$\boxed{\text{HOJA 5}}$

$\boxed{3.}$ i) Calcular determinante de f:

$$f: M_{2\times2}(\mathbb{R}) \longrightarrow M_{2\times2}(\mathbb{R})$$

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+5b & b+3c+2d \\ c-d & d \end{pmatrix}$$

matriz triangular $\Rightarrow$ det(M) prod. diagon

$$M_{cc}(f) = \begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \det(M_{cc}(f)) = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

ii) Calcula la matriz A de f respecto de la base

$$B = \left\{ V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ V_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ V_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ V_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

así como su determinante:

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} = -V_1 - V_2 + 2V_3$$

$$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} = -3V_1 + V_2 + 3V_3$$

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} = -2V_1 - 6V_2 - 2V_3 + 5V_4$$

$$f\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 0 & 1 \end{pmatrix} = -5V_1 - 6V_2 + 6V_4$$

a ojo $\Rightarrow$ $A = \begin{pmatrix} -1 & -3 & -2 & -5 \\ -1 & 1 & -6 & -6 \\ 2 & 3 & -2 & 0 \\ 0 & 0 & 5 & 6 \end{pmatrix}$

$|A| = \rightsquigarrow = 1$

5. A matriz definida por $a_{ij} = |i-j|$. Calcula $|A|$

$$\begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{vmatrix} \overset{\text{cada columna se le resta la anterior}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 1 & 1 \\ 3 & -1 & -1 & -1 & 1 \\ 4 & -1 & -1 & -1 & -1 \end{vmatrix} \overset{\text{a cada fila se le resta la anterior}}{=} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{vmatrix} =$$

$$\frac{1}{2}\begin{vmatrix} 0 & \overbrace{1 \quad 1 \quad 1 \quad 1}^{n-1 \text{ unos}} \\ 2 & -2 \quad 0 \quad 0 \quad 0 \\ 2 & 0 \quad -2 \quad 0 \quad 0 \\ 2 & 0 \quad 0 \quad -2 \quad 0 \\ 2 & 0 \quad 0 \quad 0 \quad -2 \end{vmatrix} \overset{C_1 = C_1+C_2+C_3+C_4}{=} \frac{1}{2}\begin{vmatrix} n-1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = (n-1)\cdot\frac{1}{2}\cdot(-2)^{n-1} =$$

$$= \underbrace{(-1)^{n+1}(n-1)\,2^{n-2}} \checkmark$$

6. Demostrar la igualdad:

$$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = \left|\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & c_{n1} & \cdots & c_{nm} \\ \hline 0 & \cdots & 0 & b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{m1} & \cdots & b_{mm} \end{array}\right| = |A|\cdot|B|$$

Razonando como sigue:

i) $D: \underbrace{\mathbb{K}^n \times \overset{n}{\cdots} \times \mathbb{K}^n}_{n \text{ veces}} \longrightarrow \mathbb{K}$

$$D\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \cdots \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}\right) = \left|\begin{array}{ccc|c} x_{11} & \cdots & x_{1n} & \\ \vdots & & \vdots & C \\ x_{n1} & \cdots & x_{nn} & \\ \hline & 0 & & B \end{array}\right|$$

demostrar que es multilineal alternada, luego $D = \lambda \det_{(e_1,\dots,e_n)}$, con $\lambda = D(e_1,\dots,e_n) = |B|$, siendo $\{e_1,\dots,e_n\}$ la base canónica de $\mathbb{K}^n$.

MULTILINEAL:

$$D\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \dots, \begin{pmatrix} \alpha x_{1j} + \beta y_{1j} \\ \alpha x_{nj} + \beta y_{nj} \end{pmatrix}, \dots, \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}\right) =$$

$$= \alpha D\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{nn} \end{pmatrix}, \dots, \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix}, \dots, \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}\right) + \beta D\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \dots, \begin{pmatrix} y_{1j} \\ \vdots \\ y_{nj} \end{pmatrix}, \dots, \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}\right)$$

ALTERNADA : $D\left(\begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \dots, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \dots, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \dots, \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}\right) = 0$

porque en el determinante $\left|\begin{array}{c|c} \begin{matrix} x_{11} \cdots x_{1n} \\ \vdots \\ x_{n1} \cdots x_{nn} \end{matrix} & C \\ \hline 0 & B \end{array}\right|$

hay dos columnas iguales $\Rightarrow$ det =

El espacio vectorial de aplicaciones multilineales alternadas en $\mathbb{K}^n \dots \hat{} \dots \mathbb{K}^n$ en $\mathbb{K}$ tiene dimensión 1 y $\{\det_{(e_1,\dots,e_n)}\}$ es una base

Luego $D = \lambda \det_{(e_1,\dots,e_n)}$, con $\lambda = D(e_1,\dots,e_n) = |B|$ ⇝

⇝ Porque $D(e_1,\dots,e_n) = \lambda \underbrace{\det_{(e_1,\dots,e_n)}(e_1,\dots,e_n)}_{\text{determinante identidad}} = \lambda . 1$

$$D(e_1,\dots,e_n) = \left|\begin{array}{c|c} \begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & 1 \end{matrix} & C \\ \hline 0 & B \end{array}\right| = |B| \text{ desarrollando por columnas.}$$

ii) Si ponemos $V_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, V_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}$ entonces:

$$\left|\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right| = \underbrace{D(V_1,\dots,V_n)}_{\shortparallel} = |B| \det_{(e_1,\dots,e_n)}(V_1,\dots,V_n) = |B| . |A|$$

$$\left|\begin{array}{c|c} \begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{matrix} & C \\ \hline 0 & B \end{array}\right| = |B| . |A|$$

$\boxed{8.}$

iv) $f: V \longrightarrow V$ lineal

$F \subset V$ un subespacio invariante $\left(f(F) \subset F\right) \to$ vectores de F van a F

La matriz de f tiene una forma por cajas

- Aparecen $f_{|F}: F \to F$ y $\bar{f}: V/F \longrightarrow V/F$
- Relación entre los determinantes

$$\boxed{\begin{aligned} &f_{|F}: F \longrightarrow F \\ &f_{|F}(u) = f(u), \quad u \in F \end{aligned}}$$

$$\bar{f}: V/F \longrightarrow V/F$$

$$\bar{f}([u]) = [f(u)], \quad [u] \in V/F$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & V \arrow[d] \\ V/F \arrow[r, dashed, "\bar{f}"] & V/F \end{tikzcd}$$

¿Esto está bien definido?

Hay que ver que si $[u_1] = [u_2] \rightsquigarrow \bar{f}([u_1]) = \bar{f}([u_2])$

$$\bar{f}([u_1]) = [f(u_1)] = [f(u_2 + (u_1 - u_2))] = [f(u_2) + \underbrace{f(u_1-u_2)}_{\in F}] =$$

$[u_1] = [u_2] \Rightarrow u_1 - u_2 \in F \to$ sub. invariante $\Rightarrow$

$$= [f(u_2)] = \bar{f}([u_2]) \Rightarrow \text{Bien definido} \qquad \Rightarrow f(u_1 - u_2) \in F$$

Tomamos $\{v_1, \dots, v_k\} = B_F$ base de F, ampliamos a $\{v_1, \dots, v_k, \dots, v_n\} = B_V$ base de V. Entonces $\{[v_{k+1}], \dots, [v_n]\}$ ~~so~~ $= \bar{B}$ base de V/F.

$M_{B_V}(f)$?

$\to v_1 \in F \subset V$ entonces no aparecen $v_{k+1}, \dots, v_n$

$$f(v_1) = \alpha_{11} v_1 + \dots + \alpha_{k1} v_k$$
$$\vdots$$
$$f(v_k) = \alpha_{1k} v_1 + \dots + \alpha_{kk} v_k$$

$$\Rightarrow M_{B_V}(f) = \left(\begin{array}{ccc|c} \alpha_{11} & & \alpha_{1k} & \\ \vdots & & \vdots & \\ \alpha_{k1} & \cdots & \alpha_{kk} & \\ \hline 0 & \cdots & 0 & \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \end{array}\right)$$

$$\rightsquigarrow f(v_{k+1}) = \alpha_{1,k+1} v_1 + \dots + \alpha_{n,k+1} v_n$$
$$\vdots$$
$$f(v_n) = \alpha_{1n} v_1 + \dots + \alpha_{nn} v_n$$

$$M_{B_F}(f_{|F}) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1K} \\ \vdots & & \\ \alpha_{K1} & \cdots & \alpha_{KK} \end{pmatrix} \quad ; \quad M_{\bar{B}}(\bar{f}) = \begin{pmatrix} \alpha_{K+1,K+1} & \cdots & \alpha_{K+1,n} \\ \vdots & & \\ \alpha_{n,K+1} & \cdots & \alpha_{nn} \end{pmatrix}$$

$$f_{|F}(v_1) = f(v_1) = \alpha_{11} v_1 + \cdots + \alpha_{K+1} v_1$$

$$\bar{f}([v_{K+1}]) = [f(v_{K+1})] = \alpha_{K+1,K+1}[v_{K+1}] + \cdots + \alpha_{n,K+1}[v_n]$$

$$\Rightarrow M_{B_V}(f) = \left(\begin{array}{ccc|ccc} \alpha_{11} & \cdots & \alpha_{1K} & \alpha_{1,K+1} & \cdots & \alpha_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{K1} & \cdots & \alpha_{KK} & \alpha_{K,K+1} & \cdots & \alpha_{Kn} \\ \hline 0 & \cdots & 0 & \alpha_{K+1,K+1} & \cdots & \alpha_{K+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \alpha_{n,K+1} & \cdots & \alpha_{nn} \end{array}\right) = \left(\begin{array}{c|c} M_{B_F}(f_{|F}) & * \\ \hline 0 & M_{\bar{B}}(\bar{f}) \end{array}\right)$$

- RELACIÓN DE LOS DETERMINANTES:

$$\det f = \det f_{|F} \cdot \det \bar{f}$$

i-iii) Comentarios rápidos:

$$f\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} = \begin{pmatrix} 2a & 2b & 3c \\ 2a' & 2b' & 3c' \end{pmatrix} \qquad F = \left\{ \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix} : \begin{array}{l} a + b = 0 \\ a' + b' = 0 \\ c + c' = 0 \end{array} \right\}$$

Comprobamos que $f(F) \subset F$

$\boxed{4.}$ DETERMINANTE DE VANDERMONDE

$x_1, \dots, x_n \in \mathbb{K}$ demuestra:

$$V_n(x_1, x_2, \dots, x_n) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i<j} (x_j - x_i)$$

$$\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & & \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} \overset{\substack{\text{restamos a} \\ \text{cada columna} \\ \text{la anterior} \\ \text{multiplicada} \\ \text{por } x_1}}{=} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2(x_2-1) & & x_2^{n-2}(x_2 \ -1) \\ & x_3 - x_1 & x_3(x_3-1) & & x_3^{n-2}(x_3 \ -1) \\ & & \vdots & & \vdots \\ 1 & x_n - x_1 & x_n(x_n-1) & & x_n^{n-2}(x_n \ -1) \end{vmatrix} =$$

$$= (x_2 - x_1)(x_3 - x_1) \cdots (x_n - x_1) \underbrace{\begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & & \\ 1 & x_n & \cdots & x_n^{n-2} \end{vmatrix}}_{V_{n-1}(x_2, \dots, x_n)} = \left(\prod_{j>1}^{n} (x_j - x_1) \right) \cdot V_{n-1}(x_2, x_3, \dots, x_n)$$

$$\Rightarrow V_n(x_1, \dots, x_n) = \left(\prod_{j>1}^{n} (x_j - x_1) \right) \cdot V_{n-1}(x_2, x_3, \dots x_n) = \left(\prod_{j>1} (x_j - x_1) \right) \left(\prod_{j>2} (x_j - x_2) \right) \cdot V_{n-2}(x_3, \dots, x_n)$$

$$= \cdots = \left(\prod_{\substack{j>i \\ i \leq 2}} (x_j - x_i) \right) V_{n-2}(x_3, \dots, x_n) = \left(\prod_{\substack{j>i \\ i \leq k}} (x_j - x_i) \right) V_{n-k}(x_{k+1}, \dots, x_n) = \cdots$$

```
for (i=1; i<=2; i++){
  for (j=i+1; j≤n; j++){
```

$$\overset{n-2}{=} \left[\prod_{\substack{j>i \\ i \leq n-2}} (x_j - x_i) \right] \underbrace{V_2(x_{n-1}, x_n)}_{\substack{\parallel \\ \begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix} = x_n - x_{n-1}}} = \prod_{\substack{j>i \\ (i \leq n-1)}} (x_j - x_i)$$

$\boxed{7.}$ Calcula:

$$\begin{vmatrix} 2 & 2 & 3 & 4 & \cdots & n-1 & n \\ 1 & 3 & 3 & 4 & \cdots & n-1 & n \\ 1 & 2 & 4 & 4 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 1 & 2 & 3 & 4 & \cdots & n & n \\ 1 & 2 & 3 & 4 & \cdots & n-1 & n+1 \end{vmatrix}$$

Suma primero todas las columnas

$$= \begin{vmatrix} \frac{(n+1)n}{2}+1 & 2 & 3 & \cdots & n-1 & n \\ \vdots & 3 & 3 & \cdots & n-1 & n \\ \vdots & 2 & 4 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \\ \vdots & \vdots & \vdots & & n & n \\ \frac{(n+1)n}{2}+1 & 2 & 3 & \cdots & n-1 & n+1 \end{vmatrix} = 1+\frac{(n+1)n}{2} \begin{vmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 3 & \cdots & n-1 & n \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & n & n \\ 1 & 2 & \cdots & n-1 & n+1 \end{vmatrix} =$$

$$= \frac{(n+1)n}{2}+1 \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 1 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{vmatrix} = \frac{(n+1)n}{2}+1$$

$\boxed{9.}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad adj(A) = \begin{pmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{pmatrix}$$

matriz adjunta o adjugada

i) $A \cdot adj(A)^t = |A| \cdot I = \begin{pmatrix} |A| & 0 & 0 \\ 0 & \boxed{|A|} & 0 \\ 0 & 0 & |A| \end{pmatrix}$

$$|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$$

desarrollo por esta fila

$$\boxed{|A|} = -a_{21}|A_{21}| + a_{22}|A_{22}| - a_{23}|A_{23}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$$

desarrollando por esta fila

$$|A| = a_{31}|A_{31}| - a_{32}|A_{32}| + a_{33}|A_{33}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A|$$

desarrollado por esta fila

$$0 = a_{11}(-|A_{21}|) + a_{12}|A_{22}| + a_{13}(-|A_{23}|) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ ya que } F_1 = F_2$$

desarrollado por esta fila

10.

i) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$X_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$$

ii) $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \frac{1}{|A|} \cdot adj(A)^t \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{pmatrix}$$

$$X_1 = \frac{1}{|A|} \left[b_{1} |A_{11}| - b_2 |A_{21}| + b_3 |A_{31}| \right] =$$

$$= \frac{1}{|A|} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

desarrollado

ii) Demuestra que A es invertible $\Longleftrightarrow |A| \neq 0$, ~~es~~ y que en tal caso: $A^{-1} = \frac{adj(A)^t}{|A|}$

$$|A| \neq 0 \rightsquigarrow A \cdot \left(\frac{1}{|A|} \cdot adj(A)^t \right) = I$$

$\boxed{\text{Proposición: Si } A.B = I \text{, entonces } A \text{ es invertible y } A^{-1} = B.}$

Prueba

$$Im(A) \supset Im(AB) \overset{\text{por el enunciado}}{=} Im(I)$$

porque si $ABx \in Im(AB)$, entonces $A(Bx) \in Im(A)$

A es sobreyectivo, como es endomorfismo, entonces es invertible.

Ahora $AB = I \Rightarrow A^{-1}A\,B = A^{-1} \Rightarrow B = A^{-1}$

Si A es invertible: $1 = det(I) = det(A.A^{-1}) = det(A).det(A$

$\Rightarrow det(A) \neq 0$