Postulates of Quantum Mechanics

Exercise 1: Stern-Gerlach experiment

A Stern-Gerlach (**SG**) device is able to separate particles according to their spin along a given axis. Consider a beam of spin 1/2 particles. We call *positive filter* along the z axis a **SG** that selects spin $+\frac{\hbar}{2}$, while a *measurer* **SG**$\hat{z}$ measures the value of the spin along the same axis. We perform the following experiments.

i) We apply a positive filter along the z axis followed by a measure with **SG**$\hat{z}$.

ii) We apply a filter along z followed by a measurer **SG**$\hat{x}$ along the x axis.

iii) We apply a positive filter along z followed by another positive filter along x, and we finally measure with **SG**$\hat{z}$.

Discuss the result in each case.

Exercise 2: Commutation and diagonalization of observables

Consider two observables A and B acting on the Hilbert space of a quantum system.

i) Show that if they commute (i.e. $[A, B] \equiv AB - BA = 0$), they can be simultaneously diagonalized. Is the converse also true?
[**Hint**: First show that the eigenspaces of A are invariant under B.]

ii) Can they be simultaneously diagonalized if they anticommute, i.e., $\{A, B\} \equiv AB + BA = 0$?

Exercise 3: Uncertainty relations

i) Given two observables A and B, deduce the uncertainty relation

$$\Delta_\psi A \; \Delta_\psi B \geq \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle|,$$

where the expectation value is $\langle A\rangle_\psi \equiv \langle\psi|A|\psi\rangle$ and the *uncertainty* (average quadratic dispersion) is $\Delta_\psi A \equiv [\langle\psi|(A - \langle A\rangle_\psi)^2|\psi\rangle]^{1/2} = [\langle A^2\rangle_\psi - \langle A\rangle_\psi^2]^{1/2}$.
[**Hint**: Compute $\|(A' + \mathrm{i}\lambda B')\,|\psi\rangle\|^2$ where the prime means $X' \equiv X - \langle X\rangle_\psi$ and λ is an arbitrary real number.]

ii) Use the commutation relation $[X, P] = \mathrm{i}\hbar I$ to prove Heisenberg's relation

$$\Delta_\psi X \, \Delta_\psi P \geq \frac{\hbar}{2}\,.$$

iii) Consider an observable A that does not depend explicitly on time (i.e. $\frac{\partial A}{\partial t} = 0$). Derive the *energy–time* uncertainty relation

$$\tau_\psi \, \Delta_\psi E \geq \frac{\hbar}{2} \,,$$

where $\tau_\psi \equiv \Delta_\psi A \Big/ \left| \frac{\mathrm{d}\langle A \rangle_\psi}{\mathrm{d}t} \right|$ is the time scale for a change in the observable A and $\Delta_\psi E$ is the uncertainty in the energy of $|\psi\rangle$.

Exercise 4: Eigenvalues, eigenvectors, expectation value and density matrix

The Hamiltonian H and the physical observables A and B are given in certain basis by the matrices

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix} , \quad B = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix} ,$$

where λ and μ are real numbers different from zero.

i) Find the eigenvalues and eigenvectors of H, A and B.

ii) Determine all the subsets of operators that define a CSCO and find in each case the common basis of eigenstates.

iii) Consider a system in the state

$$|\psi\rangle = c \begin{pmatrix} 2 \\ 0 \\ i \end{pmatrix} .$$

with respect to the same basis we used previously for the given operators.

(a) Find the normalization constant c.

(b) What are the possible values of the energy and their probability when we measure it to ψ? What is the state after the measure in each case?

(c) Find the expectation values of H, A and B for the state ψ. What is the uncertainty in the energy?

iv) Consider a pure ensemble with all its elements in the state ψ.

(a) Find the density matrix that describes this ensemble.

(b) Find the density matrix that describes the ensemble *after* the measurement of the energy to all its elements.

Exercise 5: Test exercise, September 2014 [3P]

Consider a system in the state $|\psi\rangle$ and the observables A y B given in a certain basis by

$$|\psi\rangle = c \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}; \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where c in the normalization constant.

i) If we measure A and right after that we measure B, find the probability to obtain the values -1 and -1, respectively.

ii) Find the probability to obtain the same values -1 and -1 if the sequence is taken in the opposite order, first we measure B and then A.

iii) In a pure statistical ensemble with all its elements in the state $|\psi\rangle$ we measure A and B in a random order, and we select those states where we find -1 and -1 disregarding any other possibility. Find the density matrix that describes the ensemble that we have selected.

Exercise 6: Baker-Campbell-Hausdorff formula and quatization rules

Consider two operators A and B that satisfy $[A,[A,B]] = [B,[A,B]] = 0$ and a function f that admits a Taylor expansion.

i) Show that $[A, f(B)] = [A,B]\,\dfrac{\mathrm{d}f(B)}{\mathrm{d}B}$.

ii) If $G(t) = \mathrm{e}^{tA}\mathrm{e}^{tB}$ being t a real variable, apply the previous result to $\dfrac{\mathrm{d}G(t)}{\mathrm{d}t}$ and prove the relation

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{A+B+[A,B]/2}.$$

iii) Consider the quantization rules in the Weyl form:

$$U_\alpha V_\beta = \mathrm{e}^{-\frac{i}{\hbar}\delta_{ij}\alpha\beta} V_\beta U_\alpha, \quad \text{where} \quad U_\alpha \equiv \mathrm{e}^{-\frac{i}{\hbar}\alpha X_i},\; V_\beta \equiv \mathrm{e}^{-\frac{i}{\hbar}\beta P_j},$$

with α and β real, X_i the position cartesian coordinates and P_j the conjugate momenta. Assuming the hypothesis $[X_i,[X_j,P_k]] = 0 = [P_i,[X_j,P_k]]$ and using the previous result, show that the infinitesimal form of these rules correspond to the canonical quantization rules

$$[X_i, P_j] = i\hbar\,\delta_{ij} I.$$

Does the result contradict the hypothesis?

Exercise 7: Neutrino oscillations

Consider two families of neutrinos with a non-zero mass. Let us assume that the neutrinos that are produced through weak interactions, ν_e and ν_μ, do not coincide with the mass eigenstates, ν_1 and ν_2:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

Suppose that at $t = 0$ a source produces a neutrino ν_e of momentum p, $|\psi(0)\rangle = |\nu_e\rangle$,

i) Solve Schrödinger equation for $|\psi(t)\rangle$ if the energy of a state of well defined mass and momentum is

$$E_i = \sqrt{p^2c^2 + m_i^2c^4} \simeq pc\left(1 + \frac{m_i^2c^2}{2p^2}\right).$$

ii) What is the probability to observe an electron neutrino ν_e at a distance L from the source? And a muon neutrino ν_μ?

iii) What is the optimal distance from the source to observe these neutrino *oscillations*? Compute the optimal distance for the particular case $p \simeq E/c = 1$ MeV$/c$, taking into account that $|m_2^2 - m_1^2| \simeq 8 \times 10^{-5}$ eV$^2/c^4$.

Exercise 8: The neutral kaon system

Neglecting CP violation, the interaction eigenstates K^0 and $\bar{K}^0$ of the neutral kaons can be expressed in terms of the mass eigenstates K_S y K_L as

$$\begin{aligned} |K^0\rangle &= \frac{1}{\sqrt{2}}(\ \ |K_S\rangle + |K_L\rangle), \\ |\bar{K}^0\rangle &= \frac{1}{\sqrt{2}}(-|K_S\rangle + |K_L\rangle). \end{aligned}$$

These two unstable states can be described by an *effective* hamiltonian that is not self-adjoint and has then complex eigenvalues:

$$E_S^0 = m_S c^2 - \frac{\mathrm{i}}{2}\Gamma_S, \quad E_L^0 = m_L c^2 - \frac{\mathrm{i}}{2}\Gamma_L,$$

where $E^0_{S,L}$ are the energy eigenvalues at rest, $m_{S,L}$ the masses, $\Gamma_{S,L} = \hbar/\tau_{S,L}$ the decay widths and $\tau_{S,L}$ the lifetimes of $K_{S,L}$. If we assume that the mass eigenstates to be orthonormal and at $t = 0$ we produce a kaon at rest in the state $|\psi(0)\rangle = |K^0\rangle$,

i) What is the probability to find the kaon in the same state K^0 after a time t? And in the state $\bar{K}^0$?

ii) Is the addition of both probabilities constant? Why?
[**Hint**: Revise the hypothesis to derive the probability conservation $\frac{\mathrm{d}}{\mathrm{d}t}\||\psi(t)\rangle\|^2 = 0$.]

iii) Discuss the similarities and differences between these kaons and the neutrino system studied in the previous exercise.

Exercise 9: Test exercise, February 2011 [2.5P] + extra question iv)

Consider a quantum system associated to a 3-dimensional Hilbert space with basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$. The matrices of the Hamiltonian H and of the observable A in this basis are

$$H = \hbar\,\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad ; \qquad A = a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; .$$

The initial state of the system is $|\psi(0)\rangle = c\left(|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle\right)$, where c is a normalization constant.

i) Find the expectation value of A in that instant. What's the probability to obtain $-a$ if we measure A?

ii) Suppose that we measure A and obtain $-a$. What values may we find if we measure the energy to the resulting state? With what probability?

iii) Determine this state (with measured value $-a$) at an arbitrary time t. Does the expectation value of the energy change with time? And the expectation value of A?

iv) Find the density matrix describing the state of the system obtained after measuring A to all the elements of an ensemble in the pure state $|\psi(0)\rangle$. Will the density matrix of the resulting ensemble change with time?

Exercise 9 must be turned in [deadline: 15 October 2020 at 15:00]