Systems of identical particles

Exercise 1: Symmetrization and parity conservation

Slow neutrons are captured in s-wave by the ^{7}Be. If $J^P(n) = \frac{1}{2}^+$, $J^P(^7\text{Be}) = \frac{1}{2}^-$ and $J^P(^4\text{He}) = 0^+$, determine whether we can expect to see the strong reaction

$$n + {}^7\text{Be} \to {}^4\text{He} + {}^4\text{He}.$$

Exercise 2: Spin of a system of two identical particles

Find the total spin of a system of two identical spin 1 particles depending on their orbital wave function.

Exercise 3: Isospin of the deuteron

A *nucleon* is an isospín $T = 1/2$ state whose $T_3 = +1/2$ and $T_3 = -1/2$ components correspond to the proton and the neutron, respectively.

i) Find the eigenvalues and eigenvectors of the isospin operator for a system of two nucleons.

ii) If the ground state of a *deuteron* (a proton and a neutron strongly bounded; the atomic state is called deuterium, not hydrogen) is a combination of $^3S_1$ y $^3D_1$, show that the isospin of the deuteron is 0. [In spectroscopic notation $^{2s+1}\ell_j$ labels a state of spin s, orbital angular momentum ℓ and total angular momentum j. To indicate $\ell = 0, 1, 2$, etc. we can use the letters S, P, D, etc.]

Exercise 4: Symmetrization of three spin $1/2$ fermions

Consider three identical spin 1/2 particles moving in 1 dimension with Hamiltonian

$$H = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) - \frac{g^2}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}.$$

The lowest energy levels and their wave functions for this *hydrogen-like atom* are

$$E_0 = -\frac{Mg^4}{2\hbar^2} \quad \text{(non degenerate)} \qquad \xi(x_1, x_2, x_3) = \frac{\mathrm{e}^{-r/a}}{\sqrt{\pi a^3}}$$

$$E_1 = -\frac{Mg^4}{8\hbar^2} \quad \text{(degeneracy 4)} \qquad \begin{cases} \phi(x_1, x_2, x_3) = \dfrac{(1 - r/2a)\,\mathrm{e}^{-r/2a}}{\sqrt{8\pi a^3}} \\ \psi_i(x_1, x_2, x_3) = \dfrac{x_i \mathrm{e}^{-r/2a}}{\sqrt{32\pi a^5}}, \quad i = 1, 2, 3. \end{cases}$$

where $a \equiv \dfrac{\hbar^2}{Mg^2}$ and $r \equiv (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Find the energy, the degeneracy and the wave function of the ground state.

Exercise 5: Total spin of three spin 1 identical particles

Consider a system of three spin 1 identical particles. We denote $|{+}0{-}\rangle$ a state where the 1st particle is at $m_s = +1$, the 2nd one at $m_s = 0$ and the 3rd one at $m_s = -1$, with an analogous notation for the rest of the possible spin combinations.
Suppose that the system is at the orbital ground state with a symmetric wave function under the exchange of any of them.

i) Find the normalized spin state of the system in the following cases.

(a) The three particles with spin $|+\rangle$.
(b) Two of them at $|+\rangle$ and one at $|0\rangle$.
(c) All of them in a different spin state.

ii) What is the total spin in each case?

Exercise 6: Normalization of the state of n particles

Normalize the state $(a^\dagger)^n |0\rangle$ using $a|0\rangle = 0$ and the commutation relation $[a, a^\dagger] = \mathbb{1}$.

Exercise 7: Test exercise, February 2014 [2P]

Consider three identical particles moving in the plane under a potential $V = \frac{1}{2}m\omega^2(x^2 + y^2)$. The orbital of each particle is specified by two integers $n_x, n_y \geq 0$:

$$\mathcal{H}^{\text{orb}} \ni |\phi^{\text{orb}}_{n_x n_y}\rangle \to E_{n_x n_y} = (n_x + n_y + 1)\hbar\omega\,.$$

Find the ground state $|\Phi\rangle \in \mathcal{H}^{\text{orb}} \otimes \mathcal{H}^{\text{spin}}$ of the system formed by the three particles and its energy *(i)* if they are spin 0 particles and *(ii)* if their spin is 1/2.

Exercise 8: Coherent states

We define a *coherent state* of bosons as an eigestate of the annihilation operator a, i.e., $a|z\rangle = z|z\rangle$, where z is a complex number.

i) Find the normalized state $|z\rangle = \displaystyle\sum_{n=0}^{\infty} c_n |n\rangle$, where $|n\rangle \equiv \dfrac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$.

ii) Determine the expectation value of the number operator for a coherent normalized state and its uncertainty, $(\Delta N)^2 = \langle z|N^2|z\rangle - \langle z|N|z\rangle^2$.

Exercise 9: Schwinger's oscillator method

Consider a boson that can occupy two states $|+\rangle$ and $|-\rangle$.

i) Show that the operators

$$J_3 = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \quad J_\pm = \hbar\, a_\pm^\dagger a_\mp$$

satisfy the algebra of the angular momentum if $a_\pm$ ($a_\pm^\dagger$) are the annihilation (creation) operators of $|\pm\rangle$.

ii) Prove that J^2 can be written in terms of the number operator $N = N_+ + N_-$ as $J^2 = \hbar^2(N^2 + 2N)/4$

iii) Show that $|j\ m\rangle = |n_+\ n_-\rangle$ with $n_+ = j + m$ and $n_- = j - m$.

Exercise 5 must be turned in [deadline: 4 December 2020 at 15:00]