CS 162 Programming languages

Lecture 5: λ-calculus

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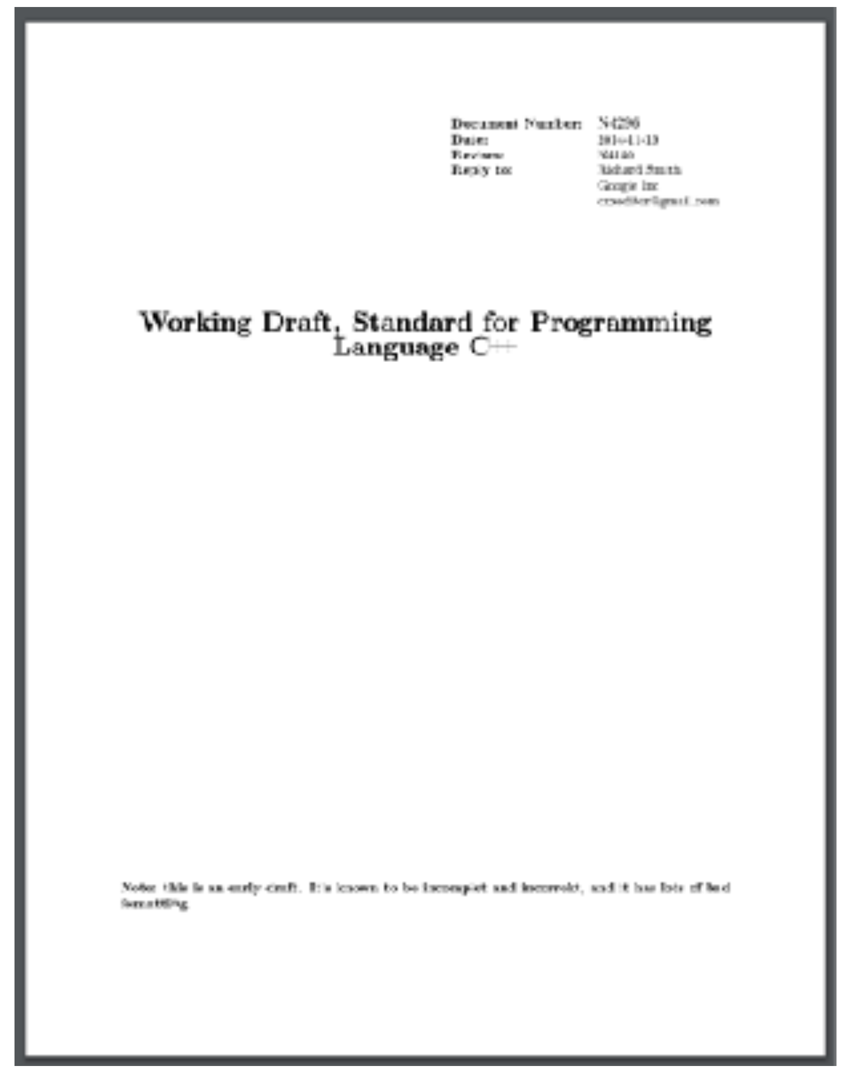
Winter 2023

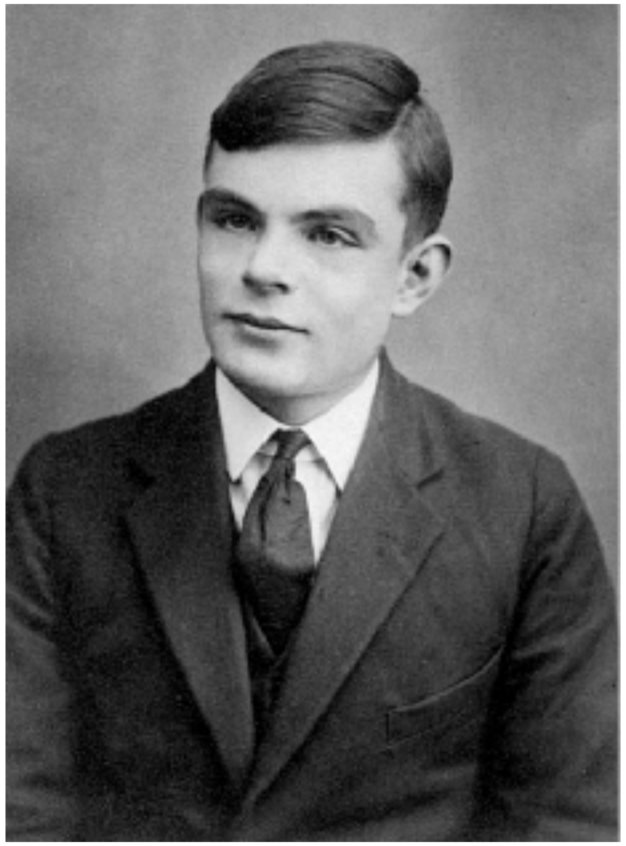
Your favorite language?

- Assignment
- Booleans, integers, characters, strings, …
- Conditionals
- Loops
- Functions
- Recursion
- References / pointers
- Objects and classes
- Inheritance


Working Draft, Standard for Programming Language C++

1368 pages in 2014!

The smallest universal language



Alan Turing

Alonzo Church

The Calculi of Lambda-Conversion, 1936
ENIAC, 1943

The next 700 languages

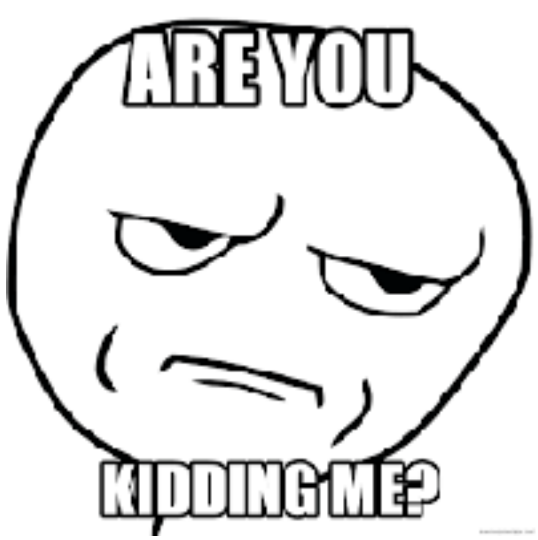
"Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus."

Peter Landin 1966

The λ-Calculus

Has one and ONLY one feature

- Functions



- ~~Assignment~~
- ~~Booleans, integers, characters, strings, …~~
- ~~Conditionals~~
- ~~Loops~~
- Functions
- ~~Recursion~~
- ~~References / pointers~~
- ~~Objects and classes~~
- ~~Inheritance~~

The λ-Calculus

More precisely, the only things you can do are:

Define a function

Call a function

“Free your mind”

Design a programming language

- Syntax: what do programs look like?
  - Grammar: what programs are we allowed to write?
- Semantics: what do programs mean?
  - Operational semantics: how do programs execute step-by-step?

Syntax: what programs look like

$$
\begin{aligned}
e ::= \; & x \\
\mid \; & \lambda x.\, e \\
\mid \; & e_1 \; e_2
\end{aligned}
$$

\x → e (Haskell)

fun x → e (OCaml)

lambda x. e (λ+)

- Programs are expressions e (also called λ-terms) of one of three kinds:
  - Variable x, y, z
  - Abstraction (i.e. nameless function definition)
    - $\lambda x.\, e$
    - x is the formal parameter, e is the function body
  - Application (i.e. function call)
    - $e_1 \; e_2$
    - e_1 is the function, e_2 is the argument

Running examples

$\lambda x.\, x$ — The identity function

$\lambda x.\, (\lambda y.\, y)$ — A function that returns the identity function

$\lambda f.\, f\, (\lambda x.\, x)$ — A function that applies its argument to the identity

Semantics: what programs mean

- How do I execute a λ-term?
- "Execute": rewrite step-by-step following simple rules, until no more rules apply

Similar to simplifying (x+1) * (2x -2) using middle-school algebra

$$
\begin{aligned}
e ::= \; & x \\
| \; & \lambda x.\, e \\
| \; & e_1 \; e_2
\end{aligned}
$$

What are the rewrite rules for λ-calculus?

Operational semantics

$$(\lambda x\,.\,t_1)\ t_2 \rightarrow [x \mapsto t_2]t_1$$

β-reduction
(function call)

$[x \mapsto t_2]t_1$ means "t_1 with all ***free occurrences*** of x replaced with t_2"

```
inc1(int x ) {
  return x+1
}

inc1(2);
```

$$(\lambda x\,.\,x + 1)\ 2 \rightarrow [x \mapsto 2]x + 1 = 3$$

$$[x \mapsto y]\lambda x\,.\,x = \lambda x\,.\,y$$

What does free occurrences mean?

Semantics: variable scope

The part of a program where a variable is visible

In the expression λx. e

- x is the newly introduced variable
- e is the scope of x
- any occurrence of x in λx. e is bound (by the binder λx)

x y

λx. x

λy. x y

λx. (λy. x)

(λx. λy. y) x

x is bounded

x is free

An occurrence of x in e is **free** if it's *not bound* by an enclosing abstraction

Semantics: free variables

An variable x is free in e if there exists a free occurrence of x in e

We use "FV" to represent the set of all free variables in a term:

$FV(x) = x$
$FV(\lambda x.\ e) = FV(e) \setminus x$
$FV(e_1\ e_2) = FV(e_1) \cup FV(e_2)$

$FV(x\ y) = \{x, y\}$
$FV(\lambda y.\ x\ y) = \{x\}$
$FV((\lambda x.\ \lambda y.\ y)\ x) = \{x\}$

If e has no free variables it is said to be closed, or combinators

Semantics: β-reduction

$$(\lambda x\,.\,t_1)\ t_2 \to [x \mapsto t_2]t_1$$

β-reduction
(function call)

$[x \mapsto t_2]t_1$ means "t_1 with all ***free occurrences*** of x replaced with t_2"

$$[x \mapsto s]x = s$$
$$[x \mapsto s]y = y(x \neq y)$$
$$[x \mapsto s]\lambda y\,.\,t_1 = \lambda y\,.\,[x \mapsto s]t_1(y \neq x \wedge y \notin FV(s))$$
$$[x \mapsto s]t_1\ t_2 = [x \mapsto s]t_1\ [x \mapsto s]t_2$$

Semantics: α-renaming

$$\lambda x \,.\, e =_\alpha \lambda y \,.\, [x \mapsto y]e$$

- Rename a formal parameter and replace all its occurrences in the body

$$\lambda x \,.\, x =_\alpha \lambda y \,.\, y =_\alpha \lambda z \,.\, z$$

$$[x \mapsto y]\lambda x \,.\, x = \lambda x \,.\, y$$

$$[x \mapsto y]\lambda x \,.\, x =_\alpha [x \mapsto y]\lambda z \,.\, z = \lambda z \,.\, z$$

Currying: multiple arguments

$$\lambda(x, y) \,.\, e = \lambda x \,.\, \lambda y \,.\, e$$

$$(\lambda(x, y) \,.\, x + y)\ 2\ 3 =$$

$$(\lambda x \,.\, \lambda y \,.\, x + y)\ 2\ 3 = (\lambda y.2 + y)\ 3 = [y \mapsto 3]2 + y = 5$$

Transformation of multi-arguments functions to higher-order functions is called currying (in the honor of Haskell Curry)

TODOs by next lecture

- Install OCaml/λ^+ on your laptop
- Continue λ calculus next Wed