Lecture 13: Type Inference II

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Type sytem in λ$^+$

$$\frac{\Gamma, x : \mathsf{T}_1 \vdash e : \mathsf{T}_2}{\Gamma \vdash \mathsf{lambda}\ x : \mathsf{T}_1.\ e : \mathsf{T}_1 \to \mathsf{T}_2}\ \textsc{T-Lambda}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{T}_1 \to \mathsf{T}_2 \qquad \Gamma \vdash e_2 : \mathsf{T}_1}{\Gamma \vdash (e_1\ e_2) : \mathsf{T}_2}\ \textsc{T-App}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Int} \qquad \Gamma \vdash e_2 : \mathsf{T} \qquad \Gamma \vdash e_3 : \mathsf{T}}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \mathsf{T}}\ \textsc{T-If} \qquad \frac{\Gamma \vdash e : \mathsf{List}[\mathsf{T}]}{\Gamma \vdash \mathsf{isnil}\ e : \mathsf{Int}}\ \textsc{T-IsNil}$$

$$\frac{x : \mathsf{T} \in \Gamma}{\Gamma \vdash x : \mathsf{T}}\ \textsc{T-Var} \qquad \frac{\Gamma, x : \mathsf{T}_1 \vdash e_1 : \mathsf{T}_1 \qquad \Gamma, x : \mathsf{T}_1 \vdash e_2 : \mathsf{T}_2}{\Gamma \vdash \mathsf{let}\ x : \mathsf{T}_1 = e_1\ \mathsf{in}\ e_2 : \mathsf{T}_2}\ \textsc{T-Let}$$

$$\frac{}{\Gamma \vdash \mathsf{Nil} : \mathsf{List}[\mathsf{T}]}\ \textsc{T-Nil} \qquad \frac{\Gamma \vdash e_1 : \mathsf{T} \qquad \Gamma \vdash e_2 : \mathsf{List}[\mathsf{T}]}{\Gamma \vdash e_1\ @\ e_2 : \mathsf{List}[\mathsf{T}]}\ \textsc{T-Cons}$$

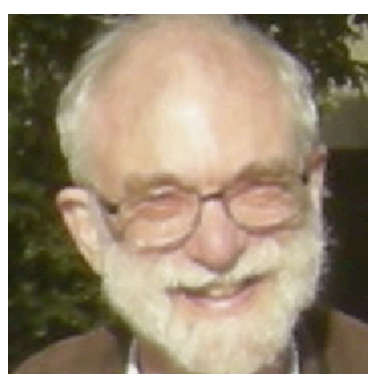
Type inference

- Goal of type inference: Automatically deduce the type for each expression

- Automatically inferring types: This means the programmer has to write no types, but still gets all the benefit from static typing

Hindley-Milner type inference

Develop an algorithm that can compute the most general type for any expression without any type annotations



J. Roger Hindley

Robin Milner
Turing Award (1991)

Type variables

- Big idea: Replace the concrete type ***Int*** annotated with a type variable and collect all constraints on this type variable.
- Specifically, pretend that the type of the argument is just some type variable called ***a***
- And for all rules that have preconditions on ***a***, write these preconditions as constraints

$$\dfrac{\dfrac{\begin{array}{c}\mathit{identifer}\ x\\ \Gamma(x) = \mathit{Int}\end{array}}{\Gamma[x \leftarrow \mathit{Int}] \vdash x : \mathit{Int}} \qquad \dfrac{\mathit{integer}\ 2}{\Gamma[x \leftarrow \mathit{Int}] \vdash 2 : \mathit{Int}}}{\dfrac{\Gamma[x \leftarrow \cancel{\mathit{Int}}] \vdash x + 2 : \mathit{Int}}{\Gamma \vdash \lambda x{:}\underset{\boldsymbol{a}}{\cancel{\mathit{Int}}}.x + 2 : \mathit{Int} \to \mathit{Int}}}$$

Type variables

- Here is the type derivation tree for this expression using type variable ***a***:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\textit{identifer } x\\ \Gamma(x) = a\end{array}}{\Gamma[x \leftarrow a] \vdash x : a} \qquad {\color{red} a = \mathit{Int}} \qquad \dfrac{\textit{integer } 2}{\Gamma[x \leftarrow a] \vdash 2 : \mathit{Int}}}{\Gamma[x \leftarrow a] \vdash x + 2 : \mathit{Int}}}{\Gamma \vdash \lambda x{:}a.x + 2 : a \rightarrow \mathit{Int}}
$$

- Observe that we have one additional precondition on the plus rule: The type variable a must be equal to Int for this rule to apply.

- We now obtain the type: ***a*** → Int and the constraint ***a*** = Int

- Final type: Int → Int

Type variables in typing rules

- We dealt with not knowing the type of $\boldsymbol{x}$ in the following way:
- We introduced a type variable $\boldsymbol{a}$ for the type of $\boldsymbol{x}$
- Every time a rule uses the type of $\boldsymbol{x}$, we use $\boldsymbol{a}$
- Since the plus rule has the precondition that both operands must be of type ***Int***, we introduced a constraint $\boldsymbol{a} = \boldsymbol{Int}$
- After we typed the expression, we had a the type $\boldsymbol{a} \rightarrow \boldsymbol{Int}$ and the constraint $\boldsymbol{a} = \boldsymbol{Int}$
- Solving the collected constraint yields: $\boldsymbol{Int} \rightarrow \boldsymbol{Int}$

Generalizing this example

- This strategy generalizes!
- Introduce type variables for every type annotation
- Collect constraints on type variables during type checking
- Solve this type with respect to the collected constraints

Hindley–Milner Type Inference

Constraint typing rules

$$\frac{}{\Gamma \vdash i : \mathsf{Int}} \text{ CT-INT}$$

Any number has type int

$$\frac{\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \qquad \Box \in \{+, -, *\} \qquad \mathsf{T}_1 = \mathsf{Int} \qquad \mathsf{T}_2 = \mathsf{Int}}{\Gamma \vdash e_1 \Box e_2 : \mathsf{Int}} \text{ CT-ARITH}$$

e_1 and e_2 are of type int

Constraint typing rules

$$\frac{\mathsf{X}\ \mathsf{fresh} \qquad \Gamma, x : \mathsf{X} \vdash e : \mathsf{T}}{\Gamma \vdash \mathsf{lambda}\, x.\ e : \mathsf{X} \to \mathsf{T}}\ \textsc{CT-Lambda}$$

Introduce a ***fresh*** type variable for parameter ***x***

The ones circled in red are constraints

$$\frac{\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \\ \mathsf{X}_1, \mathsf{X}_2\ \mathsf{fresh} \qquad \mathsf{T}_1 = \mathsf{X}_1 \to \mathsf{X}_2 \qquad \mathsf{T}_2 = \mathsf{X}_1}{\Gamma \vdash (e_1\ e_2) : \mathsf{X}_2}\ \textsc{CT-App}$$

Introduce ***fresh*** type variables for functions e_1 and argument e_2

Constraint typing rules

$$\frac{\mathsf{X}\ \text{fresh} \qquad \Gamma, x : \mathsf{X} \vdash e : \mathsf{T}}{\Gamma \vdash \mathsf{lambda}\, x.\ e : \mathsf{X} \to \mathsf{T}}\ \text{CT-Lambda}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \\ \mathsf{X}_1, \mathsf{X}_2\ \text{fresh} \qquad \mathsf{T}_1 = \mathsf{X}_1 \to \mathsf{X}_2 \qquad \mathsf{T}_2 = \mathsf{X}_1\end{array}}{\Gamma \vdash (e_1\ e_2) : \mathsf{X}_2}\ \text{CT-App}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \mathsf{T}_1 \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \qquad \Box \in \{+, -, *\} \\ \mathsf{T}_1 = \mathsf{Int} \qquad \mathsf{T}_2 = \mathsf{Int}\end{array}}{\Gamma \vdash e_1 \Box e_2 : \mathsf{Int}}\ \text{CT-Arith}$$

$$\frac{}{\Gamma \vdash i : \mathsf{Int}}\ \text{CT-Int}$$

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(lambda x. x + 2) 5
```

constraint generation

$\text{Int} = T_2$ (CT-INT)
$T_2 = X_1$ (CT-APP, CT-INT)
$T_1 = X_1 \rightarrow X_2$ (CT-APP)
$X_2 = \text{Int}$ (CT-ARITH)

constraint solving

$X_1 = X_2 = T_2 = \text{Int}$
$T_1 = \text{Int} \rightarrow \text{Int}$

Constraint solving

Algorithm 1 Unification Algorithm

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procedure UNIFY(C)
    if $C = \emptyset$ **then**
        success
    else
        $C = \{S = T\} \cup C'$
        if $S = T$ **then**
            unify(C')
        else if $S = \tau \wedge \tau \notin FV(T)$ **then**
            unify($C'[\tau \mapsto T]$) $\circ [\tau \mapsto T]$
        else if $T = \tau \wedge \tau \notin FV(S)$ **then**
            unify($C'[\tau \mapsto S]$) $\circ [\tau \mapsto S]$
        else if $S = S_1 \to S_2 \wedge T = T_1 \to T_2$ **then**
            unify($C' \cup \{S_1 = T_1, S_2 = T_2\}$)
        else
            fail

Choose a constraint S=T from C and let C' denote the remaining constraints

Occur check to avoid generating a cyclic substitution such as [X ↦ X →X]

○ performs substitutions over the ***type environment***

Perform substitutions over the ***remaining constraints*** C'

Constraint solving

`(lambda x. x + 2) 5`

constraint generatio

Int = T_2 (CT-INT)
$T_2 = X_1$ (CT-APP, CT-INT)
$T_1 = X_1 \rightarrow X_2$ (CT-APP)
X_2 = Int (CT-ARITH)

constraint solving

$X_1 = X_2 = T_2 =$ Int
T_1 = Int $\rightarrow$ Int

Int $= T_2$, $T_2 = X_1$, $T_1 = X_1 \rightarrow X_2$, $X_2 =$ Int

1) $T_2 \mapsto$ Int   Int $= X_1$, $T_1 = X_1 \rightarrow X_2$, $X_2 =$ Int

2) $X_1 \mapsto$ Int, $T_2 \mapsto$ Int $T_1 =$ Int $\rightarrow X_2$, $X_2 =$ Int

3) $X_1 \mapsto$ Int $T_2 \mapsto$ Int $X_2 =$ Int
$T_1 \mapsto$ Int $\rightarrow X_2$

4) $X_1 \mapsto$ Int, $T_2 \mapsto$ Int $\emptyset$ (done!)
$X_2 \mapsto$ Int $(T_1 \mapsto (\text{Int} \rightarrow X_2))$
$\circ\ (X_2 \mapsto \text{Int})$
$\Downarrow$
$T_1 \mapsto$ Int $\rightarrow$ Int