Angular Momentum

Exercise 1: Commutation relations for the orbital angular momentum

Using the commutation relation between X_i and P_j, show that $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$.

Exercise 2: Angular momentum in the position representation

i) Find the differential operators $\vec{L}$, L_+, L_- and L^2 in spherical coordinates.

ii) Show that $L^2 = x^2p^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar\vec{x}\cdot\vec{p}$ and then deduce the relation between L^2 and the angular part of the Laplace operator, ∇^2.

Exercise 3: Parametrizations of rotations: axis-angle and Euler angles

i) Show that a finite rotation of angle ϕ around the axis determined by a unit vector $\hat{n}(\theta,\varphi)$ in three dimensions is given by

$$[R_{\hat{n}}(\phi)]_{ij} = \delta_{ij}\cos\phi + (1-\cos\phi)\hat{n}_i\hat{n}_j - \sin\phi\,\epsilon_{ijk}\hat{n}_k.$$

Use that $\{\hat{n}, \vec{x}, \hat{n}\times\vec{x}\}$ is a complete set of vectors.

ii) Compare the rotations around an axis, $R_{\hat{n}}(\phi)$, with the Euler rotations:

$$\begin{aligned}
R(\alpha,\beta,\gamma) &\equiv R_{\hat{z}}(\alpha)R_{\hat{y}}(\beta)R_{\hat{z}}(\gamma) \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}$$

to show that

$$\varphi = \frac{\pi+\alpha-\gamma}{2}, \quad \tan\theta = \frac{\tan\frac{\beta}{2}}{\sin\frac{\alpha+\gamma}{2}}, \quad \cos\phi = 2\cos^2\frac{\beta}{2}\cos^2\frac{\alpha+\gamma}{2} - 1.$$

iii) Show that one can also write $R(\alpha,\beta,\gamma) = R_{\hat{z}'}(\gamma)R_{\hat{y}'}(\beta)R_{\hat{z}}(\alpha)$, where these primed rotations are around *body axes*, unlike the previous unprimed rotations that are performed around space-fixed axes.

Exercise 4: Representations of the rotation group

i) Find the generators of the rotations in the $j = 2$ representation.

ii) Write the matrix of a rotation around the $\hat{z}$ axis.

iii) If we rotate around any axis, what angle leaves all the vectors in this Hilbert space unchanged?

Exercise 5: State with an arbitrary polarization

Let $\mathcal{B} = \{|+\rangle, |-\rangle\}$ be the basis of the Hilbert space $\mathcal{H}^{j=1/2}$ built with the orthonormal vectors $|+\rangle \equiv \left|\frac{1}{2}\ +\frac{1}{2}\right\rangle$ and $|-\rangle \equiv \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$.

i) If a quantum system is initially at the state $|+\rangle$, find the state $|+\rangle_{\hat{n}}$ that we obtain after two consecutive rotations of it around the axis $\hat{y}$ and $\hat{z}$, i.e.

$$|+\rangle_{\hat{n}} \equiv R_{\hat{z}}(\varphi) R_{\hat{y}}(\theta) |+\rangle .$$

[**Hint**: Remember that

$$R_{\hat{k}}(\alpha) = \exp\left(-iJ_k\alpha/\hbar\right),$$

and that the matrix expression of the generators J_k is $J_k \doteq \frac{\hbar}{2}\sigma_k$ in the basis $\mathcal{B}$, where σ_k are the Pauli matrices]

ii) Show that $|+\rangle_{\hat{n}}$ is an eigenstate of $\vec{J} \cdot \hat{n}$ with eigenvalue $+\hbar/2$.

Exercise 6: Rotations and eigenstates of the position operator

Consider the position eigenstate $|\vec{x}\rangle$. Study whether this state is still a position eigenstate after the rotation

$$|\vec{x}\rangle \to \exp\left(-i\hat{n} \cdot \vec{L}\theta/\hbar\right) |\vec{x}\rangle .$$

Exercise 7: Eigenstates of the angular momentum in the position representation

i) Suppose that $|\psi\rangle$ has a spherically symmetric wave function $\psi(\vec{x})$. Show that a state with orbital wave function

$$(x_1 + ix_2)^\ell \, \psi(\vec{x})$$

is an eigenstate of L^2 and L_z with eigenvalues $\hbar^2\ell(\ell+1)$ and $\hbar\ell$, respectively.

ii) Is $f(\theta, \varphi) = e^{i\varphi}/\sin\theta$ an eigenfunction of L^2, L_z? Can it represent a physical state (*i.e.*, is it normalizable)?

Exercise 8: Eigenfunctions of the angular momentum: spherical harmonics

Consider a scalar particle with orbital wave function

$$\psi(\vec{x}) = K(x + y + 2z)\mathrm{e}^{-\alpha r},$$

where K and α are constants and $r = \sqrt{x^2 + y^2 + z^2}$.

i) Find $\psi(\vec{x})$ in the basis of spherical harmonics.

ii) Prove that the operator L^2 has a well-defined value in this state and compute that value. What is the corresponding uncertainty?

iii) Find the expectation value and the uncertainty of L_z.

iv) If we measure L_z, what is the probability to obtain $+\hbar$?

v) Determine the probability to find the particle within a solid angle $\mathrm{d}\Omega$.

Useful formulae. Spherical harmonics for $\ell = 0, 1$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathrm{e}^{\pm i\varphi}.$$

Exercise 9: Test exercise, September 2009 [2.5P]

Two distinguishable particles of spin $1/2$ are bound in s wave. Find the possible values of the total spin of the system. When we apply a uniform magnetic field $\vec{B} = B\hat{z}$ along the z axis, the Hamiltonian that describes their interaction with this field is

$$H = -\alpha\, \vec{B} \cdot (\vec{S}_1 + \vec{S}_2).$$

Find the maximum variation of energy that the system may experience.

Exercise 10: Addition of angular momenta: Clebsch-Gordan coefficients

i) Using $J_\pm \left| j\, m \right\rangle = \hbar[j(j+1) - m(m \pm 1)]^{1/2} \left| j\, m \pm 1 \right\rangle$, find the Clebsh-Gordan coefficients for the sum of $j_1 = 1$ and $j_2 = 1/2$.

ii) Consider a system of two particles with spin $s_1 = 0$ and $s_2 = 1/2$ in orbital p-wave ($\ell = 1$).

(a) What are the possible values of the total angular momentum j?

(b) For each value of L_z and S_z, what is the probability to find the different possible values of J^2 and J_z?

Exercise 10 must be turned in [deadline: 13 November 2020 at 15:00]