Metric-Affine Frameworks
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Restrictions in quadratic metric-affine gravity from the stability of the vector sector

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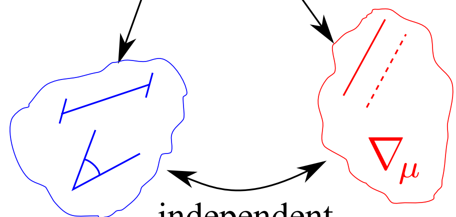
1. Introduction

$$(M, g_{\mu\nu}) \quad \rightsquigarrow \quad (M, g_{\mu\nu}, \Gamma_{\mu\nu}{}^{\rho})$$

Metric-Affine framework

$$(M, g_{\mu\nu}) \xrightarrow{\text{induces}} \mathring{\Gamma}_{\mu\nu}{}^{\rho}$$



independent

Γ

g

Curvature

$$R_{\mu\nu\rho}{}^{\lambda} := \partial_\mu \Gamma_{\nu\rho}{}^{\lambda} - \partial_\nu \Gamma_{\mu\rho}{}^{\lambda} + \Gamma_{\mu\sigma}{}^{\lambda}\Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\lambda}\Gamma_{\mu\rho}{}^{\sigma}$$

Torsion

$$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}$$

Nonmetricity

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}$$

For a given metric, a general connection can be split as follows:

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \Xi_{\mu\nu}{}^{\rho}\,, \tag{1.1}$$

where the distorsion $\Xi_{\mu\nu}{}^{\rho}$ is given by

$$\Xi_{\mu\nu}{}^{\rho} := \frac{1}{2}g^{\rho\sigma}(T_{\mu\nu\sigma} + T_{\sigma\mu\nu} - T_{\nu\sigma\mu}) + \frac{1}{2}g^{\rho\sigma}(Q_{\mu\nu\sigma} + Q_{\nu\sigma\mu} - Q_{\sigma\mu\nu})\,. \tag{1.2}$$

Post-riemannian expansion of the curvature

$$\begin{aligned} R_{\mu\nu\rho}{}^{\lambda} &= \mathring{R}_{\mu\nu\rho}{}^{\lambda} + \mathring{\nabla}_{\mu}\Xi_{\nu\rho}{}^{\lambda} - \mathring{\nabla}_{\nu}\Xi_{\mu\rho}{}^{\lambda} + \Xi_{\mu\sigma}{}^{\lambda}\Xi_{\nu\rho}{}^{\sigma} - \Xi_{\nu\sigma}{}^{\lambda}\Xi_{\mu\rho}{}^{\sigma} && (1.3)\\ &= \mathring{R}_{\mu\nu\rho}{}^{\lambda} + (\mathring{\nabla}T) + (\mathring{\nabla}Q) + (TT) + (TQ) + (QQ) && (1.4) \end{aligned}$$

Irreducible parts of the torsion

$$T_{\mu\nu}{}^{\rho} = \underbrace{\frac{1}{3}(T_{\mu}\delta^{\rho}_{\nu} - T_{\nu}\delta^{\rho}_{\mu})}_{\text{trace}} \underbrace{-\frac{1}{3}\mathcal{E}_{\mu\nu}{}^{\rho\lambda}S_{\lambda}}_{\text{axial part}} + \underbrace{\mathsf{t}_{\mu\nu}{}^{\rho}}_{\text{rest}} \tag{1.5}$$

Irreducible parts of the nonmetricity

$$Q_{\mu\nu\rho} = \underbrace{\frac{1}{4}Q_{\mu}g_{\nu\rho}}_{\text{Weyl trace}} + \underbrace{\frac{1}{9}\left[2\Lambda_{\nu}g_{\rho\mu} + 2\Lambda_{\rho}g_{\nu\mu} - \Lambda_{\mu}g_{\nu\rho}\right]}_{\text{remaining trace}} + \underbrace{\underbrace{s_{\mu\nu\rho}}_{\text{tot. sym.}} + \underbrace{\Omega_{\mu\nu\rho}}_{\text{rest}}}_{\mathsf{q}_{\mu\nu\rho}} \tag{1.6}$$

In some gauge theories of gravity...

	T_{μ}	S_{μ}	$\mathsf{t}_{\mu\nu}{}^{\rho}$	Q_{μ}	Λ_{μ}	$\mathsf{q}_{\mu\nu\rho}$	
MAG	✓	✓	✓	✓	✓	✓	partially today
Weyl-Cartan	✓	✓	✓	✓	×	×	today
Poincaré	✓	✓	✓	×	×	×	already studied

Requirements

1. only algebraic invariants in T, Q and R (+ cosmological constant)
2. at most quadratic in these quantities
3. no odd-parity invariants

Result: quadratic MAG Lagrangian

$$
\begin{aligned}
2\kappa\mathcal{L}_{\text{MAG}} &:= -2\kappa\Lambda + a_0 R \qquad \leftarrow \text{Einstein-Palatini} \\
\text{TT terms} \rightarrow \quad & +a_1 \mathsf{t}_{\mu\nu\rho}\mathsf{t}^{\mu\nu\rho} + a_2 T_\mu T^\mu + a_3 S_\mu S^\mu \\
\text{QQ terms} \rightarrow \quad & +b_1 \mathsf{q}_{\mu\nu\rho}\mathsf{q}^{\mu\nu\rho} + b_2 \mathsf{q}_{\mu\nu\rho}\mathsf{q}^{\rho\mu\nu} + b_3 Q_\mu Q^\mu + b_4 \Lambda_\mu \Lambda^\mu + b_5 Q_\mu \Lambda^\mu \\
\text{TQ terms} \rightarrow \quad & +c_1 \mathsf{q}_{\mu\nu\rho}\mathsf{t}^{\mu\nu\rho} + c_2 Q_\mu T^\mu + c_3 \Lambda_\mu T^\mu \\
\text{RR terms} \rightarrow \quad & +\ell^2 \Big[R^{\mu\nu\rho\lambda}\big(d_1 R_{\mu\nu(\rho\lambda)} + d_2 R_{\mu\nu[\rho\lambda]} + d_3 R_{\mu(\rho\lambda)\nu} + d_4 R_{\mu[\rho\lambda]\nu} + d_5 R_{\rho\lambda\mu\nu}\big) \\
& \quad + R^{\mu\nu}\big(d_6 R_{(\mu\nu)} + d_7 R_{[\mu\nu]}\big) + \mathcal{C}^{\mu\nu}\big(d_8 R_{(\mu\nu)} + d_9 R_{[\mu\nu]}\big) + \mathcal{C}^{\mu\nu}\big(d_{10}\mathcal{C}_{(\mu\nu)} + d_{11}\mathcal{C}_{[\mu\nu]}\big) \\
& \quad + \mathcal{P}^{\mu\nu}\big(d_{12} R_{\mu\nu} + d_{13}\mathcal{C}_{\mu\nu} + d_{14}\mathcal{P}_{\mu\nu}\big) + d_{15} R^2 + d_{16}\mathcal{H}^2 \Big]
\end{aligned} \tag{1.7}
$$

(27 parameters beyond Einstein-Palatini!)

Notation:

$$R_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda}\,, \qquad \mathcal{P}_{\mu\nu} := R_{\mu\nu\lambda}{}^{\lambda}\ \left(= \tfrac{1}{2}(\partial_\mu Q_\nu - \partial_\nu Q_\mu)\right), \qquad \mathcal{C}_\mu{}^\nu := g^{\rho\lambda} R_{\mu\rho\lambda}{}^{\nu}\,, \tag{1.8}$$

$$R := g^{\mu\nu} R_{\mu\nu}\,, \qquad \mathcal{H} := \mathcal{E}_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}\,. \tag{1.9}$$

Quadratic MAG Lagrangian

$$\boxed{\mathcal{L}_{\text{MAG}} = -\Lambda + \frac{a_0}{2\kappa} R + (TT \,\text{terms}) + (QQ \,\text{terms}) + (TQ \,\text{terms}) + (RR \,\text{terms})} \tag{1.10}$$

The idea now is to take:

$$R_{\mu\nu\rho}{}^{\lambda} \to \mathring{R}_{\mu\nu\rho}{}^{\lambda} + (\mathring{\nabla} T) + (\mathring{\nabla} Q) + (TT) + (TT) + (QQ) \tag{1.11}$$

and then

$$T_{\mu\nu}{}^{\rho} \to T_{\mu}, S_{\mu}, \mathsf{t}_{\mu\nu}{}^{\rho}\,, \qquad \text{and} \qquad Q_{\mu\nu\rho} \to Q_{\mu}, \Lambda_{\mu}, \mathsf{q}_{\mu\nu\rho}. \tag{1.12}$$

Goal

- Trying to get a stable theory containing at least one of the spin-1 in $\{T_{\mu}, S_{\mu}, Q_{\mu}, \Lambda_{\mu}\}$ and the graviton.

Method. Analysis based on:

- Previous results on stability of Proca fields in curved spacetime and multi-Proca. e.g. [Heisenberg 2017] [Beltran, Heisenberg 2016] [Beltran, Heisenberg 2017]

- A similar study performed in the Poincaré case: [Beltran, Maldonado 2020] (in agreement with Hamiltonian analysis: [Yo, Nester 1999] [Yo, Nester 2002])

2. The vector sector of quadratic MAG gravity

Goal

- Trying to get a stable theory containing at least one of the spin-1 in $\{T_\mu, S_\mu, Q_\mu, \varLambda_\mu\}$ and the graviton.

Structure of the Lagrangian

Everything in terms of $(\mathring{R}, T_\mu, S_\mu, \mathsf{t}_{\mu\nu}{}^{\rho}, Q_\mu, \varLambda_\mu, \mathsf{q}_{\mu\nu\rho})$

$$\begin{aligned} \mathcal{L}_{\text{MAG}} \quad \to \quad & \mathcal{L}_{\text{GR}} && = -\Lambda + \frac{a_0}{2\kappa}\mathring{R} \\ & + \mathcal{L}_{\mathring{R}\mathring{R}} && = \frac{\ell^2}{2\kappa}\left[\beta \mathring{R}^2 + \alpha \mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu}\right] \ + \ (\ldots)\mathring{\mathcal{L}}_{\text{GB}} \\ & + \mathcal{L}_{\text{v}} && = \mathcal{L}_T + \mathcal{L}_S + \mathcal{L}_Q + \mathcal{L}_\varLambda + \mathcal{L}_{\text{v, mix}} \\ & + \mathcal{L}_* && = \text{ten-(ten, vec, } g\text{) couplings}\,, \end{aligned} \tag{2.1}$$

Steps to follow:

1. Stable graviton sector (Einstein-Hilbert + Gauss-Bonnet limit)
2. Stable self-interactions for the vectors
3. Stable interactions between vectors
4. Correct signs in the kinetic matrix to avoid ghosts

$\rightarrow$ To ensure that no ghosts come from the graviton sector, we study the Riemannian limit:

$$\begin{aligned} 2\kappa\mathcal{L}_{\text{MAG}} \quad &\rightarrow \quad 2\kappa\big(\mathcal{L}_{\text{GR}} + \mathcal{L}_{\mathring{R}\mathring{R}}\big) && (2.2) \\ &= -2\kappa\Lambda + a_0\mathring{R} + \ell^2\left[\beta\mathring{R}^2 + \alpha\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu}\right] + \,(\ldots)\mathring{\mathcal{L}}_{\text{GB}}. && (2.3) \end{aligned}$$

To recover Einstein-Gauss-Bonnet, one needs to impose:

$$\alpha = 0\,, \qquad \beta = 0\,. \tag{2.4}$$

Equivalent to:

$$(\text{I}) := \begin{cases} d_6 \rightarrow -d_{10} - 4d_2 + 2d_4 - 4d_5 + d_8, \\ d_2 \rightarrow d_{15} - d_5 + \frac{d_4}{2}. \end{cases} \tag{2.5}$$

- We define:

$$ {}^{V}\!H_{\mu\nu} := 2\mathring{\nabla}_{(\mu}V_{\nu)}, \qquad {}^{V}\!H := 2\mathring{\nabla}_{\mu}V^{\mu}, \qquad {}^{V}\!F_{\mu\nu} := 2\mathring{\nabla}_{[\mu}V_{\nu]} \quad (\equiv 2\partial_{[\mu}V_{\nu]}). \tag{2.6} $$

- Boundary terms to eliminate ${}^{V}\!H_{\mu\nu} \to$ Lagrangian depending on ${}^{V}\!H$ and ${}^{V}\!F_{\mu\nu}$.
  Example:

$$ {}^{V}\!H_{\mu\nu}{}^{W}\!H^{\mu\nu} = 2\mathring{R}\,V_{\mu}W^{\mu} + {}^{V}\!H\,{}^{W}\!H + 4\mathring{G}_{\mu\nu}V^{\mu}W^{\nu} + {}^{V}\!F_{\mu\nu}{}^{W}\!F^{\mu\nu} + \text{b.t.} \tag{2.7} $$

The self-interacting Lagrangian for each of the vectors are:

$$ \mathcal{L}_{T}|_{(\mathrm{I})} = (...)\,{}^{T}\!F_{\mu\nu}{}^{T}\!F^{\mu\nu} + (...)T_{\mu}T^{\mu} \tag{2.8} $$

$$ \mathcal{L}_{Q}|_{(\mathrm{I})} = (...)\,{}^{Q}\!F_{\mu\nu}{}^{Q}\!F^{\mu\nu} + (...)Q_{\mu}Q^{\mu} \tag{2.9} $$

$$ \mathcal{L}_{S}|_{(\mathrm{I})} = (...)\,{}^{S}\!F_{\mu\nu}{}^{S}\!F^{\mu\nu} + (...)S_{\mu}S^{\mu} + (...)\mathring{G}_{\mu\nu}S^{\mu}S^{\nu}\boxed{+(...)\,{}^{S}\!H^{2} + (...)\mathring{R}\,S_{\mu}S^{\mu}} \tag{2.10} $$

$$ \begin{aligned} \mathcal{L}_{\Lambda}|_{(\mathrm{I})} = {} & (...)\,{}^{\Lambda}\!F_{\mu\nu}{}^{\Lambda}\!F^{\mu\nu} + (...)\Lambda_{\mu}\Lambda^{\mu} + (...)(\Lambda_{\mu}\Lambda^{\mu})^{2} + (...)\,{}^{\Lambda}\!H\,\Lambda_{\mu}\Lambda^{\mu} + (...)\mathring{G}_{\mu\nu}\Lambda^{\mu}\Lambda^{\nu} \\ & \boxed{+(...)\,{}^{\Lambda}\!H^{2} + (...)\mathring{R}\,\Lambda_{\mu}\Lambda^{\mu} + (...)\mathring{R}\,{}^{\Lambda}\!H} \end{aligned} \tag{2.11} $$

$\Rightarrow$ 5 more conditions (7 in total):

$$ \begin{aligned} (\mathrm{II}) := \Big\{ & d_2 \to d_5,\ d_3 \to -2(d_1 + 4d_{10}),\ d_4 \to 4d_5,\ d_6 \to d_{10}, \\ & d_8 \to 2d_{10},\ d_{15} \to 0,\ d_{16} \to \frac{1}{4}d_5 \Big\}. \end{aligned} \tag{2.12} $$

Under these conditions the sector that mixes the vectors reads:

$$
\begin{aligned}
2\kappa\mathcal{L}_{\text{v,mix}}\big|_{\text{(II)}} = {} & (\ldots)\,{}^{\Lambda}F_{\mu\nu}\,{}^{Q}F^{\mu\nu} + (\ldots)\,{}^{\Lambda}F_{\mu\nu}\,{}^{T}F^{\mu\nu} + (\ldots)\,{}^{Q}F_{\mu\nu}\,{}^{T}F^{\mu\nu} + (\ldots)\,{}^{S}F_{\mu\nu}\Lambda^{\mu}S^{\nu} \\
& + (\ldots)(\star{}^{Q}F)_{\mu\nu}\Lambda^{\mu}S^{\nu} + (\ldots)(\star{}^{\Lambda}F)_{\mu\nu}\Lambda^{\mu}S^{\nu} + (\ldots)(\star{}^{T}F)_{\mu\nu}\Lambda^{\mu}S^{\nu} \\
& + (\ldots)Q_{\mu}\Lambda^{\mu} + (\ldots)Q_{\mu}T^{\mu} + (\ldots)\Lambda_{\mu}T^{\mu} \\
& + (\ldots)(\Lambda_{\mu}S^{\mu})^{2} + (\ldots)\Lambda_{\mu}\Lambda^{\mu}S_{\nu}S^{\nu} + (\ldots)\Lambda_{\mu}T^{\mu}S_{\nu}S^{\nu} \\
& + (\ldots)S_{\mu}T^{\mu}S_{\nu}\Lambda^{\nu} + (\ldots)S_{\mu}Q^{\mu}S_{\nu}\Lambda^{\nu} + (\ldots)\Lambda_{\mu}Q^{\mu}S_{\nu}S^{\nu} \\
& \boxed{-\frac{1}{81}\ell^{2}(d_1+2d_5+4d_{10})({}^{S}H\,\Lambda_{\mu}S^{\mu}+2\,{}^{\Lambda}H\,S_{\mu}S^{\mu})}
\end{aligned}
\tag{2.13}
$$

Here $(\star{}^{V}F)^{\mu\nu} := \frac{1}{2}{}^{V}F_{\alpha\beta}\mathcal{E}^{\alpha\beta\mu\nu}$.

$\Rightarrow$ 1 more condition (8 in total):

$$
\begin{aligned}
\text{(III)} := \Big\{ & d_2 \to d_5,\ d_3 \to 4d_5,\ d_4 \to 4d_5,\ d_6 \to -\frac{1}{4}(d_1+2d_5), \\
& d_8 \to -\frac{1}{2}(d_1+2d_5),\ d_{10} \to -\frac{1}{4}(d_1+2d_5),\ d_{15} \to 0,\ d_{16} \to \frac{1}{4}d_5 \Big\}.
\end{aligned}
\tag{2.14}
$$

$$\mathcal{L}_{\mathrm{v}}\big|_{\text{(III)}} \supset ({}^{S}F_{\mu\nu}, {}^{T}F_{\mu\nu}, {}^{Q}F_{\mu\nu}, {}^{A}F_{\mu\nu}) \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & -4\zeta & K_1 & K_2 \\ 0 & K_1 & K_3 & K_4 \\ 0 & K_2 & K_4 & K_5 \end{pmatrix} \begin{pmatrix} {}^{S}F^{\mu\nu} \\ {}^{T}F^{\mu\nu} \\ {}^{Q}F^{\mu\nu} \\ {}^{A}F_{\mu\nu} \end{pmatrix} \tag{2.15}$$

$\Rightarrow$ No ghost-free theory with 4 vectors $\Rightarrow \zeta = 0$ (one more condition!)

Under $\zeta = 0$

$$\mathcal{L}_{\mathrm{v}}\big|_{\text{(III)},\zeta=0} \supset ({}^{S}F_{\mu\nu}, {}^{T}F_{\mu\nu}, {}^{Q}F_{\mu\nu}, {}^{A}F_{\mu\nu}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K_1 & K_2 \\ 0 & K_1 & K_3 & K_4 \\ 0 & K_2 & K_4 & K_5 \end{pmatrix} \begin{pmatrix} {}^{S}F^{\mu\nu} \\ {}^{T}F^{\mu\nu} \\ {}^{Q}F^{\mu\nu} \\ {}^{A}F_{\mu\nu} \end{pmatrix} \tag{2.16}$$

$\Rightarrow$ Ghostly unless $K_1 = K_2 = 0$ (2 more conditions!)

To sum up:

$\Rightarrow$ 1+2 more conditions (11 in total):

$$\begin{aligned}\text{(IV)} = \Big\{ & d_2 \to d_5,\ d_3 \to 4d_5,\ d_4 \to 4d_5,\ d_6 \to -\frac{1}{4}(d_1 + 2d_5), \\ & d_8 \to -\frac{1}{2}(d_1 + 2d_5),\ d_9 \to 2d_7,\ d_{10} \to -\frac{1}{4}(d_1 + 2d_5), \\ & d_{11} \to d_7,\ d_{13} \to d_{12},\ d_{15} \to 0,\ d_{16} \to \frac{1}{4}d_5 \Big\}. \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{v}}\big|_{(\mathrm{IV})} = -\frac{1}{4}({}^{Q}F_{\mu\nu}, {}^{\Lambda}F_{\mu\nu}) \begin{pmatrix} (\ldots) & (\ldots) \\ (\ldots) & (\ldots) \end{pmatrix} \begin{pmatrix} {}^{Q}F^{\mu\nu} \\ {}^{\Lambda}F_{\mu\nu} \end{pmatrix} \\
&+ \text{masses} \\
&+ \text{quartic terms} \\
&+ \text{coupling vector-}(\star F) \\
&+ \text{terms with } T_{\mu}, S_{\mu} \text{ (as auxiliary fields/Lagrange multipliers)} \qquad (2.18)
\end{aligned}
$$

Important remarks:

- 16 parameters in the RR sector $\to$ 5 surviving ones.
- This is a partial analysis: we still need to analyze the tensor sector!
- The conditions (IV) are necessary in the full theory.
- $\{T_{\mu}, S_{\mu}\}$ can be dynamical in the full theory (coupling to tensor parts!).

3. The case of quadratic Weyl-Cartan gravity

The starting point (now $\Lambda_\mu = \mathsf{q}_{\mu\nu\rho} = 0$) is a Lagrangian with just 7 parameters in the RR sector.

We again follow:

1. Stable graviton sector (Einstein-Hilbert + Gauss-Bonnet limit)
2. Stable self-interactions for the vectors
3. Stable interactions between vectors
4. Correct signs in the kinetic matrix to avoid ghosts

Result:

- The RR sector is reduced to just 1 parameter:

$$\mathcal{P}_{\mu\nu}\mathcal{P}^{\mu\nu} \quad \sim \quad {}^{Q}F_{\mu\nu}{}^{Q}F^{\mu\nu}\,. \tag{3.1}$$

- S_μ and $\mathsf{t}_{\mu\nu}{}^{\rho}$ only enter quadratically $\Rightarrow$ non-dynamical / vanishing.
- T_μ can be eliminated $\Rightarrow$ GR + Proca(Q_μ) (!)

(Non-trivial possibilities! This does not happen in Poincaré gravity)

4. Conclusions and limitations

Conclusion:

- The stability analysis of $\{T_\mu, S_\mu, Q_\mu, \Lambda_\mu\}$ is enough to reduce the par. of the RR sector...
  - from 16 to 5 in quadratic MAG $\to$ (IV) necessary for stable spin-1 (incomplete!)
  - from 7 to 1 in quadratic Weyl-Cartan $\to$ GR+Proca (complete!)

Limitations of this work / future work

- We follow the standard 'gauge' prescription for the Lagrangian $\to$ we are ignoring other dimension-4 invariants
- The method is oblivious to stable theories without vectors: Ricci based, purely scalar theories ($f(R)$, Holst...).
- The analysis of MAG is incomplete $\rightsquigarrow$ tensor sector! $\Rightarrow$ further restrictions + cross-check: [Percacci, Sezgin 2020] [Marzo 2021]
- Odd parity sector? $\leftarrow$ we do not expect it to solve the problems [Beltran, Maldonado 2020]

Thanks for your attention!

Aitäh!