HAWKING SINGULARITY THEOREM

Alejandro Jiménez Cano

ALEJIMCAN@GMAIL.COM

GRANADA, 19-20/12/2023

CONTENTS

REFERENCES

• Original publications

[1] A. Raychaudhuri. "Relativistic cosmology I". Physical Review 98, 1123 (1955).

[2] R. Penrose. "Gravitational Collapse and Space-Time Singularities". Phys. Rev. Lett 14 (1965).

[3] S.W. Hawking. "The occurrence of Singularities in Cosmology II". Proc. R. Soc. Lond. A 295, 490-493 (1966).

[4] S.W. Hawking and R. Penrose. "The singularities of gravitational collapse and cosmology". Proc. R. Soc. Lond. A 314, 529-548 (1970).

• Text books and reviews

[5] J.M.M. Senovilla. "Singularity theorems and their consequences". Gen. Rel. and Gravitation, 29, No. 5 (1997). [arXiv 1801.04912]

[6] M.A. Javaloyes and M. Sánchez. "An introduction to Lorentzian Geometry and its applications". XVI Escola de Geometria Diferencial, Sao Paulo (2010).

[7] R.M. Wald. "General Relativity". The University of Chicago Press (1984).

[8] F.J. Maldonado Torralba. "Teoremas de Singularidad en Relatividad General". Trabajo Fin de Grado (UGR, 2015).

1 INTRODUCTION

Notion of singularity (?): "Wherever variables, observables etc. diverge."

Ex: In electrostatics: $|\vec{F}| \propto \frac{1}{r^2} \xrightarrow{r \to 0} \infty$

Ex: In General Relativity (G.R.)

(a) Schwarzschild spacetime

$g_{rr} \xrightarrow{r \to 2M} \infty$ $\rightsquigarrow$ not physical (artifact of the coordinates)

$R \xrightarrow{r \to 0} \infty$ $\rightsquigarrow$ curvature singularity (tidal forces $\to \infty$ $\Rightarrow$ physical)

invariant

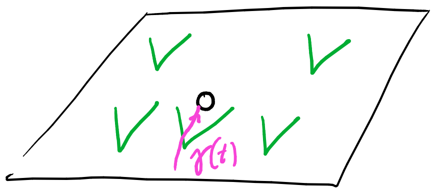
- Singularities often indicate we are going beyond the regime of validity of the theory

As $r \to 0$ G.R. breaks down $\Rightarrow$ no longer valid as effective description

(corrections at scales $R \sim \ell_P^{-2}$)

$\ell_P \sim 10^{-33}$ cm

(b) Minkowski without one point



$\gamma(t)$ incomplete timelike curve

observer

$\Leftarrow$ no curvature blowup but observers appear/disappear

The latter includes the previous case of curvature singularities.

Reasonable definition in gravity:

(M, g) is singular if there are incomplete causal curves (or at least geodesics)[1].

---

1. Some causal curves can be physically impossible to be followed by real observers

(background stability issues $\leftrightsquigarrow$ backreaction)

⚠

* History of singularity theorems — Refs

1955 Raychaudhuri → [1]

1965 Penrose ⇝ Collapse (trapped surfaces) → [2]

1966 Hawking ⇝ FLRW Big bang → [3]

1970 Hawking-Penrose ⇝ Relaxed causality condition → [4]

Learn about them → Senovilla → [5]

Pattern (Senovilla, 1998)

- Condition on the curvature
- Causality condition
- Initial / boundary condition

$\Rightarrow$ $\exists$ null/timelike inextensible incomplete geodesics.

Th (Hawking, 1966)

If spacetime satisfies ⚠ Bert's convention for the Ricci

1) Convergency condition: $R_{ab} v^a v^b \leq 0$ $\forall$ timelike v^a

2) (M, g) globally hyperbolic

3) There exists a Cauchy hypersurface Σ with expansion $\theta \geq \underset{\mathbb{R}}{C} > 0$ for the future-directed normal geodesic congruence.

$\Rightarrow$ all inextensible geodesics starting on Σ and directed to the past are incomplete and their lenghts are bounded from above by $\frac{3}{|C|}$.

Remark: these theorems are geometrical, i.e. theory-independent. In the case of G.R. the condition on R_{ab} can be translated into a condition on the matter: $R_{ab} v^a v^b \leq 0 \overset{\text{EINSTEIN EQ.}}{\Longleftrightarrow} \left(T_{ab} - \frac{1}{2} g_{ab} T_c^c\right) v^a v^b \geq 0$ (STRONG ENERGY CONDITION)

2 GLOBALLY HYPERBOLIC SPACETIMES

Let (M,g) be a spacetime (a Lorentzian time-oriented smooth manifold)

Def: Causal future/past of $p \in M$

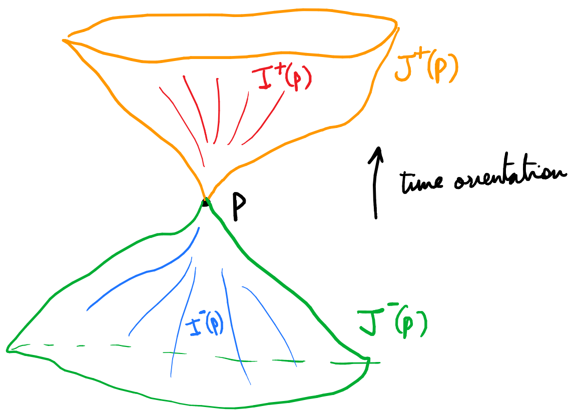
$J^+(p) = \{ q \in M \mid q$ is in the image of a future-oriented causal (timelike or lightlike) curve starting from $p \}$

$J^-(p) = \{$ (same but the curve is PAST-oriented) $\}$

Chronological past/future of $p \in M$:

$I^+(p) = \{ q \in M \mid q$ is in the image of a future-oriented timelike curve starting from $p \}$

$I^-(p) = \{$ (same but the curve is PAST-oriented) $\}$



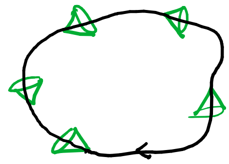
For subsets $S \subset M$, $\quad J^+(S) := \bigcup_{p \in S} J^+(p) \quad$ (same with J^-, I^+, I^-)

Prop. $I^\pm(S) \subset J^\pm(S)$

Def: (M,g) is <u>causal</u> if $\forall p \in M$, $p \notin J^+(p)$.

$\hookrightarrow$ "no point is in its own future"

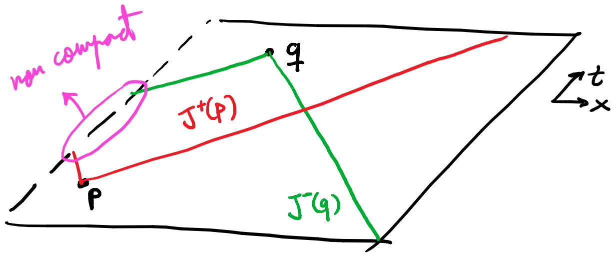
- causal $\Leftrightarrow$ no closed causal curves



not allowed

Prop: (M,g) has no naked singularities (i.e. not protected by horizon or time-reversed) iff $\forall p,q \in M$ $J^+(p) \cap J^-(q)$ is compact

<u>Ex</u>: (causal <u>with</u> naked singularities) half-Minkowski

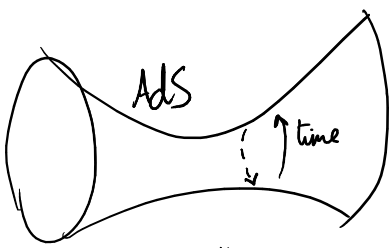


Def: (M,g) is <u>globally hyperbolic</u> if it is causal and there is no naked singularities.

<u>Ex</u>:

Globally hyperbolic: Minkowski, de Sitter, Schwarzschild, FLRW

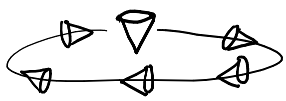
Non-globally hyperbolic: anti-de Sitter, Gödel



$\Downarrow$

not causal

(no naked singularities)



$\Downarrow$

not causal

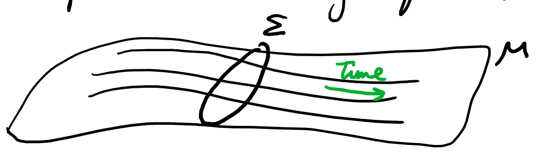
(no naked singularities)

Def: A Cauchy hypersurface Σ is a subset $\Sigma \subset M$ such that any inextensible timelike curve crosses it only once.

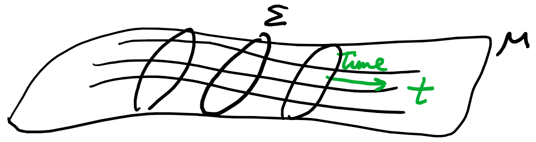
Th.: The following are equivalent:

① (M,g) globally hyperbolic.

② (M,g) admits a spacelike Cauchy hypersurface.

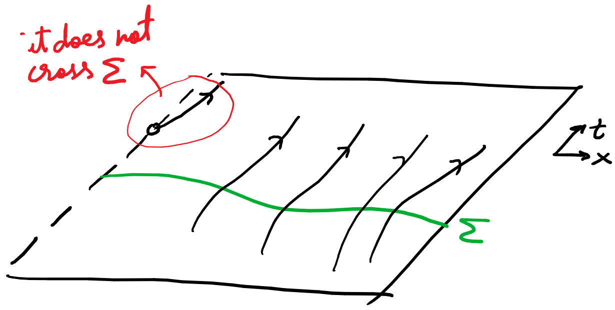


③ (M,g) admits a differentiable function strictly increasing on any causal curve $t: M \to \mathbb{R}$ such that the hypersurfaces $\Sigma_{c_0} = \{ p \in \Sigma \mid t(p) = c_0 \}$ are spacelike and Cauchy.



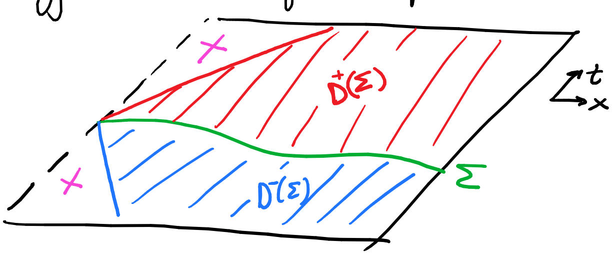
$(\Rightarrow M = \mathbb{R} \times \Sigma)$

Back to the example



$\to$ there is no Cauchy surface

✱ EXTRA: Σ Cauchy iff its Cauchy development is M:



Prop (M,g) globally hyperbolic, then:

① The Lorentzian distance

$$d(p,q) := \begin{cases} 0 & \text{if } C_{pq} = \phi \\ \sup\{\text{lenght}(\gamma),\ \gamma \in C_{pq}\} & \text{elsewhere} \end{cases}$$

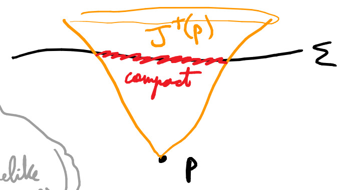
set of all piecewise smooth future-directed causal curves joining p and q.

is finite $\forall p,q \in M$ and continuous.

② $\forall p,q$ causally connected, $\exists$ causal geodesic connecting them so that its lenght is $d(p,q)$ (i.e., the maximal one)

③ Σ Cauchy hypersurface and $\forall p \in I^-(\Sigma)$ then $J^+(p) \cap \Sigma$ is compact.

chronological past (like J^- but only taking timelike curves)



Proof: See ref. [6].

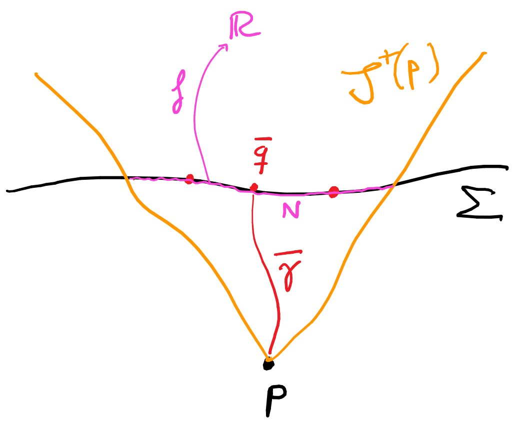
LEMMA 1 (M,g) globally hyperbolic.

For any $\left\{ \begin{array}{l} p \in M \\ \Sigma \text{ Cauchy} \end{array} \right\}$, $\exists \gamma$ curve that maximizes the distance between p and Σ.

Proof:

② $f(q) \equiv d(q, \overset{\text{fixed}}{p})$ $\quad q \in N$ continuous and finite

③ N compact



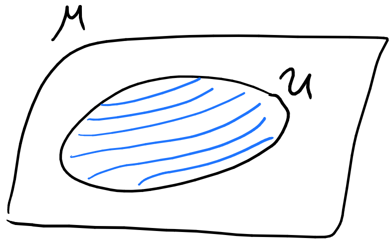
$\Rightarrow \exists \bar{q} \in N$ for which f reaches the absolute maximum

p and $\bar{q}$ are causally connected $\overset{②}{\Longrightarrow}$ $\exists \bar{\gamma}$ from p to N maximizing the lenght ∎

3 TEMPORAL CONVERGENCE CONDITION

$$\left(R_{ab} v^a v^b \le 0 \qquad \forall \text{ timelike } v^a \right)$$

> **Def** $\mathcal{U} \subset \mathcal{M}$ open.
> A congruence in $\mathcal{U}$ is a family of curves such that any $p \in \mathcal{U}$ is contained only in the image of one of them.



↳ It defines a vector field in $\mathcal{U}$ (velocity field v^a).

From now on consider a congruence of geodesics with $v^a v_a = \epsilon \, (\pm 1)$

$$B_{ab} := \nabla_b v_a$$

Let's see some properties:

$$B_{ab} v^b = v^b \nabla_b v_a \overset{\text{geodesic}}{=} 0$$

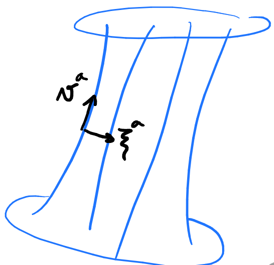
$$B_{ab} v^a = v^a \nabla_b v_a = \frac{1}{2} \nabla_b (v^a v_a) \propto \partial_b \epsilon = 0$$

$\Rightarrow$ B_{ab} is "transversal" to the congruence: $\quad h^a{}_c h^b{}_d B_{ab} = B_{cd}$

where the projector is:

$$h^a{}_b = \delta^a_b - \epsilon v^a v_b \qquad \left(h^a{}_b h^b{}_c = h^a{}_c , \quad h^a{}_b v_a = h^a{}_b v^b = 0 \right)$$

Consider any deviation vector ξ^a : $\quad \xi_a v^a = 0$



$$v^b \nabla_b \xi^a = \xi^b \nabla_b v^a = B^a{}_b \xi^b$$

$v^\mu = \dfrac{dx^\mu}{d\tau}$, $\xi^\mu = \dfrac{dx^\mu}{d\lambda}$

some parameter

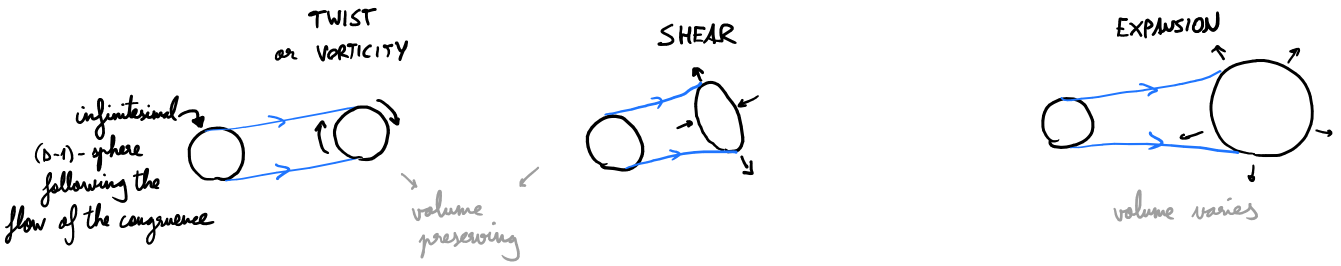
In adapted coordinates $\{x^\mu\} = \{\tau, \lambda, \dots\}$
$\{v^a, \xi^a\}$ are coordinated vectors.

$\Downarrow$

ξ^a is not parallelly transported by v^a

Decomposition of B_{ab}:

$$B_{ab} = \underbrace{\tfrac{1}{2}(B_{ab}-B_{ba})}_{\omega_{ab}} + \underbrace{\tfrac{1}{2}(B_{ab}+B_{ba}) - \frac{1}{D-1} h_{ab} B_c{}^c}_{\sigma_{ab}} + \frac{1}{D-1} h_{ab} \underbrace{B_c{}^c}_{\theta}$$



$$\boxed{\textbf{Prop} \qquad v^c \nabla_c B_{ab} = - B^c{}_b B_{ac} + R_{cadb}\, v^c v^d}$$

Proof

$$\begin{aligned}
v^c \nabla_c B_{ab} &= v^c \nabla_c \nabla_b v_a \\
&= v^c \left(\nabla_b \nabla_c v_a - R_{cba}{}^d v_d \right) \\
&= \nabla_b \big(v^c \underbrace{\cancel{\nabla_c v_a}}_{\substack{0 \\ \text{(geodesic)}}} \big) - \nabla_b v^c \nabla_c v_a - R_{cba}{}^d v^c v_d \\
&= - B^c{}_b B_{ac} + R_{cadb}\, v^c v^d
\end{aligned}$$

■

Let's take the trace:

RAYCHAUDHURI EQUATION

$$\boxed{\textbf{Prop} \qquad v^c \nabla_c \theta = \omega_{ab}\omega^{ab} - \sigma_{ab}\sigma^{ab} - \frac{1}{D-1}\theta^2 + R_{ab}\, v^a v^b}$$

Proof

$$\begin{aligned}
v^c \nabla_c \theta &= g^{ab} v^c \nabla_c B_{ab} \\
&\overset{\text{previous result}}{=} - B^{cb} B_{bc} + R_{ca d}{}^{a}\, v^c v^d \\
&= -\left(\omega^{cb} + \sigma^{cb} + \frac{1}{D-1}\theta h^{cb}\right)\left(\omega_{bc} + \sigma_{bc} + \frac{1}{D-1}\theta h_{bc}\right) + R_{ab}\, v^a v^b \\
&= + \omega_{ab}\omega^{ab} - \sigma_{ab}\sigma^{ab} - \frac{1}{D-1}\theta^2 + R_{ab}\, v^a v^b
\end{aligned}$$

■

We take v^a timelike.

$$v^c \nabla_c \theta = \underbrace{\omega_{ab}\omega^{ab}}_{\geqslant 0 \, *} - \underbrace{\sigma_{ab}\sigma^{ab}}_{\geqslant 0 \, *} + \underbrace{\frac{1}{D-1}\theta^2}_{\geqslant 0} + R_{ab} v^a v^b$$

Proof of $*$: let me call X either ω or σ

At any $p \in \mathcal{U}$, $X_{ab}X^{ab}$ is a scalar $\Rightarrow$ coordinate independent

I take normal coordinates $\{x^\mu\} = \{\tau, y^i\}$ at p, i.e. $g_{\mu\nu}(p) = \eta_{\mu\nu}$

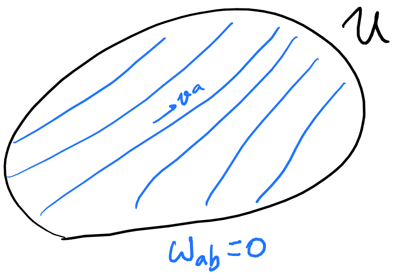
$$(X_{ab}X^{ab})(p) = \eta^{\mu\rho}\eta^{\nu\sigma} X_{\mu\nu}(p) X_{\rho\sigma}(p) = \delta^{ik}\delta^{jl} X_{ij}(p) X_{kl}(p) = \sum_{i,j} (X_{ij}(p))^2 \geqslant 0$$

$\left.\begin{matrix} v^a \text{ timelike} \\ X_{ab} \text{ transversal} \end{matrix}\right\} \Rightarrow X_{ab}$ spatial

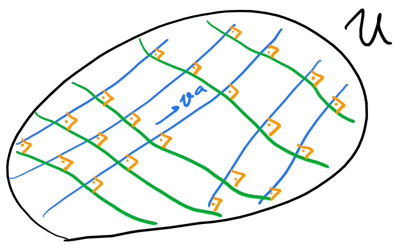
I drop Einstein summation convention

$\blacksquare$

<u>Prop.</u> let $\mathcal{U} \subset \mathcal{M}$ be an open set, C a timelike (or spacelike) congruence in $\mathcal{U}$ and ω_{ab} the vorticity of C. $\omega_{ab}\|_{\mathcal{U}} = 0 \iff \mathcal{U}$ can be foliated by hypersurfaces orthogonal to C.



$\iff$



4 INITIAL EXPANSION

(Condition of initial positive expansion $\theta \geq c > 0$)

This Cauchy hypersurface can be seen as defining the initial conditions.

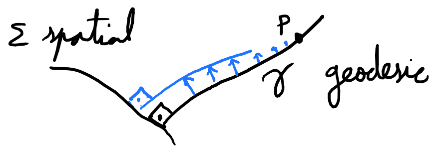
So this condition means that we are initially expanding $\theta \geq c > 0$.

Notice that $\theta \geq c > 0$ must hold $\forall p \in \Sigma$ for a certain (p-independent) c.

Realistic since $\left\{ \begin{array}{l} \text{We observe expansion from Earth} \\ \text{Cosmological principle} \end{array} \right\} \Rightarrow$ it happens everywhere

5 FOCAL POINTS

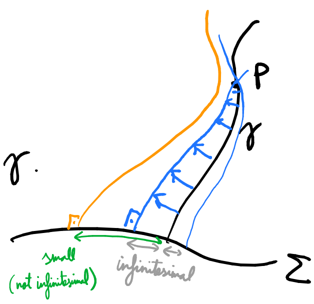
Consider the situation in the figure



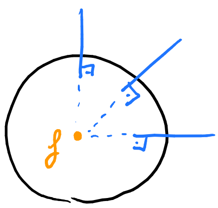
Def p is called a focal point if for infinitesimal (geodesic) deviations starting $\perp \Sigma$ the deviation field goes to zero as we approach p.

Jacobian field

Prop In this situation $\theta \to -\infty$ as we approach p along γ.



Ex. $\Sigma = S^2$ in $\mathbb{R}^3$

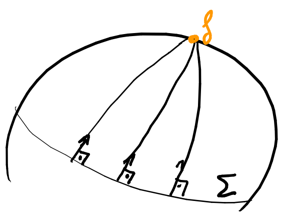


Congruence

$$\left\{ (r, \theta, \varphi)(\lambda) = (\underset{\text{curve parameter}}{\lambda}, \theta_0, \varphi_0), \ (\theta_0, \varphi_0) \in S^2 \right\}$$

focal point: $f \equiv$ center of the sphere

Ex Equator (Σ) in S^2



Congruence: North-directed geodesics

focal point: $f \equiv$ North pole

LEMMA 2 (M, g) fulfill

1) $R_{ab} v^a v^b \leq 0 \quad \forall v^a$ timelike (temporal convergence)

2) $\exists$ a congruence $\perp \Sigma$ (spatial hypersurface) such that $\theta|_\Sigma \leq C < 0$

$\Rightarrow$ In a proper time $\tau \leq \frac{3}{|C|}$ there is a point p focal to Σ.

Σ not necessarily Cauchy!

Proof: orthogonal to $\Sigma \Rightarrow \omega_{ab} = 0$, so by Raychaudhuri eq:

$$\underbrace{v^c \nabla_c \theta}_{\frac{d\theta}{d\tau}} = 0 - \underbrace{\sigma_{ab}\sigma^{ab}}_{\geq 0} - \frac{1}{3}\theta^2 + \underbrace{R_{ab} v^a v^b}_{\leq 0}$$

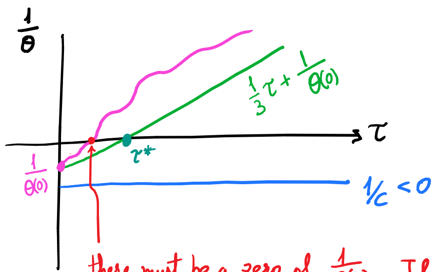
D=4 assumed

$$\Rightarrow \frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2 \Rightarrow \frac{d}{d\tau}(\theta^{-1}) \geq \frac{1}{3} \xRightarrow{\int_0^\tau} \frac{1}{\theta(\tau)} \geq \frac{1}{3}\tau + \frac{1}{\theta(0)}$$

$$-\infty < \overset{\theta|_\Sigma}{\overset{\shortparallel}{\theta(0)}} \leq C < 0$$

$$\Downarrow$$

$$0 > \frac{1}{\theta(0)} \geq \frac{1}{C} > -\infty$$



there must be a zero of $\frac{1}{\theta(\tau)}$. If $\frac{1}{\theta(\tau)} \to 0^- \Rightarrow \theta(\tau) \to -\infty$

This value of τ is bounded by

$$\tau^* = \left(\text{Sol. } 0 = \tfrac{1}{3}\tau + \tfrac{1}{\theta(0)}\right) = -\frac{3}{\underset{<0}{\theta(0)}} = \frac{3}{|\theta(0)|} \qquad \blacksquare$$

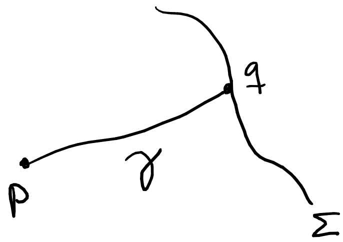
$\rightarrow$ Not an actual singularity is implied, just the existence of a caustic-like behavior. (Until we include global hyperbolicity).

Ex Minkowski fulfill $R_{ab} v^a v^b = 0$ ✓

Construct any congruence with initially negative θ

LEMMA 3

Let γ be a timelike curve connecting $p \in M$ with $q \in \Sigma$ (smooth spacelike hypersurface).



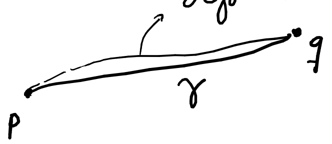
γ is locally (lenght-) maximal $\Leftrightarrow$ $\begin{cases} \gamma \text{ is geodesic } \perp \Sigma \\ \text{There is no focal point between } p \text{ and } \Sigma \text{ in } \gamma. \end{cases}$

→ Σ not necessarily Cauchy!

Proof: See [6,7,8].

Structure of the proof (following [7])

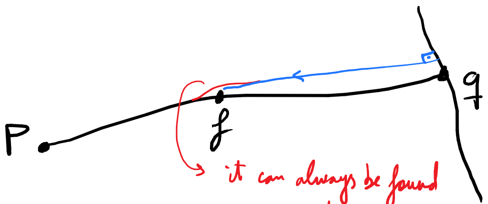
1) γ not a geodesic $\perp \Sigma$ $\Rightarrow$ infinitesimal deformation with $\left.\frac{d\,\text{lenght}}{d\alpha}\right|_{\alpha=0} > 0$



$\Downarrow$

γ not maximal (contradiction)

2) γ geodesic $\perp \Sigma$ and there is a focal point in γ between p and Σ



it can always be found with higher (and lower) lenght

↳ saddle point

$\Rightarrow \gamma$ not maximal (contradiction)

3) γ geodesic $\perp \Sigma$ without focal points

$$\ldots \Longrightarrow \left.\frac{d^2\,\text{lenght}}{d\alpha^2}\right|_{\alpha=0} < 0 \Rightarrow \text{maximum}$$

6 PROOF OF HAWKING SINGULARITY THEOREM

Th (Hawking, 1966)

If spacetime satisfies

1) Convergency condition: $R_{ab} v^a v^b \leq 0 \quad \forall$ timelike v^a
2) (M,g) globally hyperbolic
3) There exists a Cauchy hypersurface Σ with expansion $\theta \geq C > 0$ $(\theta \leq C < 0)$ for the future-directed (past) normal geodesic congruence.

Then:

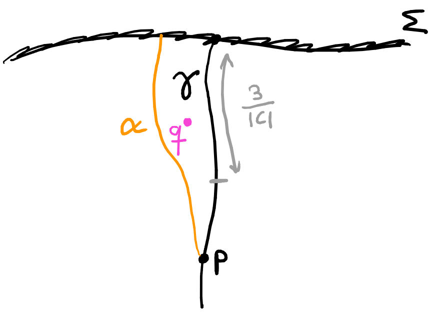
(4) all inextensible geodesics starting on Σ and directed to the past are incomplete and their lenghts are bounded from above by $\frac{3}{|C|}$.

Proof We assume (1), (2), (3).

I will assume $\neg(4)$ and reach a contradiction.

So, there is a curve γ from Σ to the past with length $> \frac{3}{|C|}$.

Take $p \in \text{image}(\gamma)$ further than $\frac{3}{|C|}$ from Σ to the past.



$(2) \xRightarrow{\text{LEMMA 1}} \exists$ curve (α) connecting p and Σ maximizing the lenght.

$\xRightarrow{\text{LEMM 3}}$ α is geodesic $\perp \Sigma$ and does not contain focal points and lenght(α) $\geq \frac{3}{|C|}$ ✳

But $\left.\begin{matrix}(1)\\(3)\end{matrix}\right\} \xRightarrow{\text{LEMMA 2}} \exists$ focal point q for the normal congruence to the past at a distance $\leq \frac{3}{|C|}$

$\Longrightarrow$ α contains a focal point #

$\boxed{\text{Contradiction ✳ v.s #}}$

$\Rightarrow$ γ cannot exist $\Rightarrow \neg(4)$ false $\Rightarrow$ (4) true ∎