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Gravitational wave solutions in metric-affine gravity

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Structure of this presentation

A. Jimenez-Cano, Yu. N. Obukhov [AJC, Obukhov 2021]
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1. Introduction to metric-affine geometry

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

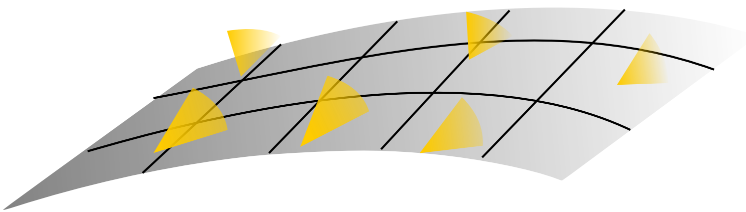
Geometric structures

- *Metric tensor:* $g_{\mu\nu}$
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_0^\sigma \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^\mu(\sigma')\dot{x}^\nu(\sigma')|}\,\mathrm{d}\sigma'\,. \tag{1.1}$$

$$\mathrm{vol}(\mathcal{U}) = \int_{\mathcal{U}} \boldsymbol{\omega}_{\mathrm{vol}}\,, \qquad \boldsymbol{\omega}_{\mathrm{vol}} := \sqrt{|g|}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \qquad D := \dim(\mathcal{U}). \tag{1.2}$$

  - Module of a vector (not necessarily non-negative) $\Rightarrow$ light cones $\Rightarrow$ causality.



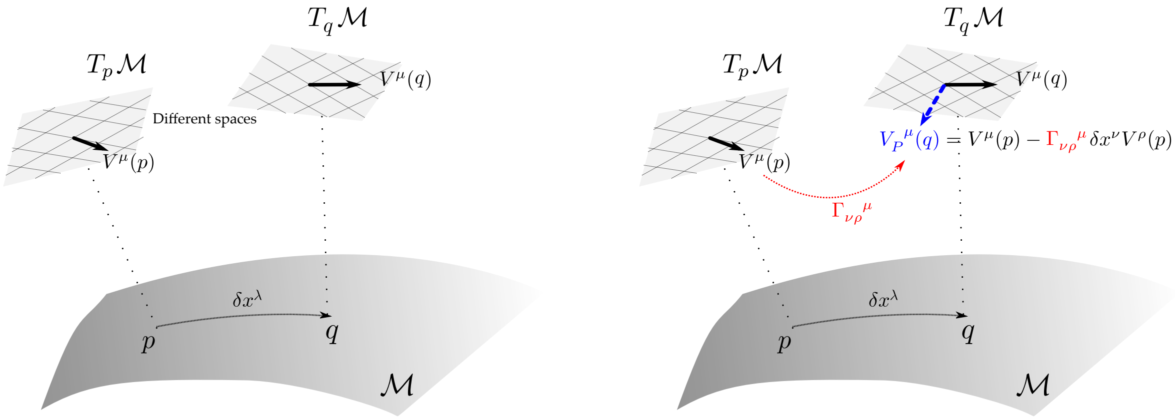
  - Notion of scale (conformal transformations...)

$$g_{\mu\nu} \to \mathrm{e}^{2\Omega} g_{\mu\nu}\,. \tag{1.3}$$

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

- *Connection:* $\Gamma_{\mu\nu}{}^{\rho}$
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_μ



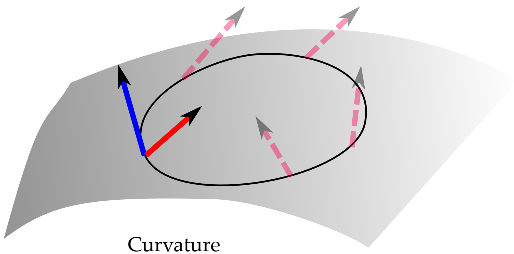
Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

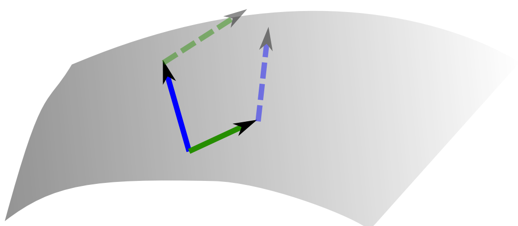
- *Connection:* $\Gamma_{\mu\nu}{}^{\rho}$
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_μ
  - Geometrical objects:

Curvature:
$$R_{\mu\nu\lambda}{}^{\rho} := \partial_\mu \Gamma_{\nu\lambda}{}^{\rho} - \partial_\nu \Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\lambda}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\lambda}{}^{\sigma}\,, \tag{1.4}$$

Torsion:
$$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}\,. \tag{1.5}$$



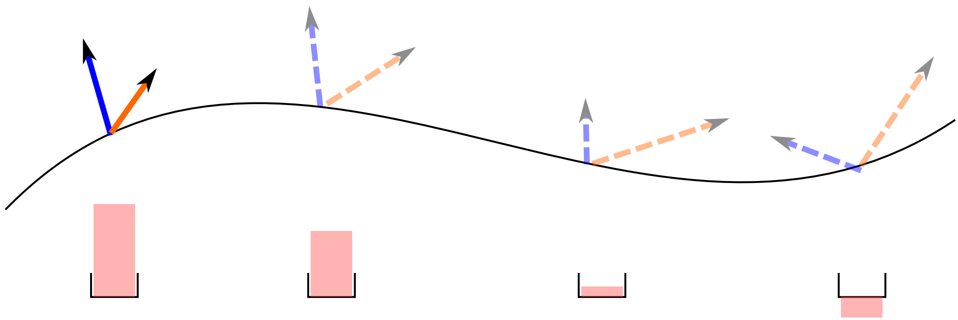
Curvature



Torsion

Def.: In the presence of metric and connection we define the *nonmetricity tensor*:

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}\,. \tag{1.6}$$



Theorem. Given $g_{\mu\nu}$, there is only one connection that satisfies

$$T_{\mu\nu}{}^{\rho} = 0 \quad \text{(torsionless condition)}, \tag{1.7}$$
$$Q_{\mu\nu\rho} = 0 \quad \text{(compatibility condition)}, \tag{1.8}$$

the *Levi-Civita connection:*

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{1.9}$$

Metric-affine theories
Instead of choosing $\mathring{\Gamma}$, they consider the metric and the (general) connection as independent fields.

Metric-affine geometry arises naturally when formulating a gauge theory of the affine group ($\mathrm{Aff}(4,\mathbb{R}) = \mathrm{Tr}_4 \rtimes \mathrm{GL}(4,\mathbb{R})$). [Hehl, McCrea, Mielke, Ne'eman 1995]

Three fundamental objects: coframe, metric and connection 1-form.

- **Coframe**. Arbitrary basis of the cotangent space pointwise smooth:

$$\boldsymbol{e}_a = e^{\mu}{}_a \boldsymbol{\partial}_\mu \,, \qquad \boxed{\boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu} \qquad [\boldsymbol{\vartheta}^a(\boldsymbol{e}_b) = \delta^a_b \quad \Leftrightarrow \quad e_\mu{}^a e^\mu{}_b = \delta^a_b] \,. \tag{1.10}$$

- **Metric**. Components of the metric in the arbitrary basis:

$$\boxed{g_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}} \,. \tag{1.11}$$

For a D-dimensional manifold:

- Canonical volume form

$$\boldsymbol{\omega}_{\mathrm{vol}} := \frac{1}{D!}\mathcal{E}_{a_1 \ldots a_D} \boldsymbol{\vartheta}^{a_1} \wedge \ldots \wedge \boldsymbol{\vartheta}^{a_D} = \sqrt{|g|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \,, \qquad |g| \equiv |\det(g_{\mu\nu})| \,. \tag{1.12}$$

- Hodge star of an arbitrary k-form $\boldsymbol{\alpha} = \frac{1}{k!}\alpha_{a_1\ldots a_k}\boldsymbol{\vartheta}^{a_1} \wedge \ldots \wedge \boldsymbol{\vartheta}^{a_k}$

$$\begin{aligned} \star \,:\, \Omega^k(\mathcal{M}) &\longrightarrow \Omega^{D-k}(\mathcal{M}) \\ \boldsymbol{\alpha} \quad &\longmapsto \quad \star\boldsymbol{\alpha} := \frac{1}{(D-k)!k!}\alpha^{b_1\ldots b_k}\mathcal{E}_{b_1\ldots b_k c_1 \ldots c_{D-k}} \boldsymbol{\vartheta}^{c_1} \wedge \ldots \wedge \boldsymbol{\vartheta}^{c_{D-k}} \,. \end{aligned} \tag{1.13}$$

Connection 1-form

$$\boxed{\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu}\,. \tag{1.14}$$

where $\omega_{\mu a}{}^b$ are the components of the affine connection in the anholonomic basis:

$$\omega_{\mu a}{}^b = e^\nu{}_a e_\lambda{}^b \Gamma_{\mu\nu}{}^\lambda + e_\sigma{}^b \partial_\mu e^\sigma{}_a\,. \tag{1.15}$$

N.B. $\Gamma_{\mu\nu}{}^\lambda$ and $\omega_{\mu a}{}^b$ contain the same information (for a given frame/coframe).

- Exterior covariant derivative (of algebra-valued forms)

$$\mathbf{D}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} = \mathrm{d}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} + \quad \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\alpha}_{a\ldots}{}^{c\ldots} + \ldots \quad - \boldsymbol{\omega}_a{}^c \wedge \boldsymbol{\alpha}_{c\ldots}{}^{b\ldots} - \ldots \quad , \tag{1.16}$$

- Curvature, torsion and non-metricity forms:

$$\boldsymbol{R}_a{}^b := \mathrm{d}\boldsymbol{\omega}_a{}^b + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\omega}_a{}^c = \frac{1}{2} R_{\mu\nu a}{}^b \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.17}$$

$$\boldsymbol{T}^a := \mathbf{D}\boldsymbol{\vartheta}^a = \frac{1}{2} T_{\mu\nu}{}^a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.18}$$

$$\boldsymbol{Q}_{ab} := -\mathbf{D} g_{ab} = Q_{\mu ab} \mathrm{d}x^\mu\,. \tag{1.19}$$

They can be decomposed according to irreps of $\mathrm{GL}(4,\mathbb{R})$:

$$\boldsymbol{T}^a = \underbrace{{}^{(1)}\boldsymbol{T}^a}_{\text{tensor}} + \underbrace{{}^{(2)}\boldsymbol{T}^a}_{\text{trace}} + \underbrace{{}^{(3)}\boldsymbol{T}^a}_{\text{axial}}, \qquad \boldsymbol{Q}_{ab} = \underbrace{{}^{(1)}\boldsymbol{Q}_{ab}}_{\text{tot. symm}} + \underbrace{{}^{(2)}\boldsymbol{Q}_{ab}}_{\text{tens.}} + \underbrace{{}^{(3)}\boldsymbol{Q}_{ab} + {}^{(4)}\boldsymbol{Q}_{ab}}_{\text{traces}}$$

$$\boldsymbol{R}^{ab} = \boldsymbol{W}^{[ab]} + \boldsymbol{Z}^{(ab)} \quad \Rightarrow \begin{cases} \boldsymbol{W}^{ab} = {}^{(1)}\boldsymbol{W}^{ab} + {}^{(2)}\boldsymbol{W}^{ab} + \underbrace{{}^{(3)}\boldsymbol{W}^{ab}}_{\text{tot. antis.}} + {}^{(4)}\boldsymbol{W}^{ab} + {}^{(5)}\boldsymbol{W}^{ab} + \underbrace{{}^{(6)}\boldsymbol{W}^{ab}}_{\text{Ric. scalar}} \\ \boldsymbol{Z}^{ab} = {}^{(1)}\boldsymbol{Z}^{ab} + {}^{(2)}\boldsymbol{Z}^{ab} + {}^{(3)}\boldsymbol{Z}^{ab} + {}^{(4)}\boldsymbol{Z}^{ab} + \underbrace{{}^{(5)}\boldsymbol{Z}^{ab}}_{\text{trace}} \end{cases}$$

2. Metric-Affine gauge theory and dynamics

[Hehl, McCrea, Mielke, Ne'eman 1995]

- The fundamental gravitational fields in metric-affine gravity are: g_{ab}, $\boldsymbol{\vartheta}^a$ and $\boldsymbol{\omega}_a{}^b$.

- Gauge symmetry $\Rightarrow$ the most general metric-affine Lagrangian D-form is:

$$\boldsymbol{L} = \boldsymbol{L}(g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{Q}_{ab}, \boldsymbol{T}^a, \boldsymbol{R}_a{}^b). \tag{2.1}$$

- Equations of motion

$$\mathbf{D}\boldsymbol{M}^{ab} - \boldsymbol{m}^{ab} = 0 \qquad [\text{EoM } g_{ab}]\,, \tag{2.2}$$

$$\mathbf{D}\boldsymbol{H}_a - \boldsymbol{E}_a = 0 \qquad [\text{EoM } \boldsymbol{\vartheta}^a]\,, \tag{2.3}$$

$$\mathbf{D}\boldsymbol{H}^a{}_b - \boldsymbol{E}^a{}_b = 0 \qquad [\text{EoM } \boldsymbol{\omega}_a{}^b]\,, \tag{2.4}$$

where

$$\begin{aligned}
\boldsymbol{M}^{ab} &:= -2\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ab}}\,, & \boldsymbol{H}_a &:= -\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^a}\,, & \boldsymbol{H}^a{}_b &:= -\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b}\,,\\
\boldsymbol{m}^{ab} &:= 2\frac{\partial \boldsymbol{L}}{\partial g_{ab}} & \boldsymbol{E}_a &:= \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a}\,, & \boldsymbol{E}^a{}_b &:= -\boldsymbol{\vartheta}^a \wedge \boldsymbol{H}_b - g_{bc}\boldsymbol{M}^{ac}\,.
\end{aligned} \tag{2.5}$$

[Hehl, McCrea, Mielke, Ne'eman 1995]

- Noether identity under $\mathrm{GL}(4,\mathbb{R})_{\text{local}} \Rightarrow$ EoM of g_{ab} is redundant.

$$\begin{aligned} \mathbf{D}\boldsymbol{M}^{ab} - \boldsymbol{m}^{ab} &= 0 \qquad [\text{EoM } g_{ab}]\,, \\ \mathbf{D}\boldsymbol{H}_a - \boldsymbol{E}_a &= 0 \qquad [\text{EoM } \boldsymbol{\vartheta}^a]\,, &(2.6)\\ \mathbf{D}\boldsymbol{H}^a{}_b - \boldsymbol{E}^a{}_b &= 0 \qquad [\text{EoM } \boldsymbol{\omega}_a{}^b]\,. &(2.7)\end{aligned}$$

In practice, one fixes the gauge (e.g. by taking $g_{ab} \equiv \eta_{ab}$).

- Noether identity under $\mathrm{Diff}(\mathcal{M}) \Rightarrow \boldsymbol{E}_a$ is determined by $\boldsymbol{H}^a{}_b$, $\boldsymbol{H}_a$ and $\boldsymbol{M}^{ab}$ via:

$$\boldsymbol{E}_a = \boldsymbol{e}_a \lrcorner \boldsymbol{L} + (\boldsymbol{e}_a \lrcorner \boldsymbol{T}^b) \wedge \boldsymbol{H}_b + (\boldsymbol{e}_a \lrcorner \boldsymbol{R}_b{}^c) \wedge \boldsymbol{H}^b{}_c + \frac{1}{2}(\boldsymbol{e}_a \lrcorner \boldsymbol{Q}_{bc})\boldsymbol{M}^{bc}\,. \tag{2.8}$$

Standard procedure to find solutions

1. Compute

$$\boldsymbol{M}^{ab} := -2\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{Q}_{ab}}\,, \qquad \boldsymbol{H}_a := -\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{T}^a}\,, \qquad \boldsymbol{H}^a{}_b := -\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b}\,. \tag{2.9}$$

2. Determine

$$\boldsymbol{E}^a{}_b := -\boldsymbol{\vartheta}^a \wedge \boldsymbol{H}_b - g_{bc}\boldsymbol{M}^{ac}\,, \tag{2.10}$$

$$\boldsymbol{E}_a = \boldsymbol{e}_a \lrcorner \boldsymbol{L} + (\boldsymbol{e}_a \lrcorner \boldsymbol{T}^b) \wedge \boldsymbol{H}_b + (\boldsymbol{e}_a \lrcorner \boldsymbol{R}_b{}^c) \wedge \boldsymbol{H}^b{}_c + \frac{1}{2}(\boldsymbol{e}_a \lrcorner \boldsymbol{Q}_{bc})\boldsymbol{M}^{bc}\,. \tag{2.11}$$

3. Evaluate $\boldsymbol{H}_a$, $\boldsymbol{H}^a{}_b$, $\boldsymbol{E}^a{}_b$ and $\boldsymbol{E}_a$ in the chosen Ansatz (fix the gauge a take a constant anholonomic metric g_{ab}, e.g. Minkowski).

4. Solve the differential equations

$$\mathbf{D}\boldsymbol{H}_a - \boldsymbol{E}_a = 0 \qquad [\text{EoM } \boldsymbol{\vartheta}^a]\,, \tag{2.12}$$

$$\mathbf{D}\boldsymbol{H}^a{}_b - \boldsymbol{E}^a{}_b = 0 \qquad [\text{EoM } \boldsymbol{\omega}_a{}^b]\,. \tag{2.13}$$

3. Gravitational waves in (quadratic) MAG

(Quadratic) MAG Lagrangian

The most general one containing linear and quadratic invariants of $\boldsymbol{Q}_{ab}$, $\boldsymbol{T}^a$ and $\boldsymbol{R}_a{}^b$:

$$\begin{aligned}
\boldsymbol{L} = \frac{1}{2\kappa c}\Big\{ & a_0 \star(\boldsymbol{\vartheta}_a \wedge \boldsymbol{\vartheta}_b) \wedge \boldsymbol{R}^{ab} - \boldsymbol{T}^a \wedge \sum_{I=1}^{3} a_I \star({}^{(I)}\boldsymbol{T}_a) && \sim R + TT \\
& - \boldsymbol{Q}_{ab} \wedge \sum_{I=1}^{4} b_I \star({}^{(I)}\boldsymbol{Q}^{ab}) - 2b_5({}^{(3)}\boldsymbol{Q}_{ac} \wedge \boldsymbol{\vartheta}^a) \wedge \star({}^{(4)}\boldsymbol{Q}^{bc} \wedge \boldsymbol{\vartheta}_b) && \sim QQ \\
& - 2\boldsymbol{\vartheta}^a \wedge \star\boldsymbol{T}^b \wedge \sum_{I=1}^{3} c_I\, {}^{(I+1)}\boldsymbol{Q}_{ab} \Big\} && \sim QT \\
- \frac{\ell_\rho^2}{2\kappa c} \boldsymbol{R}^{ab} \wedge \star\Big[& \sum_{I=1}^{6} w_I\, {}^{(I)}\boldsymbol{W}_{ab} + v_1\, \boldsymbol{\vartheta}_a \wedge (\boldsymbol{e}_c \rfloor {}^{(5)}\boldsymbol{W}^c{}_b) && \sim RR \\
& + \sum_{I=1}^{5} z_I\, {}^{(I)}\boldsymbol{Z}_{ab} + v_2\, \boldsymbol{\vartheta}_c \wedge (\boldsymbol{e}_a \rfloor {}^{(2)}\boldsymbol{Z}^c{}_b) + \sum_{I=3}^{5} v_I\, \boldsymbol{\vartheta}_a \wedge (\boldsymbol{e}_c \rfloor {}^{(I)}\boldsymbol{Z}^c{}_b)\Big]. && \sim RR \qquad (3.1)
\end{aligned}$$

(we do not consider the cosmological constant term)

- κ and ℓ_ρ are the gravitational couplings.
- Term with a_0 is the metric-affine version of the Einstein term.
- This Lagrangian has in total a_I (3) $+b_I$ (5) $+c_I$ (3) $+w_I$ (6) $+z_I$ (5) $+v_I$ (5) $= 27$ parameters.

- **Metric**: we fix the gauge $g_{ab} = \mathrm{diag}(+1, -1, -1, -1)$ (Minkowski metric).
- **Coframe**

$$\boldsymbol{\vartheta}^{\widehat{0}} = \frac{1}{2}(U + 1)\mathrm{d}\sigma + \frac{1}{2}\mathrm{d}\rho, \tag{3.2}$$

$$\boldsymbol{\vartheta}^{\widehat{1}} = \frac{1}{2}(U - 1)\mathrm{d}\sigma + \frac{1}{2}\mathrm{d}\rho, \tag{3.3}$$

$$\boldsymbol{\vartheta}^{\widehat{A}} = \mathrm{d}x^A, \qquad A = 2, 3. \tag{3.4}$$

where $U = U(\sigma, x^A)$. This implies that $g_{\mu\nu}$ is of the Brinkmann type:

$$\mathrm{d}s^2 = \mathrm{d}\sigma\mathrm{d}\rho + U\mathrm{d}\sigma^2 - \underbrace{\delta_{AB}\mathrm{d}x^A\mathrm{d}x^B}_{\text{transversal 2D space}} . \tag{3.5}$$

We introduce

$$\boxed{\boldsymbol{k} := \mathrm{d}\sigma = \boldsymbol{\vartheta}^{\widehat{0}} - \boldsymbol{\vartheta}^{\widehat{1}}} \qquad \text{(wave 1-form)} \qquad \to \qquad \text{dual to } \partial_\rho = k^\mu \partial_\mu . \tag{3.6}$$

- **Connection**

$$\boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c, \tag{3.7}$$

where W_a, V_a and u_a depend on σ and x^A and are transversal:

$$W^a = \delta^a_A W^A(\sigma, x^B), \quad V^a = \delta^a_A V^A(\sigma, x^B), \quad u_a = \delta^A_a u_A(\sigma, x^B), \qquad A = 2, 3. \tag{3.8}$$

Unknowns: Wave's profile determined by 7 variables: U, W^A, V^A, and u_A.

Ansatz :

$$\{\boldsymbol{\vartheta}^{\widehat{0}}, \boldsymbol{\vartheta}^{\widehat{1}}, \boldsymbol{\vartheta}^{\widehat{A}}\} = \{\tfrac{1}{2}(U+1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \tfrac{1}{2}(U-1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \mathrm{d}x^A\},$$

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1), \qquad \boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c,$$

- Torsion

$$\boldsymbol{T}^a = -\boldsymbol{k}\wedge k^a\left[\tfrac{1}{2}\partial_A U - \delta_{AB} W^B + u_A\right]\boldsymbol{\vartheta}^A \qquad = {}^{(1)}\boldsymbol{T}^a \qquad (3.9)$$

$$\Big(\underbrace{{}^{(2)}\boldsymbol{T}^a}_{\text{trace}} = \underbrace{{}^{(3)}\boldsymbol{T}^a}_{\text{axial}} = 0\Big).$$

Purely irreducible.

- Nonmetricity

$$\boldsymbol{Q}_{ab} = -2\boldsymbol{k}\, k_{(a}(W_{b)} + V_{b)}) + 2k_a k_b u_A \boldsymbol{\vartheta}^A \qquad = {}^{(1)}\boldsymbol{Q}_{ab} + {}^{(2)}\boldsymbol{Q}_{ab} \qquad (3.10)$$

$$\Big(\underbrace{{}^{(3)}\boldsymbol{Q}_{ab} = {}^{(4)}\boldsymbol{Q}_{ab}}_{\text{traces}} = 0\Big).$$

where

$${}^{(1)}\boldsymbol{Q}_{ab} = -\frac{4}{3}\boldsymbol{k}k_{(a}(W_{b)} + V_{b)}) - \frac{2}{3}k_a k_b (W_c + V_c)\boldsymbol{\vartheta}^c + \frac{4}{3}\boldsymbol{k}k_{(a}u_{b)} + \frac{2}{3}k_a k_b u_A \boldsymbol{\vartheta}^A, \qquad (3.11)$$

$${}^{(2)}\boldsymbol{Q}_{ab} = -\frac{2}{3}\boldsymbol{k}k_{(a}(W_{b)} + V_{b)}) + \frac{2}{3}k_a k_b (W_c + V_c)\boldsymbol{\vartheta}^c - \frac{4}{3}\boldsymbol{k}k_{(a}u_{b)} + \frac{4}{3}k_a k_b u_A \boldsymbol{\vartheta}^A. \qquad (3.12)$$

Ansatz :

$$\{\boldsymbol{\vartheta}^{\hat{0}}, \boldsymbol{\vartheta}^{\hat{1}}, \boldsymbol{\vartheta}^{\hat{A}}\} = \{\tfrac{1}{2}(U+1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \tfrac{1}{2}(U-1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \mathrm{d}x^A\},$$

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1), \qquad \boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c,$$

❒ Curvature ($\underline{\mathrm{d}} := \boldsymbol{\vartheta}^A \boldsymbol{e}_A \lrcorner \mathrm{d} = \mathrm{d}x^A \partial_A$)

$$\boldsymbol{R}_a{}^b = \boldsymbol{k} \wedge (k_a \underline{\mathrm{d}} V^b + k^b \underline{\mathrm{d}} W_a) + k_a k^b \mathrm{d}(u_A \boldsymbol{\vartheta}^A)\,. \tag{3.13}$$

If we introduce

$$\overset{(\pm)}{\Omega}{}^a := \underline{\mathrm{d}}(W^a \pm V^a) = \sum_{I=1,2,4} {}^{(I)}\overset{(\pm)}{\Omega}{}^a \quad , \begin{cases} {}^{(1)}\overset{(\pm)}{\Omega}{}^a := \frac{1}{2}\left(\overset{(\pm)}{\Omega}{}^a + \boldsymbol{\vartheta}^b \boldsymbol{e}^a \lrcorner \overset{(\pm)}{\Omega}{}_b - \boldsymbol{\vartheta}^a \boldsymbol{e}_b \lrcorner \overset{(\pm)}{\Omega}{}^b\right), \\ {}^{(2)}\overset{(\pm)}{\Omega}{}^a := \frac{1}{2}\left(\overset{(\pm)}{\Omega}{}^a - \boldsymbol{\vartheta}^b \boldsymbol{e}^a \lrcorner \overset{(\pm)}{\Omega}{}_b\right), \\ {}^{(4)}\overset{(\pm)}{\Omega}{}^a := \frac{1}{2}\, \boldsymbol{\vartheta}^a \boldsymbol{e}_b \lrcorner \overset{(\pm)}{\Omega}{}^b. \end{cases} \tag{3.14}$$

The transversal components of these objects are, if ${}^{(I)}\overset{(\pm)}{\Omega}{}^A = {}^{(I)}\overset{(\pm)}{\Omega}{}^A{}_B\, \boldsymbol{\vartheta}^B$,

$${}^{(1)}\overset{(\pm)}{\Omega}{}^A{}_B = \frac{1}{2}\left[\partial_B(W^A \pm V^A) + \partial^A(W_B \pm V_B) - \delta^A_B\, \partial_C(W^C \pm V^C)\right], \tag{3.15}$$

$${}^{(2)}\overset{(\pm)}{\Omega}{}^A{}_B = \frac{1}{2}\left[\partial_B(W^A \pm V^A) - \partial^A(W_B \pm V_B)\right], \tag{3.16}$$

$${}^{(4)}\overset{(\pm)}{\Omega}{}^A{}_B = \frac{1}{2}\, \delta^A_B\, \partial_C(W^C \pm V^C). \tag{3.17}$$

Ansatz :

$$\{\boldsymbol{\vartheta}^{\widehat{0}}, \boldsymbol{\vartheta}^{\widehat{1}}, \boldsymbol{\vartheta}^{\widehat{A}}\} = \{\tfrac{1}{2}(U+1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \tfrac{1}{2}(U-1)\mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\rho \ , \ \mathrm{d}x^{A}\},$$

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1), \qquad \boldsymbol{\omega}_a{}^b = -\boldsymbol{k}\left(k_a V^b + k^b W_a\right) + k_a k^b u_c \boldsymbol{\vartheta}^c,$$

❒ Curvature

$$\boldsymbol{R}_a{}^b = \underbrace{\boldsymbol{k}\wedge k^{[b}\overset{(-)}{\Omega}{}^{a]}}_{\boldsymbol{W}^{ab}} + \underbrace{\boldsymbol{k}\wedge k^{(b}\overset{(+)}{\Omega}{}^{a)} + k_a k^b \mathrm{d}(u_A \boldsymbol{\vartheta}^A)}_{\boldsymbol{Z}^{ab}}, \tag{3.18}$$

Non-trivial irreds in terms of $\overset{(\pm)}{\Omega}{}^a$ and its irreds:

$$\begin{aligned}
{}^{(1)}\boldsymbol{W}^{ab} &= \boldsymbol{k}\wedge {}^{(1)}\overset{(-)}{\Omega}{}^{[a}k^{b]},\\
{}^{(2)}\boldsymbol{W}^{ab} &= \boldsymbol{k}\wedge {}^{(2)}\overset{(-)}{\Omega}{}^{[a}k^{b]},\\
{}^{(4)}\boldsymbol{W}^{ab} &= \boldsymbol{k}\wedge {}^{(4)}\overset{(-)}{\Omega}{}^{[a}k^{b]},\\
{}^{(1)}\boldsymbol{Z}^{ab} &= \frac{1}{2}\boldsymbol{k}\wedge {}^{(1)}\overset{(+)}{\Omega}{}^{(a}k^{b)} + \frac{1}{4}k^a k^b\,\boldsymbol{\vartheta}_c\wedge \overset{(+)}{\Omega}{}^{c} + \frac{1}{2}\boldsymbol{k}\wedge k^{(a}e^{b)}\lrcorner\mathrm{d}(u_A\boldsymbol{\vartheta}^A) + \frac{1}{2}k^a k^b \mathrm{d}(u_A\boldsymbol{\vartheta}^A) + \frac{1}{2}\boldsymbol{k}\wedge \overset{(+)}{\Omega}{}^{(a}k^{b)},\\
{}^{(2)}\boldsymbol{Z}^{ab} &= \frac{1}{2}\boldsymbol{k}\wedge {}^{(2)}\overset{(+)}{\Omega}{}^{(a}k^{b)} - \frac{1}{4}k^a k^b\boldsymbol{\vartheta}_c\wedge \overset{(+)}{\Omega}{}^{c} - \frac{1}{2}\boldsymbol{k}\wedge k^{(a}e^{b)}\lrcorner\mathrm{d}(u_A\boldsymbol{\vartheta}^A) + \frac{1}{2}k^a k^b \mathrm{d}(u_A\boldsymbol{\vartheta}^A),\\
{}^{(4)}\boldsymbol{Z}^{ab} &= \frac{1}{2}\boldsymbol{k}\wedge {}^{(4)}\overset{(+)}{\Omega}{}^{(a}k^{b)}.
\end{aligned} \tag{3.19}$$

Trivial irreds:

$${}^{(3)}\boldsymbol{W}^{ab} = {}^{(5)}\boldsymbol{W}^{ab} = {}^{(6)}\boldsymbol{W}^{ab} = 0 \qquad {}^{(3)}\boldsymbol{Z}^{ab} = {}^{(5)}\boldsymbol{Z}^{ab} = 0 \tag{3.20}$$

4. Field equations for our the GW Ansatz

$$[\text{EoM } \boldsymbol{\vartheta}^a] \qquad 0 = -\frac{a_1}{2}\Delta U + \left[\frac{a_0}{2} + a_1 - c_1\right]\partial_A W^A - \left[\frac{a_0}{2} + c_1\right]\partial_A V^A - (a_1 - 2c_1)\partial_A \underline{u}^A \,, \tag{4.1}$$

$$\begin{aligned} [\text{EoM } \boldsymbol{\omega}_{[ab]}] \qquad 0 = {} & \frac{a_0 + a_1}{2}\partial_A U + \left[-\frac{a_0}{2} - a_1 + c_1\right]\underline{W}_A + \left[\frac{a_0}{2} + c_1\right]\underline{V}_A + (a_1 - 2c_1)u_A \\ & - \frac{\ell_\rho^2}{4}\left[2w_1\Delta \underline{W}_A - 2w_1\Delta \underline{V}_A + (v_4 + 2w_4)\partial_A\partial_B W^B + (v_4 - 2w_4)\partial_A\partial_B V^B\right] \\ & - \frac{\ell_\rho^2}{4}\epsilon_{AB}\epsilon^{CD}\underline{\partial}^B\left[(v_2 - 2w_2)\partial_C \underline{W}_D + (v_2 + 2w_2)\partial_C \underline{V}_D - 2v_2\partial_C u_D\right]. \end{aligned} \tag{4.2}$$

$$\begin{aligned} [\text{EoM } \boldsymbol{\omega}_{(ab)}] \qquad 0 = {} & \frac{a_1 - 2c_1}{2}\partial_A U + \left[\frac{a_0}{2} - a_1 - \frac{4(2b_1 + b_2)}{3} + 3c_1\right]\underline{W}_A \\ & + \left[\frac{a_0}{2} - \frac{4(2b_1 + b_2)}{3} + c_1\right]\underline{V}_A + \left[a_0 + a_1 - 4c_1 - \frac{8(b_1 - b_2)}{3}\right]u_A \\ & - \frac{\ell_\rho^2}{4}\left[2z_1\Delta \underline{W}_A + 2z_1\Delta \underline{V}_A + (z_1 + z_4 + 3v_4)\partial_A\partial_B W^B + (z_1 + z_4 + v_4)\partial_A\partial_B V^B\right] \\ & - \frac{\ell_\rho^2}{4}\epsilon_{AB}\epsilon^{CD}\underline{\partial}^B\left[(2v_2 - z_1 - z_2)\partial_C \underline{W}_D - (z_1 + z_2)\partial_C \underline{V}_D - 2(z_1 - z_2 + v_2)\partial_C u_D\right]. \end{aligned} \tag{4.3}$$

$$\begin{aligned} 0 = {} & \frac{2c_1 - a_1}{2}\partial_A U + \left[\frac{a_0}{2} + a_1 - \frac{4(b_1 - b_2)}{3} - 3c_1\right]\underline{W}_A \\ & + \left[\frac{a_0}{2} - \frac{4(b_1 - b_2)}{3} - c_1\right]\underline{V}_A + \left[4c_1 - a_1 - \frac{4(b_1 + 2b_2)}{3}\right]u_A \\ & + \frac{\ell_\rho^2}{4}\epsilon_{AB}\epsilon^{CD}\underline{\partial}^B\left[(z_1 - z_2 + 2v_2)\partial_C \underline{W}_D + (z_1 - z_2)\partial_C \underline{V}_D + 2(z_1 + z_2 - v_2)\partial_C u_D\right]. \end{aligned} \tag{4.4}$$

$$0 = \partial_\sigma\left[(z_4 - z_1 + 3v_4)\partial_A W^A + (z_4 - z_1 + v_4)\partial_A V^A - 4z_1\partial_A \underline{u}^A\right]. \tag{4.5}$$

$$[\text{EoM } \boldsymbol{\vartheta}^a] \qquad 0 = (...)\Delta U + (...)\partial_A W^A + (...)\partial_A V^A + (...)\partial_A \underline{u}^A\,, \tag{4.6}$$

$$\begin{aligned}[\text{EoM } \boldsymbol{\omega}_{[ab]}] \qquad 0 =& (...)\partial_A U + (...)\underline{W}_A + (...)\underline{V}_A + (...)u_A\\ &- \frac{\ell_\rho^2}{4}\left[(...)\Delta\underline{W}_A + (...)\Delta\underline{V}_A + (...)\partial_A\partial_B W^B + (...)\partial_A\partial_B V^B\right]\\ &- \frac{\ell_\rho^2}{4}\epsilon_{AB}\underline{\partial}^B\left\{\epsilon^{CD}\left[(...)\partial_{[C}\underline{W}_{D]} + (...)\partial_{[C}\underline{V}_{D]} + (...)\partial_{[C}u_{D]}\right]\right\}\,.\end{aligned} \tag{4.7}$$

$$[\text{EoM } \boldsymbol{\omega}_{(ab)}] \qquad 0 = (\text{same structure as the previous one}), \tag{4.8}$$

$$0 = (\text{same structure as the previous one without the second line}), \tag{4.9}$$

$$0 = \partial_\sigma\left[(...)\partial_A W^A + (...)\partial_A V^A + (...)\partial_A\underline{u}^A\right]\,. \tag{4.10}$$

where

$$\underline{W}_A := \delta_{AB}W^B, \qquad \underline{V}_A := \delta_{AB}V^B, \qquad \underline{u}^A := \delta^{AB}u_B, \qquad \underline{\partial}^A := \delta^{AB}\partial_B, \qquad \Delta := \delta^{AB}\partial_A\partial_B \tag{4.11}$$

and ϵ_{AB}, ϵ^{CD} correspond to the 2-dimensional Levi-Civita symbol (convention: $\epsilon_{23} := 1$, $\epsilon^{23} := 1$).

5. Solutions of the field equations

Conditions: Nullity of torsion and nonmetricity is equivalent to

$$u_A = 0, \qquad W^A = -V^A = \frac{1}{2}\delta^{AB}\partial_B U. \tag{5.1}$$

Non-trivial equations

$$\begin{aligned} a_0 \Delta U &= 0, &(5.2)\\ v_4\, \partial_\sigma \Delta U &= 0, &(5.3)\\ v_4 \ell_\rho^2 \partial_A \Delta U &= 0, &(5.4)\\ (w_1 + w_4)\ell_\rho^2 \partial_A \Delta\, U &= 0. &(5.5) \end{aligned}$$

Remarks

- The solution of GR, $\Delta U = 0$ (and Levi-Civita), is a solution of all MAG models.
- For $\boldsymbol{L}$ without the RR sector ($w_I = z_I = v_I = 0$):

$$u_A = 0, \qquad W^A = -V^A = \frac{1}{2}\delta^{AB}\partial_B U, \qquad \frac{a_0}{2}\Delta U = 0 \tag{5.6}$$

is the general solution, except for very specific parameters: [Obukhov, Vlachynsky, Esser, Hehl 1997]

$$-a_1 = \frac{a_2}{2} = 2a_3 = 2c_1 = -c_2 = -c_3 = a_0, \tag{5.7}$$

$$4b_1 = 2b_2 = -8b_3 = \frac{8b_4}{3} = 2b_5 = a_0. \tag{5.8}$$

Conditions: Nullity of curvature is equivalent to

$$W^A = W^A(\sigma), \qquad V^A = V^A(\sigma), \qquad u_A = \frac{1}{2}\partial_A \mathcal{U} \quad (\mathcal{U} = \mathcal{U}(x^B)). \tag{5.9}$$

Coframe equation

$$a_1 \Delta U + (a_1 - 2c_1)\,\Delta\mathcal{U} = 0\,. \tag{5.10}$$

Connection equation

→ New variables

$$\Theta_A = \frac{1}{2}\partial_A U - \underline{W}_A + u_A, \tag{5.11}$$

$$\Phi_A := \underline{W}_A + \underline{V}_A + u_A, \tag{5.12}$$

$$\Psi_A := \underline{W}_A + \underline{V}_A - 2u_A. \tag{5.13}$$

→ Non-trivial equations

$$\begin{pmatrix} 0 & (a_0 - 4b_1) & 0 \\ 2(a_0 + a_1) & 0 & (a_0 + 2c_1) \\ 3(a_0 + 2c_1) & 0 & 2(a_0 + 2b_2) \end{pmatrix} \begin{pmatrix} \Theta_A \\ \Phi_A \\ \Psi_A \end{pmatrix} = 0. \tag{5.14}$$

Remarks

- If the determinant vanishes we have non-trivial solutions.

Conditions: Nullity of curvature and nonmetricity is equivalent to

$$-V^A = W^A = W^A(\sigma), \qquad u_A = 0. \tag{5.15}$$

Coframe equation

$$a_1 \Delta U = 0\,. \tag{5.16}$$

Connection equation

$\rightarrow$ New variables become

$$\Theta_A = \frac{1}{2}\partial_A U - \underline{W}_A, \qquad \Phi_A = \Psi_A = 0. \tag{5.17}$$

$\rightarrow$ Non-trivial equations

$$(a_0 + a_1)\Theta_A = 0 \qquad (a_0 + 2c_1)\Theta_A = 0 \tag{5.18}$$

Remarks

- Non-trivial solutions for

$$a_0 + a_1 = 0, \qquad a_0 + 2c_1 = 0. \tag{5.19}$$

so

$$a_0 \Delta U = 0. \tag{5.20}$$

Same metric structure as in the GR solution (but $\Gamma \neq$ Levi-Civita).

Exploring special cases: Teleparallel case ($R = 0$) and $T = 0$ (Symmetric teleparallelism)

Conditions: Nullity of curvature and torsion is equivalent to

$$W^A = W^A(\sigma), \qquad V^A = V^A(\sigma), \qquad u_A = \frac{1}{2}\partial_A \mathcal{U}(x^B), \qquad \underbrace{\partial_A(U+\mathcal{U}) - 2W_A}_{\xrightarrow[W^A(\sigma)]{} \Delta(U+\mathcal{U})=0} = 0. \tag{5.21}$$

Coframe equation

$$c_1 \Delta U = 0\,. \tag{5.22}$$

Connection equation

→ New variables become

$$\Theta_A = 0, \qquad \Phi_A = W_A + V_A + \frac{1}{2}\partial_A \mathcal{U}, \qquad \Psi_A = W_A + V_A - \partial_A \mathcal{U}. \tag{5.23}$$

→ Non-trivial equations

$$(a_0 - 4b_1)\Phi_A = 0, \qquad (a_0 + 2b_2)\Psi_A = 0, \qquad (a_0 + 2c_1)\Psi_A = 0. \tag{5.24}$$

Remarks

- There are non-trivial solutions for some values of the parameters.

- **Step 1. Potential-copotential decomposition**

$$W^A =: \frac{1}{2}\left(\delta^{AB}\partial_B\mathcal{W} + \epsilon^{AB}\partial_B\overline{\mathcal{W}}\right), \tag{5.25}$$

$$V^A =: \frac{1}{2}\left(\delta^{AB}\partial_B\mathcal{V} + \epsilon^{AB}\partial_B\overline{\mathcal{V}}\right), \tag{5.26}$$

$$u_A =: \frac{1}{2}\left(\partial_A\mathcal{U} + \epsilon_{AB}\delta^{BC}\partial_C\overline{\mathcal{U}}\right). \tag{5.27}$$

Useful, since

$$F^A = \frac{1}{2}\left(\delta^{AB}\partial_B\mathcal{F} + \epsilon^{AB}\partial_B\overline{\mathcal{F}}\right) \quad \Rightarrow \quad \boxed{\partial_A F^A = \frac{1}{2}\Delta\mathcal{F} \quad \text{and} \quad \epsilon_{AB}\delta^{BC}\partial_C F^A = \frac{1}{2}\Delta\overline{\mathcal{F}}}\,. \tag{5.28}$$

- **Step 2. Splitting of the equations**. By a similar decomposition, the equations can be split into even and odd parts. [Blagojević, Cvetković, Obukhov 2017]

- **Step 3. Convenient change of variables**

$$\boxed{\begin{aligned} \mathcal{X}_0 &= \mathcal{W} - \mathcal{V} \\ \mathcal{X}_1 &= U - \mathcal{W} + \mathcal{U}, \\ \mathcal{X}_2 &= \mathcal{W} + \mathcal{V} + \mathcal{U}, \\ \mathcal{X}_3 &= \mathcal{W} + \mathcal{V} - 2\mathcal{U}, \end{aligned}} \quad \Leftrightarrow \quad \begin{aligned} U &= \tfrac{1}{2}\mathcal{X}_0 + \mathcal{X}_1 + \tfrac{1}{2}\mathcal{X}_3, \\ \mathcal{W} &= \tfrac{1}{2}\mathcal{X}_0 + \tfrac{1}{3}\mathcal{X}_2 + \tfrac{1}{6}\mathcal{X}_3, \\ \mathcal{V} &= -\tfrac{1}{2}\mathcal{X}_0 + \tfrac{1}{3}\mathcal{X}_2 + \tfrac{1}{6}\mathcal{X}_3, \\ \mathcal{U} &= \tfrac{1}{3}\mathcal{X}_2 - \tfrac{1}{3}\mathcal{X}_3, \end{aligned} \tag{5.29}$$

$$\boxed{\begin{aligned} \overline{\mathcal{X}}_1 &= -\overline{\mathcal{W}} + \overline{\mathcal{U}}, \\ \overline{\mathcal{X}}_2 &= \overline{\mathcal{W}} + \overline{\mathcal{V}} + \overline{\mathcal{U}}, \\ \overline{\mathcal{X}}_3 &= \overline{\mathcal{W}} + \overline{\mathcal{V}} - 2\overline{\mathcal{U}}. \end{aligned}} \quad \Leftrightarrow \quad \begin{aligned} \overline{\mathcal{W}} &= -\overline{\mathcal{X}}_1 + \tfrac{1}{3}\overline{\mathcal{X}}_2 - \tfrac{1}{3}\overline{\mathcal{X}}_3, \\ \overline{\mathcal{V}} &= \overline{\mathcal{X}}_1 + \tfrac{1}{3}\overline{\mathcal{X}}_2 + \tfrac{2}{3}\overline{\mathcal{X}}_3, \\ \overline{\mathcal{U}} &= \tfrac{1}{3}\overline{\mathcal{X}}_2 - \tfrac{1}{3}\overline{\mathcal{X}}_3. \end{aligned} \tag{5.30}$$

ODD SECTOR

- By combining the equations: $\overline{\mathcal{X}}_2$ is decoupled:

$$(a_0 - 4b_1)\,\overline{\mathcal{X}}_2 - \ell_\rho^2\, z_1\,\Delta\,\overline{\mathcal{X}}_2 = 0, \tag{5.31}$$

whereas $\overline{\mathcal{X}}_1$ and $\overline{\mathcal{X}}_3$ verify

$$\begin{aligned}(a_0 + 2c_1)\,\overline{\mathcal{X}}_1 + \frac{2}{3}(a_0 + 2b_2)\,\overline{\mathcal{X}}_3 - \frac{\ell_\rho^2}{4}\Big\{&-2\,[2w_1 + 2w_2 + v_2]\;\Delta\,\overline{\mathcal{X}}_1\\ &-\left[2w_1 + 2w_2 + v_2 + \frac{1}{3}\,(z_1 + 3z_2)\right]\Delta\,\overline{\mathcal{X}}_3\Big\} = 0,\end{aligned} \tag{5.32}$$

$$(a_0 + a_1)\,\overline{\mathcal{X}}_1 + \left(\frac{a_0}{2} + c_1\right)\overline{\mathcal{X}}_3 - \frac{\ell_\rho^2}{4}\left[-\,4(w_1 + w_2)\,\Delta\,\overline{\mathcal{X}}_1 - (2w_1 + 2w_2 + v_2)\Delta\overline{\mathcal{X}}_3\right] = 0 \tag{5.33}$$

ODD SECTOR

- If we take $\overline{\mathcal{X}}_I = \overline{\mathcal{X}}_I^{(0)}(\sigma)\mathrm{e}^{\mathrm{i}\overline{q}_A x^A}$:

$$\begin{pmatrix} 0 & a_0 - 4b_1 + 4z_1\overline{\mathcal{Q}}^2 & 0 \\ a_0 + 2c_1 - 2\overline{\mathcal{Q}}^2\Lambda_2 & 0 & \frac{2}{3}(a_0 + 2b_2) - \overline{\mathcal{Q}}^2(\Lambda_2 + \Lambda_3) \\ a_0 + a_1 - \overline{\mathcal{Q}}^2\Lambda_1 & 0 & \frac{a_0}{2} + c_1 - \overline{\mathcal{Q}}^2\Lambda_2 \end{pmatrix} \begin{pmatrix} \overline{\mathcal{X}}_1^{(0)} \\ \overline{\mathcal{X}}_2^{(0)} \\ \overline{\mathcal{X}}_3^{(0)} \end{pmatrix} = 0. \qquad (5.34)$$

Abbreviations:

$$\overline{\mathcal{Q}}^2 := \frac{\ell_\rho^2}{4}\overline{q}_A\overline{q}_B\delta^{AB}, \qquad \Lambda_1 := 4(w_1+w_2), \qquad \Lambda_2 := 2(w_1+w_2)+v_2, \qquad \Lambda_3 := \frac{1}{3}(z_1+3z_2)\,. \qquad (5.35)$$

- The three modes propagate if

$$a_0 - 4b_1 + 4z_1\overline{\mathcal{Q}}^2 = 0, \qquad \mathcal{A}\overline{\mathcal{Q}}^4 + \mathcal{B}\overline{\mathcal{Q}}^2 + \mathcal{C} = 0, \qquad (5.36)$$

where we denoted the combinations of the coupling constants

$$\mathcal{A} := 2\Lambda_2^2 + \Lambda_1(\Lambda_2 + \Lambda_3), \qquad (5.37)$$

$$\mathcal{B} := -4\left(\frac{a_0}{2} + c_1\right)\Lambda_2 + (a_0 + a_1)(\Lambda_2 + \Lambda_3) - \frac{2}{3}(a_0 + 2b_2)\Lambda_1, \qquad (5.38)$$

$$\mathcal{C} := 2\left(\frac{a_0}{2} + c_1\right)^2 - \frac{2}{3}(a_0 + 2b_2)(a_0 + a_1). \qquad (5.39)$$

- The amplitudes $\overline{\mathcal{X}}_I^{(0)}(\sigma)$ are arbitrary functions of σ.

EVEN SECTOR

- The equations are

$$
\begin{aligned}
(2c_1 - a_1)\,\mathcal{X}_1 + \frac{1}{3}(a_0 - 4b_1)\,\mathcal{X}_2 + \left[-\frac{a_0}{2} - c_1 + \frac{2}{3}(a_0 + 2b_2)\right]\mathcal{X}_3 &= 0, & (5.40)\\
(a_0 + a_1)\,\mathcal{X}_1 + \left(\frac{a_0}{2} + c_1\right)\mathcal{X}_3 - \frac{\ell_\rho^2}{4}\left[2(w_1 + w_4)\,\Delta\,\mathcal{X}_0 + \frac{2}{3}\,v_4\,\Delta\,\mathcal{X}_2 + \frac{1}{3}\,v_4\,\Delta\,\mathcal{X}_3\right] &= 0, & (5.41)\\
(a_1 - 2c_1)\,\mathcal{X}_1 + \frac{2}{3}(a_0 - 4b_1)\,\mathcal{X}_2 + \left[\frac{a_0}{2} + c_1 - \frac{2}{3}(a_0 + 2b_2)\right]\mathcal{X}_3 & \\
-\frac{\ell_\rho^2}{4}\left[v_4\,\Delta\,\mathcal{X}_0 + \frac{2}{3}\,(3z_1 + z_4 + 2v_4)\,\Delta\,\mathcal{X}_2 + \frac{1}{3}\,(3z_1 + z_4 + 2v_4)\,\Delta\,\mathcal{X}_3\right] &= 0, & (5.42)\\
\frac{a_0}{2}\,\Delta\,\mathcal{X}_0 - a_1\,\Delta\,\mathcal{X}_1 - c_1\,\Delta\,\mathcal{X}_3 &= 0, & (5.43)\\
\partial_\sigma\left\{v_4\,\Delta\,\mathcal{X}_0 + \frac{2}{3}(z_4 - 3z_1 + 2v_4)\,\Delta\,\mathcal{X}_2 + \frac{1}{3}(z_4 + 3z_1 + 2v_4)\,\Delta\,\mathcal{X}_3\right\} &= 0. & (5.44)
\end{aligned}
$$

- Special solution

$$\mathcal{X}_2 = \mathcal{X}_3 = 0 \tag{5.45}$$

If we assume $v_4 \neq 0$ and $a_1 \neq 2c_1$, the system becomes

$$\Delta U = 0, \qquad \mathcal{W} = -\mathcal{V} = U, \qquad \mathcal{U} = 0. \tag{5.46}$$

(massless graviton mode).

EVEN SECTOR

We take $\mathcal{X}_I = \mathcal{X}_I^{(0)}(\sigma)\mathrm{e}^{\mathrm{i}q_A x^A}$

- **First four equations**

$$\begin{pmatrix} -\frac{a_0}{2} & a_1 & 0 & c_1 \\ 0 & 2c_1 - a_1 & \frac{1}{3}(a_0 - 4b_1) & -\frac{a_0}{2} - c_1 + \frac{2}{3}(a_0 + 2b_2) \\ 2(w_1 + w_4)\mathcal{Q}^2 & a_0 + a_1 & \frac{2}{3}v_4\mathcal{Q}^2 & \frac{a_0}{2} + c_1 + \frac{1}{3}v_4\mathcal{Q}^2 \\ \mathcal{Q}^2 v_4 & 0 & a_0 - 4b_1 + \frac{2}{3}\mathcal{Q}^2\Lambda_0 & \frac{1}{3}\mathcal{Q}^2\Lambda_0 \end{pmatrix} \begin{pmatrix} \mathcal{X}_0^{(0)} \\ \mathcal{X}_1^{(0)} \\ \mathcal{X}_2^{(0)} \\ \mathcal{X}_3^{(0)} \end{pmatrix} = 0. \tag{5.47}$$

Abbreviations:

$$\mathcal{Q}^2 := \frac{\ell_\rho^2}{4} q_A q_B \delta^{AB}, \qquad \Lambda_0 := 3z_1 + z_4 + 2v_4\,. \tag{5.48}$$

- This is a 4×4 matrix but its determinant is a 2nd degree polynomial in $\mathcal{Q}^2$.
- Solutions for $\mathcal{Q}^2 \rightsquigarrow 2$ propagating massive modes. [Blagojević, Cvetković, Obukhov 2017]

- **Last equation**. It constrains the σ dependence:

$$v_4\,\partial_\sigma \mathcal{X}_0^{(0)} + \frac{2}{3}(\Lambda_0 - 6z_1)\,\partial_\sigma \mathcal{X}_2^{(0)} + \frac{1}{3}\Lambda_0\,\partial_\sigma \mathcal{X}_3^{(0)} = 0. \tag{5.49}$$

In [Vassiliev 2002]:

- $\boldsymbol{L} \sim RR$ (v's are omitted).
- Def. of pseudo-instantons: $\boldsymbol{Q}_{ab} = 0$ irreducible curvature solving vacuum EoM.

So:

- Vanishing nonmetricity for our Ansatz means that

$$\begin{aligned} \mathcal{U} = \overline{\mathcal{U}} &= 0\,, \\ \mathcal{W} &= -\mathcal{V}, \\ \overline{\mathcal{W}} &= -\overline{\mathcal{V}}. \end{aligned} \qquad \Leftrightarrow \qquad \left\{ \begin{array}{rlrl} \mathcal{X}_0 &= 2\mathcal{W}, & & \\ \mathcal{X}_1 &= U - \mathcal{W}, & \overline{\mathcal{X}}_1 &= -\overline{\mathcal{W}}, \\ \mathcal{X}_2 = \mathcal{X}_3 &= 0, & \overline{\mathcal{X}}_2 = \overline{\mathcal{X}}_3 &= 0. \end{array} \right. \tag{5.50}$$

Eqs. for the purely quadratic model (only w_I, z_J, v_K are nonvanishing):

$$\boxed{\Delta\mathcal{W} = 0, \quad \Delta\overline{\mathcal{W}} = 0} \tag{5.51}$$

This automatically imply

$$\boldsymbol{R}^{ab} = {}^{(1)}\boldsymbol{W}^{ab}\,, \tag{5.52}$$

which is irreducible.

6. Summary and conclusions

Results

- Quadratic (even) metric-affine action in vacuum
- Ansatz with 7 independent functions
- Particular solutions explored: Riemannian, teleparallel...
- Method to find the general solutions $\rightsquigarrow$ potential + co-potential decomp.
- Solutions for large families of MAG theories.

Limitations of this work / future work

- Solutions with matter
- Non-trivial cosmological constant
- Odd parity invariants:

$$\boldsymbol{R}^{ab} \wedge \boldsymbol{R}_{ab}, \qquad {}^{(I)}\boldsymbol{T}^{a} \wedge {}^{(J)}\boldsymbol{T}_{a} \qquad \ldots$$

- Different Ansatz (Kundt metric, other non-trivial irreds for $\boldsymbol{T}^a$ and $\boldsymbol{Q}_{ab}$).

Thanks for your attention!

Aitäh!

References used in this presentation

- F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, [Hehl, McCrea, Mielke, Ne'eman 1995]
  Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors and breaking of dilation invariance,
  Phys. Rep. **258**, 1-177 (1995).
- A. Jimenez-Cano and Yu. N. Obukhov, [AJC, Obukhov 2021]
  Gravitational waves in metric-affine gravity theory,
  Phys. Rev. D **103**, 024018 (2021).
- Yu. N. Obukhov, E. J. Vlachynsky, W. Esser, and F. W. Hehl, [Obukhov, Vlachynsky, Esser, Hehl 1997]
  Effective Einstein theory from metric-affine gravity models via irreducible decompositions,
  Phys. Rev. D **56**, 7769-7778 (1997).
- D. Vassiliev, [Vassiliev 2002]
  Pseudoinstantons in metric-affine theory,
  Gen. Rel. Grav. **34**, 1239-1265 (2002).
- M. Blagojević, B. Cvetković, and Yu. N. Obukhov, [Blagojević, Cvetković, Obukhov 2017]
  Generalized plane waves in Poincaré gauge theory of gravity,
  Phys. Rev. D **96**, 064031 (2017).