Geometric Foundations
of Gravity 2019

June, 2019
Tartu, Estonia

Non-trivial solutions of the Einstein-Hilbert and Gauss-Bonnet metric-affine lagrangians

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Structure of this presentation

B. Janssen, A. Jiménez-Cano, J. A. Orejuela [Janssen, Jiménez, Orejuela 2019]
A non-trivial connection for the metric-affine Gauss-Bonnet theory in $D = 4$.
Physics Letters B **795** (2019) 42 – 48

B. Janssen, A. Jiménez-Cano, J. A. Orejuela [Work in progress]
The role of the non-metricity in critical Lovelock theories in the metric-affine formulation. (?)

A. Jiménez-Cano, [My PhD Thesis – Still in progress]
Metric-Affine Gauge theory of gravity. Foundations, perturbations and gravitational wave solutions.

1. Introduction (geometry and notation)

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

- *Metric structure:* $g_{\mu\nu}$ (**metric tensor**)
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_{0,\,\gamma}^{\sigma} \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^{\mu}(\sigma')\dot{x}^{\nu}(\sigma')|}\mathrm{d}\sigma' \,. \tag{1.1}$$

$$\mathrm{vol}(\mathcal{U}) = \int_{\mathcal{U}} \boldsymbol{\omega}_{\mathrm{vol}}. \tag{1.2}$$

  - Module of a vector (not necessarily non-negative) $\Rightarrow$ light cones $\Rightarrow$ causality.
  - Notion of scale (conformal transformations...)

$$g_{\mu\nu} \to \mathrm{e}^{2\Omega} g_{\mu\nu} \,. \tag{1.3}$$

- *Affine structure*: $\Gamma_{\mu\nu}{}^{\rho}$ (**affine connection**)
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_{μ}
  - Geometrical objects:

Curvature: $$R_{\mu\nu\lambda}{}^{\rho} := \partial_{\mu}\Gamma_{\nu\lambda}{}^{\rho} - \partial_{\nu}\Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\lambda}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\lambda}{}^{\sigma} \,, \tag{1.4}$$

Torsion: $$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho} \,. \tag{1.5}$$

Def.: In the presence of metric and affine connection we define the *non-metricity tensor*:

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho} \,. \tag{1.6}$$

Theorem. Given $g_{\mu\nu}$, there is only one connection that satisfies

$$T_{\mu\nu}{}^{\rho} = 0 \quad \text{(torsionless condition)}, \tag{1.7}$$

$$Q_{\mu\nu\rho} = 0 \quad \text{(compatibility condition)}, \tag{1.8}$$

the *Levi-Civita connection:*

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right] \,. \tag{1.9}$$

Notation. Objects associated to the Levi-Civita connection: $\mathring{R}_{\mu\nu\lambda}{}^{\rho}$, $\mathring{R}_{\mu\nu}$, $\mathring{\nabla}_\mu$...

Three fundamental objects: coframe, metric and connection 1-form.

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- **Coframe**. We can fix a general frame in the manifold and the corresponding dual basis (coframe):

$$\boldsymbol{e}_a = e^\mu{}_a \boldsymbol{\partial}_\mu\,, \qquad \boxed{\boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu} \qquad [\boldsymbol{\vartheta}^a(\boldsymbol{e}_b) = \delta^a_b \quad \Leftrightarrow \quad e_\mu{}^a e^\mu{}_b = \delta^a_b]\,. \tag{1.10}$$

Notation:

$$\boldsymbol{\vartheta}^{a_1 \ldots a_k} \equiv \boldsymbol{\vartheta}^{a_1} \wedge \ldots \wedge \boldsymbol{\vartheta}^{a_k}\,. \tag{1.11}$$

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- **Metric**. Components of the metric in the arbitrary basis:

$$\boxed{g_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}} \,. \tag{1.12}$$

  - Canonical volume form

$$\boldsymbol{\omega}_{\mathrm{vol}} := \frac{1}{D!} \mathcal{E}_{a_1 \ldots a_D} \boldsymbol{\vartheta}^{a_1 \ldots a_D} = \sqrt{|g|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \,, \qquad |g| \equiv |\det(g_{\mu\nu})| \,. \tag{1.13}$$

  - Hodge star of an arbitrary k-form $\boldsymbol{\alpha} = \frac{1}{k!} \alpha_{a_1 \ldots a_k} \boldsymbol{\vartheta}^{a_1 \ldots a_k}$

$$\begin{aligned} \star \,:\, \Omega^k(\mathcal{M}) &\longrightarrow \Omega^{D-k}(\mathcal{M}) \\ \boldsymbol{\alpha} \quad &\longmapsto \quad \star\boldsymbol{\alpha} := \frac{1}{(D-k)!k!} \alpha^{b_1 \ldots b_k} \mathcal{E}_{b_1 \ldots b_k c_1 \ldots c_{D-k}} \boldsymbol{\vartheta}^{c_1 \ldots c_{D-k}} \,. \end{aligned} \tag{1.14}$$

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- **Connection 1-form**

$$\boxed{\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu} \,. \tag{1.15}$$

where $\omega_{\mu a}{}^b$ are the components of the affine connection in the anholonomic basis:

$$\omega_{\mu a}{}^b = e^\nu{}_a e_\lambda{}^b \Gamma_{\mu\nu}{}^\lambda + e_\sigma{}^b \partial_\mu e^\sigma{}_a \,. \tag{1.16}$$

N.B. $\Gamma_{\mu\nu}{}^\lambda$ and $\omega_{\mu a}{}^b$ contain the same information.

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- Exterior covariant derivative (of algebra-valued forms)

$$\mathbf{D}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} = \mathrm{d}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} + \quad \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\alpha}_{a\ldots}{}^{c\ldots} + \ldots \quad - \boldsymbol{\omega}_a{}^c \wedge \boldsymbol{\alpha}_{c\ldots}{}^{b\ldots} - \ldots \quad , \tag{1.17}$$

- Curvature, torsion and non-metricity forms:

$$\boldsymbol{R}_a{}^b := \mathrm{d}\boldsymbol{\omega}_a{}^b + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\omega}_a{}^c = \frac{1}{2} R_{\mu\nu a}{}^b \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.18}$$

$$\boldsymbol{T}^a := \mathbf{D}\boldsymbol{\vartheta}^a = \frac{1}{2} T_{\mu\nu}{}^a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.19}$$

$$\boldsymbol{Q}_{ab} := -\mathbf{D} g_{ab} = Q_{\mu ab} \mathrm{d}x^\mu\,. \tag{1.20}$$

- Notation for Levi-Civita: $\mathring{\boldsymbol{\omega}}_a{}^b$, $\mathring{\boldsymbol{R}}_a{}^b$.

2. Metric-Affine Lovelock theory

Def. (Metric) Lovelock term of order k in D dimensions:

$$S[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|}\mathrm{d}^D x\,, \tag{2.1}$$

where

$$\boxed{\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\delta_{\mu_1}^{[\nu_1}...\delta_{\mu_{2k}}^{\nu_{2k}]}\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}...\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}}\,. \tag{2.2}$$

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Properties

- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]

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- Total derivative in $D = 2k$ dimensions (*critical dimension*).

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- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]
- Total derivative in $D = 2k$ dimensions (*critical dimension*).

Example I. Case $k = 1$, Einstein(-Hilbert) lagrangian

$$\mathrm{sgn}(g)\mathring{\mathcal{L}}_1^{(D)} = \delta^{[\nu_1}_{\mu_1}\delta^{\nu_2]}_{\mu_2}\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2} = \quad \mathring{R}\,, \tag{2.3}$$

$$\Rightarrow [\mathrm{EoM}\, g_{\mu\nu}] \qquad 0 = \mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R}\,. \tag{2.4}$$

In the critical dimension ($D = 2$):

- Conformal symmetry of the theory
- In $D = 2$ all the metrics are conformally flat

So the equation reduces to:

$$0 = 0 \qquad \text{No conditions.} \tag{2.5}$$

Def. (Metric) Lovelock term of order k in D dimensions:

$$S[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \,, \tag{2.1}$$

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Properties

- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]
- Total derivative in $D = 2k$ dimensions (*critical dimension*).

Example II. Case $k = 2$, Gauss-Bonnet lagrangian

$$\operatorname{sgn}(g) \mathring{\mathcal{L}}_2^{(D)} = 3! \delta_{\mu_1}^{[\nu_1} \delta_{\mu_2}^{\nu_2} \delta_{\mu_3}^{\nu_3} \delta_{\mu_4}^{\nu_4]} \mathring{R}_{\nu_1 \nu_2}{}^{\mu_1 \mu_2} \mathring{R}_{\nu_3 \nu_4}{}^{\mu_3 \mu_4} = \quad \mathring{R}^2 - 4 \mathring{R}_{\mu\nu} \mathring{R}^{\mu\nu} + \mathring{R}_{\mu\nu\rho\lambda} \mathring{R}^{\mu\nu\rho\lambda} \,. \tag{2.3}$$

Equation of motion of the metric in critical dimension $D = 4$:

$$\begin{aligned} 0 &= \mathring{R}_{\alpha\beta} \mathring{R} + 2 \mathring{R}_{\mu\alpha\beta\nu} \mathring{R}^{\mu\nu} - 2 \mathring{R}_{\mu\alpha} \mathring{R}^{\mu}{}_{\beta} + \mathring{R}_{\mu\nu\alpha}{}^{\lambda} \mathring{R}^{\mu\nu}{}_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} \left(\mathring{R}^2 - 4 \mathring{R}_{\mu\nu} \mathring{R}^{\mu\nu} + \mathring{R}_{\mu\nu\rho\lambda} \mathring{R}^{\mu\nu\rho\lambda} \right) \\ &= \mathring{C}_{\alpha}{}^{\mu\nu\rho} \mathring{C}_{\beta\mu\nu\rho} - \frac{1}{4} g_{\alpha\beta} \mathring{C}_{\mu\nu\rho\lambda} \mathring{C}^{\mu\nu\rho\lambda} \,, \qquad \mathring{C}_{\mu\nu\rho\lambda} \equiv \text{Weyl tensor} \end{aligned} \tag{2.4}$$

And this is a known property of the Weyl tensor of ANY metric in $D = 4 \Rightarrow$ no conditions.

- The D-dimensional (metric) Lovelock lagrangian of order k,

$$\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}...\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\,. \tag{2.5}$$

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In the language of differential forms:

$$\boldsymbol{L}_k^{(D)} \equiv \mathcal{L}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \qquad \Leftrightarrow \qquad \boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge ... \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star \boldsymbol{\vartheta}_{a_1 ... a_{2k}}} \,, \tag{2.7}$$

Metric-affine Lovelock term of order k as the lagrangian D-form:

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General properties

- Levi-Civita is a solution of the palatini formalism EoM. [Borunda, Janssen, Bastero 2008]
- Projective symmetry:

$$\boldsymbol{\omega}_a{}^b \to \boldsymbol{\omega}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \left(\Leftrightarrow \quad \Gamma_{\mu\nu}{}^\rho \to \Gamma_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\right)\,, \tag{2.9}$$

$$\Rightarrow \qquad \boldsymbol{R}_{ab} \to \boldsymbol{R}_{ab} + \mathrm{d}\boldsymbol{A} g_{ab} \quad \left(\Leftrightarrow \quad R_{\mu\nu\rho}{}^\lambda \to R_{\mu\nu\rho}{}^\lambda + 2\partial_{[\mu} A_{\nu]} \delta_\rho^\lambda\right)\,. \tag{2.10}$$

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Critical dimension $D = 2k$

- The lagrangian becomes:

$$\boldsymbol{L}_{D/2}^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-1} a_D} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_D} \quad \equiv \quad \mathcal{E}_{a_1 \ldots a_D} \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-1} a_D}\ , \tag{2.11}$$

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  Yes for the Riemann-Cartan case (metric-compatible)

  ⇨ Two examples (orthonormal frame chosen, i.e. $g_{ab} \equiv \eta_{ab}$): [Hehl, McCrea, Mielke, Ne'eman 1995]

$$\boldsymbol{L}_1^{(2)}|_{Q=0} \propto \mathrm{d}\left[\mathcal{E}^a{}_b \boldsymbol{\omega}_a{}^b\right] \,, \tag{2.12}$$

$$\boldsymbol{L}_2^{(4)}|_{Q=0} \propto \mathrm{d}\left[\mathcal{E}^a{}_b{}^c{}_d \left(\boldsymbol{R}_a{}^b \wedge \boldsymbol{\omega}_c{}^d + \tfrac{1}{3}\boldsymbol{\omega}_a{}^b \wedge \boldsymbol{\omega}_c{}^e \wedge \boldsymbol{\omega}_e{}^d\right)\right] \,. \tag{2.13}$$

  (Exterior derivative of Chern-Simons like terms).

3. The Einstein metric-affine lagrangian. General and critical ($D = 2$) theory

- **Einstein lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = g_{cb}\boldsymbol{R}_a{}^b \wedge \star\boldsymbol{\vartheta}^{ac} = \mathrm{sgn}(g) e^\nu{}_b e^\mu{}_c g^{ca} R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\mathrm{d}^D x\,, \tag{3.1}$$

- **N.B.** In $D > 2$, the solution of the EoM of the connection is:

$$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\,. \tag{3.2}$$

Unphysical projective mode $\rightarrow$ can be eliminated using a symmetry of the theory.

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Unphysical projective mode $\to$ can be eliminated using a symmetry of the theory.

- **Critical dimension** $D = 2$.
  - Equations of motion

$$\boxed{0 = \mathbf{D}\mathcal{E}^a{}_b} = -\check{\boldsymbol{Q}}^{ca}\mathcal{E}_{bc} \qquad \text{where} \quad \check{\boldsymbol{Q}}_{ab} = \boldsymbol{Q}_{ab} - \frac{1}{2}g_{ab}\boldsymbol{Q}_c{}^c\,. \tag{3.3}$$

Therefore the general solution is one that verifies:

$$\boxed{\check{\boldsymbol{Q}}_{ab} = 0}\,. \tag{3.4}$$

- **Einstein lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = g_{cb}\boldsymbol{R}_a{}^b \wedge \star\boldsymbol{\vartheta}^{ac} = \operatorname{sgn}(g) e^\nu{}_b e^\mu{}_c g^{ca} R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\mathrm{d}^D x\,, \tag{3.1}$$

- **N.B.** In $D > 2$, the solution of the EoM of the connection is:

$$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu\delta_\nu^\rho\,. \tag{3.2}$$

Unphysical projective mode $\rightarrow$ can be eliminated using a symmetry of the theory.

- **Critical dimension** $D = 2$.
  - Equations of motion

$$\boxed{0 = \mathbf{D}\mathcal{E}^a{}_b} = -\check{\boldsymbol{Q}}^{ca}\mathcal{E}_{bc} \qquad \text{where} \quad \check{\boldsymbol{Q}}_{ab} = \boldsymbol{Q}_{ab} - \frac{1}{2}g_{ab}\boldsymbol{Q}_c{}^c\,. \tag{3.3}$$

Therefore the general solution is one that verifies:

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Tensor	d.o.f. in D dim.	d.o.f. in 2 dim.	Condition imposed by EoM
$T_{\mu\nu}{}^\rho$	$\frac{1}{2}D^2(D-1)$	2 (pure trace)	[Nothing]
$Q_{\mu\lambda}{}^\lambda$	D	2	[Nothing] (in any D due to proj. symmetry)
$\check{Q}_{\mu\nu\rho}$	$\frac{1}{2}D(D+2)(D-1)$	4	They are zero

  - **Conclusion**: There are conditions over the connection. So the theory CANNOT be topological.

4. The Gauss-Bonnet metric-affine lagrangian. General and critical ($D = 4$) theory

- **Gauss-Bonnet lagrangian** (arbitrary dimension)

$$\boldsymbol{L}_2^{(D)} = g_{mb}g_{nd}\boldsymbol{R}_a{}^b \wedge \boldsymbol{R}_c{}^d \wedge \star\boldsymbol{\vartheta}^{amcn} \tag{4.1}$$

$$= \mathrm{sgn}(g)\left[R^2 - R^{(1)}{}_{\mu\nu}R^{(1)\nu\mu} + 2R^{(1)}{}_{\mu\nu}R^{(2)\nu\mu} - R^{(2)}{}_{\mu\nu}R^{(2)\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\right]\boldsymbol{\omega}_{\mathrm{vol}}\,, \tag{4.2}$$

where

$$R^{(1)}{}_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda}\,, \qquad R := g^{\mu\nu}R^{(1)}{}_{\mu\nu}\,, \qquad R^{(2)}{}_{\mu}{}^{\nu} := g^{\lambda\sigma}R_{\mu\lambda\sigma}{}^{\nu}\,. \tag{4.3}$$

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where

$$R^{(1)}{}_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda} \,, \qquad R := g^{\mu\nu} R^{(1)}{}_{\mu\nu} \,, \qquad R^{(2)}{}_{\mu}{}^{\nu} := g^{\lambda\sigma} R_{\mu\lambda\sigma}{}^{\nu} \,. \tag{4.3}$$

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- **N.B.** In general D, the most general solution is not known. But we know there should be a free (unphysical) projective mode.
- Let us try with the Ansatz:

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu\delta_\nu^\rho + B_\nu\delta_\mu^\rho - C^\rho g_{\mu\nu}\,. \tag{4.4}$$

  - ⇨ The EoM of the connection forces:

$$B_\mu = C_\mu\,. \tag{4.5}$$

Particular Ansatz in D dimensional metric-affine Gauss-Bonnet. EoM

- ⇨ Remember our Ansatz:

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu \delta^\rho_\nu + B_\nu \delta^\rho_\mu - B^\rho g_{\mu\nu} \,. \tag{4.6}$$

  ⇨ With that in mind, EoM of the connection and the metric (or vielbein) read: [Janssen, Jiménez, Orejuela 2019]

- Remember our Ansatz:

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- With that in mind, EoM of the connection and the metric (or vielbein) read: [Janssen, Jiménez, Orejuela 2019]

$$\begin{aligned}
[\text{EoM}\,\omega] \quad 0 = {} & \tfrac{1}{12}(D-4)\Big[2B_{[\beta}\mathring{R}\,\delta^\nu_{\alpha]} + 4B^\lambda \mathring{R}_{\lambda[\alpha}\delta^\nu_{\beta]} - 2B^\lambda \mathring{R}_{\alpha\beta\lambda}{}^\nu\Big] \\
& + \tfrac{1}{6}(D-4)(D-3)\Big[2B_\rho \mathring{\nabla}_{[\alpha}B^\rho \delta^\nu_{\beta]} - 2B_{[\alpha}\mathring{\nabla}_{|\rho|}B^\rho\delta^\nu_{\beta]} + 2B_{[\alpha}\mathring{\nabla}_{\beta]}B^\nu\Big] \\
& - \tfrac{1}{6}(D-4)(D-3)(D-2)\, B_\sigma B^\sigma B_{[\alpha}\delta^\nu_{\beta]}\,,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
[\text{EoM}\,g,\ (\sim e)] \quad 0 = {} & [\text{EoM of } g \text{ in metric-Gauss-Bonnet theory}]_{\alpha\beta} \\
& + \tfrac{1}{3}(D-4)\Big[\mathring{\nabla}_{(\alpha}B_{\beta)}\mathring{R} + 2\mathring{\nabla}_\mu B^\mu(\mathring{R}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\mathring{R}) + 2\mathring{\nabla}^\mu B^\nu \mathring{R}_{\mu(\alpha\beta)\nu} \\
& \qquad - 2\mathring{\nabla}_{(\alpha}B^\mu \mathring{R}_{\beta)\mu} - 2\mathring{\nabla}_\mu B_{(\alpha}\mathring{R}_{\beta)}{}^\mu + 2\mathring{\nabla}_\mu B_\nu \mathring{R}^{\mu\nu}g_{\alpha\beta}\Big] \\
& + \tfrac{1}{3}(D-4)\Big[(D-5)B_\mu B^\mu(\mathring{R}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\mathring{R}) - B_\alpha B_\beta \mathring{R} - 2B^\mu B^\nu \mathring{R}_{\mu\nu}g_{\alpha\beta} \\
& \qquad + 4(D-3)B^\mu B_{(\alpha}\mathring{R}_{\beta)\mu} - 2B^\mu B^\nu \mathring{R}_{\mu(\alpha\beta)\nu}\Big] \\
& + \tfrac{1}{3}(D-4)(D-3)\Big[2\mathring{\nabla}_{(\alpha}B_{\beta)}\mathring{\nabla}_\mu B^\mu - 2\mathring{\nabla}_\mu B_{(\alpha}\mathring{\nabla}_{\beta)}B^\mu - 2\mathring{\nabla}_\mu B^{[\mu}\mathring{\nabla}_\nu B^{\nu]}g_{\alpha\beta}\Big] \\
& + \tfrac{1}{3}(D-4)(D-3)\Big[(D-4)\mathring{\nabla}_{(\alpha}B_{\beta)}B_\mu B^\mu - 2\mathring{\nabla}_\mu B^\mu B_\alpha B_\beta + 2\mathring{\nabla}_\mu B_{(\alpha}B_{\beta)}B^\mu \\
& \qquad + 2B^\mu B_{(\alpha}\mathring{\nabla}_{\beta)}B_\mu - 2B^\mu B^\nu \mathring{\nabla}_\mu B_\nu g_{\alpha\beta} + (D-4)B_\mu B^\mu \mathring{\nabla}_\nu B^\nu g_{\alpha\beta}\Big] \\
& + \tfrac{1}{12}(D-4)(D-3)(D-2)\Big[4B_\mu B^\mu B_\alpha B_\beta + (D-5)B_\mu B^\mu B_\nu B^\nu g_{\alpha\beta}\Big],
\end{aligned} \tag{4.8}$$

- In $D=4$ (critical dimension!!) our Ansatz solves the equations. But, is it 'trivial'?

Triviality of the solution in $D = 4$(?)

- It would be trivial if, for example,

$$\Phi\ :\quad \Gamma_{\mu\nu}{}^{\rho} \longmapsto \Gamma_{\mu\nu}{}^{\rho} + B_{\nu}\delta^{\rho}_{\mu} - B^{\rho} g_{\mu\nu}\,, \quad \Leftrightarrow \quad \Phi\ :\ \begin{cases} T_{\mu\nu}{}^{\rho} & \longmapsto T_{\mu\nu}{}^{\rho} - 2B_{[\mu}\delta^{\rho}_{\nu]} \\ Q_{\mu\nu\rho} & \longmapsto Q_{\mu\nu\rho} \end{cases} \tag{4.9}$$

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  were a symmetry.

  - Let's vary the action (remember that we are in $D = 4$): [Janssen, Jiménez, Orejuela 2019]

$$\begin{aligned}
\delta_{\Phi}\mathcal{L}_2^{(4)} = {} & -4B^{\mu}B^{\nu}g^{\rho\lambda}\left[2\nabla_{(\mu}Q_{\rho)\nu\lambda} + T_{\mu\rho}{}^{\sigma}Q_{\sigma\nu\lambda}\right] \\
& - 2Q^{\mu\nu\rho}\Big[B_{\mu}(R^{(1)}{}_{\nu\rho} + R^{(2)}{}_{\nu\rho}) + B^{\lambda}(R_{\lambda\nu\mu\rho} + R_{\lambda\nu\rho\mu}) \\
& \qquad - B^{\lambda}B_{\nu}(Q_{\lambda\mu\rho} - 2Q_{\rho\lambda\mu}) - B_{\mu}B_{\nu}(Q_{\rho\lambda}{}^{\lambda} - Q^{\lambda}{}_{\lambda\rho}) \\
& \qquad + 2B_{\mu}\nabla_{\nu}B_{\rho} + 4B_{\nu}\nabla_{\rho}B_{\mu} + 2B_{\nu}B^{\lambda}T_{\lambda\rho\mu} + 2B_{\mu}B_{\nu}T_{\rho\lambda}{}^{\lambda}\Big] \\
& - 2Q^{\mu\sigma}{}_{\sigma}\Big[B^{\nu}(R^{(1)}{}_{\nu\mu} - R^{(2)}{}_{\nu\mu} - g_{\nu\mu}R) - 2B_{\mu}B_{\nu}B^{\nu} \\
& \qquad - 2B_{\mu}\nabla_{\nu}B^{\nu} + 2B^{\nu}\nabla_{\nu}B_{\mu} + 3B_{\mu}B^{\nu}Q^{\lambda}{}_{\lambda\nu}\Big] \\
& + 2Q_{\sigma}{}^{\sigma\mu}\left[B^{\nu}(R^{(1)}{}_{\nu\mu} + R^{(2)}{}_{\nu\mu}) + 2B_{\mu}\nabla_{\nu}B^{\nu} + 2B^{\nu}\nabla_{\nu}B_{\mu} + 2B_{\mu}B^{\nu}T_{\nu\lambda}{}^{\lambda}\right],
\end{aligned} \tag{4.10}$$

  - The non-metricity prevents Φ from being a symmmetry.

5. Final discussion (general critical Lovelock theory) and conclusions

- Consider the Lovelock theory in critical dimension:

$$\boldsymbol{L}^{(D)}_{D/2} = \mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a_{D-1}}{}_{a_D} \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-1}}{}^{a_D} \,. \tag{5.1}$$

- The general equation of motion for the connection can be writen:

$$0 = \mathbf{D}\mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a}{}_{b} \wedge \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-3}}{}^{a_{D-2}} \qquad \Leftrightarrow \tag{5.2}$$

$$\Leftrightarrow \qquad \boxed{\begin{aligned} 0 = \; & \Big[\check{\boldsymbol{Q}}^{c}{}_{a_1}\mathcal{E}_{ca_2\ldots a_{D-2}ab} + \ldots + \check{\boldsymbol{Q}}^{c}{}_{a_{D-3}}\mathcal{E}_{a_1\ldots a_{D-4}ca_{D-2}ab} \\ & \quad + \check{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{a_1\ldots a_{D-2}cb}\Big] \wedge \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}} \end{aligned}} \tag{5.3}$$

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  - Particular cases:

$$(k=1) \quad \text{Einstein:} \qquad 0 = \check{\boldsymbol{Q}}^{c}{}_{a} \mathcal{E}_{bc} \qquad \Leftrightarrow \qquad \boxed{\check{\boldsymbol{Q}}_{ab} = 0} \,\text{(general sol.)} . \tag{5.4}$$

$$(k=2) \quad \text{Gauss-Bonnet:} \quad 0 = \left[\check{\boldsymbol{Q}}^{c}{}_{a} \mathcal{E}_{bcpq} + \check{\boldsymbol{Q}}^{c}{}_{p} \mathcal{E}_{qabc} \right] \wedge \boldsymbol{R}^{pq} \qquad \Rightarrow \qquad ? . \tag{5.5}$$

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- Families of solutions for arbitrary $k > 1$: [Work in progress]
  - Teleparallel $\boldsymbol{R}_c{}^d = 0$.
  - Any connection such that $\check{\boldsymbol{Q}}_{ab} \wedge \boldsymbol{R}_c{}^d = 0$.

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- Families of solutions for arbitrary $k > 1$: [Work in progress]
  - Teleparallel $\boldsymbol{R}_c{}^d = 0$.
  - Any connection such that $\check{\boldsymbol{Q}}_{ab} \wedge \boldsymbol{R}_c{}^d = 0$.
- Families of solutions for arbitrary $k > 2$:
  - Any connection such that $\boldsymbol{R}_{ab} = \boldsymbol{\alpha}_{ab} \wedge \boldsymbol{k}$ for certain 1-forms $\boldsymbol{\alpha}_{ab}$ and $\boldsymbol{k}$ (due to $\boldsymbol{k} \wedge \boldsymbol{k} \equiv 0$).

    Example. Ansatz of grav. wave: $\boldsymbol{k}$ is the dual form of the wave vector. [My PhD Thesis - still in progress]

Conclusion

Ideas to remember. Consider metric-affine Lovelock in the critical dimension:

- The simplest one (Einstein) cannot be topologically trivial (it imposes conditions!).
- In Gauss-Bonnet there are solutions unrelated through symmetries with LC: our Ansatz with B_μ; grav. wave Ansatz (ask me)...
- The conclusion is that *these lagrangians (with non-vanishing Q) cannot be total derivatives*.

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Open questions

- Analysis of the equation of the metric/coframe.
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Thanks for your attention!

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  Class. Quant. Grav. **29** (2012), 133001.

Consider a gravitational lagrangian (vacuum) depending on the connection exclusively through the curvature:

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or, in components,

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$$0 = \frac{-1}{2\sqrt{|g|}}\frac{\delta S}{\delta\omega_{\mu a}{}^b} \equiv \left(\nabla_\lambda - \frac{1}{2}Q_{\lambda\sigma}{}^\sigma + T_{\lambda\sigma}{}^\sigma\right)\left(\frac{\partial\mathcal{L}}{\partial R_{\lambda\mu a}{}^b}\right) - \frac{1}{2}T_{\lambda\sigma}{}^\mu\frac{\partial\mathcal{L}}{\partial R_{\lambda\sigma a}{}^b}\,. \tag{6.5}$$

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- Noether identity of projective symmetry $\Rightarrow$ the connection EoM is traceless (in a,b indices).