Scattering Theory

Units and constants:

$$1\ \text{b} = 10^{-24}\ \text{cm}^2 = 100\ \text{fm}^2\,, \qquad \hbar c \simeq 200\ \text{MeV fm}\,.$$

Exercise 1: Cross Section

A flux of 10^{24} particles cm^{-2} s^{-1} with momentum $p = 200$ MeV$/c$ scatters off a fixed target. Knowing that the scattering amplitude is

$$f(p,\theta,\varphi) = \frac{3\hbar}{2p}(\sin\theta\sin\varphi + i\cos\theta)\,,$$

i) Find the differential cross section $\dfrac{d\sigma}{d\Omega}$ and the total cross section σ. Give the result in barn (1 b $= 10^{-24}$ cm^2).

ii) Determine the forward amplitude $f(\vec{p} \leftarrow \vec{p})$ in fm. Check the optical theorem.

iii) Find the number of particles that scatter in direction $\Omega = (\theta,\varphi)$ per unit of time and solid angle and the total flux of particles that scatter.

iv) Calculate the total number of particles scattered per unit of time in the following regions: (a) the backward hemisphere, (b) the right hemisphere.

Exercise 2: Optical Theorem

Electrons of $p = 10$ keV$/c$ scatter off an attractive central potential $V(r)$. It is found that the forward differential cross section is 1 mb/sr and that $\sigma_{\text{tot}} = 15$ mb. Find the forward scattering amplitude. Does the scatter of the s-wave saturate the cross section?

Exercise 3: Born approximation

i) Find the phase shift associated to a central potential in the Born approximation in terms of the scattering amplitude $f(\vec{p}' \leftarrow \vec{p})$.

ii) Show that the low-energy behaviour of the phase shift is $\delta_\ell \sim p^{2\ell+1}$.

Hint: $\displaystyle\int_{-1}^{1} \mathrm{d}y\ (1-y)^\ell\ P_{\ell'}(y) \propto \delta_{\ell\ell'}$.

Exercise 4: Optical Theorem

A neutron beam ($mc^2 = 940$ MeV) with kinetic energy $E = 0.2$ MeV scatters off a repulsive central potential of range $a = 3\times 10^{-13}$ cm. We measure a total cross section $\sigma = 2.75$ b and $\mathrm{d}\sigma/\mathrm{d}\Omega = 0.58$ b/sr in the forward direction. Use the scattering of the partial waves $\ell = 0, 1$ to estimate the differential cross section at $\theta = \pi/2$.

Exercise 5: Test exercise, September 2014 [3P]

A beam of particles with mass m and momentum $\vec{p}$ scatters of the potential

$$V(r) = V_0\, \delta(r - R)\ .$$

i) Find the scattering amplitude (f) in the Born approximation.

ii) Consider the limit with $qR/\hbar \ll 1$, with $\vec{q} = \vec{p}' - \vec{p}$. Find to the lowest non-vanishing order the total cross section (σ) and the amplitude (f_ℓ) and the phase shift (δ_ℓ) in the partial waves s and p.

Hint: $P_0(x) = 1, \quad P_1(x) = x, \quad \sin^2 \dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{2}.$

Exercise 6: Diatomic molecule

A neutron beam of kinetic energy E along the $\hat{z}$ axis scatters off a diatomic molecule of identical atoms. The interaction potential is $V(\vec{r}) = A\delta(x)\delta(z)[\delta(y-b) + \delta(y+b)]$, where $y = \pm b$ gives the position of the nuclei in the two atoms. Find the scattering amplitude in the Born approximation and discuss the validity of that approximation.

Exercise 7: Test exercise, February 2013 [2.5P]

A beam of particles with mass m and kinetic energy E scatters off the potential

$$V(r) = \frac{\alpha}{r}\, e^{-\frac{r}{r_0}}\ ,$$

where α and r_0 are constants.

i) Find the scattering amplitude ($f(E,\theta)$) in the Born approximation. Determine the phase shifts ($\delta_{0,1}$) in the partial waves s and p.

ii) Consider the limit $r_0 \to \infty$. Find the total cross section ($\sigma_T(E)$) in the Born approximation in that limit. Interpret the result.

Hint. Take into account $\sin^2 \dfrac{\theta}{2} = \dfrac{1-\cos\theta}{2}$, as well as the following useful integrals

$$\int \mathrm{d}x\, e^{ax} \sin bx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}\ , \quad \int \mathrm{d}x\, \frac{x}{1 + a(1-x)} = -\frac{ax + (a+1)\ln(1 + a(1-x))}{a^2}\ .$$

Exercise 8: Identical particles

Consider the scattering $\pi^0 + \pi^0 \to \pi^0 + \pi^0$ only in the s and p waves. In the center of mass frame we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = p^{-2}(A + B\cos\theta + C\cos^2\theta).$$

Keeping in mind that the scattering amplitude must be symmetric under particle exchange, what can be said about the coefficients B and C?

Exercise 9: Identical particles

Find the non-polarized differential cross section for the collision of two identical spin 1 bosons in the center of mass frame. Suppose that the interaction potential is spin independent.

Hint. For a system of two identical particles with well-defined total spin (s and m_s), the scattering amplitude exhibits the same symmetry as the spatial part under particle exchange.

Exercise 10: Spin dependent potential

A beam of spin 1/2 particles with mass m scatters off a heavy spin 1/2 nucleus. The interaction potential is

$$V = c\,\vec{s}_1 \cdot \vec{s}_2\,\delta(\vec{x}_1 - \vec{x}_2),$$

where c is a small constant, $\vec{s}_1$ and $\vec{s}_2$ are the spin of the projectile and the nucleus, respectively, and $\vec{x}_1$ and $\vec{x}_2$ are their positions.

i) Find the differential and the total cross section in the Born approximation doing the average over the initial and the sum over the final spin configurations.

ii) If the spin of the projectile is $|+\rangle$ and the nucleus is not polarized, find the probability that the particles after the scattering still have spin $|+\rangle$.

Exercise 11: Time-independent formalism

A particle of mass m and energy E scatters off the central potential

$$V(r) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(r/a)},$$

where a is a constant. If the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + c^2 y + \frac{2}{\cosh^2 x} y = 0$$

has as solutions $y = \mathrm{e}^{\pm \mathrm{i}cx}(\tanh x \mp \mathrm{i}c)$, find the phase shift and the s wave contribution to the total cross section.

Exercise 7 must be turned in [deadline: 12 January 2021 at 23:00]