

$\sum_{j=0}^k \alpha_j y_{n+j} = h \phi(t_n, y_n, \dots, y_{n+k-1}; h) \quad (MN)$

Un MN es 0-estable si $\forall \epsilon > 0 \exists \tau > 0$

$\forall \{u_n\}_{n=0}^N, \{v_n\}_{n=0}^N$ que satisfacen $\sum_{j=0}^k \alpha_j u_{n+j} - h \phi(-, u_{n+k-1}; h) = h \delta_n$
 $\sum_{j=0}^k \alpha_j v_{n+j} - h \phi(-, v_{n+k-1}; h) = h \delta_n$

$\Rightarrow \max_{K \leq n \leq N} \|u_n - v_n\| \leq C [\max_{0 \leq n \leq K-1} \|u_n - v_n\| + \max_{0 \leq n \leq N-K} \|\delta_n - \delta_n\|]$
 0-estable quiere decir que $y(t_n)$ e y_n estarán cerca si las perturb. δ_n, δ_n y R_n son peq.
 0-estable \Leftrightarrow cont. raíz \Leftrightarrow raíz 1er pol. tienen mód. < 1 y si = 1 simples

Un mn de k pasos es convergente si $\forall \epsilon > 0 \lim_{N \rightarrow \infty} \max_{K \leq n \leq N} \|y(t_n) - y_n\| = 0$
 si $\lim_{N \rightarrow \infty} \max_{0 \leq n \leq K-1} \|y(t_n) - y_n\| = 0$.

Un mn de k pasos es converg. ord. p si $\forall \epsilon > 0 \lim_{N \rightarrow \infty} \max_{K \leq n \leq N} \|y(t_n) - y_n\| = O(h^p)$
 si $\lim_{N \rightarrow \infty} \max_{0 \leq n \leq K-1} \|y(t_n) - y_n\| = O(h^p)$.

Un mn de k pasos es consist. si $\forall \epsilon > 0 \lim_{h \rightarrow 0^+} \tau(h) = 0$

Un mn de k pasos es consist. de ord. p si $\forall \epsilon > 0 \lim_{h \rightarrow 0^+} \tau(h) = O(h^p)$

CORO: Si mn (MN) es 0-estable y $\tau := \max_{0 \leq n \leq N-K} \frac{\|R_n\|}{h} \rightarrow 0 \Rightarrow$
 \Rightarrow convergente.

Obs: $R_n = \sum_{j=0}^k \alpha_j y_{n+j} - h \phi(t_n, y_n, \dots, y_{n+k-1}; h)$
 si $\forall \epsilon > 0 \tau = O(h^p), f \in C^q, \Rightarrow$ converg. al menos orden p.

$K_1 = f(t + c_1 h, y_n + h \sum_{j=1}^s a_{1j} K_j)$
 $K_s = f(t + c_s h, y_n + h \sum_{j=1}^s a_{sj} K_j)$
 $y_{n+1} = y_n + h \sum_{j=1}^s b_j K_j$

Los RK son siempre 0-estables \Rightarrow [converg. \Leftrightarrow consist.]

Un m.n. RK explícito s etapas no puede tener ord. $> s$.
 Para $p \geq 5$, \nexists RK explícito de ord. p y $s = p$ etapas.
 Para $p \geq 7$, \nexists RK explícito de ord. p y $s = p+1$ etapas.
 Para $p \geq 8$, \nexists RK explícito de ord. p y $s = p+2$ etapas.

$R(z) = 1 + z b^T (I - zA)^{-1} e, z \in \mathbb{C}$
 RK A-estable $\Leftrightarrow |R(z)| < 1 \quad \forall z \in \mathbb{C}^-$

Obs: ningún RK explícito es A-estable.

$R(z) = \frac{p(z)}{q(z)}$. Entonces $|R(z)| < 1 \quad \forall z \in \mathbb{C}^-$

si \forall cero de q tiene parte real positiva.
 $|R(it)| \leq 1 \quad \forall t \in \mathbb{R}$.

RK orden p. $\Rightarrow R(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^p}{p!} + O(|z|^{p+1})$
 RK explícito ord. p $\Rightarrow R(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^s}{s!}$ y $s = p$ etapas

- i) ϕ continua
- ii) $\|\phi - \hat{\phi}\| \leq L \sum_{j=1}^k \|y_{n+j}\|$
- iii) $f=0 \Rightarrow \phi=0$

Un MN es consist. \Leftrightarrow cond. 1 & 2
 Cond 1: $\sum \alpha_j = 0$
 Cond 2: $\phi(t_n, y_n, \dots, y_n) = (\sum \alpha_j) f(t, y)$
 Tm. Equiv: $\forall m, n$ (MN) que H_{mn} converg. \Leftrightarrow consist + 0-estab.
 0-estab. \Leftrightarrow cont. raíz \Leftrightarrow converg.

- CS: $\sum a_{ij} = c_i \quad \forall i \quad b^T A^2 c$
- 1: $\sum b_i = 1$
- 2: $\sum b_i c_i = \frac{1}{2}$
- 3: $\sum b_i c_i^2 = \frac{1}{3}$
- 4: $\sum b_i c_i^3 = \frac{1}{4}$
- 5: $\sum b_i c_i^4 = \frac{1}{5}$
- 6: $\sum b_i c_i^5 = \frac{1}{6}$
- 7: $\sum b_i c_i^6 = \frac{1}{7}$
- 8: $\sum b_i c_i^7 = \frac{1}{8}$
- 9: $\sum b_i c_i^8 = \frac{1}{9}$
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- 100: $\sum b_i c_i^{99} = \frac{1}{100}$

mn A-estable si $\mathbb{C}^- \subseteq D$
 $D = \{z \in \mathbb{C} : |R(z)| < 1\}$

$\|F(k)_i - F(\hat{k})_i\| = \|f(t_n + c_i h, y_n + h \sum_{j=1}^s a_{ij} K_j) - f(t_n + c_i h, \hat{y}_n + h \sum_{j=1}^s \hat{a}_{ij} \hat{K}_j)\|$
 $\Rightarrow \|F(k)_i - F(\hat{k})_i\| = \sum_{j=1}^s |F(k)_i - F(\hat{k})_i| \leq \sum_{j=1}^s \|y_n - \hat{y}_n + h \sum_{j=1}^s a_{ij} K_j - \hat{a}_{ij} \hat{K}_j\| \leq L \|y_n - \hat{y}_n\| \sum_{j=1}^s |K_j - \hat{K}_j| \leq L \|y_n - \hat{y}_n\| \sum_{j=1}^s |K_j - \hat{K}_j| \leq L \|y_n - \hat{y}_n\| \|K - \hat{K}\|$

It. punto fijo $F(\phi) = \dots$
 F contract. $h < 0$
 f cont. $\Rightarrow \phi^{(k)}$ cont.
 k infinito cont. unip.
 $\phi^{(k)} = \sum_{j=1}^s \phi_j^{(k)}$
 converg. compactos
 $\|\phi^{(k)} - \phi^{(k-1)}\| = \|F(\phi^{(k-1)}) - F\|$
 $\leq O(\|\phi^{(k-1)} - \phi\|)$
 $\| \cdot \| \leq O^{-1} \| \cdot \|$
 Weierstrass

MLM k pasos si $\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f(t_{n+j}, y_{n+j})$ con $\alpha_k = 1, |\alpha_0| + |\beta_0| \neq 0$
 Si $\beta_k \neq 0 \Rightarrow$ implícito
 Si $\beta_k = 0 \Rightarrow$ explícito $\left| \rho(x) = \sum_{j=0}^k \alpha_j x^j \right| \sigma(x) = \sum_{j=0}^k \beta_j x^j \left| \begin{array}{l} \text{CONSISTENTE} \iff \rho(1) = 0 \\ \text{y } \rho'(1) = \sigma(1) \end{array} \right.$

$$R_n = C_0 Y(t_n) + C_1 h Y'(t_n) + \dots + C_p h^p Y^{(p)}(t_n) + O(h^{p+1}) \text{ donde } C_q = \frac{1}{q!} \left[\sum_{j=0}^k \alpha_j j^q - q \sum_{j=0}^k \beta_j j^{q-1} \right] = 0$$

Si $C_q = 0 \quad \forall q = 0, \dots, p \Rightarrow$ al menos orden p .
 Si $C_{p+1} \neq 0 \Rightarrow$ el método no puede ser consist. ord $p+1$

Barrera Dahlquist: el orden de convergencia de un MLM 0-estab. de k pasos satisface:
 1) $p \leq k+2$ si k par; 2) $p \leq k+1$ si k impar; $p \leq k$ si $\beta_k \leq 0$ (particular, explícito)

Pol. estabilidad $\rightarrow \Pi(r, z) = \rho(r) - z\sigma(r) \rightarrow D = \{z \in \mathbb{C} : |r_i(z)| < 1, r_i \text{ raíces de } \Pi(r, z)\}$
 $F_r(D) \subset \mathcal{F} := \{z \in \mathbb{C} : \exists \text{ una raíz de } \Pi(r, z) \text{ mod } 1\} \quad D \cap \mathcal{F} = \emptyset$

$\mathcal{F} = \{z \in \mathbb{C} : z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, 0 \leq \theta \leq 2\pi\}$ Obs: \mathcal{F} imagen curva cerrada divide \mathbb{C} en dos. cada una de ellas está en D o en $\mathbb{C} \setminus D$

Cogemos $\bar{z} \in A_1$ si alguna raíz de $\Pi(r, \bar{z})$ tiene módulo $> 1 \Rightarrow D = A_2$.
 Si todas las raíces tienen mod. $< 1 \Rightarrow D = A_1$. $\left[\begin{array}{l} \text{RK y MLM consist.} \iff \text{consist. ord } 1 \\ y_{n+1} = y_n + hf + h^{3/2} \text{ consist pero no } \uparrow \end{array} \right.$

Obs: No existen MLM explícitos A-estables.
Obs: Si la región de estabilidad D es acotada no es A-estable.

Obs: Si la región de estabilidad D es acotada no es A-estable es $p=2$.
2ª Barrera de Dahlquist: El mayor orden de consist. MLM A-estable es $p=2$.

Existen MLM con ord. consist. $p=2k$ (maximales) por ej. Simpson / Gauss-Legendre $\rightarrow s=2, p=4$, A-estab

No son convergentes salvo $k=1$ y $k=2$.

Matriz M de un RK: $m_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j$. Obs: M simétrica.
 Si M es semidef. pos. y $b_i \geq 0 \quad \forall i \Rightarrow$ RK algebraicamente estab.

Un RK algebraicamente estable \Rightarrow es A-estable.

$$P_s(\tau) = \prod_{j=1}^s (\tau - c_j) ; q_i(\tau) = \frac{P_s(\tau)}{(\tau - c_i)} \quad i=1, \dots, s ; a_{ij} = \int_0^1 \frac{q_j(\tau)}{q_j(c_j)} d\tau ; b_i = \int_0^1 \frac{q_j(\tau)}{q_j(c_j)} d\tau$$

$$y(t_{n+k}) - y(t_{n+k-1}) = \int_{t_{n+k-1}}^{t_{n+k}} f(t, y(t)) dt = \sum_{j=0}^{k-1} f(t_{n+j}, y(t_{n+j})) \cdot h \int_{t_{n+k-1}}^{t_{n+k}} L_j(t) dt = \sum_{j=0}^{k-1} f(t_{n+j}, y(t_{n+j})) \cdot h \int_{t_{n+k-1}}^{t_{n+k}} \prod_{i=0}^{k-1} \frac{t - t_{n+i}}{t_{n+j} - t_{n+i}} dt$$

$$R_n = y(t_{n+k}) - y(t_{n+k-1}) - \int_{t_{n+k-1}}^{t_{n+k}} P_k(t) dt \Rightarrow \|R_n\| = Ch^{k+1} \left[\begin{array}{l} \leftarrow \text{grado} \leq k-1 \text{ (interp. } k-1 \text{ nodos)} \right] + 2$$

$I(f) = b_1 f(c_1) + \dots + b_s f(c_s)$ integra de manera exacta pol. de grado $2s$ en $[0, 1]$
 $b_1 f(c_1) + \dots + b_s f(c_s) = \int_0^1 f(x) dx \rightarrow f(x) = 1, f(x) = x, \dots, f(x) = x^{s-1} \rightarrow$ obtenemos c 's \rightarrow colocación b 's coinciden