$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2}$$

$U^{n \to \text{tiempo}}_{i,j \to \text{espacio}}$ (i para x y j para y)

Entonces,
$$\frac{U^{n+1}_{i,j} - 2U^n_{i,j} + U^{n-1}_{i,j}}{(\Delta t)^2} = c^2\left(\frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{(\Delta x)^2}\right) + c^2\left(\frac{U^n_{i,j+1} - 2U^n_{i,j} + U^n_{i,j-1}}{(\Delta y)^2}\right).$$

Tome $r_x \equiv \frac{(\Delta t)^2 c^2}{(\Delta x)^2}$ y $r_y \equiv \frac{(\Delta t)^2 c^2}{(\Delta y)^2}$,

$$U^{n+1}_{i,j} = 2U^n_{i,j} - U^{n-1}_{i,j} + r_x U^n_{i+1,j} - 2r_x U^n_{i,j} + r_x U^n_{i-1,j} + r_y U^n_{i,j+1} - 2r_y U^n_{i,j} + r_y U^n_{i,j-1}$$

Tome $a \equiv 2(1 - r_x - r_y)$,

$$U^{n+1}_{i,j} = a U^n_{i,j} - U^{n-1}_{i,j} + r_x U^n_{i+1,j} + r_x U^n_{i-1,j} + r_y U^n_{i,j+1} + r_y U^n_{i,j-1}$$

$$\therefore U^{n+1}_{i,j} = a U^n_{i,j} - U^{n-1}_{i,j} + r_x\left(U^n_{i+1,j} + U^n_{i-1,j}\right) + r_y\left(U^n_{i,j+1} + U^n_{i,j-1}\right). \quad (1)$$

Ahora vea que $\left.\frac{\partial u}{\partial t}\right|_{t=0} = V_0(x,y)$ (esto por condiciones de frontera).

Con ello,
$$\frac{U^1_{i,j} - U^{-1}_{i,j}}{2\Delta t} = V^0_{i,j} \Rightarrow -U^{-1}_{i,j} = 2V^0_{i,j}\Delta t - U^1_{i,j}. \quad (2)$$

Para $n=0$ en (1), se obtiene que

$$U^1_{i,j} = a U^0_{i,j} - U^{-1}_{i,j} + r_x\left(U^0_{i+1,j} + U^0_{i-1,j}\right) + r_y\left(U^0_{i,j+1} + U^0_{i,j-1}\right). \quad (3)$$

Reemplazando (2) en (3),

$$U^1_{i,j} = a U^0_{i,j} + 2V^0_{i,j}\Delta t - U^1_{i,j} + r_x\left(U^0_{i+1,j} + U^0_{i-1,j}\right) + r_y\left(U^0_{i,j+1} + U^0_{i,j-1}\right)$$

$$\Rightarrow 2U^1_{i,j} = a U^0_{i,j} + 2V^0_{i,j}\Delta t + r_x\left(U^0_{i+1,j} + U^0_{i-1,j}\right) + r_y\left(U^0_{i,j+1} + U^0_{i,j-1}\right)$$

$$\therefore U^1_{i,j} = \frac{a}{2} U^0_{i,j} + V^0_{i,j}\Delta t + \frac{r_x}{2}\left(U^0_{i+1,j} + U^0_{i-1,j}\right) + \frac{r_y}{2}\left(U^0_{i,j+1} + U^0_{i,j-1}\right).$$

Finalmente, queda demostrado que

$$U^{n+1}_{i,j} = a U^n_{i,j} - U^{n-1}_{i,j} + r_x\left(U^n_{i+1,j} + U^n_{i-1,j}\right) + r_y\left(U^n_{i,j+1} + U^n_{i,j-1}\right) \quad \text{y}$$

$$U^1_{i,j} = \frac{a}{2} U^0_{i,j} + V^0_{i,j}\Delta t + \frac{r_x}{2}\left(U^0_{i+1,j} + U^0_{i-1,j}\right) + \frac{r_y}{2}\left(U^0_{i,j+1} + U^0_{i,j-1}\right). \blacksquare$$