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Differential forms

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Abstract

Personal notes and collection of useful formulas about exterior algebra.

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1 Notation, conventions and previous comments

❒ Symmetrization and antisymmetrization of indices

$$H_{(\mu_1...\mu_k)} \equiv \frac{1}{k!}\left[H_{\mu_1\mu_2...\mu_k} + H_{\mu_2\mu_1...\mu_k} + ...\right] , \qquad H_{[\mu_1...\mu_k]} \equiv \frac{1}{k!}\left[H_{\mu_1\mu_2...\mu_k} - H_{\mu_2\mu_1...\mu_k} + ...\right] . \tag{1.1}$$

❒ **Abbreviations**

$$\mathrm{d}x^{\otimes\mu\nu\rho...} \equiv \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\rho \otimes ... \qquad \boldsymbol{\vartheta}^{\otimes abc...} \equiv \boldsymbol{\vartheta}^a \otimes \boldsymbol{\vartheta}^b \otimes \boldsymbol{\vartheta}^c \otimes ... \tag{1.2}$$

$$\mathrm{d}x^{\mu\nu\rho...} \equiv \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge ... \qquad \boldsymbol{\vartheta}^{abc...} \equiv \boldsymbol{\vartheta}^a \wedge \boldsymbol{\vartheta}^b \wedge \boldsymbol{\vartheta}^c \wedge ... \tag{1.3}$$

❒ Consider that the manifold is equipped with a *metric* $\boldsymbol{g}$ of arbitrary signature and whose determinant in the coordinate basis will be denoted as $g \equiv \det(g_{\mu\nu})|$. The sign of this determinant is given by $\mathrm{sgn}(g) \equiv (-1)^{\mathrm{Ind}(g)}$ where $\mathrm{Ind}(g)$ is the index of the metric.[2] The metric and its *inverse metric* $\boldsymbol{g}^{-1}$, induced in the cotangent space, are expressed as follows in the previously defined basis:

$$\boldsymbol{g} = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = g_{ab}\boldsymbol{\vartheta}^a \otimes \boldsymbol{\vartheta}^b , \qquad \boldsymbol{g}^{-1} = g^{\mu\nu}\boldsymbol{\partial}_\mu \otimes \boldsymbol{\partial}_\nu = g^{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b ,$$

which fulfill $g_{ab}g^{bc} = \delta^c_a$ and $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$.

❒ **Index criteria**. The metrics raise and lower indices, and the *vielbein* (the objects $e^\mu{}_a$ and $e_\mu{}^a$) transform indices from one type to the other:

$$g_{\mu\nu}H^\mu \equiv H_\nu , \qquad g^{\mu\nu}H_\mu \equiv H^\nu , \tag{1.4}$$

$$e_\mu{}^a H^\mu \equiv H^a , \qquad e_\mu{}^a H_a \equiv H_\mu , \qquad e^\mu{}_a H_\mu \equiv H_a , \qquad e^\mu{}_a H^a \equiv H^\mu . \tag{1.5}$$

❒ We define the *anholonomy coefficients* as the Lie brackets of the basis elements:

$$\Omega_{bc}{}^a := -\left[\boldsymbol{e}_b,\, \boldsymbol{e}_c\right]^a = -\left(2e^\rho{}_{[b}\partial_{|\rho|}e^\lambda{}_{c]}\right) e_\lambda{}^a \quad \Rightarrow \quad \left(e_\mu{}^b e_\nu{}^c \Omega_{bc}{}^a \equiv\right) \quad \Omega_{\mu\nu}{}^a := 2\partial_{[\mu}e_{\nu]}{}^a . \tag{1.6}$$

❒ Let $\Gamma_{\mu\nu}{}^\rho$ be an arbitrary *linear connection* with covariant derivative ∇_μ. In the anholonomic basis:

$$\omega_{\mu a}{}^b = e^\nu{}_a e_\rho{}^b \Gamma_{\mu\nu}{}^\rho + e_\lambda{}^b \partial_\mu e^\lambda{}_a , \tag{1.7}$$

which implies $\nabla_\mu e_\nu{}^a = 0 = \nabla_\mu e^\nu{}_a$ (∇ acts on both kind of indices).
The *torsion* and the *nonmetricity* of the connection are given by

$$T_{\mu\nu}{}^\rho := 2\Gamma_{[\mu\nu]}{}^\rho \qquad \Rightarrow \qquad (e_\rho{}^a T_{\mu\nu}{}^\rho \equiv) \quad T_{\mu\nu}{}^a = 2\omega_{[\mu|b}{}^a e_{|\nu]}{}^b + \Omega_{\mu\nu}{}^a , \tag{1.8}$$

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho} \qquad \Rightarrow \qquad (e^\nu{}_a e^\rho{}_b Q_{\mu\nu\rho} \equiv) \quad Q_{\mu ab} = -\nabla_\mu g_{ab} . \tag{1.9}$$

and the *curvature* (Riemann tensor, Ricci tensor and Ricci scalar, respectively):

$$R_{\mu\nu\rho}{}^\lambda := 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^\lambda + 2\Gamma_{[\mu|\sigma|}{}^\lambda \Gamma_{\nu]\rho}{}^\sigma \quad \Rightarrow \quad \left(e_\lambda{}^b e^\rho{}_a R_{\mu\nu\rho}{}^\lambda \equiv\right) \quad R_{\mu\nu a}{}^b = 2\partial_{[\mu}\omega_{\nu]a}{}^b + 2\omega_{[\mu|c|}{}^b \omega_{\nu]a}{}^c , \tag{1.10}$$

$$R_{\mu\nu} := R_{\mu\lambda\nu}{}^\lambda \qquad , \tag{1.11}$$

$$R := R_{\mu\nu}g^{\mu\nu} \qquad . \tag{1.12}$$

The *Levi-Civita connection* (zero torsion and zero nonmetricity) and its associated objects will be denoted as $\mathring{\Gamma}_{\mu\nu}{}^\rho$, $\mathring{\omega}_{\mu a}{}^b$, $\mathring{\nabla}_\mu$, $\mathring{R}_{\mu\nu\rho}{}^\lambda$,...

[2] *Index of a metric*: the number of (-1)'s that appear in its reduced diagonal form according to Sylvester's law of inertia.

❒ If we introduce the *connection 1-form* and its associated *exterior covariant derivative,*

$$\boldsymbol{\omega}_a{}^b := \omega_{\mu a}{}^b \mathrm{d}x^\mu \,, \tag{1.13}$$

$$\mathbf{D}\boldsymbol{\alpha}_{a...}{}^{b...} := \mathrm{d}\boldsymbol{\alpha}_{a...}{}^{b...} + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\alpha}_{a...}{}^{c...} + ... - \boldsymbol{\omega}_a{}^c \wedge \boldsymbol{\alpha}_{c...}{}^{b...} - ... \,, \tag{1.14}$$

we can define the *curvature, torsion and nonmetricity forms* as

$$\boldsymbol{R}_a{}^b := \mathrm{d}\boldsymbol{\omega}_a{}^b + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\omega}_a{}^c \qquad \equiv \frac{1}{2} R_{\mu\nu a}{}^b \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \tag{1.15}$$

$$\boldsymbol{T}^a := \mathbf{D}\boldsymbol{\vartheta}^a \qquad \equiv \frac{1}{2} T_{\mu\nu}{}^a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \tag{1.16}$$

$$\boldsymbol{Q}_{ab} := -\mathbf{D} g_{ab} \qquad \equiv Q_{\mu ab} \mathrm{d}x^\mu \,. \tag{1.17}$$

2 k-forms on arbitrary vector spaces

In this section we will see some general results valid in any vector space (not necessarily a tangent space of the considered manifold). Let V an arbitrary vector space of finite dimension $\mathtt{D}$ over a field $\mathbb{K}$ (which represents either $\mathbb{C}$ or $\mathbb{R}$). Let $\{\boldsymbol{e}_{\mathtt{M}}\}$ be an arbitrary basis V, and $\{\boldsymbol{\vartheta}^{\mathtt{M}}\}$ its dual basis.

2.1 Exterior product of multilinear forms

- A *multilinear form* (also called covariant tensor) of order s is a map $\boldsymbol{A}\,:\,V\times \overset{s\text{ times}}{\cdots}\times V\longrightarrow\mathbb{K}$.
- The multilinear forms with $s=1$ are called 1-forms and generate the dual of V. In particular, $\{\boldsymbol{\vartheta}^{\mathtt{M}}\}$ is a basis of 1-forms.
- By using the tensor product, we can construct forms of higher order. Hence, a covariant tensor of order s can be expanded in the basis $\{\boldsymbol{\vartheta}^{\otimes\mathtt{MN}\ldots}\equiv\boldsymbol{\vartheta}^{\mathtt{M}}\otimes\boldsymbol{\vartheta}^{\mathtt{N}}\otimes\ldots\}$ as

$$\boldsymbol{A}=A_{\mathtt{M}_1\ldots\mathtt{M}_s}\boldsymbol{\vartheta}^{\otimes\mathtt{M}_1\ldots\mathtt{M}_s}\,. \tag{2.1}$$

Exterior product of two multilinear forms

Given two covariant tensors, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, of order k and l respectively, we define theit exterior product as the $(k+l)$-covariant tensor that acts as follows

$$\boxed{(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})(\boldsymbol{v}_1,\,\ldots,\,\boldsymbol{v}_{k+l}):=\frac{1}{k!l!}\sum_{\sigma\in\mathrm{S}_{k+l}}\mathrm{sgn}(\sigma)\boldsymbol{\alpha}(\boldsymbol{v}_{\sigma(1)},\,\ldots,\,\boldsymbol{v}_{\sigma(k)})\boldsymbol{\beta}(\boldsymbol{v}_{\sigma(k+1)},\,\ldots,\,\boldsymbol{v}_{\sigma(k+l)})}\,, \tag{2.2}$$

where S_{k+l} represents the symmetric group of $(k+l)!$ element (the group of permutations of $k+l$ objects), and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ (a minus sign for each transposition needed to connect it with the identity).

Let us see the components of this object in the basis $\left\{\boldsymbol{\vartheta}^{\otimes\mathtt{M}_1\ldots\mathtt{M}_{k+l}}\equiv\boldsymbol{\vartheta}^{\mathtt{M}_1}\otimes\ldots\otimes\boldsymbol{\vartheta}^{\mathtt{M}_{k+l}}\right\}_{\mathtt{M}_i=1}^{\mathtt{D}}$ of the tensorial space:

$$\begin{aligned}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})(\boldsymbol{e}_{\mathtt{M}_1},\,\ldots,\,\boldsymbol{e}_{\mathtt{M}_{k+l}})&=\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}\mathrm{sgn}(\sigma)\;\boldsymbol{\alpha}(\boldsymbol{e}_{\sigma(\mathtt{M}_1)},\,\ldots,\,\boldsymbol{e}_{\sigma(\mathtt{M}_k)})\;\boldsymbol{\beta}(\boldsymbol{e}_{\sigma(\mathtt{M}_{k+1})},\,\ldots,\,\boldsymbol{e}_{\sigma(\mathtt{M}_{k+l})})\\&=\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}\mathrm{sgn}(\sigma)\;\alpha_{\sigma(\mathtt{M}_1)\ldots\sigma(\mathtt{M}_k)}\beta_{\sigma(\mathtt{M}_{k+1})\ldots\sigma(\mathtt{M}_{k+l})}\\&=\frac{(k+l)!}{k!l!}\alpha_{[\mathtt{M}_1\ldots\mathtt{M}_k}\beta_{\mathtt{M}_{k+1}\ldots\mathtt{M}_{k+l}]}\,,\end{aligned} \tag{2.3}$$

i.e.,

$$\boxed{\boldsymbol{\alpha}\wedge\boldsymbol{\beta}=\frac{(k+l)!}{k!l!}\alpha_{[\mathtt{M}_1\ldots\mathtt{M}_k}\beta_{\mathtt{M}_{k+1}\ldots\mathtt{M}_{k+l}]}\boldsymbol{\vartheta}^{\otimes\mathtt{M}_1\ldots\mathtt{M}_{k+l}}}\,. \tag{2.4}$$

Example 1. Exterior product of two covariant tensors of order 1:

$$\left(\alpha_{\mathtt{M}}\boldsymbol{\vartheta}^{\mathtt{M}}\right)\wedge\left(\beta_{\mathtt{N}}\boldsymbol{\vartheta}^{\mathtt{N}}\right)=\frac{2!}{1!1!}\alpha_{[\mathtt{M}}\beta_{\mathtt{N}]}\boldsymbol{\vartheta}^{\mathtt{M}}\otimes\boldsymbol{\vartheta}^{\mathtt{N}}=\left(\alpha_{\mathtt{M}}\beta_{\mathtt{N}}-\alpha_{\mathtt{N}}\beta_{\mathtt{M}}\right)\boldsymbol{\vartheta}^{\mathtt{M}}\otimes\boldsymbol{\vartheta}^{\mathtt{N}} \tag{2.5}$$

Properties of the exterior product

The exterior product is bilinear, anticommutative and associative. Symbolically, given the tensors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\beta}'$ and $\boldsymbol{\gamma}$, respectively of order k, l, l y m and $\lambda \in \mathbb{K}$,

$$\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} + \boldsymbol{\beta}') = \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + \boldsymbol{\alpha} \wedge \boldsymbol{\beta}' \,, \tag{2.6}$$
$$\boldsymbol{\alpha} \wedge (\lambda\boldsymbol{\beta}) = (\lambda\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} = \lambda(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \,, \tag{2.7}$$
$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-1)^{kl} \boldsymbol{\beta} \wedge \boldsymbol{\alpha} \,, \tag{2.8}$$
$$\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma} \,. \tag{2.9}$$

(The linearity in the first variable can be deduced from the first and the third properties).

Proof. The bilinearity is immediate; for the anticommutativity,

$$\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\beta} &= \frac{(k+l)!}{k!l!} \alpha_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \beta_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l}} \\
&= \frac{(k+l)!}{k!l!} \beta_{[\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}} \alpha_{\mathtt{M}_1 \ldots \mathtt{M}_k]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l}} \\
&= (-1)^{kl} \frac{(k+l)!}{k!l!} \beta_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \alpha_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l}} \\
&= (-1)^{kl} \boldsymbol{\beta} \wedge \boldsymbol{\alpha} \,,
\end{aligned}$$

and for the associativity (let $\boldsymbol{\gamma}$ be a covariant tensor of order m),

$$\begin{aligned}
\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) &= \boldsymbol{\alpha} \wedge \left(\frac{(l+m)!}{l!m!} \beta_{[\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}} \gamma_{\mathtt{M}_{k+l+1} \ldots \mathtt{M}_{k+l+m}]} \boldsymbol{\vartheta}^{\mathtt{M}_{k+1}} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l+m}} \right) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \alpha_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \beta_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}} \gamma_{\mathtt{M}_{k+l+1} \ldots \mathtt{M}_{k+l+m}]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l+m}} \\
&= \frac{(k+l+m)!}{k!l!m!} \alpha_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \beta_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}} \gamma_{\mathtt{M}_{k+l+1} \ldots \mathtt{M}_{k+l+m}]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l+m}} \\
&= \frac{(k+l+m)!}{m!(k+l)!} \frac{(k+l)!}{k!l!} \alpha_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \beta_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}} \gamma_{\mathtt{M}_{k+l+1} \ldots \mathtt{M}_{k+l+m}]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l+m}} \\
&= \left(\frac{(k+l)!}{k!l!} \alpha_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \beta_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}]} \boldsymbol{\vartheta}^{\mathtt{M}_1} \otimes \ldots \otimes \boldsymbol{\vartheta}^{\mathtt{M}_{k+l}} \right) \wedge \boldsymbol{\gamma} \\
&= (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma} \,.
\end{aligned}$$

■

Exterior product of an arbitrary number of multilinear forms

When proving the associativity of the exterior product we found that

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma} = \frac{(k+l+m)!}{k!l!m!} \alpha_{[\mathtt{M}_1 \ldots \mathtt{M}_k} \beta_{\mathtt{M}_{k+1} \ldots \mathtt{M}_{k+l}} \gamma_{\mathtt{M}_{k+l+1} \ldots \mathtt{M}_{k+l+m}]} \boldsymbol{\vartheta}^{\otimes \mathtt{M}_1 \ldots \mathtt{M}_{k+l+m}} \,. \tag{2.10}$$

Indeed, in general, for any family a $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\alpha}^{(2)}$, ..., $\boldsymbol{\alpha}^{(N)}$ of N covariant tensors of order k_1, ..., k_N respectively, it follows that

$$\boxed{\boldsymbol{\alpha}^{(1)} \wedge \boldsymbol{\alpha}^{(2)} \wedge \ldots \wedge \boldsymbol{\alpha}^{(N)} = \frac{(k_1 + k_2 + \ldots + k_N)!}{k_1! k_2! \ldots k_N!} \alpha^{(1)}_{[\mathtt{M}_1 \ldots \mathtt{M}_{k_1}} \quad \ldots \quad \alpha^{(N)}_{\mathtt{M}_{1+s(N-1)} \ldots \mathtt{M}_{s(N)}]} \boldsymbol{\vartheta}^{\otimes \mathtt{M}_1 \ldots \mathtt{M}_{s(N)}}} \,, \tag{2.11}$$

Example 2. Consider the particular case in which all the involved covariant tensors are of order 1 ($k_1 = ... = k_N = 1$):

$$\boldsymbol{\alpha}^{(1)} \wedge \boldsymbol{\alpha}^{(2)} \wedge ... \wedge \boldsymbol{\alpha}^{(N)} = N! \alpha^{(1)}_{[\mathtt{M}_1} \quad ... \quad \alpha^{(N)}_{\mathtt{M}_N]} \boldsymbol{\vartheta}^{\otimes \mathtt{M}_1 ... \mathtt{M}_N} \,, \tag{2.12}$$

If all of them were elements of the basis, then we arrive at the following interesting sub-case:

$$\boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_N} \equiv \qquad \boldsymbol{\vartheta}^{\mathtt{M}_1} \wedge ... \wedge \boldsymbol{\vartheta}^{\mathtt{M}_N} = N! \boldsymbol{\vartheta}^{\otimes [\mathtt{M}_1 ... \mathtt{M}_N]} \,. \tag{2.13}$$

And, for here, we can extract the components of the tensors $\boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_N}$:

$$\boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_N} = \left(\boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_N}\right)_{\mathtt{N}_1 ... \mathtt{N}_N} \boldsymbol{\vartheta}^{\otimes \mathtt{N}_1 ... \mathtt{N}_N} \,, \qquad \text{where} \qquad \boxed{\left(\boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_N}\right)_{\mathtt{N}_1 ... \mathtt{N}_N} = N! \delta^{[\mathtt{M}_1}_{\mathtt{N}_1} ... \delta^{\mathtt{M}_N]}_{\mathtt{N}_N}} \,. \tag{2.14}$$

2.2 k-forms

A *k-form*, is a covariant tensor of order k, $\boldsymbol{\alpha} = \alpha_{\mathtt{M}_1 ... \mathtt{M}_k} \boldsymbol{\vartheta}^{\otimes \mathtt{M}_1 ... \mathtt{M}_k}$, which is completely antisymmetric, $\alpha_{\mathtt{M}_1 ... \mathtt{M}_k} = \alpha_{[\mathtt{M}_1 ... \mathtt{M}_k]}$. It can then be expressed as

$$\boldsymbol{\alpha} = \frac{1}{k!} \alpha_{\mathtt{M}_1 ... \mathtt{M}_k} \boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_k} \,. \tag{2.15}$$

The set of k-forms over V constitutes a vector space over the same field that we will denote as $\Lambda^k V$. This space has dimension $\frac{\mathtt{D}!}{k!(\mathtt{D}-k)!}$ and the set $\{\boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_k}\}$ is a basis of it.

The exterior product once restricted to totally antisymmetric tensors becomes a map

$$\wedge \,:\, \Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V \,. \tag{2.16}$$

2.3 Interior product of k-forms

Given a vector $\boldsymbol{v} = v^{\mathtt{M}} \boldsymbol{e}_{\mathtt{M}}$ and a k-form $\boldsymbol{\alpha} = \frac{1}{k!} \alpha_{\mathtt{M}_1 ... \mathtt{M}_k} \boldsymbol{\vartheta}^{\mathtt{M}_1 ... \mathtt{M}_k}$, we define the *interior product* by the vector $\boldsymbol{v}$ (or *interior derivative* with respect to $\boldsymbol{v}$) as the $(k-1)$-form

$$(\boldsymbol{v} \lrcorner \boldsymbol{\alpha}) (\boldsymbol{u}_1, \, ..., \, \boldsymbol{u}_{k-1}) \coloneqq \boldsymbol{\alpha} (\boldsymbol{v}, \, \boldsymbol{u}_1, \, ..., \, \boldsymbol{u}_k) \,. \tag{2.17}$$

In components

$$(\boldsymbol{v} \lrcorner \boldsymbol{\alpha})_{\mathtt{N}_1 ... \mathtt{N}_{k-1}} = v^{\mathtt{C}} \alpha_{\mathtt{C}\mathtt{N}_1 ... \mathtt{N}_{k-1}} \,. \tag{2.18}$$

Proof.

$$\boldsymbol{v} \lrcorner \boldsymbol{\alpha} = \alpha_{\mathtt{M}_1 ... \mathtt{M}_k} \left(\boldsymbol{\vartheta}^{\mathtt{M}_1} (v^{\mathtt{C}} \boldsymbol{e}_{\mathtt{C}})\right) \otimes \boldsymbol{\vartheta}^{\otimes \mathtt{M}_2 ... \mathtt{M}_k} = v^{\mathtt{C}} \alpha_{\mathtt{C}\mathtt{N}_1 ... \mathtt{N}_{k-1}} \boldsymbol{\vartheta}^{\otimes \mathtt{N}_1 ... \mathtt{N}_{k-1}} = \frac{1}{(k-1)!} v^{\mathtt{C}} \alpha_{\mathtt{C}\mathtt{N}_1 ... \mathtt{N}_{k-1}} \boldsymbol{\vartheta}^{\mathtt{N}_1 ... \mathtt{N}_{k-1}} \,.$$

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As we can see, this constitutes a map $\boldsymbol{v} \lrcorner \,:\, \Lambda^k V \to \Lambda^{k-1} V$.

Basic properties of the interior product

Let $\boldsymbol{u}$, $\boldsymbol{v}$ be vectors, $\boldsymbol{\alpha}$ a k-form and $\boldsymbol{\beta}$ a l-form, and $\lambda_1,\ \lambda_2 \in \mathbb{K}$. The interior derivative fulfills:

$$(\lambda_1 \boldsymbol{u} + \lambda_2 \boldsymbol{v} \lrcorner)\boldsymbol{\alpha} = \lambda_1\ \boldsymbol{u}\lrcorner\boldsymbol{\alpha} + \lambda_2\ \boldsymbol{v}\lrcorner\boldsymbol{\alpha}\,, \qquad \text{(Linear in the 1st variable)}, \tag{2.19}$$

$$\boldsymbol{u}\lrcorner(\lambda_1 \boldsymbol{\alpha} + \lambda_2 \boldsymbol{\beta}) = \lambda_1\ \boldsymbol{u}\lrcorner\boldsymbol{\alpha} + \lambda_2\ \boldsymbol{u}\lrcorner\boldsymbol{\beta}\,, \qquad \text{(Linear in the 2nd variable)}, \tag{2.20}$$

$$\boldsymbol{u}\lrcorner\,(\boldsymbol{u}\lrcorner\boldsymbol{\alpha}) = 0\,, \qquad \text{(Nilpotent)}, \tag{2.21}$$

$$\boldsymbol{u}\lrcorner\,(\boldsymbol{v}\lrcorner\boldsymbol{\alpha}) = -\boldsymbol{v}\lrcorner\,(\boldsymbol{u}\lrcorner\boldsymbol{\alpha}) \qquad \text{(Antisymmetric)}, \tag{2.22}$$

$$\boldsymbol{u}\lrcorner(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) = (\boldsymbol{u}\lrcorner\boldsymbol{\alpha})\wedge\boldsymbol{\beta} + (-1)^k\boldsymbol{\alpha}\wedge(\boldsymbol{u}\lrcorner\boldsymbol{\beta})\ . \tag{2.23}$$

Useful formulae

$$\boxed{\boldsymbol{e}_{\mathtt{M}}\lrcorner\boldsymbol{\vartheta}^{\mathtt{N}_1\ldots\mathtt{N}_k} = k\delta_{\mathtt{M}}^{[\mathtt{N}_1}\delta_{\mathtt{C}_1}^{\mathtt{N}_2}\ldots\delta_{\mathtt{C}_{k-1}}^{\mathtt{N}_k]}\boldsymbol{\vartheta}^{\mathtt{C}_1\ldots\mathtt{C}_{k-1}}\,,} \tag{2.24}$$

Proof.

$$\begin{aligned}\boldsymbol{e}_{\mathtt{M}}\lrcorner\boldsymbol{\vartheta}^{\mathtt{M}_1\ldots\mathtt{M}_k} &= \boldsymbol{e}_{\mathtt{M}}\lrcorner\left(k!\delta_{\mathtt{N}_1}^{[\mathtt{M}_1}\ldots\delta_{\mathtt{N}_k}^{\mathtt{M}_k]}\boldsymbol{\vartheta}^{\otimes\mathtt{N}_1\ldots\mathtt{N}_k}\right)\\ &= \frac{1}{(k-1)!}k!\delta_{\mathtt{M}}^{[\mathtt{M}_1}\delta_{\mathtt{N}_2}^{\mathtt{M}_2}\ldots\delta_{\mathtt{N}_k}^{\mathtt{M}_k]}\boldsymbol{\vartheta}^{\mathtt{N}_2\ldots\mathtt{N}_k}\,.\end{aligned}$$

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Particular cases:

$$\boldsymbol{v}\lrcorner\boldsymbol{\vartheta}^{\mathtt{M}_1\ldots\mathtt{M}_k} = kv^{[\mathtt{M}_1}\delta_{\mathtt{N}_1}^{\mathtt{M}_2}\ldots\delta_{\mathtt{N}_{k-1}}^{\mathtt{M}_k]}\boldsymbol{\vartheta}^{\mathtt{N}_1\ldots\mathtt{N}_{k-1}}\,, \tag{2.25}$$

$$\begin{aligned}\boldsymbol{e}_{\mathtt{M}}\lrcorner\boldsymbol{\alpha} &= \alpha_{\mathtt{M}\mathtt{N}_1\ldots\mathtt{N}_{k-1}}\boldsymbol{\vartheta}^{\otimes\mathtt{N}_1\ldots\mathtt{N}_{k-1}}\\ &= \frac{1}{(k-1)!}\alpha_{\mathtt{M}\mathtt{N}_1\ldots\mathtt{N}_{k-1}}\boldsymbol{\vartheta}^{\mathtt{N}_1\ldots\mathtt{N}_{k-1}}\,,\end{aligned} \tag{2.26}$$

$$\boldsymbol{e}_{\mathtt{M}}\lrcorner\boldsymbol{\vartheta}^{\mathtt{N}} = \delta_{\mathtt{M}}^{\mathtt{N}}\,. \tag{2.27}$$

For all k-form $\boldsymbol{\alpha}$ we have

$$\boldsymbol{\vartheta}^{\mathtt{M}}\wedge(\boldsymbol{e}_{\mathtt{N}}\lrcorner\boldsymbol{\alpha}) = \frac{1}{(k-1)!}\alpha_{\mathtt{N}\mathtt{C}_1\ldots\mathtt{C}_{k-1}}\boldsymbol{\vartheta}^{\mathtt{M}\mathtt{C}_1\ldots\mathtt{C}_{k-1}}\,, \tag{2.28}$$

$$\boldsymbol{e}_{\mathtt{N}}\lrcorner\left(\boldsymbol{\vartheta}^{\mathtt{M}}\wedge\boldsymbol{\alpha}\right) = \delta_{\mathtt{N}}^{\mathtt{M}}\boldsymbol{\alpha} - \boldsymbol{\vartheta}^{\mathtt{M}}\wedge(\boldsymbol{e}_{\mathtt{N}}\lrcorner\boldsymbol{\alpha})\ , \tag{2.29}$$

whose contractions are:

$$\boldsymbol{\vartheta}^{\mathtt{M}}\wedge(\boldsymbol{e}_{\mathtt{M}}\lrcorner\boldsymbol{\alpha}) = k\boldsymbol{\alpha}\,, \tag{2.30}$$

$$\boldsymbol{e}_{\mathtt{M}}\lrcorner\left(\boldsymbol{\vartheta}^{\mathtt{M}}\wedge\boldsymbol{\alpha}\right) = (\mathtt{D}-k)\boldsymbol{\alpha}\,. \tag{2.31}$$

3 Differential forms. Basic definitions

From now on we will work in an arbitrary manifold. So, let $\mathcal{M}$ be a *smooth manifold* of dimension $\mathtt{D}$. Let $\{\boldsymbol{\partial}_\mu\}$ and $\{\mathrm{d}x^\mu\}$ be the basis (fields) of the tangent and the cotangent spaces at every point, respectively, with respect to certain coordinates x^μ. We will consider also an arbitrary frame (not necessarily orthogonal!):

$$\boldsymbol{e}_a = e^\mu{}_a \boldsymbol{\partial}_\mu \,, \qquad \boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu \,. \tag{3.1}$$

We will formulate everything in terms of the general basis $\{\boldsymbol{\vartheta}^a\}$, so all expressions will be valid for a coordinate basis by taking $\boldsymbol{\vartheta}^a = \delta^a_\mu \mathrm{d}x^\mu$. Latin and Greek indices are, respectively, referred to the general and the coordinate basis.

We can consider at each point the k-forms over the tangent space at $p \in \mathcal{M}$, which we will denote $\Lambda^k T_p\mathcal{M}$ and construct the bundle of k-forms

$$\Lambda^k T\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} \Lambda^k T_p\mathcal{M} \,. \tag{3.2}$$

The smooth sections of this bundle are called differential forms or k-form fields over $\mathcal{M}$ (we will omit the word "field" from now on) and the set of all of them will be denoted as $\Omega^k(\mathcal{M})$.[3] In other words:

Definition 3. A *differential form of rank k* (or *k-form*), is a smooth covariant tensor field of order k, $\boldsymbol{\alpha} = \alpha_{a_1\ldots a_k} \boldsymbol{\vartheta}^{\otimes a_1\ldots a_k}$, which is completely antisymmetric, $\alpha_{a_1\ldots a_k} = \alpha_{[a_1\ldots a_k]}$. It can then be expressed as

$$\boldsymbol{\alpha} = \frac{1}{k!} \alpha_{a_1\ldots a_k} \boldsymbol{\vartheta}^{a_1\ldots a_k} \,. \tag{3.3}$$

3.1 Exterior derivative

The *exterior derivative* $\mathrm{d} : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ of a k-form $\boldsymbol{\alpha}$ is defined as the $(k+1)$-form that acts as follows over some vector fields $\boldsymbol{V}_1$, ...,$\boldsymbol{V}_{k+1}$:

$$\begin{aligned} \mathrm{d}\boldsymbol{\alpha}\left(\boldsymbol{V}_1, \ldots, \boldsymbol{V}_{k+1}\right) = &\sum_{i=1}^{k+1} (-1)^{i+1} \boldsymbol{V}_i \left(\boldsymbol{\alpha}\left(\boldsymbol{V}_1, \ldots, \hat{\boldsymbol{V}}_i, \ldots, \boldsymbol{V}_{k+1}\right)\right) \\ &+ \sum_{i<j} (-1)^{i+j} \boldsymbol{\alpha}\left(\left[\boldsymbol{V}_i, \boldsymbol{V}_j\right], \boldsymbol{V}_1, \ldots, \hat{\boldsymbol{V}}_i, \ldots, \hat{\boldsymbol{V}}_j, \ldots, \boldsymbol{V}_{k+1}\right) \end{aligned} \tag{3.4}$$

where $\hat{}$ means that the element in that position has been omitted. For $k = 1$ we trivially get:

$$\mathrm{d}\boldsymbol{\alpha}\left(\boldsymbol{V}_1, \boldsymbol{V}_2\right) = \boldsymbol{V}_1\left(\boldsymbol{\alpha}\left(\boldsymbol{V}_2\right)\right) - \boldsymbol{V}_2\left(\boldsymbol{\alpha}\left(\boldsymbol{V}_1\right)\right) - \boldsymbol{\alpha}\left(\left[\boldsymbol{V}_1, \boldsymbol{V}_2\right]\right) \,. \tag{3.5}$$

Therefore,

$$\begin{aligned} \mathrm{d}\boldsymbol{\alpha} &= \underbrace{(k+1)\left[\partial_{[a_1}\alpha_{a_2\ldots a_{k+1}]} + \frac{k}{2}\Omega_{[a_1 a_2}{}^c \alpha_{|c|a_3\ldots a_{k+1}]}\right]}_{(\mathrm{d}\boldsymbol{\alpha})_{a_1\ldots a_{k+1}}} \boldsymbol{\vartheta}^{\otimes a_1\ldots a_{k+1}} \\ &= (k+1)\left(\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]} e^{\mu_1}{}_{a_1} \ldots e^{\mu_{k+1}}{}_{a_{k+1}}\right) \boldsymbol{\vartheta}^{\otimes a_1\ldots a_{k+1}} \\ &= \underbrace{(k+1)\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}}_{(\mathrm{d}\boldsymbol{\alpha})_{\mu_1\ldots\mu_{k+1}}} \mathrm{d}x^{\otimes \mu_1\ldots\mu_{k+1}} \end{aligned}$$

[3]This set is a C^∞-module.

or, in terms of the exterior product,

$$\begin{aligned} \mathrm{d}\boldsymbol{\alpha} &= \frac{1}{k!}\left[\partial_{a_1}\alpha_{a_2\ldots a_{k+1}} + \frac{k}{2}\Omega_{a_1 a_2}{}^{c}\alpha_{c a_3\ldots a_{k+1}}\right]\boldsymbol{\vartheta}^{a_1\ldots a_{k+1}} \\ &= \frac{1}{k!}\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}\mathrm{d}x^{\mu_1\ldots\mu_{k+1}} \end{aligned} \tag{3.6}$$

Proof. See Appendix B.1 ■

It can also be expressed, trivially, in a explicitly covariant form by using an arbitrary torsionless connection $\bar{\nabla}$ (in particular, the Levi-Civita one):

$$\mathrm{d}\boldsymbol{\alpha} = \frac{1}{k!}\bar{\nabla}_{[a_1}\alpha_{a_2\ldots a_{k+1}]}\boldsymbol{\vartheta}^{a_1\ldots a_{k+1}}\,. \tag{3.7}$$

Basic properties of the exterior derivative

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be differential forms and $\lambda_1,\ \lambda_2 \in \mathbb{K}$. The exterior derivative satisfies:

$$\mathrm{d}(\lambda_1\boldsymbol{\alpha} + \lambda_2\boldsymbol{\beta}) = \lambda_1\mathrm{d}\boldsymbol{\alpha} + \lambda_2\mathrm{d}\boldsymbol{\beta}\,, \qquad \text{(Linear)} \tag{3.8}$$

$$\mathrm{d}(\mathrm{d}\boldsymbol{\alpha}) = 0 \qquad \text{(Nilpotent)} \tag{3.9}$$

$$\mathrm{d}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) = \mathrm{d}\boldsymbol{\alpha}\wedge\boldsymbol{\beta} + (-1)^k(\boldsymbol{\alpha}\wedge\mathrm{d}\boldsymbol{\beta})\,, \qquad \text{where } k = \mathrm{rank}(\boldsymbol{\alpha}). \tag{3.10}$$

Another (equivalent) way to define the exterior derivative

Another way to define the exterior differential is by defining first the exterior derivative of a 0-form just as the partial derivative ($\mathrm{d}f := \partial_\mu f\,\mathrm{d}x^\mu$) and, then, extend this to an arbitrary k-form as (Important! The frame should be a coordinate frame!):

$$\mathrm{d}\boldsymbol{\alpha} = \mathrm{d}\left(\frac{1}{k!}\alpha_{\mu_2\ldots\mu_{k+1}}\mathrm{d}x^{\mu_2\ldots\mu_{k+1}}\right) := \frac{1}{k!}(\mathrm{d}\alpha_{\mu_2\ldots\mu_{k+1}})\wedge\mathrm{d}x^{\mu_2\ldots\mu_{k+1}}\,, \tag{3.11}$$

where in the last expression we are treating $\alpha_{\mu_2\ldots\mu_{k+1}}$ as a 0-form whose d has been previously defined.

Exact and closed forms. De Rham cohomology

- ❐ A k-form $\boldsymbol{\alpha}$ such that $\mathrm{d}\boldsymbol{\alpha} = 0$ is called *closed*. All of them form a vector space that we will denote as $Z^k_{\mathrm{d}}(\mathcal{M})$.
- ❐ A k-form $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta}$ for certain $(k-1)$-form $\boldsymbol{\beta}$ is said to be *exact*. All of them form a vector space that we will denote as $B^k_{\mathrm{d}}(\mathcal{M})$.
- ❐ Since the exterior derivative is nilpotent, any exact form is closed, i.e. $B^k_{\mathrm{d}}(\mathcal{M}) \subset Z^k_{\mathrm{d}}(\mathcal{M})$.
- ❐ If we introduce the equivalence relation "being equal up to an exact form" in the set $Z^k_{\mathrm{d}}(\mathcal{M})$, the resulting equivalence classes constitute the *de Rham cohomology group*:

  $$H^k_{\mathrm{d}}(\mathcal{M}) = Z^k_{\mathrm{d}}(\mathcal{M})/B^k_{\mathrm{d}}(\mathcal{M})\,, \tag{3.12}$$

  whose dimension is called the k-th Betti number:

  $$b^k(\mathcal{M}) = \dim\left(H^k_{\mathrm{d}}(\mathcal{M})\right)\,. \tag{3.13}$$

❒ The *Euler characteristic* of a manifold is given by:

$$\chi(\mathcal{M}) = \sum_{k=0}^{\mathtt{D}} (-1)^k b^k = b^0 - b^1 + b^2 - \dots . \tag{3.14}$$

❒ As a consequence of the Poincaré duality $H^k_{\mathrm{d}}(\mathcal{M}) \simeq H_{\mathtt{D}-k}(\mathcal{M})$ which relates cohomology with homology, we get

$$H^k_{\mathrm{d}}(\mathcal{M}) \simeq H^{\mathtt{D}-k}_{\mathrm{d}}(\mathcal{M}) . \tag{3.15}$$

A corollary is

$$b^k(\mathcal{M}) = b^{\mathtt{D}-k}(\mathcal{M}) , \tag{3.16}$$

and, therefore, for a manifold of odd dimension $\chi(\mathcal{M}^{\mathtt{D}\,\mathrm{odd}}) = 0$, whereas if the dimension is even, $\chi(\mathcal{M}^{\mathtt{D}\,\mathrm{even}}) = (-1)^{\mathtt{D}/2} b^{\mathtt{D}/2}$.

3.2 Interior and exterior derivatives. Lie derivative of forms

The Lie derivative of differential forms with respect to a vector $\boldsymbol{V} = V^a \boldsymbol{e}_a$ can be expressed:

$$\boxed{\mathfrak{L}_{\boldsymbol{V}} = \mathrm{d} \circ (\boldsymbol{V} \lrcorner) + (\boldsymbol{V} \lrcorner) \circ \mathrm{d}} . \tag{3.17}$$

Proof. See Appendix B.2. ■

From here, we can easily show that:

$$[\boldsymbol{V}, \boldsymbol{W}] \lrcorner = [\mathfrak{L}_{\boldsymbol{V}}, \boldsymbol{W} \lrcorner] . \tag{3.18}$$

Proof. See Appendix B.3. ■

3.3 Volume form

In a $\mathtt{D}$-dimensional manifold, a *volume form* is a $\mathtt{D}$-form which is everywhere non-zero. Given a positively oriented basis $\{\boldsymbol{\vartheta}^a\}$, and a scalar density $\mathfrak{f}$ of weight -1, the associated volume form is

$$\mathbf{vol} := \mathfrak{f}\boldsymbol{\vartheta}^1 \wedge ... \wedge \boldsymbol{\vartheta}^{\mathtt{D}} = \frac{1}{\mathtt{D}!}\mathfrak{f}\epsilon_{a_1...a_{\mathtt{D}}}\boldsymbol{\vartheta}^{a_1...a_{\mathtt{D}}} = \mathfrak{f}\epsilon_{a_1...a_{\mathtt{D}}}\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}\,, \tag{3.19}$$

where $\epsilon_{a_1...a_{\mathtt{D}}} := \mathtt{D}!\delta^1_{[a_1}...\delta^{\mathtt{D}}_{a_{\mathtt{D}}]}$ is the Levi-Civita pseudotensor (de peso 1) whose components are in any frame: 1 for even permutations of $\{1,\,2,\,...,\,\mathtt{D}\}$, -1 for odd ones and 0 if there are repetitions.

❒ Is the manifold is not orientable, it is not possible to globally define a volume form.

❒ By using the Lie derivative, we can check that:

$$\mathfrak{L}_{\boldsymbol{V}}\mathbf{vol} \equiv \mathrm{d}\left(\boldsymbol{V}\lrcorner\mathbf{vol}\right) = \left[\frac{1}{\mathfrak{f}}\partial_c\left(\mathfrak{f}V^c\right) + \Omega_{cd}{}^d V^c\right]\mathbf{vol} \tag{3.20}$$

And, if we take a coordinate basis $\{\mathrm{d}x^\mu\}$, the anholonomy coefficients disappear,

$$\mathfrak{L}_{\boldsymbol{V}}\mathbf{vol} = \left[\frac{1}{\mathfrak{f}}\partial_\mu\left(\mathfrak{f}V^\mu\right)\right]\mathbf{vol} \tag{3.21}$$

Proof. Consider $\mathbf{vol} = \mathfrak{f}\epsilon_{a_1...a_{\mathtt{D}}}\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}$

$$\begin{aligned}
\mathrm{d}\left(\boldsymbol{V}\lrcorner\mathbf{vol}\right) &= \mathtt{D}\left[\partial_{[a_1}\left(V^c\mathfrak{f}\epsilon_{|c|a_2...a_{\mathtt{D}}]}\right) + \frac{\mathtt{D}-1}{2}\Omega_{[a_1a_2}{}^dV^c\mathfrak{f}\epsilon_{|cd|a_3...a_{\mathtt{D}}]}\right]\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}\\
&= \mathtt{D}\left[\partial_{[a_1}\left(V^c\mathfrak{f}\right)\epsilon_{|c|a_2...a_{\mathtt{D}}]} + \frac{\mathtt{D}-1}{2}\Omega_{[a_1a_2}{}^dV^c\mathfrak{f}\epsilon_{|cd|a_3...a_{\mathtt{D}}]}\right]\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}\\
&= \mathtt{D}\left[(-1)^{\mathtt{D}-1}\epsilon_{c[a_1...a_{\mathtt{D}-1}}\partial_{a_{\mathtt{D}}]}\left(V^c\mathfrak{f}\right) + \frac{\mathtt{D}-1}{2}\epsilon_{cd[a_1...a_{\mathtt{D}-2}}\Omega_{a_{\mathtt{D}-1}a_{\mathtt{D}}]}{}^dV^c\mathfrak{f}\right]\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}\\
&= \mathtt{D}\left[\epsilon_{[a_1...a_{\mathtt{D}-1}|c|}\partial_{a_{\mathtt{D}}]}\left(V^c\mathfrak{f}\right) + \frac{\mathtt{D}-1}{2}\epsilon_{[a_1...a_{\mathtt{D}-2}|cd|}\Omega_{a_{\mathtt{D}-1}a_{\mathtt{D}}]}{}^dV^c\mathfrak{f}\right]\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}\\
! &= \left[\partial_c\left(\mathfrak{f}V^c\right) + \Omega_{cd}{}^dV^c\mathfrak{f}\right]\epsilon_{a_1...a_{\mathtt{D}}}\boldsymbol{\vartheta}^{\otimes a_1...a_{\mathtt{D}}}\\
&= \left[\frac{1}{\mathfrak{f}}\partial_c\left(\mathfrak{f}V^c\right) + \Omega_{cd}{}^dV^c\right]\mathbf{vol}
\end{aligned}$$

where in ! we have used that for any totally antisymmetric object C with $\mathtt{D}$ indices:

$$A^cC_{[a_1...a_{\mathtt{D}-1}|c|}B_{a_{\mathtt{D}}]} = \frac{1}{\mathtt{D}}C_{a_1...a_{\mathtt{D}}}A^cB_c\,. \tag{3.22}$$

$$A^{bc}C_{[a_1...a_{\mathtt{D}-2}|bc|}B_{a_{\mathtt{D}-1}a_{\mathtt{D}}]} = \frac{2}{\mathtt{D}(\mathtt{D}-1)}C_{a_1...a_{\mathtt{D}}}A^{bc}B_{bc}\,. \tag{3.23}$$

Finally we use $\mathfrak{L}_{\boldsymbol{V}}\mathbf{vol} = \boldsymbol{V}\lrcorner\cancel{(\mathrm{d}\mathbf{vol})} + \mathrm{d}\left(\boldsymbol{V}\lrcorner\mathbf{vol}\right)$. ∎

❒ Any two volume forms are proportional (the space of $\mathtt{D}$-forms is 1-dimensional). To be precise, given two arbitrary volume forms,

$$\mathbf{vol} = \frac{1}{\mathtt{D}!}\mathfrak{f}\epsilon_{a_1\dots a_\mathtt{D}}\boldsymbol{\vartheta}^{a_1\dots a_\mathtt{D}} \qquad \text{and} \qquad \mathbf{vol}' = \frac{1}{\mathtt{D}!}\mathfrak{f}'\epsilon_{m_1\dots m_\mathtt{D}}\boldsymbol{\alpha}^{m_1}\wedge\dots\wedge\boldsymbol{\alpha}^{m_\mathtt{D}}\,, \tag{3.24}$$

and if we call $M^a{}_m$ the basis transformation matrix (from $\boldsymbol{\alpha}\to\boldsymbol{\vartheta}$, i.e., $\boldsymbol{\vartheta}^a = M^a{}_m\boldsymbol{\alpha}^m$), then we find

$$\boldsymbol{\vartheta}^{a_1\dots a_\mathtt{D}} = \det(M)\,\bar{\epsilon}^{a_1\dots a_\mathtt{D}}\boldsymbol{\alpha}^1\wedge\dots\wedge\boldsymbol{\alpha}^\mathtt{D}\,. \tag{3.25}$$

where we have introduced $\bar{\epsilon}^{a_1\dots a_\mathtt{D}} := \mathtt{D}!\delta_1^{[a_1}\dots\delta_\mathtt{D}^{a_\mathtt{D}]} = \det(g_{ab})\; g^{m_1a_1}\dots g^{m_\mathtt{D}a_\mathtt{D}}\epsilon_{m_1\dots m_\mathtt{D}}$.

Proof.

$$\begin{aligned}
\boldsymbol{\vartheta}^{a_1\dots a_\mathtt{D}} &= M^{a_1}{}_{[m_1}\dots M^{a_\mathtt{D}}{}_{m_\mathtt{D}]}\boldsymbol{\alpha}^{m_1}\wedge\dots\wedge\boldsymbol{\alpha}^{m_\mathtt{D}}\\
! &= \mathtt{D}!M^{[a_1}{}_1\dots M^{a_\mathtt{D}]}{}_\mathtt{D}\boldsymbol{\alpha}^1\wedge\dots\wedge\boldsymbol{\alpha}^\mathtt{D}\\
&= \mathtt{D}!\delta_{b_1}^{[a_1}\dots\delta_{b_\mathtt{D}}^{a_\mathtt{D}]}M^{[b_1}{}_1\dots M^{b_\mathtt{D}]}{}_\mathtt{D}\boldsymbol{\alpha}^1\wedge\dots\wedge\boldsymbol{\alpha}^\mathtt{D}\\
! &= \underbrace{\mathtt{D}!\delta_1^{[a_1}\dots\delta_\mathtt{D}^{a_\mathtt{D}]}}_{\bar{\epsilon}^{a_1\dots a_\mathtt{D}}}\underbrace{\mathtt{D}!M^{[1}{}_1\dots M^{\mathtt{D}]}{}_\mathtt{D}}_{\det(M)}\boldsymbol{\alpha}^1\wedge\dots\wedge\boldsymbol{\alpha}^\mathtt{D}
\end{aligned}$$

where in the steps ! we have used that an antisymmetrization of $\mathtt{D}$ indices (m's in one case and b's in the other) is non-zero only if each of them take a different value in $\{1, \dots, \mathtt{D}\}$. Notice that the factor $\mathtt{D}!$ that appears whenever we use this property is due to the fact that the choice $1\dots\mathtt{D}$ is not unique, because we have to take into account all possible permutations. ■

This leads us to the following particular cases:

- Case $a_i = i$, i.e., $\boldsymbol{\vartheta}^{1\dots\mathtt{D}} = \boldsymbol{\vartheta}^1\wedge\dots\wedge\boldsymbol{\vartheta}^\mathtt{D} = \frac{1}{\mathfrak{f}}\mathbf{vol}$:

$$\boldsymbol{\vartheta}^1\wedge\dots\wedge\boldsymbol{\vartheta}^\mathtt{D} = \det(M)\,\boldsymbol{\alpha}^1\wedge\dots\wedge\boldsymbol{\alpha}^\mathtt{D} \qquad\Leftrightarrow\qquad \mathbf{vol} = \frac{\mathfrak{f}}{\mathfrak{f}'}\det(M)\,\mathbf{vol}'\,. \tag{3.26}$$

- Case $\boldsymbol{\alpha}^a = \boldsymbol{\vartheta}^a$ (M is the identity):

$$\boldsymbol{\vartheta}^{a_1\dots a_\mathtt{D}} = \bar{\epsilon}^{a_1\dots a_\mathtt{D}}\boldsymbol{\vartheta}^1\wedge\dots\wedge\boldsymbol{\vartheta}^\mathtt{D} = \frac{1}{\mathfrak{f}}\bar{\epsilon}^{a_1\dots a_\mathtt{D}}\mathbf{vol}\,. \tag{3.27}$$

Canonical volume form associated to a metric

Given a metric (and a particular orientation), there exists a volume form canonically associated. It is constructed just by taking $\mathfrak{f} = \sqrt{|\det(g_{ab})|}$ in (3.19):

$$\mathbf{vol}_g \coloneqq \mathcal{E}_{a_1\ldots a_\mathtt{D}} \boldsymbol{\vartheta}^{\otimes a_1\ldots a_\mathtt{D}} = \frac{1}{\mathtt{D}!}\mathcal{E}_{a_1\ldots a_\mathtt{D}} \boldsymbol{\vartheta}^{a_1\ldots a_\mathtt{D}}\,, \qquad \mathcal{E}_{a_1\ldots a_\mathtt{D}} \coloneqq \sqrt{|\det(g_{ab})|}\epsilon_{a_1\ldots a_\mathtt{D}}\,. \tag{3.28}$$

The components $\mathcal{E}_{a_1\ldots a_\mathtt{D}}$ define the so-called *Levi-Civita tensor* associated to the metric.

❒ The expression of the canonical volume form is the same in any basis with the same orientation, one just has to change the elements of the basis and the determinant for the corresponding ones consistently. For example if we take a coordinate basis $\{\mathrm{d}x^\mu\}$ with the same orientation as $\{\boldsymbol{\vartheta}^a\}$ (if not, in ! we would have to introduce a minus sign):

$$\mathbf{vol}_g = \sqrt{|\det(g_{ab})|}\boldsymbol{\vartheta}^1 \wedge \ldots \wedge \boldsymbol{\vartheta}^\mathtt{D} \;=\; \sqrt{|\det(g_{ab})|}\,\det(e_\mu{}^a)^{-1}\mathrm{d}^\mathtt{D}x \;\stackrel{!}{=}\; \sqrt{|\det(g_{\mu\nu})|}\mathrm{d}^\mathtt{D}x\,. \tag{3.29}$$

where $\mathrm{d}^\mathtt{D}x \equiv \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^\mathtt{D}$.
Moreover, in the coordinate basis the Levi-Civita tensor can be written

$$\mathcal{E}_{\mu_1\ldots\mu_\mathtt{D}} = \sqrt{|\det(g_{\mu\nu})|}\epsilon_{\mu_1\ldots\mu_\mathtt{D}}\,, \tag{3.30}$$

where we have used the pseudotensorial character of ϵ, i.e., $[\det(e_\mu{}^a)\, e_{\mu_1}{}^{a_1}\ldots e_{\mu_\mathtt{D}}{}^{a_\mathtt{D}}]\,\epsilon_{a_1\ldots a_\mathtt{D}} = \epsilon_{\mu_1\ldots\mu_\mathtt{D}}$.

❒ If we use $\mathfrak{f} = \sqrt{|\det(g_{ab})|}$, the expression (3.27) can be re-written:

$$\boldsymbol{\vartheta}^{a_1\ldots a_\mathtt{D}} = \mathrm{sgn}(g)\,\mathcal{E}^{a_1\ldots a_\mathtt{D}}\mathbf{vol}_g\,. \tag{3.31}$$

❒ If the manifold is not orientable, the Levi-Civita tensor is not globally defined.

❒ An interesting property is:

$$\mathbf{D}\mathcal{E}_{a_1\ldots a_\mathtt{D}} = -\frac{1}{2}\boldsymbol{Q}_c{}^c\mathcal{E}_{a_1\ldots a_\mathtt{D}}\,. \tag{3.32}$$

> *Proof.* $\mathbf{D}\mathcal{E}_{a_1\ldots a_\mathtt{D}} = \mathbf{D}\left(\sqrt{|\det(g_{ab})|}\epsilon_{a_1\ldots a_\mathtt{D}}\right) = \frac{\mathbf{D}\sqrt{|\det(g_{ab})|}}{\sqrt{|\det(g_{ab})|}}\mathcal{E}_{a_1\ldots a_\mathtt{D}} = -\frac{1}{2}\boldsymbol{Q}_c{}^c\mathcal{E}_{a_1\ldots a_\mathtt{D}}$. ■

NOTE. To abbreviate, from now on we will call $\sqrt{|\det(g_{ab})|} \equiv \sqrt{|g|}$, i.e. the symbol g refers to the general basis that we are using. Depending on the context it can represent $\sqrt{|\det(g_{\mu\nu})|}$.

4 Hodge dual. Definition and basic properties

Consider a $\mathtt{D}$-dimensional orientable manifold equipped with a metric. In this context, the Levi-Civita tensor of the metric allows to establish a duality between k-forms and $(\mathtt{D}-k)$-forms as follows: given a k-form $\boldsymbol{\alpha}$, we define its *Hodge dual* (via the *Hodge star operator*) as the $(\mathtt{D}-k)$-form

$$\boxed{\star\boldsymbol{\alpha} \;:=\; \frac{1}{(\mathtt{D}-k)!k!}\mathcal{E}_{b_1\ldots b_k a_1\ldots a_{\mathtt{D}-k}}\alpha^{b_1\ldots b_k}\;\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}}}\,, \tag{4.1}$$

i.e., its compoents in the tensor basis $\{\boldsymbol{\vartheta}^{\otimes a_1\ldots a_{\mathtt{D}-k}}\}$ are

$$(\star\boldsymbol{\alpha})_{a_1\ldots a_{\mathtt{D}-k}} = \frac{1}{k!}\mathcal{E}_{b_1\ldots b_k a_1\ldots a_{\mathtt{D}-k}}\alpha^{b_1\ldots b_k}\,. \tag{4.2}$$

Basic properties of the Hodge dual

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two k-form and $\lambda\in\mathbb{K}$.

- ❒ **Linearity**

$$\star(\boldsymbol{\alpha}+\boldsymbol{\beta}) = \star\boldsymbol{\alpha}+\star\boldsymbol{\beta}\,, \qquad\qquad \star(\lambda\boldsymbol{\alpha}) = \lambda\star\boldsymbol{\alpha}\,. \tag{4.3}$$

- ❒ **Double star.** The composition of the Hodge star with itself is the identity up to a sign:

$$\star\star\boldsymbol{\alpha} = (-1)^{k(\mathtt{D}-k)}\,\mathrm{sgn}(g)\boldsymbol{\alpha}\,. \tag{4.4}$$

Proof.

$$\begin{aligned}
(\star\star\boldsymbol{\alpha})_{a_1\ldots a_k} &= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{b_1\ldots b_{\mathtt{D}-k}a_1\ldots a_k}\,(\star\boldsymbol{\alpha})^{b_1\ldots b_{\mathtt{D}-k}} \\
&= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{b_1\ldots b_{\mathtt{D}-k}a_1\ldots a_k}g^{b_1d_1}\ldots g^{b_{\mathtt{D}-k}d_{\mathtt{D}-k}}\,(\star\boldsymbol{\alpha})_{d_1\ldots d_{\mathtt{D}-k}} \\
&= \frac{1}{(\mathtt{D}-k)!k!}\mathcal{E}_{b_1\ldots b_{\mathtt{D}-k}a_1\ldots a_k}g^{b_1d_1}\ldots g^{b_{\mathtt{D}-k}d_{\mathtt{D}-k}}\frac{1}{k!}\mathcal{E}_{c_1\ldots c_k d_1\ldots d_{\mathtt{D}-k}}\alpha^{c_1\ldots c_k} \\
&= \frac{1}{(\mathtt{D}-k)!k!}\mathcal{E}_{b_1\ldots b_{\mathtt{D}-k}a_1\ldots a_k}\mathcal{E}^{c_1\ldots c_k}{}_{d_1\ldots d_{\mathtt{D}-k}}g^{b_1d_1}\ldots g^{b_{\mathtt{D}-k}d_{\mathtt{D}-k}}\alpha_{c_1\ldots c_k} \\
&= \frac{1}{(\mathtt{D}-k)!k!}(-1)^{k(\mathtt{D}-k)}\mathcal{E}_{b_1\ldots b_{\mathtt{D}-k}a_1\ldots a_k}\mathcal{E}^{b_1\ldots b_{\mathtt{D}-k}c_1\ldots c_k}\alpha_{c_1\ldots c_k} \\
&= \frac{1}{(\mathtt{D}-k)!k!}(-1)^{k(\mathtt{D}-k)}\left(\mathrm{sgn}(g)(\mathtt{D}-k)!k!\delta^{[c_1}_{a_1}\ldots\delta^{c_k]}_{a_k}\right)\alpha_{c_1\ldots c_k} \\
&= (-1)^{k(\mathtt{D}-k)}\,\mathrm{sgn}(g)\alpha_{a_1\ldots a_k}
\end{aligned}$$

■

- ❒ **Inverse star.** From the previous result one can immediately deduce that

$$\star^{-1}\boldsymbol{\alpha} = (-1)^{k(\mathtt{D}-k)}\,\mathrm{sgn}(g)\star\boldsymbol{\alpha} \tag{4.5}$$

Interesting examples

❐ Dual of the elements of the k-form basis

$$\star\boldsymbol{\vartheta}^{c_1\ldots c_k} = \frac{1}{(\mathtt{D}-k)!k!}\mathcal{E}_{b_1\ldots b_k a_1\ldots a_{\mathtt{D}-k}}\left(k!g^{c_1 b_1}\ldots g^{c_k b_k}\right)\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}} \tag{4.6}$$

$$= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}^{c_1\ldots c_k}{}_{a_1\ldots a_{\mathtt{D}-k}}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}} \tag{4.7}$$

$$= \mathcal{E}^{c_1\ldots c_k}{}_{a_1\ldots a_{\mathtt{D}-k}}\boldsymbol{\vartheta}^{\otimes a_1\ldots a_{\mathtt{D}-k}}\,. \tag{4.8}$$

In particular,

$$\star\boldsymbol{\vartheta}_c = \frac{1}{(\mathtt{D}-1)!}\mathcal{E}_{c a_1\ldots a_{\mathtt{D}-1}}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-1}}\,. \tag{4.9}$$

❐ Dual of the canonical volume form

$$\star\mathbf{vol}_g = \frac{1}{\mathtt{D}!}\mathcal{E}_{b_1\ldots b_k}\mathcal{E}^{b_1\ldots b_k} = \frac{1}{\mathtt{D}!}\,\mathrm{sgn}(g)\mathtt{D}! = \mathrm{sgn}(g)\,. \tag{4.10}$$

Consequently, any $\mathtt{D}$-form can be written

$$\boldsymbol{\alpha} = \mathrm{sgn}(g)\;(\star\boldsymbol{\alpha})\;\;\mathbf{vol}_g\,. \tag{4.11}$$

❐ Dual of a 0-form f and particular case $f = 1$

$$(\star f)_{a_1\ldots a_{\mathtt{D}}} = f\mathcal{E}_{a_1\ldots a_{\mathtt{D}}} \;\Leftrightarrow\; \star f = f\mathbf{vol}_g \qquad \overset{f\equiv 1}{\Rightarrow} \qquad \star 1 = \mathbf{vol}_g\,, \tag{4.12}$$

from which we can see that for any k-form $\boldsymbol{\alpha}$,

$$\star(f\boldsymbol{\alpha}) = f\star\boldsymbol{\alpha}\,, \tag{4.13}$$

i.e., the star is $C^\infty(\mathcal{M})$-linear.

5 Useful formulae to manipulate expressions with differential forms

5.1 Hodge dual and interior/exterior products

5.1.1 Exterior product of a form with the Hodge dual of another one of the same rank

For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Omega^k(\mathcal{M})$, we have

$$\boxed{\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta} = \frac{1}{k!}\alpha^{a_1\dots a_k}\beta_{a_1\dots a_k}\mathbf{vol}_g}\,. \tag{5.1}$$

Two special cases are:

- Case $\boldsymbol{\alpha} = \boldsymbol{\beta}$:

$$\boldsymbol{\alpha} \wedge \star\boldsymbol{\alpha} = \frac{1}{k!}\alpha^{a_1\dots a_k}\alpha_{a_1\dots a_k}\mathbf{vol}_g\,, \tag{5.2}$$

- Case $\boldsymbol{\alpha} = \boldsymbol{\vartheta}^{a_1\dots a_k}$, $\boldsymbol{\beta} = \boldsymbol{\vartheta}_{b_1\dots b_k}$

$$\boldsymbol{\vartheta}^{a_1\dots a_k} \wedge \star\boldsymbol{\vartheta}_{b_1\dots b_k} = k!\delta^{a_1}_{[b_1}\dots\delta^{a_k}_{b_k]}\mathbf{vol}_g\,. \tag{5.3}$$

Proof.

$$\begin{aligned}
k!\boldsymbol{\alpha} \wedge (\star\boldsymbol{\beta}) &= k!\left(\frac{1}{k!}\alpha_{a_1\dots a_k}\boldsymbol{\vartheta}^{a_1\dots a_k}\right) \wedge \left(\frac{1}{(\mathtt{D}-k)!k!}\mathcal{E}_{b_1\dots b_k a_{k+1}\dots a_{\mathtt{D}}}\beta^{b_1\dots b_k}\boldsymbol{\vartheta}^{a_{k+1}\dots a_{\mathtt{D}}}\right)\\
&= \frac{1}{(\mathtt{D}-k)!k!}\beta_{b_1\dots b_k}\alpha_{a_1\dots a_k}\mathcal{E}^{b_1\dots b_k}{}_{a_{k+1}\dots a_{\mathtt{D}}}\boldsymbol{\vartheta}^{a_1\dots a_{\mathtt{D}}}\\
&= \frac{1}{\mathtt{D}!}\frac{\mathtt{D}!}{(\mathtt{D}-k)!k!}\beta_{b_1\dots b_k}\alpha_{[a_1\dots a_k}\mathcal{E}^{b_1\dots b_k}{}_{a_{k+1}\dots a_{\mathtt{D}}]}\boldsymbol{\vartheta}^{a_1\dots a_{\mathtt{D}}}\\
&= \frac{1}{\mathtt{D}!}\binom{\mathtt{D}}{k}\beta_{b_1\dots b_k}\alpha_{[a_1\dots a_k}\mathcal{E}_{a_{k+1}\dots a_{\mathtt{D}}]}{}^{b_1\dots b_k}(-1)^{k(\mathtt{D}-k)}\boldsymbol{\vartheta}^{a_1\dots a_{\mathtt{D}}}\\
&= \frac{1}{\mathtt{D}!}\binom{\mathtt{D}}{k}\beta_{b_1\dots b_k}\mathcal{E}_{[a_1\dots a_{\mathtt{D}-k}}{}^{b_1\dots b_k}\alpha_{a_{\mathtt{D}-k+1}\dots a_{\mathtt{D}}]}\boldsymbol{\vartheta}^{a_1\dots a_{\mathtt{D}}}\\
\overset{(A.1)}{=}& \frac{1}{\mathtt{D}!}\beta_{b_1\dots b_k}\alpha^{b_1\dots b_k}\mathcal{E}_{a_1\dots a_{\mathtt{D}}}\boldsymbol{\vartheta}^{a_1\dots a_{\mathtt{D}}}\\
&= \beta_{b_1\dots b_k}\alpha^{b_1\dots b_k}\mathbf{vol}_g
\end{aligned}$$

The first particular case is immediate. For the other one we use $(\boldsymbol{\vartheta}^{a_1\dots a_k})_{c_1\dots c_k} = k!\delta^{[a_1}_{c_1}\dots\delta^{a_k]}_{c_k}$ so

$$\begin{aligned}
\boldsymbol{\alpha} \wedge (\star\boldsymbol{\beta}) &= \frac{1}{k!}\alpha_{c_1\dots c_k}\beta^{c_1\dots c_k}\mathbf{vol}_g\\
&= \frac{1}{k!}k!\delta^{[a_1}_{c_1}\dots\delta^{a_k]}_{c_k}k!\delta^{c_1}_{[b_1}\dots\delta^{c_k}_{b_k]}\mathbf{vol}_g\\
&= k!\delta^{a_1}_{[b_1}\dots\delta^{a_k}_{b_k]}\mathbf{vol}_g
\end{aligned}$$

∎

From the previous expressions one can easily check the following symmetry property:

$$\boldsymbol{\alpha} \wedge \star \boldsymbol{\beta} = \boldsymbol{\beta} \wedge \star \boldsymbol{\alpha} \qquad \xrightarrow{\times(-1)^{k(\mathtt{D}-k)}} \qquad (\star \boldsymbol{\alpha}) \wedge \boldsymbol{\beta} = (\star \boldsymbol{\beta}) \wedge \boldsymbol{\alpha}\,. \tag{5.4}$$

5.1.2 Hodge dual of a exterior product VS interior product of a Hodge dual

Consider $\boldsymbol{\alpha}_{(k)} \in \Omega^k(\mathcal{M})$ and $\boldsymbol{\beta}_{(p)} \in \Omega^p(\mathcal{M})$ such that $k+p \leq \mathtt{D}$. Then:

$$\star\left(\boldsymbol{\alpha}_{(k)} \wedge \boldsymbol{\beta}_{(p)}\right) = \frac{1}{k!p!} \mathcal{E}_{a_1\ldots a_{k+p} b_1 \ldots b_{\mathtt{D}-(k+p)}} \alpha^{a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{k+p}} \boldsymbol{\vartheta}^{\otimes b_1 \ldots b_{\mathtt{D}-(k+p)}}\,. \tag{5.5}$$

Proof. Let us omit the subscripts $\boldsymbol{\alpha}_{(k)} \equiv \boldsymbol{\alpha}$, $\boldsymbol{\beta}_{(p)} \equiv \boldsymbol{\beta}$:

$$\begin{aligned}
\star(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) &= \star\left(\frac{1}{k!}\frac{1}{p!} \alpha_{a_1\ldots a_k} \beta_{a_{k+1}\ldots a_{k+p}} \boldsymbol{\vartheta}^{a_1\ldots a_{k+p}}\right) \\
&= \star\left(\frac{(k+p)!}{k!p!} \alpha_{[a_1\ldots a_k} \beta_{a_{k+1}\ldots a_{k+p}]} \boldsymbol{\vartheta}^{\otimes a_1\ldots a_{k+p}}\right) \\
&= \frac{(k+p)!}{k!p!} \frac{1}{(k+p)!} \mathcal{E}_{a_1\ldots a_{k+p} b_1\ldots b_{\mathtt{D}-(k+p)}} \alpha^{[a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{k+p}]} \boldsymbol{\vartheta}^{\otimes b_1\ldots b_{\mathtt{D}-(k+p)}} \\
&= \frac{1}{k!p!} \mathcal{E}_{a_1\ldots a_{k+p} b_1\ldots b_{\mathtt{D}-(k+p)}} \alpha^{a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{k+p}} \boldsymbol{\vartheta}^{\otimes b_1\ldots b_{\mathtt{D}-(k+p)}}
\end{aligned}$$

■

❒ Case $p = \mathtt{D} - k$:

$$\star\left(\boldsymbol{\alpha}_{(k)} \wedge \boldsymbol{\beta}_{(\mathtt{D}-k)}\right) = \frac{1}{k!(\mathtt{D}-k)!} \mathcal{E}_{a_1\ldots a_{\mathtt{D}}} \alpha^{a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{\mathtt{D}}}\,, \tag{5.6}$$

$$\boldsymbol{\alpha}_{(k)} \wedge \boldsymbol{\beta}_{(\mathtt{D}-k)} = \frac{1}{k!(\mathtt{D}-k)!} \mathcal{E}_{a_1\ldots a_{\mathtt{D}}} \alpha^{a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{\mathtt{D}}}\, \mathrm{sgn}(g) \mathbf{vol}_g\,. \tag{5.7}$$

Proof. The first one is trivially obtained from (5.5) with $p = \mathtt{D} - k$. Finally we take the dual in both sides of the equation:

$$\star^2\left(\boldsymbol{\alpha}_{(k)} \wedge \boldsymbol{\beta}_{(\mathtt{D}-k)}\right) = \frac{1}{k!(\mathtt{D}-k)!} \mathcal{E}_{a_1\ldots a_{\mathtt{D}}} \alpha^{a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{\mathtt{D}}} (\star 1)$$

$$\boldsymbol{\alpha}_{(k)} \wedge \boldsymbol{\beta}_{(\mathtt{D}-k)} = \frac{1}{k!(\mathtt{D}-k)!}\, \mathrm{sgn}(g) \mathcal{E}_{a_1\ldots a_{\mathtt{D}}} \alpha^{a_1\ldots a_k} \beta^{a_{k+1}\ldots a_{\mathtt{D}}} \mathbf{vol}_g$$

■

❒ Case $p = 1$:[4]

$$\star\left(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}_{(1)}\right) = \left(\boldsymbol{\beta}_{(1)}{}^{\sharp}\right)\lrcorner\,(\star\boldsymbol{\alpha})\ . \tag{5.8}$$

Here we see that dual of the exterior product with a 1-form coincides with the interior product with respect to the vector associated to the 1-form (through the metric) of the dual. If we apply this to the basis elements we get a very important and useful equation:

$$\boxed{\star\,(\boldsymbol{\alpha}\wedge\boldsymbol{\vartheta}^{a}) = \boldsymbol{e}^{a}\lrcorner\,(\star\boldsymbol{\alpha})}\,. \tag{5.9}$$

Proof. We take $p = 1$ in (5.5):

$$\begin{aligned}
\star\left(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}_{(1)}\right) &= \frac{1}{k!}\mathcal{E}_{a_1\ldots a_k c b_1\ldots b_{\mathrm{D}-(k+1)}}\,\alpha^{a_1\ldots a_k}\,\beta_{(1)}{}^{c}\boldsymbol{\vartheta}^{\otimes b_1\ldots b_{\mathrm{D}-(k+1)}}\\
&= \frac{1}{k!}\mathcal{E}_{a_1\ldots a_k b_1 b_2\ldots b_{\mathrm{D}-k}}\,\alpha^{a_1\ldots a_k}\,\beta_{(1)}{}^{b_1}\boldsymbol{\vartheta}^{\otimes b_2\ldots b_{\mathrm{D}-k}}\\
&= \frac{1}{k!}\mathcal{E}_{a_1\ldots a_k b_1 b_2\ldots b_{\mathrm{D}-k}}\,\alpha^{a_1\ldots a_k}\boldsymbol{\vartheta}^{b_1}\left(\beta_{(1)}{}^{c}\boldsymbol{e}_{c}\right)\otimes\boldsymbol{\vartheta}^{\otimes b_2\ldots b_{\mathrm{D}-k}}\\
&= \boldsymbol{\beta}_{(1)}{}^{\sharp}\lrcorner\left[\frac{1}{k!}\mathcal{E}_{a_1\ldots a_k b_1 b_2\ldots b_{\mathrm{D}-k}}\,\alpha^{a_1\ldots a_k}\boldsymbol{\vartheta}^{\otimes b_1\ldots b_{\mathrm{D}-k}}\right]\\
&= \boldsymbol{\beta}_{(1)}{}^{\sharp}\lrcorner\,(\star\boldsymbol{\alpha})
\end{aligned}$$

■

From here we can extract two immediate corollaries (we omit the proofs):

- Sub-case $\boldsymbol{\alpha} = \boldsymbol{\vartheta}^{b_1\ldots b_k}$

$$\boldsymbol{e}^{a}\lrcorner\star\boldsymbol{\vartheta}^{b_1\ldots b_k} = \star\boldsymbol{\vartheta}^{b_1\ldots b_k a}\,. \tag{5.10}$$

- By successively applying the property one can prove:

$$\star\left(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}_{(1)}\wedge\ldots\wedge\boldsymbol{\gamma}_{(1)}\right) = \left(\boldsymbol{\gamma}_{(1)}{}^{\sharp}\right)\lrcorner\ldots\left(\boldsymbol{\beta}_{(1)}{}^{\sharp}\right)\lrcorner(\star\boldsymbol{\alpha})\,, \tag{5.11}$$

  Again, if we apply this to the basis elements:

$$\star\left(\boldsymbol{\alpha}\wedge\boldsymbol{\vartheta}^{a\ldots b}\right) = \boldsymbol{e}^{b}\lrcorner\ldots\boldsymbol{e}^{a}\lrcorner(\star\boldsymbol{\alpha})\,. \tag{5.12}$$

Interior products of the canonical volume form

It is worth remarking the sub-case of (5.12) when $\boldsymbol{\alpha} = 1$ (0-form):

$$\boxed{\boldsymbol{e}_{a_k}\lrcorner\ldots\boldsymbol{e}_{a_1}\lrcorner\ \mathbf{vol}_{g} = \star\boldsymbol{\vartheta}_{a_1\ldots a_k}}\,. \tag{5.13}$$

This relates k successive interior products of the canonical volume form with the dual of an element of the k-form basis.

[4] The *sharp* map is known in physics as "raising indices". Formally, the sharp of a 1-form $\boldsymbol{\alpha}_{(1)}$ is defined as the (only) vector $\boldsymbol{\alpha}_{(1)}{}^{\sharp}$ satisfying for all 1-forms $\boldsymbol{\beta}_{(1)}$ the condition

$$\boldsymbol{\beta}_{(1)}(\boldsymbol{\alpha}_{(1)}{}^{\sharp}) = \boldsymbol{g}^{-1}(\boldsymbol{\alpha}_{(1)},\boldsymbol{\beta}_{(1)})\,.$$

In components, what happens is:

$$(\boldsymbol{\alpha}_{(1)})_a = \alpha_a \qquad\Rightarrow\qquad (\boldsymbol{\alpha}_{(1)}{}^{\sharp})^a = g^{ab}\alpha_b\,.$$

So one can easily check the previous relation:

$$\boldsymbol{\beta}_{(1)}(\boldsymbol{\alpha}_{(1)}{}^{\sharp}) = \beta_a(\boldsymbol{\alpha}_{(1)}{}^{\sharp})^a = g^{ab}\beta_a\alpha_b \equiv \boldsymbol{g}^{-1}(\boldsymbol{\alpha}_{(1)},\boldsymbol{\beta}_{(1)})\,.$$

Similarly for lowering the indices one defines the *flat* map $\boldsymbol{V}^{\flat}$ that gives the corresponding 1-form when acting on a vector (lowering the index with the metric).

5.1.3 Hodge dual of a interior product VS exterior product of a Hodge dual

Let $\boldsymbol{\alpha}$ be a k-form and $p \in \mathbb{N}$ such that $p < k$ (so that the expressions do not trivialize). Then the following relations are fulfilled

$$\boldsymbol{\vartheta}^{a_1\dots a_p} \wedge \star\boldsymbol{\alpha} = (-1)^{p(k+1)} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \boldsymbol{\alpha}) \,, \tag{5.14}$$

$$\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p} = \operatorname{sgn}(g)(-1)^{(\mathtt{D}+1)(k+p)} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner (\star\boldsymbol{\alpha})) \,. \tag{5.15}$$

Proof. We start with (5.12) and take dual in both sides:

$$\star^2 (\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p}) = \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner (\star\boldsymbol{\alpha}))$$

❒ Case $\boldsymbol{\alpha} \to \star\boldsymbol{\alpha}$,

$$\begin{aligned}
\star^2 (\star\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p}) &= \star \left(\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \left(\star^2\boldsymbol{\alpha}\right)\right) \\
\operatorname{sgn}(g)(-1)^{(\mathtt{D}-k+p)(k-p)} (\star\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p}) &= \star \left(\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \left(\operatorname{sgn}(g)(-1)^{k(\mathtt{D}-k)}\boldsymbol{\alpha}\right)\right) \\
\star\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p} &= (-1)^{p(k-p)}(-1)^{(\mathtt{D}-k)(k-p)}(-1)^{k(\mathtt{D}-k)} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \boldsymbol{\alpha}) \\
\star\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p} &= (-1)^{p(\mathtt{D}-p)} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \boldsymbol{\alpha}) \\
(-1)^{p(\mathtt{D}-k)}\boldsymbol{\vartheta}^{a_1\dots a_p} \wedge \star\boldsymbol{\alpha} &= (-1)^{p(\mathtt{D}-p)} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \boldsymbol{\alpha}) \\
\boldsymbol{\vartheta}^{a_1\dots a_p} \wedge \star\boldsymbol{\alpha} &= (-1)^{p(k+1)} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner \boldsymbol{\alpha})
\end{aligned}$$

❒ Case $\boldsymbol{\alpha} \to \boldsymbol{\alpha}$:

$$\begin{aligned}
\operatorname{sgn}(g)(-1)^{(\mathtt{D}-k+p)(k+p)} (\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p}) &= \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner (\star\boldsymbol{\alpha})) \\
\operatorname{sgn}(g)(-1)^{\mathtt{D}(k+p)+k+p+2kp} (\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p}) &= \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner (\star\boldsymbol{\alpha})) \\
\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a_1\dots a_p} &= \operatorname{sgn}(g)(-1)^{(\mathtt{D}+1)(k+p)+kp} \star (\boldsymbol{e}^{a_p} \lrcorner \dots \boldsymbol{e}^{a_1} \lrcorner (\star\boldsymbol{\alpha}))
\end{aligned}$$

Note: in this proof we have used $(-1)^{n^2} = (-1)^n$ (for all $n \in \mathbb{Z}$).

■

❒ Case $p = 1$:

$$\boxed{\boldsymbol{\vartheta}^{a} \wedge \star\boldsymbol{\alpha} = (-1)^{k+1} \star (\boldsymbol{e}^{a} \lrcorner \boldsymbol{\alpha})}\,, \tag{5.16}$$

$$\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{a} = \operatorname{sgn}(g)(-1)^{(\mathtt{D}+1)(k+1)} \star (\boldsymbol{e}^{a} \lrcorner (\star\boldsymbol{\alpha})) \,. \tag{5.17}$$

These expression are of course valid for any 1-form $\boldsymbol{\beta}$ (just by contracting the free index with the components β_a):

$$\boldsymbol{\beta} \wedge \star\boldsymbol{\alpha} = (-1)^{k+1} \star \left(\boldsymbol{\beta}^\sharp \lrcorner \boldsymbol{\alpha}\right) \,, \tag{5.18}$$

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \operatorname{sgn}(g)(-1)^{(k+1)(\mathtt{D}+1)} \star \left(\boldsymbol{\beta}^\sharp \lrcorner \star \boldsymbol{\alpha}\right) \,. \tag{5.19}$$

❒ If we evaluate $\boldsymbol{\alpha} \to \boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{b_1\dots b_p}$ (our hypothesis is still valid since $p < k+p$)

$$\boldsymbol{\vartheta}_{a_1\dots a_p} \wedge \star \left(\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{b_1\dots b_p}\right) = (-1)^{p(k+p+1)} \star \left[\boldsymbol{e}_{a_p} \lrcorner \dots \boldsymbol{e}_{a_1} \lrcorner \left(\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{b_1\dots b_p}\right)\right] \,, \tag{5.20}$$

For instance for the cases $p = 1$ and $p = 2$ we obtain

$$\boldsymbol{\vartheta}_{a} \wedge \star \left(\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{b}\right) = \star \left[\delta_a^b \boldsymbol{\alpha} - \boldsymbol{\vartheta}^{b} \wedge (\boldsymbol{e}_a \lrcorner \boldsymbol{\alpha})\right] \,, \tag{5.21}$$

$$\boldsymbol{\vartheta}_{ac} \wedge \star \left(\boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}^{bd}\right) = \star \left[4\delta_{[a}^{[b}\delta_{c]}^{d]} \boldsymbol{\alpha} - 4\delta_{[a}^{[b} \boldsymbol{\vartheta}^{d]} \wedge (\boldsymbol{e}_{c]} \lrcorner \boldsymbol{\alpha}) + (\boldsymbol{e}_c \lrcorner \boldsymbol{e}_a \lrcorner \boldsymbol{\alpha}) \wedge \boldsymbol{\vartheta}^{bd}\right] \,. \tag{5.22}$$

❒ Useful related formulas involving the basis elements are

$$\boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\star\boldsymbol{\vartheta}^{c_1\ldots c_k b_1\ldots b_p} = \frac{(k+p)!}{k!}\delta^{[b_1}_{a_1}\ldots\delta^{b_p}_{a_p}\star\boldsymbol{\vartheta}^{c_1\ldots c_k]}\,, \tag{5.23}$$

$$= \frac{1}{(\mathtt{D}-k-p)!}\mathcal{E}^{c_1\ldots c_k b_1\ldots b_p}{}_{a_{p+1}\ldots a_{\mathtt{D}-k}}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}}\,. \tag{5.24}$$

The contracted version of the previous equation is

$$\boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\star\boldsymbol{\vartheta}^{c_1\ldots c_k a_1\ldots a_p} = \frac{(\mathtt{D}-k)!}{(\mathtt{D}-k-p)!}\star\boldsymbol{\vartheta}^{c_1\ldots c_k}\,. \tag{5.25}$$

Proof. For the first one

$$\begin{aligned}\boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\star\boldsymbol{\vartheta}^{c_1\ldots c_k b_1\ldots b_p} &= (-1)^{p(k+p+1)}\star\left[\boldsymbol{e}_{a_p}\lrcorner\ldots\boldsymbol{e}_{a_1}\lrcorner\boldsymbol{\vartheta}^{c_1\ldots c_k b_1\ldots b_p}\right]\\ &= (-1)^{p(k+p+1)}(-1)^{kp}\star\left[\boldsymbol{e}_{a_p}\lrcorner\ldots\boldsymbol{e}_{a_1}\lrcorner\boldsymbol{\vartheta}^{b_1\ldots b_p c_1\ldots c_k}\right]\\ &= \cancel{(-1)^{p(p+1)}}(k+p)(k+p-1)\ldots(k+1)\delta^{[b_1}_{a_1}\ldots\delta^{b_p}_{a_p}\star\boldsymbol{\vartheta}^{c_1\ldots c_k]}\\ &= \frac{(k+p)!}{k!}\delta^{[b_1}_{a_1}\ldots\delta^{b_p}_{a_p}\star\boldsymbol{\vartheta}^{c_1\ldots c_k]}\end{aligned}$$

For the alternative expression we use the definition of the Hodge dual at the beginning

$$\begin{aligned}&\boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\star\boldsymbol{\vartheta}^{c_1\ldots c_k b_1\ldots b_p}\\ &= \boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\star\left[(k+p)!\delta^{[c_1}_{m_1}\ldots\delta^{c_k}_{m_k}\delta^{b_1}_{n_1}\ldots\delta^{b_p]}_{n_p}\boldsymbol{\vartheta}^{\otimes m_1\ldots m_k n_1\ldots n_p}\right]\\ &= \boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\left[\frac{1}{(k+p)!(\mathtt{D}-k-p)!}\mathcal{E}^{m_1\ldots m_k n_1\ldots n_p}{}_{d_1\ldots d_{\mathtt{D}-k-p}}(k+p)!\delta^{[c_1}_{m_1}\ldots\delta^{c_k}_{m_k}\delta^{b_1}_{n_1}\ldots\delta^{b_p]}_{n_p}\boldsymbol{\vartheta}^{d_1\ldots d_{\mathtt{D}-k-p}}\right]\\ &= \boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\left[\frac{1}{(\mathtt{D}-k-p)!}\mathcal{E}^{c_1\ldots c_k b_1\ldots b_p}{}_{d_1\ldots d_{\mathtt{D}-k-p}}\boldsymbol{\vartheta}^{d_1\ldots d_{\mathtt{D}-k-p}}\right]\\ &= \frac{1}{(\mathtt{D}-k-p)!}\mathcal{E}^{c_1\ldots c_k b_1\ldots b_p}{}_{a_{p+1}\ldots a_{\mathtt{D}-k}}\boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\boldsymbol{\vartheta}^{a_{p+1}\ldots a_{\mathtt{D}-k}}\end{aligned}$$

For the contraction it is easier to use the second one:

$$\boldsymbol{\vartheta}_{a_1\ldots a_p}\wedge\star\left(\boldsymbol{\vartheta}^{c_1\ldots c_k b_1\ldots b_p}\right) = \frac{1}{(\mathtt{D}-k-p)!}\mathcal{E}^{c_1\ldots c_k}{}_{a_1\ldots a_{\mathtt{D}-k}}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}} = \frac{(\mathtt{D}-k)!}{(\mathtt{D}-k-p)!}\star\boldsymbol{\vartheta}^{c_1\ldots c_k}$$

■

Some examples are

$$\boldsymbol{\vartheta}_a\wedge\star\boldsymbol{\vartheta}^{mnc} = 3\delta^{[m}_a\star\boldsymbol{\vartheta}^{nc]}\,, \tag{5.26}$$

$$\boldsymbol{\vartheta}_{ab}\wedge\star\boldsymbol{\vartheta}^{mncd} = 12\delta^{[m}_a\delta^n_b\star\boldsymbol{\vartheta}^{cd]}\,, \tag{5.27}$$

$$\boldsymbol{\vartheta}_a\wedge\star\boldsymbol{\vartheta}^{ca} = (\mathtt{D}-1)\star\boldsymbol{\vartheta}^c\,, \tag{5.28}$$

$$\boldsymbol{\vartheta}_{ab}\wedge\star\boldsymbol{\vartheta}^{cab} = (\mathtt{D}-1)(\mathtt{D}-2)\star\boldsymbol{\vartheta}^c\,. \tag{5.29}$$

5.2 Hodge dual and exterior derivative

Consider $\boldsymbol{\alpha}^A \in \Omega^k(\mathcal{M};V)$ a differential form with values in some vector space. If we call $\mathbf{D}$ and ∇ the exterior and ordinary total covariant derivatives (including not only the gravitational but also the other internal connections), we have

$$\begin{aligned}\mathbf{D}\star\boldsymbol{\alpha}^A =&\ \frac{(-1)^{k+1}}{(k-1)!(\mathtt{D}-k+1)!}\mathcal{E}_{a_1\ldots,a_{k-1}c_1\ldots c_{\mathtt{D}-k+1}}\boldsymbol{\vartheta}^{c_1\ldots c_{\mathtt{D}-k+1}}\\ &\times\left[\left(\nabla_b-\frac{1}{2}Q_{bc}{}^c+T_{bc}{}^c\right)\alpha^{ba_1\ldots,a_{k-1}A}-\frac{k-1}{2}T_{bc}{}^{[a_1}\alpha^{|bc|a_2\ldots,a_{k-1}]A}\right], \end{aligned}\tag{5.30}$$

$$\begin{aligned}\star\mathbf{D}\star\boldsymbol{\alpha}^A =&\ \frac{(-1)^{\mathtt{D}(k-1)}\,\mathrm{sgn}(g)}{(k-1)!}\boldsymbol{\vartheta}_{a_1\ldots a_{k-1}}\\ &\times\left[\left(\nabla_b-\frac{1}{2}Q_{bc}{}^c+T_{bc}{}^c\right)\alpha^{ba_1\ldots a_{k-1}A}-\frac{k-1}{2}T_{bc}{}^{[a_1}\alpha^{|bc|a_2\ldots a_{k-1}]A}\right]. \end{aligned}\tag{5.31}$$

> *Proof.* See page Appendix B.4. ∎

If $\boldsymbol{\alpha}$ has no external indices the conection plays no role (since $\mathbf{D}=\mathrm{d}$), so the connection is just an auxiliary object in the right hand side that will disappear if we expand all the covariant derivatives and simplify. Hence, we can take the simplest connection (Levi-Civita) without loss of generality:

$$\mathrm{d}\star\boldsymbol{\alpha} = \frac{(-1)^{k+1}}{(k-1)!(\mathtt{D}-k+1)!}\mathcal{E}_{a_1\ldots,a_{k-1}c_1\ldots c_{\mathtt{D}-k+1}}\mathring{\nabla}_b\alpha^{ba_1\ldots,a_{k-1}}\boldsymbol{\vartheta}^{c_1\ldots c_{\mathtt{D}-k+1}}, \tag{5.32}$$

$$\boxed{\star\mathrm{d}\star\boldsymbol{\alpha} = \frac{(-1)^{\mathtt{D}(k-1)}}{(k-1)!}\,\mathrm{sgn}(g)\mathring{\nabla}_b\alpha^b{}_{a_1\ldots a_{k-1}}\boldsymbol{\vartheta}^{a_1\ldots a_{k-1}}}\,. \tag{5.33}$$

Interesting example: gradient, divergence and curl operators

If we take the 3-dimensional Euclidean metric $\mathrm{sgn}(g)=1$,

$$\begin{aligned}\mathrm{grad}(f) &= \partial^a f\,\boldsymbol{e}_a && = (\mathrm{d}f)^\sharp\,,\\ \mathrm{div}(\boldsymbol{V}) &= \mathring{\nabla}_b V^b && = \star\mathrm{d}\star\boldsymbol{V}^\flat\,,\\ \mathrm{rot}(\boldsymbol{V}) &= \mathcal{E}^{abc}\partial_b V_c\,\boldsymbol{e}_a\\ &= 2\left[\frac{1}{2}\mathcal{E}^{bca}\partial_b V_c\,\boldsymbol{\vartheta}_a\right]^\sharp && = 2\left(\star\mathrm{d}\boldsymbol{V}^\flat\right)^\sharp\,. \end{aligned}$$

Another interesting formula

$$\mathbf{D}\star\boldsymbol{\vartheta}_{a_1\ldots a_k} = -\frac{1}{2}\boldsymbol{Q}_c{}^c\wedge\star\boldsymbol{\vartheta}_{a_1\ldots a_k} + \boldsymbol{T}^c\wedge\star\boldsymbol{\vartheta}_{a_1\ldots a_k c}\,. \tag{5.34}$$

Notice that $\mathbf{D}\star\boldsymbol{\vartheta}^{a_1\ldots a_k}$ would not have given the same result with the indices raised, because the $\mathbf{D}$ acts on the metrics (recall $\mathbf{D}$ has non-vanishing nonmetricity).

Proof. We introduce the multi-index $\mathfrak{A}\equiv a_1 \ldots a_k$.

$$\begin{aligned}\mathbf{D}\star\boldsymbol{\vartheta}_{\mathfrak{A}} &= \mathbf{D}\left[\frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k}}\right] \\ &= \frac{1}{(\mathtt{D}-k)!}\mathbf{D}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\wedge\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k}} + \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\mathbf{D}\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k}} \\ &= \underbrace{\frac{1}{(\mathtt{D}-k)!}\left(-\frac{1}{2}\boldsymbol{Q}_c{}^c\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\right)\wedge\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k}}}_{(1)} + \underbrace{\frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\mathbf{D}\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k}}}_{(2)}\end{aligned}$$

We compute each term separately

$$\begin{aligned}(1) &= \frac{1}{(\mathtt{D}-k)!}\left(-\frac{1}{2}\boldsymbol{Q}_c{}^c\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\right)\wedge\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k}} = -\frac{1}{2}\boldsymbol{Q}_c{}^c\wedge\star\boldsymbol{\vartheta}_{\mathfrak{A}}\,, \\ (2) &= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\left[\boldsymbol{T}^{b_1}\wedge\boldsymbol{\vartheta}^{b_2\ldots b_{\mathtt{D}-k}} - \boldsymbol{\vartheta}^{b_1}\wedge\boldsymbol{T}^{b_2}\wedge\boldsymbol{\vartheta}^{b_3\ldots b_{\mathtt{D}-k}}\ldots + (-1)^{\mathtt{D}-k-1}\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k-1}}\wedge\boldsymbol{T}^{b_{\mathtt{D}-k}}\right] \\ &= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\left[\boldsymbol{T}^{b_1}\wedge\boldsymbol{\vartheta}^{b_2\ldots b_{\mathtt{D}-k}} - \boldsymbol{T}^{b_2}\wedge\boldsymbol{\vartheta}^{b_1 b_3\ldots b_{\mathtt{D}-k}}\ldots + (-1)^{\mathtt{D}-k-1}\boldsymbol{T}^{b_{\mathtt{D}-k}}\wedge\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k-1}}\right] \\ &= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}\underbrace{\left[\boldsymbol{T}^{[b_1}\wedge\boldsymbol{\vartheta}^{b_2\ldots b_{\mathtt{D}-k}]} + \boldsymbol{T}^{[b_1}\wedge\boldsymbol{\vartheta}^{b_2\ldots b_{\mathtt{D}-k}]}\ldots + \boldsymbol{T}^{[b_1}\wedge\boldsymbol{\vartheta}^{b_2\ldots b_{\mathtt{D}-k}]}\right]}_{(\mathtt{D}-k)\text{ terms}} \\ &= \frac{1}{(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{A}b_1\ldots b_{\mathtt{D}-k}}(\mathtt{D}-k)\boldsymbol{T}^{b_1}\wedge\boldsymbol{\vartheta}^{b_2\ldots b_{\mathtt{D}-k}} \\ &= \boldsymbol{T}^c\wedge\left[\frac{1}{(\mathtt{D}-k-1)!}\mathcal{E}_{\mathfrak{A}cb_1\ldots b_{\mathtt{D}-k-1}}\boldsymbol{\vartheta}^{b_1\ldots b_{\mathtt{D}-k-1}}\right] \\ &= \boldsymbol{T}^c\wedge\star\boldsymbol{\vartheta}_{\mathfrak{A}c}\end{aligned}$$

■

6 Hypersurface elements and Stokes theorem

6.1 Hypersurface elements

Consider a submanifold of codimension n and a frame that is adapted to the submanifold $\{\boldsymbol{e}_a\} = \{\boldsymbol{e}_A, \boldsymbol{e}_i\}_{A=1,\dots,n}^{i=n+1,\dots,\mathtt{D}}$, where the last ones generate the tangent of the submanifold and the rest (labeled with indices A, B, C...) are chosen to be normal directions; namely, he metric is block-diagonal in this frame ($g_{Ai} = 0$). We can also introduce the corresponding dual coframe $\{\boldsymbol{\vartheta}^a\} = \{\boldsymbol{\vartheta}^A, \boldsymbol{\vartheta}^i\}_{A=1,\dots,n}^{i=n+1,\dots,\mathtt{D}}$.

We can define the (hyper-)surface element:[5]

$$\overset{(\mathtt{D}-n)}{\boldsymbol{\Sigma}}{}^{A_1\dots A_n} := \star\boldsymbol{\vartheta}^{A_1\dots A_n} . \tag{6.1}$$

Some particular cases are:

❒ For $n = 0$

$$\overset{(\mathtt{D})}{\boldsymbol{\Sigma}} = \frac{1}{\mathtt{D}!}\mathcal{E}_{a_1\dots a_\mathtt{D}}\boldsymbol{\vartheta}^{a_1\dots a_\mathtt{D}} \qquad = \mathbf{vol}_g . \tag{6.2}$$

❒ For $n = 1$. Let us call n^μ the normalized normal vector field which we assume to be spacelike or timelike ($n^\mu n_\mu = \epsilon$). Then we can make the choice

$$\boldsymbol{e}_{A=1} = n^a\boldsymbol{e}_a ,$$

or, in other words, the frame in which $n^a \overset{*}{=} \delta^a_1$. The inverse metric can then be written as

$$g^{ab} = \epsilon n^a n^b + \gamma^{ij}P^a{}_i P^b{}_j$$

so

$$\overset{(\mathtt{D}-1)}{\boldsymbol{\Sigma}}{}^{A} = \star\boldsymbol{\vartheta}^A = \epsilon n^A \mathbf{vol}_\gamma . \tag{6.3}$$

Proof.

$$\begin{aligned}
\overset{(\mathtt{D}-1)}{\boldsymbol{\Sigma}}{}^{A} = \star\boldsymbol{\vartheta}^A &= \frac{1}{(\mathtt{D}-1)!}g^{Aa}\mathcal{E}_{ai_1\dots i_{\mathtt{D}-1}}\boldsymbol{\vartheta}^{i_1\dots i_{\mathtt{D}-1}} \\
&= \frac{1}{(\mathtt{D}-1)!}g^{AB}\mathcal{E}_{Bi_1\dots i_{\mathtt{D}-1}}\boldsymbol{\vartheta}^{i_1\dots i_{\mathtt{D}-1}} \\
&= \frac{1}{(\mathtt{D}-1)!}\epsilon n^A n^B\mathcal{E}_{Bi_1\dots i_{\mathtt{D}-1}}\boldsymbol{\vartheta}^{i_1\dots i_{\mathtt{D}-1}} \\
&= \frac{1}{(\mathtt{D}-1)!}\epsilon n^A\sqrt{|\det(g_{ab})|}\epsilon_{1i_1\dots i_{\mathtt{D}-1}}\boldsymbol{\vartheta}^{i_1\dots i_{\mathtt{D}-1}} \\
&= \frac{\sqrt{|\det(g_{ab})|}}{\sqrt{|\det(\gamma_{ij})|}}\epsilon n^A\mathbf{vol}_\gamma \\
&= \epsilon n^A\mathbf{vol}_\gamma
\end{aligned}$$

■

6.2 Stokes theorem

In a manifold of dimension D, only in the integration of D-forms is defined.

[5] They differ in a factor of $(-1)^{n(\mathtt{D}-n)}$ from those of [Ortín].

Theorem 4. (de Stokes) *Consider a compact orientable manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, and the inclusion map $i:\partial\mathcal{M}\to\mathcal{M}$. Then, for all $(\mathtt{D}-1)$-form, $\boldsymbol{\alpha}$,*

$$\int_{\mathcal{M}} \mathrm{d}\boldsymbol{\alpha} = \int_{\partial\mathcal{M}} i^*\boldsymbol{\alpha}\,. \tag{6.4}$$

If we work in terms of the dual of $\boldsymbol{\alpha}$, i.e., $\boldsymbol{\alpha}=\star\boldsymbol{\beta}$, this can be written as

$$\int_{\mathcal{M}} \mathrm{d}\star\boldsymbol{\beta} = \int_{\partial\mathcal{M}} i^*(\star\boldsymbol{\beta})\,. \tag{6.5}$$

If we introduce components notation we get the so-called *Gauss-Ostrogadski theorem (for non-null boundaries)*

$$\int_{\mathcal{V}} \mathring{\nabla}_a\alpha^a\boldsymbol{\Sigma}^{(\mathtt{D})} = \int_{\partial\mathcal{V}}\alpha_A\boldsymbol{\Sigma}^{(\mathtt{D}-1)A} \qquad\equiv\qquad \boxed{\int_{\mathcal{V}} \mathring{\nabla}_a\alpha^a\mathbf{vol}_g = \epsilon\int_{\partial\mathcal{V}}\alpha_A n^A\mathbf{vol}_\gamma}\,. \tag{6.6}$$

7 Adjoint exterior derivative, Laplacian and harmonic forms

7.1 Scalar product of differential forms

7.1.1 'Basic' scalar product

From (5.1), we can define the following scalar product of two k-forms:

$$\boxed{\langle\boldsymbol{\alpha},\,\boldsymbol{\beta}\rangle \coloneqq \star(\boldsymbol{\alpha}\wedge(\star\boldsymbol{\beta}))} \qquad \equiv \frac{1}{k!}\,\mathrm{sgn}(g)\alpha^{a_1\ldots a_k}\beta_{a_1\ldots a_k}\,. \tag{7.1}$$

This is very related to the notion of determinant, as we will see later.

Observation. In fact, we could have introduced this product, just as $\langle\boldsymbol{\alpha},\,\boldsymbol{\beta}\rangle \coloneqq \frac{1}{k!}\,\mathrm{sgn}(g)\alpha^{a_1\ldots a_k}\beta_{a_1\ldots a_k}$ (without mentioning the star), and the Hodge star of a k-form $\boldsymbol{\beta}$ could be defined (a posteriori) as the unique $(\mathtt{D}-k)$-form $\star\boldsymbol{\beta}$ that fulfills:

$$\boldsymbol{\alpha}\wedge(\star\boldsymbol{\beta}) = \langle\boldsymbol{\alpha},\,\boldsymbol{\beta}\rangle\,\mathbf{vol}_g\,, \qquad \forall\boldsymbol{\alpha}\in\Omega^{\mathtt{D}-k}(\mathcal{M})\,. \tag{7.2}$$

7.1.2 Scalar product of forms and determinent

Consider the following scalar product

$$\left\langle\boldsymbol{\vartheta}^{a_1}\wedge\ldots\wedge\boldsymbol{\vartheta}^{a_k},\,\boldsymbol{\vartheta}^{b_1}\wedge\ldots\wedge\boldsymbol{\vartheta}^{b_k}\right\rangle_{\det} \coloneqq \det\begin{pmatrix} g^{a_1b_1} & \ldots & g^{a_1b_k}\\ \vdots & \ddots & \vdots\\ g^{a_kb_1} & \ldots & g^{a_kb_k}\end{pmatrix} \tag{7.3}$$

$$= \tfrac{1}{k!}\epsilon_{i_1\ldots i_k}\epsilon_{j_1\ldots j_k}g^{a_{i_1}b_{j_1}}\ldots g^{a_{i_k}b_{j_k}} \tag{7.4}$$

$$! = k!\delta^1_{[i_1}\ldots\delta^k_{i_k]}g^{a_{i_1}b_1}\ldots g^{a_{i_k}b_k} \tag{7.5}$$

In ! we employed $\epsilon_{j_1\ldots j_k}\equiv k!\delta^1_{[j_1}\ldots\delta^k_{j_k]}$. By using the linearity, it can be proved that for arbitrary k-forms this product coincides with the one defined in the previous subsection up to a sign:

$$\langle\boldsymbol{\alpha},\,\boldsymbol{\beta}\rangle_{\det} = \mathrm{sgn}(g)\,\langle\boldsymbol{\alpha},\,\boldsymbol{\beta}\rangle\,. \tag{7.6}$$

Notice that if the metric is Riemannian (i.e., $\mathrm{sgn}(g)=1$) both coincide.

Proof.

$$\begin{aligned}
\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle_{\det} &= \left\langle \tfrac{1}{k!}\alpha_{a_1\ldots a_k}\boldsymbol{\vartheta}^{a_1}\wedge\ldots\wedge\boldsymbol{\vartheta}^{a_k},\, \tfrac{1}{k!}\beta_{b_1\ldots b_k}\boldsymbol{\vartheta}^{b_1}\wedge\ldots\wedge\boldsymbol{\vartheta}^{b_k}\right\rangle_{\det}\\
&= \tfrac{1}{k!}\tfrac{1}{k!}\alpha_{a_1\ldots a_k}\beta_{b_1\ldots b_k}\left\langle \boldsymbol{\vartheta}^{a_1}\wedge\ldots\wedge\boldsymbol{\vartheta}^{a_k},\, \boldsymbol{\vartheta}^{b_1}\wedge\ldots\wedge\boldsymbol{\vartheta}^{b_k}\right\rangle_{\det}\\
&= \tfrac{1}{k!}\tfrac{1}{k!}\alpha_{a_1\ldots a_k}\beta_{b_1\ldots b_k}k!\delta^1_{[i_1}\ldots\delta^k_{i_k]}g^{a_{i_1}b_1}\ldots g^{a_{i_k}b_k}\\
&= \tfrac{1}{k!}\alpha_{a_1\ldots a_k}\delta^1_{[i_1}\ldots\delta^k_{i_k]}\beta^{a_{i_1}\ldots a_{i_k}}\\
&= \tfrac{1}{k!}\alpha_{a_1\ldots a_k}\beta^{a_{[1}\ldots a_{k]}} \qquad = \tfrac{1}{k!}\alpha_{a_1\ldots a_k}\beta^{a_1\ldots a_k} \qquad \equiv \mathrm{sgn}(g)\,\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle
\end{aligned}$$

■

7.2 Scalar product L^2 of forms

The L^2 *scalar product* of two k-forms is defined:[6]

$$\boxed{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \coloneqq \int \boldsymbol{\alpha}\wedge(\star\boldsymbol{\beta})}\,. \tag{7.7}$$

Proof. That it is linear and hermitic (symmetry in case of $g_{\mu\nu}$ is a real metric) are immediate.

Being positive definite or not depends on the metric $g_{\mu\nu}$. If this is positive definite (Riemannian), then the product $(\boldsymbol{\alpha}, \boldsymbol{\alpha}) \geq 0$ will be positive definite and null if and only if $\boldsymbol{\alpha} = 0$. For non-Riemannian metrics this is not true and the result is a non-Riemannian scalar product. ■

The following equivalent expressions can be proven:

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \quad = \quad \frac{1}{k!}\int \alpha^{a_1\ldots a_k}\beta_{a_1\ldots a_k}\mathbf{vol}_g \quad = \quad \mathrm{sgn}(g)\int\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle\,\mathbf{vol}_g \quad = \quad \int\langle\boldsymbol{\alpha}, \boldsymbol{\beta}\rangle_{\det}\mathbf{vol}_g\,. \tag{7.8}$$

7.3 Adjoint exterior derivative (codifferencial)

Let $\mathcal{M}$ be a compact and orientable manifold. In addition, we assume that either it has no boundary or the considered forms have compact support and vanish at the boundary.

We can now define the adjoint operator of d with respect to the scalar product of k-forms (the

[6] There is another convention used by some authors,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_2 \coloneqq \int(\star\boldsymbol{\alpha})\wedge\boldsymbol{\beta}\,,$$

which is related to our convention via:

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_2 = \int(\star\boldsymbol{\alpha})\wedge\boldsymbol{\beta} = \int(\star\boldsymbol{\beta})\wedge\boldsymbol{\alpha} = (-1)^{(\mathtt{D}-k)k}(\boldsymbol{\alpha}, \boldsymbol{\beta})\,.$$

Normally this is used with the other convention for the Hodge star, which compensates this sign. Consequently, the expressions in both conventions can be mapped as:

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \coloneqq \int\boldsymbol{\alpha}\wedge(\star\boldsymbol{\beta}) \qquad \Rightarrow \qquad (\boldsymbol{\alpha}, \boldsymbol{\beta})_2 \coloneqq \int\boldsymbol{\alpha}\wedge(\star_2\boldsymbol{\beta})\,.$$

subscript indicates the rank of the form) as follows[7]

$$\big(\boldsymbol{\alpha}_k,\, \mathrm{d}\boldsymbol{\beta}_{k-1}\big) = \big(\delta\boldsymbol{\alpha}_k,\, \boldsymbol{\beta}_{k-1}\big)\ . \tag{7.9}$$

The result is the map $\delta\ :\ \Omega^k(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M})$ given by

$$\boxed{\delta := (-1)^k \star^{-1} \mathrm{d}\star = (-1)^{\mathtt{D}(k-1)-1}\,\mathrm{sgn}(g)\star\mathrm{d}\star}\,, \tag{7.10}$$

which we call *codifferencial* or *adjoint exterior derivative*.

Proof. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be $(k-1)$-forms. Then,

$$\begin{aligned} 0 &= \int \mathrm{d}\left(\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta}\right) \\ &= \int \left[\mathrm{d}\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta} + (-1)^{k-1}(\boldsymbol{\alpha}\wedge(\mathrm{d}\star\boldsymbol{\beta}))\right] \\ &= (\mathrm{d}\boldsymbol{\alpha},\,\boldsymbol{\beta}) - \big(\boldsymbol{\alpha},\quad (-1)^k\star^{-1}(\mathrm{d}\star\boldsymbol{\beta})\quad\big) \equiv (\mathrm{d}\boldsymbol{\alpha},\,\boldsymbol{\beta}) - (\boldsymbol{\alpha},\,\delta\boldsymbol{\beta}) \end{aligned}$$

where the initial expression is vanishing due to Stokes theorem.

To derive the other expression we substitute $\star^{-1}$:

$$\begin{aligned} \delta\boldsymbol{\alpha} &= (-1)^k\star^{-1}\underbrace{\mathrm{d}\star\boldsymbol{\alpha}}_{\text{rank } k'=\mathtt{D}-k+1} \\ &= (-1)^k\left[(-1)^{k'(\mathtt{D}-k')}\,\mathrm{sgn}(g)\star\right]\mathrm{d}\star\boldsymbol{\alpha} \\ &= (-1)^k(-1)^{(\mathtt{D}-k+1)\mathtt{D}-(\mathtt{D}-k+1)^2}\,\mathrm{sgn}(g)\star\mathrm{d}\star\boldsymbol{\alpha} \\ &= (-1)^{k+(\mathtt{D}-k+1)(\mathtt{D}-1)}\,\mathrm{sgn}(g)\star\mathrm{d}\star\boldsymbol{\alpha} \\ &= (-1)^{k+k(\mathtt{D}-1)+(\mathtt{D}-1)^2}\,\mathrm{sgn}(g)\star\mathrm{d}\star\boldsymbol{\alpha} \\ &= (-1)^{k\mathtt{D}+\mathtt{D}-1}\,\mathrm{sgn}(g)\star\mathrm{d}\star\boldsymbol{\alpha} \\ &= (-1)^{\mathtt{D}(k-1)-1}\,\mathrm{sgn}(g)\star\mathrm{d}\star\boldsymbol{\alpha} \end{aligned}$$

■

- ❒ As we did for d, we can analogously define coexact k-forms coexactas (those which are the codifferential of a certain $(k-1)$-form) and coclosed (thouse with vanishing δ). The corresponding spaces are denoted, respectively, as $B^k_\delta(\mathcal{M})$ and $Z^k_\delta(\mathcal{M})$.
- ❒ The codifferential fulfills the same properties as the ordinary exterior derivative:

$$\delta(\lambda_1\boldsymbol{\alpha}+\lambda_2\boldsymbol{\beta}) = \lambda_1\delta\boldsymbol{\alpha}+\lambda_2\delta\boldsymbol{\beta} \qquad \text{(Linear)}\,, \tag{7.11}$$

$$\delta(\delta\boldsymbol{\alpha}) = 0 \qquad \text{(Nilpotent)}\,. \tag{7.12}$$

[7] With the other convention, $(\boldsymbol{\alpha},\,\boldsymbol{\beta})_2 := \int(\star\boldsymbol{\alpha})\wedge\boldsymbol{\beta}$,

$$\begin{aligned} (\mathrm{d}\boldsymbol{\alpha}_{k-1},\,\boldsymbol{\beta}_k)_2 &= (-1)^{(\mathtt{D}-k)k}(\mathrm{d}\boldsymbol{\alpha}_{k-1},\,\boldsymbol{\beta}_k) \\ &= (-1)^{(\mathtt{D}-k)k}(\boldsymbol{\alpha}_{k-1},\,\delta\boldsymbol{\beta}_k) \\ &= (-1)^{(\mathtt{D}-k)k}(-1)^{(\mathtt{D}-(k-1))(k-1)}(\boldsymbol{\alpha}_{k-1},\,\delta\boldsymbol{\beta}_k)_2 \\ &= (\boldsymbol{\alpha}_{k-1},\,(-1)^{\mathtt{D}+1}\delta\boldsymbol{\beta}_k)_2\,. \end{aligned}$$

So the associated adjoint exterior derivative is related with the one of our convention as:

$$\delta_2 = (-1)^{\mathtt{D}+1}\delta \qquad \Rightarrow \qquad \delta_2 = (-1)^{\mathtt{D}k}\,\mathrm{sgn}(g)\star\mathrm{d}\star\ .$$

Proof. The linearity is trivial due to the properties of the star and the exterior derivative. Nilpotency: $\delta^2 = \left(\star^{-1}\mathrm{d}\star\right)\left(\star^{-1}\mathrm{d}\star\right) = \star^{-1}\mathrm{d}\mathrm{d}\star = 0.$ ■

❒ Exact and coexact forms are orthogonal under the L^2 scalar product.

Proof. $(\mathrm{d}\boldsymbol{\alpha},\, \delta\boldsymbol{\beta}) = (\boldsymbol{\alpha},\, \delta\delta\boldsymbol{\beta}) = (\boldsymbol{\alpha},\, 0) = 0\,.$ ■

❒ Due to the previous results:[8]

$$\delta\boldsymbol{\alpha} = -\frac{1}{(k-1)!}\left[\left(\nabla_b - \frac{1}{2}Q_{bc}{}^c + T_{bc}{}^c\right)\alpha^{ba_1\ldots a_{k-1}} - \frac{k-1}{2}T_{bc}{}^{[a_1}\alpha^{|bc|a_2\ldots a_{k-1}]}\right]\boldsymbol{\vartheta}_{a_1\ldots a_{k-1}} \tag{7.13}$$

$$= -\frac{1}{(k-1)!}\mathring{\nabla}_b\alpha^b{}_{a_1\ldots a_{k-1}}\boldsymbol{\vartheta}^{a_1\ldots a_{k-1}} \tag{7.14}$$

Proof. Immediate consequence of (5.33) and (7.10). ■

7.4 Laplace-de Rham operator

We introduce the *Laplace-de Rham operator* as the map $\Delta\,:\,\Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})$, given by

$$\Delta = (\mathrm{d}+\delta)^2 \tag{7.15}$$

$$= \mathrm{d}\delta + \delta\mathrm{d} \tag{7.16}$$

$$= (-1)^{\mathtt{D}(k-1)-1}\,\mathrm{sgn}(g)\left[(\mathrm{d}\star\mathrm{d}\star) + (\star\mathrm{d}\star\mathrm{d})\right] \tag{7.17}$$

Properties

$$(\Delta\boldsymbol{\alpha},\, \boldsymbol{\beta}) = (\boldsymbol{\alpha},\, \Delta\boldsymbol{\beta}) \qquad \text{(Symmetry)}\,. \tag{7.18}$$

$$\Delta\star\boldsymbol{\alpha} = \star\Delta\boldsymbol{\alpha}\,. \tag{7.19}$$

Proof. First property

$$\begin{aligned}(\boldsymbol{\alpha},\, \mathrm{d}\delta\boldsymbol{\beta}+\delta\mathrm{d}\boldsymbol{\beta}) &= (\boldsymbol{\alpha},\, \mathrm{d}\delta\boldsymbol{\beta}) + (\boldsymbol{\alpha},\, \delta\mathrm{d}\boldsymbol{\beta})\\ &= (\delta\boldsymbol{\alpha},\, \delta\boldsymbol{\beta}) + (\mathrm{d}\boldsymbol{\alpha},\, \mathrm{d}\boldsymbol{\beta})\\ &= (\mathrm{d}\delta\boldsymbol{\alpha},\, \boldsymbol{\beta}) + (\delta\mathrm{d}\boldsymbol{\alpha},\, \boldsymbol{\beta})\\ &= (\mathrm{d}\delta\boldsymbol{\alpha}+\delta\mathrm{d}\boldsymbol{\alpha},\, \boldsymbol{\beta})\end{aligned}$$

Second property. For a k-form $\boldsymbol{\alpha}$

$$\begin{aligned}\Delta\star\boldsymbol{\alpha} &= (\mathrm{d}\star\mathrm{d}\star + \star\mathrm{d}\star\mathrm{d})\star\boldsymbol{\alpha}\\ &= \mathrm{d}\star\mathrm{d}\,(\star\star\boldsymbol{\alpha}) + \star\mathrm{d}\star\mathrm{d}\star\boldsymbol{\alpha}\\ &= (-1)^{k(\mathtt{D}-k)}\,\mathrm{sgn}(g)\mathrm{d}\star\mathrm{d}\boldsymbol{\alpha} + \star\mathrm{d}\star\mathrm{d}\star\boldsymbol{\alpha}\\ &= \star\,(\star\mathrm{d}\star\mathrm{d} + \mathrm{d}\star\mathrm{d}\star)\,\boldsymbol{\alpha}\end{aligned}$$

[8] With the other convention for $\star$ (that we denoted $\star_2$), instead of the -1, we would have obtained a $(-1)^{\mathtt{D}}$.

■

Expression in components

Δ is independent of any connection (but depends on the metric since δ is metric-dependent). However, we can use a connection, e.g. the Levi-Civita one, to write down an expression for this operator in components in a explicitly covariant way:

$$\Delta\boldsymbol{\alpha} = \frac{1}{k!}\left[-\mathring{\nabla}_b\mathring{\nabla}^b\alpha_{a_1\dots a_k} + k\left[\mathring{\nabla}_b,\, \mathring{\nabla}_{[a_1}\right]\alpha^b{}_{a_2\dots a_k]}\right]\boldsymbol{\vartheta}^{a_1\dots a_k}\,. \tag{7.20}$$

$$= \frac{1}{k!}\left[-\mathring{\nabla}_b\mathring{\nabla}^b\alpha_{a_1\dots a_k} - k\mathring{R}_{ca_1}\alpha^c{}_{a_2\dots a_k} + \tfrac{1}{2}k!\mathring{R}_{[a_1a_2|bc|}\alpha^{bc}{}_{a_3\dots a_k]}\right]\boldsymbol{\vartheta}^{a_1\dots a_k}\,. \tag{7.21}$$

Proof. We compute each term of $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ separately:

$$\begin{aligned}
\mathrm{d}\delta\boldsymbol{\alpha} &= \mathrm{d}\left[\frac{1}{(k-1)!}\left(-\mathring{\nabla}_b\alpha^b{}_{a_1\dots a_{k-1}}\right)\boldsymbol{\vartheta}^{a_1\dots a_{k-1}}\right]\\
&= \frac{1}{(k-1)!}\mathring{\nabla}_{[a_1|}\left(-\mathring{\nabla}_b\alpha^b{}_{|a_2\dots a_k]}\right)\boldsymbol{\vartheta}^{a_1\dots a_k}\\
&= -\frac{1}{k!}\left[k\mathring{\nabla}_{a_1}\mathring{\nabla}_b\alpha^b{}_{a_2\dots a_k}\right]\boldsymbol{\vartheta}^{a_1\dots a_k}\,,\\
\delta\mathrm{d}\boldsymbol{\alpha} &= \delta\left[\frac{1}{k!}\mathring{\nabla}_{[a_1}\alpha_{a_2\dots a_{k+1}]}\boldsymbol{\vartheta}^{a_1\dots a_{k+1}}\right]\\
&= \delta\left[\frac{1}{(k+1)!}\left((k+1)\mathring{\nabla}_{[a_1}\alpha_{a_2\dots a_{k+1}]}\right)\boldsymbol{\vartheta}^{a_1\dots a_{k+1}}\right]\\
&= -\frac{1}{k!}\mathring{\nabla}_b\left((k+1)\mathring{\nabla}^{[b}\alpha^{a_1\dots a_k]}\right)\boldsymbol{\vartheta}_{a_1\dots a_k}\\
&= -\frac{1}{k!}\mathring{\nabla}_b\left(\mathring{\nabla}^b\alpha_{a_1\dots a_k} - k\mathring{\nabla}_{a_1}\alpha^b{}_{a_2\dots a_k}\right)\boldsymbol{\vartheta}^{a_1\dots a_k}\,.
\end{aligned}$$

We add them and obtain the first expression to be proven. Now we expand the commutator:

$$\begin{aligned}
\left[\mathring{\nabla}_b,\, \mathring{\nabla}_{a_1}\right]\alpha^b{}_{a_2\dots a_k}\boldsymbol{\vartheta}^{a_1\dots a_k} &= \left[\mathring{R}_{ba_1c}{}^b\alpha^c{}_{a_2\dots a_k} - \mathring{R}_{ba_1a_2}{}^c\alpha^b{}_{ca_3\dots a_k} - \mathring{R}_{ba_1a_3}{}^c\alpha^b{}_{a_2c\dots a_k}\cdots\right]\boldsymbol{\vartheta}^{a_1\dots a_k}\\
&= \left[-\mathring{R}_{a_1c}\alpha^c{}_{a_2\dots a_k} - (k-1)!\mathring{R}_{b[a_1a_2}{}^c\alpha^b{}_{|c|a_3\dots a_k]}\right]\boldsymbol{\vartheta}^{a_1\dots a_k}\\
&\overset{!}{=} \left[-\mathring{R}_{ca_1}\alpha^c{}_{a_2\dots a_k} + \tfrac{1}{2}(k-1)!\mathring{R}_{[a_1a_2|b}{}^c\alpha^b{}_{c|a_3\dots a_k]}\right]\boldsymbol{\vartheta}^{a_1\dots a_k}
\end{aligned}$$

In ! we used the Bianchi identity $-\mathring{R}_{\rho[\mu\nu]}{}^\sigma = \frac{1}{2}\mathring{R}_{\mu\nu\rho}{}^\sigma$. ■

Particular cases:

❒ For 0-forms, 1-forms and 2-forms:

$$\Delta f \equiv \delta\mathrm{d}f = -\mathring{\nabla}_b\mathring{\nabla}^b f \qquad = -\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu f\right)\,, \tag{7.22}$$

$$\Delta\boldsymbol{\alpha}_{(1)} = \left[-\mathring{\nabla}_\rho\mathring{\nabla}^\rho\alpha_\mu - \mathring{R}_{\rho\mu}\alpha^\rho\right]\mathrm{d}x^\mu\,, \tag{7.23}$$

$$\Delta\boldsymbol{\alpha}_{(2)} = \frac{1}{2}\left[-\mathring{\nabla}_\rho\mathring{\nabla}^\rho\alpha_{\mu\nu} - 2\mathring{R}_{\sigma[\mu}\alpha_{\nu]}{}^\sigma + \mathring{R}_{\mu\nu\rho\sigma}\alpha^{\rho\sigma}\right]\mathrm{d}x^{\mu\nu} \tag{7.24}$$

❒ For k-forms in Minkowski space:

$$\Delta\boldsymbol{\alpha} = -\frac{1}{k!}\partial_\sigma\partial^\sigma\alpha_{\mu_1\dots\mu_k}\mathrm{d}x^{\mu_1\dots\mu_k}\,. \tag{7.25}$$

7.5 Particular case: Riemannian metric and Hodge decomposition

If the metric is Riemannian, the Laplace-de Rham operator has special properties. We assume in this section the signature $(++\ldots+)$ and we will call Δ the "Laplacian".

A certain differential form $\boldsymbol{\alpha}$ is called *harmonic* if its Laplacian vanishes. The vector space of harmonic k-forms is denoted as $H^k_\Delta(\mathcal{M})$.

New properties of the Laplacian in the Riemannian case

$$(\Delta\boldsymbol{\alpha},\, \boldsymbol{\alpha}) \geq 0 \qquad \text{(Positivity)}\,, \tag{7.26}$$

$$\Delta\boldsymbol{\alpha} = 0 \qquad \Leftrightarrow \qquad \mathrm{d}\boldsymbol{\alpha} = 0,\, \delta\boldsymbol{\alpha} = 0\,. \tag{7.27}$$

Proof. First property

$$(\Delta\boldsymbol{\alpha},\, \boldsymbol{\alpha}) = (\delta\boldsymbol{\alpha},\, \delta\boldsymbol{\alpha}) + (\mathrm{d}\boldsymbol{\alpha},\, \mathrm{d}\boldsymbol{\alpha}) = \|\mathrm{d}\boldsymbol{\alpha}\|^2 + \|\delta\boldsymbol{\alpha}\|^2 \geq 0$$

Second propery.

$\Leftarrow$) immediate

$\Rightarrow$)

$$\begin{aligned}\Delta\boldsymbol{\alpha} = 0 \quad \Rightarrow \quad 0 = (\Delta\boldsymbol{\alpha}|\boldsymbol{\alpha}) &= (\mathrm{d}\boldsymbol{\alpha},\, \mathrm{d}\boldsymbol{\alpha}) + (\delta\boldsymbol{\alpha},\, \delta\boldsymbol{\alpha}) \\ &= \|\mathrm{d}\boldsymbol{\alpha}\|^2 + \|\delta\boldsymbol{\alpha}\|^2 \quad \Rightarrow \quad \mathrm{d}\boldsymbol{\alpha} = 0,\, \delta\boldsymbol{\alpha} = 0\end{aligned}$$

■

- ❒ The functions (0-forms) that are harmonic in a oriented, connected and compact Riemannian manifold are constant.
- ❒ (Hodge decomposition) Any form over a orientable closed (compact without coundary) Riemannian manifold admits a unique (orthogonal) decomposition:

  $$\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta} + \delta\gamma + \boldsymbol{\eta}\,, \qquad \Delta\boldsymbol{\eta} = 0\,. \tag{7.28}$$

  In other words, the space of differential forms of a certain rank k can be split in orthogonal direct sum as:

  $$\begin{aligned}\Omega^k(\mathcal{M}) &= \mathrm{Im}\,(\mathrm{d}|_{\Lambda^{k-1}\mathcal{M}}) \quad \oplus^\perp \quad \mathrm{Im}\,(\delta|_{\Lambda^{k+1}\mathcal{M}}) \quad \oplus^\perp \quad H^k_\Delta(\mathcal{M}) &(7.29)\\ &= B^k_{\mathrm{d}}(\mathcal{M}) \quad \oplus^\perp \quad B^k_\delta(\mathcal{M}) \quad \oplus^\perp \quad H^k_\Delta(\mathcal{M})\,. &(7.30)\end{aligned}$$

  " Form = exact + coexact + harmonic".
  If $\boldsymbol{\alpha}$ is closed, then: $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta} + \boldsymbol{\eta}$ with $\boldsymbol{\eta}$ harmonic.

  Proof. Due to the orthogonality of exact and coexact forms, any one can in principle be written as $\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta} + \delta\gamma + \boldsymbol{\eta}$. We only need to check that $\boldsymbol{\eta}$ is harmonic. The previous sum separates orthogonal terms, so $\boldsymbol{\eta}$ is orthogonal to both exact and coexact forms. Let us investigate this subspace:

  $$0 = (\mathrm{d}\boldsymbol{\beta},\, \boldsymbol{\eta}) = (\boldsymbol{\beta},\, \delta\boldsymbol{\eta}) = \int \boldsymbol{\beta} \wedge (\star\delta\boldsymbol{\eta})$$

  $$0 = (\boldsymbol{\eta},\, \delta\gamma) = (\mathrm{d}\boldsymbol{\eta},\, \gamma) = \int \mathrm{d}\boldsymbol{\eta} \wedge (\star\gamma)$$

These conditions are verified if and only if $\mathrm{d}\boldsymbol{\eta} = 0$ and $\delta\boldsymbol{\eta} = 0$. In Riemannian manifolds, we have already seen that this only happens for harmonic $\boldsymbol{\eta}$.

For closed forms:

$$0 = \mathrm{d}\boldsymbol{\alpha} = \mathrm{d}^2\boldsymbol{\beta} + \mathrm{d}\delta\boldsymbol{\gamma} + \mathrm{d}\boldsymbol{\eta} = 0 + \mathrm{d}\delta\boldsymbol{\gamma} + 0$$

so,

$$0 = (\mathrm{d}\delta\boldsymbol{\gamma}, \boldsymbol{\gamma}) = (\delta\boldsymbol{\gamma}, \delta\boldsymbol{\gamma}) \qquad \Leftrightarrow \qquad \delta\boldsymbol{\gamma} = 0\,.$$

■

❒ (Hodge Theorem) The space of harmonic k-forms is isomorphic to the k-th de Rham cohomology group.

Proof. Consider two forms $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ in the same equivalence class within the de Rham cohomology group. They should be closed, so:

$$\boldsymbol{\alpha} = \mathrm{d}\boldsymbol{\beta} + \boldsymbol{\eta}\,, \qquad \boldsymbol{\alpha}' = \mathrm{d}\boldsymbol{\beta}' + \boldsymbol{\eta}'\,.$$

Their difference is an exact form, so $\boldsymbol{\eta} = \boldsymbol{\eta}'$ and, therefore $\boldsymbol{\eta}$ characterizes the equivalence class. As we see, each equivalence class has an associated harmonic form. As a consequence of this, there is an injective map $H^k_{\mathrm{d}}(\mathcal{M}) \hookrightarrow H^k_{\Delta}(\mathcal{M})$.

Now, given a harmonic form $\boldsymbol{\eta}$, it is closed but NOT exact. Consequently, it must be a non-trivial element of $H^k_{\mathrm{d}}(\mathcal{M})$. So both spaces must be isomorphic. ■

❒ $\star$ establishes an isomorphism $H^k_{\mathrm{d}}(\mathcal{M}) \simeq H^{\mathtt{D}-k}_{\mathrm{d}}(\mathcal{M})$. Thanks to the Poincaré duality, a similar isomorphism can also be establishes between the homology groups, $H_k(\mathcal{M}) \simeq H_{\mathtt{D}-k}(\mathcal{M})$.

Proof. The property $\Delta\star = \star\Delta$ says that the isomorphism $\star$ satisfies

$$\Delta\left(\star\boldsymbol{\eta}\right) = \star\left(\Delta\boldsymbol{\eta}\right) = \star 0 = 0\,,$$

(i.e., $\star$ transforms harmonic forms into harmonic forms). Then:

$$\star\ :\ H^k_{\Delta}(\mathcal{M}) \to H^{\mathtt{D}-k}_{\Delta}(\mathcal{M})\,,$$

that gives $H^k_{\mathrm{d}}(\mathcal{M}) \simeq H^{\mathtt{D}-k}_{\mathrm{d}}(\mathcal{M})$. The other one is immediate via Poincaré duality. ■

Remark. Semi-Riemannian case. The Hodge decomposition is nor valid in that case. In fact, there is no relation between the cohomology and the dimension of the space of harmonic forms. For instance, in a closed orientable Riemannian manifold, the only harmonic functions are the constant functions. However, for example in the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with metric $\mathrm{d}t^2 - \mathrm{d}x^2$, all the functions $\{\sin(kt)\sin(kx)\}$ for all k are harmonics and linearly independents in L^2. The key point of this is the fact that the Laplacian, which is an elliptic operator, becomes an hyperbolic one in the semi-Riemannian case.

8 Pullback of forms

Preliminaries. Pushforward of vectors and pullback of forms

Consider a smooth map between two manifolds, $\phi : \mathcal{M} \to \mathcal{N}$. We define:

- The pullback of a function $f : \mathcal{N} \to \mathbb{R}$ is the function $\phi^* f : \mathcal{M} \to \mathbb{R}$ given by:
$$\phi^* f \coloneqq f \circ \phi\,. \tag{8.1}$$
- The pushforward of a vector $\boldsymbol{v} \in T_p(\mathcal{M})$ is the vector $(\phi_* \boldsymbol{v}) \in T_{\phi(p)}(\mathcal{N})$ that acts on functions as follows:
$$(\phi_* \boldsymbol{v})(f) \coloneqq \boldsymbol{v}(\phi^* f)\ . \tag{8.2}$$
- The pullback of a k-form $\boldsymbol{\alpha} \in \wedge^k T_q^*(\mathcal{N})$ on a $p \in \mathcal{M}$ such that $\phi(p) = q$ is the k-form $(\phi^* \boldsymbol{\alpha}) \in T_p^*(\mathcal{M})$ that acts on vectors as follows:
$$(\phi^* \boldsymbol{\alpha})(\boldsymbol{v}_1, ..., \boldsymbol{v}_k) \coloneqq \boldsymbol{\alpha}(\phi_* \boldsymbol{v}_1, ..., \phi_* \boldsymbol{v}_k)\ . \tag{8.3}$$

The first one is always well-defined. Generalizing the last two definitions to vector fields and k-forms is not trivial.

- One could think that the pushforward associates to any vector field on $\mathcal{M}$, vector fields on $\mathcal{N}$, but this is not true. In principle, if ϕ is not surjective, at most we can get a vector field on $\operatorname{im}\phi$ (and there could be no way to naturally extend it to the whole $\mathcal{N}$). But, in fact, if the map ϕ is not injective, it is not guaranteed that the result is a vector field on $\operatorname{im}\phi$, since given a vector field in $\mathcal{M}$, $\boldsymbol{X}$, and two distinct points with the same image p_1 and p_2, there would be an ambiguity when selecting a vector in the point $\phi(p_1) = \phi(p_2)$ (the image of $\boldsymbol{X}_{p_1}$ or the image of $\boldsymbol{X}_{p_2}$ by the pushforward?).
- If the map ϕ is an embedding:[9] the image is a submanifold that inherits a differential structure and it is sensible to talk about smooth vector fields on $\operatorname{im}\phi$. In this case, given the injectivity, the pushforwards associates to any vector field on $\mathcal{M}$ a vector field on $\operatorname{im}\phi$. Regarding the pullback of forms, the clarification "on a point $p \in \mathcal{M}$..." is not needed because it is unique (due to injectivity); and, consequently, the pullback extends naturally the k-forms on the image manifold $\boldsymbol{\alpha} \in \Omega^k(\operatorname{im}\phi)$. So, in the case in which ϕ is an embedding:
  - for any vector field $\boldsymbol{X} \in \mathfrak{X}(\mathcal{M})$, we can build the pushforward vector field $(\phi_* \boldsymbol{X}) \in \mathfrak{X}(\operatorname{im}\phi)$ acting on functions as:
  $$(\phi_* \boldsymbol{X})(f) \coloneqq \boldsymbol{X}(\phi^* f)\ , \tag{8.4}$$
  i.e., point-wise:
  $$(\phi_* \boldsymbol{X})_n(f) \coloneqq \boldsymbol{X}_{\phi^{-1}(n)}(\phi^* f) \qquad \text{on each} \quad n \in \operatorname{im}\phi \subseteq \mathcal{N}\,; \tag{8.5}$$
  - for any k-form $\boldsymbol{\alpha} \in \Omega^k(\operatorname{im}\phi)$, we can define the k-form $(\phi^* \boldsymbol{\alpha}) \in \Omega^k(\mathcal{M})$ that acts on vector fields as follows:
  $$(\phi^* \boldsymbol{\alpha})((\boldsymbol{X}_1, ..., \boldsymbol{X}_k) \coloneqq \boldsymbol{\alpha}(\phi_* \boldsymbol{X}_1, ..., \phi_* \boldsymbol{X}_k)\ , \tag{8.6}$$
  i.e., point-wise:
  $$(\phi^* \boldsymbol{\alpha})_p((\boldsymbol{X}_1)_p, ..., (\boldsymbol{X}_k)_p) \coloneqq \boldsymbol{\alpha}_{\phi(p)}(\phi_*(\boldsymbol{X}_1)_p, ..., \phi_*(\boldsymbol{X}_k)_p) \qquad \text{on each} \quad p \in \mathcal{M}\,. \tag{8.7}$$
- A particular case of the latter point is the one in which $\operatorname{im}\phi$ coincides with $\mathcal{N}$ (ϕ surjective) or, equivalently, that ϕ is a diffeomorphism. In this case, one can even define the pullback of vector fields and the pushforward of forms (analogous definitions but using the inverse map ϕ^{-1} instead).

[9] Recall that an embedding is a diffeomorphism between the first manifold and its image (so, in particular, it is injective).

Properties of the pullback of differential forms

For any differential forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on $\mathcal{N}$, and assuming that their pullbacks are well-defined, the following properties hold:

$$\phi^* (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\phi^* \boldsymbol{\alpha}) \wedge (\phi^* \boldsymbol{\beta}) \ , \tag{8.8}$$

$$\phi^* \mathrm{d}\boldsymbol{\alpha} = \mathrm{d} (\phi^* \boldsymbol{\alpha}) \ . \tag{8.9}$$

Proof.

❒ Pullback of the exterior product:

$$\begin{aligned}
&(\phi^* (\boldsymbol{\alpha} \wedge \boldsymbol{\beta})) (\boldsymbol{V}_1, ..., \boldsymbol{V}_{k+l}) \\
&\quad = (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) (\phi_* \boldsymbol{V}_1, ..., \phi_* \boldsymbol{V}_{k+l}) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in \mathrm{S}_{k+l}} \mathrm{sgn}(\sigma) \boldsymbol{\alpha} \left(\phi_* \boldsymbol{V}_{\sigma(1)}, ..., \phi_* \boldsymbol{V}_{\sigma(k)}\right) \boldsymbol{\beta} \left(\phi_* \boldsymbol{V}_{\sigma(k+1)}, ..., \phi_* \boldsymbol{V}_{\sigma(k+l)}\right) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in \mathrm{S}_{k+l}} \mathrm{sgn}(\sigma) (\phi^* \boldsymbol{\alpha}) \left(\boldsymbol{V}_{\sigma(1)}, ..., \boldsymbol{V}_{\sigma(k)}\right) (\phi^* \boldsymbol{\beta}) \left(\boldsymbol{V}_{\sigma(k+1)}, ..., \boldsymbol{V}_{\sigma(k+l)}\right) \\
&\quad = ((\phi^* \boldsymbol{\alpha}) \wedge (\phi^* \boldsymbol{\beta})) (\boldsymbol{V}_1, ..., \boldsymbol{V}_{k+l})
\end{aligned}$$

❒ Pullback of the exterior differential.
For a 0-form it is easy to prove if we act with it on an arbitrary vector:

$$(\phi^* \mathrm{d}\boldsymbol{\alpha}) (\boldsymbol{V}) = \mathrm{d}\boldsymbol{\alpha} (\phi_* \boldsymbol{V}) = (\phi_* \boldsymbol{V}) (\boldsymbol{\alpha}) = \boldsymbol{V} (\phi^* \boldsymbol{\alpha}) = \mathrm{d} (\phi^* \boldsymbol{\alpha}) (\boldsymbol{V})$$

where we have used: the definition of the pullback of forms (step 1); derivative of a function seen as its differential acting on the vector (step 2 and 4); and definition of the push-forward of vectors (step 3).
To abbreviate, we denote $\bar{\alpha}_{\mu_1 ... \mu_{k+1}} = \frac{1}{k!} \alpha_{\mu_2 ... \mu_{k+1}}$. So,

$$\begin{aligned}
\phi^* \mathrm{d}\boldsymbol{\alpha} &= \phi^* \left(\mathrm{d}\bar{\alpha}_{\mu_1 ... \mu_k} \wedge \mathrm{d}x^{\mu_1 \cdots \mu_k}\right) \\
&= \phi^* \left(\mathrm{d}\bar{\alpha}_{\mu_1 ... \mu_k}\right) \wedge \phi^* \left(\mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right) \\
! &= \mathrm{d} \left(\phi^* \bar{\alpha}_{\mu_1 ... \mu_k}\right) \wedge \phi^* \left(\mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right) \\
&= \mathrm{d} \left(\phi^* \bar{\alpha}_{\mu_1 ... \mu_k} \phi^* \left(\mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right)\right) - \phi^* \bar{\alpha}_{\mu_1 ... \mu_k} \wedge \mathrm{d} \left(\phi^* \left(\mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right)\right) \\
!! &= \mathrm{d} \left(\phi^* \left(\bar{\alpha}_{\mu_1 ... \mu_k} \mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right)\right) - 0 \\
&= \mathrm{d} (\phi^* \boldsymbol{\alpha})
\end{aligned}$$

where in ! we used the previous computation for 0-forms and in !! we employed:

$$\mathrm{d} \left(\phi^* \mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right) (...) = \mathrm{d} \left(\mathrm{d}x^{\mu_1 \cdots \mu_{k+1}}\right) (\phi_* ...) = 0$$

■

9 Algebra-valued differential forms

Let $\{\boldsymbol{X}_A\}_{A=1}^m$ be a basis of a Lie algebra $\mathfrak{g}$. A $\mathfrak{g}$-*valued k-form* is an object:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^A \otimes \boldsymbol{X}_A = \alpha_{a_1 \ldots a_k}{}^A \boldsymbol{\vartheta}^{a_1 \ldots a_k} \otimes \boldsymbol{X}_A \tag{9.1}$$

($\boldsymbol{\alpha}^A$ are usual k-forms over the manifold $\mathcal{M}$). The real vector space in which they live is $\Omega^k(\mathcal{M}) \otimes \mathfrak{g}$, and will be denoted as $\Omega^k(\mathcal{M}, \mathfrak{g})$. When they act on vectors, it is understood that the part in $\Omega^k(\mathcal{M})$ is the one that operates. Namely: $\boldsymbol{\alpha} \in \Omega^k(\mathcal{M}, \mathfrak{g})$ is an object of the type:

$$\boldsymbol{\alpha} : T\mathcal{M} \longrightarrow C^\infty(\mathcal{M}) \otimes \mathfrak{g}\,, \qquad \boldsymbol{\alpha}|_p : T_p\mathcal{M} \longrightarrow \mathfrak{g}\,. \tag{9.2}$$

9.1 Lie bracket of algebra-valued differential forms

$\Omega^k(\mathcal{M}, \mathfrak{g})$ has also a Lie algebra structure given by the bracket:

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}]_\Omega \coloneqq \left(\boldsymbol{\alpha}^A \wedge \boldsymbol{\beta}^B\right) \otimes [\boldsymbol{X}_A, \boldsymbol{X}_B] \tag{9.3}$$

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be two vector fields. Then, for 1-forms we have the following interesting expression:

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}]_\Omega (\boldsymbol{U}, \boldsymbol{V}) = [\boldsymbol{\alpha}(\boldsymbol{U}), \boldsymbol{\beta}(\boldsymbol{V})] - [\boldsymbol{\alpha}(\boldsymbol{V}), \boldsymbol{\beta}(\boldsymbol{U})]\ . \tag{9.4}$$

implying

$$[\boldsymbol{\alpha}, \boldsymbol{\alpha}]_\Omega (\boldsymbol{U}, \boldsymbol{V}) = 2\,[\boldsymbol{\alpha}(\boldsymbol{U}), \boldsymbol{\alpha}(\boldsymbol{V})]\ . \tag{9.5}$$

Proof.

$$\begin{aligned}
[\boldsymbol{\alpha}, \boldsymbol{\beta}]_\Omega (\boldsymbol{U}, \boldsymbol{V}) &= \left(\alpha_a{}^A \beta_b{}^B \boldsymbol{\vartheta}^{ab}\right)(\boldsymbol{U}, \boldsymbol{V}) \otimes [\boldsymbol{X}_A, \boldsymbol{X}_B] \\
&= \left(2\alpha_{[a}{}^A \beta_{b]}{}^B \boldsymbol{\vartheta}^a \otimes \boldsymbol{\vartheta}^b\right)(\boldsymbol{U}, \boldsymbol{V}) \otimes [\boldsymbol{X}_A, \boldsymbol{X}_B] \\
&= 2\alpha_{[a}{}^A \beta_{b]}{}^B U^a V^b\, [\boldsymbol{X}_A, \boldsymbol{X}_B] \\
&= \left[\alpha_a{}^A U^a \boldsymbol{X}_A, \beta_b{}^B V^b \boldsymbol{X}_B\right] - \left[\alpha_b{}^A V^b \boldsymbol{X}_A, \beta_a{}^B U^a \boldsymbol{X}_B\right] \\
&\equiv [\boldsymbol{\alpha}(\boldsymbol{U}), \boldsymbol{\beta}(\boldsymbol{V})] - [\boldsymbol{\alpha}(\boldsymbol{V}), \boldsymbol{\beta}(\boldsymbol{U})]
\end{aligned}$$

■

9.2 Exterior derivative of algebra-valued differential forms

We define the map $\mathrm{d}_\mathfrak{g} : \Omega^k(\mathcal{M}, \mathfrak{g}) \longrightarrow \Omega^{k+1}(\mathcal{M}, \mathfrak{g})$,

$$\mathrm{d}_\mathfrak{g}\boldsymbol{\alpha} \coloneqq \left(\mathrm{d}\boldsymbol{\alpha}^A\right) \otimes \boldsymbol{X}_A\,. \tag{9.6}$$

If we apply it on a Lie braket we get

$$\mathrm{d}_\mathfrak{g}\,[\boldsymbol{\alpha}, \boldsymbol{\beta}]_\Omega = [\mathrm{d}_\mathfrak{g}\boldsymbol{\alpha}, \boldsymbol{\beta}]_\Omega + (-1)^k\,[\boldsymbol{\alpha}, \mathrm{d}_\mathfrak{g}\boldsymbol{\beta}]_\Omega\ , \tag{9.7}$$

where k is the rank of $\boldsymbol{\alpha}$.

Proof.

$$\begin{aligned}
\mathrm{d}_\mathfrak{g}\,[\boldsymbol{\alpha}, \boldsymbol{\beta}]_\Omega &= \mathrm{d}\left(\boldsymbol{\alpha}^A \wedge \boldsymbol{\beta}^B\right) \otimes [\boldsymbol{X}_A, \boldsymbol{X}_B] \\
&= \left(\mathrm{d}\boldsymbol{\alpha}^A \wedge \boldsymbol{\beta}^B + (-1)^k\,\boldsymbol{\alpha}^A \wedge \mathrm{d}\boldsymbol{\beta}^B\right) \otimes [\boldsymbol{X}_A, \boldsymbol{X}_B]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\mathrm{d}\boldsymbol{\alpha}^A \wedge \boldsymbol{\beta}^B\right) \otimes [\boldsymbol{X}_A,\, \boldsymbol{X}_B] + (-1)^k \left(\boldsymbol{\alpha}^A \wedge \mathrm{d}\boldsymbol{\beta}^B\right) \otimes [\boldsymbol{X}_A,\, \boldsymbol{X}_B] \\
&= [\mathrm{d}_{\mathfrak{g}}\boldsymbol{\alpha},\, \boldsymbol{\beta}]_\Omega + (-1)^k [\boldsymbol{\alpha},\, \mathrm{d}_{\mathfrak{g}}\boldsymbol{\beta}]_\Omega
\end{aligned}
$$

■

References

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A Useful tools

Here we collect some useful formulae that we use in this text:

- For any two tensors $A_{a_1...a_k}$ and $B_{b_1...b_k}$ (not necessarily antisymmetric) and any $C_{a_1...a_{\mathtt{D}}} = C_{[a_1...a_{\mathtt{D}}]}$ (notice that the number of indices coincides with the dimension), we have

$$
A_{b_1...b_k} C_{[a_1...a_{\mathtt{D}-k}}{}^{b_1...b_k} B_{a_{\mathtt{D}-k+1}...a_{\mathtt{D}}]} = \binom{\mathtt{D}}{k}^{-1} C_{a_1...a_{\mathtt{D}}} A_{[b_1...b_k]} B^{b_1...b_k} \,. \tag{A.1}
$$

- For any tensor $A^{a_1...a_k}$ (not necessarily antisymmetric) the following is true:

$$
\delta^{[a_1}_{a_1} ... \delta^{a_p}_{a_p} A^{b_1...b_k]} = \frac{(\mathtt{D}-k)!k!}{(\mathtt{D}-(k+p))!(k+p)!} A^{[b_1...b_k]} \qquad \forall p \leq \mathtt{D}-k \,. \tag{A.2}
$$

The proofs of these expressions are left to the reader as exercise. Hint for the second one: show it first with $p = 1$, then with $p = 2$, etc. It is not difficult to figure out the sequence that leads to iteratively to the general formula.

B Lengthy proofs

B.1 Proof of (3.6)

We want to prove

$$\begin{aligned}\mathrm{d}\boldsymbol{\alpha} &= \frac{1}{k!}\left[\partial_{a_1}\alpha_{a_2\ldots a_{k+1}} + \frac{k}{2}\Omega_{a_1a_2}{}^{c}\alpha_{ca_3\ldots a_{k+1}}\right]\boldsymbol{\vartheta}^{a_1\ldots a_{k+1}}\\ &= \frac{1}{k!}\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}\mathrm{d}x^{\mu_1\ldots\mu_{k+1}}\end{aligned}$$

Proof. We work separately with both terms in the definition of the exterior derivative (3.4) $\mathrm{d}\boldsymbol{\alpha}\left(\boldsymbol{e}_{a_1}, \ldots, \boldsymbol{e}_{a_{k+1}}\right) = (1) + (2)$ (recall that we are using the abbreviation $\partial_a = e^{\mu}{}_{a}\partial_{\mu}$):

$$\begin{aligned}(1) &= \sum_{i=1}^{k+1}(-1)^{i+1}\boldsymbol{e}_{a_i}\left(\boldsymbol{\alpha}\left(\boldsymbol{e}_{a_1}, \ldots, \hat{\boldsymbol{e}}_{a_i}, \ldots, \boldsymbol{e}_{a_{k+1}}\right)\right)\\ &= \sum_{i=1}^{k+1}(-1)^{i+1}\partial_{a_i}\alpha_{a_1\ldots\hat{a}_i\ldots a_{k+1}}\\ &= \partial_{a_1}\alpha_{a_2\ldots a_{k+1}} - \partial_{a_2}\alpha_{a_1\ldots a_{k+1}} + \ldots + (-1)^{k+2}\,\partial_{a_{k+1}}\alpha_{a_1\ldots a_k}\\ &= (k+1)\partial_{[a_1}\alpha_{a_2\ldots a_{k+1}]}\end{aligned}$$

$$\begin{aligned}(2) &= \sum_{i<j}(-1)^{i+j}\boldsymbol{\alpha}\left(\left[\boldsymbol{e}_{a_i}, \boldsymbol{e}_{a_j}\right], \boldsymbol{e}_{a_1}, \ldots, \hat{\boldsymbol{e}}_{a_i}, \ldots, \hat{\boldsymbol{e}}_{a_j}, \ldots, \boldsymbol{e}_{a_{k+1}}\right)\\ &= \sum_{i<j}(-1)^{i+j}\boldsymbol{\alpha}\left(-\Omega_{a_ia_j}{}^{c}\boldsymbol{e}_c, \boldsymbol{e}_{a_1}, \ldots, \hat{\boldsymbol{e}}_{a_i}, \ldots, \hat{\boldsymbol{e}}_{a_j}, \ldots, \boldsymbol{e}_{a_{k+1}}\right)\\ &= \sum_{i<j}(-1)^{i+j+1}\Omega_{a_ia_j}{}^{c}\alpha_{ca_1\ldots\hat{a}_i\ldots\hat{a}_j\ldots a_{k+1}}\\ &= \sum_{i<j}(-1)^{i+j+1}\Omega_{a_ia_j}{}^{c}\alpha_{ca_1\ldots\hat{a}_i\ldots\hat{a}_j\ldots a_{k+1}}\\ &= \sum_{i<j}(-1)^{(i-1)+(j-2)}\Omega_{a_ia_j}{}^{c}\alpha_{ca_1\ldots\hat{a}_i\ldots\hat{a}_j\ldots a_{k+1}}\\ &= \frac{(k+1)!}{2!(k-1)!}\Omega_{[a_1a_2}{}^{c}\alpha_{|c|a_3\ldots a_{k+1}]}\\ &= (k+1)\frac{k}{2}\Omega_{[a_1a_2}{}^{c}\alpha_{|c|a_3\ldots a_{k+1}]}\end{aligned}$$

It is convenient to introduce the abbreviation $e^{\mu\ldots\nu}_{a\ldots b} \equiv e^{\mu}{}_{[a}\ldots e^{\nu}{}_{b]}$. We find then:

$$\begin{aligned}\frac{(\mathrm{d}\boldsymbol{\alpha})_{a_1\ldots a_{k+1}}}{(k+1)} &= \partial_{[a_1}\alpha_{a_2\ldots a_{k+1}]} + \frac{k}{2}\Omega_{[a_1a_2}{}^{c}\alpha_{|c|a_3\ldots a_{k+1}]}\\ &= \partial_{[a_1}\alpha_{a_2\ldots a_{k+1}]} - ke_{\lambda}{}^{c}\partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|c|a_3\ldots a_{k+1}]}\\ &= \partial_{[a_1}\alpha_{a_2\ldots a_{k+1}]} - k\partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]}\\ &= \partial_{[a_1}\left(e^{\mu_1\ldots\mu_k}_{a_2\ldots a_{k+1}]}\alpha_{\mu_1\ldots\mu_k}\right) - k\partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]}\\ &= \partial_{[a_1}\left(e^{\mu_2\ldots\mu_{k+1}}_{a_2\ldots a_{k+1}]}\right)\alpha_{\mu_2\ldots\mu_{k+1}} - k\partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]}\\ &\quad + e^{\mu_1\ldots\mu_{k+1}}_{a_1\ldots a_{k+1}}\partial_{\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}}\end{aligned}$$

$$! = \quad e^{\mu_1\ldots\mu_{k+1}}_{a_1\ldots a_{k+1}}\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_{k+1}]}$$

where in ! we used

$$\begin{aligned}
&\partial_{[a_1}\left(e^{\mu_2}{}_{a_2}\ldots e^{\mu_{k+1}}{}_{a_{k+1}]}\right)\alpha_{\mu_2\ldots\mu_{k+1}} \\
&= \partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]} + \partial_{[a_1}e^{\lambda}{}_{a_3}\alpha_{a_2|\lambda|a_4\ldots a_{k+1}]} + \ldots + \partial_{[a_1}e^{\lambda}{}_{a_{k+1}}\alpha_{a_2\ldots a_k]\lambda} \\
&= \partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]} + (-1)^1\,\partial_{[a_1}e^{\lambda}{}_{a_3}\alpha_{|\lambda|a_2 a_4\ldots a_{k+1}]} + \ldots + (-1)^{k-1}\,\partial_{[a_1}e^{\lambda}{}_{a_{k+1}}\alpha_{|\lambda|a_2\ldots a_k]} \\
&= \partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]} + \partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]} + \ldots + \partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]} \\
&= k\partial_{[a_1}e^{\lambda}{}_{a_2}\alpha_{|\lambda|a_3\ldots a_{k+1}]}
\end{aligned}$$

■

B.2 Proof of (3.17)

We want to prove

$$\mathfrak{L}_{\boldsymbol{V}} = \mathrm{d}\circ(\boldsymbol{V}\lrcorner) + (\boldsymbol{V}\lrcorner)\circ\mathrm{d}$$

Proof. To show this, we use a coordinate frame for simplicity. Consider an arbitrary k-form $\boldsymbol{\alpha}$. We compute each term separately:

$$\begin{aligned}
&k!\mathrm{d}\left(\boldsymbol{V}\lrcorner\boldsymbol{\alpha}\right) \\
&= k!\mathrm{d}\left(V^{\lambda}\alpha_{\lambda\mu_2\ldots\mu_k}\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\right) \\
&= k!k\partial_{[\mu_1|}\left(V^{\lambda}\alpha_{\lambda|\mu_2\ldots\mu_k]}\right)\mathrm{d}x^{\otimes\mu_1\ldots\mu_k} \qquad = k\partial_{[\mu_1|}\left(V^{\lambda}\alpha_{\lambda|\mu_2\ldots\mu_k]}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= k\left(\partial_{[\mu_1|}V^{\lambda}\alpha_{\lambda|\mu_2\ldots\mu_k]} + V^{\lambda}\partial_{[\mu_1|}\alpha_{\lambda|\mu_2\ldots\mu_k]}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(k\partial_{\mu_1}V^{\lambda}\alpha_{\lambda\mu_2\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} + \left(kV^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(\partial_{\mu_1}V^{\lambda}\alpha_{\lambda\mu_2\ldots\mu_k} - \partial_{\mu_1}V^{\lambda}\alpha_{\mu_2\lambda\ldots\mu_k} - \ldots - \partial_{\mu_1}V^{\lambda}\alpha_{\mu_k\mu_2\ldots\mu_{k-1}\lambda}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \;+\; \left(kV^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(\partial_{\mu_1}V^{\lambda}\alpha_{\lambda\mu_2\ldots\mu_k} + \partial_{\mu_2}V^{\lambda}\alpha_{\mu_1\lambda\ldots\mu_k} + \ldots + \partial_{\mu_k}V^{\lambda}\alpha_{\mu_1\ldots\mu_{k-1}\lambda}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \;+\; k\left(V^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k}
\end{aligned}$$

$$\begin{aligned}
&k!\boldsymbol{V}\lrcorner\left(\mathrm{d}\boldsymbol{\alpha}\right) \\
&= k!\boldsymbol{V}\lrcorner\left((k+1)\partial_{[\mu_0}\alpha_{\mu_1\ldots\mu_k]}\mathrm{d}x^{\otimes\mu_0\mu_1\ldots\mu_k}\right) \\
&= k!(k+1)V^{\lambda}\partial_{[\lambda}\alpha_{\mu_1\ldots\mu_k]}\mathrm{d}x^{\otimes\mu_1\ldots\mu_k} \\
&= (k+1)V^{\lambda}\partial_{[\lambda}\alpha_{\mu_1\ldots\mu_k]}\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(V^{\lambda}\partial_{\lambda}\alpha_{[\mu_1\ldots\mu_k]} - V^{\lambda}\partial_{[\mu_1}\alpha_{|\lambda|\mu_2\ldots\mu_k]} + \ldots + (-1)^k V^{\lambda}\partial_{[\mu_1}\alpha_{\mu_2\ldots\mu_k]\lambda}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(V^{\lambda}\partial_{\lambda}\alpha_{\mu_1\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \quad + \quad \left(-V^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k} + \ldots + (-1)^k V^{\lambda}\partial_{\mu_1}\alpha_{\mu_2\ldots\mu_k\lambda}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(V^{\lambda}\partial_{\lambda}\alpha_{\mu_1\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \quad + \quad \left(-V^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k} - \ldots - V^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \\
&= \left(V^{\lambda}\partial_{\lambda}\alpha_{\mu_1\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \quad - \quad k\left(V^{\lambda}\partial_{\mu_1}\alpha_{\lambda\mu_2\ldots\mu_k}\right)\mathrm{d}x^{\mu_1\ldots\mu_k}
\end{aligned}$$

When adding them up, the terms with the factor k cancel, and the result is:

$$\begin{aligned}
&\left(\mathrm{d}\circ\boldsymbol{V}\lrcorner + \boldsymbol{V}\lrcorner\circ\mathrm{d}\right)\boldsymbol{\alpha} \\
&= \frac{1}{k!}\left(V^{\lambda}\partial_{\lambda}\alpha_{\mu_1\ldots\mu_k} + \partial_{\mu_1}V^{\lambda}\alpha_{\lambda\mu_2\ldots\mu_k} + \partial_{\mu_2}V^{\lambda}\alpha_{\mu_1\lambda\ldots\mu_k} + \ldots + \partial_{\mu_k}V^{\lambda}\alpha_{\mu_1\ldots\mu_{k-1}\lambda}\right)\mathrm{d}x^{\mu_1\ldots\mu_k} \equiv \mathfrak{L}_{\boldsymbol{V}}\boldsymbol{\alpha}
\end{aligned}$$

■

B.3 Proof of (3.18)

We want to prove

$$[\boldsymbol{V},\, \boldsymbol{W}]\lrcorner = [\mathfrak{L}_{\boldsymbol{V}},\, \boldsymbol{W}\lrcorner]$$

Proof.

$$\begin{aligned}
&[\mathfrak{L}_{\boldsymbol{V}},\, \boldsymbol{W}\lrcorner]\, \boldsymbol{\alpha}\\
&= (\mathfrak{L}_{\boldsymbol{V}}\, (\boldsymbol{W}\lrcorner\boldsymbol{\alpha}) \quad - \quad \boldsymbol{W}\lrcorner\, (\mathfrak{L}_{\boldsymbol{V}}\boldsymbol{\alpha}))\\
&= (\mathrm{d}\circ \boldsymbol{V}\lrcorner + \boldsymbol{V}\lrcorner\circ \mathrm{d})\, (\boldsymbol{W}\lrcorner\boldsymbol{\alpha}) \quad - \quad (\boldsymbol{W}\lrcorner\circ (\mathrm{d}\circ \boldsymbol{V}\lrcorner + \boldsymbol{V}\lrcorner\circ \mathrm{d}))\, \boldsymbol{\alpha}\\
&= \mathrm{d}\, (\boldsymbol{V}\lrcorner\boldsymbol{W}\lrcorner\boldsymbol{\alpha}) \quad + \quad \boldsymbol{V}\lrcorner\mathrm{d}\, (\boldsymbol{W}\lrcorner\boldsymbol{\alpha})\\
&\quad - \boldsymbol{W}\lrcorner\mathrm{d}\, (\boldsymbol{V}\lrcorner\boldsymbol{\alpha}) \quad - \quad \boldsymbol{W}\lrcorner\boldsymbol{V}\lrcorner\mathrm{d}\boldsymbol{\alpha}\\
&= \mathrm{d}\left(W^\nu V^\rho \alpha_{\nu\rho\mu_3\ldots\mu_k} \mathrm{d}x^{\otimes\mu_3\ldots\mu_k}\right) + \boldsymbol{V}\lrcorner\mathrm{d}\left(W^\nu \alpha_{\nu\mu_2\mu_3\ldots\mu_k} \mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\right)\\
&\quad - \boldsymbol{W}\lrcorner\mathrm{d}\left(V^\nu \alpha_{\nu\mu_2\mu_3\ldots\mu_k} \mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\right) - \boldsymbol{W}\lrcorner\boldsymbol{V}\lrcorner\left((k+1)\partial_{[\mu_0}\alpha_{\mu_1\mu_2\ldots\mu_{k+1}]}\mathrm{d}x^{\otimes\mu_0\mu_1\ldots\mu_k}\right)\\
&= (k-1)\partial_{[\mu_2}\left(W^\nu V^\rho \alpha_{|\nu\rho|\mu_3\ldots\mu_k]}\right)\mathrm{d}x^{\otimes\mu_2\ldots\mu_k} + kV^\rho\partial_{[\rho}\left(W^\nu\alpha_{|\nu|\mu_2\mu_3\ldots\mu_k]}\right)\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&\quad - kW^\nu\partial_{[\nu}\left(V^\rho\alpha_{|\rho|\mu_2\mu_3\ldots\mu_k]}\right)\mathrm{d}x^{\otimes\mu_2\ldots\mu_k} - W^\nu V^\rho\left((k+1)\partial_{[\rho}\alpha_{\nu\mu_2\ldots\mu_k]}\right)\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&= \Big[(k-1)\partial_{[\mu_2}\left(W^\nu V^\rho \alpha_{|\nu\rho|\mu_3\ldots\mu_k]}\right) + kV^\rho\partial_{[\rho}\left(W^\nu\alpha_{|\nu|\mu_2\mu_3\ldots\mu_k]}\right)\\
&\quad -kW^\nu\partial_{[\nu}\left(V^\rho\alpha_{|\rho|\mu_2\mu_3\ldots\mu_k]}\right) - (k+1)W^\nu V^\rho\partial_{[\nu}\alpha_{\rho\mu_2\ldots\mu_k]}\Big]\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&= \Big[(k-1)\partial_{[\mu_2}\left(W^\nu V^\rho\right)\alpha_{|\nu\rho|\mu_3\ldots\mu_k]} + kV^\rho\partial_{[\rho}W^\nu\alpha_{|\nu|\mu_2\mu_3\ldots\mu_k]} - kW^\nu\partial_{[\nu}\left(V^\rho\right)\alpha_{|\rho|\mu_2\mu_3\ldots\mu_k]}\Big]\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&\quad + \Big[(k-1)W^\nu V^\rho\partial_{[\mu_2}\alpha_{|\nu\rho|\mu_3\ldots\mu_k]} + kV^\rho W^\nu\partial_{[\rho}\alpha_{|\nu|\mu_2\mu_3\ldots\mu_k]}\\
&\qquad -kW^\nu V^\rho\partial_{[\nu}\left(\alpha_{|\rho|\mu_2\mu_3\ldots\mu_k]}\right) - (k+1)W^\nu V^\rho\partial_{[\rho}\alpha_{\nu\mu_2\ldots\mu_k]}\Big]\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}
\end{aligned}$$

The second bracket is zero, as we can check,

$$\begin{aligned}
&(k-1)W^\nu V^\rho\partial_{[\mu_2}\alpha_{|\nu\rho|\mu_3\ldots\mu_k]} \underbrace{+kV^\rho W^\nu\partial_{[\rho}\alpha_{|\nu|\mu_2\mu_3\ldots\mu_k]}}_{(a)} \underbrace{-kW^\nu V^\rho\partial_{[\nu}\alpha_{|\rho|\mu_2\mu_3\ldots\mu_k]}}_{(b)} \underbrace{-(k+1)W^\nu V^\rho\partial_{[\rho}\alpha_{\nu\mu_2\ldots\mu_k]}}_{(c)}\\
&= (k-1)W^\nu V^\rho\partial_{[\mu_2}\alpha_{|\nu\rho|\mu_3\ldots\mu_k]} \underbrace{+V^\rho W^\nu\partial_\rho\alpha_{\nu\mu_2\mu_3\ldots\mu_k} - (k-1)V^\rho W^\nu\partial_{[\mu_2}\alpha_{|\nu\rho|\mu_3\ldots\mu_k]}}_{(a)}\\
&\quad \underbrace{-W^\nu V^\rho\partial_\nu\alpha_{\rho\mu_2\mu_3\ldots\mu_k} + (k-1)W^\nu V^\rho\partial_{[\mu_2}\alpha_{|\rho\nu|\mu_3\ldots\mu_k]}}_{(b)}\\
&\quad \underbrace{-W^\nu V^\rho\partial_\rho\alpha_{\nu\mu_2\ldots\mu_k} + W^\nu V^\rho\partial_\nu\alpha_{\rho\mu_2\ldots\mu_k} - (k-1)W^\nu V^\rho\partial_{[\mu_2}\alpha_{|\rho\nu|\ldots\mu_k]}}_{(c)} \qquad = 0
\end{aligned}$$

So, finally,

$$\begin{aligned}
&[\mathfrak{L}_{\boldsymbol{V}},\, \boldsymbol{W}\lrcorner]\, \boldsymbol{\alpha}\\
&= \Big[(k-1)\partial_{[\mu_2}\left(W^\nu V^\rho\right)\alpha_{|\nu\rho|\mu_3\ldots\mu_k]} + kV^\rho\partial_{[\rho}W^\nu\alpha_{|\nu|\mu_2\mu_3\ldots\mu_k]} - kW^\nu\partial_{[\nu}\left(V^\rho\right)\alpha_{|\rho|\mu_2\mu_3\ldots\mu_k]}\Big]\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&= \Big[(k-1)\partial_{[\mu_2}\left(W^\nu V^\rho\right)\alpha_{|\nu\rho|\mu_3\ldots\mu_k]} + V^\rho\partial_\rho W^\nu\alpha_{\nu\mu_2\mu_3\ldots\mu_k} - (k-1)V^\rho\partial_{[\mu_2}W^\nu\alpha_{|\nu\rho|\mu_2\mu_3\ldots\mu_k]}
\end{aligned}$$

$$
\begin{aligned}
&\qquad -W^\nu\partial_\nu\left(V^\rho\right)\alpha_{\rho\mu_2\mu_3\ldots\mu_k} - (k-1)W^\nu\partial_{[\mu_2}\left(V^\rho\right)\alpha_{|\nu\rho|\mu_2\mu_3\ldots\mu_k]}\Big]\,\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&= \left[V^\rho\partial_\rho W^\nu\alpha_{\nu\mu_2\mu_3\ldots\mu_k} - W^\rho\partial_\rho V^\nu\alpha_{\nu\mu_2\mu_3\ldots\mu_k}\right]\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&= \left(V^\rho\partial_\rho W^\nu - W^\rho\partial_\rho V^\nu\right)\alpha_{\nu\mu_2\mu_3\ldots\mu_k}\mathrm{d}x^{\otimes\mu_2\ldots\mu_k}\\
&= [\boldsymbol{V},\,\boldsymbol{W}]\lrcorner\boldsymbol{\alpha}
\end{aligned}
$$

■

B.4 Proof of (5.31)

We want to prove

$$
\begin{aligned}
\star\mathbf{D}\star\boldsymbol{\alpha}^A = \frac{(-1)^{\mathtt{D}(k-1)}\operatorname{sgn}(g)}{(k-1)!}&\boldsymbol{\vartheta}_{a_1\ldots a_{k-1}}\\
&\times\left[\left(\nabla_b - \frac{1}{2}Q_{bc}{}^c + T_{bc}{}^c\right)\alpha^{ba_1\ldots a_{k-1}A} - \frac{k-1}{2}T_{bc}{}^{[a_1}\alpha^{|bc|a_2\ldots a_{k-1}]A}\right]
\end{aligned}
$$

Proof. We omit the external index (A) to alleviate notation and it will play no role in practice. Let us also introduce a convenient (ordered) multi-index $\mathfrak{B}\equiv b_1 \ldots b_k$ to shorten some expressions

$$
\begin{aligned}
&\mathbf{D}\star\boldsymbol{\alpha}\\
&= \mathbf{D}\left[\frac{1}{k!(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{B}a_1\ldots a_{\mathtt{D}-k}}\alpha^{\mathfrak{B}}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}}\right]\\
&= \frac{1}{k!(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{B}a_1\ldots a_{\mathtt{D}-k}}\left[-\frac{1}{2}\boldsymbol{Q}_d{}^d\wedge\alpha^{\mathfrak{B}}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}} + \mathbf{D}\alpha^{\mathfrak{B}}\wedge\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}} + \alpha^{\mathfrak{B}}\mathbf{D}\boldsymbol{\vartheta}^{a_1\ldots a_{\mathtt{D}-k}}\right]\\
!!&= \frac{1}{k!(\mathtt{D}-k)!}\mathcal{E}_{\mathfrak{B}a_1\ldots a_{\mathtt{D}-k}}\left[\left(\mathbf{D}-\frac{1}{2}\boldsymbol{Q}_d{}^d\right)\alpha^{\mathfrak{B}}\wedge\boldsymbol{\vartheta}^{a_1} + (\mathtt{D}-k)\alpha^{\mathfrak{B}}\boldsymbol{T}^{a_1}\right]\wedge\boldsymbol{\vartheta}^{a_2\ldots a_{\mathtt{D}-k}}\\
&= \frac{1}{k!(\mathtt{D}-k)}\left[\left(\mathbf{D}-\frac{1}{2}\boldsymbol{Q}_d{}^d\right)\alpha^{\mathfrak{B}}\wedge\boldsymbol{\vartheta}^{a_1} + (\mathtt{D}-k)\alpha^{\mathfrak{B}}\boldsymbol{T}^{a_1}\right]\wedge\left(\frac{1}{(\mathtt{D}-k-1)!}\mathcal{E}_{\mathfrak{B}a_1\ldots a_{\mathtt{D}-k}}\boldsymbol{\vartheta}^{a_2\ldots a_{\mathtt{D}-k}}\right)\\
&= \frac{1}{k!(\mathtt{D}-k)}\left[\left(\mathbf{D}-\frac{1}{2}\boldsymbol{Q}_d{}^d\right)\alpha^{\mathfrak{B}}\wedge\boldsymbol{\vartheta}^{b_{k+1}} + (\mathtt{D}-k)\alpha^{\mathfrak{B}}\boldsymbol{T}^{b_{k+1}}\right]\wedge\star\boldsymbol{\vartheta}_{\mathfrak{B}b_{k+1}}\\
&= \frac{1}{k!(\mathtt{D}-k)}\left[\left(\nabla_b-\frac{1}{2}Q_{bd}{}^d\right)\alpha^{\mathfrak{B}}\delta_c^{b_{k+1}} + (\mathtt{D}-k)\alpha^{\mathfrak{B}}\frac{1}{2}T_{bc}{}^{b_{k+1}}\right]\boldsymbol{\vartheta}^{bc}\wedge\star\boldsymbol{\vartheta}_{\mathfrak{B}b_{k+1}}\\
&= \frac{1}{k!(\mathtt{D}-k)}\left[\left(\nabla_b-\frac{1}{2}Q_{bd}{}^d\right)\alpha^{\mathfrak{B}}\delta_c^{b_{k+1}} + (\mathtt{D}-k)\alpha^{\mathfrak{B}}\frac{1}{2}T_{bc}{}^{b_{k+1}}\right]\\
&\qquad\qquad\times(-1)^{k+2}(-1)^{k+1}(k+1)k\star\delta^c_{[b_1}\delta^b_{b_2}\boldsymbol{\vartheta}_{b_3\ldots.b_{k+1}]}\\
!!!&= -\frac{k+1}{(k-1)!(\mathtt{D}-k)}\left[\left(\nabla_b-\frac{1}{2}Q_{bd}{}^d\right)\alpha^{[cbb_3\ldots.b_k}\delta_c^{b_{k+1}]} + (\mathtt{D}-k)\frac{1}{2}T_{bc}{}^{[b_{k+1}}\alpha^{cbb_3\ldots.b_k]}\right]\star\boldsymbol{\vartheta}_{b_3\ldots.b_{k+1}}\\
&= \frac{k+1}{(k-1)!(\mathtt{D}-k)}(-1)^{k+1}\left[\left(\nabla_b-\frac{1}{2}Q_{bd}{}^d\right)\alpha^{[ba_1\ldots.a_{k-1}]}\frac{\mathtt{D}-k}{k+1} + (\mathtt{D}-k)\frac{1}{2}T_{bc}{}^{[a_1}\alpha^{cba_2\ldots.a_{k-1}]}\right]\star\boldsymbol{\vartheta}_{a_1\ldots.a_{k-1}}\\
&= \frac{1}{(k-1)!}(-1)^{k+1}\left[\left(\nabla_b-\frac{1}{2}Q_{bd}{}^d\right)\alpha^{ba_1\ldots.a_{k-1}} + \frac{1}{2}(k+1)T_{bc}{}^{[a_1}\alpha^{cba_2\ldots.a_{k-1}]}\right]\star\boldsymbol{\vartheta}_{a_1\ldots.a_{k-1}}\\
!!!!&= \frac{1}{(k-1)!}(-1)^{k+1}\left[\left(\nabla_b-\frac{1}{2}Q_{bd}{}^d + T_{bc}{}^c\right)\alpha^{ba_1\ldots.a_{k-1}} - \frac{1}{2}(k-1)T_{bc}{}^{[a_1}\alpha^{|bc|a_2\ldots.a_{k-1}]}\right]\star\boldsymbol{\vartheta}_{a_1\ldots.a_{k-1}}
\end{aligned}
$$

We have used:

❒ In ! the property (3.32)

- In !!

$$
\begin{aligned}
&\mathcal{E}_{b_1\dots b_k a_1\dots a_{\mathtt{D}-k}} \mathbf{D}\boldsymbol{\vartheta}^{a_1\dots a_{D-k}} \\
&= \mathcal{E}_{b_1\dots b_k a_1\dots a_{\mathtt{D}-k}} \left[\boldsymbol{T}^{a_1} \wedge \boldsymbol{\vartheta}^{a_2\dots a_{\mathtt{D}-k}} - \boldsymbol{\vartheta}^{a_1} \wedge \boldsymbol{T}^{a_2} \wedge \boldsymbol{\vartheta}^{a_3\dots a_{\mathtt{D}-k}} + \dots\right] \\
&= \mathcal{E}_{b_1\dots b_k [a_1\dots a_{\mathtt{D}-k}]} \left[\boldsymbol{T}^{a_1} \wedge \boldsymbol{\vartheta}^{a_2\dots a_{\mathtt{D}-k}} - \boldsymbol{T}^{a_2} \wedge \boldsymbol{\vartheta}^{a_1 a_3\dots a_{\mathtt{D}-k}} + \dots\right] \\
&= \mathcal{E}_{b_1\dots b_k [a_1\dots a_{\mathtt{D}-k}]} (\mathtt{D}-k) \boldsymbol{T}^{a_1} \wedge \boldsymbol{\vartheta}^{a_2\dots a_{\mathtt{D}-k}}
\end{aligned}
$$

- In !!! a particular case of (A.2):

$$
\delta_a^{[a} A^{b_1\dots b_k]} = \frac{(\mathtt{D}-k)!k!}{(\mathtt{D}-(k+1))!(k+1)!} A^{[b_1\dots b_k]} = \frac{\mathtt{D}-k}{k+1} A^{[b_1\dots b_k]}\,. \tag{B.1}
$$

- In !!!!:

$$
\begin{aligned}
&(k+1) T_{bc}{}^{[a_1} \alpha^{cba_2\dots a_{k-1}]} \\
&= -T_{bc}{}^{c} \alpha^{[a_1 ba_2\dots a_{k-1}]} + T_{bc}{}^{[a_1} \alpha^{|c|ba_2\dots a_{k-1}]} - T_{bc}{}^{[a_1} \alpha^{b|c|a_2\dots a_{k-1}]} + \dots \\
&= -T_{bc}{}^{c} \alpha^{[a_1 ba_2\dots a_{k-1}]} + k T_{bc}{}^{[a_1} \alpha^{|c|ba_2\dots a_{k-1}]} \\
&= -T_{bc}{}^{c} \alpha^{[a_1 ba_2\dots a_{k-1}]} + \left(-T_{bc}{}^{b} \alpha^{c[a_1 a_2\dots a_{k-1}]} + T_{bc}{}^{[a_1} \alpha^{|cb|a_2\dots a_{k-1}]} - T_{bc}{}^{[a_1} \alpha^{|c|a_2|b|a_3\dots a_{k-1}]} + \dots\right) \\
&= -T_{bc}{}^{c} \alpha^{[a_1 ba_2\dots a_{k-1}]} - T_{bc}{}^{b} \alpha^{c[a_1 a_2\dots a_{k-1}]} + (k-1) T_{bc}{}^{[a_1} \alpha^{|cb|a_2\dots a_{k-1}]} \\
&= 2 T_{bc}{}^{c} \alpha^{ba_1 a_2\dots a_{k-1}} - (k-1) T_{bc}{}^{[a_1} \alpha^{|bc|a_2\dots a_{k-1}]}
\end{aligned}
$$

Now we can do two things:

- expand $\star\boldsymbol{\vartheta}_{a_1\dots a_{k-1}} = \frac{1}{(\mathtt{D}-k+1)!}(-1)^{k+1} \mathcal{E}_{a_1\dots a_{k-1} c_1\dots c_{\mathtt{D}-k+1}} \boldsymbol{\vartheta}^{c_1\dots c_{\mathtt{D}-k+1}}$. The first proof ends;
- take Hodge dual and use $(-1)^{k+1}(-1)^{(k-1)(\mathtt{D}-k+1)} = (-1)^{\mathtt{D}(k-1)}$ since $k(k-1)$ is always an even number. The second proof ends.

Trivially, for the case of Levi-Civita we drop the torsion and the nonmetricity. ■