Seminar

10 September 2019
Institute of Physics
Tartu, Estonia

Are critical Lovelock Lagrangians topological in the metric-affine formulation?

Alejandro Jiménez Cano

Universidad de Granada
Dpto. de Física Teórica y del Cosmos

alejandrojc@ugr.es

www.ugr.es/~alejandrojc

Structure of this presentation

1. Introduction (metric-affine formalism and geometry)
2. Metric-Affine Lovelock theory
3. The metric-affine Einstein Lagrangian in $D = 2$
4. The metric-affine Gauss-Bonnet Lagrangian in $D = 4$
5. Discussion of the general critical Lovelock term
6. Summary and conclusions

B. Janssen, A. Jiménez-Cano, J. A. Orejuela [Janssen, Jiménez, Orejuela 2019]
A non-trivial connection for the metric-affine Gauss-Bonnet theory in $D = 4$.
Physics Letters B **795** (2019) 42 – 48

B. Janssen, A. Jiménez-Cano [Janssen, Jiménez 2019]
On the topological character of metric-affine Lovelock Lagrangians in critical dimensions.
arXiv:1907.12100 [gr-qc]

A. Jiménez-Cano, [My PhD Thesis – Still in progress]
Metric-Affine Gauge theory of gravity. Foundations, perturbations and gravitational wave solutions.

1. Introduction (metric-affine formalism and geometry)

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

- *Metric structure:* $g_{\mu\nu}$ (**metric tensor**)
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_0^\sigma \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^\mu(\sigma')\dot{x}^\nu(\sigma')|}\, \mathrm{d}\sigma' \,. \tag{1.1}$$

$$\mathrm{vol}(\mathcal{U}) = \int_{\mathcal{U}} \boldsymbol{\omega}_{\mathrm{vol}} \,, \qquad \boldsymbol{\omega}_{\mathrm{vol}} := \sqrt{|g|}\, \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^D \,. \tag{1.2}$$

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

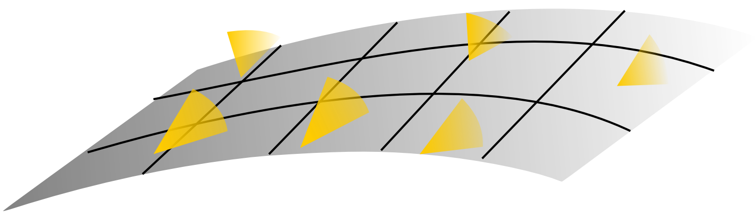
Geometric structures

- *Metric structure:* $g_{\mu\nu}$ (**metric tensor**)
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_0^\sigma \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^\mu(\sigma')\dot{x}^\nu(\sigma')|}\, \mathrm{d}\sigma' \,. \tag{1.1}$$

$$\mathrm{vol}(\mathcal{U}) = \int_{\mathcal{U}} \boldsymbol{\omega}_{\mathrm{vol}} \,, \qquad \boldsymbol{\omega}_{\mathrm{vol}} := \sqrt{|g|}\, \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^D \,. \tag{1.2}$$

  - Module of a vector (not necessarily non-negative) $\Rightarrow$ light cones $\Rightarrow$ causality.



Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

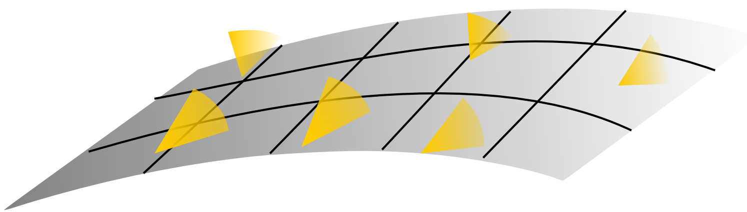
Geometric structures

- *Metric structure:* $g_{\mu\nu}$ (**metric tensor**)
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_0^\sigma \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^\mu(\sigma')\dot{x}^\nu(\sigma')|}\, \mathrm{d}\sigma' \,. \tag{1.1}$$

$$\mathrm{vol}(\mathcal{U}) = \int_{\mathcal{U}} \boldsymbol{\omega}_{\mathrm{vol}} \,, \qquad \boldsymbol{\omega}_{\mathrm{vol}} := \sqrt{|g|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \,. \tag{1.2}$$

  - Module of a vector (not necessarily non-negative) $\Rightarrow$ light cones $\Rightarrow$ causality.



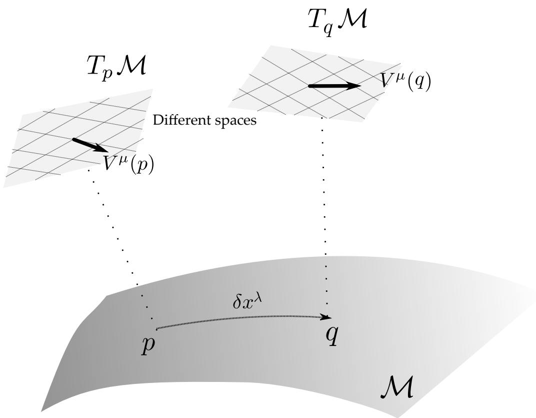
  - Notion of scale (conformal transformations...)

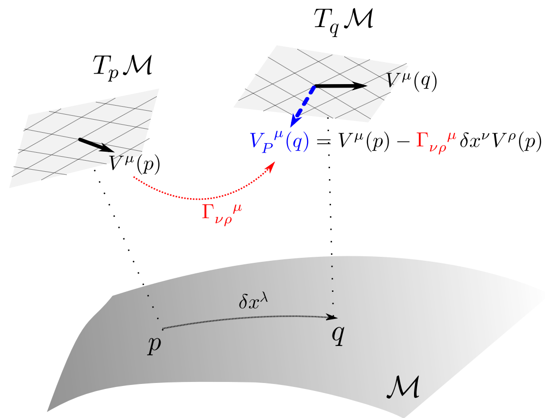
$$g_{\mu\nu} \to \mathrm{e}^{2\Omega} g_{\mu\nu} \,. \tag{1.3}$$

Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

- *Affine structure*: $\Gamma_{\mu\nu}{}^{\rho}$ (**affine connection**)
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_μ





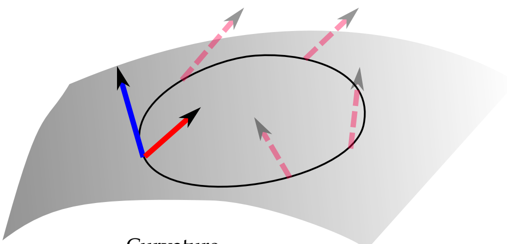
Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

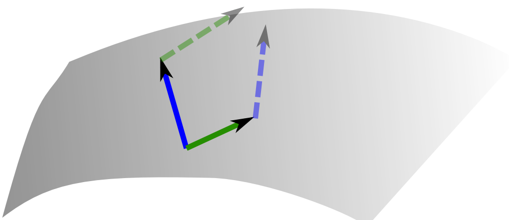
- *Affine structure*: $\Gamma_{\mu\nu}{}^{\rho}$ (**affine connection**)
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_μ
  - Geometrical objects:

$$\text{Curvature:} \quad R_{\mu\nu\lambda}{}^{\rho} := \partial_\mu \Gamma_{\nu\lambda}{}^{\rho} - \partial_\nu \Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\lambda}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\lambda}{}^{\sigma}\,, \tag{1.4}$$

$$\text{Torsion:} \quad T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}\,. \tag{1.5}$$



Curvature



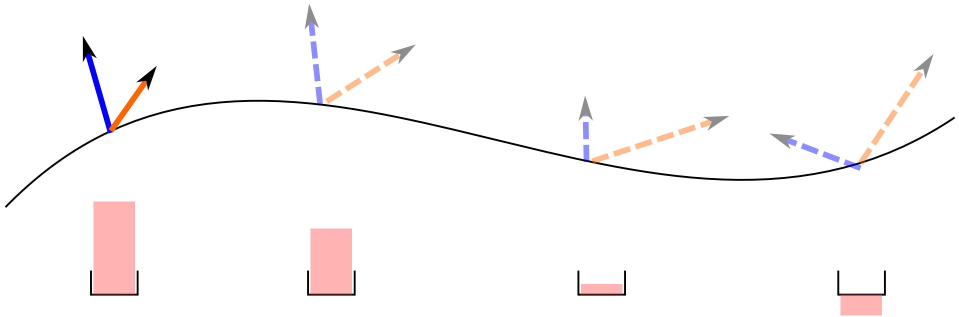
Torsion

Def.: In the presence of metric and affine connection we define the *non-metricity tensor*:

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}\,. \tag{1.6}$$

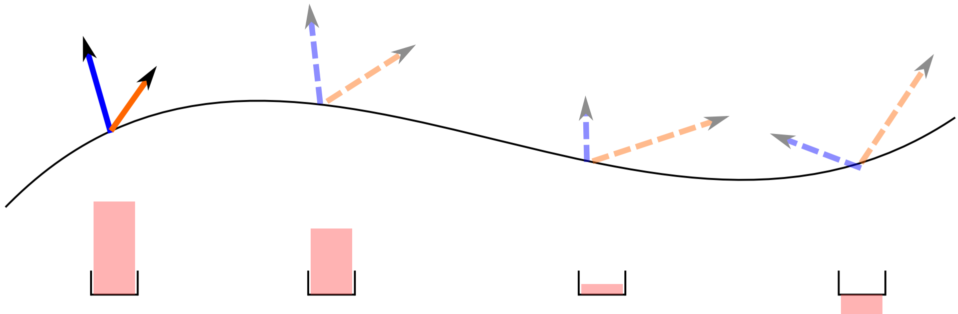
Def.: In the presence of metric and affine connection we define the *non-metricity tensor*:

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho} \,. \tag{1.6}$$



Def.: In the presence of metric and affine connection we define the *non-metricity tensor*:

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho} \,. \tag{1.6}$$



Theorem. Given $g_{\mu\nu}$, there is only one connection that satisfies

$$T_{\mu\nu}{}^{\rho} = 0 \quad \text{(torsionless condition)}, \tag{1.7}$$

$$Q_{\mu\nu\rho} = 0 \quad \text{(compatibility condition)}, \tag{1.8}$$

the *Levi-Civita connection:*

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right] \,. \tag{1.9}$$

Notation. Objects associated to the Levi-Civita connection: $\mathring{R}_{\mu\nu\lambda}{}^{\rho}$, $\mathring{R}_{\mu\nu}$, $\mathring{\nabla}_\mu$...

- Consider a theory depending on the metric structure and its associated curvature (Levi-Civita):

$$S[g, \Psi] = \int \mathcal{L}(g, \mathring{R}_{\mu\nu\rho}{}^{\lambda}(g), \Psi, \mathring{\nabla}_{\mu}\Psi, ...)\sqrt{|g|}\mathrm{d}^D x\,, \tag{1.10}$$

where Ψ are certain non-geometrical fields.

- Consider a theory depending on the metric structure and its associated curvature (Levi-Civita):

$$S[g,\Psi] = \int \mathcal{L}(g, \mathring{R}_{\mu\nu\rho}{}^{\lambda}(g), \Psi, \mathring{\nabla}_{\mu}\Psi, \ldots)\sqrt{|g|}\mathrm{d}^D x\,, \tag{1.10}$$

  where Ψ are certain non-geometrical fields.

- We are assuming a particular affine structure, the one fixed by the metric.

  $\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ put by hand $\Leftarrow$ It is natural, it is the simplest one,...

- Consider a theory depending on the metric structure and its associated curvature (Levi-Civita):

$$S[g,\Psi] = \int \mathcal{L}(g, \mathring{R}_{\mu\nu\rho}{}^{\lambda}(g), \Psi, \mathring{\nabla}_{\mu}\Psi, \ldots)\sqrt{|g|}\mathrm{d}^D x\,, \tag{1.10}$$

where Ψ are certain non-geometrical fields.

- We are assuming a particular affine structure, the one fixed by the metric.

  $\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ put by hand $\Leftarrow$ It is natural, it is the simplest one,...

- What if... $\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ were fixed by the dynamics?

- Consider a theory depending on the metric structure and its associated curvature (Levi-Civita):

$$S[g,\Psi] = \int \mathcal{L}(g, \mathring{R}_{\mu\nu\rho}{}^{\lambda}(g), \Psi, \mathring{\nabla}_{\mu}\Psi, \ldots)\sqrt{|g|}\mathrm{d}^D x\,, \tag{1.10}$$

where Ψ are certain non-geometrical fields.

- We are assuming a particular affine structure, the one fixed by the metric.

$\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ put by hand $\Leftarrow$ It is natural, it is the simplest one,...

- What if... $\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ were fixed by the dynamics?

Metric-affine (or Palatini) formulation
Promotion of $\mathring{\Gamma}_{\mu\nu}{}^{\rho}$ to a general connection $\Gamma_{\mu\nu}{}^{\rho}$ (independent field).

- Consider a theory depending on the metric structure and its associated curvature (Levi-Civita):

$$S[g,\Psi]=\int \mathcal{L}(g,\mathring{R}_{\mu\nu\rho}{}^{\lambda}(g),\Psi,\mathring{\nabla}_{\mu}\Psi,...)\sqrt{|g|}\mathrm{d}^{D}x\,, \tag{1.10}$$

where Ψ are certain non-geometrical fields.

- We are assuming a particular affine structure, the one fixed by the metric.

$\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ put by hand $\Leftarrow$ It is natural, it is the simplest one,...

- What if... $\mathring{\Gamma}_{\mu\nu}{}^{\rho}(g)$ were fixed by the dynamics?

Metric-affine (or Palatini) formulation
Promotion of $\mathring{\Gamma}_{\mu\nu}{}^{\rho}$ to a general connection $\Gamma_{\mu\nu}{}^{\rho}$ (independent field).

- Let us see what happens in the most simple case: Einstein gravity.
  The resulting action is

$$S[g,\,\Gamma,\,\Psi]=\frac{1}{2\kappa}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)\,\sqrt{|g|}\mathrm{d}^{D}x+S_{\text{matter}}[g,\,\Psi]\,. \tag{1.11}$$

(Hypothesis $S_{\text{matter}}\neq S_{\text{matter}}[\Gamma]$).

- Action for the Einstein-Palatini theory

$$S[g,\,\Gamma,\,\Psi] = S_{\mathrm{EP}}[g,\,\Gamma] + S_{\mathrm{matter}}[g,\,\Psi]\,, \qquad S_{\mathrm{EP}}[g,\,\Gamma] := \frac{1}{2\kappa}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)\,\sqrt{|g|}\mathrm{d}^D x\,. \tag{1.12}$$

- In $D > 2$ the equations of motion read

$$\text{EoM } g: \quad 0 = R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R + \kappa\mathcal{T}_{\mu\nu}\,, \tag{1.13}$$

$$\text{EoM } \Gamma: \quad 0 = \nabla_\lambda g_{\mu\nu} - T_{\nu\lambda}{}^{\sigma}g_{\mu\sigma} - \frac{1}{D-1}T_{\sigma\lambda}{}^{\sigma}g_{\mu\nu} - \frac{1}{D-1}T_{\sigma\nu}{}^{\sigma}g_{\mu\lambda}\,. \tag{1.14}$$

- Action for the Einstein-Palatini theory

$$S[g,\,\Gamma,\,\Psi]=S_{\text{EP}}[g,\,\Gamma]+S_{\text{matter}}[g,\,\Psi]\,,\qquad S_{\text{EP}}[g,\,\Gamma]:=\frac{1}{2\kappa}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)\,\sqrt{|g|}\mathrm{d}^Dx\,. \tag{1.12}$$

- In $D>2$ the equations of motion read

$$\text{EoM } g:\quad 0=R_{(\mu\nu)}-\frac{1}{2}g_{\mu\nu}R+\kappa\mathcal{T}_{\mu\nu}\,, \tag{1.13}$$

$$\text{EoM } \Gamma:\quad 0=\nabla_\lambda g_{\mu\nu}-T_{\nu\lambda}{}^{\sigma}g_{\mu\sigma}-\frac{1}{D-1}T_{\sigma\lambda}{}^{\sigma}g_{\mu\nu}-\frac{1}{D-1}T_{\sigma\nu}{}^{\sigma}g_{\mu\lambda}\,. \tag{1.14}$$

- General solutions of the EoM of Γ

$$\Gamma_{\mu\nu}{}^{\rho}=\mathring{\Gamma}_{\mu\nu}{}^{\rho}+A_\mu\delta_\nu^\rho\,. \tag{1.15}$$

- Action for the Einstein-Palatini theory

$$S[g,\,\Gamma,\,\Psi]=S_{\text{EP}}[g,\,\Gamma]+S_{\text{matter}}[g,\,\Psi]\,,\qquad S_{\text{EP}}[g,\,\Gamma]:=\frac{1}{2\kappa}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)\,\sqrt{|g|}\mathrm{d}^Dx\,. \tag{1.12}$$

- In $D>2$ the equations of motion read

$$\text{EoM }g:\quad 0=\,R_{(\mu\nu)}-\frac{1}{2}g_{\mu\nu}R+\kappa\mathcal{T}_{\mu\nu}\,, \tag{1.13}$$

$$\text{EoM }\Gamma:\quad 0=\,\nabla_\lambda g_{\mu\nu}-T_{\nu\lambda}{}^\sigma g_{\mu\sigma}-\frac{1}{D-1}T_{\sigma\lambda}{}^\sigma g_{\mu\nu}-\frac{1}{D-1}T_{\sigma\nu}{}^\sigma g_{\mu\lambda}\,. \tag{1.14}$$

- General solutions of the EoM of Γ

$$\Gamma_{\mu\nu}{}^\rho=\mathring{\Gamma}_{\mu\nu}{}^\rho+A_\mu\delta^\rho_\nu\,. \tag{1.15}$$

Torsion, non-metricity and curvature tensors

$$T_{\mu\nu}{}^\rho=A_\mu\delta^\rho_\nu-A_\nu\delta^\rho_\mu\,, \tag{1.16}$$

$$\nabla_\mu g_{\nu\rho}=-2A_\mu g_{\nu\rho}\,, \tag{1.17}$$

$$R_{\mu\nu\rho}{}^\lambda=\mathring{R}_{\mu\nu\rho}{}^\lambda+2\partial_{[\mu}A_{\nu]}\delta^\lambda_\rho\qquad\Rightarrow\qquad R_{(\mu\nu)}=\mathring{R}_{\mu\nu}\,. \tag{1.18}$$

Substituting this last condition into the metric equation one obtains the Einstein equations,

$$(\text{EoM }g)|_{\Gamma\text{on-shell}}\quad:\qquad 0\;=\;\mathring{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\mathring{R}+\kappa\mathcal{T}_{\mu\nu}\,. \tag{1.19}$$

- Action for the Einstein-Palatini theory

$$S[g,\,\Gamma,\,\Psi] = S_{\rm EP}[g,\,\Gamma] + S_{\rm matter}[g,\,\Psi]\,, \qquad S_{\rm EP}[g,\,\Gamma] := \frac{1}{2\kappa}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)\,\sqrt{|g|}\mathrm{d}^D x\,. \tag{1.12}$$

- In $D>2$ the equations of motion read

$$\text{EoM } g: \quad 0 = R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R + \kappa\mathcal{T}_{\mu\nu}\,, \tag{1.13}$$

$$\text{EoM } \Gamma: \quad 0 = \nabla_\lambda g_{\mu\nu} - T_{\nu\lambda}{}^{\sigma}g_{\mu\sigma} - \frac{1}{D-1}T_{\sigma\lambda}{}^{\sigma}g_{\mu\nu} - \frac{1}{D-1}T_{\sigma\nu}{}^{\sigma}g_{\mu\lambda}\,. \tag{1.14}$$

- General solutions of the EoM of Γ

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu\delta^\rho_\nu\,. \tag{1.15}$$

Torsion, non-metricity and curvature tensors

$$T_{\mu\nu}{}^{\rho} = A_\mu\delta^\rho_\nu - A_\nu\delta^\rho_\mu\,, \tag{1.16}$$

$$\nabla_\mu g_{\nu\rho} = -2A_\mu g_{\nu\rho}\,, \tag{1.17}$$

$$R_{\mu\nu\rho}{}^{\lambda} = \mathring{R}_{\mu\nu\rho}{}^{\lambda} + 2\partial_{[\mu}A_{\nu]}\delta^\lambda_\rho \qquad \Rightarrow \qquad R_{(\mu\nu)} = \mathring{R}_{\mu\nu}\,. \tag{1.18}$$

Substituting this last condition into the metric equation one obtains the Einstein equations,

$$(\text{EoM } g)|_{\Gamma\text{on-shell}} \quad : \qquad 0 \;=\; \mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} + \kappa\mathcal{T}_{\mu\nu}\,. \tag{1.19}$$

- A_μ is unphysical, since can be absorbed using the projective symmetry of the theory:

$$\text{proj}: \qquad \boxed{\Gamma_{\mu\nu}{}^{\rho} \to \Gamma_{\mu\nu}{}^{\rho} + k_\mu\delta^\rho_\nu} \quad (R_{\mu\nu\rho}{}^{\lambda} \to R_{\mu\nu\rho}{}^{\lambda} + 2\partial_{[\mu}k_{\nu]}\delta^\lambda_\rho) \qquad \Rightarrow \qquad \delta_{\rm proj}\mathcal{L}_{\rm EP} = 0\,. \tag{1.20}$$

Three fundamental objects: coframe, metric and connection 1-form.

Three fundamental objects: coframe, metric and connection 1-form.

- **Coframe**. We can fix a general frame in the manifold and the corresponding dual basis (coframe):

$$\boldsymbol{e}_a = e^\mu{}_a \boldsymbol{\partial}_\mu \,, \qquad \boxed{\boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu} \qquad [\boldsymbol{\vartheta}^a(\boldsymbol{e}_b) = \delta^a_b \quad \Leftrightarrow \quad e_\mu{}^a e^\mu{}_b = \delta^a_b] \,. \tag{1.21}$$

Notation:

$$\boldsymbol{\vartheta}^{a_1 \ldots a_k} \equiv \boldsymbol{\vartheta}^{a_1} \wedge \ldots \wedge \boldsymbol{\vartheta}^{a_k} \,. \tag{1.22}$$

Three fundamental objects: coframe, metric and connection 1-form.

- **Coframe**. We can fix a general frame in the manifold and the corresponding dual basis (coframe):

$$\boldsymbol{e}_a = e^\mu{}_a \boldsymbol{\partial}_\mu \,, \qquad \boxed{\boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu} \qquad [\boldsymbol{\vartheta}^a(\boldsymbol{e}_b) = \delta^a_b \quad \Leftrightarrow \quad e_\mu{}^a e^\mu{}_b = \delta^a_b] \,. \tag{1.21}$$

Notation:

$$\boldsymbol{\vartheta}^{a_1 \ldots a_k} \equiv \boldsymbol{\vartheta}^{a_1} \wedge \ldots \wedge \boldsymbol{\vartheta}^{a_k} \,. \tag{1.22}$$

- **Metric**. Components of the metric in the arbitrary basis:

$$\boxed{g_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}} \,. \tag{1.23}$$

  - Canonical volume form

$$\boldsymbol{\omega}_{\mathrm{vol}} := \frac{1}{D!} \mathcal{E}_{a_1 \ldots a_D} \boldsymbol{\vartheta}^{a_1 \ldots a_D} = \sqrt{|g|} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \,, \qquad |g| \equiv |\det(g_{\mu\nu})| \,. \tag{1.24}$$

  - Hodge star of an arbitrary k-form $\boldsymbol{\alpha} = \frac{1}{k!} \alpha_{a_1 \ldots a_k} \boldsymbol{\vartheta}^{a_1 \ldots a_k}$

$$\begin{aligned} \star \,:\, \Omega^k(\mathcal{M}) &\longrightarrow \Omega^{D-k}(\mathcal{M}) \\ \boldsymbol{\alpha} \quad &\longmapsto \quad \star\boldsymbol{\alpha} := \frac{1}{(D-k)!k!} \alpha^{b_1 \ldots b_k} \mathcal{E}_{b_1 \ldots b_k c_1 \ldots c_{D-k}} \boldsymbol{\vartheta}^{c_1 \ldots c_{D-k}} \,. \end{aligned} \tag{1.25}$$

Three fundamental objects: coframe, metric and connection 1-form.

Three fundamental objects: coframe, metric and connection 1-form.

- **Connection 1-form**

$$\boxed{\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu}\,. \tag{1.26}$$

where $\omega_{\mu a}{}^b$ are the components of the affine connection in the anholonomic basis:

$$\omega_{\mu a}{}^b = e^\nu{}_a e_\lambda{}^b \Gamma_{\mu\nu}{}^\lambda + e_\sigma{}^b \partial_\mu e^\sigma{}_a\,. \tag{1.27}$$

N.B. $\Gamma_{\mu\nu}{}^\lambda$ and $\omega_{\mu a}{}^b$ contain the same information.

Three fundamental objects: coframe, metric and connection 1-form.

- **Connection 1-form**

$$\boxed{\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu}\,. \tag{1.26}$$

where $\omega_{\mu a}{}^b$ are the components of the affine connection in the anholonomic basis:

$$\omega_{\mu a}{}^b = e^\nu{}_a e_\lambda{}^b \Gamma_{\mu\nu}{}^\lambda + e_\sigma{}^b \partial_\mu e^\sigma{}_a\,. \tag{1.27}$$

N.B. $\Gamma_{\mu\nu}{}^\lambda$ and $\omega_{\mu a}{}^b$ contain the same information.

- Exterior covariant derivative (of algebra-valued forms)

$$\mathbf{D}\boldsymbol{\alpha}_{a\dots}{}^{b\dots} = \mathrm{d}\boldsymbol{\alpha}_{a\dots}{}^{b\dots} + \quad \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\alpha}_{a\dots}{}^{c\dots} + \dots \quad - \boldsymbol{\omega}_a{}^c \wedge \boldsymbol{\alpha}_{c\dots}{}^{b\dots} - \dots \quad , \tag{1.28}$$

- Curvature, torsion and non-metricity forms:

$$\boldsymbol{R}_a{}^b := \mathrm{d}\boldsymbol{\omega}_a{}^b + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\omega}_a{}^c = \frac{1}{2} R_{\mu\nu a}{}^b \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.29}$$

$$\boldsymbol{T}^a := \mathbf{D}\boldsymbol{\vartheta}^a = \frac{1}{2} T_{\mu\nu}{}^a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.30}$$

$$\boldsymbol{Q}_{ab} := -\mathbf{D} g_{ab} = Q_{\mu ab} \mathrm{d}x^\mu\,. \tag{1.31}$$

- Notation for Levi-Civita: $\mathring{\boldsymbol{\omega}}_a{}^b$, $\mathring{\boldsymbol{R}}_a{}^b$.

2. Metric-Affine Lovelock theory

Def. (Metric) Lovelock term of order k in D dimensions:

$$\mathring{S}_k^{(D)}[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \,, \tag{2.1}$$

where

$$\boxed{\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k} \operatorname{sgn}(g) \delta^{[\nu_1}_{\mu_1} ... \delta^{\nu_{2k}]}_{\mu_{2k}} \mathring{R}_{\nu_1 \nu_2}{}^{\mu_1 \mu_2} ... \mathring{R}_{\nu_{2k-1} \nu_{2k}}{}^{\mu_{2k-1} \mu_{2k}}} \,. \tag{2.2}$$

Def. (Metric) Lovelock term of order k in D dimensions:

$$\mathring{S}_k^{(D)}[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \,, \tag{2.1}$$

where

$$\boxed{\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k} \mathrm{sgn}(g) \delta_{\mu_1}^{[\nu_1} \ldots \delta_{\mu_{2k}}^{\nu_{2k}]} \mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2} \ldots \mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}} \,. \tag{2.2}$$

Properties

- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]

Def. (Metric) Lovelock term of order k in D dimensions:

$$\mathring{S}_k^{(D)}[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \,, \tag{2.1}$$

where

$$\boxed{\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k} \operatorname{sgn}(g) \delta^{[\nu_1}_{\mu_1} ... \delta^{\nu_{2k}]}_{\mu_{2k}} \mathring{R}_{\nu_1 \nu_2}{}^{\mu_1 \mu_2} ... \mathring{R}_{\nu_{2k-1} \nu_{2k}}{}^{\mu_{2k-1} \mu_{2k}}} \,. \tag{2.2}$$

Properties

- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]
- Total derivative in $D = 2k$ dimensions (*critical dimension*).

Def. (Metric) Lovelock term of order k in D dimensions:

$$\mathring{S}_k^{(D)}[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \,, \tag{2.1}$$

where

$$\boxed{\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k} \operatorname{sgn}(g) \delta^{[\nu_1}_{\mu_1} ... \delta^{\nu_{2k}]}_{\mu_{2k}} \mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2} ... \mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}} \,. \tag{2.2}$$

Properties

- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]
- Total derivative in $D = 2k$ dimensions (*critical dimension*).

Example I. Case $k = 1$, Einstein(-Hilbert) lagrangian

$$\operatorname{sgn}(g) \mathring{\mathcal{L}}_1^{(D)} = \delta^{[\nu_1}_{\mu_1} \delta^{\nu_2]}_{\mu_2} \mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2} = \mathring{R} \,, \tag{2.3}$$

$$\Rightarrow [\text{EoM}\, g_{\mu\nu}] \qquad 0 = \mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} \,. \tag{2.4}$$

In the critical dimension ($D = 2$):

- Conformal symmetry of the theory
- In $D = 2$ all the metrics are conformally flat

So the equation reduces to:

$$0 = 0 \qquad \text{No conditions.} \tag{2.5}$$

Def. (Metric) Lovelock term of order k in D dimensions:

$$\mathring{S}_k^{(D)}[\boldsymbol{g}] = \int \mathring{\mathcal{L}}_k^{(D)} \sqrt{|g|} \mathrm{d}^D x \,, \tag{2.1}$$

where

$$\boxed{\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k} \operatorname{sgn}(g) \delta^{[\nu_1}_{\mu_1} ... \delta^{\nu_{2k}]}_{\mu_{2k}} \mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2} ... \mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}} \,. \tag{2.2}$$

Properties

- 2nd order differential equations for the metric (by constr.) [Lovelock 1971]
- Total derivative in $D = 2k$ dimensions (*critical dimension*).

Example II. Case $k = 2$, Gauss-Bonnet lagrangian

$$\operatorname{sgn}(g) \mathring{\mathcal{L}}_2^{(D)} = 3! \delta^{[\nu_1}_{\mu_1} \delta^{\nu_2}_{\mu_2} \delta^{\nu_3}_{\mu_3} \delta^{\nu_4]}_{\mu_4} \mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2} \mathring{R}_{\nu_3\nu_4}{}^{\mu_3\mu_4} = \mathring{R}^2 - 4\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + \mathring{R}_{\mu\nu\rho\lambda}\mathring{R}^{\mu\nu\rho\lambda} \,. \tag{2.3}$$

Equation of motion of the metric in critical dimension $D = 4$:

$$\begin{aligned} 0 &= \mathring{R}_{\alpha\beta}\mathring{R} + 2\mathring{R}_{\mu\alpha\beta\nu}\mathring{R}^{\mu\nu} - 2\mathring{R}_{\mu\alpha}\mathring{R}^{\mu}{}_{\beta} + \mathring{R}_{\mu\nu\alpha}{}^{\lambda}\mathring{R}^{\mu\nu}{}_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} \left(\mathring{R}^2 - 4\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + \mathring{R}_{\mu\nu\rho\lambda}\mathring{R}^{\mu\nu\rho\lambda} \right) \\ &= \mathring{C}_{\alpha}{}^{\mu\nu\rho}\mathring{C}_{\beta\mu\nu\rho} - \frac{1}{4} g_{\alpha\beta} \mathring{C}_{\mu\nu\rho\lambda}\mathring{C}^{\mu\nu\rho\lambda} \,, \qquad \mathring{C}_{\mu\nu\rho\lambda} \equiv \text{Weyl tensor} \end{aligned} \tag{2.4}$$

And this is a known property of the Weyl tensor of ANY metric in $D = 4 \Rightarrow$ no conditions.

- The D-dimensional (metric) Lovelock lagrangian of order k,

$$\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\,\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}\,\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}...\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\,. \tag{2.5}$$

- The D-dimensional (metric) Lovelock lagrangian of order k,

$$\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\,\delta^{[\nu_1}_{\mu_1}\ldots\delta^{\nu_{2k}]}_{\mu_{2k}}\,\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}\ldots\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\,. \tag{2.5}$$

⚠.................. Jump to metric-affine ....⚠

- The D-dimensional (metric) Lovelock lagrangian of order k,

$$\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}...\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\ . \tag{2.5}$$

⚠................. Jump to metric-affine ....⚠

Def. D dimensional (metric-affine) Lovelock term of order k:

$$\mathcal{L}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}R_{\nu_1\nu_2}{}^{\mu_1\mu_2}...R_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\ . \tag{2.6}$$

- The D-dimensional (metric) Lovelock lagrangian of order k,

$$\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\,\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}\,\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}...\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\ . \tag{2.5}$$

⚠................. Jump to metric-affine ....⚠

Def. D dimensional (metric-affine) Lovelock term of order k:

$$\mathcal{L}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\,\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}\,R_{\nu_1\nu_2}{}^{\mu_1\mu_2}...R_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\ . \tag{2.6}$$

⚠................. Jump to exterior algebra notation ....⚠

- The D-dimensional (metric) Lovelock lagrangian of order k,

$$\mathring{\mathcal{L}}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}\mathring{R}_{\nu_1\nu_2}{}^{\mu_1\mu_2}...\mathring{R}_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\,. \tag{2.5}$$

⚠................. Jump to metric-affine ....⚠

Def. D dimensional (metric-affine) Lovelock term of order k:

$$\mathcal{L}_k^{(D)} = \frac{(2k)!}{2^k}\,\mathrm{sgn}(g)\delta^{[\nu_1}_{\mu_1}...\delta^{\nu_{2k}]}_{\mu_{2k}}R_{\nu_1\nu_2}{}^{\mu_1\mu_2}...R_{\nu_{2k-1}\nu_{2k}}{}^{\mu_{2k-1}\mu_{2k}}\,. \tag{2.6}$$

⚠................. Jump to exterior algebra notation ....⚠

In the language of differential forms:

$$\boldsymbol{L}_k^{(D)} \equiv \mathcal{L}_k^{(D)}\sqrt{|g|}\mathrm{d}^D x \qquad \Leftrightarrow \qquad \boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1a_2}\wedge ... \wedge \boldsymbol{R}^{a_{2k-1}a_{2k}}\wedge \star\boldsymbol{\vartheta}_{a_1...a_{2k}}}\,. \tag{2.7}$$

Metric-affine Lovelock term of order k as the lagrangian D-form:

$$\boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_{2k}}} \,. \tag{2.8}$$

Metric-affine Lovelock term of order k as the lagrangian D-form:

$$\boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_{2k}}} \,. \tag{2.8}$$

General properties

- Levi-Civita is a solution of the palatini formalism EoM. [Borunda, Janssen, Bastero 2008]
- Projective symmetry:

$$\boldsymbol{\omega}_a{}^b \to \boldsymbol{\omega}_a{}^b + \boldsymbol{A}\delta_a^b \qquad (\Leftrightarrow \quad \Gamma_{\mu\nu}{}^\rho \to \Gamma_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho) \;, \tag{2.9}$$

$$\Rightarrow \qquad \boldsymbol{R}_{ab} \to \boldsymbol{R}_{ab} + \mathrm{d}\boldsymbol{A} g_{ab} \quad (\Leftrightarrow \quad R_{\mu\nu\rho}{}^\lambda \to R_{\mu\nu\rho}{}^\lambda + 2\partial_{[\mu} A_{\nu]} \delta_\rho^\lambda) \;. \tag{2.10}$$

Metric-affine Lovelock term of order k as the lagrangian D-form:

$$\boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_{2k}}}\,. \tag{2.8}$$

General properties

- Levi-Civita is a solution of the palatini formalism EoM. [Borunda, Janssen, Bastero 2008]
- Projective symmetry:

$$\boldsymbol{\omega}_a{}^b \to \boldsymbol{\omega}_a{}^b + \boldsymbol{A}\delta_a^b \qquad (\Leftrightarrow \quad \Gamma_{\mu\nu}{}^\rho \to \Gamma_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho)\ , \tag{2.9}$$

$$\Rightarrow \qquad \boldsymbol{R}_{ab} \to \boldsymbol{R}_{ab} + \mathrm{d}\boldsymbol{A} g_{ab} \quad \left(\Leftrightarrow \quad R_{\mu\nu\rho}{}^\lambda \to R_{\mu\nu\rho}{}^\lambda + 2\partial_{[\mu} A_{\nu]} \delta_\rho^\lambda\right)\ . \tag{2.10}$$

Critical dimension $D = 2k$

- The Lagrangian becomes:

$$\boldsymbol{L}_k^{(2k)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_{2k}} \quad \equiv \quad \mathcal{E}_{a_1 \ldots a_{2k}} \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}}\ , \tag{2.11}$$

Metric-affine Lovelock term of order k as the lagrangian D-form:

$$\boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1}a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1\ldots a_{2k}}} . \tag{2.8}$$

General properties

- Levi-Civita is a solution of the palatini formalism EoM. [Borunda, Janssen, Bastero 2008]
- Projective symmetry:

$$\boldsymbol{\omega}_a{}^b \to \boldsymbol{\omega}_a{}^b + \boldsymbol{A}\delta_a^b \qquad (\Leftrightarrow \quad \Gamma_{\mu\nu}{}^\rho \to \Gamma_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho) , \tag{2.9}$$

$$\Rightarrow \qquad \boldsymbol{R}_{ab} \to \boldsymbol{R}_{ab} + \mathrm{d}\boldsymbol{A} g_{ab} \quad (\Leftrightarrow \quad R_{\mu\nu\rho}{}^\lambda \to R_{\mu\nu\rho}{}^\lambda + 2\partial_{[\mu}A_{\nu]}\delta_\rho^\lambda) . \tag{2.10}$$

Critical dimension $D = 2k$

- The Lagrangian becomes:

$$\boldsymbol{L}_k^{(2k)} = \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1}a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1\ldots a_{2k}} \quad \equiv \quad \mathcal{E}_{a_1\ldots a_{2k}} \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1}a_{2k}} , \tag{2.11}$$

- *Question*: Is this a total derivative?

Metric-affine Lovelock term of order k as the lagrangian D-form:

$$\boxed{\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_{2k}}} \,. \tag{2.8}$$

General properties

- Levi-Civita is a solution of the palatini formalism EoM. [Borunda, Janssen, Bastero 2008]
- Projective symmetry:

$$\boldsymbol{\omega}_a{}^b \to \boldsymbol{\omega}_a{}^b + \boldsymbol{A}\delta_a^b \qquad (\Leftrightarrow \quad \Gamma_{\mu\nu}{}^\rho \to \Gamma_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho) \,, \tag{2.9}$$

$$\Rightarrow \qquad \boldsymbol{R}_{ab} \to \boldsymbol{R}_{ab} + \mathrm{d}\boldsymbol{A} g_{ab} \quad (\Leftrightarrow \quad R_{\mu\nu\rho}{}^\lambda \to R_{\mu\nu\rho}{}^\lambda + 2\partial_{[\mu} A_{\nu]} \delta_\rho^\lambda) \,. \tag{2.10}$$

Critical dimension $D = 2k$

- The Lagrangian becomes:

$$\boldsymbol{L}_k^{(2k)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \wedge \star\boldsymbol{\vartheta}_{a_1 \ldots a_{2k}} \quad \equiv \quad \mathcal{E}_{a_1 \ldots a_{2k}} \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1} a_{2k}} \,, \tag{2.11}$$

- *Question*: Is this a total derivative?

  Yes for the Riemann-Cartan case (metric-compatible)

  - Two examples (orthonormal frame chosen, i.e. $g_{ab} \equiv \eta_{ab}$): [Hehl, McCrea, Mielke, Ne'eman 1995]

$$\boldsymbol{L}_1^{(2)}|_{Q=0} \propto \mathrm{d}\left[\mathcal{E}^a{}_b \boldsymbol{\omega}_a{}^b\right] \,, \tag{2.12}$$

$$\boldsymbol{L}_2^{(4)}|_{Q=0} \propto \mathrm{d}\left[\mathcal{E}^a{}_b{}^c{}_d \left(\boldsymbol{R}_a{}^b \wedge \boldsymbol{\omega}_c{}^d + \tfrac{1}{3}\boldsymbol{\omega}_a{}^b \wedge \boldsymbol{\omega}_c{}^e \wedge \boldsymbol{\omega}_e{}^d\right)\right] \,. \tag{2.13}$$

  (Exterior derivative of Chern-Simons like terms).

3. The metric-affine Einstein Lagrangian in $D = 2$

- **Einstein Lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = g_{cb}\boldsymbol{R}_a{}^b \wedge \star\boldsymbol{\vartheta}^{ac} = \mathrm{sgn}(g) e^\nu{}_b e^\mu{}_c g^{ca} R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\mathrm{d}^D x\,, \tag{3.1}$$

> **Reminder.** In $D > 2$, the solution of the EoM of the connection is:
>
> $$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\,. \tag{3.2}$$
>
> Unphysical projective mode $\leftarrow$ can be eliminated using a symmetry of the theory.

The metric-affine Einstein Lagrangian

- **Einstein Lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = g_{cb}\boldsymbol{R}_a{}^b \wedge \star\boldsymbol{\vartheta}^{ac} = \mathrm{sgn}(g) e^\nu{}_b e^\mu{}_c g^{ca} R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\mathrm{d}^D x\,, \tag{3.1}$$

> **Reminder.** In $D > 2$, the solution of the EoM of the connection is:
>
> $$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\,. \tag{3.2}$$
>
> Unphysical projective mode $\leftarrow$ can be eliminated using a symmetry of the theory.

- **Critical dimension** $D = 2$.
  - Equation of motion of the connection

$$\boxed{0 = \mathbf{D}\mathcal{E}^a{}_b} = -\cancel{\boldsymbol{Q}}^{ca}\mathcal{E}_{bc} \qquad \text{where} \quad \cancel{\boldsymbol{Q}}_{ab} = \boldsymbol{Q}_{ab} - \frac{1}{2}g_{ab}\boldsymbol{Q}_c{}^c\,. \tag{3.3}$$

Therefore the general solution is one that verifies

$$\boxed{\cancel{\boldsymbol{Q}}_{ab} = 0}\,. \tag{3.4}$$

- **Einstein Lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = g_{cb}\boldsymbol{R}_a{}^b \wedge \star\boldsymbol{\vartheta}^{ac} = \mathrm{sgn}(g) e^\nu{}_b e^\mu{}_c g^{ca} R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\mathrm{d}^D x\,, \tag{3.1}$$

> **Reminder.** In $D > 2$, the solution of the EoM of the connection is:
>
> $$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\,. \tag{3.2}$$
>
> Unphysical projective mode $\leftarrow$ can be eliminated using a symmetry of the theory.

- **Critical dimension** $D = 2$.
  - Equation of motion of the connection

$$\boxed{0 = \mathbf{D}\mathcal{E}^a{}_b} = -\cancel{\boldsymbol{Q}}^{ca}\mathcal{E}_{bc} \qquad \text{where} \quad \cancel{\boldsymbol{Q}}_{ab} = \boldsymbol{Q}_{ab} - \frac{1}{2}g_{ab}\boldsymbol{Q}_c{}^c\,. \tag{3.3}$$

Therefore the general solution is one that verifies

$$\boxed{\cancel{\boldsymbol{Q}}_{ab} = 0}\,. \tag{3.4}$$

  - But, is this trivial? Or are there conditions over the $D^3 = 8$ degrees of freedom of the connection?

- **Einstein Lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = g_{cb}\boldsymbol{R}_a{}^b \wedge \star\boldsymbol{\vartheta}^{ac} = \operatorname{sgn}(g) e^\nu{}_b e^\mu{}_c g^{ca} R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\mathrm{d}^D x\,, \tag{3.1}$$

> **Reminder.** In $D > 2$, the solution of the EoM of the connection is:
>
> $$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\,. \tag{3.2}$$
>
> Unphysical projective mode ← can be eliminated using a symmetry of the theory.

- **Critical dimension** $D = 2$.
  - Equation of motion of the connection

$$\boxed{0 = \mathbf{D}\mathcal{E}^a{}_b} = -\cancel{\boldsymbol{Q}}^{ca}\mathcal{E}_{bc} \qquad \text{where} \quad \cancel{\boldsymbol{Q}}_{ab} = \boldsymbol{Q}_{ab} - \frac{1}{2}g_{ab}\boldsymbol{Q}_c{}^c\,. \tag{3.3}$$

Therefore the general solution is one that verifies

$$\boxed{\cancel{\boldsymbol{Q}}_{ab} = 0}\,. \tag{3.4}$$

  - But, is this trivial? Or are there conditions over the $D^3 = 8$ degrees of freedom of the connection?

Tensor	d.o.f. in D dim.	d.o.f. in 2 dim.	Condition imposed by EoM
$T_{\mu\nu}{}^\rho$	$\frac{1}{2}D^2(D-1)$	2 (pure trace)	[None]
$Q_{\mu\lambda}{}^\lambda$	D	2	[None] (in any D due to proj. symmetry)
$\cancel{Q}_{\mu\nu\rho}$	$\frac{1}{2}D(D+2)(D-1)$	4	They are zero

Appendix. Useful decomposition of the connection.
In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\cancel{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab}\boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{3.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{3.6}$$

Appendix. Useful decomposition of the connection.
In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\not{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab}\boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{3.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{3.6}$$

- Instead of working in terms of $\boldsymbol{\omega}_a{}^b$ we can work with the independent fields $\bar{\boldsymbol{\omega}}_a{}^b$, $\boldsymbol{Q}_c{}^c$ and $\not{\boldsymbol{Q}}_{ab}$.

Appendix. Useful decomposition of the connection.
In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} & = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\not{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab}\boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} & =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{3.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{3.6}$$

- Instead of working in terms of $\boldsymbol{\omega}_a{}^b$ we can work with the independent fields $\bar{\boldsymbol{\omega}}_a{}^b$, $\boldsymbol{Q}_c{}^c$ and $\not{\boldsymbol{Q}}_{ab}$.
- Plugging this into the action we obtain

$$S_1^{(2)} = \int \mathcal{E}^a{}_b\, \boldsymbol{R}_a{}^b(\omega) = \int \mathcal{E}^a{}_b \left[\bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) - \tfrac{1}{4}\not{\boldsymbol{Q}}_a{}^c \wedge \not{\boldsymbol{Q}}_c{}^b\right], \tag{3.7}$$

- If we choose a orthonormal gauge, i.e. $g_{ab} = \eta_{ab}$, one can use

$$\mathcal{E}_{ab}\, \bar{\boldsymbol{R}}^{ab}(\bar{\omega}) = \mathcal{E}_{ab}\, \mathrm{d}\bar{\boldsymbol{\omega}}^{ab} = \mathrm{d}(\mathcal{E}_{ab}\, \bar{\boldsymbol{\omega}}^{ab})\,, \tag{3.8}$$

  to rewrite the Riemann-Cartan part of the action as a total derivative,

$$S_1^{(2)} = \int \mathrm{d}(\mathcal{E}_{ab}\, \bar{\boldsymbol{\omega}}^{ab}) - \tfrac{1}{4}\int \mathcal{E}_{ab} \not{\boldsymbol{Q}}^{ac} \wedge \not{\boldsymbol{Q}}_c{}^b\,. \tag{3.9}$$

- The action can be rewritten in terms of the new fields as

$$S_1^{(2)}[g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{\omega}_a{}^b] = \int \mathcal{E}^a{}_b \, \boldsymbol{R}_a{}^b(\omega) \tag{3.10}$$

$$\Updownarrow \text{ equivalent}$$

$$\hat{S}_1^{(2)}[g_{ab}, \boldsymbol{\vartheta}^a, \bar{\boldsymbol{\omega}}_a{}^b, \not{\boldsymbol{Q}}_{ab}, \boldsymbol{Q}_c{}^c] = \text{(boundary term)} - \frac{1}{4} \int \mathcal{E}_{ab} \not{\boldsymbol{Q}}^{ac} \wedge \not{\boldsymbol{Q}}_c{}^b \, . \tag{3.11}$$

- The action can be rewritten in terms of the new fields as

$$S_1^{(2)}[g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{\omega}_a{}^b] = \int \mathcal{E}^a{}_b \, \boldsymbol{R}_a{}^b(\omega) \tag{3.10}$$

$\updownarrow$ equivalent

$$\hat{S}_1^{(2)}[g_{ab}, \boldsymbol{\vartheta}^a, \bar{\boldsymbol{\omega}}_a{}^b, \not{\boldsymbol{Q}}_{ab}, \boldsymbol{Q}_c{}^c] = (\text{boundary term}) - \tfrac{1}{4}\int \mathcal{E}_{ab} \not{\boldsymbol{Q}}^{ac} \wedge \not{\boldsymbol{Q}}_c{}^b \,. \tag{3.11}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0 = 0 \quad \leftarrow (\text{due to projective symmetry})\,, \tag{3.12}$$

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = 0\,, \tag{3.13}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0 = \mathcal{E}_{ab} \not{\boldsymbol{Q}}_c{}^b \quad \Leftrightarrow \quad \boxed{\not{\boldsymbol{Q}}_{ab} = 0}\,. \tag{3.14}$$

- The action can be rewritten in terms of the new fields as

$$S_1^{(2)}[g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{\omega}_a{}^b] = \int \mathcal{E}^a{}_b \, \boldsymbol{R}_a{}^b(\omega) \tag{3.10}$$

$\updownarrow$ equivalent

$$\hat{S}_1^{(2)}[g_{ab}, \boldsymbol{\vartheta}^a, \bar{\boldsymbol{\omega}}_a{}^b, \not{\boldsymbol{Q}}_{ab}, \boldsymbol{Q}_c{}^c] = \text{(boundary term)} - \frac{1}{4} \int \mathcal{E}_{ab} \not{\boldsymbol{Q}}^{ac} \wedge \not{\boldsymbol{Q}}_c{}^b \,. \tag{3.11}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0 = 0 \quad \leftarrow \text{(due to projective symmetry)}\,, \tag{3.12}$$

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = 0\,, \tag{3.13}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0 = \mathcal{E}_{ab} \not{\boldsymbol{Q}}_c{}^b \quad \Leftrightarrow \quad \boxed{\not{\boldsymbol{Q}}_{ab} = 0}\,. \tag{3.14}$$

- **Conclusion**:
  There are conditions over the connection. So the Lagrangian CANNOT be a total derivative.
  As we have seen a term quadratic in the non-metricity survives.

4. The metric-affine Gauss-Bonnet Lagrangian in $D = 4$

- **Gauss-Bonnet Lagrangian** (arbitrary dimension)

$$\boldsymbol{L}_2^{(D)} = g_{mb} g_{nd} \boldsymbol{R}_a{}^b \wedge \boldsymbol{R}_c{}^d \wedge \star\boldsymbol{\vartheta}^{amcn} \tag{4.1}$$

$$= \mathrm{sgn}(g) \left[R^2 - R_{\mu\nu} R^{\nu\mu} + 2 R_{\mu\nu} \tilde{R}^{\nu\mu} - \tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda} R^{\rho\lambda\mu\nu} \right] \sqrt{|g|} \mathrm{d}^D x \,, \tag{4.2}$$

where

$$R_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda} \,, \qquad R := g^{\mu\nu} R_{\mu\nu} \,, \qquad \tilde{R}_{\mu}{}^{\nu} := g^{\lambda\sigma} R_{\mu\lambda\sigma}{}^{\nu} \,. \tag{4.3}$$

- **Gauss-Bonnet Lagrangian** (arbitrary dimension)

$$\boldsymbol{L}_2^{(D)} = g_{mb} g_{nd} \boldsymbol{R}_a{}^b \wedge \boldsymbol{R}_c{}^d \wedge \star\boldsymbol{\vartheta}^{amcn} \tag{4.1}$$

$$= \mathrm{sgn}(g) \left[R^2 - R_{\mu\nu}R^{\nu\mu} + 2R_{\mu\nu}\tilde{R}^{\nu\mu} - \tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\right] \sqrt{|g|}\mathrm{d}^D x\,, \tag{4.2}$$

where

$$R_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda}\,, \qquad R := g^{\mu\nu}R_{\mu\nu}\,, \qquad \tilde{R}_{\mu}{}^{\nu} := g^{\lambda\sigma}R_{\mu\lambda\sigma}{}^{\nu}\,. \tag{4.3}$$

- **N.B.** In general D, the most general solution is not known. But we know there should be a free (unphysical) projective mode.

- **Gauss-Bonnet Lagrangian** (arbitrary dimension)

$$\boldsymbol{L}_2^{(D)} = g_{mb} g_{nd} \boldsymbol{R}_a{}^b \wedge \boldsymbol{R}_c{}^d \wedge \star\boldsymbol{\vartheta}^{amcn} \tag{4.1}$$

$$= \mathrm{sgn}(g) \left[R^2 - R_{\mu\nu} R^{\nu\mu} + 2 R_{\mu\nu} \tilde{R}^{\nu\mu} - \tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} + R_{\mu\nu\rho\lambda} R^{\rho\lambda\mu\nu} \right] \sqrt{|g|} \mathrm{d}^D x \,, \tag{4.2}$$

where

$$R_{\mu\nu} := R_{\mu\lambda\nu}{}^{\lambda} \,, \qquad R := g^{\mu\nu} R_{\mu\nu} \,, \qquad \tilde{R}_{\mu}{}^{\nu} := g^{\lambda\sigma} R_{\mu\lambda\sigma}{}^{\nu} \,. \tag{4.3}$$

- **N.B.** In general D, the most general solution is not known. But we know there should be a free (unphysical) projective mode.

- **Critical dimension** $D = 4$.

$$D = 4 \qquad \Rightarrow \qquad \star\boldsymbol{\vartheta}^{amcn} = \mathcal{E}^{amcn} \qquad \Rightarrow \qquad \boldsymbol{L}_2^{(4)} = \mathcal{E}^a{}_b{}^c{}_d \, \boldsymbol{R}_a{}^b \wedge \boldsymbol{R}_c{}^d \tag{4.4}$$

Appendix. Useful decomposition of the connection.
In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\cancel{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab} \boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{4.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{4.6}$$

Appendix. Useful decomposition of the connection.
In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\cancel{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab}\boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{4.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{4.6}$$

- Instead of working in terms of $\boldsymbol{\omega}_a{}^b$ we can work with the independent fields $\bar{\boldsymbol{\omega}}_a{}^b$, $\boldsymbol{Q}_c{}^c$ and $\cancel{\boldsymbol{Q}}_{ab}$.

Appendix. Useful decomposition of the connection.

In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\not{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab} \boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{4.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{4.6}$$

- Instead of working in terms of $\boldsymbol{\omega}_a{}^b$ we can work with the independent fields $\bar{\omega}_a{}^b$, $\boldsymbol{Q}_c{}^c$ and $\not{\boldsymbol{Q}}_{ab}$.
- Plugging this into the action we obtain

$$\begin{aligned} S_2^{(4)} &= \int \mathcal{E}^a{}_b{}^c{}_d \boldsymbol{R}_a{}^b(\omega) \wedge \boldsymbol{R}_c{}^d(\omega) \\ &= \int \mathcal{E}^a{}_b{}^c{}_d \Big[\bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \bar{\boldsymbol{R}}_c{}^d(\bar{\omega}) - \tfrac{1}{2} \bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d + \tfrac{1}{16} \not{\boldsymbol{Q}}_a{}^e \wedge \not{\boldsymbol{Q}}_e{}^b \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d \Big]. \end{aligned} \tag{4.7}$$

Appendix. Useful decomposition of the connection.
In the presence of a metric, an arbitrary $\boldsymbol{\omega}_a{}^b$ can be split

$$\boldsymbol{\omega}_{ab} = \boldsymbol{\omega}_{[ab]} + \boldsymbol{\omega}_{(ab)} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{\omega}_{(ab)} = \frac{1}{2}\boldsymbol{Q}_{ab} = \frac{1}{2}\left(\not{\boldsymbol{Q}}_{ab} + \frac{1}{D} g_{ab}\boldsymbol{Q}_c{}^c\right) \\ \boldsymbol{\omega}_{[ab]} =: \bar{\boldsymbol{\omega}}_{ab} \end{cases} \tag{4.5}$$

It can be proved that $\bar{\boldsymbol{\omega}}_{ab}$ is also a connection with

$$\bar{\boldsymbol{Q}} = 0\,, \qquad \bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}(\boldsymbol{T}, \boldsymbol{Q})\,. \tag{4.6}$$

- Instead of working in terms of $\boldsymbol{\omega}_a{}^b$ we can work with the independent fields $\bar{\boldsymbol{\omega}}_a{}^b$, $\boldsymbol{Q}_c{}^c$ and $\not{\boldsymbol{Q}}_{ab}$.
- Plugging this into the action we obtain

$$\begin{aligned} S_2^{(4)} &= \int \mathcal{E}^a{}_b{}^c{}_d \boldsymbol{R}_a{}^b(\omega) \wedge \boldsymbol{R}_c{}^d(\omega) \\ &= \int \mathcal{E}^a{}_b{}^c{}_d \Big[\bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \bar{\boldsymbol{R}}_c{}^d(\bar{\omega}) - \tfrac{1}{2}\bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d + \tfrac{1}{16}\not{\boldsymbol{Q}}_a{}^e \wedge \not{\boldsymbol{Q}}_e{}^b \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d \Big]. \end{aligned} \tag{4.7}$$

- Choosing the orthonormal gauge, i.e. $g_{ab} = \eta_{ab}$, the first term (the purely Riemann-Cartan one) can be expressed as the Euler density and therefore,

$$S_2^{(4)} = \int \mathrm{d}\boldsymbol{C} - \int \mathcal{E}^a{}_b{}^c{}_d \Big[\tfrac{1}{2}\bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d - \tfrac{1}{16}\not{\boldsymbol{Q}}_a{}^e \wedge \not{\boldsymbol{Q}}_e{}^b \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d \Big]. \tag{4.8}$$

- The action can be rewritten in terms of the new fields as

$$S_2^{(4)}[g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{\omega}_a{}^b] = \int \mathcal{E}^a{}_b{}^c{}_d \boldsymbol{R}_a{}^b(\omega) \wedge \boldsymbol{R}_c{}^d(\omega) \tag{4.9}$$

$\updownarrow$ equivalent

$$\hat{S}_2^{(4)}[g_{ab}, \boldsymbol{\vartheta}^a, \bar{\boldsymbol{\omega}}_a{}^b, \cancel{\boldsymbol{Q}}_{ab}, \boldsymbol{Q}_c{}^c] = \text{(boundary term)} \\ - \int \mathcal{E}^a{}_b{}^c{}_d \Big[\tfrac{1}{2} \bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \cancel{\boldsymbol{Q}}_c{}^f \wedge \cancel{\boldsymbol{Q}}_f{}^d - \tfrac{1}{16} \cancel{\boldsymbol{Q}}_a{}^e \wedge \cancel{\boldsymbol{Q}}_e{}^b \wedge \cancel{\boldsymbol{Q}}_c{}^f \wedge \cancel{\boldsymbol{Q}}_f{}^d \Big]. \tag{4.10}$$

- The action can be rewritten in terms of the new fields as

$$S_2^{(4)}[g_{ab}, \boldsymbol{\vartheta}^a, \boldsymbol{\omega}_a{}^b] = \int \mathcal{E}^a{}_b{}^c{}_d \boldsymbol{R}_a{}^b(\omega) \wedge \boldsymbol{R}_c{}^d(\omega) \tag{4.9}$$

$\updownarrow$ equivalent

$$\hat{S}_2^{(4)}[g_{ab}, \boldsymbol{\vartheta}^a, \bar{\boldsymbol{\omega}}_a{}^b, \not{\boldsymbol{Q}}_{ab}, \boldsymbol{Q}_c{}^c] = (\text{boundary term}) - \int \mathcal{E}^a{}_b{}^c{}_d \Big[\tfrac{1}{2}\bar{\boldsymbol{R}}_a{}^b(\bar{\omega}) \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d - \tfrac{1}{16}\not{\boldsymbol{Q}}_a{}^e \wedge \not{\boldsymbol{Q}}_e{}^b \wedge \not{\boldsymbol{Q}}_c{}^f \wedge \not{\boldsymbol{Q}}_f{}^d\Big]. \tag{4.10}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0 = 0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{4.11}$$

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = \bar{\mathbf{D}}\left[\not{\boldsymbol{Q}}_c{}^a \wedge \not{\boldsymbol{Q}}^{bc}\right]\,, \tag{4.12}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0 = \mathcal{E}_{abcd}\left[\bar{\boldsymbol{R}}^{ab} - \tfrac{1}{4}\not{\boldsymbol{Q}}_f{}^a \wedge \not{\boldsymbol{Q}}^{bf}\right] \wedge \not{\boldsymbol{Q}}^{dm}\,. \tag{4.13}$$

- The action can be rewritten in terms of the new fields as

$$S_2^{(4)}[g_{ab},\boldsymbol{\vartheta}^a,\boldsymbol{\omega}_a{}^b] = \int \mathcal{E}^a{}_b{}^c{}_d \boldsymbol{R}_a{}^b(\omega)\wedge \boldsymbol{R}_c{}^d(\omega) \tag{4.9}$$

$\updownarrow$ equivalent

$$\begin{aligned}\hat{S}_2^{(4)}[g_{ab},\boldsymbol{\vartheta}^a,\bar{\boldsymbol{\omega}}_a{}^b,\not{\boldsymbol{Q}}_{ab},\boldsymbol{Q}_c{}^c] &= (\text{boundary term})\\ &- \int \mathcal{E}^a{}_b{}^c{}_d\left[\tfrac{1}{2}\bar{\boldsymbol{R}}_a{}^b(\bar{\omega})\wedge\not{\boldsymbol{Q}}_c{}^f\wedge\not{\boldsymbol{Q}}_f{}^d - \tfrac{1}{16}\not{\boldsymbol{Q}}_a{}^e\wedge\not{\boldsymbol{Q}}_e{}^b\wedge\not{\boldsymbol{Q}}_c{}^f\wedge\not{\boldsymbol{Q}}_f{}^d\right].\end{aligned} \tag{4.10}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0=0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{4.11}$$

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0=\bar{\mathrm{D}}\left[\not{\boldsymbol{Q}}_c{}^a\wedge\not{\boldsymbol{Q}}^{bc}\right]\,, \tag{4.12}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0=\mathcal{E}_{abcd}\left[\bar{\boldsymbol{R}}^{ab} - \tfrac{1}{4}\not{\boldsymbol{Q}}_f{}^a\wedge\not{\boldsymbol{Q}}^{bf}\right]\wedge\not{\boldsymbol{Q}}^{dm}\,. \tag{4.13}$$

- **Counterexample**. The following field configuration violates EoM of $\bar{\boldsymbol{\omega}}_a{}^b$:

$$\begin{aligned} g_{ab} &= \eta_{ab}\,, & \bar{\boldsymbol{\omega}}^{ab} &= \mathring{\boldsymbol{\omega}}^{ab} + f\boldsymbol{\alpha}^{[a}\delta_t^{b]}\,,\\ \boldsymbol{\vartheta}^a &= \mathrm{d}x^a\,, & \not{\boldsymbol{Q}}^{ab} &= 2\boldsymbol{\alpha}^{(a}\delta_t^{b)}\,,\end{aligned} \quad \text{where} \quad \begin{cases} f & \text{is an arbitrary function,}\\ \boldsymbol{\alpha}^a & = \mathrm{e}^t\left(\delta_y^a\,\mathrm{d}y + \delta_z^a\,\mathrm{d}z\right)\,.\end{cases} \tag{4.14}$$

since

$$\bar{\mathrm{D}}\left[\not{\boldsymbol{Q}}_c{}^a\wedge\not{\boldsymbol{Q}}^{bc}\right] = \mathrm{d}\left[\boldsymbol{\alpha}^a\wedge\boldsymbol{\alpha}^b\right] = 2\mathrm{e}^{2t}\left(\delta_y^a\delta_z^b - \delta_y^b\delta_z^a\right)\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z \neq 0 \quad \text{in the entire } \mathcal{M}. \tag{4.15}$$

The critical metric-affine Gauss-Bonnet Lagrangian in $D=4$. Equivalent formulation

- The action can be rewritten in terms of the new fields as

$$S_2^{(4)}[g_{ab},\boldsymbol{\vartheta}^a,\boldsymbol{\omega}_a{}^b] \;=\; \int \mathcal{E}^a{}_b{}^c{}_d \boldsymbol{R}_a{}^b(\omega)\wedge \boldsymbol{R}_c{}^d(\omega) \tag{4.9}$$

$\updownarrow$ equivalent

$$\begin{aligned}\hat{S}_2^{(4)}[g_{ab},\boldsymbol{\vartheta}^a,\bar{\boldsymbol{\omega}}_a{}^b,\not{\boldsymbol{Q}}_{ab},\boldsymbol{Q}_c{}^c] &= (\text{boundary term})\\ &\quad - \int \mathcal{E}^a{}_b{}^c{}_d\Big[\tfrac{1}{2}\bar{\boldsymbol{R}}_a{}^b(\bar{\omega})\wedge\not{\boldsymbol{Q}}_c{}^f\wedge\not{\boldsymbol{Q}}_f{}^d - \tfrac{1}{16}\not{\boldsymbol{Q}}_a{}^e\wedge\not{\boldsymbol{Q}}_e{}^b\wedge\not{\boldsymbol{Q}}_c{}^f\wedge\not{\boldsymbol{Q}}_f{}^d\Big].\end{aligned} \tag{4.10}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0=0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{4.11}$$

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0=\bar{\mathrm{D}}\left[\not{\boldsymbol{Q}}_c{}^a\wedge\not{\boldsymbol{Q}}^{bc}\right]\,, \tag{4.12}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0=\mathcal{E}_{abcd}\left[\bar{\boldsymbol{R}}^{ab} - \tfrac{1}{4}\not{\boldsymbol{Q}}_f{}^a\wedge\not{\boldsymbol{Q}}^{bf}\right]\wedge\not{\boldsymbol{Q}}^{dm}\,. \tag{4.13}$$

- **Counterexample**. The following field configuration violates EoM of $\bar{\boldsymbol{\omega}}_a{}^b$:

$$\begin{aligned} g_{ab} &= \eta_{ab}\,, & \bar{\boldsymbol{\omega}}^{ab} &= \mathring{\boldsymbol{\omega}}^{ab} + f\boldsymbol{\alpha}^{[a}\delta_t^{b]}\,,\\ \boldsymbol{\vartheta}^a &= \mathrm{d}x^a\,, & \not{\boldsymbol{Q}}^{ab} &= 2\boldsymbol{\alpha}^{(a}\delta_t^{b)}\,, \end{aligned} \qquad \text{where} \quad \begin{cases} f & \text{is an arbitrary function,}\\ \boldsymbol{\alpha}^a & = \mathrm{e}^t\left(\delta_y^a\,\mathrm{d}y + \delta_z^a\,\mathrm{d}z\right)\,. \end{cases} \tag{4.14}$$

since

$$\bar{\mathrm{D}}\left[\not{\boldsymbol{Q}}_c{}^a\wedge\not{\boldsymbol{Q}}^{bc}\right] = \mathrm{d}\left[\boldsymbol{\alpha}^a\wedge\boldsymbol{\alpha}^b\right] = 2\mathrm{e}^{2t}\left(\delta_y^a\delta_z^b - \delta_y^b\delta_z^a\right)\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z \neq 0 \quad \text{in the entire } \mathcal{M}. \tag{4.15}$$

- **Conclusion**: The Lagrangian CANNOT be a total derivative.

5. Discussion of the general critical Lovelock term

- **Critical dimension** $D = 2k$.

$$\boldsymbol{L}_k^{(2k)} = \mathcal{E}^{a_1}{}_{a_2 \ldots}{}^{a_{2k-1}}{}_{a_{2k}} \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1}}{}^{a_{2k}} \,. \tag{5.1}$$

- The action can be rewritten in terms of the new fields as

$$\boldsymbol{L}_k^{(2k)} = \mathcal{E}_{a_1 \ldots a_{2k}} \sum_{m=0}^{k} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_1 a_2} \wedge \ \ldots \ \wedge \bar{\boldsymbol{R}}^{a_{2m-1} a_{2m}} \wedge \tag{5.2}$$

$$\wedge \, \not{\boldsymbol{Q}}^{a_{2m+1} f_1} \wedge \not{\boldsymbol{Q}}_{f_1}{}^{a_{2m+2}} \wedge \ \ldots \ \wedge \not{\boldsymbol{Q}}^{a_{2k-1} f_{k-m}} \wedge \not{\boldsymbol{Q}}_{f_{k-m}}{}^{a_{2k}} \tag{5.3}$$

- **Critical dimension** $D = 2k$.

$$\boldsymbol{L}_k^{(2k)} = \mathcal{E}^{a_1}{}_{a_2}{}^{\dots a_{2k-1}}{}_{a_{2k}} \boldsymbol{R}_{a_1}{}^{a_2} \wedge \dots \wedge \boldsymbol{R}_{a_{2k-1}}{}^{a_{2k}} \,. \tag{5.1}$$

- The action can be rewritten in terms of the new fields as

$$\boldsymbol{L}_k^{(2k)} = \mathcal{E}_{a_1 \dots a_{2k}} \sum_{m=0}^{k} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_1 a_2} \wedge \dots \wedge \bar{\boldsymbol{R}}^{a_{2m-1} a_{2m}} \wedge \tag{5.2}$$

$$\wedge \not{\boldsymbol{Q}}^{a_{2m+1} f_1} \wedge \not{\boldsymbol{Q}}_{f_1}{}^{a_{2m+2}} \wedge \dots \wedge \not{\boldsymbol{Q}}^{a_{2k-1} f_{k-m}} \wedge \not{\boldsymbol{Q}}_{f_{k-m}}{}^{a_{2k}} \tag{5.3}$$

basically,

$$\begin{aligned} \boldsymbol{L}_k^{(2k)} = \mathcal{E}_{a_1 \dots a_{2k}} \Big[& \bar{\boldsymbol{R}} \wedge \bar{\boldsymbol{R}} \wedge \dots \wedge \bar{\boldsymbol{R}} \wedge \bar{\boldsymbol{R}} \qquad \leftarrow \qquad \text{boundary term} \\ & + \bar{\boldsymbol{R}} \wedge \bar{\boldsymbol{R}} \wedge \dots \wedge \bar{\boldsymbol{R}} \wedge \not{\boldsymbol{Q}}^2 \\ & + \bar{\boldsymbol{R}} \wedge \bar{\boldsymbol{R}} \wedge \dots \wedge \not{\boldsymbol{Q}}^2 \wedge \not{\boldsymbol{Q}}^2 \\ & \quad \vdots \\ & + \bar{\boldsymbol{R}} \wedge \not{\boldsymbol{Q}}^2 \wedge \dots \wedge \not{\boldsymbol{Q}}^2 \wedge \not{\boldsymbol{Q}}^2 \\ & + \not{\boldsymbol{Q}}^2 \wedge \not{\boldsymbol{Q}}^2 \wedge \dots \wedge \not{\boldsymbol{Q}}^2 \wedge \not{\boldsymbol{Q}}^2 \Big] \qquad \text{where} \qquad \not{\boldsymbol{Q}}^2 \equiv \not{\boldsymbol{Q}}^{a_i f} \wedge \not{\boldsymbol{Q}}_f{}^{a_{i+1}} \end{aligned} \tag{5.4}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0 = 0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{5.5}$$

$$\begin{aligned} \text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = \mathcal{E}_{aba_3\ldots a_{2k}} \sum_{m=1}^{k-1} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_3a_4} \wedge \ldots \wedge \bar{\boldsymbol{R}}^{a_{2m-1}a_{2m}} \wedge \\ \wedge \bar{\mathbf{D}}\left[\cancel{\boldsymbol{Q}}^{a_{2m+1}f_1} \wedge \cancel{\boldsymbol{Q}}_{f_1}{}^{a_{2m+2}} \wedge \ldots \wedge \cancel{\boldsymbol{Q}}^{a_{2k-1}f_{k-m}} \wedge \cancel{\boldsymbol{Q}}_{f_{k-m}}{}^{a_{2k}}\right], \end{aligned} \tag{5.6}$$

$$\text{EoM for } \cancel{\boldsymbol{Q}}_{ab} \qquad 0 = \text{...omitted...}\,. \tag{5.7}$$

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0 = 0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{5.5}$$

$$\begin{aligned}\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = \mathcal{E}_{aba_3\ldots a_{2k}} \sum_{m=1}^{k-1} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_3a_4} \wedge \ldots \wedge \bar{\boldsymbol{R}}^{a_{2m-1}a_{2m}} \wedge \\ \wedge \bar{\mathrm{D}}\left[\not{\boldsymbol{Q}}^{a_{2m+1}f_1} \wedge \not{\boldsymbol{Q}}_{f_1}{}^{a_{2m+2}} \wedge \ldots \wedge \not{\boldsymbol{Q}}^{a_{2k-1}f_{k-m}} \wedge \not{\boldsymbol{Q}}_{f_{k-m}}{}^{a_{2k}}\right],\end{aligned} \tag{5.6}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0 = \text{...omitted...}\,. \tag{5.7}$$

- **Counterexample**. Consider the following field configuration:

$$\begin{aligned} g_{ab} &= \eta_{ab}\,, & \bar{\boldsymbol{\omega}}^{ab} &= \mathring{\boldsymbol{\omega}}^{ab}\,, \\ \boldsymbol{\vartheta}^a &= \mathrm{d}x^a\,, & \not{\boldsymbol{Q}}^{ab} &= 2\boldsymbol{\alpha}^{(a}\delta_t^{b)}\,, \end{aligned} \qquad \text{where} \quad \boldsymbol{\alpha}^a = \mathrm{e}^t\left(\delta_3^a \mathrm{d}x^3 + \ldots + \delta_{2k}^a \mathrm{d}x^{2k}\right)\,. \tag{5.8}$$

An immediate consequence is $\bar{\boldsymbol{R}}^{ab} = 0$,

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0=0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{5.5}$$

$$\begin{aligned}\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = \mathcal{E}_{aba_3\dots a_{2k}} \sum_{m=1}^{k-1} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_3a_4}\wedge\dots\wedge\bar{\boldsymbol{R}}^{a_{2m-1}a_{2m}}\wedge \\ \wedge\,\bar{\mathrm{D}}\left[\not{\boldsymbol{Q}}^{a_{2m+1}f_1}\wedge\not{\boldsymbol{Q}}_{f_1}{}^{a_{2m+2}}\wedge\dots\wedge\not{\boldsymbol{Q}}^{a_{2k-1}f_{k-m}}\wedge\not{\boldsymbol{Q}}_{f_{k-m}}{}^{a_{2k}}\right],\end{aligned} \tag{5.6}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0 = \text{...omitted...}\,. \tag{5.7}$$

- **Counterexample**. Consider the following field configuration:

$$\begin{aligned} g_{ab} &= \eta_{ab}\,, & \bar{\boldsymbol{\omega}}^{ab} &= \mathring{\boldsymbol{\omega}}^{ab}\,, \\ \boldsymbol{\vartheta}^a &= \mathrm{d}x^a\,, & \not{\boldsymbol{Q}}^{ab} &= 2\boldsymbol{\alpha}^{(a}\delta_t^{b)}\,, \end{aligned} \qquad \text{where} \quad \boldsymbol{\alpha}^a = \mathrm{e}^t\left(\delta_3^a\mathrm{d}x^3 + \dots + \delta_{2k}^a\mathrm{d}x^{2k}\right). \tag{5.8}$$

An immediate consequence is $\bar{\boldsymbol{R}}^{ab}=0$, so

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = \mathcal{E}_{aba_3\dots a_{2k}}\mathring{\mathrm{D}}\left[\not{\boldsymbol{Q}}^{a_3f_1}\wedge\not{\boldsymbol{Q}}_{f_1}{}^{a_4}\wedge\dots\wedge\not{\boldsymbol{Q}}^{a_{2k-1}f_{k-1}}\wedge\not{\boldsymbol{Q}}_{f_{k-1}}{}^{a_{2k}}\right], \tag{5.9}$$

The ansatz (5.8) gives:

$$\begin{aligned} 0 &= \mathrm{d}\left(\boldsymbol{\alpha}^{a_3}\wedge\dots\wedge\boldsymbol{\alpha}^{a_{2k}}\right) \\ &= 2(k-1)\,\mathrm{e}^{2(k-1)t}\,(2k-2)!\,\delta_3^{[a_3}\dots\delta_{2k}^{a_{2k}]}\,\mathrm{d}t\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\dots\wedge\mathrm{d}x^{2k}\,. \end{aligned} \tag{5.10}$$

General Lovelock in critical dimensions ($k > 1$). NOT a boundary term

- Instead of an equation of motion for ${\boldsymbol{\omega}_a}^b$, now our splitting gives rise to three equations

$$\text{EoM for } {\boldsymbol{Q}_c}^c \qquad 0 = 0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{5.5}$$

$$\begin{aligned}\text{EoM for } {\bar{\boldsymbol{\omega}}_a}^b \qquad 0 &= \mathcal{E}_{aba_3...a_{2k}} \sum_{m=1}^{k-1} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_3a_4} \wedge ... \wedge \bar{\boldsymbol{R}}^{a_{2m-1}a_{2m}} \wedge \\ &\quad \wedge \bar{\mathrm{D}} \left[\not{\boldsymbol{Q}}^{a_{2m+1}f_1} \wedge {\not{\boldsymbol{Q}}_{f_1}}^{a_{2m+2}} \wedge ... \wedge \not{\boldsymbol{Q}}^{a_{2k-1}f_{k-m}} \wedge {\not{\boldsymbol{Q}}_{f_{k-m}}}^{a_{2k}} \right], \end{aligned} \tag{5.6}$$

$$\text{EoM for } \not{\boldsymbol{Q}}_{ab} \qquad 0 = \text{...omitted...}\,. \tag{5.7}$$

- **Counterexample**. Consider the following field configuration:

$$\begin{aligned} g_{ab} &= \eta_{ab}\,, & \bar{\boldsymbol{\omega}}^{ab} &= \mathring{\boldsymbol{\omega}}^{ab}\,, \\ \boldsymbol{\vartheta}^a &= \mathrm{d}x^a\,, & \not{\boldsymbol{Q}}^{ab} &= 2\boldsymbol{\alpha}^{(a}\delta_t^{b)}\,, \end{aligned} \qquad \text{where} \quad \boldsymbol{\alpha}^a = \mathrm{e}^t \left(\delta_3^a \mathrm{d}x^3 + ... + \delta_{2k}^a \mathrm{d}x^{2k}\right). \tag{5.8}$$

An immediate consequence is $\bar{\boldsymbol{R}}^{ab} = 0$, so

$$\text{EoM for } {\bar{\boldsymbol{\omega}}_a}^b \qquad 0 = \mathcal{E}_{aba_3...a_{2k}} \mathring{\mathrm{D}} \left[\not{\boldsymbol{Q}}^{a_3f_1} \wedge {\not{\boldsymbol{Q}}_{f_1}}^{a_4} \wedge ... \wedge \not{\boldsymbol{Q}}^{a_{2k-1}f_{k-1}} \wedge {\not{\boldsymbol{Q}}_{f_{k-1}}}^{a_{2k}} \right], \tag{5.9}$$

The ansatz (5.8) gives:

$$\begin{aligned} 0 &= \mathrm{d}\left(\boldsymbol{\alpha}^{a_3} \wedge ... \wedge \boldsymbol{\alpha}^{a_{2k}}\right) \\ &= 2(k-1)\,\mathrm{e}^{2(k-1)t}\,(2k-2)!\,\delta_3^{[a_3}...\delta_{2k}^{a_{2k}]}\,\mathrm{d}t \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge ... \wedge \mathrm{d}x^{2k}\,. \end{aligned} \tag{5.10}$$

If $k > 1$ we get a contradiction:

$$0 = \boxed{\delta_3^{[a_3}...\delta_{2k}^{a_{2k}]}\,\mathrm{d}t \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \wedge ... \wedge \mathrm{d}x^{2k}} \neq 0 \quad \forall p \in \mathcal{M}. \tag{5.11}$$

- **Conclusion**:

- Instead of an equation of motion for $\boldsymbol{\omega}_a{}^b$, now our splitting gives rise to three equations

$$\text{EoM for } \boldsymbol{Q}_c{}^c \qquad 0=0 \qquad \leftarrow \text{(due to projective symmetry)}\,, \tag{5.5}$$

$$\begin{aligned}\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 &= \mathcal{E}_{aba_3\ldots a_{2k}} \sum_{m=1}^{k-1} \frac{1}{4^{k-m}} \frac{k!}{m!(k-m)!} \bar{\boldsymbol{R}}^{a_3a_4} \wedge \ldots \wedge \bar{\boldsymbol{R}}^{a_{2m-1}a_{2m}} \wedge \\ &\wedge \bar{\mathrm{D}}\left[\cancel{\boldsymbol{Q}}^{a_{2m+1}f_1} \wedge \cancel{\boldsymbol{Q}}_{f_1}{}^{a_{2m+2}} \wedge \ldots \wedge \cancel{\boldsymbol{Q}}^{a_{2k-1}f_{k-m}} \wedge \cancel{\boldsymbol{Q}}_{f_{k-m}}{}^{a_{2k}}\right], \end{aligned} \tag{5.6}$$

$$\text{EoM for } \cancel{\boldsymbol{Q}}_{ab} \qquad 0 = \ldots\text{omitted}\ldots\,. \tag{5.7}$$

- **Counterexample**. Consider the following field configuration:

$$\begin{aligned} g_{ab} &= \eta_{ab}\,, & \bar{\boldsymbol{\omega}}^{ab} &= \mathring{\boldsymbol{\omega}}^{ab}\,, \\ \boldsymbol{\vartheta}^a &= \mathrm{d}x^a\,, & \cancel{\boldsymbol{Q}}^{ab} &= 2\boldsymbol{\alpha}^{(a}\delta_t^{b)}\,, \end{aligned} \qquad \text{where} \quad \boldsymbol{\alpha}^a = \mathrm{e}^t\left(\delta_3^a \mathrm{d}x^3 + \ldots + \delta_{2k}^a \mathrm{d}x^{2k}\right). \tag{5.8}$$

An immediate consequence is $\bar{\boldsymbol{R}}^{ab}=0$, so

$$\text{EoM for } \bar{\boldsymbol{\omega}}_a{}^b \qquad 0 = \mathcal{E}_{aba_3\ldots a_{2k}} \mathring{\mathrm{D}}\left[\cancel{\boldsymbol{Q}}^{a_3f_1} \wedge \cancel{\boldsymbol{Q}}_{f_1}{}^{a_4} \wedge \ldots \wedge \cancel{\boldsymbol{Q}}^{a_{2k-1}f_{k-1}} \wedge \cancel{\boldsymbol{Q}}_{f_{k-1}}{}^{a_{2k}}\right], \tag{5.9}$$

The ansatz (5.8) gives:

$$\begin{aligned} 0 &= \mathrm{d}\left(\boldsymbol{\alpha}^{a_3} \wedge \ldots \wedge \boldsymbol{\alpha}^{a_{2k}}\right) \\ &= 2(k-1)\,\mathrm{e}^{2(k-1)t}\,(2k-2)!\,\delta_3^{[a_3}\ldots\delta_{2k}^{a_{2k}]}\,\mathrm{d}t\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\ldots\wedge\mathrm{d}x^{2k}\,. \end{aligned} \tag{5.10}$$

If $k>1$ we get a contradiction:

$$0 = \boxed{\delta_3^{[a_3}\ldots\delta_{2k}^{a_{2k}]}\,\mathrm{d}t\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\ldots\wedge\mathrm{d}x^{2k}} \neq 0 \quad \forall p\in\mathcal{M}. \tag{5.11}$$

- **Conclusion**: The Lagrangian (with $k>1$) CANNOT be a total derivative. [Janssen, Jiménez 2019]

[Janssen, Jiménez 2019]

- The general equation of motion for the connection can be written:

$$0 = \mathbf{D}\mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a}{}_{b} \wedge \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-3}}{}^{a_{D-2}} \qquad \Leftrightarrow \tag{5.12}$$

$$\Leftrightarrow \qquad \boxed{\begin{aligned} 0 = \quad & \Big[\cancel{Q}^{c}{}_{a_1}\mathcal{E}_{ca_2\ldots a_{D-2}ab} + \ldots + \cancel{Q}^{c}{}_{a_{D-3}}\mathcal{E}_{a_1\ldots a_{D-4}ca_{D-2}ab} \\ & \quad +\cancel{Q}^{c}{}_{a}\mathcal{E}_{a_1\ldots a_{D-2}cb}\Big] \wedge \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}} \end{aligned}} \tag{5.13}$$

[Janssen, Jiménez 2019]

- The general equation of motion for the connection can be written:

$$0 = \mathbf{D}\mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a}{}_{b} \wedge \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-3}}{}^{a_{D-2}} \qquad \Leftrightarrow \tag{5.12}$$

$$\Leftrightarrow \qquad \boxed{\begin{aligned} 0 = \; & \Big[\not{\boldsymbol{Q}}^{c}{}_{a_1}\mathcal{E}_{ca_2\ldots a_{D-2}ab} + \ldots + \not{\boldsymbol{Q}}^{c}{}_{a_{D-3}}\mathcal{E}_{a_1\ldots a_{D-4}ca_{D-2}ab} \\ & + \not{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{a_1\ldots a_{D-2}cb}\Big] \wedge \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}} \end{aligned}} \tag{5.13}$$

- Particular cases:

$$(k=1) \quad \text{Einstein:} \quad 0 = \not{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{bc} \quad \Leftrightarrow \quad \boxed{\not{\boldsymbol{Q}}_{ab} = 0}\,\text{(general sol.)}\,. \tag{5.14}$$

$$(k=2) \quad \text{Gauss-Bonnet:} \quad 0 = [\not{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{bcpq} + \not{\boldsymbol{Q}}^{c}{}_{p}\mathcal{E}_{qabc}] \wedge \boldsymbol{R}^{pq} \quad \Rightarrow \quad ?\,. \tag{5.15}$$

[Janssen, Jiménez 2019]

- The general equation of motion for the connection can be written:

$$0 = \mathbf{D}\mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a}{}_{b} \wedge \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-3}}{}^{a_{D-2}} \qquad \Leftrightarrow \tag{5.12}$$

$$\Leftrightarrow \qquad \boxed{\begin{aligned} 0 = \quad & \Big[\not{Q}^{c}{}_{a_1}\mathcal{E}_{ca_2\ldots a_{D-2}ab} + \ldots + \not{Q}^{c}{}_{a_{D-3}}\mathcal{E}_{a_1\ldots a_{D-4}ca_{D-2}ab} \\ & +\not{Q}^{c}{}_{a}\mathcal{E}_{a_1\ldots a_{D-2}cb}\Big] \wedge \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}} \end{aligned}} \tag{5.13}$$

  - Particular cases:

$$(k=1) \quad \text{Einstein:} \qquad 0 = \not{Q}^{c}{}_{a}\mathcal{E}_{bc} \qquad \Leftrightarrow \qquad \boxed{\not{Q}_{ab} = 0}\,\text{(general sol.)}\,. \tag{5.14}$$

$$(k=2) \quad \text{Gauss-Bonnet:} \quad 0 = [\not{Q}^{c}{}_{a}\mathcal{E}_{bcpq} + \not{Q}^{c}{}_{p}\mathcal{E}_{qabc}] \wedge \boldsymbol{R}^{pq} \qquad \Rightarrow \qquad ?\,. \tag{5.15}$$

- Families of solutions for arbitrary k:
  - connection with $Q_{\mu\nu\rho} = V_{\mu}g_{\nu\rho}$ (i.e. $\not{Q}_{ab} = 0$).

[Janssen, Jiménez 2019]

- The general equation of motion for the connection can be written:

$$0 = \mathbf{D}\mathcal{E}^{a_1}{}_{a_2 \dots}{}^{a}{}_{b} \wedge \boldsymbol{R}_{a_1}{}^{a_2} \wedge \dots \wedge \boldsymbol{R}_{a_{D-3}}{}^{a_{D-2}} \qquad \Leftrightarrow \tag{5.12}$$

$$\Leftrightarrow \quad \boxed{\begin{aligned} 0 = \quad & \Big[\not{\boldsymbol{Q}}^{c}{}_{a_1}\mathcal{E}_{ca_2\dots a_{D-2}ab} + \dots + \not{\boldsymbol{Q}}^{c}{}_{a_{D-3}}\mathcal{E}_{a_1\dots a_{D-4}ca_{D-2}ab} \\ & + \not{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{a_1\dots a_{D-2}cb}\Big] \wedge \boldsymbol{R}^{a_1a_2} \wedge \dots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}} \end{aligned}} \tag{5.13}$$

  - Particular cases:

$$(k=1) \quad \text{Einstein:} \qquad 0 = \not{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{bc} \qquad \Leftrightarrow \qquad \boxed{\not{\boldsymbol{Q}}_{ab} = 0}\,\text{(general sol.)}\,. \tag{5.14}$$

$$(k=2) \quad \text{Gauss-Bonnet:} \qquad 0 = [\not{\boldsymbol{Q}}^{c}{}_{a}\mathcal{E}_{bcpq} + \not{\boldsymbol{Q}}^{c}{}_{p}\mathcal{E}_{qabc}] \wedge \boldsymbol{R}^{pq} \qquad \Rightarrow \qquad ?\,. \tag{5.15}$$

- Families of solutions for arbitrary k:
  - connection with $Q_{\mu\nu\rho} = V_\mu g_{\nu\rho}$ (i.e. $\not{\boldsymbol{Q}}_{ab} = 0$).

- Families of solutions for arbitrary $k > 1$:
  - Teleparallel $\boldsymbol{R}_c{}^d = 0$.
  - Any connection such that $\not{\boldsymbol{Q}}_{ab} \wedge \boldsymbol{R}_c{}^d = 0$.

[Janssen, Jiménez 2019]

- The general equation of motion for the connection can be written:

$$0 = \mathbf{D}\mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a}{}_{b} \wedge \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-3}}{}^{a_{D-2}} \qquad \Leftrightarrow \tag{5.12}$$

$$\Leftrightarrow \qquad \boxed{\begin{aligned} 0 = \; & \Big[\cancel{Q}^{c}{}_{a_1}\mathcal{E}_{ca_2\ldots a_{D-2}ab} + \ldots + \cancel{Q}^{c}{}_{a_{D-3}}\mathcal{E}_{a_1\ldots a_{D-4}ca_{D-2}ab} \\ & + \cancel{Q}^{c}{}_{a}\mathcal{E}_{a_1\ldots a_{D-2}cb}\Big] \wedge \boldsymbol{R}^{a_1a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}} \end{aligned}} \tag{5.13}$$

  - Particular cases:

$$(k=1) \quad \text{Einstein:} \qquad 0 = \cancel{Q}^{c}{}_{a}\mathcal{E}_{bc} \qquad \Leftrightarrow \qquad \boxed{\cancel{Q}_{ab} = 0}\,\text{(general sol.)}\,. \tag{5.14}$$

$$(k=2) \quad \text{Gauss-Bonnet:} \qquad 0 = [\cancel{Q}^{c}{}_{a}\mathcal{E}_{bcpq} + \cancel{Q}^{c}{}_{p}\mathcal{E}_{qabc}] \wedge \boldsymbol{R}^{pq} \qquad \Rightarrow \qquad ?\,. \tag{5.15}$$

- Families of solutions for arbitrary k:
  - connection with $Q_{\mu\nu\rho} = V_\mu g_{\nu\rho}$ (i.e. $\cancel{Q}_{ab} = 0$).

- Families of solutions for arbitrary $k > 1$:
  - Teleparallel $\boldsymbol{R}_c{}^d = 0$.
  - Any connection such that $\cancel{Q}_{ab} \wedge \boldsymbol{R}_c{}^d = 0$.

- Families of solutions for arbitrary $k > 2$:
  - Any connection such that $\boldsymbol{R}_{ab} = \boldsymbol{\alpha}_{ab} \wedge \boldsymbol{k}$ for certain 1-forms $\boldsymbol{\alpha}_{ab}$ and $\boldsymbol{k}$ (due to $\boldsymbol{k} \wedge \boldsymbol{k} \equiv 0$).

    Example. Ansatz of grav. wave: $\boldsymbol{k}$ is the dual form of the wave vector. [My PhD Thesis - still in progress]

- EoM in the general case

$$\frac{\delta S_k^{(D)}}{\delta \boldsymbol{\vartheta}^a} = g_{ab} \boldsymbol{R}_{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1} a_{2k}} \wedge \star \boldsymbol{\vartheta}^{a_1 \ldots a_{2k} b} \,. \tag{5.16}$$

- EoM in the general case

$$\frac{\delta S_k^{(D)}}{\delta \boldsymbol{\vartheta}^a} = g_{ab} \boldsymbol{R}_{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1} a_{2k}} \wedge \star \boldsymbol{\vartheta}^{a_1 \ldots a_{2k} b} \,. \tag{5.16}$$

- In $D = 2k$ (critical) the red object becomes a $(D+1)$-form, and the equation becomes trivial:

$$\frac{\delta S_k^{(2k)}}{\delta \boldsymbol{\vartheta}^a} = 0 \,. \tag{5.17}$$

- EoM in the general case

$$\frac{\delta S_k^{(D)}}{\delta \boldsymbol{\vartheta}^a} = g_{ab} \boldsymbol{R}_{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1} a_{2k}} \wedge \star \boldsymbol{\vartheta}^{a_1 \ldots a_{2k} b} \,. \tag{5.16}$$

- In $D = 2k$ (critical) the red object becomes a $(D+1)$-form, and the equation becomes trivial:

$$\frac{\delta S_k^{(2k)}}{\delta \boldsymbol{\vartheta}^a} = 0 \,. \tag{5.17}$$

- Another way of checking this is expanding the star in the Lagrangian to see the dependence on the coframe,

$$\boldsymbol{L}_k^{(D)} = \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1}}{}^{a_{2k}} \wedge \star \boldsymbol{\vartheta}^{a_1}{}_{a_2 \ldots}{}^{a_{2k-1}}{}_{a_{2k}} \tag{5.18}$$

$$= \boldsymbol{R}_{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1} a_{2k}} \wedge \left(\frac{1}{(D-2k)!} \mathcal{E}^{a_1 \ldots a_{2k}}{}_{b_1 \ldots b_{D-2k}} \boldsymbol{\vartheta}^{b_1 \ldots b_{D-2k}} \right) \tag{5.19}$$

In the critical case,

$$\boldsymbol{L}_k^{(2k)} = \mathcal{E}^{a_1}{}_{a_2 \ldots}{}^{a_{2k-1}}{}_{a_{2k}} \boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{2k-1}}{}^{a_{2k}} \,, \tag{5.20}$$

where

$\mathcal{E}^{a_1}{}_{a_2 \ldots}{}^{a_{2k-1}}{}_{a_{2k}}$	depends only on g_{ab} (in the orthonormal case it is a constant tensor)
$\boldsymbol{R}_a{}^b$	is only connection-dependent .

6. Summary and conclusions

Ideas to remember. Consider metric-affine Lovelock in the critical dimension:

- The equation of the coframe (Vielbein) is a trivial identity (so, no information).
- The Einstein-Palatini theory have been solved in all dimensions (even in the critical one).

Ideas to remember. Consider metric-affine Lovelock in the critical dimension:

- The equation of the coframe (Vielbein) is a trivial identity (so, no information).
- The Einstein-Palatini theory have been solved in all dimensions (even in the critical one).

- **The big lesson**:
  There are configurations that do not satisfy the EoM $\Rightarrow$ *these theories are not boundary terms.*

  Consequently, we cannot use Lovelock terms to rewrite certain powers of R in terms of others!!
  (as we can in the metric or in the Riemann-Cartan case)

Ideas to remember. Consider metric-affine Lovelock in the critical dimension:

- The equation of the coframe (Vielbein) is a trivial identity (so, no information).
- The Einstein-Palatini theory have been solved in all dimensions (even in the critical one).

- **The big lesson**:
  There are configurations that do not satisfy the EoM $\Rightarrow$ *these theories are not boundary terms.*

  Consequently, we cannot use Lovelock terms to rewrite certain powers of R in terms of others!!
  (as we can in the metric or in the Riemann-Cartan case)

Open questions / further work

- Which is the role of the non-metricity in breaking the triviality in critical dimension?
- What about the complete Lovelock Lagrangian (all of the Lovelock terms allowed in that D)?
  The most interesting case is obviously $D = 4$, which includes EP + GBP. [Work in progress]

Ideas to remember. Consider metric-affine Lovelock in the critical dimension:

- The equation of the coframe (Vielbein) is a trivial identity (so, no information).
- The Einstein-Palatini theory have been solved in all dimensions (even in the critical one).

- **The big lesson**:
  There are configurations that do not satisfy the EoM $\Rightarrow$ *these theories are not boundary terms.*

  Consequently, we cannot use Lovelock terms to rewrite certain powers of R in terms of others!!
  (as we can in the metric or in the Riemann-Cartan case)

Open questions / further work

- Which is the role of the non-metricity in breaking the triviality in critical dimension?
- What about the complete Lovelock Lagrangian (all of the Lovelock terms allowed in that D)?
  The most interesting case is obviously $D = 4$, which includes EP + GBP. [Work in progress]

Thanks for your attention!

Aitäh!

References for this presentation:

- D. Lovelock, [Lovelock 1971]
  The Einstein Tensor and Its Generalizations,
  J. Math. Phys. **12** (1971), 498–501.
- M. Borunda, B. Janssen, M. Bastero-Gil, , [Borunda, Janssen, Bastero 2008]
  Palatini versus metric formulation in higher curvature gravity,
  JCAP **0811** (2008), 008.
- B. Janssen, A. Jiménez-Cano, J. A. Orejuela, [Janssen, Jiménez, Orejuela 2019]
  A non-trivial connection for the metric-affine Gauss-Bonnet theory in $D = 4$,
  Phys. Lett. B **795** (2019) 42 – 48.
- F. W. Hehl, J. D. McCrea, E. W. Mielke, Y. Ne'eman, [Hehl, McCrea, Mielke, Ne'eman 1995]
  Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance,
  Phys. Rep. **258** (1995) 1 – 171.
- A. Jiménez-Cano, [My PhD Thesis – Still in progress]
  Metric-Affine Gauge theory of gravity. Foundations, perturbations and gravitational wave solutions.
- B. Janssen, A. Jiménez-Cano, [Janssen, Jiménez, 2019]
  On the topological character of metric-affine Lovelock Lagrangians in critical dimensions.
  arXiv:1907.12100 [gr-qc]

Other interesting references:

- D. Iosifidis, and T. Koivisto,
  Scale transformations in metric-affine geometry,
  arXiv preprint: 1810.12276.
- J. Beltrán Jiménez, L. Heisenberg, T. Koivisto,
  Cosmology for quadratic gravity in generalized Weyl geometry,
  JCAP **2016** (2016), no 04, 046.
- J. Zanelli,
  Chern-Simons Forms in Gravitation Theories,
  Class. Quant. Grav. **29** (2012), 133001.

Consider a gravitational lagrangian (vacuum) depending on the connection exclusively through the curvature:

$$S[g,\boldsymbol{\vartheta},\boldsymbol{\omega}]=\int \boldsymbol{L}(g_{ab},\boldsymbol{\vartheta}^a,\boldsymbol{R}_a{}^b(\boldsymbol{\omega}))\equiv\int \mathcal{L}(g_{ab},e_\mu{}^a,R_{\mu\nu a}{}^b(\boldsymbol{\omega}))\sqrt{|g|}\mathrm{d}^D x\,, \tag{7.1}$$

with projective symmetry.

Consider a gravitational lagrangian (vacuum) depending on the connection exclusively through the curvature:

$$S[g,\boldsymbol{\vartheta},\boldsymbol{\omega}] = \int \boldsymbol{L}(g_{ab},\boldsymbol{\vartheta}^a,\boldsymbol{R}_a{}^b(\boldsymbol{\omega})) \equiv \int \mathcal{L}(g_{ab},e_\mu{}^a,R_{\mu\nu a}{}^b(\boldsymbol{\omega}))\sqrt{|g|}\mathrm{d}^D x\,, \tag{7.1}$$

with projective symmetry.

- Noether identities of $\mathrm{Diff}(\mathcal{M})$ and $\mathrm{GL}(D,\mathbb{R}) \Rightarrow$ We only need the EoM of $\boldsymbol{\vartheta}^a$ and $\boldsymbol{\omega}_a{}^b$:

$$0 = \frac{\delta S}{\delta \boldsymbol{\vartheta}^a} \equiv \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a}\,, \tag{7.2}$$

$$0 = \frac{\delta S}{\delta \boldsymbol{\omega}_a{}^b} \equiv \mathbf{D}\left(\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b}\right)\,, \tag{7.3}$$

or, in components,

$$0 = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta e_\mu{}^a} \equiv e^\mu{}_a \mathcal{L} + \frac{\partial \mathcal{L}}{\partial e_\mu{}^a}\,, \tag{7.4}$$

$$0 = \frac{-1}{2\sqrt{|g|}}\frac{\delta S}{\delta \omega_{\mu a}{}^b} \equiv \left(\nabla_\lambda - \frac{1}{2}Q_{\lambda\sigma}{}^\sigma + T_{\lambda\sigma}{}^\sigma\right)\left(\frac{\partial \mathcal{L}}{\partial R_{\lambda\mu a}{}^b}\right) - \frac{1}{2}T_{\lambda\sigma}{}^\mu \frac{\partial \mathcal{L}}{\partial R_{\lambda\sigma a}{}^b}\,. \tag{7.5}$$

Consider a gravitational lagrangian (vacuum) depending on the connection exclusively through the curvature:

$$S[g,\boldsymbol{\vartheta},\boldsymbol{\omega}] = \int \boldsymbol{L}(g_{ab},\boldsymbol{\vartheta}^a,\boldsymbol{R}_a{}^b(\boldsymbol{\omega})) \equiv \int \mathcal{L}(g_{ab},e_\mu{}^a,R_{\mu\nu a}{}^b(\boldsymbol{\omega}))\sqrt{|g|}\mathrm{d}^D x\,, \tag{7.1}$$

with projective symmetry.

- Noether identities of $\mathrm{Diff}(\mathcal{M})$ and $\mathrm{GL}(D,\mathbb{R}) \Rightarrow$ We only need the EoM of $\boldsymbol{\vartheta}^a$ and $\boldsymbol{\omega}_a{}^b$:

$$0 = \frac{\delta S}{\delta \boldsymbol{\vartheta}^a} \equiv \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\vartheta}^a}\,, \tag{7.2}$$

$$0 = \frac{\delta S}{\delta \boldsymbol{\omega}_a{}^b} \equiv \mathbf{D}\left(\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{R}_a{}^b}\right)\,, \tag{7.3}$$

  or, in components,

$$0 = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta e_\mu{}^a} \equiv e^\mu{}_a\mathcal{L} + \frac{\partial \mathcal{L}}{\partial e_\mu{}^a}\,, \tag{7.4}$$

$$0 = \frac{-1}{2\sqrt{|g|}}\frac{\delta S}{\delta \omega_{\mu a}{}^b} \equiv \left(\nabla_\lambda - \frac{1}{2}Q_{\lambda\sigma}{}^\sigma + T_{\lambda\sigma}{}^\sigma\right)\left(\frac{\partial \mathcal{L}}{\partial R_{\lambda\mu a}{}^b}\right) - \frac{1}{2}T_{\lambda\sigma}{}^\mu \frac{\partial \mathcal{L}}{\partial R_{\lambda\sigma a}{}^b}\,. \tag{7.5}$$

- Noether identity of projective symmetry $\Rightarrow$ the connection EoM is traceless (in a, b indices).