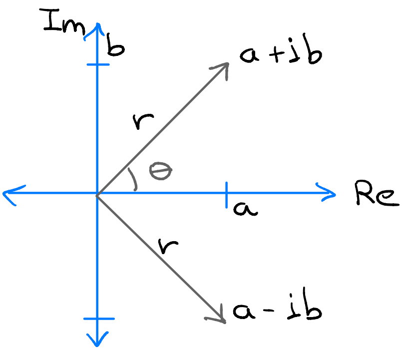
Matemática para la computación I

Matematical Background

$i = \sqrt{-1}$, $i^2 = -1$

Complex number

$z = x + iy$, complex conjugate $z^* = x - iy$



is a reflection

Addition and Substraction

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Multiplication

$$(a + ib)(c + id) = (ac + bd) + i(bc + ad)$$

$$(a + ib)(a - ib) = a^2 + b^2$$

Division

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id}$$

Modulus

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$

Trigonomery

$$e^{i\theta} = \cos\theta + i\,\mathrm{sen}\,\theta \quad , \quad e^{-i\theta} = \cos\theta - i\,\mathrm{sen}\,\theta$$

$$e^{i\pi} = -1$$

$z = x + iy \Rightarrow$ can be transformed to a polar coordinates via

$x = r\cos\theta \quad , \quad y = r\,\mathrm{sen}\,\theta$

$$\boxed{r = \sqrt{x^2 + y^2}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

if $x = \varnothing$:

- $y > \varnothing \rightarrow \frac{\pi}{2}$
- $y < \varnothing \rightarrow -\frac{\pi}{2}$

$$z = r(\cos\theta + i\sin\theta)$$

$$= r\,\mathrm{cis}\,\theta$$

Trigonometric functions are periodic:

$$\cos\theta = \cos(\theta + 2\pi)$$
$$\mathrm{sen}\,\theta = \mathrm{sen}\,(\theta + 2\pi)$$

x	y	Quadrant	range
+	+	I	$\varnothing < \theta < \frac{\pi}{2}$
−	+	II	$\frac{\pi}{2} < \theta < \pi$
−	−	III	$\pi < \theta < \frac{3\pi}{2}$
+	−	IV	$\frac{3\pi}{2} < \theta < 2\pi$ or negative

The argument of a complex number Z is the angle θ that the complex number forms with the positive real axis in the complex plane

$$\theta = \arg(Z) = \tan^{-1}\left(\frac{x}{y}\right)$$

Vector sum

$$A = A_x \hat{x} + A_y \hat{y} \quad , \quad B = B_x \hat{x} + B_y \hat{y}$$

$$A + B = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y}$$

Magnitude of a vector

$$A = \sqrt{A_x^2 + A_y^2}$$

$$B = \sqrt{B_x^2 + B_y^2}$$

Dot product

$$A \cdot B = AB \cos\theta$$

In terms of component, the dot product can be written as

$$A \cdot B = A_x B_x + A_y B_y$$

Cross product

$$C = A \times B = AB \operatorname{Sen}\theta$$

In cartesian unit:

$$A \times B = \begin{vmatrix} x & y & z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

Ejercicios de la lección 1

1. Calcule $z_1 + z_2$, $z_1 \cdot z_2$, z_1/z_2 para:

i. $z_1 = 1 + i$, $z_2 = 1 - i$

$\boxed{z_1 + z_2}$

$= (1 + i) + (1 - i)$

$= (1 + 1) + i(1 - 1)$

$= 2$

$\boxed{z_1 \cdot z_2}$

$= (1 \cdot 1 - 1[-1]) + i(1 \cdot 1 + 1 \cdot [-1])$

$= (1 + 1) + i(1 - 1)$

$= 2$

$\boxed{z_1 / z_2}$

$$= \frac{(1 \cdot 1 - 1 \cdot 1)}{1^2 + (-1)^2} + i\,\frac{(1 \cdot 1 + 1 \cdot 1)}{1^2 + (-1)^2}$$

$$= \varnothing + i\,\frac{2}{2}$$

$= i$

ii. $z_1 = 1 + 4i$, $z_2 = 2 + 7i$

$\boxed{z_1 + z_2}$

$= (1 + 2) + i(4 + 7)$

$= 3 + i11$

$\boxed{Z_1 \cdot Z_2}$

$= (2 - [4 \cdot 7]) + i([4 \cdot 2] + 7)$

$= (2 - 28) + i(8 + 7)$

$= -26 + i15$

$\boxed{Z_1 / Z_2}$

$$= \frac{(1 + 4i)}{(2 + 7i)} \cdot \frac{(2 - 7i)}{(2 - 7i)}$$

$$= \frac{2 - 7i + 8i + 28}{4 - 14i + 14i + 49}$$

$$= \frac{30 + i}{53}$$

$$= \frac{30}{53} + \frac{1}{53} i$$

iii. $Z_1 = 3 - 2i \quad , \quad Z_2 = 2 + 3i$

$\boxed{Z_1 + Z_2}$

$= 3 - 2i + 2 + 3i$

$= 5 + i$

$\boxed{Z_1 \cdot Z_2}$

$= (3 - 2i)(2 + 3i)$

$= 6 + 9i - 4i + 6$

$= 12 + 5i$

$\boxed{z_1 / z_2}$

$$= \frac{3-2i}{2+3i} \cdot \frac{2-3i}{2-3i}$$

$$= \frac{6-9i-4i-6}{4-6i+6i+9}$$

$$= \frac{-13i}{13}$$

$= -i$

iv. $z_1 = -\frac{1}{2} + i \quad , \quad z_2 = 1 - \frac{1}{2}i$

$\boxed{z_1 + z_2}$

$$= -\frac{1}{2} + i + 1 - \frac{1}{2}i$$

$= \frac{1}{2} + \frac{1}{2}i$

$\boxed{z_1 \cdot z_2}$

$$= \left(-\frac{1}{2} + i\right)\left(1 - \frac{1}{2}i\right)$$

$$= -\frac{1}{2} + \frac{1}{4}i + i + \frac{1}{2}$$

$= \frac{5}{4}i$

$\boxed{z_1 / z_2}$

$$= \frac{-\frac{1}{2} + i}{1 - \frac{1}{2} i} \cdot \frac{1 + \frac{1}{2} i}{1 + \frac{1}{2} i}$$

$$= \frac{-\frac{1}{2} - \frac{1}{4} i + i - \frac{1}{2}}{1 + \frac{1}{2} i - \frac{1}{2} i + \frac{1}{4}}$$

$$= \frac{-\frac{2}{2} + \frac{3}{4} i}{\frac{5}{4}}$$

$$= \frac{-\frac{2}{2}}{\frac{5}{4}} + \frac{\frac{3}{4}}{\frac{5}{4}} i$$

$$= \frac{-8}{10} + \frac{12}{20} i$$

$$= -\frac{4}{5} + \frac{3}{5} i$$

V. $z_1 = \dfrac{1 + \sqrt{3}\, i}{2}$, $z_2 = \dfrac{1 - \sqrt{3}\, i}{2}$

$\boxed{z_1 + z_2}$

$$= \frac{1 + \sqrt{3}\, i}{2} + \frac{1 - \sqrt{3}\, i}{2}$$

$$= \frac{2}{2}$$

$$= 1$$

$\boxed{z_1 \cdot z_2}$

$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} - \frac{3}{4}$$

$$= \frac{-2}{4}$$

$$= -\frac{1}{2}$$

$\boxed{z_1 / z_2}$

$$= \frac{\frac{1+\sqrt{3}i}{2}}{\frac{1-\sqrt{3}i}{2}}$$

$$= \frac{1+\sqrt{3}i}{1-\sqrt{3}i} \cdot \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$$

$$= \frac{4+2\sqrt{3}}{1-3}$$

$$= -2-\sqrt{3}$$

Problems

2.1. Find the real and imaginary parts of

$$z = \frac{2+3i}{5-7i}$$

$$= \frac{2+3i}{5-7i} \cdot \frac{5+7i}{5+7i}$$

$$= \frac{10+14i+15i-21}{25+35i-35i+49}$$

$$= \frac{-11+29i}{74}$$

$$= \frac{-11}{74} + \frac{29}{74}i$$

Parte real $= -\frac{11}{74}$, Parte imaginaria $= \frac{29}{74}i$

What are the real and imaginary parts of z^2?

$$= \left(\frac{-11}{74} + \frac{29}{74} i\right)^2$$

$$= \left(-\frac{11}{74}\right)^2 + 2\left(\frac{-11}{74} \cdot \frac{29}{74} i\right) - \frac{841}{5476}$$

$$= \frac{121}{5476} - \frac{319}{2738} i - \frac{841}{5476}$$

$$= -\frac{180}{1369} - \frac{319}{2738} i$$

Parte real $= -\frac{180}{1369}$, Parte imaginaria $= -\frac{319}{2738} i$

2.2. What is the result of the multiplication of $3+4i$ and its complex conjugate? What are the real and the imaginary parts of $3+4i$ divided by its complex conjugate?

$$z = 3 + 4i$$
$$z^* = 3 - 4i$$

$\boxed{z \cdot z^*}$

$$= 3^2 + 4^2$$
$$= 9 + 16$$
$$= 25$$

$\boxed{z / z^*}$

$$= \frac{3+4i}{3-4i} \cdot \frac{3+4i}{3+4i}$$

$$= \frac{9 + 2(3 \cdot 4i) - 16}{3^2 - 4^2}$$

$$= \frac{9 + 24i - 16}{25}$$

$$= \frac{-7}{25} + \frac{24}{25} i$$

Parte real $= \frac{-7}{25}$

Parte imaginaria $= \frac{24}{25} i$

2.3. Express the complex number $z = 4 + 3i$ in polar coordinates r and θ

$$r = \sqrt{4^2 + 3^2}$$

$$r = 5$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\theta = 40.9^\circ$$

2.4. Show that:

$$1 + e^{i\frac{2\pi}{4}} + e^{i\frac{4\pi}{4}} + e^{i\frac{6\pi}{4}} = \emptyset$$

$$1 + e^{i\frac{2\pi}{4}}\left(1 + e^{i\frac{2\pi}{4}} + e^{i\frac{4\pi}{4}}\right) = \emptyset$$

$$\left(1 + e^{i\frac{2\pi}{4}}\right)\left(1 + e^{i\pi}\right) = \emptyset$$

$$\left(1 + \cos[\tfrac{\pi}{2}] + i\sin[\tfrac{\pi}{2}]\right)\left(1 + \cos[\pi] + \sin[\pi]\right) =$$

$$(1 + 1)(1 - 1) =$$

$$\emptyset = \emptyset$$

$$e^{i\frac{5\pi}{4}} + e^{i\frac{7\pi}{4}} + e^{i\frac{\pi}{4}} + e^{i\frac{3\pi}{4}} = \emptyset$$

$$e^{i\frac{\pi}{4}}\left(e^{i\frac{2\pi}{2}} + e^{i\frac{3\pi}{2}} + 1 + e^{i\frac{\pi}{2}}\right) =$$

$$e^{i\frac{\pi}{4}}\left(1 + e^{i\frac{\pi}{2}}\left[e^{i\frac{\pi}{2}} + 1 + e^{i\pi}\right]\right) =$$

$$e^{i\frac{\pi}{4}}\left(\left[1 + e^{i\frac{\pi}{2}}\right]\left[1 + e^{i\pi}\right]\right) =$$

$$\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]\left(\left[1 + 0 + i\right]\left[\underbrace{1 - 1}_{\emptyset} + 0\right]\right) = \emptyset$$

$$e^{i\frac{6\pi}{4}} + e^{i\frac{2\pi}{4}} + e^{i\frac{6\pi}{4}} + e^{i\frac{2\pi}{4}} = \emptyset$$

$$e^{i\frac{3\pi}{2}} + e^{i\frac{\pi}{2}} + e^{i\frac{3\pi}{2}} + e^{i\frac{\pi}{2}} =$$

$$2e^{i\frac{3\pi}{2}} + 2e^{i\frac{\pi}{2}} =$$

$$2e^{i\frac{\pi}{2}}\underbrace{\left(e^{i\pi} + 1\right)}_{(-1+1)=\emptyset} = \emptyset$$

2.5. A vector A has a magnitude of 8 units and an angle of 60° with the x-axis. What are the x-and y-components of A?

$$r = 8 \quad , \quad \theta = 60^\circ$$

$$x = 8\cos 60^\circ \quad , \quad y = 8\,\text{Sen}\, 60^\circ$$

$$x = 4.7 \quad , \quad y = 6.47$$

2.6. Find the dot product of two vectors $A = \hat{x} + \hat{y}$ and $B = \hat{x} + 2\hat{y}$. What is the angle between the two vectors?

$$A \cdot B = 1 + 2 = 3$$

$$3 = \sqrt{1^2 + 1^2}\sqrt{1^2 + 2^2}\cos\theta$$

$$3 = \sqrt{2}\sqrt{5}\cos\theta$$

$$3 = \sqrt{10}\cos\theta$$

$$\frac{3}{\sqrt{10}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

$$\theta = 20.4^\circ$$

2.7. Find out the cross product of two vectors

$$A = \hat{x} + \hat{y} + \hat{z} \quad , \quad B = \hat{x} - \hat{y} + \hat{z}$$

$$A \times B = \begin{array}{ccc} x & y & z \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{array}$$

$$= (1+1)\hat{x} + (1-1)\hat{y} + (-1-1)\hat{z}$$

$$= 2\hat{x} - 2\hat{z}$$

2.8. Prove that

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$1 + 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\cos^2\left(\frac{\theta}{2}\right) \quad \Rightarrow \quad \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$2\cos^2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$1 - \cos\theta = 2\,\mathrm{Sen}^2\left(\frac{\theta}{2}\right)$$

$$\cos\theta = 1 - 2\,\mathrm{Sen}^2\left(\frac{\theta}{2}\right)$$

$$1 - \left(1 - 2\,\mathrm{Sen}^2\left(\frac{\theta}{2}\right)\right) =$$

$$1 - 1 + 2\,\mathrm{Sen}^2\left(\frac{\theta}{2}\right) =$$

$$2\,\mathrm{Sen}^2\left(\frac{\theta}{2}\right) = 2\,\mathrm{Sen}^2\left(\frac{\theta}{2}\right)$$

Ejercicios 1.3

En los problemas 1 a 10 escriba el número complejo dado en la forma polar usando primero un argumento $\theta \neq \arg(z)$, y después usando $\theta = \arg(z)$

1. 2

$z = 2 + i\emptyset$

$r = \sqrt{2^2 + \emptyset^2} = 2$

$\theta = \emptyset$

$z = 2\operatorname{cis}(\emptyset)$

$z = 2\operatorname{cis}(2\pi)$

2. −10

$z = -10 + i\emptyset$

$r = 10$

$\theta = \emptyset$

$z = 10\operatorname{cis}(\emptyset)$

$z = 10\operatorname{cis}(2\pi)$

3. −3i

$z = \emptyset - 3i$

$r = 3$

$\theta = \tan^{-1}\left(\frac{-3}{\emptyset}\right) = \frac{-\pi}{2}$

$z = 3\operatorname{cis}\left(\frac{-\pi}{2}\right)$

$z = 3\operatorname{cis}\left(\frac{3\pi}{2}\right)$

4. 6i

$z = \emptyset + 6i$

$r = 6$

$\theta = \tan^{-1}\left(\frac{6}{\emptyset}\right) = \frac{\pi}{2}$

$z = 6\operatorname{cis}\left(\frac{\pi}{2}\right)$

$z = 6\operatorname{cis}\left(\frac{5\pi}{2}\right)$

5. 1 + i

$r = \sqrt{2}$

$\theta = \tan^{-1}(1) = \frac{1}{4}\pi$

$z = \sqrt{2}\operatorname{cis}\left(\frac{1}{4}\pi\right)$

$z = \sqrt{2}\operatorname{cis}\left(\frac{7\pi}{4}\right)$

6. 5 − 5i

$r = 5\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-5}{5}\right) = -\frac{\pi}{4}$

$z = 5\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$

$z = 5\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{4}\right)$

7. $-\sqrt{3}+i$

$r = 2$

$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{-\pi}{\sqrt{6}}$

$Z = 2\,\text{Cis}\left(\frac{-\pi}{\sqrt{6}}\right)$

$Z = 2\,\text{Cis}\left(\frac{11\pi}{\sqrt{6}}\right)$

8. $-2-2\sqrt{3}i$

$r = 4$

$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$Z = 4\,\text{Cis}\left(\frac{\pi}{3}\right)$

$Z = 4\,\text{Cis}\left(\frac{7\pi}{3}\right)$

9. $\frac{3}{-1+i}$

$\frac{3}{-1+i} \cdot \frac{-1-i}{-1-i}$

$= \frac{-3-3i}{1^2+1^2}$

$= \frac{-3}{2} - \frac{3i}{2}$

$r = \frac{3\sqrt{2}}{2}$

$\theta = \tan^{-1}\left(\frac{\frac{-3}{2}}{\frac{-3}{2}}\right) = \frac{\pi}{4}$

$Z = \frac{3\sqrt{2}}{2}\,\text{Cis}\left(\frac{\pi}{2}\right)$

$Z = \frac{3\sqrt{2}}{2}\,\text{Cis}\left(\frac{5\pi}{2}\right)$

10. $\frac{12}{\sqrt{3}+i}$

$\frac{12}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{12\sqrt{3} - 12i}{\sqrt{3}^2+1}$

$= \frac{12\sqrt{3}}{4} - \frac{12i}{4}$

$= 3\sqrt{3} - 3i$

$r = 6$

$\theta = \tan^{-1}\left(\frac{-3}{3\sqrt{3}}\right) = \frac{-\pi}{6}$

$Z = 6\,\text{Cis}\left(\frac{-\pi}{6}\right)$

$Z = 6\,\text{Cis}\left(\frac{11\pi}{6}\right)$

En los problemas 11 y 12 utilice una calculadora para escribir el número complejo dado primero en forma polar usando un $\theta \neq \arg(z)$ y después usando $\theta = \arg(z)$

11. $-\sqrt{2} + \sqrt{7}\,i$

$r = 3$

$\theta = \tan^{-1}\left(\frac{\sqrt{7}}{-\sqrt{2}}\right) = -1.071$

$z = 3\,\mathrm{Cis}\,(-1.071)$

$z = 3\,\mathrm{Cis}\,(5.203)$

12. $-12 - 5i$

$r = 13$

$\theta = \tan^{-1}\left(\frac{-5}{-12}\right) = 0.394$

$z = 13\,\mathrm{Cis}\,(0.394)$

$z = 13\,\mathrm{Cis}\,(6.677)$

En los problemas 13 y 14 escriba el número complejo cuyas coordenadas polares (r, θ) están dadas en la forma $a + bi$.

13. $\left(4, -\frac{5\pi}{3}\right)$

$$z = 4\left[\cos\left(\frac{-5\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{-5\pi}{3}\right)\right]$$

$$= 4\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$$

$$= 2 + 2\sqrt{3}\,i$$

14. $(2, 2)$

$$z = 2\left[\cos(2) + i\,\mathrm{Sen}(2)\right]$$

$$z = 2\cos(2) + 2i\,\mathrm{Sen}(2)$$

$$= -0.832 + 1.818\,i$$

En los problemas 15 a 18 escriba el número complejo cuyas coordenadas polares (r,θ) están dadas en la forma $a+bi$.

15. $Z = 5\left(\cos\frac{7\pi}{6} + i\,\mathrm{sen}\,\frac{7\pi}{6}\right)$

$= 5\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right)$

$= \frac{-5\sqrt{3}}{2} - \frac{5i}{2}$

16. $8\sqrt{2}\left(\mathrm{Cos}\,\frac{11\pi}{4} + i\,\mathrm{Sen}\,\frac{11\pi}{4}\right)$

$= 8\sqrt{2}\left(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= \frac{-16}{2} + \frac{16i}{2}$

$= -8 + 8i$

17. $Z = 6\left(\mathrm{Cos}\,\frac{\pi}{8} + i\,\mathrm{Sen}\,\frac{\pi}{8}\right)$

$= 6\,\mathrm{Cos}\,\frac{\pi}{8} + 6i\,\mathrm{Sen}\,\frac{\pi}{8}$

$= 5.543 + 2.296i$

18. $Z = 10\left(\mathrm{Cos}\,\frac{\pi}{5} + i\,\mathrm{Sen}\,\frac{\pi}{5}\right)$

$= 10\,\mathrm{Cos}\,\frac{\pi}{5} + i\,10\,\mathrm{Sen}\,\frac{\pi}{5}$

$= 8.090 + 5.877i$

En los problemas 19 y 20 determin $z_1 \cdot z_2$ y z_1/z_2.
Escriba el número de la forma $a+bi$

15. $z_1 = 5\left(\cos\frac{7\pi}{6} + i\operatorname{Sen}\frac{7\pi}{6}\right)$, $z_2 = 4\left(\cos\frac{3\pi}{8} + i\operatorname{Sen}\frac{3\pi}{8}\right)$

$$z_1 = 5\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$z_1 = \frac{-5\sqrt{3}}{2} - \frac{5i}{2}$$

$$z_2 = 4\cos\frac{3\pi}{8} + 4i\operatorname{Sen}\frac{3\pi}{8}$$

$$z_1 \cdot z_2 = \left(\overset{a}{\frac{-5\sqrt{3}}{2}} - \overset{b}{\frac{5i}{2}}\right)\left(\overset{c}{4\cos\frac{3\pi}{8}} + \overset{d}{4i\operatorname{Sen}\frac{3\pi}{8}}\right)$$

$$= \left[\left(\frac{-5\sqrt{3}}{2}\right)\left(4\cos\frac{3\pi}{8}\right) - \left(\frac{-5}{2}\right)\left(4\operatorname{Sen}\frac{3\pi}{8}\right)\right]$$

$$+ i\left[\left(\frac{-5}{2}\right)\left(4\cos\frac{3\pi}{8}\right) + \left(\frac{-5\sqrt{3}}{2}\right)\left(4\operatorname{Sen}\frac{3\pi}{8}\right)\right]$$

$$z_1 z_2 = 2.61 - 19.828i$$

$$z_1/z_2 = \frac{\frac{-5\sqrt{3}}{2} - \frac{5i}{2}}{2.61 - 19.828i} \cdot \frac{2.61 + 19.828i}{2.61 + 19.828i}$$

$$= \frac{-60.871 - 92.382i}{399.96}$$

$$= \frac{-60.871}{399.96} - \frac{92.382}{399.96}i$$

20. $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\frac{\pi}{4}\right)$, $z_2 = \sqrt{3}\left(\cos\frac{\pi}{12} + i\,\mathrm{sen}\frac{\pi}{12}\right)$

$$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= 1 + i$$

$$= \frac{\sqrt{6}+3\sqrt{2}}{4} + \frac{-i\sqrt{6}+3\sqrt{2}}{4}$$

$$z_1 z_2 = \left(\frac{\sqrt{6}+3\sqrt{2}}{4} - \left[\frac{-\sqrt{6}+3\sqrt{2}}{4}\right]\right) + i\left(\frac{\sqrt{6}+3\sqrt{2}}{4} + \left[\frac{-\sqrt{6}+3\sqrt{2}}{4}\right]\right)$$

$$z_1 z_2 = \frac{\sqrt{6}}{2} + \frac{3\sqrt{2}}{2}$$

$$z_1/z_2 = \frac{1+i}{\frac{\sqrt{6}+3\sqrt{2}}{4} + \frac{-i\sqrt{6}+3\sqrt{2}}{4}} \cdot \frac{\frac{\sqrt{6}+3\sqrt{2}}{4} - \frac{-i\sqrt{6}+3\sqrt{2}}{4}}{\frac{\sqrt{6}+3\sqrt{2}}{4} - \frac{-i\sqrt{6}+3\sqrt{2}}{4}}$$

$$= \frac{\left(\frac{\sqrt{6}+3\sqrt{2}}{4} + \frac{\sqrt{6}-3\sqrt{2}}{4}\right) + i\left(\frac{\sqrt{6}+3\sqrt{2}}{4} + \frac{-\sqrt{6}+3\sqrt{2}}{4}\right)}{\left(\frac{\sqrt{6}+3\sqrt{2}}{4}\right)^2 + \left(\frac{-\sqrt{6}+3\sqrt{2}}{4}\right)^2}$$

$$= \frac{\frac{\sqrt{6}}{2} + \frac{3\sqrt{2}}{2}i}{3}$$

$$= \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

En los problemas 21 a 24 escriba cada número complejo en forma polar. Después utilice 6 o 7 para obtener la forma polar del número dado. Finalmente, escriba la forma polar en la forma $a + ib$

21. $(3-3i)(5+5\sqrt{3}i)$

$= (15+15\sqrt{3}) + i(-15+15\sqrt{3})$

$r = 30\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-15+15\sqrt{3}}{15+15\sqrt{3}}\right) = \frac{\pi}{12}$

$z = 30\sqrt{2}\ \text{Cis}\ \frac{\pi}{12}$

22. $(4+4i)(-1+i)$

$= -4 + 4i - 4i - 4$

$= -8$

$r = 8$

$\theta = \tan^{-1}\left(\frac{\emptyset}{8}\right) = \emptyset$

$z = 8\ \text{Cis}\ \emptyset$

23. $\frac{-i}{1+i}$

$= \frac{-i}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{-i-1}{2} = -\frac{1}{2} - \frac{i}{2}$

$r = \frac{\sqrt{2}}{2}$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

$z = \frac{\sqrt{2}}{2}\ \text{Cis}\ \frac{\pi}{4}$

24. $\dfrac{\sqrt{2} + \sqrt{6}\, i}{-1 + \sqrt{3}\, i}$

$$= \frac{\sqrt{2} + \sqrt{6}\, i}{-1 + \sqrt{3}\, i} \cdot \frac{-1 - \sqrt{3}\, i}{-1 - \sqrt{3}\, i}$$

$$= \frac{-\sqrt{2} - \sqrt{6}\, i - \sqrt{6}\, i + \sqrt{18}}{4}$$

$$= \frac{-2\sqrt{2}}{4} - \frac{2\sqrt{6}}{4}\, i$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}\, i$$

$r = \sqrt{2}$

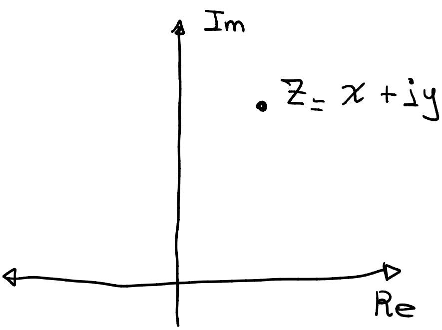
$$\theta = \tan^{-1}\left(\frac{-\frac{\sqrt{6}}{2}}{\frac{\sqrt{2}}{2}}\right) = -\frac{\pi}{3}$$

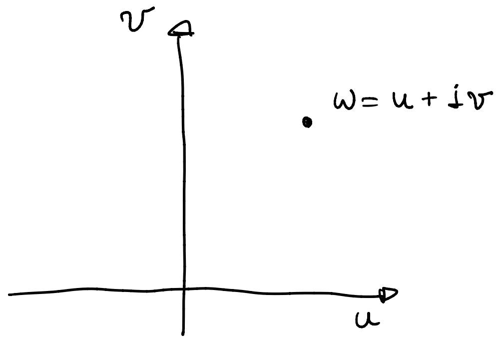
$z = \sqrt{2}\, \mathrm{Cis}\, -\frac{\pi}{3}$

Potenciación

Sea z un número complejo, $z = x + iy$, $z = re^{i\theta}$

$z^2 = ?$, $w = f_{(z)}$





Si $z = x + iy$, $z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy$

$$w = f_{(z)} = z^2 = u + iv, \quad \boxed{u = x^2 + y^2, \quad v = 2xy}$$

Para fijar ideas

$$w = f_{(z)} = z^3 = u + iv$$

hallar u y v

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(x + iy)^3 = x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3$$

$$= \underbrace{(x^3 - 3xy^2)}_{u} + i\,\underbrace{(3x^2y - y^3)}_{v}$$

En representación polar

si $z = re^{j\theta}$, $z^2 = (re^{j\theta})^2 = r^2 e^{2j\theta}$

Para recordar:

$e^{j\theta} = \cos\theta + i\,\mathrm{Sen}\,\theta$ // $e^{j\square} = \cos\square + j\,\mathrm{Sen}\,\square$

$e^{-j\theta} = \cos\theta - j\,\mathrm{sen}\,\theta$ | $e^{-j\square} = \cos\square - j\,\mathrm{sen}\,\square$

$$= r^2 \underbrace{(\cos 2\theta + j\,\mathrm{Sen}\,2\theta)} = r^2 \cos 2\theta + j r^2 \mathrm{Sen}\,2\theta$$

Otra forma:

$$e^{2j\theta} = (e^{j\theta})^2 = (\cos\theta + i\,\mathrm{Sen}\,\theta)^2$$

$$= (\cos^2\theta - \mathrm{Sen}^2\theta) + 2j\,\mathrm{Sen}\,\theta\cos\theta$$

Teorema de Moivre

$$(e^{j\theta})^2 = (\cos\theta + i\,\mathrm{Sen}\,\theta)^2 = \cos 2\theta + i\,\mathrm{Sen}\,2\theta$$

$$(e^{j\theta})^n = \cos n\theta + i\,\mathrm{Sen}\,n\theta$$

Ejercicio 1

$z = (1+i)^2$

$z^2 = 1^2 + 2i + i^2$

$= 1 + 2i - 1$

$= 2i$

//

$z^2 = (re^{j\theta})$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

$= 2\left[\cancel{\cos\frac{\pi}{2}} + i\,\mathrm{sen}\,\frac{\pi}{2}\right]$

$\varnothing$

$\boxed{= 2j}$

Ejercicio 2:

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$z^2 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 \frac{\sqrt{3}}{2} i + 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} i\right)^2 + \left(\frac{\sqrt{3}}{2} i\right)^3$$

$$= \frac{1}{8} + \frac{3\sqrt{3}}{8} i - \frac{9}{8} - \frac{3\sqrt{3}}{8} i$$

$$= \frac{1}{8} - \frac{9}{8} + i\left(\frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{8}\right)$$

$$= -1$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \frac{\pi}{3}$$

$$z^3 = 1\left(\underbrace{\cancel{\cos \pi}}_{-1} + i \overbrace{\cancel{\operatorname{Sen} \pi}}^{0}\right)$$

$$= -1$$

Radicación de números complejos

$$z^{1/n} = r^{1/n}\left(\cos\theta + i\,\text{Sen}\,\theta\right)^{1/n}$$

$$= r^{1/n}\left(\cos\frac{\theta + 2\pi k}{n} + i\,\text{Sen}\,\frac{\theta+2\pi k}{n}\right) \quad \text{donde} \quad k = 0, 1, 2, n-1$$

Ejemplo

$$\sqrt[2]{-1}$$

Paso 1: $z = -1 = e^{i\theta}$

Paso 2: $r = 1$, $\theta = \pi$

Paso 3: $n = 2$

Paso 4: Aplicar teorma de Mouvre, $k = 0$
$k = 1$

$$z_0^{1/2} = 1^{1/2}\left(\text{Cos}\,\frac{\pi + 2\pi 0}{2} + i\,\text{Sen}\,\frac{\pi + 2\pi 0}{2}\right)$$

$$= i$$

$$z_1^{1/2} = 1^{1/2}\left(\text{Cos}\,\frac{\pi + 2\pi}{2} + i\,\text{Sen}\,\frac{\pi + 2\pi}{2}\right)$$

$$= -i$$

Verificación:

$$i^2 = -1$$

$$(-i)^2 = -1$$

Ejemplo 2:

$Z = \sqrt[3]{-1}$

$r = 1$

$\theta = \pi$

$$Z_0^{1/3} = \cos\frac{\pi}{3} + i\,\text{Sen}\,\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$Z_1^{1/3} = \cos\pi + i\,\text{Sen}\,\pi = -1$$

$$Z_2^{1/3} = \cos\frac{5\pi}{3} + i\,\text{Sen}\,\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Ejemplo 3:

$Z = \sqrt[4]{-1}$ $\quad r = 1,\ \theta = \pi$

$$Z_0^{1/4} = \cos\frac{\pi}{4} + i\,\text{Sen}\,\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$Z_1^{1/4} = \cos\frac{3\pi}{4} + i\,\text{Sen}\,\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$Z_2^{1/4} = \cos\frac{5\pi}{4} + i\,\text{Sen}\,\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$Z_3^{1/4} = \cos\frac{7\pi}{4} + i\,\text{Sen}\,\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

Multiplicación de un complejo $z = x + iy$ por el complejo $c = a + ib$

$$w = f(z) = cz$$

$$w = u + iv = (ax - by) + i(bx + ay)$$

$$\begin{aligned} u &= ax - by \\ v &= bx + ay \end{aligned} \quad ; \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Para fijar ideas, sea $z = 1 + i$ y $c = i$ hallar $w = cz$

$$w = (1 + i)i = i - 1 = -1 + i$$

$$\begin{aligned} u &= -1 \\ v &= i \end{aligned} \quad ; \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Introducción al algebra linea

Sistemas de ecuaciones lineales

Para aprender a resolver sistemas de ecuaciones, vamos a empezar por lo básico, 2×2. Para esto se plantea lo siguiente:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \quad \ldots \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

$$[A]\vec{v} = \vec{b} \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$[A]^{-1}[A]\vec{v} = \vec{b}\,[A]^{-1}$$

$$[I]\vec{v} = [A]^{-1}\vec{b}$$

$$\vec{v} = [A]^{-1}\vec{b}$$

Pero notamos que se requiere recordar conceptos antes de resolver el sistema de ecuaciones, como: invertir una matriz, saber que es la matriz idendentidad, multiplicaciones de matrices

Invertir una matriz 2×2

$$[M] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad [M][M]^{-1} = [I]$$

$$[M] = \begin{pmatrix} \alpha & \gamma \\ \delta & \mu \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \delta & \mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{c|c} a\alpha + b\delta & a\gamma + b\mu \\ \hline c\alpha + d\delta & c\gamma + d\mu \end{array}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} a\alpha + b\delta &= 1 \\ c\alpha + b\delta &= 0 \end{aligned}}$$

$$\alpha = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d}{\Delta} \; ; \; \delta = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{\Delta}$$

$$\boxed{\begin{aligned} a\gamma + b\mu &= 0 \\ c\gamma + d\mu &= 1 \end{aligned}}$$

$$\gamma = \frac{\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}}{\Delta} = \frac{-b}{\Delta} \; ; \; \mu = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{\Delta} = \frac{a}{\Delta}$$

$$\boxed{[M]^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \Delta = ad - bc \neq \emptyset}$$

Para validar conocimiento resolver:

$$[M] = \begin{pmatrix} -1 & 2 \\ \emptyset & 3 \end{pmatrix}$$

$$[M]^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ \emptyset & -1 \end{pmatrix}$$

$$[M][M]^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ \emptyset & -1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ \emptyset & 3 \end{pmatrix}$$

$$= \frac{1}{-3}\begin{pmatrix} -3 & \emptyset \\ \emptyset & -3 \end{pmatrix} = \begin{pmatrix} 1 & \emptyset \\ \emptyset & 1 \end{pmatrix}$$

Con esto validamos conocimiento y nos brinda la puerta a crear un algoritmo que pueda invertir una matriz 2x2

Ejercicio

$$\begin{pmatrix} -1 & 2 \\ \emptyset & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad , \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned} -x + 2y &= 1 \\ 3y &= 3 \end{aligned}$$

$$y = 1 \; , \; -x = 1 - 2(1) = -1$$

$$x = 1$$

Ahora vamos a ver otra manera de resolver el sistema de ecuaciones, diagonalizando $[A]$

→ Diagonalizar la matriz $[A]$

La matriz $[A]$ se diagonaliza con una transformación de similitud o similaridad

$$[T]^{-1}[A][T] = \begin{pmatrix} \lambda_1 & \varnothing \\ \varnothing & \lambda_2 \end{pmatrix}, \quad [T] = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$$\vec{T}_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}, \quad \vec{T}_2 = \begin{pmatrix} T_{12} \\ T_{22} \end{pmatrix}$$

La matriz $[T]$ se llama matriz de vectores propios de $[A]$, $\vec{T}_1$ y $\vec{T}_2$ se llaman vectores propios de $[A]$.

Para hallar los vectores propios de $[A]$ seguimos los siguientes pasos:

1) Encontramos los valores propios de $[A]$, resolviendo la ecuación

$$f(\lambda) = \left| [A] - \lambda[I] \right| = 0$$

llamada ecuación característica (Dos valores propios λ_1 y λ_2)

2) A cada valor propio λ_i, $i = 1, 2$ se asocia un vector propio, es decir:

$$\lambda_1 \rightarrow \vec{T}_1 \quad \text{y} \quad \lambda_2 \rightarrow \vec{T}_2$$

y esta ecuación permite hallar $\vec{T}_1$ y $\vec{T}_2$

$$([A] - \lambda_i[I])(\vec{T}_i) = \vec{0} \quad i = 1, 2$$

Ejemplo: diagonalizar la matriz

$$[A] = \begin{pmatrix} -1 & 2 \\ \varnothing & 3 \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 & \varnothing \\ \varnothing & \lambda_2 \end{pmatrix}$$

Paso 1

$$f(x) = \left| \begin{pmatrix} -1 & 2 \\ \varnothing & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \varnothing$$

$$= \left| \begin{pmatrix} -1-\lambda & 2 \\ \varnothing & 3-\lambda \end{pmatrix} \right| \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 3 \end{matrix}$$

Paso 2

$$\begin{pmatrix} -1-\lambda_i & 2 \\ \varnothing & 3-\lambda_i \end{pmatrix} \begin{pmatrix} T_{1i} \\ T_{2i} \end{pmatrix} = \begin{pmatrix} \varnothing \\ \varnothing \end{pmatrix}$$

$i = 1$, $\lambda_1 = -1$

$$\begin{pmatrix} \varnothing & 2 \\ \varnothing & 4 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} = \begin{pmatrix} \varnothing \\ \varnothing \end{pmatrix}$$

$$\begin{matrix} 2T_{11} = \varnothing \\ 4T_{21} = \varnothing \end{matrix}, \quad \vec{T_1} = \begin{pmatrix} 1 \\ \varnothing \end{pmatrix}$$

$i = 2$
$\lambda_2 = 3$

$$\begin{pmatrix} -4 & 2 \\ \varnothing & \varnothing \end{pmatrix} \begin{pmatrix} T_{12} \\ T_{22} \end{pmatrix} = \begin{pmatrix} \varnothing \\ \varnothing \end{pmatrix}$$

$-4x + 2y = \varnothing$
$y = 2x$

$$\vec{T_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Paso 3:

$$[T] = \begin{pmatrix} 1 & 1 \\ \varnothing & 2 \end{pmatrix}, \quad [T]^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ \varnothing & 1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 2 & -1 \\ \emptyset & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ \emptyset & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \emptyset & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & \emptyset \\ \emptyset & 6 \end{pmatrix} = \begin{pmatrix} -1 & \emptyset \\ \emptyset & 3 \end{pmatrix}$$

¿Para que sirve diagonalizar una matriz?

Facilita cálculos de potencias de matrices, por ejemplo elevar una matriz A^n es complicado, pero si A está diagonalizada como $A^n = PD^nP^{-1}$

Ejemplo: Dado $[A] = \begin{pmatrix} -1 & 2 \\ \emptyset & 3 \end{pmatrix}$, hallar $[A]^2$

$$[A]^2 = [A][A] = \begin{pmatrix} 1 & 4 \\ \emptyset & 9 \end{pmatrix}$$

Series de tiempo

Ecuaciones en diferencia lineales

Aplicación de motivación

Por medio de una serie de censos en una población, y se obtuvieron los datos que se muestran en la siguiente tabla.

K en años	$y(K)$
$\emptyset$	20
1	30
2	35
3	45
4	50

Modelo

$$y(K) = 15 + y(K-2)$$

$$y(\emptyset) = 20, \quad y(1) = 30$$

Se desea establecer una expresión matemática (modelo) para determinar la población en cualquier tiempo

Definición

Una ecuación en diferencia lineal tiene la forma

$$a_{\emptyset}\, y(k+n) + a_1\, y(k+n-1) + a_2\, y(k+n-2)$$

$$\dots a_{n-2}\, y(k+2) + a_{n-1}\, y(k+1) + a_n\, y(k) = x_{(k)}$$

n: órden de la ecuación

<u>Pregunta</u>: Clasificar la ecuación $y(k+1) - 2y(k) = \emptyset$

R// Ecuación en diferencia en órden uno homogénea.

<u>Pregunta</u>: La función $y(k) = 3(2)^k$ es solución de la ecuación anterior?

$$3(2)^{k+1} - 2(3)(2)^k \overset{?}{=} \emptyset$$

$$3(2)^k 2^1 - 2(3)(2)^k \overset{?}{=} \emptyset$$

$$6(2)^k - 6(2)^k \equiv \emptyset$$

<u>Pregunta</u>: $y(k) = r^k$ es solución de $y(k+1) - 2y(k) = \emptyset$

$$r^{k+1} - 2r^k = \emptyset$$

$$r^k \underbrace{(r-2)}_{r=2} = \emptyset$$

<u>Pregunta</u>: $y(k) = c(2)^k$ es solución?

$$c(2)^{k+1} - 2c(2)^k = \emptyset$$

$$c(2)^{k+1} = c(2)^{k+1}$$

$$y(k) = c(2)^k, \quad k = 0, 1, 2 \dots$$

$$y(0) = c$$

$$y(k) = 3(2)^k$$

$$y(0) = 3$$

Problema: Clasificar la ecuación

$$\boxed{y(k+2) - 3y(k+1) + 2y(k) = 4^k}$$

y verifique que

$$y(k) = C_1 + C_2(2)^k + \frac{1}{6}(4)^k \text{ es solución}$$

Pausa!

$$y(k+2) - 3y(k+1) + 2y(k) = \emptyset$$

$$y(k) = r^k$$

$$r^{k+2} - 3r^{k+1} + 2r^k = \emptyset$$

$$r^k r^2 - 3r^k r + 2r^k = \emptyset$$

$$r^k(\underbrace{r^2 - 3r + 2}) = \emptyset$$

$$r=1$$
$$r=2$$

$$y_1(k) = C_1(1)^k = C_1 \quad ; \quad y_2(k) = C_2(2)^k$$

$$y_h(k) = C_1 + C_2(2)^k$$

$$y(k) = A(4)^k$$

$$A(4)^{k+2} - 3A(4)^{k+1} + 2A(4)^k = 1(4)^k$$

$$A(4)^k(16 - 12 + 2) = 4^k$$

$$6A = 1 \quad ; \quad A = \frac{1}{6}$$

Problema: Resolver la ecuación:

$$y(k+2) + 3y(k+1) + 2y(k) = \varnothing$$

$$y(\varnothing) = 1 \quad , \quad y(1) = 1$$

Propuesta de solución: $y(k) = r^k$

R// $\quad r^{k+2} + 3r^{r+1} + 2r^k = \varnothing$

$$r^k\underbrace{(r^2 + 3r + 2)}_{} = \varnothing$$

$$r_1 = -1 \qquad ; \qquad y_1(k) = -1^k$$

$$r_2 = -2 \qquad\qquad y_2(k) = -2^k$$

$$y_h(k) = C_1(-1)^k + C_2(-2)^k$$

$$y(\varnothing) = C_1 + C_2 = \varnothing$$

$$C_1 = 1$$

$$y(1) = -C_1 - 2C_2 = 1 \qquad C_2 = -1$$

$$\boxed{y(k) = (-1)^k - (-2)^k}$$

Ahora un cambio...

Problema: $\quad y(k+2) - y(k) = 15 \quad , \quad y(\varnothing) = 20 \quad , \quad y(1) = 30$

Solución para no homogéneas

$$y(k) = y_h(k) + y_p(k)$$

Primero encontrar la homogénea

Solución propuesta $y(k) = r^k$

$$y(k+2) - y(k) = \varnothing$$

$$r^{k+2} - r^k = \emptyset$$

$$r^k \underbrace{(r^2 - 1)} = \emptyset$$

$r_1 = 1$

$r_2 = -1$

$$y_1(k) = 1^k = 1 \quad , \quad y_2(k) = (-1)^k$$

$$\boxed{y_h(k) = C_1 + C_2(-1)^k}$$

Para $y_p(k)$, se propone como solución $y_p(k) = AK$

$$A(K+2) - AK = 15$$

$$AK + 2A - AK = 15$$

$$2A = 15$$

$$A = \frac{15}{2} \Rightarrow \boxed{y_p(k) = \frac{15}{2}K}$$

Entonces

$$y(k) = C_1 + C_2(-1)^K + \frac{15}{2}K$$

$$20 = C_1 + C_2(-1)^{\emptyset} + \frac{15}{2}(\emptyset)$$

$$\boxed{20 = C_1 + C_2}$$

$$30 = C_1 + C_2(-1)^1 + \frac{15}{2}(1)$$

$$\boxed{30 = C_1 - C_2 + \frac{15}{2}}$$

Sistema de ecuaciones

$$20 = C_1 + C_2$$

$$30 = C_1 - C_2 + \frac{15}{2}$$

$$C_1 = \frac{85}{4}$$

$$C_2 = -\frac{5}{4}$$

Solución completa

$$\boxed{y_{(k)} = \frac{85}{4} - \frac{5}{4}(-1)^k + \frac{15}{2}k}$$

Verificamos si replica el censo

$y_{(2)} = 35$ ✓ , $y_{(3)} = 45$ ✓ , $y_{(4)} = 50$ ✓

Problema

Se colocan \$60.000 en una cuenta de ahorros. Cada mes se sacan 2/3 de lo que se tiene en la cuenta. Si se reconocen intereses del 1.2% mensual, establecer una ecuación que permita calcular el saldo en la cuenta en cualquier momento. ¿Con cuanto se contará al cabo de 8 meses?

Sea S_n el sueldo en la cuenta luego del retiro. Se puede establecer la ecuación de diferencias:

$$S_{(n+1)} = S_{(n)} + i\,S_{(n)} - \frac{2}{3}S_{(n)}, \qquad \text{con} \quad S_{(\emptyset)} = 60.000$$

$$i = 1.2\% = 0.012$$

$$S_{(n+1)} = S_{(n)}\left(1 + i - \frac{2}{3}\right) = S_{(n)}\left(\frac{1}{3} + i\right)$$

$$S_{(n+1)} - S_{(n)}\left(\frac{1}{3} + i\right) = \emptyset$$

Propongo $S_{h(n)} = r^n$

$$r^{n+1} - r^n\left(\frac{1}{3} + i\right) = \emptyset$$

$$r^n r^1 - r^n\left(\frac{1}{3} + i\right) = \emptyset$$

$$r^n\underbrace{\left(r-\frac{1}{3}+i\right)}_{r=\frac{1}{3}-0.012=\frac{259}{750}}=\emptyset$$

$$S_{h(n)}=C_1\left(\frac{259}{750}\right)^n$$

$$60\_000=C_1\left(\frac{259}{750}\right)^{\emptyset}$$

$$\boxed{60\_000=C_1}$$

$$S_{(n)}=60\_000\left(\frac{259}{750}\right)^n$$

Se prueba para $S_{(8)}$

$$S_{(8)}=60\_000\left(\frac{259}{750}\right)^8=12.1355$$

Ecuaciones no-homogéneas

Sabemos como resolver ecuaciones homogeneas en diferencia de segundo orden. Las ecuaciones no-homogéneas son de la forma:

$$x_{(n)}+a\,x_{(n-1)}+b\,x_{(n-2)}=y_{(n)}$$

Para fijar ideas:

$$6x_{(n)}-5x_{(n-1)}+x_{(n-2)}=n \quad , \quad n\geq 3$$

$$x_1=1$$
$$x_2=2$$

La solución se expresa:

$$x_n = x_h(n) + x_p(n)$$

Entonces, primero solucionamos la homogenea

$$6x_{(n)} - 5x_{(n-1)} + x_{(n-2)} = \varnothing \quad , \text{ propongo } \quad x_{(n)} = r^n$$

$$6r^n - 5r^{n-1} + r^{n-2} = \varnothing$$

$$r^{n-2}(6r^2 - 5r + 1) = \varnothing$$

$$x_1 = \frac{1}{3} \quad , \quad x_2 = \frac{1}{2}$$

$$x_h(n) = C_1\left(\frac{1}{3}\right)^n + C_2\left(\frac{1}{2}\right)^n$$

Para la no-homogénea, se propone una solución; se adapta una solución: En este caso, de coeficientes indeterminados

$$x(n) = An + B$$

$$\Rightarrow x(n-1) = A(n-1) + B$$

$$\Rightarrow x(n-2) = A(n-2) + B$$

$$6(An + B) - 5(A(n-1) + B) + A(n-2) + B = n$$

$$(6A - 5A + A)n + (6B + 5A - 5B - 2A + B) = n$$

$$2A = 1$$

$$A = \frac{1}{2} \qquad B = -\frac{3}{4}$$

$$x(n) = C_1\left(\frac{1}{3}\right)^n + C_2\left(\frac{1}{2}\right)^n - \frac{3}{4} + \frac{1}{2}n$$

Se buscan C_1 y C_2

$$x_{(1)} = C_1\left(\frac{1}{3}\right) + C_2\left(\frac{1}{2}\right) - \frac{3}{4} + \frac{1}{2} = 1$$

$$x_{(2)} = C_1\left(\frac{1}{3}\right)^2 + C_2\left(\frac{1}{2}\right)^2 - \frac{3}{4} + \frac{1}{2}(2) = 2$$

$$\left.\begin{aligned} \frac{5}{4} &= C_1\left(\frac{1}{3}\right) + C_2\left(\frac{1}{2}\right) \\ \frac{7}{4} &= C_1\left(\frac{1}{9}\right) + C_2\left(\frac{1}{4}\right) \end{aligned}\right\} \quad C_1 = -\frac{81}{4}, \quad C_2 = 16$$

Solución

$$x_{(n)} = -\frac{81}{4}\left(\frac{1}{3}\right)^n + 16\left(\frac{1}{2}\right)^n - \frac{3}{4} + \frac{1}{2}n$$

La mayor dificultad estaba en la solución para la forma particular. La siguiente tabla puede ayudar

$f_{(n)}$	Solución de la forma particular
n	$a + bn$
n^2	$a + bn + cn^2$
K^n	aK^n (o anK^n en caso especiales)
constante	a

Para poner en práctica la tabla:

Ejemplo 1

$$6x_n - 5x_{n-1} + x_{n-2} = 2$$

Se resuelve la ecuación homogénea asociada

$$6x_n - 5x_{n-1} + x_{n-2} = \emptyset \quad , \text{ propongo } \quad x_{(n)} = r^n$$

$$6r^n - 5r^{n-1} + r^{n-2} = \emptyset$$

$$r^{n-2}(6r^2 - 5r + 1) = \emptyset$$

$$x_1 = \frac{1}{3} \quad , \quad x_2 = \frac{1}{2}$$

$$x_{h(n)} = C_1\left(\frac{1}{3}\right)^n + C_2\left(\frac{1}{2}\right)^n$$

Ahora la particular, siguiendo la tabla

$$x_{p(n)} = A$$

$$6A - 5A + A = 2$$

$$2A = 2 \quad , \quad A = 1$$

Entonces, la solución es

$$x_{(n)} = C_1\left(\frac{1}{3}\right)^n + C_2\left(\frac{1}{2}\right)^n + 1$$

Ejemplo 2:

$$6x_n - 5x_{n-1} + x_{n-2} = 2^n$$

Es la misma solución para la homogénea, entonces saltamos a la particular para poner en práctica la tabla

$$x_{p(n)} = A(2)^n$$

$$6A2^n - 5A2^{n-1} + A2^{n-2} = 2^n$$

$$A2^n\left(6 - 5\left(\frac{1}{2}\right) + \frac{1}{4}\right) = 2^n$$

$$A2^n \frac{15}{4} = 2^n$$

$$A = \frac{4}{15}$$

Ejemplo 3

$$6x_n - 5x_{n-1} + x_{n-2} = \left(\frac{1}{2}\right)^n$$

En este caso, se presenta un problema futuro de duplicidad, por esto se propone $x_n = A n K^n$

Taller

1) Find the general solution of the difference ecuation

$$x_{(n)} - 5x_{(n-1)} + 6x_{(n-2)} = f_{(n)}$$

a) $f_{(n)} = 2$

$$x_{(n)} - 5x_{(n-1)} + 6x_{(n-2)} = 2$$

$$x_h(n) = r^n - 5r^{n-1} + 6r^{n-2} = \emptyset$$

$$r^{n-2}(r^2 - 5r + 6) = \emptyset$$

$$r_1 = 2, \quad r_2 = 3$$

$$\boxed{x_h(n) = C_1 2^n + C_2 3^n}$$

$$x_p(n) = A - 5A + 6A = 2$$

$$A = 1$$

$$x_{(n)} = C_1 2^n + C_2 3^n + 1$$

b) $f_{(n)} = n$, ya tenemos la $x_h(n)$

$$x_p(n) = (A + Bn) - 5(A + (n-1)B) + 6(A + (n-2)B) = n$$

$$(B - 5B + 6B)n + (A - 5A + 5B + 6A - 12B) = \emptyset + 1n$$

$$2B = 1 \qquad\qquad 2A - 7B = \emptyset$$

$$B = \frac{1}{2} \qquad\qquad 2A - \frac{7}{2} = \emptyset$$

$$A = \frac{7}{4}$$

$$x_{(n)} = C_1 2^n + C_2 3^n + \frac{7}{4} + \frac{1}{2}n$$

c) $f_{(n)} = 1 + n^2$, la misma homogénea , para la particular se propone

$$\boxed{A + Bn + Cn^2}$$

$$x_n = A + Bn + Cn^2$$

$$-5x_{n-1} = -5A - 5Bn + 5B - 5Cn^2 + 10Cn - 5C$$

$$6x_{n-2} = 6A + 6Bn - 12B + 6Cn^2 - 24Cn + 24C$$

Agrupamos términos:

$$Cn^2 - 5Cn^2 + 6Cn^2 = 2Cn^2$$

$$Bn - 5Bn + 10Cn + 6Bn - 24Cn = (2B - 14C)n$$

$$A - 5A + 5B - 5C + 6A - 12B + 24C = 2A - 7B + 19C$$

Comparamos coeficientes

$$2C = 1 \quad , \quad 2B - 14\left(\frac{1}{2}\right) = \emptyset \quad , \quad 2A - 7\left(\frac{7}{2}\right) + 19\left(\frac{1}{2}\right) = 1$$

$$C = \frac{1}{2} \qquad\qquad B = \frac{7}{2} \qquad\qquad A = 8$$

$$x_{(n)} = C_1 2^n + C_2 3^n + 8 + \frac{7}{2}n + \frac{1}{2}n^2$$

d) $f_{(n)} = 5^n$, la misma homogénea...

$x_{p(n)} = A5^n$

$A5^n - 5(A5^{n-1}) + 6(A5^{n-2}) = 5^n$

$A5^n\left(1 - 1 + \frac{6}{25}\right) = 5^n$, $A = \frac{25}{6}$

$x_{(n)} = C_1 2^n + C_2 3^n + \frac{25}{6}(5^n)$

e) $f_{(n)} = 2^n$, la misma homogénea

$x_{p(n)} = An2^n$

$An2^n - 5A(n-1)2^{n-1} + 6A(n-2)2^{n-2} = 2^n$

$A2^n\left(n - 5\frac{n-1}{2} + 6\frac{n-2}{2}\right) = 2^n$

$\frac{2n}{2} - \frac{5n-5}{2} + \frac{3n-6}{2} = \frac{-1}{2}$

$x_{(n)} = C_1 2^n + C_2 3^n - \frac{1}{2}n2^n$

2) Find the general solution to $x_{(n)} - 7x_{(n-1)} + 12x_{(n-2)} = 2^n$ when $x_1 = 1$ and $x_2 = 1$

$r^n - 7r^{n-1} + 12r^{n-2} = \emptyset$

$r^n((r-3)(r-4)) = \emptyset$

$r_1 = 3$, $r_2 = 4$

$x_{h(n)} = C_1 3^n + C_2 4^n$

$x_{(n)} = -\frac{5}{3}(3)^n + \frac{1}{2}(4)^n + 2^{n+1}$

$x_{p(n)} = A2^n$

$A2^n - 7A2^{n-1} + 12A2^{n-2} = 2^n$

$A = 2$

3) $x_n + 3x_{n-1} - 10x_{n-2} = 2^n$, $x_1 = 2$
$x_2 = 1$

homogénea:

$$x_h(n) = r^n + 3r^{n-1} - 10r^{n-2} = \emptyset$$
$$r^{n-2}(r^2 + 3r - 10) = \emptyset$$
$$r_1 = 2$$
$$r_2 = -5$$

$$x_h(n) = C_1 2^n + C_2(-5)^n$$

La particular

$$x_p(n) = An2^n$$

$$An2^n + 3A(n-1)2^{n-1} - 10A(n-2)2^{n-2} = 2^n$$

$$A2^n(n + 3n2^{-1} - 3 2^{-1} - 10n2^{-2} + 20 2^{-2}) = 2^n$$

$$A2^n \frac{7}{2} = 2^n$$

$$\boxed{A = \frac{2}{7}}$$

$$x_p(n) = \frac{2}{7} n 2^n$$

$$x(n) = C_1 2^n + C_2(-5)^n + \frac{2}{7} n 2^n$$

$x(2) = 1 = C_1 4 + C_2 25 + \frac{16}{7}$ | Sistema de ecuaciones

$x(1) = 2 = C_1 2 - C_2 5 + \frac{4}{7}$ | $\frac{-9}{7} = C_1 4 + C_2 25$

| $\frac{10}{7} = C_1 2 - C_2 5$

$C_1 = \frac{41}{98}$, $C_2 = \frac{-29}{245}$

$$x(n) = \frac{41}{98}(2)^n - \frac{29}{245}(-5)^n + \frac{2}{7} n 2^n$$

Ecuación diferencial logistica, ecuación en diferencia logística, Mapa logístico, Bifurcaciones, Doble periodo, cuadruple periodo, caos

Para motivar, leer:

"Enhanced logistic map with infinite chaos and its applicability in lightweight and high-speed pseudo-random bit generation"

Encontramos que existen diagramas de bifurcación y que existe varios tipos de caos

→ Ecuación logística en diferencia

$$x(n+1) = r\,x(n)\,(1 - x(n))$$

$$x(n+1) = r\,x(n) - r\,x^2(n)$$ Importante: es cuadrática

Ecuación diferencial logística

Modelo de Malthus

Inició con: $\frac{dP}{dt} = RP$, pero si $P(t) = P_0 e^{kt}$; $k > \emptyset$, sería un modelo que siempre crece

$$\frac{dP}{dt} = kP, \quad P(0) = P(t=0)$$

$$\frac{dP}{P} = k\,dt;$$

$$\int \frac{dP}{P} = k \int dt,$$

$$\ln P = kt + C$$

$$P(t) = e^{kt + C}$$

Iteró agregando tasa de natalidad y mortalidad

Modelo de Pierre Francis Verhulst

$$\frac{dP}{dt} = rP - \alpha P^2 = P(r - \alpha P)$$

No se puede computar la ecuación diferencial, entonces

$t = kT, \quad k = 0, 1, 2 \ldots$

$$\frac{P((k+1)T) - P(kT)}{(k+1)T - kT} = rP(kT)$$

$$P(k+1) - (rT+1)P(k) = \emptyset$$

Ahora la ecuación de Verhulst

$$\frac{dP}{dt} = rp - \alpha p^2 = p(r - \alpha p)$$

$$\frac{dP}{P(r-\alpha P)} = dt$$

$$\frac{dP}{-\alpha P(P - \frac{r}{\alpha})} = dt$$

$$\frac{dP}{P(P-\frac{r}{\alpha})} = -\alpha \, dt$$

$$\int \frac{dP}{P(P-\frac{r}{\alpha})} = -\alpha t + c$$

$$\int \frac{dP}{P(P-\frac{r}{\alpha})} = \int \frac{A}{P} dp + \int \frac{B}{P - \frac{r}{\alpha}} dp$$

$$= A \ln P + B \ln (P - \tfrac{r}{\alpha})$$

$$= \ln P^A + \ln (P - \tfrac{r}{\alpha})^B$$

$$= P^A (P - \tfrac{r}{\alpha})^B = e^c e^{-\alpha t}$$

$$\frac{1}{P(P-\frac{r}{\alpha})} = \frac{A}{P} + \frac{B}{P - \frac{r}{\alpha}}$$

$$\cancel{P}\left(\frac{1}{\cancel{P}(P - \frac{r}{\alpha})}\right) = A + P\left(\frac{B}{P - \frac{r}{\alpha}}\right), \quad P = \emptyset$$

$$\frac{-\alpha}{r} = A, \quad \frac{\alpha}{r} = B$$

$$\left(\frac{P}{P - \frac{r}{\alpha}}\right)^{-\frac{\alpha}{r}} = e^c e^{-\alpha t}$$

$$\frac{P}{P - \frac{r}{\alpha}} = \left(e^c e^{-\alpha t}\right)^{-\frac{r}{\alpha}}$$

$$\frac{P}{P - \frac{r}{\alpha} - P} = \frac{(e^c e^{-\alpha t})^{\frac{-r}{\alpha}}}{1 - (e^c e^{-\alpha t})^{\frac{-r}{\alpha}}}$$

$$P = -\frac{r}{\alpha} \frac{(e^c e^{-\alpha t})^{\frac{-r}{\alpha}}}{1 - (e^c e^{-\alpha t})^{\frac{-r}{\alpha}}}$$

Partiendo de la ecuación diferencial:

$$\frac{dx}{dt} = rx - \alpha x^2$$

$$\frac{x(n+1) - x(n)}{(n+1) - n} = r\,x(n) - \alpha\,x(n)^2$$

$$x(n+1) - x(n) = rx(n) - \alpha\,x(n)^2$$

$$\varnothing = x(n) - x(n+1) + rx(n) - \alpha x(n)^2$$

Mapas logísticos

Considerando: $x_{(n+1)} = r\,x_{(n)}(1 - x_{(n)})$, normalizado

$\varnothing \leq x \leq 1$, con $\varnothing \leq r \leq 4$

Encontrar punto fijo

$$x^* = f_{(x^*)} = rx^*(1 - x^*)$$

si $x^* = \varnothing$, $x^* = \frac{r-1}{r}$

La estabilidad depende de: $f'_{(x^*)} = r - 2rx^*$, $f'_{(0)} = r$
el origen es estable para $r<1$ e inestable $r>1$

10.3.1 10.3.3 $x_{(n)}/r$

Si se grafica la ecuación logística iterada, da como resultado:

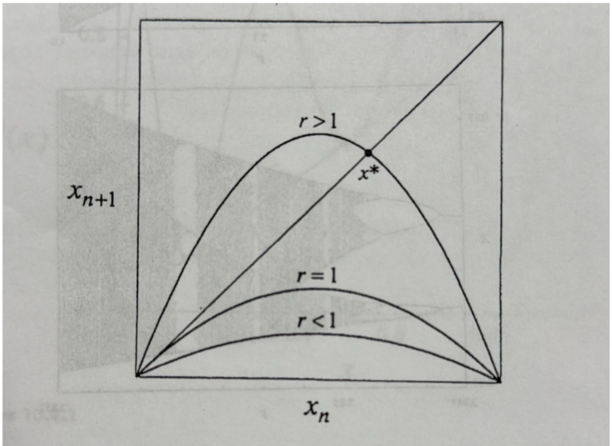


Figure 10.3.1

Para los distintos valores de r:

- $r<1$: parábola "aplastada"
- $r=1$: parábola toca la diagonal en un punto x^*
- $r>1$: parábola sobrepasa la diagonal; puede generar dinámicas complejas

Ahora, al analizar la pendiente al evaluar: $f'(x^*) = r - 2rx^*$

Si:

- $f'(x^*) < 1$, es estable
- $f'(x^*) > 1$, es inestable
- $f'(x^*) = -1$, está en el límite de una bifurcación

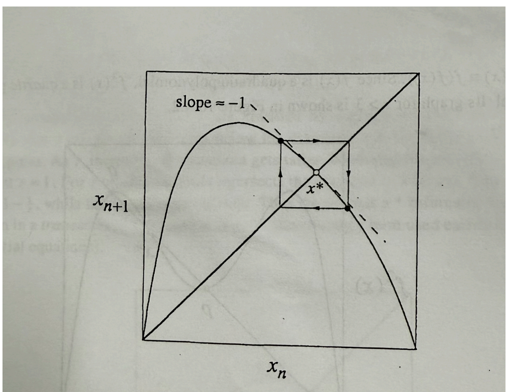


Figure 10.3.3

Diagrama de bifurcación del mapa logístico

Queremos ver cómo cambia el valor al que llega la sucesión x_n cuando el parámetro r cambia, es decir:

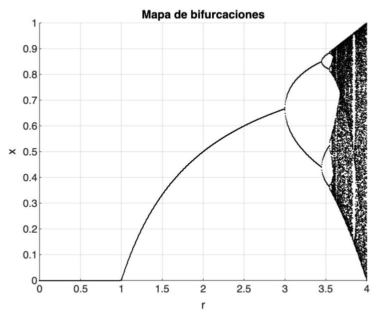
$$r \to \lim_{n \to \infty} x_n$$

Al graficar se encuentra que:

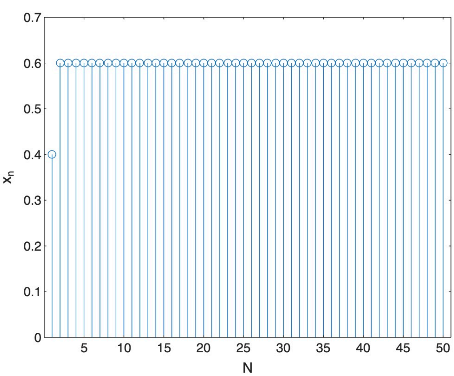
- $r < 3$, los valores convergen a un punto fijo
- $r \approx 3$, se bifurca en 2 valores
- $r > 3.57$, caos

Anexo: 3.57, el número de Feigenbaum

Al graficar en mathlab para r_min = ∅ y r_max = 4, se obtiene



Al graficar $x_{(n)}$ vs n, para r entre 2 y 3, se evidencia



Se evidencia que se estabiliza en 0,6. Lo mismo que hacer

$$\frac{r-1}{r} = \frac{3-1}{3} = 0.666$$

Problema

1) Encontrar los valores propios y vectores propios de la matriz

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$ y diagonalizar la matriz

2) Calcular $[A]^2$ y hallar sus valores propios

3) Calcular $[A]^{-1}$ y hallar sus valores propios

R_1// $f_{(\lambda)} = |[A] - \lambda[I]| = \emptyset$

$$f_{(\lambda)} = \left| \begin{pmatrix} 1-\lambda & \emptyset & \emptyset \\ \emptyset & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \right| = \emptyset$$

$$(1-\lambda)\left[(3-\lambda^2) - 1\right] = \emptyset$$

$$(1-\lambda)(3-\lambda-1)(3-\lambda+1) = \emptyset$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

Ecuación de vectores propios $\quad [A]T_i^{-1} = \lambda_i T_i^{-1} \qquad i = 1,2,3$

$$\begin{pmatrix} 1-\lambda_i & \emptyset & \emptyset \\ \emptyset & 3-\lambda_i & -1 \\ 1 & -1 & 3-\lambda_i \end{pmatrix} \begin{pmatrix} T_i \\ T_i \\ T_i \end{pmatrix} = \begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \quad ; \quad [T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Para $i = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\emptyset = \emptyset$

$2T_{21} - 1T_{31} = \emptyset$

$T_{11} - T_{21} + 2T_{31} = \emptyset$

$\Rightarrow$

$T_{11} = 1$

$T_{21} = -2/3$

$T_{31} = -1/3$

Para $\lambda = 2$

$$\begin{pmatrix} -1 & \varnothing & \varnothing \\ \varnothing & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} T_{12} \\ T_{22} \\ T_{32} \end{pmatrix} = \begin{pmatrix} \varnothing \\ \varnothing \\ \varnothing \end{pmatrix}$$

$$\begin{aligned} -T_{12} &= \varnothing \\ T_{22} - T_{32} &= \varnothing \\ T_{12} - T_{22} + T_{32} &= \varnothing \end{aligned} \quad // \quad \begin{aligned} T_{12} &= \varnothing \\ T_{22} &= 1 \\ T_{32} &= 1 \end{aligned}$$

Para $\lambda_3 = 4$

$$\begin{pmatrix} -3 & \varnothing & \varnothing \\ \varnothing & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} T_{13} \\ T_{23} \\ T_{33} \end{pmatrix} = \begin{pmatrix} \varnothing \\ \varnothing \\ \varnothing \end{pmatrix}$$

$$\begin{aligned} -T_{13} &= \varnothing \\ -T_{23} - T_{33} &= \varnothing \end{aligned} \quad // \quad \begin{aligned} T_{13} &= \varnothing \\ T_{23} &= -1 \\ T_{33} &= 1 \end{aligned}$$

$$[T] = \begin{pmatrix} 1 & \varnothing & \varnothing \\ -1/3 & 1 & -1 \\ -2/3 & 1 & 1 \end{pmatrix}$$

Para hacer $[A]^2 = \begin{bmatrix} a \cdot a + b \cdot c & ab + db \\ c \cdot a + d \cdot c & cb + dd \end{bmatrix}$

pero al diagonalizar, se puede D^2

Ejercicio

Diagonalizar y elevar al cuadrado $[A] = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & \varnothing \\ -1 & -2 & -1 \end{pmatrix}$

$$f_{(\lambda)} = \left| \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & \varnothing \\ -1 & -2 & -1-\lambda \end{pmatrix} \right| = \varnothing$$

$$= (1-\lambda)(-1-\lambda)^2 + 12(1+\lambda) + 13 - \lambda$$
$$= -\lambda(\lambda^2 + \lambda - 12) = \varnothing$$

$$// \quad \begin{aligned} \lambda_1 &= \varnothing \\ \lambda_2 &= 3 \\ \lambda_3 &= -4 \end{aligned}$$

Para $\lambda = 1$

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} T_{11} + 2T_{21} + T_{31} &= 0 \\ 6T_{11} - T_{21} &= 0 \\ -T_{11} - 2T_{21} - T_{31} &= 0 \end{aligned}$$

$$T_{11} = \frac{-1}{13}, \quad T_{21} = \frac{-6}{13}, \quad T_{31} = 1$$

Para $\lambda = 2$

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \begin{pmatrix} T_{12} \\ T_{22} \\ T_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} -2T_{12} + 2T_{22} + T_{32} &= 0 \\ 6T_{12} - 4T_{22} &= 0 \\ -T_{12} - 2T_{22} - 4T_{32} &= 0 \end{aligned}$$

$$T_{12} = -1, \quad T_{22} = \frac{-3}{2}, \quad T_{32} = 1$$

Para $\lambda = -4$

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} T_{13} \\ T_{23} \\ T_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} T_{13} &= -1 \\ T_{23} &= 2 \\ T_{33} &= 1 \end{aligned}$$

$$P = \begin{bmatrix} -1/13 & -1 & -1 \\ -6/13 & -3/2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix} \qquad A^2 = PD^2P^{-1} = \begin{bmatrix} 3 & 4 & 2 \\ 12 & 37 & 18 \\ -6 & -10 & -5 \end{bmatrix}$$

Ejercicio

$$x_{(n+3)} - 3x_{(n+2)} - 4x_{(n+1)} + 12x_n = \emptyset \; ; \; n \geq \emptyset$$

$x_\emptyset = \emptyset$
$x_1 = -11$
$x_2 = -15$

Se propone $x_{(n)} = r^n$

$$r^{n+3} - 3r^{n+2} - 4r^{n+1} + 12^n = \emptyset$$

$$r^n(r^3 - 3r^2 - 4r + 12) = \emptyset$$

$$r^n(r-3)(r+2)(r-2) = \emptyset$$

$$r_1 = 3 \; , \; r_2 = -2 \; , \; r_3 = 2$$

$$x_{(n)} = C_1 3^n + C_2(-2)^n + C_3 2^n$$

$$x_\emptyset = C_1 + C_2 + C_3 = \emptyset$$

$$x_1 = 3C_1 - 2C_2 + 2C_3 = -11$$

$$x_2 = 9C_1 + 4C_2 + 4C_3 = 15$$

$C_1 = \frac{3}{7}$

$C_2 = \frac{7}{2}$

$C_3 = \frac{-13}{2}$