HOJA 4

3. $\overbrace{Y' = \underbrace{\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}}_{A} Y}^{(*)}$, $Y(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

TEORÍA

$Y(x) = e^{Ax} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$

$\Rightarrow e^A = \boxed{P e^D} \cdot P^{-1} = P \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} P^{-1}$

matriz fundamental

↳ también es matriz fundamental

- Si A es diagonalizable: $A = PDP^{-1}$ con $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
- Si A no es diagonalizable: $A = PJP^{-1}$ con $J = D + N$ (nilpotente)

La MATRIZ FUNDAMENTAL: matriz cuyas columnas son soluciones de (*)

La matriz fundamental tal que $\Phi(0) = Id$ es $\boxed{\Phi(x) = e^{Ax}}$

Si B es una matriz con $\det B \neq 0$, y Φ es matriz fundamental $\Rightarrow$ $\Phi \cdot B$ es matriz fundamental.

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 - 9 = 0 \iff 1-\lambda = \pm 3 \iff \lambda = \pm 3 + 1$$

$$\Rightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_2 = -2 \end{cases}$$

Autovectores

$\boxed{\lambda = 4}$ $\quad$ $(A - 4I)v = 0$ ($v \in Ker(A-4I)$) $\iff \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -3v_1 + 3v_2 = 0 \iff v_1 = v_2$

$\Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = -2}$ $\quad$ $(A + 2I)w = 0$ ($w \in Ker(A+2I)$) $\iff \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 3w_1 + 3w_2 = 0 \iff w_1 = -w_2$

$\Rightarrow w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; $e^{DX} = \begin{pmatrix} e^{4x} & 0 \\ 0 & e^{-2x} \end{pmatrix}$ $\quad$ $y = Pe^{DX} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{4x} & e^{-2x} \\ e^{4x} & -e^{-2x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$y(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = 2 \end{cases}$$

$$\Rightarrow Y(x) = \begin{pmatrix} e^{4x} & e^{-2x} \\ e^{-2x} & -e^{-2x} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3e^{4x} + 2e^{-2x} \\ 3e^{4x} + -2e^{-2x} \end{pmatrix}$$

$\boxed{4.}$ Hallar una matriz fundamental tal que $\underbrace{\Phi(0) = Id}_{\Downarrow}$ para

$$X' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} X \qquad e^{AX} = Pe^{JX}P^{-1}$$

<u>Autovalores</u>:

$$\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0 \iff (3-\lambda)(-1-\lambda) + 4 = -3 - 3\lambda + \lambda + \lambda^2 + 4 = \lambda^2 - 2\lambda +$$

$$= 0 \iff (\lambda - 1)^2 = 0 \iff \lambda = 1 \text{ (doble)}$$

<u>Autovectores</u>:

$$(A - I)v = 0 \iff \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 2v_1 - 4v_2 = 0 \iff v_1 = 2v_2$$

$$\Rightarrow v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Ahora buscamos $w \in Ker(A-I)^2 \setminus \underbrace{Ker(A-I)}_{\text{anteriores potencias}}$

e.d., busco w: $(A-I)w = v$ ← anterior

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow w_1 - 2w_2 = 1 \Rightarrow w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad J = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{N}$$

$$e^{Jt} = e^{Dt} \cdot e^{Nt}$$

$$e^{Nt} = I + Nt + \cancel{\frac{N^2 t^2}{2!} + \cdots}^{\,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow e^{Jt} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

dos opciones → calcular P^{-1} y multiplicar: $Pe^{Jt}P^{-1}$

→ $\underbrace{\tilde{\Phi}(0) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\boxed{7.}$ Encontrar una matriz fundamental para $X' = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix} X$

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ -1 & -\lambda & -1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \iff \begin{cases} \lambda = 2 \text{ (doble)} \\ \lambda = 1 \text{ (simple)} \end{cases}$$

Autovalores & autovectores

$\boxed{\lambda = 2}$ $(A-2I)v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} v_1 + 2v_2 + v_3 = 0 \\ v_1 + v_2 = 0 \end{cases} \iff$

$\iff \begin{cases} v_3 + v_2 = 0 \\ v_1 = -v_2 \end{cases} \Rightarrow v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Busco w : $(A-2I) = v \iff \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \iff \begin{cases} w_2 = -w_3 \\ w_1 = 1 - w_2 \end{cases}$

$\Rightarrow w = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = 1}$ $\begin{pmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} u_3 = 0 \\ u_1 = -u_2 \end{cases} \iff u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\Rightarrow e^{Jt} = e^{Dt} \cdot e^{Nt} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}$$

$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$P = \begin{array}{c} \begin{matrix} u & v & w \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \Rightarrow \phi = Pe^{Jt}$$

2.1 Hallar la solución general de $y' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} y + \begin{pmatrix} x-1 \\ -5x-2 \end{pmatrix}$

Homogénea: $y' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} y$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-6 = 2-2\lambda+\lambda^2-\lambda-6 = \lambda^2-3\lambda-4 \Rightarrow \lambda = \frac{3\pm\sqrt{9+16}}{2}$$

$$\Rightarrow \lambda_1 = 4,\ \lambda_2 = -1 \qquad \Rightarrow D = \begin{pmatrix} e^{4x} & 0 \\ 0 & e^{-x} \end{pmatrix}$$

Autovector para $\lambda = 4$

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3x+2y=0 \qquad \text{autovector} \to \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Autovector para $\lambda = -1$

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x+y=0 \qquad \text{autovector} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

Solución homogénea:

$$\underline{y_h} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{4x} & 0 \\ 0 & e^{-x} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2e^{4x} & -e^{-x} \\ 3e^{4x} & e^{-x} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}} =$$

$$= \underbrace{C_1 e^{4x}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-x}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

expresiones de la solución válidas

$y = y_h + y_p \to$ buscamos y_p:

- Coeficientes indeterminados: $Y' - AY = \begin{pmatrix} x-1 \\ -5x-2 \end{pmatrix}$ $\qquad Y = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix}$, $\quad Y' = \begin{pmatrix} a \\ c \end{pmatrix}$

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} ax+b \\ cx+d \end{pmatrix} + \begin{pmatrix} x-1 \\ -5x-2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 0x+a = ax+b+2(cx+d)+x-1 \\ 0x+c = 3(ax+b)+2(cx+d)-5x-2 \end{cases} \Rightarrow \begin{cases} 0 = a+2c+1 \\ 0 = 3a+2c-5 \\ a = b+2d-1 \\ c = 3b+2d-2 \end{cases} \begin{matrix} 3 = b+2d-1 \\ -2 = 3b+2d-2 \end{matrix}$$

$$\Rightarrow -2a+6=0 \Rightarrow a=3 \Rightarrow c=-2$$

$$\Rightarrow 5 = -2b+1 \Rightarrow b=-2 \Rightarrow d=3$$

$$\Rightarrow \text{Sol. part.} = \begin{pmatrix} 3x-2 \\ -2x+3 \end{pmatrix} = \underline{y_p}$$

$$\boxed{y = y_h + y_p}$$

11. $X' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} \operatorname{cosec} t \\ \sec t \end{pmatrix}$ (ED)

$$X_h :\Rightarrow X' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X$$

$$\begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda)+5 = -4-2\lambda+2\lambda+\lambda^2+5 = \lambda^2+1 \Rightarrow \lambda = \pm i$$

Autovector para $\lambda = i$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (2-i)x_1 - 5x_2 = 0 \\ x_1 - (2+i)x_2 = 0 \end{cases} \Rightarrow (2-i)x_1 - (2+i)(2-i)x_2 =$$

$$\Rightarrow (2-i)x_1 - 5x_2 \Rightarrow \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = u$$

IMPORTANTE

$$\begin{cases} \dfrac{e^{it}+e^{-it}}{2} = \cos t \\ \dfrac{e^{it}-e^{-it}}{2i} = \operatorname{sen} t \end{cases}$$

$$\Uparrow$$

$$\begin{cases} e^{it} = \cos t + i \operatorname{sen} t \\ e^{-it} = \cos t - i \operatorname{sen} t \end{cases}$$

Autovector para $\lambda = -i$

$$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (2+i)x_1 - 5x_2 = 0 \\ x_1 - (2-i)x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} 5 \\ 2+i \end{pmatrix} = v$$

$$X_h = \underbrace{c_1 e^{it}\begin{pmatrix} 5 \\ 2-i \end{pmatrix}}_{X_{h_1}} + \underbrace{c_2 e^{-it}\begin{pmatrix} 5 \\ 2+i \end{pmatrix}}_{X_{h_2}}$$

$$\Rightarrow Z_1 = \frac{X_{h_1}+X_{h_2}}{2} = \frac{e^{it}\begin{pmatrix} 5 \\ 2-i \end{pmatrix} + e^{-it}\begin{pmatrix} 5 \\ 2+i \end{pmatrix}}{2} = \begin{pmatrix} 5\left(\frac{e^{it}+e^{-it}}{2}\right) \\ 2\left(\frac{e^{it}+e^{-it}}{2}\right) - \frac{i(e^{it}-e^{-it})(-i)}{2(-i)} \end{pmatrix} =$$

$$= \begin{pmatrix} 5\cos t \\ 2\cos t + \operatorname{sen} t \end{pmatrix}$$

$$\Rightarrow Z_2 = \frac{X_{h_1}-X_{h_2}}{2i} = \frac{e^{it}\begin{pmatrix} 5 \\ 2-i \end{pmatrix} - e^{-it}\begin{pmatrix} 5 \\ 2+i \end{pmatrix}}{2i} = \begin{pmatrix} 5\frac{(e^{it}-e^{-it})}{2i} \\ 2\frac{(e^{it}-e^{-it})}{2i} - \cancel{i}\frac{(e^{it}+e^{-it})}{2\cancel{i}} \end{pmatrix} =$$

$$= \begin{pmatrix} 5\operatorname{sen} t \\ 2\operatorname{sen} t - \cos t \end{pmatrix}$$

$$\Rightarrow \boxed{X_h = \underbrace{\begin{pmatrix} 5\cos t & 5\operatorname{sen} t \\ 2\cos t + \operatorname{sen} t & 2\operatorname{sen} t - \cos t \end{pmatrix}}_{\phi}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

Buscamos $X_p(t)$ que cumpla $X' = AX + \begin{pmatrix} \operatorname{cosec} t \\ \sec t \end{pmatrix}$

$X_p(t) = \phi \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}$, donde ϕ verifica $\boxed{\phi' = A\phi}$

$$X_p'(t) \overset{R.C.}{=} \phi'(t) \cdot \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} + \phi \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \underbrace{A\phi \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}}_{X_p(t)} + \phi \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix}$$

Por tanto, $X_p(t)$ verifica (ED) si y sdo si $\boxed{\phi \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\operatorname{sen} t} \\ \frac{1}{\cos t} \end{pmatrix}}$

$$\Rightarrow \overset{(*)}{\begin{pmatrix} 5\cos t & 5\operatorname{sen} t \\ 2\cos t + \operatorname{sen} t & 2\operatorname{sen} t - \cos t \end{pmatrix}} \begin{pmatrix} C_1'(t) \\ C_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\operatorname{sen} t} \\ \frac{1}{\cos t} \end{pmatrix} \qquad W = \left| (*) \right| = -5$$

$$\Rightarrow C_1'(t) = \frac{\begin{vmatrix} \frac{1}{\operatorname{sen} t} & 5\operatorname{sen} t \\ \frac{1}{\cos t} & 2\operatorname{sen} t - \cos t \end{vmatrix}}{W} = \frac{2 - \frac{\cos t}{\operatorname{sen} t} - 5\frac{\operatorname{sen} t}{\cos t}}{-5}$$

$$\Rightarrow C_1(t) = -\frac{2}{5}t + \frac{1}{5}\log|\operatorname{sen} t| - \log|\cos t|$$

$$\Rightarrow C_2'(t) = \frac{\begin{vmatrix} 5\cos t & \frac{1}{\operatorname{sen} t} \\ 2\cos t + \operatorname{sen} t & \frac{1}{\cos t} \end{vmatrix}}{W} = \frac{-1}{5}\left(5 - \frac{2\cos t}{\operatorname{sen} t} - 1\right) = \frac{-4}{5} \cdot \frac{-2}{5}\frac{\cos t}{\operatorname{sen} t}$$

$$\Rightarrow C_2(t) = -\frac{4}{5}t - \frac{2}{5}\log|\operatorname{sen} t|$$

$$\Rightarrow \boxed{X = X_h + X_p = \phi\left[\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}\right]}$$

12. $Y' = \begin{pmatrix} 1 & 0 \\ \text{sen}x & -1 \end{pmatrix} Y$

Escribir la matriz fundamental de la forma $\phi(x) = B(x)\, e^{xL}$ donde B es una matriz periódica y L matriz constante.

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \text{sen}x & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Longleftrightarrow \begin{cases} y_1' = y_1 \longrightarrow \boxed{y_1 = ce^x} \\ y_2' = \text{sen}x\, y_1 - y_2 \end{cases}$$

$$\Longrightarrow y_2' = ce^x \text{sen}x - y_2 \quad (**)$$

$$y_{2h}' = -y_2 \Longleftrightarrow \boxed{y_{2h} = ke^{-x}}$$

$\Longrightarrow$ Para la particular probamos con: $y_{2p} = e^x(A\,\text{sen}x + B\,\text{sen}x)$

$$y_{2p}' = e^x(A\cos x - B\,\text{sen}x) + e^x(A\,\text{sen}x + B\cos x)$$

Por tanto y_{2p} cumple $(**)$ si y sólo si $y_{2p}' + y_2 = ce^x \text{sen}x$, es decir, $e^x(A\cos x - B\,\text{sen}x + 2A\,\text{sen}x + 2B\cos x) = ce^x\,\text{sen}x$

$$\Longrightarrow e^x\big((A+2B)\cos x + (2A-B)\,\text{sen}x\big) = ce^x\,\text{sen}x$$

$$\Longrightarrow \begin{cases} A + 2B = 0 \\ 2A - B = c \end{cases} \Longrightarrow -5B = c \Longrightarrow B = \frac{-c}{5}$$

$$\Longrightarrow A = -2B = \frac{2c}{5}$$

$$\Longrightarrow y_{2p} = e^x(A\,\text{sen}x + B\cos x) = \boxed{\frac{2}{5}ce^x\,\text{sen}x - \frac{c}{5}e^x\cos x}$$

$$\Longrightarrow \boxed{y_2 = y_{2h} + y_{2p} = ke^{-x} + c\left(\frac{2}{5}e^x\,\text{sen}x - \frac{e^x}{5}\cos x\right)}$$

$$\Longrightarrow \boxed{Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} ce^x \\ ke^{-x} + c\left(\frac{2}{5}e^x\,\text{sen}x - \frac{e^x}{5}\cos x\right) \end{pmatrix}} \longrightarrow \text{Hay que expresar esto como nos pide el enunciado}$$

$$\boxed{Y = \begin{pmatrix} 1 & 0 \\ \frac{2}{5}\text{sen}x - \frac{1}{5}\cos x & 1 \end{pmatrix}\begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}\begin{pmatrix} c \\ k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{5}\text{sen}x - \frac{1}{5}\cos x & 1 \end{pmatrix}\begin{pmatrix} c \\ k \end{pmatrix} e^{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}x}}$$

$\boxed{13.}$ Sean X_1, X_2 soluciones de $X'' + pX' + qX = 0$ (ED)

$X_1(0) = 1, \quad X_2(0) = 0, \quad X_1'(0) = 0, \quad X_2'(0) = 1$

a) Demostrar que $X_1''(0) + q = 0, \quad X_2''(0) + p = 0$

X_1, X_2 soluciones $\Rightarrow X_1'' + pX_1' + qX_1 = 0$ Como $X_1(0) = 1 \wedge X_1'(0) = 0$

$\Rightarrow X_1''(0) + q = 0$ ■

Lo mismo con X_2

$\boxed{X_1' = -qX_2}^{(*)}, \quad X_2' = X_1 - pX_2$

$Z = X_1' + qX_2$

$Z' = X_1'' + qX_2' \overset{X_1 \text{ sol. de (ED)}}{=} -pX_1' - qX_1 + qX_2' = 0$

$Z'' = -pX_1'' - qX_1' + qX_2'' \overset{X_1, X_2 \text{ sol. de (ED)}}{=} p^2X_1' + pqX_1 - qX_1' - qpX_2' - q^2X_2 =$

$= -q(X_1' + qX_2) - p(-pX_1' - qX_1 + qX_2') = -qZ - pZ'$

Z es solución de (PVI) $\begin{cases} Z'' + qZ + pZ' = 0, \\ Z(0) = X_1'(0) + qX_2(0) = 0 \\ Z'(0) = 0 \end{cases}$ y como $Z = 0$ es solución $\Rightarrow$

Por unicidad de soluciones

$\Rightarrow Z \equiv 0.$

$\begin{cases} W = X_2' - X_1 + pX_2 \\ W' = X_2'' - X_1' + pX_2' = -\cancel{pX_2'} - qX_2 - X_1' + \cancel{pX_2'} \overset{\text{usando } (*)}{=} 0 \\ W(0) = X_2'(0) - X_1(0) + pX_2(0) = 0 \end{cases}$ (P.V.I)

W cumple $\begin{cases} W' = 0 \\ W(0) = 0 \end{cases} \Rightarrow$ Unicidad de soluciones $\Rightarrow W \equiv 0.$

b) $A_{2\times 2}$ $\qquad e^{tA} = x_1 I + x_2 A$

$$A_{2\times2} \xrightarrow[\text{Tma Cayley-Hamilton}]{} A^2 + pA + q = 0$$

e^{tA} es la única solución de $\begin{cases} X' = AX \\ X(0) = Id \end{cases}$

$Z = x_1 I + x_2 A$

$$\begin{cases} Z' = x_1' I + x_2' A = -q x_2 + A(x_1 - p x_2) \\ AZ = A x_1 + x_2 A^2 \underset{\overset{\uparrow}{C-H}}{=} A x_1 + x_2(-pA - q) \\ Z(0) = Id \end{cases}$$

$\Rightarrow$ cumple el mismo problema que la exponencial y por unicida de sol. $\Rightarrow$

$\Rightarrow Z = e^{At}$

$\boxed{1.}$ a) $X_1(t)$ y $X_2(t)$ lin. indep. $\Leftrightarrow \left(aX_1(t) + bX_2(t) = \vec{0} \Rightarrow a = b = 0 \right)$

$a\begin{pmatrix} t \\ t^2 \end{pmatrix} + b\begin{pmatrix} t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} at \\ at^2 \end{pmatrix} + \begin{pmatrix} bt^2 \\ bt^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} at + bt^2 \\ at^2 + bt^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} t(a+bt) \\ t^2(a+bt) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ se cumple para $t = 0$ y para $a = b = 0$

b) $W(t) = \begin{bmatrix} \uparrow & \uparrow \\ X_1(t) & X_2(t) \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} t & t^2 \\ t^2 & t^3 \end{bmatrix}$ Está claro que $\exists t_0$ tal que

$\forall t \in \mathbb{R}$? ¿$t = 0$? pág 50

$\begin{bmatrix} t_0 & t_0^2 \\ t_0^2 & t_0^3 \end{bmatrix} \neq 0 \Rightarrow W(t) \neq 0$

$\boxed{2.}$ $B = \{ \underbrace{\cos x - \operatorname{sen} x}_{V}, \underbrace{2\operatorname{sen} x}_{u} \}$ ¿base de $y'' + y = 0$?

↳ dimensión de solución de tamaño 2

a)

¿V y u lin. indep.?

$V = \cos x - \operatorname{sen} x$ $\quad u = 2\operatorname{sen} x$

$V' = -\operatorname{sen} x - \cos x$ $\quad u' = 2\cos x$

$\Rightarrow W(x) = \begin{bmatrix} V & u \\ V' & u' \end{bmatrix} = \begin{bmatrix} \cos x - \operatorname{sen} x & 2\operatorname{sen} x \\ -\operatorname{sen} x - \cos x & 2\cos x \end{bmatrix} =$

$= 2\cos^2 x - 2\cos x \operatorname{sen} x + (\operatorname{sen} x + \cos x) \cdot 2\operatorname{sen} x =$

$= 2\cos^2 x - 2\cos x \operatorname{sen} x + 2\operatorname{sen}^2 x + 2\operatorname{sen} x \cos x =$

$= 2 + 4\operatorname{sen}^2 x \cos^2 x$

? Para $x = 0 \Rightarrow 2 + 4\operatorname{sen}^2 0 \cos^2 0 = 2 \neq 0 \Rightarrow$ lin. indep.

↳ ¿está esto bien?

b) ~~y(0) = y(1)~~

$y(x) = C_1 V(x) + C_2 u(x) \longrightarrow y(0) = 1 \Leftrightarrow C_1(\cos 0 - \operatorname{sen} 0) + C_2 \cdot 2\operatorname{sen} 0 =$

$y'(x) = C_1 V'(x) + C_2 u'(x) \rightarrow y'(0) = 1 \Leftrightarrow C_1(-\operatorname{sen} 0 - \cos 0) + C_2\, 2\cos 0 = 1$

$\Rightarrow \begin{cases} \cancel{C_1 \cos 0} \quad C_1 + C_2 \cdot 0 = 1 \longrightarrow C_1 = 1 \\ -C_1 + 2C_2 = 1 \longrightarrow C_2 = 1 \end{cases}$

5.)

$$X' = \overbrace{\begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}}^{A} X \qquad X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda) + 16 = -7 - \lambda + 7\lambda + \lambda^2 + 16 = 0 \iff$$

$$\iff \lambda^2 + 6\lambda + \cancel{23}\,9 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36 - 4(\cancel{23})9}}{2} = \frac{-6}{2} \longrightarrow \lambda = -3 \quad \text{(doble)}$$

$\mathrm{Ker}(A+3I)$: $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4v_1 - 4v_2 = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Buscamos $(A+3I)w = v \Rightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow 4w_1 - 4w_2 = 1 \rightarrow$

$\rightarrow w = \left(\frac{1}{4}, 0\right) \rightarrow \qquad w = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

$\Rightarrow$

(5.)

$$X' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} X$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda) + 16 = -7 - \lambda + 7\lambda + \lambda^2 + 16$$

$$\iff \lambda^2 + 6\lambda + 9 = 0 \longrightarrow \lambda = \frac{-6 \pm \sqrt{36 - 4 \cdot 9}}{2}$$

$$\lambda = -3 \text{ (doble)}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow 4u_1 - 4u_2 = 1 \longrightarrow u = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

Tenemos: $J = D + N = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$e^{Jt} = e^{Dt} \cdot e^{Nt}$$

$$e^{Nt} = I + Nt + \cancel{\frac{N^2 t^2}{2!}}^{\,0} + \dots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{pmatrix}$$

Solución general: $\cancel{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-3t} & te^{-3t} + \frac{e^{-3t}}{4} \\ e^{-3t} & te^{-3t} \end{pmatrix}}_{\mathbb{F}(t)}$

$$\underline{X(t) = \mathbb{F}(t)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}}$$

$$X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \iff \cancel{\begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}} \quad \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Longrightarrow$$

$$\Longrightarrow \boxed{C_1 = 2}$$

$$C_1 + \frac{1}{4}C_2 = 3 \Longrightarrow \frac{C_2}{4} = 1 \Longrightarrow \boxed{C_2 = 4}$$

$$\boxed{X(t) = \mathbb{F}(t)\begin{pmatrix} 2 \\ 4 \end{pmatrix}}$$

6. $X' = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 6 \\ 0 & -2 & 6 \end{pmatrix}}_{A} X$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 6 \\ 0 & -2 & 6-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(6-\lambda) - \left(-2 \cdot 6 \cdot (1-\lambda)\right) =$$

$$= \left(-1 - \cancel{\lambda} + \cancel{\lambda} + \lambda^2\right)(6-\lambda) - \left(-12(1-\lambda)\right) =$$

$$= -6 + \lambda + 6\lambda^2 - \lambda^3 + 12 - 12\lambda = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

	-1	6	-11	6
-2		$+2$	$+8$	-6
	1	8	3	0

$\lambda+3$

$(\lambda+2)(-\lambda^2 + 4\lambda + 3) = 0$

$\lambda_1 = 2$

$\Longleftrightarrow \lambda = \dfrac{-4 \pm \sqrt{16 - 4 \cdot 3 \cdot (-1)}}{2} = \dfrac{-4 \pm 2}{2}$

$\lambda_2 = -1$

$\lambda_3 = -3$

Autovectores

λ_1

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$$

$v_1 = 0$

$v_2 + 6v_3 = 0 \longrightarrow v_2 = -6v_3$

$-2v_2 + 4v_3 = 0$

$\hookrightarrow -2(-6v_3) + 4v_3 = 0 \longrightarrow$

$\longrightarrow 12v_3 + 4v_3 = 0$

(6.) $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$$\begin{array}{c|cccc}
 & -1 & +6 & -11 & +6 \\
1 & & -1 & 5 & -6 \\
\hline
 & -1 & 5 & -6 & \boxed{0} \\
2 & & -2 & 6 & \\
\hline
 & -1 & 3 & \boxed{0} & \\
3 & & -3 & & \\
\hline
 & -1 & \boxed{0} & &
\end{array}$$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 3$

Autovectores

$\lambda_1 = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 6 \\ 0 & -2 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} \cancel{v_1 = 0} \\ \cancel{-2v_2 + 6v_3} = 0 \\ -2v_2 \cancel{+ 5v_3 = 0} \end{array} \right. \quad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 6 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left\{ \begin{array}{l} -u_1 = 0 \\ -3u_2 + 6u_3 = 0 \\ -2u_2 + 4u_3 = 0 \end{array} \right. \rightarrow u = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\lambda_3 = 3$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 6 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left\{ \begin{array}{l} w_1 = 0 \\ -2w_2 + 3w_3 = 0 \end{array} \right. \rightarrow w = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$X(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

(7.)

$$X' = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix} X$$

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ -1 & -\lambda & -1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(-\lambda)(2-\lambda) -1 -2 - (-\lambda -3+\lambda +2\lambda -4) =$$

$$= (6-3\lambda-2\lambda+\lambda^2)(-\lambda) \cancel{-3} + \cancel{\lambda} \cancel{+3} - \cancel{\lambda} -2\lambda+4 =$$

$$= -6\lambda + 3\lambda^2 + 2\lambda^2 - \lambda^3 - 2\lambda + 4 \rightarrow$$

$$\rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\begin{array}{r|rrrr} & -1 & 5 & -8 & 4 \\ 2 & & -2 & 6 & -4 \\ \hline & -1 & 3 & -2 & 0 \\ 1 & & -1 & 2 & \\ \hline & -1 & 2 & 0 & \\ 2 & & -2 & & \\ \hline & -1 & 0 & & \end{array}$$

$$(\lambda-2)^2(\lambda-1) = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 2 \text{ (doble)}$$

Autovectores

$\lambda_1 = 1$

$$\begin{pmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} v_3 = 0 \\ 2v_1 + 2v_2 + v_3 = 0 \end{cases} \rightarrow v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} u_1 + 2u_2 + u_3 = 0 \\ u_1 + u_2 = 0 \end{cases} \rightarrow u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \rightarrow \begin{cases} w_1 + 2w_2 + w_3 = 1 \\ w_1 + w_2 = -1 \end{cases} \rightarrow w = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$