# On the solutions of the Einstein-Hilbert and Gauss-Bonnet metric-affine Lagrangians


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**Abstract**

It is a known result that the $2k$-dimensional Lovelock Lagrangian of order $k$ is a total derivative in the Levi-Civita (metric) and in the metric-compatible (Riemann-Cartan) cases. We are indeed interested in figuring out the case with non-metricity, i.e. with a completely general connection. In this work, we focus on the first and second order of the Lovelock expansion in the metric-affne formalism, proving that the non-metricity prevents the theory from being a total derivative. We also provide a few general conclusions for the critical Lovelock Lagrangian with arbitrary $k$.


## Formalism, notation and basic objects

Three fundamental (independent) objects in metric-affine formalism:

- **Coframe**. Basis of the cotangent space in each point

$$\boldsymbol{\vartheta}^a = e_\mu{}^a \mathrm{d}x^\mu \qquad (\text{dual to} \quad \boldsymbol{e}_a = e^\mu{}_a \boldsymbol{\partial}_\mu)\,. \tag{1}$$

- **Metric**. Components of the metric in the arbitrary basis:

$$g_{ab} = e^\mu{}_a e^\nu{}_b\, g_{\mu\nu}\,. \tag{2}$$

Associated objects:

- Hodge duality between $k$-forms and $(D-k)$-forms $\rightsquigarrow \star$
- Canonical volume form:

$$\star 1 := \sqrt{|g|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^D \;\equiv\; \frac{1}{D!}\mathcal{E}_{a_1\ldots a_D}\boldsymbol{\vartheta}^{a_1\ldots a_D}\,, \qquad |g| \equiv |\det(g_{\mu\nu})|\,. \tag{3}$$

- **Connection 1-form**

$$\boldsymbol{\omega}_a{}^b = \omega_{\mu a}{}^b \mathrm{d}x^\mu\,, \qquad \text{where} \qquad \omega_{\mu a}{}^b := e^\nu{}_a e_\lambda{}^b\, \Gamma_{\mu\nu}{}^\lambda + e_\sigma{}^b\, \partial_\mu e^\sigma{}_a\,. \tag{4}$$

Associated objects

- Exterior covariant derivative (of algebra-valued forms)

$$\mathbf{D}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} = \mathrm{d}\boldsymbol{\alpha}_{a\ldots}{}^{b\ldots} + \quad \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\alpha}_{a\ldots}{}^{c\ldots} + \ldots \quad - \boldsymbol{\omega}_a{}^c \wedge \boldsymbol{\alpha}_{c\ldots}{}^{b\ldots} - \ldots \quad , \tag{5}$$

- Curvature, torsion and non-metricity forms:

$$\boldsymbol{R}_a{}^b := \mathrm{d}\boldsymbol{\omega}_a{}^b + \boldsymbol{\omega}_c{}^b \wedge \boldsymbol{\omega}_a{}^c \qquad = \left(\tfrac{1}{2}\, R_{\mu\nu\rho}{}^\lambda\, x^\mu \wedge \mathrm{d}x^\nu\right) e^\rho{}_a e_\lambda{}^b\,, \tag{6}$$

$$\boldsymbol{T}^a := \mathbf{D}\boldsymbol{\vartheta}^a \qquad = \left(\tfrac{1}{2}\, T_{\mu\nu}{}^\lambda\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) e_\lambda{}^a\,, \tag{7}$$

$$\boldsymbol{Q}_{ab} := -\mathbf{D}g_{ab} \qquad = \left(Q_{\mu\nu\rho}\, \mathrm{d}x^\mu\right) e^\nu{}_a e^\rho{}_b\,. \tag{8}$$

where

$$R_{\mu\nu\lambda}{}^\rho := \partial_\mu \Gamma_{\nu\lambda}{}^\rho - \partial_\nu \Gamma_{\mu\lambda}{}^\rho + \Gamma_{\mu\sigma}{}^\rho\, \Gamma_{\nu\lambda}{}^\sigma - \Gamma_{\nu\sigma}{}^\rho\, \Gamma_{\mu\lambda}{}^\sigma\,,$$
$$T_{\mu\nu}{}^\rho := \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho \qquad\qquad Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}\,. \tag{9}$$

*More notation*:
* Levi-Civita: $\mathring{\boldsymbol{\omega}}_a{}^b$, $\mathring{\boldsymbol{R}}_a{}^b$.
* Contractions of the curvature:

$$R^{(1)}{}_{\mu\nu} := R_{\mu\lambda\nu}{}^\lambda\,, \qquad R := g^{\mu\nu} R^{(1)}{}_{\mu\nu}\,, \qquad R^{(2)}{}_\mu{}^\nu := g^{\lambda\sigma} R_{\mu\lambda\sigma}{}^\nu\,. \tag{10}$$

## Metric-affine Lovelock gravity

- **Def.** $D$-dimensional (metric-affine) Lovelock term of order $k$:

$$\boldsymbol{L}_k^{(D)} = \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{2k-1}a_{2k}} \wedge \star(\boldsymbol{\vartheta}_{a_1} \wedge \ldots \wedge \boldsymbol{\vartheta}_{a_{2k}}) \tag{11}$$

$$= \frac{(2k)!}{2^k}\, \mathrm{sgn}(g)\; \delta^{[b_1}_{a_1} \ldots \delta^{b_{2k}]}_{a_{2k}}\; R_{b_1 b_2}{}^{a_1 a_2} \ldots R_{b_{2k-1}b_{2k}}{}^{a_{2k-1}a_{2k}}\, \sqrt{|g|}\; \mathrm{d}^D x\,, \tag{12}$$

- **General properties**
  - Levi-Civita is a solution of the Palatini formalism EoM.
  - Projective symmetry ($\boldsymbol{A} = A_\mu \mathrm{d}x^\mu$ arbitrary):

$$\boldsymbol{\omega}_a{}^b \to \boldsymbol{\omega}_a{}^b + \boldsymbol{A}\,\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho \to \Gamma_{\mu\nu}{}^\rho + A_\mu\, \delta_\nu^\rho\,. \tag{13}$$

  - In $D = 2k$ (critical dim.), the Riemann-Cartan and the metric cases are topological.

## Metric-affine Einstein Lagrangian

- **Einstein Lagrangian** (arbitrary dimension) (We drop the factor $(2\kappa)^{-1}$)

$$\boldsymbol{L}_1^{(D)} = \boldsymbol{R}_a{}^b \wedge \star(\boldsymbol{\vartheta}^a \wedge \boldsymbol{\vartheta}_b) = \mathrm{sgn}(g)\; e^\nu{}_b\, e^\mu{}_c\, g^{ca}\, R_{\mu\nu a}{}^b(\boldsymbol{\omega})\sqrt{|g|}\; \mathrm{d}^D x\,, \tag{14}$$

- In $D > 2$, the solution of the EoM of the connection is:

$$\boldsymbol{\omega}_a{}^b = \mathring{\boldsymbol{\omega}}_a{}^b + \boldsymbol{A}\delta_a^b \qquad \Leftrightarrow \qquad \Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu \delta_\nu^\rho\,. \tag{15}$$

Unphysical projective mode $\to$ can be eliminated using a symmetry of the theory.

- **Critical dimension** $D = 2$.
  - Lagrangian (we extract the Riemann-Cartan part which is exact)

$$\boldsymbol{L}_1^{(2)} \;=\; \mathrm{d}(\ldots) - \tfrac{1}{4}\mathcal{E}^a{}_b\, \check{\boldsymbol{Q}}_a{}^c \wedge \check{\boldsymbol{Q}}_c{}^b \qquad\qquad \text{where} \quad \check{\boldsymbol{Q}}_{ab} = \boldsymbol{Q}_{ab} - \frac{1}{2}g_{ab}\boldsymbol{Q}_c{}^c\,. \tag{16}$$

  - Equations of motion

$$0 = \mathbf{D}\mathcal{E}^a{}_b = -\check{\boldsymbol{Q}}^{ca}\mathcal{E}_{bc}\,. \tag{17}$$

Therefore the general solution is one that verifies:

$$\boxed{\check{\boldsymbol{Q}}_{ab} = 0}\,. \tag{18}$$

  - And this is not an identity $\rightsquigarrow$ 4 conditions over the 8 d.o.f. of the connection:

| Tensor | d.o.f. in $D$ dim. | d.o.f. in 2 dim. | Condition imposed by EoM |
|---|---|---|---|
| $T_{\mu\nu}{}^\rho$ | $\frac{1}{2}D^2(D-1)$ | 2 (pure trace) | [None] |
| $Q_{\mu\lambda}{}^\lambda$ | $D$ | 2 | [None]* |
| $\check{Q}_{\mu\nu\rho}$ | $\frac{1}{2}D(D+2)(D-1)$ | 4 | They are zero |

**Table 1:** d.o.f. of $\Gamma$ and their conditions for metric-affine Einstein gravity

* The trace of the non-metricity is undetermined in any $D$ due to proj. symmetry.

## Metric-affine Gauss-Bonnet Lagrangian

- **Gauss-Bonnet Lagrangian** (arbitrary dimension)

$$\boldsymbol{L}_2^{(D)} = \boldsymbol{R}_a{}^b \wedge \boldsymbol{R}_c{}^d \wedge \star(\boldsymbol{\vartheta}^a \wedge \boldsymbol{\vartheta}_b \wedge \boldsymbol{\vartheta}^c \wedge \boldsymbol{\vartheta}_d) \tag{19}$$

$$= \mathrm{sgn}(g)\left[R^2 - R^{(1)}{}_{\mu\nu}R^{(1)\nu\mu} + 2R^{(1)}{}_{\mu\nu}R^{(2)\nu\mu} - R^{(2)}{}_{\mu\nu}R^{(2)\nu\mu} + R_{\mu\nu\rho\lambda}R^{\rho\lambda\mu\nu}\right]\sqrt{|g|}\mathrm{d}^D x\,, \tag{20}$$

- In arbitrary $D$, the general solution is not known (up to the unphysical projective mode).
- **Critical dimension** $D = 4$.
  - Lagrangian (we extract the Riemann-Cartan part which is exact)

$$\boldsymbol{L}_2^{(4)} = \mathrm{d}(\ldots) \;-\; \tfrac{1}{4}\mathcal{E}_{abcd}\left[2\bar{\boldsymbol{R}}^{ab} \wedge \check{\boldsymbol{Q}}_e{}^c \wedge \check{\boldsymbol{Q}}^{de} - \tfrac{1}{4}\check{\boldsymbol{Q}}_e{}^a \wedge \check{\boldsymbol{Q}}^{be} \wedge \check{\boldsymbol{Q}}_f{}^c \wedge \check{\boldsymbol{Q}}^{df}\right]\,. \tag{21}$$

  - Particular solution. The Ansatz

$$\Gamma_{\mu\nu}{}^\rho = \mathring{\Gamma}_{\mu\nu}{}^\rho + A_\mu\delta_\nu^\rho + B_\nu\delta_\mu^\rho - C^\rho g_{\mu\nu}\,. \tag{22}$$

is a solution for the critical case $D = 4$ as long as $B_\mu = C_\mu$.

The transformation that connects it to Levi-Civita,

$$\Phi\;:\quad \Gamma_{\mu\nu}{}^\rho \longmapsto \Gamma_{\mu\nu}{}^\rho + B_\nu\delta_\mu^\rho - B^\rho g_{\mu\nu}\,, \qquad \Leftrightarrow \qquad \Phi\;:\; \begin{cases} T_{\mu\nu}{}^\rho \longmapsto T_{\mu\nu}{}^\rho - 2B_{[\mu}\delta^\rho_{\nu]} \\ Q_{\mu\nu\rho} \longmapsto Q_{\mu\nu\rho} \end{cases} \tag{23}$$

is not a symmetry of the theory (due to the non-vanishing $Q_{\mu\nu\sigma}$):

$$\begin{aligned}\delta_\Phi \mathcal{L}_2^{(4)} = &-4B^\mu B^\nu g^{\rho\lambda}\left[2\nabla_{(\mu}Q_{\rho)\nu\lambda} + T_{\mu\rho}{}^\sigma Q_{\sigma\nu\lambda}\right] \\ &- 2Q^{\mu\nu\rho}\left[B_\mu(R^{(1)}{}_{\nu\rho} + R^{(2)}{}_{\nu\rho}) + B^\lambda(R_{\lambda\nu\mu\rho} + R_{\lambda\nu\rho\mu}) + \ldots\right] \\ &- 2Q^{\mu\sigma}{}_\sigma\left[B^\nu(R^{(1)}{}_{\nu\mu} - R^{(2)}{}_{\nu\mu} - g_{\nu\mu}R) - 2B_\mu B_\nu B^\nu + \ldots\right] \\ &+ 2Q_\sigma{}^{\sigma\mu}\left[B^\nu(R^{(1)}{}_{\nu\mu} + R^{(2)}{}_{\nu\mu}) + 2B_\mu\nabla_\nu B^\nu + 2B^\nu\nabla_\nu B_\mu + 2B_\mu B^\nu T_{\nu\lambda}{}^\lambda\right]. \end{aligned} \tag{24}$$

  - In addition, configurations that do not verify the equations of motion can be given. For these reasons, the theory cannot be a total derivative.

## General metric-affine Lovelock Lagrangian in $D = 2k$ (critical)

- Lovelock theory in critical dimension:

$$\boldsymbol{L}_{D/2}^{(D)} = \mathcal{E}^{a_1}{}_{a_2\ldots}{}^{a_{D-1}}{}_{a_D}\boldsymbol{R}_{a_1}{}^{a_2} \wedge \ldots \wedge \boldsymbol{R}_{a_{D-1}}{}^{a_D}\,. \tag{25}$$

- Equation of motion for the connection:

$$0 = \left[\check{\boldsymbol{Q}}^c{}_{a_1}\,\mathcal{E}_{ca_2\ldots a_{D-2}ab} + \ldots + \check{\boldsymbol{Q}}^c{}_{a_{D-3}}\,\mathcal{E}_{a_1\ldots a_{D-4}ca_{D-2}ab} + \check{\boldsymbol{Q}}^c{}_a\,\mathcal{E}_{a_1\ldots a_{D-2}cb}\right] \wedge \boldsymbol{R}^{a_1 a_2} \wedge \ldots \wedge \boldsymbol{R}^{a_{D-3}a_{D-2}}\,. \tag{26}$$

- Families of solutions:
  - for all $k$: connection with $Q_{\mu\nu\rho} = V_\mu g_{\nu\rho}$ (i.e. $\check{\boldsymbol{Q}}_{ab} = 0$).
  - for $k > 1$: teleparallel (i.e. $\boldsymbol{R}_c{}^d = 0$), any connection such that $\check{\boldsymbol{Q}}_{ab} \wedge \boldsymbol{R}_c{}^d = 0$.
  - for $k > 2$: any connection such that $\boldsymbol{R}_{ab} = \boldsymbol{\alpha}_{ab} \wedge \boldsymbol{k}$ for certain 1-forms $\boldsymbol{\alpha}_{ab}$ and $\boldsymbol{k}$.

## Conclusions and forthcoming research

- **Conclusions**
  - Metric-affine Einstein gravity in critical dimension is not topological (the Lagrangian form is not exact), since the dynamics is not identically satisfied.
  - In metric-affine Gauss-Bonnet gravity, we have non-trivial solutions that are not connected with Levi-Civita through a symmetry. Indeed, it is possible to find configurations of the fields that violate the equations of motion. Therefore, the theory is not topological either.
  - (Conjecture) In arbitrary critical dimension the presence of non-metricity spoils the topological character of the Lovelock action.
- **Future work**
  - Analysis of the equation of the metric/coframe.
  - Is there an easy (systematic) way to solve the EoM of the connection for any $k$?
  - Which is the role of the non-metricity in breaking the triviality in critical dimension?


## Acknowledgements

Thanks to José Beltrán Jiménez, Tomi Koivisto, Gonzalo Olmo, Miguel Sánchez and Jorge Zanelli for useful discussions. Funding: Spanish Ministry of Economy and Competitiveness (FIS2016-78198-P), Junta de Andalucía (FQM101) and the Unidad de Excelencia UCE-PP2016-02 (UGR); A.J.C. is supported by the PhD contract FPU15/02864 (MEC); and J.A.O. by a PhD contract of the Plan Propio (UGR). We used xAct to check our calculations.


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