

Connection equation

$\rightarrow$ New variables

$$\Theta_A = \frac{1}{2}\partial_A U - \underline{W}_A + u_A, \tag{5.19}$$

$$\Phi_A := \underline{W}_A + \underline{V}_A + u_A, \tag{5.20}$$

$$\Psi_A := \underline{W}_A + \underline{V}_A - 2u_A. \tag{5.21}$$

$\rightarrow$ Non-trivial equations

$$\begin{pmatrix} 0 & (a_0 - 4b_1) & 0 \\ 2(a_0 + a_1) & 0 & (a_0 + 2c_1) \\ 3(a_0 + 2c_1) & 0 & 2(a_0 + 2b_2) \end{pmatrix} \begin{pmatrix} \Theta_A \\ \Phi_A \\ \Psi_A \end{pmatrix} = 0. \tag{5.22}$$

Remarks

- If the determinant vanishes we have non-trivial solutions.

Conditions: Nullity of curvature and nonmetricity is equivalent to

$$-V^A = W^A = W^A(\sigma), \qquad u_A = 0. \tag{5.23}$$

Coframe equation

$$a_1 \Delta U = 0\,. \tag{5.24}$$

Connection equation

$\rightarrow$ New variables become

$$\Theta_A = \frac{1}{2}\partial_A U - \underline{W}_A, \qquad \Phi_A = \Psi_A = 0. \tag{5.25}$$

$\rightarrow$ Non-trivial equations

$$(a_0 + a_1)\Theta_A = 0 \qquad (a_0 + 2c_1)\Theta_A = 0 \tag{5.26}$$

Remarks

- Non-trivial solutions for

$$a_0 + a_1 = 0, \qquad a_0 + 2c_1 = 0. \tag{5.27}$$

so

$$a_0 \Delta U = 0. \tag{5.28}$$

Same metric structure as in the GR solution (but $\Gamma \neq$ Levi-Civita).

Conditions: Nullity of curvature and torsion is equivalent to

$$W^A = W^A(\sigma), \qquad V^A = V^A(\sigma), \qquad u_A = \frac{1}{2}\partial_A \mathcal{U}(x^B), \qquad \underbrace{\partial_A(U + \mathcal{U}) - 2\underline{W}_A = 0}_{\xrightarrow[W^A(\sigma)]{} \Delta(U+\mathcal{U})=0}. \tag{5.29}$$

Coframe equation

$$c_1 \Delta U = 0\,. \tag{5.30}$$

Connection equation

→ New variables become

$$\Theta_A = 0, \qquad \Phi_A = \underline{W}_A + \underline{V}_A + \frac{1}{2}\partial_A \mathcal{U}, \qquad \Psi_A = \underline{W}_A + \underline{V}_A - \partial_A \mathcal{U}. \tag{5.31}$$

→ Non-trivial equations

$$(a_0 - 4b_1)\Phi_A = 0, \qquad (a_0 + 2b_2)\Psi_A = 0, \qquad (a_0 + 2c_1)\Psi_A = 0. \tag{5.32}$$

Remarks

- There are non-trivial solutions for some values of the parameters.

In [Vassiliev 2002]:

- $\boldsymbol{L} \sim RR$ (v's are omitted).
- Def. of pseudo-instantons: $\boldsymbol{Q}_{ab} = 0$ irreducible curvature solving vacuum EoM.

So:

- Vanishing nonmetricity for our Ansatz means that

$$\begin{aligned} \mathcal{U} = \overline{\mathcal{U}} &= 0\,, \\ \mathcal{W} &= -\mathcal{V}, \\ \overline{\mathcal{W}} &= -\overline{\mathcal{V}}. \end{aligned} \quad \Leftrightarrow \quad \left\{ \begin{aligned} \mathcal{X}_0 &= 2\mathcal{W}, & & \\ \mathcal{X}_1 &= U - \mathcal{W}, & \overline{\mathcal{X}}_1 &= -\overline{\mathcal{W}}, \\ \mathcal{X}_2 = \mathcal{X}_3 &= 0, & \overline{\mathcal{X}}_2 = \overline{\mathcal{X}}_3 &= 0. \end{aligned} \right. \tag{5.33}$$

Eqs. for the purely quadratic model (only w_I, z_J, v_K are nonvanishing):

$$\boxed{\Delta\mathcal{W} = 0, \quad \Delta\overline{\mathcal{W}} = 0} \tag{5.34}$$

This automatically imply

$$\boldsymbol{R}^{ab} = {}^{(1)}\boldsymbol{W}^{ab}\,, \tag{5.35}$$

which is irreducible.