Approximation Methods

Exercise 1: Stationary perturbations

A particle of mass m inside a 1-dimensional well of width a (*i.e.*, $V(x) = 0$ if $0 < x < a$ and $V(x) \to \infty$ if $x > a$ or $x < 0$) has been perturbed by a potential of type

$$V_{\text{pert}}(x) = a\,\omega_0\,\delta(x - a/2)\,.$$

Find the correction to the energy levels at first order in ω_0.

Hint. The eigenvalues and eigenfunctions of the 1-dimensional quantum well of width a are, respectively

$$E_n^{(0)} = \frac{\hbar^2\pi^2 n^2}{2ma^2}\,, \qquad \psi_n^{(0)}(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n}{a}x\right) \qquad \text{with} \qquad n \in \{1, 2, 3 ...\}\,.$$

Exercise 2: Perturbations in a degenerate spectrum

Consider a particle of mass m inside a 2-dimensional well of $L \times L$. The energy levels are $E^{(0)}_{n_1 n_2} = (n_1^2 + n_2^2)\hbar^2\pi^2/(2mL^2)$ with $n_{1,2} \in \{1, 2, 3...\}$, whereas the corresponding orbital wave functions are

$$\psi^{(0)}_{n_1 n_2}(x, y) = \frac{2}{L}\sin\left(\frac{\pi n_1}{L}x\right)\sin\left(\frac{\pi n_2}{L}y\right)\,.$$

Suppose that the system has been perturbed by an interaction $V(x, y) = c\,xy$, where c is a constant. At first order in c:

i) Find the shift of the energy for a non-degenerate level.

ii) Determine the energy of the first excited state.

Hint. Useful integrals:

$$\int \mathrm{d}x\, x \sin^2 ax = \frac{x^2}{4} - \frac{\cos 2ax}{8a^2} - \frac{x \sin 2ax}{4a},$$

$$\int \mathrm{d}x\, x \sin ax \sin 2ax = \frac{9\cos ax - \cos 3ax + 12ax\sin^3 ax}{18a^2}.$$

Exercise 3: Oscillator

Consider a quantum oscillator of restoring constant k and reduced mass m that has been perturbed by $V(x) = a\,x^3$, where a is some (small) positive parameter.
The energy and wave function of the level n are $E_n^{(0)} = (n + 1/2)\hbar\omega$ and

$$\psi_n^{(0)}(x) = \sqrt{\frac{\alpha^{1/2}}{2^n n! \pi^{1/2}}} e^{-\alpha x^2/2} H_n(\sqrt{\alpha} x)\,,$$

where $\alpha = m\omega/\hbar$ and $H_n(x)$ are the Hermite polynomials.

i) Show that the *first non-trivial* correction to the energy of the ground state is of second order in a and calculate such correction.
Hint. For the second-order correction, consider only the contribution of $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$ and $H_3(x) = 8x^3 - 12x$, and ignore the contributions of $H_n(x)$ with $n > 3$ (at the end of the problem you will understand why).

ii) Compute the correction to the ground state wave function at first order in a.
Hint. Again, ignore the contributions of $\psi_n^{(0)}(x) \propto H_n(\sqrt{\alpha} x)$ with $n > 3$.

iii) Check that the ground state wave function with its first-order correction fulfills the Schrödinger equation for the total Hamiltonian up to higher order terms $\sim \mathcal{O}(a^2)$.

iv) Instead of working in terms of the wave functions $\psi_n^{(0)}(x) \equiv \langle x \mid n \rangle$, consider the state $|n\rangle$. Repeat *i)* and *ii)* in the creation-annihilation formalism and prove that the contributions of $|n\rangle$ with $n > 3$ vanish identically, as suggested in the Hints.

Exercise 4: Based on Test exercise, September 2010 [2.5P]

The spectrum of a particle inside a quantum well of width a (*i.e.*, $V(x) = 0$ if $0 < x < a$ and $V(x) \to \infty$ elsewhere) is

$$E_n^{(0)} = \frac{\hbar^2 \pi^2 n^2}{2ma^2}\,, \qquad \psi_n^{(0)}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n}{a} x\right) \qquad \text{with} \qquad n \in \{1, 2, 3...\}\,,$$

where m is the mass of the particle. Suppose that the system has been perturbed

$$H \to H + \frac{V_0}{a} x\,, \quad 0 \le x \le a\,.$$

i) Find the energy shift in the three lowest energy levels at first order in V_0.

ii) We introduce three identical spin 1/2 particles inside this well. Find the energy of the lowest energy level (up to first order in V_0) if the interaction among them is negligible.

Exercise 5: Time-dependent perturbation of the quantum harmonic oscillator

Consider a 1-dimensional harmonic oscillator of angular frequency ω and electric charge q. At $t = 0$ the oscillator is at the ground state, then we apply a constant electric field: $V(x) = -qEx$ for $t \geq 0$.

i) Find at first order the transition probability to the first excited state $|n = 1\rangle$.

ii) Show that at this order the system *cannot* jump into $|n\rangle$ with $n > 1$.

Exercise 6: Time-dependent perturbation of the quantum well

Consider a particle of mass m and electric charge q in the ground state of a 1-dimensional quantum well of width a (*i.e.*, $V(x) = 0$ if $0 < x < a$ and $V(x) \to \infty$ elsewhere). The corresponding eigenvalues and eigenfunctions are

$$E_n^{(0)} = \frac{\hbar^2 \pi^2 n^2}{2ma^2}, \qquad \psi_n^{(0)}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n}{a} x\right) \qquad \text{with} \qquad n \in \{1, 2, 3 \ldots\}.$$

Suppose we apply a constant electric field:

$$V_{\text{pert}}(x,t) = \begin{cases} -qEx & \text{if } x \in (0, a)\,,\ t \geq 0 \\ 0 & \text{if } x \in (0, a)\,,\ t < 0 \\ 0 & \text{if } x \notin (0, a)\,,\ \forall t \end{cases}$$

i) Compute the transition probability to an arbitrary *excited* state $\psi_{n(>1)}^{(0)}$ at first order, showing that (at this order):

- For those with odd n, the transition is not allowed.
- For those with even n, the transition probability has a suppression $\sim \frac{n^2}{(n^2-1)^6}$.

ii) Compare these results with those obtained in Exercise 5.

Hint. Useful integral

$$\int_0^a x \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{\pi}{a} x\right) \mathrm{d}x = -\frac{2a^2}{\pi^2} \frac{n}{(n^2-1)^2} (1 + (-1)^n) \qquad n \in \{2, 3, 4, \ldots\}$$

Exercise 4 must be turned in [deadline: 17 December 2020 at 15:00]