LISTA 1

1. a) Comprobar que $y = e^{x^2}\left(c + \int_0^x e^{-t^2} dt\right)$ es solución de $y' = 2xy +$

$$y' = 2xe^{x^2}c + e^{x^2}e^{-x^2} + \int_0^x e^{-t^2} \cdot 2xe^{x^2} = 2x\underbrace{e^{x^2}\left(c + \int_0^x e^{-t^2}\right)}_{y} + 1$$

$$= 2xy + 1$$

b) Dados y_1, y_2 soluciones diferentes, calcular la ecuación diferencial que satisface $u = y_1 - y_2$

$$u = e^{x^2}\left(c_1 + \int_0^x e^{-t^2} dt\right) - e^{x^2}\left(c_2 + \int_0^x e^{-t^2} dt\right) = e^{x^2}(c_1 - c_2)$$

$$u' = 2xe^{-x^2}(c_1 - c_2) = 2xu \Longrightarrow \boxed{u' = 2xu}$$

• Otra forma: $u' = y_1' - y_2' = 2xy_1 + 1 - (2xy_2 + 1) = 2x(y_1 - y_2) =$

$$= 2xu \Longrightarrow \boxed{u' = 2xu}$$

2. a) $y = e^{mx}$ Hallar m para que se cumpla $2y''' + y'' - 5y' + 2y = 0$

$y''' = m^3 e^{mx}$ $\quad 2m^3e^{mx} + m^2e^{mx} - 5me^{mx} + 2e^{mx} = 0$

$y'' = m^2 e^{mx}$ $\quad (2m^3 + m^2 - 5m + 2)e^{mx} = 0$

$y' = me^{mx}$ $\quad (m-1)(2m^2 + 3m - 2)e^{mx} = 0$

$$\begin{array}{c|cccc} & 2 & 1 & -5 & 2 \\ 1 & & 2 & 3 & -2 \\ \hline & 2 & 3 & -2 & \boxed{0} \end{array}$$

$\boxed{m = 1}$

$$\frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4} = \begin{cases} \boxed{m = \frac{1}{2}} \\ \boxed{m = -2} \end{cases}$$

b) $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$

$y = ae^x + be^{\frac{1}{2}x} + ce^{-2x}$ ← combinación lineal de las solucione

$$\begin{cases} y(0) = a + b + c = 0 \\ y'(0) = a + \frac{b}{2} - 2c = 1 \\ y''(0) = a + \frac{b}{4} + 4c = -1 \end{cases} \Longrightarrow \boxed{b = 0} \quad \boxed{a = \frac{1}{3}} \quad \boxed{c = -\frac{1}{3}}$$

3. $Cx - y\int_0^x \frac{\operatorname{sen} t}{t}\,dt = 0$

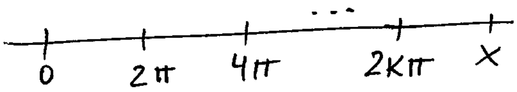
$y = \dfrac{Cx}{\int_0^x \frac{\operatorname{sen} t}{t}\,dt}$; $F(x) = \int_0^x \frac{\operatorname{sen} t}{t}\,dt$ tiene que ser $\neq 0 \quad \forall x$

- Si $x = 0$:

$$y(0) = \lim_{x\to 0} \frac{Cx}{\int_0^x \frac{\operatorname{sen} t}{t}\,dt} \overset{L'H}{=} \lim_{x\to 0} \frac{C}{\frac{\operatorname{sen} x}{x}} = \lim_{x\to 0} \frac{Cx}{\operatorname{sen} x} = C \neq 0$$

(enunciado)

- Si $x > 0$:



Sea $x > 0$ $\exists K_0 \in \mathbb{N}\cup\{0\}$ tal que $x > 2K_0\pi \ \wedge \ x \leq 2(K_0+1)\pi$

$$\int_0^x \frac{\operatorname{sen} t}{t}\,dt = \sum_{K=0}^{K_0-1} \int_{2K\pi}^{2(K+1)\pi} \frac{\operatorname{sen} t}{t}\,dt + \int_{2K_0\pi}^{x} \frac{\operatorname{sen} t}{t}\,dt =$$

$t - 2K\pi = s$ (cambio variable 1ª integral)

$t - 2K_0\pi = s$ (C.V. 2ª integral)

$$= \sum_{K=0}^{K_0-1} \int_0^{2\pi} \frac{\operatorname{sen}(s+2K\pi)}{s+2K\pi}\,ds + \int_0^{x-2K_0\pi} \frac{\operatorname{sen}(s+2K_0\pi)}{s+2K_0\pi}\,ds =$$

$\operatorname{sen}(s+2K\pi) = \operatorname{sen}(s)$

$\operatorname{sen}(s+2K_0\pi) = \operatorname{sen}(s)$

$s + 2K_0\pi \leq x$

$$= \sum_{K=0}^{K_0-1} \int_0^{2\pi} \frac{\operatorname{sen}(s)}{s+2K\pi}\,ds + \int_0^{x-2K_0\pi} \frac{\operatorname{sen}(s)}{s+2K_0\pi}\,ds > \sum_{K=0}^{K_0-1} \int_0^{2\pi} \frac{\operatorname{sen}(s)}{x}\,ds + \int_0^{x-2K_0\pi} \frac{\operatorname{sen}(s)}{x}\,ds$$

$s + 2K\pi \leq \underbrace{2\pi + 2K\pi}_{2(K+1)\pi} \leq 2K_0\pi < x$ (como $K \leq K_0 - 1$)

$\dfrac{1}{s+2K\pi} > \dfrac{1}{x}$

$$= \sum_{K=0}^{K_0-1} \int_0^{2\pi} \cancel{\frac{\operatorname{sen}(s)}{x}}^{\,0}\,ds + \int_0^{x-2K_0\pi} \frac{\operatorname{sen}(s)}{x}\,ds =$$

$$= \frac{1}{x}\left(-\cos(x - 2K_0\pi) + \cos 0\right) = \frac{1-\cos x}{x} \geq 0$$

$-1 \leq \cos x \leq 1$

$-1 \leq -\cos x \leq 1$

$\boxed{0 \leq 1 - \cos x \leq 2}$

Por tanto, si $x > 0$ $y(x)$ está bien definida

- Si $x < 0$: análogo

Concluimos que para cualquier x podemos escribir

$$y = \frac{Cx}{\int_0^x \frac{\operatorname{sen} t}{t}\,dt} \Longrightarrow C = \frac{y\int_0^x \frac{\operatorname{sen} t}{t}\,dt}{x}$$

Derivamos: $y' = \dfrac{C\int_0^{x}\frac{\text{sen}\,t}{t}dt - C\cancel{x}\cdot\frac{\text{sen}\,x}{\cancel{x}}}{\left(\int_0^x \frac{\text{sen}\,t}{t}dt\right)^2} =$

$$= \frac{C}{\int_0^x \frac{\text{sen}\,t}{t}dt} - \frac{C\,\text{sen}\,x}{\left(\int_0^x \frac{\text{sen}\,t}{t}dt\right)^2} = \frac{y}{x} - \frac{\text{sen}\,x\cdot y}{x\int_0^x \frac{\text{sen}\,t}{t}dt}$$

¿ $y'(0)$? $\quad y'(0) = C\lim_{x\to 0} \dfrac{\int_0^x \frac{\text{sen}\,t}{t}dt - \text{sen}\,x}{x\int_0^x \frac{\text{sen}\,t}{t}dt} \overset{L'H}{=} C\lim_{x\to 0} \dfrac{\frac{\text{sen}\,x}{x} - \cos x}{\cancel{x}\,\frac{\text{sen}\,x}{\cancel{x}} + \int_0^x \frac{\text{sen}\,t}{t}dt}$

$$= C\lim_{x\to 0} \frac{\frac{x\cos x + \text{sen}\,x}{x^2} + \text{sen}\,x}{\cos x + \frac{\text{sen}\,x}{x}} = C\cdot\frac{0}{2} = 0$$

14 $100° \longrightarrow 90°$ en 5 segundos a $20°$ ambiente
¿Cuánto tardará en estar a $30°C$?

$$\frac{dT}{dt} = K(T - T_{AMB})$$

$$T' = k(T - T_A)$$

$T(0) = 100°$ $\quad T(5) = 90°$ $\quad T_A = 20°$ $\quad$ ¿ $T(x) = 30°$?

$$\int \frac{dT}{T-20} = \int K\,dt \qquad \log|T-20| = Kt + C \implies |T-20| = e^{Kt}\cdot \underbrace{e^{C}}_{\text{constante}}$$

$T \geq 20$: $\quad T - 20 = e^{Kt}\cdot\tilde{C}$

$$\boxed{T(t) = 20 + \tilde{C}e^{Kt}}$$

$$100 = 20 + \tilde{C} \implies \boxed{\tilde{C} = 80}$$

$$90 = 20 + 80\cdot e^{5K}$$

$$\frac{70}{80} = e^{5K}$$

$$5K = \log\left(\tfrac{7}{8}\right)$$

$$K = \log\left(\sqrt[5]{7/8}\right)$$

$$T(x) = 20 + 80\cdot e^{\log\left(\sqrt[5]{7/8}\right)x}$$

$$\frac{10}{80} = e^{\log\left(\sqrt[5]{7/8}\right)x}$$

$$\log\left(\tfrac{1}{8}\right) = \log\left(\sqrt[5]{7/8}\right)x$$

$$\boxed{x = \frac{\log\left(\frac{1}{8}\right)}{\log\left(\sqrt[5]{7/8}\right)}}$$

19. LEY DE MALTUS

Tasa de variación proporcional al número de individuos.
Si tarda en duplicarse 24h, ¿cuánto en triplicarse?

$$\boxed{\frac{dP}{dt} = KP}$$

$$\frac{dP}{P} = Kdt \Rightarrow \int \frac{dP}{P} = \int Kdt \Rightarrow \ln P = Kt + c$$
$$P = e^{Kt} \cdot e^{c}$$

$$\left.\begin{array}{l} P(t) = e^{Kt} \cdot e^{c} \\ P(0) = P_0 \\ P(24) = 2P_0 \end{array}\right\} \longrightarrow \begin{array}{l} P_0 = 1 \cdot e^{c} \\ 2P_0 = e^{24K} \cdot \cancel{e^{c}} \nearrow P_0 \\ 2 = e^{24K} \longrightarrow \ln 2 = 24K \longrightarrow \boxed{K = \frac{\ln 2}{24}} \end{array}$$

$$P(t) = e^{\frac{\ln 2}{24} t} \cdot P_0 = 3P_0 \ ?$$
$$e^{\frac{\ln 2}{24} t} = 3 \rightarrow \frac{\ln 2}{24} t = \ln 3 \rightarrow \boxed{t = \frac{24 \ln 3}{\ln 2}} \text{ triplica}$$

16. $r(0) = 1$, $r(1) = 0'5$ [*] $\quad \frac{dV}{dt} = K.S \quad V = \frac{4}{3}\pi r^3 \quad S = 4\pi r^2$

En todo el problema $\boxed{r = r(t)}$. $\quad \boxed{V = V(r(t))}$

$$\frac{dV}{dt} = V' = \frac{dV}{dr} \cdot \frac{dr}{dt} \text{ (regla cadena)} \rightarrow V' = \cancel{4\pi r^2} \cdot \frac{dr}{dt} = K.S = K\cancel{4\pi r^2} \Rightarrow$$

$$\Rightarrow \frac{dr}{dt} = K \Rightarrow dr = Kdt$$

Variables separables:

$$\int dr = \int Kdt \Rightarrow r = Kt + c \xrightarrow{\text{utilizando [*]}} \boxed{\begin{array}{l} c = 1 \\ K = -0'5 \end{array}}$$

$$\boxed{r(t) = -0'5t + 1}$$

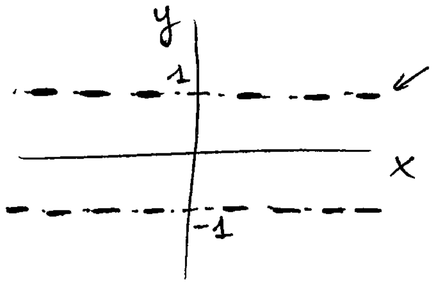
Observación: t es en meses

14. a) Antes de nada, el dibujo:

$y' = y^2 - 1 = f(x,y)$ Isoclinas: $f(x,y) = c$ ← pendiente de la solución

- $\underline{c=0}$

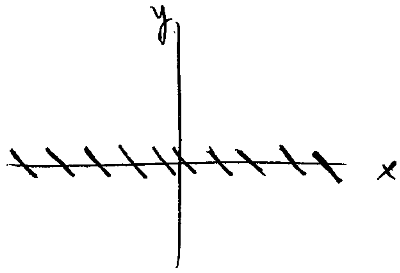
$y^2 - 1 = 0 \Rightarrow y = \pm 1$



← pendiente 0

- $\underline{c=-1}$

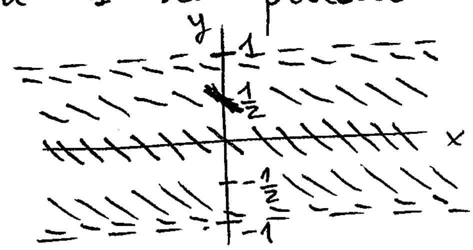
$y^2 - 1 = -1 \Rightarrow y = 0$



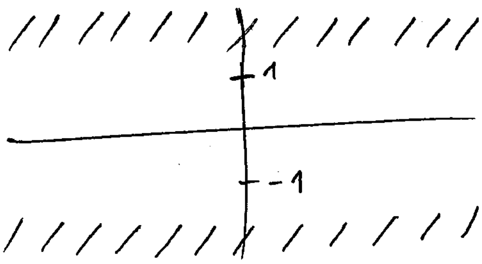
Vemos que si $-1<y<1 \Rightarrow c<0 \Rightarrow \diagdown$ (pendiente negativa)

Según nos acercamos a 1 o a -1 la pendiente se va haciendo cero.

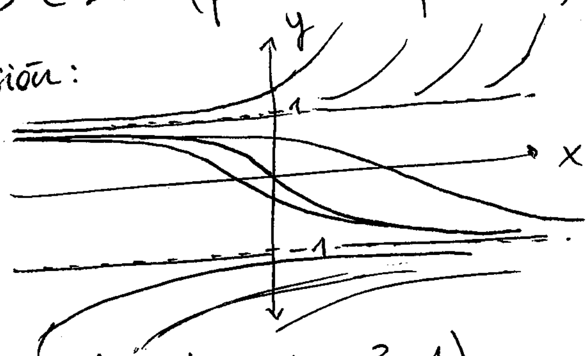
- $y = \pm 1/2 \Rightarrow y' = -3/4 = c$



- $y > 1$ ó $y < -1 \Rightarrow y' > 0 \Rightarrow c > 0$ (pendiente positiva)



En conclusión:



NOTA: $y = \pm 1$ son soluciones estacionarias (cumple $y' = y^2 - 1$)

b) resolver explícitamente:

$$\frac{dy}{dx} = y' = y^2 - 1 = (y-1)(y+1) \Rightarrow \frac{dy}{(y-1)(y+1)} = dx$$

$$\Rightarrow \underbrace{\int \frac{dy}{(y-1)(y+1)}}_{\downarrow} = \int dx \Rightarrow$$

$$\frac{1}{(y-1)(y+1)} = \frac{A}{(y-1)} + \frac{B}{(y+1)} \Rightarrow 1 = A(y+1) + B(y-1)$$

para $y=-1 \Rightarrow 1 = B(y-1) \Rightarrow B = -1/2$

para $y = 1 \Rightarrow 1 = 2A \Rightarrow A = 1/2$

$$\Rightarrow \frac{1}{2}\int\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy = x + c \Rightarrow \frac{1}{2}\left(\log|y-1| - \log|y+1|\right) = x + c$$

$$\Rightarrow \left|\frac{y-1}{y+1}\right| = e^{2x}.K \quad \text{donde } K = e^c$$

$$\left|\frac{y-1}{y+1}\right| = \begin{cases} \frac{y-1}{y+1}, & y>1 \text{ ó } y<-1 \quad (\text{RAMA } 1) \\ \frac{1-y}{y+1}, & -1<y<1 \quad (\text{RAMA } 2) \end{cases} \Rightarrow \begin{cases} \frac{y-1}{y+1} = K_2 e^{2x} & y>1 \text{ ó } y<-1 \\ \frac{1-y}{y+1} = K_1 e^{2x} & -1<y<1 \\ y = 1 & \text{(soluciones} \\ y = -1 & \text{estacionarias)} \end{cases}$$

Hallar la solución que cumple $y(0) = 0$.

Despejando de la rama 2: $\frac{1}{1} = e^0 . K_1 \Rightarrow \boxed{K_1 = 1}$

$$\Rightarrow \begin{cases} \frac{1-y}{y+1} = e^{2x} & -1<y<1 \\ \frac{y-1}{y+1} = K_2 e^{2x} & y>1 \text{ ó } y<-1 \end{cases}$$

¿cuánto tiene que valer K_2 para que sea continua?

Para sacar K_2 (utilizando la continuidad) hacemos:

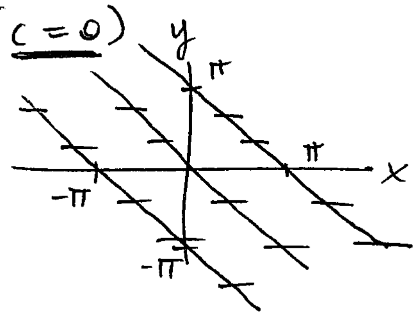
$$\frac{y-1}{y+1} = K_2\frac{1-y}{y+1} \Rightarrow y-1 = K_2(1-y) \Rightarrow \boxed{K_2 = -1}$$

5.) Esbozar las soluciones, trazando algunas isoclinas de $y' = \text{sen}($

$-1 \leq y' \leq 1$ $\quad$ $\text{sen}(x+y) = 0$ $\quad$ $(\underline{c=0})$

$$x + y = K\pi \,, \quad K \in \mathbb{Z}$$

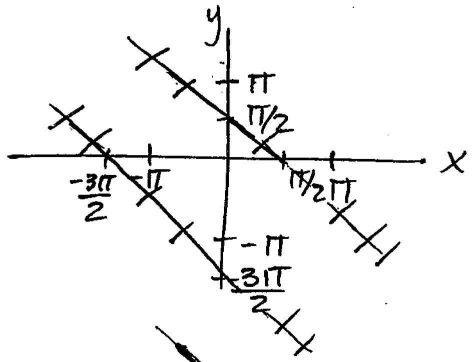
$$y = -x + K\pi \,, \quad K \in \mathbb{Z}$$



$\text{sen}(x+y) = 1$ $\quad$ $(\underline{c=1})$

$$x+y = \frac{\pi}{2} + 2K\pi \,, \quad K \in \mathbb{Z}$$

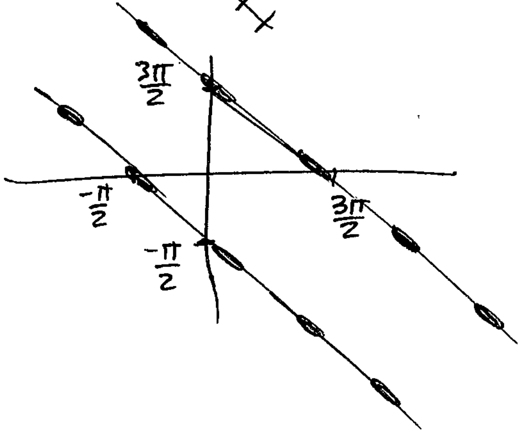
$$y = -x + \frac{\pi}{2} + 2K\pi \quad K \in \mathbb{Z}$$



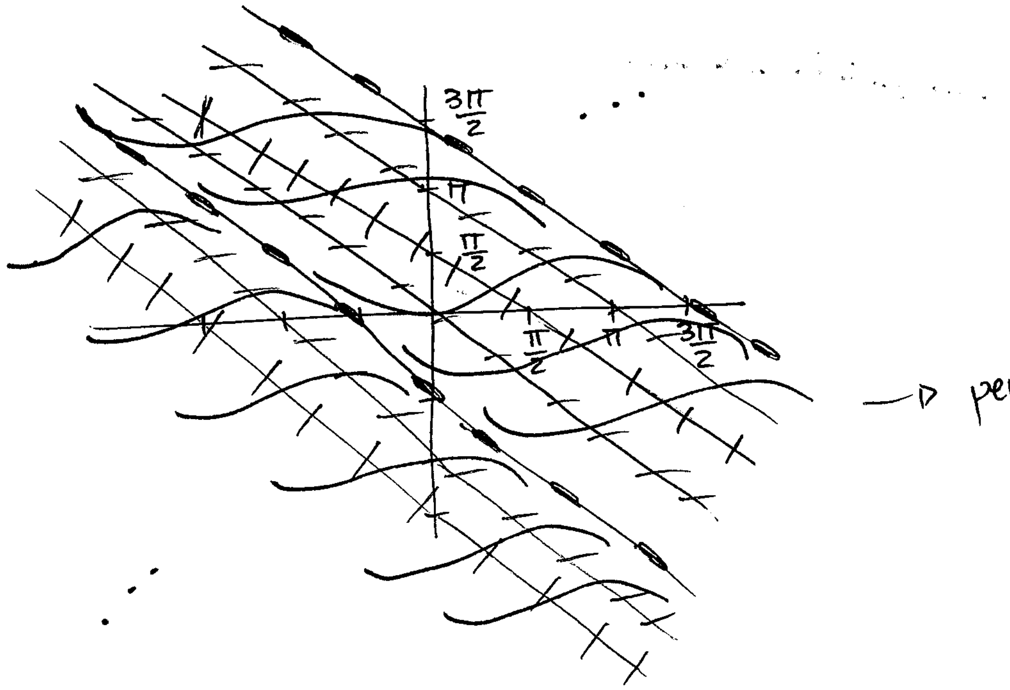
$\text{sen}(x+y) = -1$ $\quad$ $(\underline{c=-1})$

$$x+y = \frac{3\pi}{2} + 2K\pi \,, \quad K \in \mathbb{Z}$$

$$y = -x + \frac{3\pi}{2} + 2K\pi \,, \quad K \in \mathbb{Z}$$



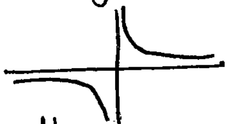
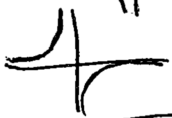
<u>En conclusión:</u>



—▷ periódicas

8. Esbozar las familias de curvas y hallar sus trayectorias ortogonales:

a) $xy = c$

- $\boxed{c = 0}$ $\quad xy = 0 \to x = 0,\ y = 0$ (las curvas son los ejes)
- $\boxed{c > 0}$ $\quad y = \frac{c}{x}$ 
- $\boxed{c < 0}$ $\quad y = \frac{c}{x}$ 

$c = xy$

ec. familia normal

ec. de la familia ortogonal

$$y(x) = \frac{c}{x} \Rightarrow \boxed{y' = \frac{-c}{x^2} = \frac{-y}{x}}$$

$$\Rightarrow \boxed{\tilde{y}' = \frac{-1}{f(x,y)} = \frac{x}{\tilde{y}}} \Rightarrow \frac{d\tilde{y}}{dx} = \frac{x}{\tilde{y}} \Rightarrow \int \tilde{y}\,d\tilde{y} = \int x\,dx \Rightarrow$$

$$\Rightarrow \frac{\tilde{y}^2}{2} = \frac{x^2}{2} + c \Rightarrow \boxed{\tilde{y}^2 = x^2 + K}$$

b) $y = ce^x$

$\quad \Rightarrow c = \frac{y}{e^x}$

$$y' = ce^x = \frac{y}{e^x} e^x = y \Rightarrow \boxed{y' = y = f(x,y)}$$ ec. familia original

ec. familia ortogonal

$$\boxed{\tilde{y}' = \frac{-1}{f(x,y)} = \frac{-1}{\tilde{y}}} \Rightarrow \frac{d\tilde{y}}{dx} = \frac{-1}{\tilde{y}} \Rightarrow \int \tilde{y}\,d\tilde{y} = -\int dx \Rightarrow$$

$$\Rightarrow \frac{\tilde{y}^2}{2} = -x + K_1 \Rightarrow \boxed{\tilde{y}^2 = -2x + K_2}$$ (donde $K_2 = 2K_1$)

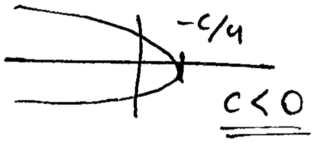
$\boxed{4.}$ Hallar curvas ortogonales de $y^2 - Cx = C^2/4$

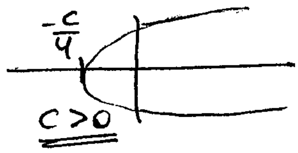
- $\underline{C=0}$, $y^2 = 0 \Rightarrow y = 0$
- $\underline{C \neq 0}$, $\dfrac{C^2}{4} + Cx - y^2 = 0 \Leftrightarrow C = \dfrac{-x \pm \sqrt{x^2 - 4 \cdot \frac{1}{4} \cdot (-y^2)}}{2\,\frac{1}{4}} \Rightarrow$

$$\Rightarrow C = -2x \pm 2\sqrt{x^2+y^2} \quad \begin{array}{l} \nearrow C_1 = -2x - 2\sqrt{x^2+y^2} < 0 \\ \searrow C_2 = -2x + 2\sqrt{x^2+y^2} > 0 \end{array}$$

Derivamos $y^2 - Cx = \dfrac{C^2}{4}$ con respecto de x:

OJO: $y^2 = Cx + \dfrac{C^2}{4} \Rightarrow x = \dfrac{-C}{4}$ para $y=0$





$$2yy' - C = 0$$

$$\cancel{2}yy' + \cancel{2}x \pm \cancel{2}\sqrt{x^2+y^2} = 0$$

$$\boxed{y' = \frac{-x \pm \sqrt{x^2+y^2}}{y}}$$ ECUACIÓN FAMILIA ORIGINAL

Ecuación de la familia ortogonal:

$$y' = \frac{-1}{f(x,y)} = \frac{-y}{-x \pm \sqrt{x^2+y^2}} = \frac{y}{x \pm \sqrt{x^2+y^2}}$$

[...] cambio de variable $z = \dfrac{x}{y}$

atajo porque se sabe la respuesta

Δ Vector tangente de las parábolas con $\oplus$:

$$\left(1, \frac{dy}{dx}\right) = \left(1, \frac{-x+\sqrt{x^2+y^2}}{y}\right) = V_1$$

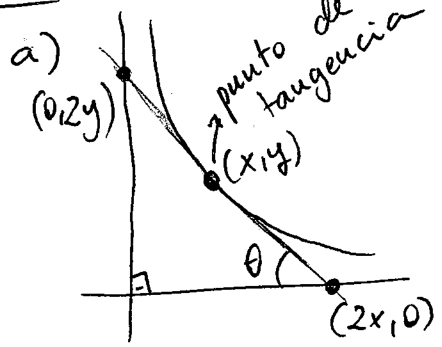
Vector tangente de las parábolas con $\ominus$:

$$\left(1, \frac{dy}{dx}\right) = \left(1, \frac{-x-\sqrt{x^2+y^2}}{y}\right) = V_2$$

$$V_1 \cdot V_2 = 1 + \frac{(-x)^2 - (x^2+y^2)}{y^2} = 0 \Rightarrow \text{ortogonales}$$

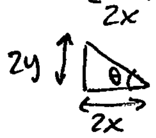
12. CASO 1 (1^{er} cuadrante) (en este caso, punto medio del segmento con los ejes)

a)



⊖ porque la pendiente es negativa

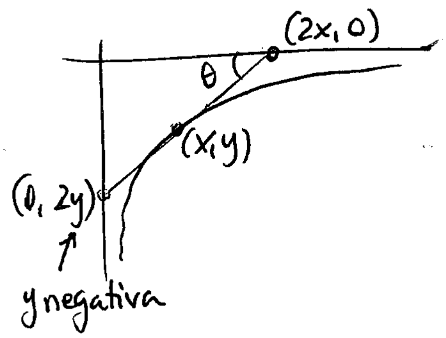
$tg\theta = \frac{2y}{2x} =$

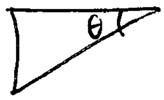


$tg\theta = \frac{-2y}{2x} =$

$$y' = \frac{-2y}{2x} = \frac{-y}{x}$$

CASO 2 (4^{o} cuadrante)





$tg\theta = \frac{-y}{x}$

$$y' = \frac{-y}{x}$$