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Bootstrapping gravity
Extension to the metric-affine framework

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1. Introduction

Bootstrapping: consistent way of generating non-linear interactions for a given linear theory.

Bootstrapping GR

- [Rosen 1940 (I)] [Rosen 1940 (II)] : GR = self-interacting field theory in flat space
- [Gupta 1954] : GR only nonlinear extension (?)
- [Deser 1970] : Deser's argument in the 1st order formalism.
- Feynman (Lec., 1996): consistency requires self-coupling to its own $T_{\mu\nu}$
- ...
- [Padmanabhan 2008] : uniqueness? GR just from Fierz-Pauli + $T_{\mu\nu}$ self-coupling?
  $\rightarrow$ clarified in [Butcher et al. 2009]

Other lines of research

- [Wald 1986] : order-by-order preservation of the Bianchi id
- ...

Beyond GR

- Higher order der. theories bootstrap in the same way [Butcher et al. 2009] [Ortin 2017]
- [Deser 2017] : 1st order formalism does not work unless one imposes $\Gamma = \mathring{\Gamma}$

Goal: extend [Butcher et al. 2009] approach beyond standard metric theories

Currents: canonical vs Hilbert prescriptions

Consider a (globally) Poincaré invariant theory, $\mathcal{L} = \mathcal{L}(\Phi^A, \partial_\mu \Phi^A)$.

Canonical currents: defined via Noether theorem

$$\text{translations} \rightsquigarrow \qquad T_{\text{can}}{}^{\mu}{}_{\nu} := \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^A} \partial_\nu \Phi^A - \mathcal{L} \delta^\mu{}_\nu \,, \tag{1.1}$$

$$\text{Lorentz} \rightsquigarrow \qquad J_{\text{can}}{}^{\mu\nu\lambda} := \underbrace{T_{\text{can}}{}^{\mu[\lambda} x^{\nu]}}_{\text{orbital}} + \underbrace{\frac{1}{2} \sum_A \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^A} (\Lambda^{\nu\lambda}_{\Phi^A}) \Phi^A}_{\text{spin} \quad S_{\text{can}}{}^{\mu\nu\lambda}} \,. \tag{1.2}$$

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Example 1. Massless free scalar field

$$S[\Phi] = -\frac{1}{2} \int \mathrm{d}^{\mathrm{D}} x \, \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi, \tag{1.3}$$

$$\Rightarrow \qquad T_{\text{can}\,\mu\nu} = -\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} \eta_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi, \qquad {S_{\text{can}}}^{\mu\nu\lambda} = 0 \,. \tag{1.4}$$

Example 2. Massive Dirac spinor

$$S[\Psi] = \int \mathrm{d}^4 x \left[\frac{\mathrm{i}}{2} \left(\overline{\Psi} \gamma^\mu \partial_\mu \Psi - \partial_\mu \overline{\Psi} \gamma^\mu \Psi \right) - m \overline{\Psi} \Psi \right] \tag{1.5}$$

$$\Rightarrow \qquad T_{\text{can}\,\mu\nu} = \frac{\mathrm{i}}{2} (\overline{\Psi} \gamma_\mu \partial_\nu \Psi - \partial_\nu \overline{\Psi} \gamma_\mu \Psi) - \mathcal{L} \eta_{\mu\nu} \,, \qquad {S_{\text{can}}}^{\mu\nu\lambda} = \frac{\mathrm{i}}{4} \overline{\Psi} \gamma^{[\mu} \gamma^\nu \gamma^{\lambda]} \Psi \,. \tag{1.6}$$

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Hilbert prescription

1. Minimal coupling: $\eta^{\mu\nu} \to g^{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$ (torsionful but metric-compatible).

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3. I define:

$$T_{\text{H}\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{M}}[g, K, \Phi]}{\delta g^{\mu\nu}} \bigg|_{g=\eta, K=0} \,, \tag{1.9}$$

$$S_{\text{H}}{}^{\mu\nu\lambda} \eta_{\lambda\rho} := \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{M}}[g, K, \Phi]}{\delta K_{\mu\nu}{}^{\rho}} \bigg|_{g=\eta, K=0} \,. \tag{1.10}$$

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Hilbert prescription in the vielbein formulation

1. Minimal coupling: $\eta^{\mu\nu} \to \eta^{ab} e^\mu{}_a e^\nu{}_b$ and $\partial_\mu \to \nabla_\mu$ (torsionful but metric-compatible), as well as $\gamma^\mu \to \gamma^a e^\mu{}_a$.
2. The resulting action is a functional $S_{\text{M}}[e, K, \Phi]$ (where $K_{\mu a}{}^b := \omega_{\mu a}{}^b - \mathring{\omega}_{\mu a}{}^b$).
3. I define:

$$T_{\text{V}\,\mu\nu} := e_\nu{}^c \eta_{ca} \frac{1}{|e|} \frac{\delta S_{\text{M}}[e, K, \Phi]}{\delta e^\mu{}_a} \bigg|_{e=\delta, K=0} \,, \tag{1.13}$$

$$S_{\text{V}}{}^{\mu\nu\lambda} := e^\nu{}_a e^\lambda{}_b \frac{1}{|e|} \frac{\delta S_{\text{M}}[e, K, \Phi]}{\delta K_{\mu ab}} \bigg|_{e=\delta, K=0} \,. \tag{1.14}$$

❒ Both approaches coincide.

- There is a connection between

superpotentials (canonical pres.) $\leftrightarrow$ minimal couplings (Hilbert's pres.)

Example. Massless free scalar field
We can add to the canonical e-m tensor:

$$\Delta T_{\mu\nu} = \alpha \left(\partial_\mu \partial_\nu \Phi - \eta_{\mu\nu} \partial^2 \Phi \right) \equiv \partial^\rho \chi_{\rho\mu\nu}, \tag{1.15}$$

where the associated superpotential is $\chi_{\rho\mu\nu} = 2\alpha \partial_{[\rho} \Phi \eta_{\mu]\nu}$.

This corresponds in Hilbert's approach with the non-minimal coupling:

$$S_{\text{nm}}[g, \Phi] = -\frac{\alpha}{2} \int \mathrm{d}^{\mathrm{D}} x \sqrt{-g}\, \Phi \mathring{R}(g). \tag{1.16}$$

(We are free to add them between the 1st and the 2nd steps of the prescription).

- Irrelevant for the physical charges.
- But an appropriate choice is crucial for the bootstrapping to succeed.

For an action that is quadratic on Φ^A, we get the EoM:

$$\mathcal{D}_{AB}\Phi^B = 0, \tag{1.17}$$

Step 1: We identify a current j^A which we want to add as a source:

$$\mathcal{D}_{AB}\Phi^B = \lambda j_A, \tag{1.18}$$

Consistent variational principle requires adding (at the action level)

$$\Delta S \sim \lambda \int \mathrm{d}^{\mathsf{D}}x \; j_A \Phi^A. \tag{1.19}$$

Step 2: $j_A = j_A(\Phi^B)$, so the new current is $j_A + \lambda \Delta j_A$...

$$\mathcal{D}_{AB}\Phi^B = \lambda(j_A + \lambda \Delta j_A), \tag{1.20}$$

Step 3...

Examples

- Finite process in Yang-Mills theories.
- Infinite process for the Fierz-Pauli action.

2. Bootstrapping GR and other metric theories

Generic action $S[\mathrm{Q}^I]$ ($\{\mathrm{Q}^I(x)\}$ are spacetime fields).

We evaluate $\mathrm{Q}^I = \bar{\mathrm{Q}}^I + \lambda \mathrm{q}^I$,

$$S[\mathrm{Q}] = \sum_{n=0}^{\infty} \lambda^n S^{(n)}[\bar{\mathrm{Q}}, \mathrm{q}], \tag{2.1}$$

where the partial actions $S^{(n)}$ are given by

$$S^{(n)}[\bar{\mathrm{Q}}, \mathrm{q}] = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} S[\bar{\mathrm{Q}} + \lambda \mathrm{q}]\bigg|_{\lambda=0}. \tag{2.2}$$

Bootstrapping recursive formula

$$\frac{\delta S^{(n)}[\bar{\mathrm{Q}}, \mathrm{q}]}{\delta \mathrm{q}^I} = \frac{\delta S^{(n-1)}[\bar{\mathrm{Q}}, \mathrm{q}]}{\delta \bar{\mathrm{Q}}^I}. \tag{2.3}$$

Reconstruction formula

$$S^{(n)}[\bar{\mathrm{Q}}, \mathrm{q}] = \frac{2}{n!} \left[\int \mathrm{d}^{\mathrm{D}}x\, \mathrm{q}^I(x) \frac{\delta}{\delta \bar{\mathrm{Q}}^I(x)}\right]^{n-2} S^{(2)}[\bar{\mathrm{Q}}, \mathrm{q}]\,. \tag{2.4}$$

We consider the reconstruction of the equations of a diff-invariant metric theory $S[g]$.

* We do not know $S^{(2)}[\bar{g}, h]$ for any arbitrary background metric $\bar{g}^{\mu\nu}$

Bootstrapping (à la [Butcher et al. 2009])

1. Starting point: $S^{(2)}[\eta, h]$.
2. Promote the metric to a general one.
3. Add the appropriate non-minimal couplings.
4. Apply the *recursive formulae* to generate the source of the next order.

$$\frac{\lambda^n}{\sqrt{-\bar{g}}}\frac{\delta S^{(n)}[\bar{g}, h]}{\delta h^{\mu\nu}} = \lambda t^{(n-1)}_{\mu\nu}. \tag{2.5}$$

with

$$t^{(n)}_{\mu\nu} := -\frac{\lambda^n}{\sqrt{-\bar{g}}}\frac{\delta S^{(n)}[\bar{g}, h]}{\delta \bar{g}^{\mu\nu}}, \tag{2.6}$$

5. Use the *reconstruction formula* to obtain the next order action.
6. Go to step 4 (or 3) and iterate.

The Einstein-Hilbert action is given by

$$S[g] = \frac{1}{2\kappa_{(\mathtt{D})}} \int \mathrm{d}^{\mathtt{D}}x \sqrt{-g}\, \mathring{R}(g). \tag{2.7}$$

We expand $g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu}$,

$$S[\bar{g} + \lambda h] = \frac{1}{4\kappa_{(\mathtt{D})}} \int \mathrm{d}^{\mathtt{D}}x \sqrt{-\bar{g}} \sum_{n=2}^{\infty} \Big[M_{(n)}{}^{\alpha_1\alpha_2}{}_{\mu_1\nu_1\ldots\mu_n\nu_n} \bar{\nabla}_{\alpha_1} h^{\mu_1\nu_1} \bar{\nabla}_{\alpha_2} h^{\mu_2\nu_2} h^{\mu_3\nu_3} \ldots h^{\mu_n\nu_n} \\ + H_{(n)\mu_1\nu_1\ldots\mu_n\nu_n} h^{\mu_1\nu_1} \ldots h^{\mu_n\nu_n} \Big], \tag{2.8}$$

Only the term $n = 2$ is required to reconstruct the whole dynamics.

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We expand $g^{\mu\nu} = \bar{g}^{\mu\nu} + \lambda h^{\mu\nu}$,

$$S[\bar{g} + \lambda h] = \frac{1}{4\kappa_{(\mathtt{D})}} \int \mathrm{d}^{\mathtt{D}}x\sqrt{-\bar{g}}\Big[{M_{(2)}}^{\alpha\beta}{}_{\mu\nu\rho\lambda}\bar{\nabla}_{\alpha}h^{\mu\nu}\bar{\nabla}_{\beta}h^{\rho\lambda} + H_{(2)\mu\nu\rho\lambda}h^{\mu\nu}h^{\rho\lambda}\Big] + \ldots \tag{2.10}$$

where

$$\begin{aligned} {M_{(2)}}^{\alpha\beta}{}_{\mu\nu\rho\lambda} &= -\frac{1}{2}\Big[\bar{g}^{\alpha\beta}\bar{g}_{\mu(\rho}\bar{g}_{\lambda)\nu} - \bar{g}^{\alpha\beta}\bar{g}_{\mu\nu}\bar{g}_{\rho\lambda} - 2\delta^{\alpha}{}_{(\rho}\bar{g}_{\lambda)(\mu}\delta^{\beta}{}_{\nu)} + \delta^{\alpha}{}_{(\rho}\delta^{\beta}{}_{\lambda)}\bar{g}_{\mu\nu} + \delta^{\beta}{}_{(\mu}\delta^{\alpha}{}_{\nu)}\bar{g}_{\rho\lambda}\Big]\,, \\ H_{(2)\mu\nu\rho\lambda} &= \frac{1}{2}\bar{R}\left(\bar{g}_{\mu\rho}\bar{g}_{\lambda\nu} + \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}_{\rho\lambda}\right) - \bar{R}_{\mu\nu}\bar{g}_{\rho\lambda} \end{aligned} \tag{2.11}$$

Both,

- the Fierz-Pauli term $M_{(2)}\bar{\nabla}h\bar{\nabla}h$
- **and the non-minimal coupling** $H_{(2)}hh$

are needed to reconstruct Einstein eqs.! [Butcher et al. 2009]

In [Butcher et al. 2009] only quadratic matter actions were considered.

Generalization

$$S[g,\Phi]=S_{\mathrm{g}}[g]+S_{\mathrm{M}}[g,\Phi],\qquad S_{\mathrm{M}}[g,\Phi]=\sum_{p=2}^{N}\mathcal{A}_{\mathrm{M}}^{(p)}[g,\Phi],\tag{2.12}$$

where $\mathcal{A}_{\mathrm{M}}^{(p)}\propto\Phi^p$ (including derivatives).

Everything works the same way, but now:

$$t_{\mu\nu}^{(n)}=t_{\mathrm{g}\,\mu\nu}^{(n)}+t_{\mathrm{M}\,\mu\nu}^{(n)}\qquad\qquad t_{\mathrm{g,M}\,\mu\nu}^{(n)}:=-\frac{\lambda^n}{\sqrt{-\bar{g}}}\frac{\delta S_{\mathrm{g,M}}^{(n)}[\bar{g},h,\bar{\Phi},\phi]}{\delta\bar{g}^{\mu\nu}}.\tag{2.13}$$

Particular case: small matter perturbations ($\Phi=0+\lambda\phi$)

$$t_{\mathrm{M}\,\mu\nu}^{(n)}|_{\bar{\Phi}=0}=-\frac{\lambda^n}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left[\sum_{p=0}^{n}\frac{1}{(n-p)!}\left(\int\mathrm{d}^{\mathrm{D}}x\,h^{\mu\nu}\frac{\delta}{\delta\bar{g}^{\mu\nu}(x)}\right)^{n-p}\mathcal{A}_{\mathrm{M}}^{(p)}[\bar{g}+\lambda h,\phi]\right]_{\lambda=0}\tag{2.14}$$

Every order of λ mixes terms $\mathcal{A}^{(p)}$ and $\mathcal{A}^{(q)}$ with $p+q=n$.

Alternative (ghostfree) Lagrangian for the massless spin-2 field in Minkowski spacetime:

$$\mathcal{L}_{\text{WTDiff}} \sim \frac{1}{2}\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} - \partial_\rho h^{\rho\mu}\partial_\sigma h^\sigma{}_\mu + \frac{2}{\mathtt{D}}\partial_\sigma h^{\sigma\mu}\partial_\mu h - \frac{\mathtt{D}+2}{2\mathtt{D}^2}\partial_\mu h \partial^\mu h\,. \tag{2.15}$$

WTDiff symmetry:

$$\delta_\xi h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} \qquad \text{with} \qquad \partial_\mu \xi^\mu = 0 \tag{2.16}$$

$$\delta_\phi h_{\mu\nu} = \phi\eta_{\mu\nu}, \tag{2.17}$$

Coupling to the traceless energy-momentum tensor!

$$\frac{\delta S^{(n)}[\bar{g},h]}{\delta h^{\mu\nu}} = -\sqrt{-\bar{g}}\left(t^{(n-1)}_{\mu\nu} - \frac{1}{\mathtt{D}}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}t^{(n-1)}_{\alpha\beta}\right), \tag{2.18}$$

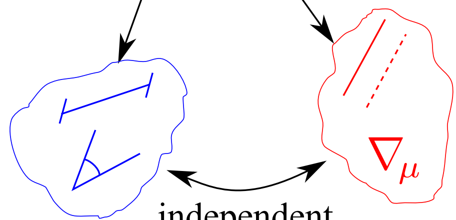
$\Rightarrow$ we reconstruct the eqs. of Unimodular gravity [Carballo et al. 2022]

3. Extension to the metric-affine framework

$$(M, g_{\mu\nu}) \quad \rightsquigarrow \quad (M, g_{\mu\nu}, \Gamma_{\mu\nu}{}^{\rho})$$

Metric-Affine framework

$$(M, g_{\mu\nu}) \xrightarrow{\text{induces}} \mathring{\Gamma}_{\mu\nu}{}^{\rho}$$



independent

$\Gamma \to$ ***Curvature***

$$R_{\mu\nu\rho}{}^{\lambda} := \partial_\mu \Gamma_{\nu\rho}{}^{\lambda} - \partial_\nu \Gamma_{\mu\rho}{}^{\lambda} + \Gamma_{\mu\sigma}{}^{\lambda}\Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\lambda}\Gamma_{\mu\rho}{}^{\sigma}$$

$\Gamma \to$ ***Torsion***

$$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}$$

$\Gamma, g \to$ ***Nonmetricity***

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}$$

* Everything works in the vielbein formulation.

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We add the contorsion $K_{\mu ab} := \omega_{\mu ab} - \mathring{\omega}_{\mu ab}$ (we assume $\nabla g = 0$):

$$S[e, K, \Phi] = S_{\rm g}[g(e)] + S_{\rm F}[e, K, \Phi], \qquad S_{\rm F}[g, \Phi] = \sum_{p=2}^{N} \mathcal{A}_{\rm F}^{(p)}[e, K, \Phi], \tag{3.1}$$

where $\mathcal{A}_{\rm F}^{(p)}$ is the sector $\propto \Phi^p$ (including derivatives).

- EoM of e reconstructed order by order by:

$$\mathfrak{t}^{(n)}{}_{\mu}{}^{a} := -\frac{\tilde{\lambda}^n}{|\bar{e}|} \frac{\delta S^{(n)}[\bar{e}, \epsilon, \bar{K}, k, \bar{\Phi}, \phi]}{\delta \bar{e}^{\mu}{}_{a}} \tag{3.2}$$

- EoM of K reconstructed order by order by:

$$\mathfrak{s}^{(n)\mu ab} = \mathfrak{s}_{\rm F}^{(n)\mu ab} := \frac{\tilde{\lambda}^n}{|\bar{e}|} \frac{\delta S_{\rm F}^{(n)}[\bar{e}, \epsilon, \bar{K}, k, \bar{\Phi}, \phi]}{\delta \bar{K}_{\mu ab}}. \tag{3.3}$$

$$S_{\text{Dirac}}[e, K, \Psi, \overline{\Psi}] = \int \mathrm{d}^4x|e| \left[\frac{\mathrm{i}}{2} e^\mu{}_c \left(\overline{\Psi}\gamma^c \hat{\nabla}_\mu \Psi - \text{h.c.}\right) - m\overline{\Psi}\Psi\right] \tag{3.4}$$

$$= S_{\text{Dirac}}[e, 0, \Psi, \overline{\Psi}] + \int \mathrm{d}^4x|e| \left[-\frac{1}{4} K_{\mu ab} e^\mu{}_c \left(\mathrm{i}\overline{\Psi}\gamma^{[a}\gamma^b\gamma^{c]}\Psi\right)\right], \tag{3.5}$$

- $K_{\mu ab}$ only enters linearly and without derivatives
  $\Rightarrow$ the bootstrapping of $K_{\mu ab}$ closes after 1st non-trivial iteration.

- For small matter perturbations $\Psi = 0 + \tilde{\lambda}\psi$

$$\sum_{n=0}^{\infty} \mathfrak{s}^{(n)\mu ab}|_{\bar{\Psi}=0} = \mathfrak{s}^{(2)\mu ab}|_{\bar{\Psi}=0} = \frac{1}{|\bar{e}|} \frac{\delta S_{\text{Dirac}}[\bar{e}, \bar{K}, \psi, \overline{\psi}]}{\delta \bar{K}_{\mu ab}} = \frac{\mathrm{i}}{4} \bar{e}^\mu{}_c \overline{\psi}\gamma^{[c}\gamma^a\gamma^{b]}\psi\,, \tag{3.6}$$

4. Summary

* Matter fields are just spectators.

Bootstrapping process with metric (à la [Butcher et al. 2009])

1. Starting point: $S^{(2)}[\eta, h]$.
2. Promote the metric to a general one.
3. Add the appropriate non-minimal couplings.
4. Apply the *recursive formulae* to generate the source of the next order.
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New results / extensions

- Vielbein formulation.
- Arbitrary analytic matter.
- WTDiff: coupling to the traceless part of $T_{\mu\nu}$ (see also [Carballo et al. 2022]).
- Torsionful case (a coupling to the partial spin density tensors appear order by order).
- Analogously, the nonmetricity couples to the corresponding currents (GL theories).

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Thanks for your attention!