CS 162 Programming languages

Lecture 11: Type Inference I

Yu Feng

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Type sytem in λ⁺

$$\frac{\Gamma, x : \mathsf{T}_1 \vdash e : \mathsf{T}_2}{\Gamma \vdash \mathsf{lambda}\ x : \mathsf{T}_1.\ e : \mathsf{T}_1 \to \mathsf{T}_2}\ \textsc{T-Lambda}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{T}_1 \to \mathsf{T}_2 \qquad \Gamma \vdash e_2 : \mathsf{T}_1}{\Gamma \vdash (e_1\ e_2) : \mathsf{T}_2}\ \textsc{T-App}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Int} \qquad \Gamma \vdash e_2 : \mathsf{T} \qquad \Gamma \vdash e_3 : \mathsf{T}}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \mathsf{T}}\ \textsc{T-If} \qquad \frac{\Gamma \vdash e : \mathsf{List}[\mathsf{T}]}{\Gamma \vdash \mathsf{isnil}\ e : \mathsf{Int}}\ \textsc{T-IsNil}$$

$$\frac{x : \mathsf{T} \in \Gamma}{\Gamma \vdash x : \mathsf{T}}\ \textsc{T-Var} \qquad \frac{\Gamma, x : \mathsf{T}_1 \vdash e_1 : \mathsf{T}_1 \qquad \Gamma, x : \mathsf{T}_1 \vdash e_2 : \mathsf{T}_2}{\Gamma \vdash \mathsf{let}\ x : \mathsf{T}_1 = e_1\ \mathsf{in}\ e_2 : \mathsf{T}_2}\ \textsc{T-Let}$$

$$\frac{}{\Gamma \vdash \mathsf{Nil} : \mathsf{List}[\mathsf{T}]}\ \textsc{T-Nil} \qquad \frac{\Gamma \vdash e_1 : \mathsf{T} \qquad \Gamma \vdash e_2 : \mathsf{List}[\mathsf{T}]}{\Gamma \vdash e_1 @ e_2 : \mathsf{List}[\mathsf{T}]}\ \textsc{T-Cons}$$

Type annotations

- So far when we studied typing, we always assumed that the programmer annotated some types

- Example: We gave types to let bindings and lambda variables in class

- But annotating types can be cumbersome!

- Anyone who has ever written C++ code can really empathize:
  `vector<Map<int,string>>::const_iterator it...`

Type inference

- Goal of type inference: Automatically deduce the type for each expression
- Automatically inferring types: This means the programmer has to write no types, but still gets all the benefit from static typing

Type inference example 1

- Do we really need these type annotations?
- Consider the following example:

```
let f = lambda x.x+2 in ..
```

- Here, we know that function `f` adds two to its argument
- We also know that plus is only defined on integers
- Therefore, the type of `f` must be $Int \rightarrow Int$

Type inference example 2

- Consider the following example:

```
let f = lambda x.lambda y.x+y in ..
```

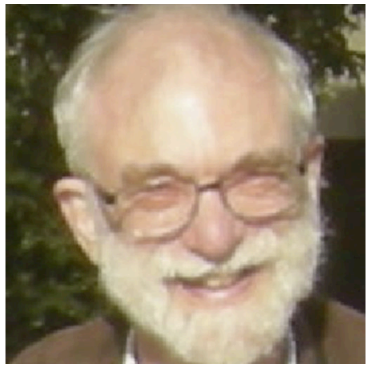
- Here, we know that function `f` has two (curried) arguments, x and y

- We also know that plus is only defined on integers

- Therefore, the type of `f` must be $Int \rightarrow Int \rightarrow Int$

Hindley-Milner type inference

Develop an algorithm that can compute the most general type for any expression without any type annotations



J. Roger Hindley

Robin Milner
Turing Award (1991)

Type variables

- Big idea: Replace the concrete type ***Int*** annotated with a type variable and collect all constraints on this type variable.
- Specifically, pretend that the type of the argument is just some type variable called ***a***
- And for all rules that have preconditions on ***a***, write these preconditions as constraints

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}identifer\ x\\ \Gamma(x) = Int\end{array}}{\Gamma[x \leftarrow Int] \vdash x : Int} \qquad \dfrac{integer\ 2}{\Gamma[x \leftarrow Int] \vdash 2 : Int}}{\Gamma[x \leftarrow \cancel{Int}] \vdash x + 2 : Int}}{\Gamma \vdash \lambda x{:}\underset{\boldsymbol{a}}{\cancel{Int}}.x + 2 : Int \rightarrow Int}$$

Type variables

- Here is the type derivation tree for this expression using type variable ***a***:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\textit{identifer } x\\ \Gamma(x) = a\end{array}}{\Gamma[x \leftarrow a] \vdash x : a} \qquad a = \mathit{Int} \qquad \dfrac{\textit{integer } 2}{\Gamma[x \leftarrow a] \vdash 2 : \mathit{Int}}}{\Gamma[x \leftarrow a] \vdash x + 2 : \mathit{Int}}}{\Gamma \vdash \lambda x{:}a.x + 2 : a \to \mathit{Int}}$$

- Observe that we have one additional precondition on the plus rule: The type variable a must be equal to Int for this rule to apply.

- We now obtain the type: ***a*** → Int and the constraint ***a*** = Int

- Final type: Int → Int

Type variables in typing rules

- We dealt with not knowing the type of $\boldsymbol{x}$ in the following way:
- We introduced a type variable $\boldsymbol{a}$ for the type of $\boldsymbol{x}$
- Every time a rule uses the type of $\boldsymbol{x}$, we use $\boldsymbol{a}$
- Since the plus rule has the precondition that both operands must be of type ***Int***, we introduced a constraint $\boldsymbol{a} = \boldsymbol{Int}$
- After we typed the expression, we had a the type $\boldsymbol{a} \rightarrow \boldsymbol{Int}$ and the constraint $\boldsymbol{a} = \boldsymbol{Int}$
- Solving the collected constraint yields: $\boldsymbol{Int} \rightarrow \boldsymbol{Int}$

Generalizing this example

- This strategy generalizes!
- Introduce type variables for every type annotation
- Collect constraints on type variables during type checking
- Solve this type with respect to the collected constraints

Hindley–Milner Type Inference

Will talk about this in the next lecture

Constraint typing rules

$$\frac{}{\Gamma \vdash i : \mathsf{Int}} \ \textsc{CT-Int}$$

Any number has type int

$$\frac{\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \qquad \Box \in \{+, -, *\} \qquad \mathsf{T}_1 = \mathsf{Int} \qquad \mathsf{T}_2 = \mathsf{Int}}{\Gamma \vdash e_1 \Box e_2 : \mathsf{Int}} \ \textsc{CT-Arith}$$

e_1 and e_2 are of type int

Constraint typing rules

$$\frac{\mathsf{X}\ \text{fresh} \qquad \Gamma, x : \mathsf{X} \vdash e : \mathsf{T}}{\Gamma \vdash \mathsf{lambda}\ x.\ e : \mathsf{X} \to \mathsf{T}}\ \text{CT-LAMBDA}$$

Introduce a ***fresh*** type variable for parameter ***x***

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \\ \mathsf{X}_1, \mathsf{X}_2\ \text{fresh} \qquad \mathsf{T}_1 = \mathsf{X}_1 \to \mathsf{X}_2 \qquad \mathsf{T}_2 = \mathsf{X}_1\end{array}}{\Gamma \vdash (e_1\ e_2) : \mathsf{X}_2}\ \text{CT-APP}$$

The ones circled in red are constraints

Introduce ***fresh*** type variables for functions e_1 and argument e_2

Constraint typing rules

$$\frac{\mathsf{X}\ \mathsf{fresh} \qquad \Gamma, x : \mathsf{X} \vdash e : \mathsf{T}}{\Gamma \vdash \mathsf{lambda}\, x.\ e : \mathsf{X} \rightarrow \mathsf{T}}\ \text{CT-LAMBDA}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \\ \mathsf{X}_1, \mathsf{X}_2\ \mathsf{fresh} \qquad \mathsf{T}_1 = \mathsf{X}_1 \rightarrow \mathsf{X}_2 \qquad \mathsf{T}_2 = \mathsf{X}_1\end{array}}{\Gamma \vdash (e_1\ e_2) : \mathsf{X}_2}\ \text{CT-APP}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \qquad \Box \in \{+, -, *\} \\ \mathsf{T}_1 = \mathsf{Int} \qquad \mathsf{T}_2 = \mathsf{Int}\end{array}}{\Gamma \vdash e_1 \Box e_2 : \mathsf{Int}}\ \text{CT-ARITH}$$

$$\frac{}{\Gamma \vdash i : \mathsf{Int}}\ \text{CT-INT}$$

```
(lambda x. x + 2) 5
```

constraint generation

$\mathrm{Int} = T_2$ (CT-INT)
$T_2 = X_1$ (CT-APP, CT-INT)
$T1 = X_1 \rightarrow X_2$ (CT-APP)
$X_2 = \mathrm{Int}$ (CT-ARITH)

constraint solving

$X_1 = X_2 = T_2 = \mathrm{Int}$
$T_1 = \mathrm{Int} \rightarrow \mathrm{Int}$

Constraint typing rules

$$\frac{x : \mathsf{T} \in \Gamma}{\Gamma \vdash x : \mathsf{T}} \text{ CT-VAR}$$

$\boxed{\text{Look up on the type environment}}$

$$\frac{\mathsf{X}\ \text{fresh} \qquad \Gamma, x : \mathsf{X} \vdash e_1 : \mathsf{T}_1 \qquad \Gamma, x : \mathsf{X} \vdash e_2 : \mathsf{T}_2 \qquad \mathsf{X} = \mathsf{T}_1}{\Gamma \vdash \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 : \mathsf{T}_2} \text{ CT-LET}$$

$\boxed{\text{Introduce fresh type variable for } \boldsymbol{x}}$