$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2}$$

$U_{i,j}^{n}$ $n \to$ tiempo, $i,j \to$ espacio (i para x y j para y)

Entonces,

$$\frac{U_{i,j}^{n+1} - 2U_{i,j}^{n} + U_{i,j}^{n-1}}{(\Delta t)^2} = c^2\left(\frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{(\Delta x)^2}\right) + c^2\left(\frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(\Delta y)^2}\right).$$

Tome $r_x \equiv \frac{(\Delta t)^2 c^2}{(\Delta x)^2}$ y $r_y \equiv \frac{(\Delta t)^2 c^2}{(\Delta y)^2}$,

$$U_{i,j}^{n+1} = 2U_{i,j}^{n} - U_{i,j}^{n-1} + r_x U_{i+1,j}^{n} - 2r_x U_{i,j}^{n} + r_x U_{i-1,j}^{n} + r_y U_{i,j+1}^{n} - 2r_y U_{i,j}^{n} + r_y U_{i,j-1}^{n}.$$

Tome $a \equiv 2(1 - r_x - r_y)$,

$$U_{i,j}^{n+1} = a U_{i,j}^{n} - U_{i,j}^{n-1} + r_x U_{i+1,j}^{n} + r_x U_{i-1,j}^{n} + r_y U_{i,j+1}^{n} + r_y U_{i,j-1}^{n}$$

$$\therefore U_{i,j}^{n+1} = a U_{i,j}^{n} - U_{i,j}^{n-1} + r_x\left(U_{i+1,j}^{n} + U_{i-1,j}^{n}\right) + r_y\left(U_{i,j+1}^{n} + U_{i,j-1}^{n}\right). \quad (1)$$

Ahora vea que $\left.\frac{\partial u}{\partial t}\right|_{t=0} = V_0(x,y)$ (esto por condiciones de frontera).

Con ello, $$\frac{U_{i,j}^{1} - U_{i,j}^{-1}}{2\Delta t} = V_{i,j}^{0} \Rightarrow -U_{i,j}^{-1} = 2V_{i,j}^{0}\Delta t - U_{i,j}^{1}. \quad (2)$$

Para $n=0$ en (1), se obtiene que

$$U_{i,j}^{1} = a U_{i,j}^{0} - U_{i,j}^{-1} + r_x\left(U_{i+1,j}^{0} + U_{i-1,j}^{0}\right) + r_y\left(U_{i,j+1}^{0} + U_{i,j-1}^{0}\right). \quad (3)$$

Reemplazando (2) en (3),

$$U_{i,j}^{1} = a U_{i,j}^{0} + 2V_{i,j}^{0}\Delta t - U_{i,j}^{1} + r_x\left(U_{i+1,j}^{0} + U_{i-1,j}^{0}\right) + r_y\left(U_{i,j+1}^{0} + U_{i,j-1}^{0}\right)$$

$$\Rightarrow 2U_{i,j}^{1} = a U_{i,j}^{0} + 2V_{i,j}^{0}\Delta t + r_x\left(U_{i+1,j}^{0} + U_{i-1,j}^{0}\right) + r_y\left(U_{i,j+1}^{0} + U_{i,j-1}^{0}\right)$$

$$\therefore U_{i,j}^{1} = \frac{a}{2} U_{i,j}^{0} + V_{i,j}^{0}\Delta t + \frac{r_x}{2}\left(U_{i+1,j}^{0} + U_{i-1,j}^{0}\right) + \frac{r_y}{2}\left(U_{i,j+1}^{0} + U_{i,j-1}^{0}\right).$$

Finalmente, queda demostrado que

$$U_{i,j}^{n+1} = a U_{i,j}^{n} - U_{i,j}^{n-1} + r_x\left(U_{i+1,j}^{n} + U_{i-1,j}^{n}\right) + r_y\left(U_{i,j+1}^{n} + U_{i,j-1}^{n}\right) \quad \text{y}$$

$$U_{i,j}^{1} = \frac{a}{2} U_{i,j}^{0} + V_{i,j}^{0}\Delta t + \frac{r_x}{2}\left(U_{i+1,j}^{0} + U_{i-1,j}^{0}\right) + \frac{r_y}{2}\left(U_{i,j+1}^{0} + U_{i,j-1}^{0}\right). \blacksquare$$