CS 162 Programming languages

Lecture 8: Operational Semantics II

Yu Feng

Winter 2022

What we have by far

- Given a program as an input string
- First, we separate a string into words (Lexer)
- Second, we understand sentence structure by diagramming the string (Parser)
- Finally, we assign meanings to the structure sentence (Operational semantics)

Operational Semantics

$$\frac{}{\mathbb{E} \vdash \mathsf{lambda}\, x.\ e \Downarrow \mathsf{lambda}\, x.\ e}\ \textsc{Lambda}$$

Lambda abstractions just evaluate to themselves

$$\frac{\mathbb{E} \vdash e_1 \Downarrow \mathsf{lambda}\, x.\ e_1' \qquad \mathbb{E} \vdash [x \mapsto e_2]e_1' \Downarrow v}{\mathbb{E} \vdash (e_1\ e_2) \Downarrow v}\ \textsc{App}$$

To evaluate the application (e_1 e_2), we first evaluate the expression e_1. The operational semantics "get stuck" if e_1 is not a lambda abstraction. This notion of "getting stuck" in the operational semantics corresponds to a runtime error. Assuming the expression e_1 evaluates to a lambda expression, we evaluate the application expression by binding e_2 to x and then evaluating the expression $[x \rightarrow e_2]e'_1$ as in β-reduction in lambda calculus.

The Lambda rule

- Question: What would change if we write the hypothesis as

$$\frac{\mathbb{E} \vdash e_1 \overset{=}{\not\Downarrow} \mathsf{lambda}\, x.\ e_1' \qquad \mathbb{E} \vdash [x \mapsto e_2] e_1' \Downarrow v}{\mathbb{E} \vdash (e_1\ e_2) \Downarrow v}\ \textsc{App}$$

- Answer: This would still give semantics to (lambda x.x 3), but no longer to let y=lambda x.x in (y 3)

The Lambda rule

- Question: What would change if we write the hypothesis as

$$\frac{\mathbb{E} \vdash e_1 \Downarrow \mathsf{lambda}\, x.\ e_1' \qquad \mathbb{E} \vdash [x \mapsto \cancel{e_2}]e_1' \Downarrow v}{\mathbb{E} \vdash (e_1\ e_2) \Downarrow v}\ \textsc{App}$$

$e_2 \Downarrow v_2$ (above the premise) and v_2 (above the substituted e_2)

- Answer: This is also correct: you will eagerly evaluate e_2 before passing it to the lambda abstraction (call-by-value)

Booleans: implementation

Boolean implementation

- let TRUE = λx y. x -- Returns its first argument
- let FALSE = λx y. y -- Returns its second argument
- let ITE = λb x y. b x y -- Applies condition to branches

Why they are correct?

Booleans: examples

eval ite_true:

ITE TRUE e_1 e_2

= (λb x y. b x y) TRUE e_1 e_2 -- expand def ITE

$=_\beta$ (λx y. TRUE x y) e_1 e_2 -- beta-step

$=_\beta$ (λy. TRUE e_1 y) e_2 -- beta-step

$=_\beta$ TRUE e_1 e_2 -- expand def TRUE

= (λx y. x) e_1 e_2 -- beta-step

$=_\beta$ (λy. e_1) e_2 -- beta-step

$=_\beta$ e_1

Other boolean API:

let NOT = λb. ITE b FALSE TRUE

let AND = λb_1 b_2. ITE b_1 b_2 FALSE

let OR = λb_1 b_2. ITE b_1 TRUE b_2

λ-calculus:Numbers

- Church numerals: a number N is encoded as a combinator that calls a function on an argument N times

let ONE = \f x → f x

let TWO = \f x → f (f x)

let THREE = \f x → f (f (f x))

let FOUR = \f x → f (f (f (f x)))

let FIVE = \f x → f (f (f (f (f x))))

let SIX = \f x → f (f (f (f (f (f x)))))

let ZERO = \f x → x

https://en.wikipedia.org/wiki/Church_encoding

λ-calculus:Numbers API

- Numbers API
  - let INC = (\n f x -> f (n f x)) -- Call `f` on `x` one more time than `n` does

- let ADD = \n m -> n INC m. -- Call `f` on `x` exactly `n + m` times

eval inc_zero :
 INC ZERO
 = (\n f x -> f (n f x)) ZERO
 $=_\beta$ \f x -> f (ZERO f x)
 = \f x -> f x
 = ONE

eval add_one_zero :
ADD ONE ZERO = ONE

https://en.wikipedia.org/wiki/Church_encoding

Recursion

- Recursion can not be directly applied with β-reduction

$$(\lambda x\,.\,x\ x)\ (\lambda x\,.\,x\ x)\ \rightarrow (\lambda x\,.\,x\ x)\ (\lambda x\,.\,x\ x)$$

- Fixed-point combinator is defined to evaluate recursive functions

$$\mathsf{fix} = \lambda f\,.\,(\lambda x\,.f\ (\lambda y\,.\,x\ x\ y))\ (\lambda x\,.f\ (\lambda y\,.\,x\ x\ y))$$

- If we define a recursive function g, then invoking function g on argument a is equivalent to applying fixed-point combinator on g:

$$\mathsf{factorial}\ n = \mathsf{fix}\ g\ n$$

The Fix-point operator

- A fixed-point combinator is a higher-order function that returns some fixed point of its argument

$$\mathsf{fix}\ f = f\ (\mathsf{fix}\ f) \qquad \mathsf{fix}\ f = f(f(\ldots f(\mathsf{fix}\ f)\ldots))$$

- To evaluate a fixed-point expression *fix e*, we first evaluate e to a lambda expression lambda f.e′, which is the generator of a recursive function that refers to itself as f. We then apply the lambda expression to a copy of the fixed-point expression (i.e. substituting fix (lambda f. e′) for f in e′), essentially unrolling the body of the recursive function once.

$$\frac{e \Downarrow \mathsf{lambda}\ f.\ e' \qquad [f \mapsto \mathsf{fix}\ (\mathsf{lambda}\ f.\ e')]e' \Downarrow v}{\mathsf{fix}\ \ e \Downarrow v}\ \textsc{Fix}$$

Call-by-name v.s. call-by-value

- Not evaluating the argument before substitution is known as call-by name, evaluating the argument before substitution as call-by-value.

- Languages with call-by-name: classic lambda calculus, ALGOL 60

- Languages with call-by-value: C, C++, Java, Python, FORTRAN, . . .

- Advantage of call-by-name: If argument is not used, it will not be evaluated

- Disadvantage: If argument is uses k times, it will be evaluated k times!

Operational Semantics

$$\frac{\mathbb{E}(x) = v}{\mathbb{E} \vdash x \Downarrow v}\ \textsc{Var}$$

If variable x is bound to some value v in the environment E, then x evaluates to its bound value.

$$\frac{\mathbb{E} \vdash e_1 \Downarrow v_1 \qquad \mathbb{E}[x \mapsto v_1] \vdash e_2 \Downarrow v_2}{\mathbb{E} \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 \Downarrow v_2}\ \textsc{Let}$$

First evaluate the initial expression e_1 in environment E, which yields value v_1. Then we obtain a new environment E′ by binding identifier x to value v_1, i.e., E′ = E[x ← v_1]. Next, we evaluate the body e_2 in this new environment E′, which yields value v_2 , which is also the result of evaluating the entire let expression.

Environments example

$$\frac{\mathbb{E} \vdash e_1 \Downarrow v_1 \qquad \mathbb{E}[x \mapsto v_1] \vdash e_2 \Downarrow v_2}{\mathbb{E} \vdash \mathsf{let}\, x = e_1 \,\mathsf{in}\, e_2 \Downarrow v_2}\ \textsc{Let} \qquad \frac{\mathbb{E}(x) = v}{\mathbb{E} \vdash x \Downarrow v}\ \textsc{Var}$$

- Consider the λ⁺ program: let x = 3 in x
- Here is the proof that this program evaluates to 3:

$$\frac{E \vdash 3 : 3 \qquad \dfrac{E[x \leftarrow 3](\mathsf{x}) = 3}{E[x \leftarrow 3] \vdash \mathsf{x} : 3}}{E \vdash \mathsf{let}\ x = 3\ \mathsf{in}\ x : 3}$$

Operational Semantics

$$\frac{}{\mathbb{E} \vdash \mathsf{Nil} \Downarrow \mathsf{Nil}}\ \textsc{Nil} \qquad \frac{\mathbb{E} \vdash e_1 \Downarrow v_1 \qquad \mathbb{E} \vdash e_2 \Downarrow v_2}{\mathbb{E} \vdash e_1 \mathbin{@} e_2 \Downarrow v_1 \mathbin{@} v_2}\ \textsc{Cons}$$

a list is either the empty list Nil, or it is a cons cell(e_1@e_2)where e_1 is the head of the list and e_2 is the tail of the list.

$$\frac{\mathbb{E} \vdash e \Downarrow \mathsf{Nil}}{\mathbb{E} \vdash \mathsf{isnil}\, e \Downarrow 1}\ \textsc{IsNilTrue} \qquad \frac{\mathbb{E} \vdash e \Downarrow v_1 \mathbin{@} v_2}{\mathbb{E} \vdash \mathsf{isnil}\, e \Downarrow 0}\ \textsc{IsNilFalse}$$

Since any list value can either be Nil or a cons cell, we have two cases which rule matches triggered will depend on whether e evaluates to Nil or not. If e is not a list, then the evaluation will get stuck.

$$\frac{\mathbb{E} \vdash e \Downarrow v_1 \mathbin{@} v_2}{\mathbb{E} \vdash\, !\, e \Downarrow v_1}\ \textsc{Head} \qquad \frac{\mathbb{E} \vdash e \Downarrow v_1 \mathbin{@} v_2}{\mathbb{E} \vdash\, \#\, e \Downarrow v_2}\ \textsc{Tail}$$

We define similar rules for head and tail

Congratulations!

- You can now understand every page in the λ^+ reference manual
- For HW3&4, you will need to refer to the operational semantics of λ^+ in the manual to implement your interpreter
- The manual is the official source for the semantics of λ^+

Operational semantics

- The rules we have written are known as ***big-step*** operational semantics
- They are called big step because each rule completely evaluates an expression, taking *as many steps as necessary*.
- Example: The plus rule

$$\frac{\mathbb{E} \vdash e_1 \Downarrow i_1 \qquad \mathbb{E} \vdash e_2 \Downarrow i_2}{\mathbb{E} \vdash e_1 + e_2 \Downarrow i_1 + i_2}\ \textsc{Add}$$

- Here, we evaluate both e_1 and e_2 to compute the final value in one (***big***) step
- Alternate formalism for giving semantics: ***small-step*** operational semantics

Small step operational semantics

- Small step operational semantics (denoted as "→") perform *only one step* of computation *per rule* invocation

- You can think of SSOS as "decomposing" all operations that happen in one rule in LSOS into individual steps

- This means: Each rule in SSOS has at most one precondition

$$t \longrightarrow^* v \text{ iff } t \Downarrow v$$

Small step operational semantics

- Consider the plus rule in λ^+ written in SSOS
- Rule 1: Adding two integers

$$\frac{}{\langle c_1 + c_2, E\rangle \rightarrow \langle c_1 + c_2, E\rangle}$$

- Rule 2: Reducing first expression to an integer

$$\frac{\langle e_1, E\rangle \rightarrow \langle c, E'\rangle}{\langle e_1 + e_2, E\rangle \rightarrow \langle c + e_2, E'\rangle}$$

- Rule 3: Reducing second expression to an integer

$$\frac{\langle e, E\rangle \rightarrow \langle c_2, E'\rangle}{\langle c_1 + e, E\rangle \rightarrow \langle c_1 + c_2, E'\rangle}$$

SSOS in action

- Let's use these rules to prove what the value of (2+4)+(6-1) is:
- ⟨(2+4)+(6-1), E⟩→⟨6+(6-1), E⟩→⟨6+5, E⟩→⟨11, E⟩

One atomic step at a time!

Small-step v.s. Big-step

- In big-step semantics, any rule may invoke any number of other rules in the hypothesis

- This means any derivation is a tree.

- In small-step semantics, each rule only performs one step of computation

- This means any derivation is a line

TODOs by next lecture

- HW4 is out. Please start ASAP!
- Will switch to type checking next week