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Electromagnetism induced by projective symmetry in metric-affine gravity

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Structure of this presentation

B. Janssen, A. Jiménez-Cano. [Janssen, Jiménez 2018]
Projective symmetries and induced electromagnetism in metric-affine gravity.
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1. Introduction

Geometric structures

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- *Metric structure:* $g_{\mu\nu}$ (**metric tensor**)
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_{0,\,\gamma}^{\sigma} \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^{\mu}(\sigma')\dot{x}^{\nu}(\sigma')|}\mathrm{d}\sigma' \,. \tag{1.1}$$

$$\mathrm{vol}(\Omega) = \int_{\Omega} \boldsymbol{\omega}_{\mathrm{vol}}. \tag{1.2}$$

  - Module of a vector (not necessarily non-negative)
    $\Rightarrow$ light cones $\Rightarrow$ causality.
  - Notion of scale (conformal transformations...)

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- *Affine structure*: $\Gamma_{\mu\nu}{}^{\rho}$ (**affine connection**)
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_{μ}
  - Geometrical objects:

Curvature: $$R_{\mu\nu\lambda}{}^{\rho} := \partial_{\mu}\Gamma_{\nu\lambda}{}^{\rho} - \partial_{\nu}\Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\lambda}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\lambda}{}^{\sigma} \,, \tag{1.4}$$

Torsion: $$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho} \,. \tag{1.5}$$

- **Def.:** In the presence of metric and affine connection we define the *non-metricity tensor*:

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho} \,. \tag{1.6}$$

- **Theorem**. Given $g_{\mu\nu}$, there is only one connection that satisfies

$$T_{\mu\nu}{}^{\rho} = 0 \quad \text{(torsionless condition)}, \tag{1.7}$$

$$Q_{\mu\nu\rho} = 0 \quad \text{(compatibility condition)}, \tag{1.8}$$

the *Levi-Civita connection:*

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right] \,. \tag{1.9}$$

Notation. Objects associated to the Levi-Civita connection: $\mathring{R}_{\mu\nu\lambda}{}^{\rho}$, $\mathring{R}_{\mu\nu}$, $\mathring{\nabla}_\mu$...

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- We can fix a general frame in the manifold and the corresponding dual basis (coframe):

$$\boldsymbol{e}_a = e_a{}^\mu \boldsymbol{\partial}_\mu \,, \qquad \boldsymbol{\vartheta}^a = e^a{}_\mu \mathrm{d}x^\mu \qquad \left[\boldsymbol{\vartheta}^a\left(\boldsymbol{e}_b\right) = \delta^a_b \quad \Leftrightarrow \quad e^a{}_\mu e_b{}^\mu = \delta^a_b\right]. \tag{1.10}$$

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We can easily obtain the components of the metric in the coframe:

$$g_{ab} = e_a{}^{\mu} e_b{}^{\nu} g_{\mu\nu}\,, \tag{1.11}$$

and the components of the *connection 1-form*, $\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu$, associated to the affine connection $\Gamma_{\mu\nu}{}^{\lambda}$:

$$\omega_{\mu a}{}^{b} = e_a{}^{\nu} e^b{}_{\lambda}\Gamma_{\mu\nu}{}^{\lambda} + e^b{}_{\sigma}\partial_\mu e_a{}^{\sigma}\,. \tag{1.12}$$

N.B. The objects $\Gamma_{\mu\nu}{}^{\lambda}$ and $\omega_{\mu a}{}^{b}$ contain the same information.

- Action

$$S[g,\,\Gamma,\,\chi]=\frac{1}{2\kappa}\int g^{\mu\nu}R_{\mu\nu}(\Gamma)\,\sqrt{|g|}\mathrm{d}^{D}x+S_{\mathrm{matter}}[g,\,\chi]\,. \tag{1.13}$$

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- Equations of motion (we assume $D > 2$) ($D = 2$ case [Deser 1996])

$$[g]:\qquad 0 = \frac{2\kappa}{\sqrt{|g|}}\frac{\delta S}{\delta g_{\mu\nu}} = \; -\left(R^{(\mu\nu)} - \frac{1}{2}g^{\mu\nu}R\right) + \kappa\mathcal{T}_{\boldsymbol{g}}{}^{\mu\nu}\,, \qquad \left[\mathcal{T}_{\boldsymbol{g}}{}^{\mu\nu} := \frac{2}{\sqrt{|g|}}\frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}\right] \tag{1.14}$$

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Remarks

- This theory exhibits a projective symmetry $\Gamma_{\mu\nu}{}^{\rho} \to \Gamma_{\mu\nu}{}^{\rho} + k_\mu\delta_\nu^\rho \quad (\forall \boldsymbol{k} = k_\mu \mathrm{d}x^\mu \in \Omega^1(\mathcal{M}))$.

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  (Noether 2nd Th.) $\Rightarrow$ The variation $\frac{\delta S}{\delta\Gamma_{\mu\nu}{}^{\sigma}}$ must have zero δ_ν^σ-trace. So the equation $[\Gamma]$ has a trivial trace:

$$\delta_\nu^\sigma[\Gamma]:\quad 0=-\underbrace{T_{\sigma}{}^{\sigma\mu}}_{0}-2\underbrace{Q_{\lambda}{}^{[\lambda\mu]}}_{0}+2\underbrace{g^{[\rho\mu]}}_{0}\left(\frac{1}{2}Q_{\rho\lambda}{}^{\lambda}-T_{\rho\lambda}{}^{\lambda}\right)\qquad\Leftrightarrow\qquad 0=0\,. \tag{1.16}$$

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- General solutions of $[\Gamma]$: [Dadhich, Pons 2012]

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + V_\mu\delta_\nu^\rho\,. \tag{1.17}$$

  Torsion, non-metricity and curvature tensors (let us define $\mathcal{F}_{\mu\nu} := 2\partial_{[\mu}V_{\nu]}$):

$$T_{\mu\nu}{}^{\rho} = 2V_{[\mu}\delta_{\nu]}^{\rho}\,, \qquad Q_{\mu\nu\rho} = 2V_\mu g_{\nu\rho}\,, \qquad R_{\mu\nu\rho}{}^{\lambda} = \mathring{R}_{\mu\nu\rho}{}^{\lambda} + \mathcal{F}_{\mu\nu}\delta_\rho^\lambda\,. \tag{1.18}$$

  V_μ does not have physical effects. [Bernal et al. 2017]

- A metric-compatible connection is called a *Riemann-Cartan connection*:

$$\nabla^{\mathrm{RC}}{}_{\rho} g_{\mu\nu} = 0\,. \tag{1.19}$$

It only contains torsion terms:

$$T^{\mathrm{RC}}{}_{\mu\nu}{}^{a} \quad = \quad 2\Gamma^{\mathrm{RC}}{}_{[\mu\nu]}{}^{\rho} \quad = \quad 2\omega^{\mathrm{RC}}{}_{[\mu\nu]}{}^{\rho} + e^{a}{}_{\mu} e^{b}{}_{\nu} \Omega_{ab}{}^{\rho}\,, \qquad \Omega_{ab}{}^{\rho} \equiv -\left[\boldsymbol{e}_a\,,\,\boldsymbol{e}_b\right]^{\rho}$$

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- Einstein-Cartan action:

$$S[e,\, \omega^{\mathrm{RC}},\, \chi] = \int \tfrac{1}{2\kappa} \eta^{ab} e_{c}{}^{\mu} e_{b}{}^{\nu} R^{\mathrm{RC}}{}_{\mu\nu a}{}^{b}(\omega^{\mathrm{RC}}) \, |e| \mathrm{d}^{D} x + S_{\text{matter}}[e,\, \chi,\, \nabla^{\mathrm{RC}} \chi] \,. \tag{1.20}$$

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- Equations of motion:

$$0 = \frac{\kappa}{|e|} e^{a}{}_{\mu} g_{\nu\rho} \frac{\delta S}{\delta e^{a}{}_{\rho}} = \quad -\left(R^{\mathrm{RC}}{}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\mathrm{RC}}\right) + \kappa \mathcal{T}_{\mu\nu} \,, \tag{1.21}$$

$$0 = \frac{2\kappa}{|e|} e_{a}{}^{\nu} e^{b}{}_{\sigma} \frac{\delta S}{\delta \omega^{\mathrm{RC}}{}_{\mu a}{}^{b}} = \quad - T^{\mathrm{RC}}{}_{\sigma}{}^{\nu\mu} - 2\delta_{\sigma}^{[\mu} g^{\rho]\nu} T^{\mathrm{RC}}{}_{\rho\lambda}{}^{\lambda} - 2\kappa \Sigma^{\mu\nu}{}_{\sigma} \,. \tag{1.22}$$

where we have introduced the *(canonical) energy-momentum tensor* and the *spin density*:

$$\mathcal{T}_{a}{}^{\rho} := \frac{1}{|e|} \frac{\delta S_{\mathrm{matter}}}{\delta e^{a}{}_{\rho}} \,, \qquad \Sigma^{\mu a}{}_{b} := -\frac{1}{|e|} \frac{\delta S_{\mathrm{matter}}}{\delta \omega^{\mathrm{RC}}{}_{\mu a}{}^{b}} \,. \tag{1.23}$$

- It is a particular Poincaré gauge theory (probably the most successful).

2. Spinor covariant derivative

- Decomposition of a general affine connection (in the anholonomic frame):

$$\omega_{\mu ab} = \underbrace{\mathring{\omega}_{\mu ab}}_{\text{L-C}} + \underbrace{\frac{1}{2} e_a{}^\nu e_b{}^\rho \left(T_{\mu\nu\rho} + T_{\rho\mu\nu} - T_{\nu\rho\mu}\right)}_{\text{torsion part (contorsion)}} + \underbrace{\frac{1}{2} e_a{}^\nu e_b{}^\rho \left(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}\right)}_{\text{non-metricity part (disformation(?))}} \tag{2.1}$$

$$= \underbrace{\mathring{\omega}_{\mu ab} + \frac{1}{2} e_{[a}{}^\nu e_{b]}{}^\rho \left(2T_{\mu\nu\rho} + T_{\rho\nu\mu} + 2Q_{\nu\mu\rho}\right)}_{\omega_{\mu[ab]}} + \underbrace{\frac{1}{2} Q_{\mu ab}}_{\omega_{\mu(ab)}} \tag{2.2}$$

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Notation: $\boldsymbol{\gamma}^{ab\ldots c} \equiv \boldsymbol{\gamma}^{[a}\boldsymbol{\gamma}^{b}\ldots\boldsymbol{\gamma}^{c]}$. Hypothesis: $g_{ab} \equiv e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}$.

- Natural Lorentzian prescription:

$$\nabla^{\text{Lor}}{}_\mu \psi = \partial_\mu \psi - \frac{1}{2} \omega_{\mu ab} M^{(s)ab} \psi = \partial_\mu \psi - \frac{1}{4} \omega_{\mu ab} \boldsymbol{\gamma}^{ab} \psi \,, \quad (2.3)$$

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Only the antisymmetric part $\omega_{\mu[ab]}$ contributes to this spinor covariant derivative.

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Only the antisymmetric part $\omega_{\mu[ab]}$ contributes to this spinor covariant derivative.

- **Prop.** $\bar{\omega}_{\mu ab} := \omega_{\mu[ab]}$ is a metric-compatible affine connection ($\bar{Q}_{\mu\nu\rho} = 0$) with torsion given by:

$$\bar{T}_{\mu\nu\rho} = 2T_{\mu\nu\rho} - T_{\rho[\mu\nu]} - 2Q_{[\mu\nu]\rho} \,. \quad (2.4)$$

Proof. Inmediate by direct substitution in the general decomposition of a metric-compatible affine connection:

$$\bar{\omega}_{\mu ab} = \mathring{\omega}_{\mu ab} + \frac{1}{2} e_a{}^\nu e_b{}^\rho \left(\bar{T}_{\mu\nu\rho} + \bar{T}_{\rho\mu\nu} - \bar{T}_{\nu\rho\mu}\right) \,. \quad (2.5)$$

Spinor convariant derivative

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- Extension 1: [Adak, Dereli, Ryder 2003]

$$\nabla^{(1)}_{\mu}\psi = \partial_\mu\psi - \frac{1}{4}\omega_{\mu ab}\boldsymbol{\gamma}^{a}\boldsymbol{\gamma}^{b}\psi \tag{2.7}$$

$$= \nabla^{\mathrm{Lor}}{}_{\mu}\psi - \frac{1}{4}\omega_{\mu c}{}^{c}\psi \tag{2.8}$$

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This is a particular case ($k = -1/8$) of the following one.

- Extension 2 : [Hurley, Vandyck 1994]

$$\nabla^{(2)}_{\mu}\psi = \nabla^{\mathrm{Lor}}{}_{\mu}\psi + kQ_{\mu c}{}^{c}\psi\,, \qquad k \in \mathbb{R}\,. \tag{2.10}$$

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$$\nabla^{\mathrm{Lor}}{}_{\mu}\psi = \partial_\mu\psi - \frac{1}{4}\bar{\omega}_{\mu ab}\boldsymbol{\gamma}^{ab}\psi\,, \qquad \bar{\omega}_{\mu ab} \equiv \omega_{\mu[ab]}\,. \tag{2.6}$$

- Extension 1: [Adak, Dereli, Ryder 2003]

$$\nabla^{(1)}_{\mu}\psi = \partial_\mu\psi - \frac{1}{4}\omega_{\mu ab}\boldsymbol{\gamma}^{a}\boldsymbol{\gamma}^{b}\psi \tag{2.7}$$

$$= \nabla^{\mathrm{Lor}}{}_{\mu}\psi - \frac{1}{4}\omega_{\mu c}{}^{c}\psi \tag{2.8}$$

$$= \nabla^{\mathrm{Lor}}{}_{\mu}\psi - \frac{1}{8}Q_{\mu\rho}{}^{\rho}\psi\,. \tag{2.9}$$

This is a particular case ($k = -1/8$) of the following one.

- Extension 2 : [Hurley, Vandyck 1994]

$$\nabla^{(2)}_{\mu}\psi = \nabla^{\mathrm{Lor}}{}_{\mu}\psi + kQ_{\mu c}{}^{c}\psi\,, \qquad k \in \mathbb{R}\,. \tag{2.10}$$

- Inspired by these works and [Koivisto 2018], we will consider:

$$\boxed{\nabla_\mu\psi = \nabla^{\mathrm{Lor}}{}_{\mu}\psi - \mathrm{i}kQ_{\mu c}{}^{c}\psi\,, \qquad k \in \mathbb{R}}\,. \tag{2.11}$$

3. Induced electromagnetism from non-metricity

Consider the action (in the first order-vielbein formalism):

$$S[e,\,\omega,\,\psi] = \int \mathrm{d}^D x |e| \left\{ \frac{1}{2\kappa}\eta^{ab} e_c{}^{\mu} e_b{}^{\nu} R_{\mu\nu a}{}^{b}(\omega) - \frac{1}{\rho^2} g^{\mu\rho} g^{\nu\lambda} R_{\mu\nu a}{}^{a}(\omega) R_{\rho\lambda c}{}^{c}(\omega) \right.$$
$$\left. + \frac{\mathrm{i}\hbar}{2} e_a{}^{\mu} \left(\bar{\psi}\boldsymbol{\gamma}^a \nabla_\mu \psi - \nabla_\mu \bar{\psi}\boldsymbol{\gamma}^a \psi \right) - m\bar{\psi}\psi \right\}, \quad \rho^2 \in \mathbb{R}^+ . \qquad (3.1)$$

where

$$\nabla_\mu \psi \quad = \quad \nabla^{\mathrm{Lor}}{}_\mu \psi - \mathrm{i} k Q_{\mu c}{}^{c} \psi \,, \qquad k \equiv \frac{e}{\rho}\,, \text{ with } \quad e \in \mathbb{R}\,. \qquad (3.2)$$

Consider the action (in the first order-vielbein formalism):

$$S[e,\,\omega,\,\psi] = \int \mathrm{d}^D x |e| \left\{ \tfrac{1}{2\kappa}\eta^{ab} e_c{}^{\mu} e_b{}^{\nu} R_{\mu\nu a}{}^{b}(\omega) - \tfrac{1}{\rho^2} g^{\mu\rho} g^{\nu\lambda} R_{\mu\nu a}{}^{a}(\omega) R_{\rho\lambda c}{}^{c}(\omega) \right.$$
$$\left. + \tfrac{\mathrm{i}\hbar}{2} e_a{}^{\mu} \left(\bar{\psi}\boldsymbol{\gamma}^a \nabla_\mu \psi - \nabla_\mu \bar{\psi}\boldsymbol{\gamma}^a \psi \right) - m\bar{\psi}\psi \right\}, \quad \rho^2 \in \mathbb{R}^+ . \tag{3.1}$$

where

$$\nabla_\mu \psi \quad = \quad \nabla^{\mathrm{Lor}}{}_{\mu}\psi - \mathrm{i}kQ_{\mu c}{}^{c}\psi\,, \qquad k \equiv \tfrac{e}{\rho}\,, \text{ with } \quad e \in \mathbb{R}\,. \tag{3.2}$$

- This lagrangian is invariant under combined (integrable) projective transformations of the affine structure and $\mathrm{U}(1)$ transformations of the spinor field:

$$\omega_{\mu a}{}^{b} \to \omega_{\mu a}{}^{b} + \frac{\rho}{2D}\partial_\mu \Lambda \delta_a^b \qquad \Rightarrow \qquad Q_{\mu c}{}^{c} \to Q_{\mu c}{}^{c} + \rho\partial_\mu\Lambda\,, \tag{3.3}$$

$$\psi \to \mathrm{e}^{\mathrm{i}e\Lambda}\psi\,. \tag{3.4}$$

Consider the action (in the first order-vielbein formalism):

$$
\begin{aligned}
S[e,\,\omega,\,\psi] = \int \mathrm{d}^D x |e| \Big\{ &\tfrac{1}{2\kappa}\eta^{ab} e_c{}^{\mu} e_b{}^{\nu} R_{\mu\nu a}{}^{b}(\omega) - \tfrac{1}{\rho^2} g^{\mu\rho} g^{\nu\lambda} R_{\mu\nu a}{}^{a}(\omega) R_{\rho\lambda c}{}^{c}(\omega) \\
&+ \tfrac{\mathrm{i}\hbar}{2} e_a{}^{\mu}\left(\bar\psi \boldsymbol{\gamma}^a \nabla_\mu \psi - \nabla_\mu \bar\psi \boldsymbol{\gamma}^a \psi\right) - m\bar\psi\psi \Big\}\,, \quad \rho^2 \in \mathbb{R}^+\,.
\end{aligned} \tag{3.1}
$$

where

$$
\nabla_\mu \psi = \nabla^{\mathrm{Lor}}{}_{\mu}\psi - \mathrm{i}k Q_{\mu c}{}^{c}\psi\,, \qquad k \equiv \tfrac{e}{\rho}\,, \text{ with } \quad e \in \mathbb{R}\,. \tag{3.2}
$$

- This lagrangian is invariant under combined (integrable) projective transformations of the affine structure and $\mathrm{U}(1)$ transformations of the spinor field:

$$
\omega_{\mu a}{}^{b} \to \omega_{\mu a}{}^{b} + \frac{\rho}{2D}\partial_\mu \Lambda \delta_a^b \qquad \Rightarrow \qquad Q_{\mu c}{}^{c} \to Q_{\mu c}{}^{c} + \rho \partial_\mu \Lambda\,, \tag{3.3}
$$

$$
\psi \to \mathrm{e}^{\mathrm{i}e\Lambda}\psi\,. \tag{3.4}
$$

- **N.B.**
  The square-curvature term is a Maxwell term for $Q_{\mu c}{}^{c}$:

$$
R_{\mu\nu a}{}^{a} R^{\mu\nu}{}_{b}{}^{b} = \frac{1}{4} F_{\mu\nu}(Q) F^{\mu\nu}(Q)\,, \qquad \text{where} \qquad F_{\mu\nu}(Q) = 2\partial_{[\mu} Q_{\nu]c}{}^{c}\,. \tag{3.5}
$$

- **N.B.** Due to the presence of $R_{\mu\nu a}{}^{a}$, our action is not a Ricci-based theory. [Afonso, Olmo, Rubiera 2018]

Equation of motion of the connection

$$[\omega]: \quad 0=\frac{2\kappa}{|e|}e_a{}^\nu e^b{}_\sigma\frac{\delta S}{\delta\omega_{\mu a}{}^b}= \quad -T_\sigma{}^{\nu\mu}-2\delta_\sigma^{[\mu}Q_\lambda{}^{\lambda]\nu}+2\delta_\sigma^{[\mu}g^{\rho]\nu}\left(\tfrac{1}{2}Q_{\rho\lambda}{}^\lambda-T_{\rho\lambda}{}^\lambda\right)$$
$$+4\kappa\left[\quad\delta_\sigma^\nu\left(\rho^{-2}\mathring{\nabla}_\lambda F^{\mu\lambda}(Q)-k\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi\right)\quad-\tfrac{\mathrm{i}\hbar}{8}\bar{\psi}\boldsymbol{\gamma}^{\mu\nu}{}_\sigma\psi\right]. \qquad (3.6)$$

Equation of motion of the connection

$$[\omega]: \quad 0 = \frac{2\kappa}{|e|} e_a{}^\nu e^b{}_\sigma \frac{\delta S}{\delta \omega_{\mu a}{}^b} = \quad - T_\sigma{}^{\nu\mu} - 2\delta_\sigma^{[\mu} Q_\lambda{}^{\lambda]\nu} + 2\delta_\sigma^{[\mu} g^{\rho]\nu} \left(\tfrac{1}{2} Q_{\rho\lambda}{}^\lambda - T_{\rho\lambda}{}^\lambda\right)$$
$$+ 4\kappa \left[\quad \delta_\sigma^\nu \left(\rho^{-2} \mathring{\nabla}_\lambda F^{\mu\lambda}(Q) - k\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi\right) \quad - \tfrac{\mathrm{i}\hbar}{8}\bar{\psi}\boldsymbol{\gamma}^{\mu\nu}{}_\sigma\psi\right]. \tag{3.6}$$

- δ_ν^σ-trace

$$\rho^{-2}\mathring{\nabla}_\lambda F^{\mu\lambda}(Q) = k\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi \tag{3.7}$$
$$\Updownarrow_{\rho^{-1}Q_{\mu c}{}^c \equiv A_\mu} \tag{3.8}$$
$$\mathring{\nabla}_\lambda F^{\mu\lambda}(A) = e\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi \qquad \text{(Maxwell equation)}\,. \tag{3.9}$$

$$[\omega]: \quad 0 = \frac{2\kappa}{|e|} e_a{}^\nu e^b{}_\sigma \frac{\delta S}{\delta \omega_{\mu a}{}^b} = \quad - T_\sigma{}^{\nu\mu} - 2\delta_\sigma^{[\mu} Q_\lambda{}^{\lambda]\nu} + 2\delta_\sigma^{[\mu} g^{\rho]\nu} \left(\tfrac{1}{2} Q_{\rho\lambda}{}^\lambda - T_{\rho\lambda}{}^\lambda\right)$$
$$+ 4\kappa \left[\quad \delta_\sigma^\nu \left(\rho^{-2} \mathring{\nabla}_\lambda F^{\mu\lambda}(Q) - k\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi\right) \quad - \tfrac{\mathrm{i}\hbar}{8} \bar{\psi}\boldsymbol{\gamma}^{\mu\nu}{}_\sigma\psi \right] . \qquad (3.6)$$

- δ_ν^σ-trace

$$\rho^{-2} \mathring{\nabla}_\lambda F^{\mu\lambda}(Q) = k\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi \qquad (3.7)$$
$$\Updownarrow_{\rho^{-1} Q_{\mu c}{}^c \equiv A_\mu} \qquad (3.8)$$
$$\mathring{\nabla}_\lambda F^{\mu\lambda}(A) = e\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi \qquad \text{(Maxwell equation)}\,. \qquad (3.9)$$

- **Def.**: $S^{\mu\nu\rho} := \frac{\mathrm{i}\hbar}{4}\bar{\psi}\boldsymbol{\gamma}^{\mu\nu\rho}\psi$.

- Extracting the trace, the equation of motion of the connection becomes:

$$\underbrace{-T_\sigma{}^{\nu\mu} - 2\delta_\sigma^{[\mu} Q_\lambda{}^{\lambda]\nu} + 2\delta_\sigma^{[\mu} g^{\rho]\nu} \left(\tfrac{1}{2} Q_{\rho\lambda}{}^\lambda - T_{\rho\lambda}{}^\lambda\right)}_{\text{Einstein-Hilbert-Palatini}} = 2\kappa \underbrace{S^{\mu\nu}{}_\sigma}_{\text{Antis. hyperm.}} \qquad (3.10)$$

the general solution is:

$$\boxed{\omega_{\mu a}{}^b = \mathring{\omega}_{\mu a}{}^b + \kappa e_a{}^\nu e^b{}_\mu S_{\mu\nu}{}^\rho + V_\mu \delta_a^b}, \qquad \text{where} \qquad V_\mu \equiv \frac{1}{2D} Q_{\mu c}{}^c = \frac{\rho}{2D} A_\mu\,. \qquad (3.11)$$

$$[\omega]: \quad 0 = \frac{2\kappa}{|e|} e_a{}^\nu e^b{}_\sigma \frac{\delta S}{\delta \omega_{\mu a}{}^b} = -T_\sigma{}^{\nu\mu} - 2\delta_\sigma^{[\mu} Q_\lambda{}^{\lambda]\nu} + 2\delta_\sigma^{[\mu} g^{\rho]\nu} \left(\tfrac{1}{2} Q_{\rho\lambda}{}^\lambda - T_{\rho\lambda}{}^\lambda\right) + 4\kappa \left[\delta_\sigma^\nu \left(\rho^{-2} \mathring{\nabla}_\mu F^{\nu\mu}(Q) - k\hbar \bar{\psi} \boldsymbol{\gamma}^\mu \psi\right) - \tfrac{\mathrm{i}\hbar}{8} \bar{\psi} \boldsymbol{\gamma}^{\mu\nu}{}_\sigma \psi\right]. \quad (3.12)$$

- The general solution is:

$$\boxed{\omega_{\mu a}{}^b = \mathring{\omega}_{\mu a}{}^b + \kappa e_a{}^\nu e^b{}_\mu S_{\mu\nu}{}^\rho + V_\mu \delta_a^b}, \qquad \text{where} \qquad \begin{cases} V_\mu & \equiv \frac{1}{2D} Q_{\mu c}{}^c = \frac{\rho}{2D} A_\mu \\ S_{\mu\nu}{}^\rho & := \frac{\mathrm{i}\hbar}{4} \bar{\psi} \boldsymbol{\gamma}_{\mu\nu}{}^\rho \psi \end{cases}. \quad (3.13)$$

Equation of motion of the connection

$$\boxed{\begin{aligned}[\omega]:\quad 0=\frac{2\kappa}{|e|}e_a{}^\nu e^b{}_\sigma\frac{\delta S}{\delta\omega_{\mu a}{}^b} = \quad & -T_\sigma{}^{\nu\mu}-2\delta_\sigma^{[\mu}Q_\lambda{}^{\lambda]\nu}+2\delta_\sigma^{[\mu}g^{\rho]\nu}\left(\tfrac{1}{2}Q_{\rho\lambda}{}^\lambda-T_{\rho\lambda}{}^\lambda\right)\\ & +4\kappa\left[\quad \delta_\sigma^\nu\left(\rho^{-2}\mathring{\nabla}_\mu F^{\nu\mu}(Q)-k\hbar\bar{\psi}\boldsymbol{\gamma}^\mu\psi\right)\quad -\tfrac{\mathrm{i}\hbar}{8}\bar{\psi}\boldsymbol{\gamma}^{\mu\nu}{}_\sigma\psi\right]. \qquad (3.12)\end{aligned}}$$

- The general solution is:

$$\boxed{\omega_{\mu a}{}^b=\mathring{\omega}_{\mu a}{}^b+\kappa e_a{}^\nu e^b{}_\mu S_{\mu\nu}{}^\rho+V_\mu\delta_a^b}, \qquad \text{where} \qquad \left\{\begin{array}{lll} V_\mu & \equiv \frac{1}{2D}Q_{\mu c}{}^c=\frac{\rho}{2D}A_\mu \\ S_{\mu\nu}{}^\rho & :=\frac{\mathrm{i}\hbar}{4}\bar{\psi}\boldsymbol{\gamma}_{\mu\nu}{}^\rho\psi \end{array}\right. . \qquad (3.13)$$

- Compare with the Riemann-Cartan connection obtained as the general solution of *Einstein-Cartan-(Maxwell)-Dirac:*

$$\boxed{\omega^{\mathrm{RC}}{}_{\mu a}{}^b=\mathring{\omega}_{\mu a}{}^b+\kappa\Sigma_{\mu a}{}^b}, \qquad (3.14)$$

where we have introduced the *spin density*:

$$\Sigma^{\mu a}{}_b:=-\frac{1}{|e|}\frac{\delta S_{\mathrm{Dirac}}}{\delta\omega^{\mathrm{RC}}{}_{\mu a}{}^b} \quad = \quad \frac{\mathrm{i}\hbar}{4}\bar{\psi}\boldsymbol{\gamma}^{\mu a}{}_b\psi\,. \qquad (3.15)$$

Their solutions coincide (with the identification $S_{\mu\nu}{}^\rho\leftrightarrow\Sigma_{\mu\nu}{}^\rho$) up to the projective term.

$$[\psi]: \qquad 0 = -\frac{1}{|e|}\frac{\delta S}{\delta \psi} = i\hbar\left(\nabla_\mu - \tfrac{1}{2}Q_{[\mu\sigma]}{}^{\sigma} + \tfrac{1}{2}T_{\mu\sigma}{}^{\sigma}\right)\bar{\psi}\boldsymbol{\gamma}^\mu + m\bar{\psi}\,, \tag{3.16}$$

$$[\bar{\psi}]: \qquad 0 = \frac{1}{|e|}\frac{\delta S}{\delta \bar{\psi}} = i\hbar\boldsymbol{\gamma}^\mu\left(\nabla_\mu - \tfrac{1}{2}Q_{[\mu\sigma]}{}^{\sigma} + \tfrac{1}{2}T_{\mu\sigma}{}^{\sigma}\right)\psi - m\psi\,, \tag{3.17}$$

$$[e]: \qquad 0 = \frac{\kappa}{|e|}e^a{}_\mu g_{\tau\nu}\frac{\delta S}{\delta e^a{}_\tau} = \tfrac{1}{2}\left(R_{\mu\lambda}{}^{\lambda}{}_{\nu} - R_{\mu\nu}\right) + \kappa\left[F_\mu{}^\lambda(A)F_{\nu\lambda}(A) - \tfrac{i\hbar}{2}\left(\bar{\psi}\boldsymbol{\gamma}_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\boldsymbol{\gamma}_\nu\psi\right)\right] + \kappa g_{\mu\nu}\mathcal{L}\,. \tag{3.18}$$

Expanding $\mathcal{L}$:

$$[\bar{\psi}]: \qquad 0 = i\hbar\boldsymbol{\gamma}^\mu\left(\nabla_\mu - \tfrac{1}{2}Q_{[\mu\sigma]}{}^{\sigma} + \tfrac{1}{2}T_{\mu\sigma}{}^{\sigma}\right)\psi - m\psi\,, \tag{3.19}$$

$$\begin{aligned}[e]: \qquad \tfrac{1}{2}\left(R_{\mu\nu} - R_{\mu\lambda}{}^{\lambda}{}_{\nu}\right) - \tfrac{1}{2}g_{\mu\nu}R &= \kappa\left[F_\mu{}^\lambda(A)F_{\nu\lambda}(A) - \tfrac{1}{4}g_{\mu\nu}F^{\lambda\sigma}(A)F_{\lambda\sigma}(A)\right] + \kappa g_{\mu\nu}\mathcal{L}_{\text{Dirac}} \\ &\quad - \kappa\left[\tfrac{i\hbar}{2}\left(\bar{\psi}\boldsymbol{\gamma}_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\boldsymbol{\gamma}_\nu\psi\right)\right]\,.\end{aligned} \tag{3.20}$$

$$[\psi]: \qquad 0 = -\frac{1}{|e|}\frac{\delta S}{\delta \psi} = i\hbar \left(\nabla_\mu - \tfrac{1}{2}Q_{[\mu\sigma]}{}^\sigma + \tfrac{1}{2}T_{\mu\sigma}{}^\sigma\right)\bar{\psi}\boldsymbol{\gamma}^\mu + m\bar{\psi}\,, \tag{3.16}$$

$$[\bar{\psi}]: \qquad 0 = \frac{1}{|e|}\frac{\delta S}{\delta \bar{\psi}} = i\hbar\boldsymbol{\gamma}^\mu \left(\nabla_\mu - \tfrac{1}{2}Q_{[\mu\sigma]}{}^\sigma + \tfrac{1}{2}T_{\mu\sigma}{}^\sigma\right)\psi - m\psi\,, \tag{3.17}$$

$$[e]: \qquad 0 = \frac{\kappa}{|e|}e^a{}_\mu g_{\tau\nu}\frac{\delta S}{\delta e^a{}_\tau} = \tfrac{1}{2}\left(R_{\mu\lambda}{}^\lambda{}_\nu - R_{\mu\nu}\right) + \kappa\left[F_\mu{}^\lambda(A)F_{\nu\lambda}(A) - \tfrac{i\hbar}{2}\left(\bar{\psi}\boldsymbol{\gamma}_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\boldsymbol{\gamma}_\nu\psi\right)\right] + \kappa g_{\mu\nu}\mathcal{L}\,. \tag{3.18}$$

Expanding $\mathcal{L}$:

$$[\bar{\psi}]: \qquad 0 = i\hbar\boldsymbol{\gamma}^\mu \left(\nabla_\mu - \tfrac{1}{2}Q_{[\mu\sigma]}{}^\sigma + \tfrac{1}{2}T_{\mu\sigma}{}^\sigma\right)\psi - m\psi\,, \tag{3.19}$$

$$\begin{aligned}[e]: \qquad \tfrac{1}{2}\left(R_{\mu\nu} - R_{\mu\lambda}{}^\lambda{}_\nu\right) - \tfrac{1}{2}g_{\mu\nu}R &= \kappa\left[F_\mu{}^\lambda(A)F_{\nu\lambda}(A) - \tfrac{1}{4}g_{\mu\nu}F^{\lambda\sigma}(A)F_{\lambda\sigma}(A)\right] + \kappa g_{\mu\nu}\mathcal{L}_{\text{Dirac}} \\ &\quad - \kappa\left[\tfrac{i\hbar}{2}\left(\bar{\psi}\boldsymbol{\gamma}_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\boldsymbol{\gamma}_\nu\psi\right)\right]\,. \end{aligned} \tag{3.20}$$

If we now split the vielbein equation into its symmetric and antisymmetric parts and use the solution of the equation of the connection:

$$[\bar{\psi}]^{\omega\text{ on-shell}}: \qquad \tfrac{i\hbar}{4}\kappa S_{\mu\nu\rho}\boldsymbol{\gamma}^{\mu\nu\rho}\psi = i\hbar\boldsymbol{\gamma}^\mu\left(\mathring{\nabla}_\mu - ieA_\mu\psi\right)\psi - m\psi\,, \tag{3.21}$$

$$\begin{aligned}[e]^{\omega\text{ on-shell}}_{\text{sim}}: \qquad \mathring{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathring{R} &= \tfrac{1}{2}\kappa^2 g_{\mu\nu}S^{\rho\lambda\sigma}S_{\rho\lambda\sigma} + \kappa\left[F_\mu{}^\lambda(A)F_{\nu\lambda}(A) - \tfrac{1}{4}g_{\mu\nu}F^{\lambda\sigma}(A)F_{\lambda\sigma}(A)\right] + \kappa g_{\mu\nu}\mathcal{L}_{\text{Dirac}} \\ &\quad - \tfrac{i\hbar}{2}\kappa\left[\bar{\psi}\boldsymbol{\gamma}_{(\mu}\left(\mathring{\nabla}_{\nu)}\psi - ieA_{\nu)}\psi\right) - \left(\mathring{\nabla}_{(\mu}\bar{\psi} + ieA_{(\mu}\bar{\psi}\right)\boldsymbol{\gamma}_{\nu)}\psi\right]\,, \end{aligned} \tag{3.22}$$

$$[e]^{\omega\text{ on-shell}}_{\text{antis}}: \qquad \mathring{\nabla}_\lambda S_{\mu\nu}{}^\lambda = -\tfrac{i\hbar}{2}\left[\bar{\psi}\boldsymbol{\gamma}_{[\mu}\left(\mathring{\nabla}_{\nu]}\psi - ieA_{\nu]}\psi\right) - \left(\mathring{\nabla}_{[\nu}\bar{\psi} + ieA_{[\nu}\bar{\psi}\right)\boldsymbol{\gamma}_{\mu]}\psi\right]\,. \tag{3.23}$$

Our equations:

$$[\bar{\psi}]^{\omega\text{ on-shell}}: \quad \tfrac{\mathrm{i}\hbar}{4}\kappa S_{\mu\nu\rho}\boldsymbol{\gamma}^{\mu\nu\rho}\psi = \mathrm{i}\hbar\boldsymbol{\gamma}^{\mu}\left(\mathring{\nabla}_{\mu} - \mathrm{i}eA_{\mu}\psi\right)\psi - m\psi\,, \tag{3.24}$$

$$\begin{aligned}[e]^{\omega\text{ on-shell}}_{\text{sim}}: \quad \mathring{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathring{R} = {} & \tfrac{1}{2}\kappa^2 g_{\mu\nu}S^{\rho\lambda\sigma}S_{\rho\lambda\sigma} + \kappa\left[F_{\mu}{}^{\lambda}(A)F_{\nu\lambda}(A) - \tfrac{1}{4}g_{\mu\nu}F^{\lambda\sigma}(A)F_{\lambda\sigma}(A)\right] \\ & + \kappa\left\{-\tfrac{\mathrm{i}\hbar}{2}\left[\bar{\psi}\boldsymbol{\gamma}_{(\mu}\left(\mathring{\nabla}_{\nu)}\psi - \mathrm{i}eA_{\nu)}\psi\right) - \left(\mathring{\nabla}_{(\mu}\bar{\psi} + \mathrm{i}eA_{(\mu}\bar{\psi}\right)\boldsymbol{\gamma}_{\nu)}\psi\right] + g_{\mu\nu}\mathcal{L}_{\text{Dirac}}\right\}\,,\end{aligned} \tag{3.25}$$

$$[e]^{\omega\text{ on-shell}}_{\text{antis}}: \quad \mathring{\nabla}_{\lambda}S_{\mu\nu}{}^{\lambda} = -\tfrac{\mathrm{i}\hbar}{2}\left[\bar{\psi}\boldsymbol{\gamma}_{[\mu}\left(\mathring{\nabla}_{\nu]}\psi - \mathrm{i}eA_{\nu]}\psi\right) - \left(\mathring{\nabla}_{[\nu}\bar{\psi} + \mathrm{i}eA_{[\nu}\bar{\psi}\right)\boldsymbol{\gamma}_{\mu]}\psi\right]\,. \tag{3.26}$$

Our equations:

$$[\bar{\psi}]^{\omega \text{ on-shell}} : \quad \tfrac{i\hbar}{4}\kappa S_{\mu\nu\rho}\boldsymbol{\gamma}^{\mu\nu\rho}\psi = i\hbar\boldsymbol{\gamma}^{\mu}\left(\mathring{\nabla}_{\mu} - ieA_{\mu}\psi\right)\psi - m\psi\,, \tag{3.24}$$

$$\begin{aligned}[e]^{\omega \text{ on-shell}}_{\text{sim}} : \quad \mathring{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathring{R} = {}& \tfrac{1}{2}\kappa^2 g_{\mu\nu}S^{\rho\lambda\sigma}S_{\rho\lambda\sigma} + \kappa\left[F_{\mu}{}^{\lambda}(A)F_{\nu\lambda}(A) - \tfrac{1}{4}g_{\mu\nu}F^{\lambda\sigma}(A)F_{\lambda\sigma}(A)\right] \\ &+ \kappa\left\{-\tfrac{i\hbar}{2}\left[\bar{\psi}\boldsymbol{\gamma}_{(\mu}\left(\mathring{\nabla}_{\nu)}\psi - ieA_{\nu)}\psi\right) - \left(\mathring{\nabla}_{(\mu}\bar{\psi} + ieA_{(\mu}\bar{\psi}\right)\boldsymbol{\gamma}_{\nu)}\psi\right] + g_{\mu\nu}\mathcal{L}_{\text{Dirac}}\right\}\,,\end{aligned} \tag{3.25}$$

$$[e]^{\omega \text{ on-shell}}_{\text{antis}} : \quad \mathring{\nabla}_{\lambda}S_{\mu\nu}{}^{\lambda} = -\tfrac{i\hbar}{2}\left[\bar{\psi}\boldsymbol{\gamma}_{[\mu}\left(\mathring{\nabla}_{\nu]}\psi - ieA_{\nu]}\psi\right) - \left(\mathring{\nabla}_{[\nu}\bar{\psi} + ieA_{[\nu}\bar{\psi}\right)\boldsymbol{\gamma}_{\mu]}\psi\right]\,. \tag{3.26}$$

Compare these results with the equations of motion of *Einstein-Cartan-Maxwell-Dirac*:

$$[\bar{\psi}]^{\omega^{\text{RC}} \text{ on-shell}} : \quad \tfrac{i\hbar}{4}\kappa\Sigma_{\mu\nu\rho}\boldsymbol{\gamma}^{\mu\nu\rho}\psi = i\hbar\boldsymbol{\gamma}^{\mu}\left(\mathring{\nabla}_{\mu} - ieA_{\mu}\psi\right)\psi - m\psi\,, \tag{3.27}$$

$$\begin{aligned}[e]^{\omega^{\text{RC}} \text{ on-shell}}_{\text{sim}} : \quad \mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = {}& \kappa^2\left\{\tfrac{1}{2}g_{\mu\nu}\Sigma^{\rho\lambda\sigma}\Sigma_{\rho\lambda\sigma} - \Sigma_{\mu\rho\sigma}\Sigma_{\nu}{}^{\rho\sigma}\right\} \\ &+ \kappa\left\{\mathcal{T}^{\text{em}}{}_{\mu\nu}(F) + \mathcal{T}^{\psi}{}_{(\mu\nu)}(\psi,\, \nabla\psi)\right\}\end{aligned} \tag{3.28}$$

$$\begin{aligned}= {}& \tfrac{1}{2}\kappa^2 g_{\mu\nu}\Sigma^{\rho\lambda\sigma}\Sigma_{\rho\lambda\sigma} \\ &+ \kappa\left\{\mathcal{T}^{\text{em}}{}_{\mu\nu}(F) + \mathcal{T}^{\psi}{}_{(\mu\nu)}\left(\psi,\, (\mathring{\nabla} - ieA)\psi\right)\right\}\,,\end{aligned} \tag{3.29}$$

$$[e]^{\omega^{\text{RC}} \text{ on-shell}}_{\text{antis}} : \quad \mathring{\nabla}_{\lambda}\Sigma_{\mu\nu}{}^{\lambda} = \mathcal{T}^{\psi}{}_{[\mu\nu]}\,. \tag{3.30}$$

where

$$\mathcal{T}^{\text{em}}{}_{\mu}{}^{\nu} := \tfrac{1}{|e|}e^{a}{}_{\mu}\frac{\delta S_{\text{Maxwell}}}{\delta e^{a}{}_{\nu}}\,, \qquad \mathcal{T}^{\psi}{}_{\mu}{}^{\nu} := \tfrac{1}{|e|}e^{a}{}_{\mu}\frac{\delta S_{\text{Dirac}}}{\delta e^{a}{}_{\nu}} \qquad \text{and} \qquad \Sigma^{\mu\nu}{}_{\rho} := -\tfrac{1}{|e|}e_{a}{}^{\nu}e^{b}{}_{\rho}\frac{\delta S_{\text{Dirac}}}{\delta \omega^{\text{RC}}{}_{\mu a}{}^{b}}\,. \tag{3.31}$$

- **Conclusion.**
  Our approach must be equivalent (at least with ω on-shell) to Einstein-Cartan-Maxwell-Dirac theory, with S playing the role of the *spin density* Σ.

4. Final discussion

Action

$$S[e,\,\omega,\,\psi]=\int \mathrm{d}^D x\,|e|\left\{\frac{1}{2\kappa}R(\omega)-\frac{1}{4\rho^2}F_{\mu\nu}(Q)F^{\mu\nu}(Q)+\mathcal{L}_{\text{Dirac}}\right\}\,. \tag{4.1}$$

It is invariant under combined (integrable) projective transformations of the affine structure and $\mathrm{U}(1)$ transformations of the spinor field:

$$\omega_{\mu a}{}^{b}\to\omega_{\mu a}{}^{b}+\frac{\rho}{2D}\partial_\mu\Lambda\delta_a^b \qquad\Rightarrow\qquad Q_{\mu c}{}^{c}\to Q_{\mu c}{}^{c}+\rho\partial_\mu\Lambda\,, \tag{4.2}$$

$$\psi\to\mathrm{e}^{\mathrm{i}e\Lambda}\psi\,. \tag{4.3}$$

Action

$$S[e,\,\omega,\,\psi]=\int \mathrm{d}^D x\,|e|\left\{\frac{1}{2\kappa}R(\omega)-\frac{1}{4\rho^2}F_{\mu\nu}(Q)F^{\mu\nu}(Q)+\mathcal{L}_{\text{Dirac}}\right\}\,. \tag{4.1}$$

It is invariant under combined (integrable) projective transformations of the affine structure and $\mathrm{U}(1)$ transformations of the spinor field:

$$\omega_{\mu a}{}^{b}\to\omega_{\mu a}{}^{b}+\frac{\rho}{2D}\partial_\mu\Lambda\delta_a^b \qquad \Rightarrow \qquad Q_{\mu c}{}^{c}\to Q_{\mu c}{}^{c}+\rho\partial_\mu\Lambda\,, \tag{4.2}$$

$$\psi\to \mathrm{e}^{\mathrm{i}e\Lambda}\psi\,. \tag{4.3}$$

....

Consider the decomposition of the general connection $\omega_{\mu a}{}^{b}$ (where $\omega_{\mu[ab]}\equiv\bar{\omega}_{\mu ab}$):

$$\omega_{\mu a}{}^{b}=\bar{\omega}_{\mu a}{}^{b}+\tfrac{1}{2}Q_{\mu a}{}^{b} \tag{4.4}$$

$$=\bar{\omega}_{\mu a}{}^{b}+V_\mu\delta_a^b+\tilde{Q}_{\mu a}{}^{b} \qquad \left(\tilde{Q}_{\mu c}{}^{c}=0\right)\,. \tag{4.5}$$

If we then separate the degrees of freedom $\omega_{\mu a}{}^{b}$ into the equivalent set $\left\{\bar{\omega}_{\mu a}{}^{b},\,V_\mu,\,\tilde{Q}_{\mu a}{}^{b}\right\}$:

Action

$$S[e,\,\omega,\,\psi]=\int \mathrm{d}^D x\,|e|\left\{\frac{1}{2\kappa}R(\omega)-\frac{1}{4\rho^2}F_{\mu\nu}(Q)F^{\mu\nu}(Q)+\mathcal{L}_{\text{Dirac}}\right\}\,. \tag{4.1}$$

It is invariant under combined (integrable) projective transformations of the affine structure and $\mathrm{U}(1)$ transformations of the spinor field:

$$\omega_{\mu a}{}^{b}\to\omega_{\mu a}{}^{b}+\frac{\rho}{2D}\partial_\mu\Lambda\delta_a^b \qquad \Rightarrow \qquad Q_{\mu c}{}^{c}\to Q_{\mu c}{}^{c}+\rho\partial_\mu\Lambda\,, \tag{4.2}$$

$$\psi\to \mathrm{e}^{\mathrm{i}e\Lambda}\psi\,. \tag{4.3}$$

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Consider the decomposition of the general connection $\omega_{\mu a}{}^{b}$ (where $\omega_{\mu[ab]}\equiv\bar{\omega}_{\mu ab}$):

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$$R(\omega)=\bar{R}(\bar{\omega})+\tilde{Q}_{\mu\nu\rho}\tilde{Q}^{\nu\mu\rho}-\tilde{Q}_{\sigma}{}^{\sigma\lambda}\tilde{Q}^{\tau}{}_{\tau\lambda}\,. \tag{4.6}$$

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$$\tilde{Q}_{\mu a}{}^{b}\to\tilde{Q}_{\mu a}{}^{b} \qquad \text{(invariant)}\,, \tag{4.8}$$

$$V_\mu\to V_\mu+\frac{\rho}{2D}\partial_\mu\Lambda\,, \tag{4.9}$$

$$\psi\to\mathrm{e}^{\mathrm{i}e\Lambda}\psi\,. \tag{4.10}$$

Separating the degrees of freedom $\omega_{\mu a}{}^{b}$ into the equivalent set $\left\{\bar{\omega}_{\mu a}{}^{b},\, V_{\mu},\, \tilde{Q}_{\mu a}{}^{b}\right\}$:

- The action becomes:

$$S[e,\, \bar{\omega},\, V,\, \tilde{Q},\, \psi] = \int \mathrm{d}^{D}x\, |e| \left\{ \tfrac{1}{2\kappa} \bar{R}(\bar{\omega}) + \tfrac{1}{2\kappa} \left(\tilde{Q}_{\mu\nu\rho} \tilde{Q}^{\nu\mu\rho} - \tilde{Q}_{\sigma}{}^{\sigma\lambda} \tilde{Q}^{\tau}{}_{\tau\lambda} \right) - \tfrac{1}{4\rho^{2}} F_{\mu\nu}(Q) F^{\mu\nu}(Q) + \mathcal{L}_{\mathrm{Dirac}} \right\} . \tag{4.11}$$

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Taking $\tilde{Q}_{\mu\nu\rho}$ on-shell, the gravitational part of the action is reduced to the EC gravity coupled to the usual Maxwell field:

$$S|_{\tilde{Q}\text{ on-shell}} \equiv \quad \hat{S}[e,\,\bar{\omega},\,V,\,\psi] = \int \mathrm{d}^{D}x\,|e|\left\{\tfrac{1}{2\kappa}\bar{R}(\bar{\omega}) - \tfrac{1}{4}F_{\mu\nu}(A)F^{\mu\nu}(A) + \mathcal{L}_{\text{Dirac}}\right\}\,. \tag{4.12}$$

We are reinterpreting the projective transformation of $\omega_{\mu ab}$, which does not affect $\bar{\omega}_{\mu ab}$, as a $\mathrm{U}(1)$ transformation of

$$A_{\mu} \quad := \quad \rho^{-1}Q_{\mu c}{}^{c} \quad = \quad 2D\rho^{-1}V_{\mu}\,. \tag{4.13}$$

In EC-Maxwell-Dirac theory there is a way to...

- ... interpret the electromagnetic potential and the "physical" connection $\bar{\omega}$ as parts of one "fundamental" connection $\omega_{\mu ab}$,
- ... and encode the $\mathrm{U}(1)$ transformation within the projective symmetry of the generalized theory. For this purpose, the theory needs a complex extension of the usual Lorentzian (Riemann-Cartan) connection for spinors.

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Weyl's dream?

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Weyl’s dream?

Thanks for your attention!

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Lorentz derivative acting on vectors

- Generators of the Lorentz group in spinor and vector representation:

$$M^{(\mathrm{s})ab} = \frac{1}{2}\gamma^{ab}\,, \qquad \left(M^{(\mathrm{v})ab}\right)^{c}{}_{d} = -2\delta_d^{[a}\eta^{b]c}$$

- Lorentzian derivative acting on vectors:

$$\begin{aligned}
\nabla^{\mathrm{Lor}}{}_{\mu}V^c &= \partial_\mu V^c - \frac{1}{2}\omega_{\mu ab}\left(M^{(\mathrm{v})ab}\right)^{c}{}_{d}V^d \\
&= \partial_\mu V^c + \omega_{\mu[db]}\eta^{bc}V^d \\
&= \partial_\mu V^c + \bar{\omega}_{\mu d}{}^{c}V^d \\
&= \partial_\mu V^c + \omega_{\mu d}{}^{c}V^d - \frac{1}{2}Q_{\mu d}{}^{c}V^d \\
&= \nabla_\mu^{(\omega)}V^c - \frac{1}{2}Q_{\mu d}{}^{c}V^d
\end{aligned}$$

where $\nabla_\mu^{(\omega)}V^c$ is the usual general spacetime covariant derivative ($\mathrm{GL}(D)$).

Kosmann lift

- Generalized Lie derivative of a spinor through the Kosmann lift:

$$\mathfrak{L}_{\boldsymbol{k}}\psi = k^\mu \mathring{\nabla}_\mu \psi - \mathring{\nabla}_\mu k_\nu \boldsymbol{\gamma}^\mu \boldsymbol{\gamma}^\nu \psi$$

where $\boldsymbol{k}$ is a Killing vector field.

[Hurley, Vandyck 1994]

- The symmetric part in

$$\begin{aligned}\omega_{\mu ab} &= \mathring{\omega}_{\mu ab} + \frac{1}{2}e_a{}^\nu e_b{}^\rho \left(T_{\mu\nu\rho} + T_{\rho\mu\nu} - T_{\nu\rho\mu}\right) + \frac{1}{2}e_a{}^\nu e_b{}^\rho \left(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}\right) & (5.1)\\ &= \underbrace{\mathring{\omega}_{\mu ab} + \frac{1}{2}e_{[a}{}^\nu e_{b]}{}^\rho \left(2T_{\mu\nu\rho} + T_{\rho\nu\mu} + 2Q_{\nu\mu\rho}\right)}_{\omega_{\mu[ab]}} + \underbrace{\frac{1}{2}Q_{\mu ab}}_{\omega_{\mu(ab)}} & (5.2)\end{aligned}$$

  cannot be included in the Lorentz connection.
- Hurley and Vandyck showed that the trace of $\omega_{\mu(ab)}$ (i.e. $Q_{\mu c}{}^c$), can be consistently lifted the bundle of spin frames.
- Considering

$$\nabla^{(2)}_\mu \psi \quad = \quad \nabla^{\mathrm{Lor}}{}_\mu \psi + k Q_{\mu c}{}^c \psi\,, \qquad k \in \mathbb{R}\,, \tag{5.3}$$

  we are extending the Lie algebra $\mathfrak{so}(1, D-1)$.
- The value of k is not fixed by the lift.

$$
\begin{array}{rll}
 & (c=1) & (c,\,\hbar=1) \\
[\Gamma_{\mu\nu}{}^{\rho}] = \left[\omega_{\mu a}{}^{b}\right] = [Q_{\mu ab}] = \left[T_{\mu\nu}{}^{b}\right] & = \mathrm{L}^{-1} & = \mathrm{M} \\
\left[R_{\mu\nu a}{}^{b}\right] & = \mathrm{L}^{-2} & = \mathrm{M}^{2} \\
[\kappa] & = \mathrm{M}^{-1}\mathrm{L}^{D-3} & = \mathrm{M}^{-(D-2)} \\
[k]\ _{\text{(eq.(3.2))}} & = 1 & = 1 \\
[e] = \left[\rho^{2}\right] & = \mathrm{M}^{-1}\mathrm{L}^{D-5} & = \mathrm{M}^{-(D-4)} \\
[\hbar] & = \mathrm{ML} & = 1 \\
[\psi] & = \mathrm{L}^{-(D-1)/2} & = \mathrm{M}^{(D-1)/2}
\end{array}
$$