# Hoja 9

1. Sea $S$ una superficie regular y sea $p \in S$. Demuestra que la curvatura gaussiana en $p$ (con respecto a la primera forma fundamental) verifica:

$$K(p) = \lim_{r \to 0} \frac{12}{\pi} \frac{\pi r^2 - A}{r^4}$$

donde $A$ es el área del disco geodésico centrado en $p$ y de radio $r$.

2. Sea $S$ una superficie regular con métrica de Riemann $Q$ y sea $c > 0$ una constante.
   (a) Demuestra que las geodésicas para $cQ$ son las mismas que para $Q$.
   (b) Halla la relación entre la curvatura gaussiana de $Q$ y de $cQ$.

3. A una superficie $S$, con parametrización regular $\mathbf{X}(u, v)$, $u > 0, v > 1$, le damos la siguiente métrica de Riemann:

$$Q = (du)^2 + \frac{2}{v}(du)(dv) + e^u (dv)^2 .$$

   (a) Comprueba que las curvas $\alpha(t) = \mathbf{X}(t, \text{cte})$ son geodésicas unitarias para $Q$.
   (b) Halla la trayectoria $Q$-ortogonal, pasando por el punto $\mathbf{X}(1, e)$, de esas geodésicas. Parametrízala como $\mathbf{X}(h(v), v)$ para cierta función $h(v)$. $\rightarrow$ curva $u + \log v = 2$
   (c) Utiliza lo obtenido en los apartados (a) y (b) para construir coordenadas $(\widetilde{u}, \widetilde{v})$ en las cuales tengamos:

$$Q = (d\widetilde{u})^2 + C(\widetilde{u}, \widetilde{v})(d\widetilde{v})^2 ,$$

   y calcula explícitamente la función $C(\widetilde{u}, \widetilde{v})$.

4. Consideremos el plano hiperbólico: $\mathbb{H} = \{(u, v) : v > 0\}$ con $Q = \dfrac{(du)^2 + (dv)^2}{v^2}$

Sea $C(c, R)$ el círculo de centro $c = (u_0, v_0)$ y radio $R$ en $\mathbb{H}$, es decir,

$$C(c, R) := \{(u, v) \in \mathbb{H} : d_{\mathbb{H}}((u, v), c_0) = R\}$$

   (a) Comprueba que $C(c, R)$ es un círculo euclídeo de centro $(u_0, v_0 \cosh R)$ y radio $v_0 \operatorname{senh} R$.
   (b) Demuestra que $C(c, R)$ tiene curvatura geodésica $\coth R$.

5. Sea $S$ una superficie regular con parametrización $\mathbf{X}(u, v)$ y con métrica de Riemann $Q$. Demuestra:

$$Q = (du)^2 + h(u, v)^2 (dv)^2 \implies K = \frac{-h_{uu}}{h}.$$

$$Q = a^2 (du)^2 + b^2 (dv)^2 \implies K = \frac{-1}{ab}\left[\left(\frac{a_v}{b}\right)_v + \left(\frac{b_u}{a}\right)_u\right].$$

$$Q = e^{2h(u,v)}[(du)^2 + (dv)^2] \implies K = \frac{-h_{uu} - h_{vv}}{e^{2h}}.$$

$$Q = (du)^2 + 2\cos\theta (du)(dv) + (dv)^2 \implies K = \frac{-\theta_{uv}}{\operatorname{sen}\theta}.$$

6. Una superficie $S$ tiene primera forma fundamental

$$I = (du)^2 + 2u(du)(dv) + (dv)^2, \qquad \text{con} \qquad |u| < 1$$

Demuestra que $S$ es localmente isométrica al plano euclídeo.

7. En el plano $xy$ consideramos la fórmula:

$$\frac{(dx)^2 + (dy)^2}{(x^2 + y^2 + c)^2} \qquad \text{con } c = cte \in \mathbb{R}$$

Describe, según el valor de $c$, el dominio del plano donde esta fórmula define una métrica y calcula la curvatura gaussiana de dicha métrica.

8. Sea $\Sigma$ con parametrización $\mathbf{X}(u, v)$ y métrica de Riemann

$$Q = \frac{(du)^2}{1 - u^2} + u^2\,(dv)^2 .$$

Demuestra que $(\Sigma, Q)$ es isométrica a una esfera (con primera forma fundamental). Calcula el radio de la esfera y determina la isometría.

9. Sea $S$ el helicoide, parametrizado por $\Phi(u, v) \equiv (\,u\cos v\,,\ u\operatorname{sen} v\,,\ v\,)$. Halla todas las isometrías del helicoide consigo mismo respecto de la primera forma fundamental.

10. Tenemos dos parametrizaciones de las mismas variables: $\Phi(u, v)$ y $\Psi(u, v)$, con $u > 0$. Sean $S_1$ y $S_2$ las superficies definidas por ellas. Sabiendo que:

$$\begin{aligned} I_\Phi &\equiv \frac{1}{2u}\,(du)^2 + u^2\,(dv)^2 , \\ I_\Psi &\equiv \frac{1}{2u}\,(du)^2 + \frac{1}{2u}\,(dv)^2 , \end{aligned}$$

demuestra que no hay ninguna isometría $S_1 \to S_2$.

OTRO (CREO QUE EL 8)

TEOREMA DE MINDING

misma curvatura de Gauss const.
$\Rightarrow$ son localm. isométricas

$$E = \frac{1}{1-u^2} \qquad F = 0 \qquad G = u^2 \qquad 0 < u < 1$$

$$E = \lambda^2 \qquad G = \mu^2 \qquad F = 0$$

$$K = \frac{-1}{\lambda\mu}\left(\left(\frac{\lambda_v}{\mu}\right)_v + \left(\frac{\mu_u}{\lambda}\right)_u\right) = \frac{-1}{\lambda\mu}(-\lambda\mu) = 1$$

$$\lambda = (1-u^2)^{-1/2} \qquad \mu = u$$

$$\lambda_v = 0 \qquad \mu_u = 1 \qquad \frac{\mu_u}{\lambda} = (1-u^2)^{1/2}$$

$$\left(\frac{\mu_u}{\lambda}\right)_u = \frac{1}{2}(1-u^2)^{-1/2}(-2u) = \frac{-u}{\sqrt{1-u^2}} = -u\lambda = -\mu\lambda$$

$$d\bar{u} = \lambda\, du = \frac{1}{\sqrt{1-u^2}}\, du$$

$\bar{v} = v$ $\qquad$ $u = \operatorname{sen}\bar{u}$ $\qquad$ $0 < \bar{u} < \pi/2$

$\bar{u} = \arcsin u$ $\qquad$ $\bar{v} = v$

$$\frac{1}{1-u^2}\, du^2 + dv^2 = d\bar{u}^2 + \operatorname{sen}^2\bar{u}\, d\bar{v}^2$$

otro ejercicio : HOJA 9 - EJ. 8

$$\lambda = \mu = \frac{\varepsilon}{x^2+y^2+c} \qquad \varepsilon(x^2+y^2+c) = |x^2+y^2+c|$$

$$E = \lambda^2 \qquad F = 0 \qquad G = \mu^2$$

Sabemos: $$K = \frac{-1}{\lambda\mu}\left(\left(\frac{\lambda_v}{\mu}\right)_v + \left(\frac{\mu_u}{\lambda}\right)_u\right)$$ Laplaciano

Si $\lambda = \mu$ : $$K = \frac{-1}{\lambda^2}\Delta(\log\lambda) = \frac{-1}{\lambda^2}\left((\log\lambda)_{uu} + (\log\lambda)_{vv}\right)$$

$$\log\lambda = -\log|x^2+y^2+c|$$

$$(\log\lambda)_x = \frac{2x}{x^2+y^2+c}$$

$$(\log\lambda)_{xx} = \frac{2}{x^2+y^2+c} - \frac{2x\,2x}{(x^2+y^2+c)^2} = \frac{2}{(x^2+y^2+c)^2}\cdot(x^2+y^2+c-2x^2) =$$

$$= \frac{2}{(x^2+y^2+c)^2}\cdot(-x^2+y^2+c)$$

$$\Rightarrow K = \frac{-1}{\lambda^2}\Delta(\log\lambda) = \frac{-1}{\lambda^2}\left((\log\lambda)_{uu} + (\log\lambda)_{vv}\right) =$$

$$= \frac{-1}{\frac{1}{(x^2+y^2+c)^2}}\cdot\frac{4c}{(x^2+y^2+c)^2} = -4c$$

$u>0 \qquad v>1$

$E=1 \qquad F=\frac{1}{v} \qquad G=e^u$

Demostrar:

$\alpha(t)=\mathbb{X}(t, cte.)$ son geodésicas unitarias.

$$\left.\begin{array}{ll} u(t)=t & v(t)=v_0 \\ u'(t)=1 & v'(t)=0 \end{array}\right\} \Rightarrow \|\alpha'(t)\|^2=1 \Rightarrow \|\alpha'(t)\|=1$$

$$\Gamma=\frac{1}{e^u-\frac{1}{v^2}}\begin{pmatrix} e^u & -1/v \\ -1/v & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & \frac{-1}{v^2}\cdot\left(\frac{-1}{2}e^4\right) \\ 0 & \frac{1}{2}e^u & 0 \end{pmatrix}=$$

$$=\frac{1}{e^u-\frac{1}{v^2}}\begin{pmatrix} 0 & \frac{-1}{2}e^u & -e^u\left(e^u+\frac{1}{v^2}\right) \\ 0 & \frac{-1}{2}e^u & \frac{1}{v}\left(e^u+\frac{1}{v^2}\right) \end{pmatrix}$$

$$\begin{cases} u''+\Gamma^u_{uu}u'^2+2\Gamma^u_{uv}u'v'+\Gamma^u_{vv}v'^2=0 \\ v''+\Gamma^v_{uu}u'^2+2\Gamma^v_{uv}u'v'+\Gamma^v_{vv}v'^2=0 \end{cases}$$

$$\Gamma^u_{uu}=0 \qquad \Gamma^v_{uu}=0$$

$\beta(t)=\mathbb{X}(u(t), v(t))$

$$\langle\beta'(t), \mathbb{X}_u(\beta(t))\rangle=0=\langle u'\mathbb{X}_u+v'\mathbb{X}_v, \mathbb{X}_u\rangle=u'+\frac{v'}{v}=($$

$(u+\log \bar{v})'=0 \qquad u+\log v=-c$

$u=c+\log v=h(v)$

$(1,e) \qquad h(e)=1=c+\log e=c+1 \Longrightarrow c=0$

$u=\log v$

19 - EJ. 10 | $\Phi(u,v)$ , $\psi(u,v)$

$S_1$ y $S_2$ sup. def. por ellas, sabiendo que:

$$I_\Phi \equiv \frac{1}{2u}(du)^2 + u^2(dv)^2$$

$$I_\psi \equiv \frac{1}{2u}(du)^2 + \frac{1}{2u}(dv)^2$$

demostrar que no hay ninguna isometría $S_1 \to S_2$.

De $I_\Phi$ sacamos $\lambda = (2u)^{-1/2}$ $\quad \mu = u$

$\lambda_v = 0 \qquad \mu_u = 1 \qquad \frac{\mu_u}{\lambda} = (2u)^{1/2}$

$$\left(\frac{\mu_u}{\lambda}\right)_u = \frac{1}{2}(2u)^{-1/2} \cdot 2 = (2u)^{-1/2} = \lambda$$

$$\Rightarrow K_1(u,v) = \frac{-1}{\lambda} \cdot \frac{1}{u}(\lambda) = \frac{-1}{u}$$

De $I_\psi$ sacamos $\lambda = (2u)^{-1/2} \quad \mu = (2u)^{-1/2} = \lambda$

$\log\lambda = \frac{-1}{2}\log(2u) \qquad (\log\lambda)_v = 0$

$(\log\lambda)_u = \frac{-1}{2} \cdot \frac{2}{2u} = \frac{-1}{2u} \qquad (\log\lambda)_{uu} = \frac{1}{2u^2}$

$$\Rightarrow K_2(\bar u,\bar v) = \frac{-1}{\lambda^2}\Delta\log\lambda = -2\bar u\left(\frac{1}{2\bar u^2}\right) = \frac{-1}{\bar u}$$

$$\left.\begin{array}{l} u = u(\bar u,\bar v) \\ v = v(\bar u,\bar v)\end{array}\right\} \Rightarrow \frac{-1}{u} = \frac{-1}{\bar u} \Rightarrow u = \bar u \Rightarrow u_u = 1 \; ; \; u_v = 0$$

$$\frac{1}{2u}du + u^2 dv^2 = \frac{1}{2\bar u}\underbrace{(d\bar u)^2}_{\substack{\text{porque } u=\bar u \\ \Rightarrow du = d\bar u}} + \bar u^2\underbrace{(v_{\bar u}d\bar u + v_{\bar v}d\bar v)}_{d\bar v \text{ derivando } v = v(\bar u,\bar v)}{}^2 =$$

$$= \frac{1}{2\bar u}(d\bar u)^2 + \bar u^2\left(v_{\bar u}^2 d\bar u^2 + 2v_{\bar u}v_{\bar v}\,d\bar u\,d\bar v + v_{\bar v}^2 d\bar v^2\right) =$$

$$= \left(\frac{1}{2\bar u} + \bar u^2 v_{\bar u}^2\right)d\bar u^2 + 2\bar u v_{\bar u}v_{\bar v}\,d\bar u\,d\bar v + \bar u^2 v_{\bar v}^2 d\bar v^2 =$$

$$= \frac{1}{2\bar u}d\bar u^2 + \frac{1}{2\bar u}d\bar v^2$$

$2\bar u v_{\bar u} v_{\bar v} = 0 \qquad \frac{1}{2\bar u} + \bar u^2 v_{\bar u}^2 = \frac{1}{2\bar u} \Rightarrow v_{\bar u} = 0 \Rightarrow v = v(\bar v) \Rightarrow$

$\Rightarrow v_{\bar v}(\bar v) = v'(\bar v)$ sólo depende de $\bar v$

$\bar u^2 v_{\bar v}^2 = \frac{1}{2\bar u} \Rightarrow v_{\bar v}^2 = 2\bar u^3$

contradicción $\Rightarrow$ no pueden ser isométricas

H9- EJ.10

$\mathbb{X}(u,v) \quad u>0 \qquad I_{\mathbb{X}} \equiv \frac{1}{2u}(du)^2 + u^2(dv)^2$

$\mathbb{Y}(\bar{u},\bar{v}) \quad \bar{u}>0 \qquad I_{\mathbb{Y}} \equiv \frac{1}{2\bar{u}}(d\bar{u})^2 + \frac{1}{2\bar{u}}(d\bar{v})^2$

$f(\mathbb{X}(u,v)) = \mathbb{Y}(h(u,v))$ es una isometría (suponemos esto)

$\mathbb{Y}^{-1} \circ f \circ \mathbb{X} = h : \bar{\mathcal{U}} \longrightarrow \mathcal{U}$

$h(u,v) = (\bar{u}(u,v), \bar{v}(u,v))$

$K_{\mathbb{X}}(u,v) = K_{\mathbb{Y}}(h(u,v))$

$\mathbb{X}: \quad \lambda = \frac{1}{\sqrt{2u}} \qquad \mu = u \qquad E = \lambda^2 \qquad F=0 \qquad G = \mu^2$

$$K_{\mathbb{X}} = \frac{-1}{\lambda\mu}\left(\left(\frac{\lambda_v}{\mu}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right)$$

$\lambda\mu = \frac{1}{\sqrt{2u}} \qquad u = \sqrt{\frac{u}{2}}$

$\lambda = \frac{1}{\sqrt{2}} u^{-1/2} \longrightarrow \lambda_u = \frac{1}{\sqrt{2}} \cdot \frac{-1}{2} u^{-3/2} = \frac{-1}{2\sqrt{2}\, u\sqrt{2}} \qquad \lambda_v = 0$

$\Longrightarrow \mu = u \longrightarrow \mu_u = 1 \qquad \mu_v = 0$

$\frac{\mu_u}{\lambda} = \sqrt{2u} = \sqrt{2}\sqrt{u} \Longrightarrow \left(\frac{\mu_u}{\lambda}\right)_u = \frac{\sqrt{2}}{2\sqrt{u}} = \frac{1}{\sqrt{2u}}$

$K_{\mathbb{X}} = -\sqrt{\frac{u}{2}} \cdot \frac{1}{\sqrt{2u}} = \frac{-1}{2}$

$\mathbb{Y}: \quad \lambda = \mu = \frac{1}{\sqrt{2u}} \qquad \log\lambda = \frac{-1}{2}\log(2\bar{u}) \qquad (\log\lambda)_{\bar{u}} = \frac{-1}{2}\frac{2}{2\bar{u}} = \frac{-1}{2\bar{u}}$

$(\log\lambda)_{\bar{v}} = 0 \qquad (\log\lambda)_{\bar{u}\bar{u}} = \frac{1}{2\bar{u}^2}$

$K_{\mathbb{Y}} = \frac{-1}{2}\Delta\log\lambda = -2\bar{u}^2 \cdot \frac{1}{2\bar{u}^2} = -1.$

HOJA 9 - EJ. 6

$E=1, \quad F=u, \quad G=1 \qquad |u|<1$

Demostrar que es localmente isométrico al plano euclídeo.

$$D_t w = \left( \left( a_u + \Gamma_{uu}^{u} a + \Gamma_{uv}^{u} b \right) u' + \left( a_v + \Gamma_{uv}^{u} a + \Gamma_{vv}^{u} b \right) v' \right) X_u +$$

$$+ \left( \left( b_u + \Gamma_{uu}^{v} a + \Gamma_{uv}^{v} b \right) u' + \left( b_v + \Gamma_{uv}^{v} a + \Gamma_{vv}^{v} b \right) \right) X_v$$

$$K = \frac{\langle [D_v, D_u] X_u, X_v \rangle}{EG - F^2} \qquad w = a X_u + b X_v$$

$$\begin{pmatrix} \Gamma_{uu}^{u} & \Gamma_{uv}^{u} & \Gamma_{vv}^{u} \\ \Gamma_{uu}^{v} & \Gamma_{uv}^{v} & \Gamma_{vv}^{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_u & \frac{1}{2} E_v & F_v - \frac{1}{2} G_u \\ F_u - \frac{1}{2} E_v & \frac{1}{2} G_u & \frac{1}{2} G_v \end{pmatrix} =$$

$$= \frac{1}{1-u^2} \begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-u}{1-u^2} & 0 & 0 \\ \frac{1}{1-u^2} & 0 & 0 \end{pmatrix}$$

$$w = a X_u + b X_v$$

$$D_u w = \left( a_u - \frac{u}{1-u^2} a \right) X_u + \left( b_u + \frac{1}{1+u^2} a \right) X_v$$

$$D_v w = a_v X_v + b_v X_v$$

$$D_u X_u = \frac{-u}{1-u^2} X_u \qquad D_v D_u X_u = 0$$

$$D_v X_u = 0 \implies [D_v, D_u] X_u = D_v D_u X_u - D_u D_v X_u = 0 \implies K = 0 \ \blacksquare$$

TEOREMA DE MINDING

$$\exp_p u = 0 \quad (1)$$

$$\begin{cases} \gamma^u(0) = p \\ \gamma^{u\prime}(0) = u \end{cases} \quad \text{geodésica}$$

$f : S_1 \longrightarrow S_2$ es isometría local.

$\gamma$ geodésica a $S_1 \Longrightarrow f \circ \gamma$ es geodésica en $S_2$

$$\exp_{f(p)} T_p f u = (f \circ \gamma^u)(1)$$

$$\exp_{g(u)} T_p g(u) = g(\exp_p u)$$

$$\parallel \qquad\qquad \parallel$$

$$\exp_{f(p)} T_p f u = f \exp_p u$$

$f, g : S_1 \longrightarrow S_2$ isometrías locales

conexo

$$\left. \begin{array}{l} f(p) = g(p) \\ T_p f = T_p g \end{array} \right\} \Longrightarrow f = g$$

Isometrías del helicoide sobre sí mismo:

$$\mathbb{X}(u,v) = (v\cos u,\ v\,\mathrm{sen}\,u,\ u)$$

$$E = 1+v^2 \qquad F = 0 \qquad G = 1 \qquad \mu = 1 = \sqrt{G}$$

$$\lambda = \sqrt{1+v^2} \qquad \lambda_v = \frac{v}{\sqrt{1+v^2}}$$

$$\frac{\lambda_v}{\cancel{\mu}} = \lambda_v \qquad \lambda_{vv} = (1+v^2)^{-1/2} + v\left(-\frac{1}{2}(1+v^2)^{-3/2}\cdot 2v\right) =$$

$$= \frac{1+v^2-v^2}{(1+v^2)^{3/2}} = \frac{1}{(1+v^2)^{3/2}}$$

$$EG - F^2 = 1+v^2$$

$$\frac{1}{1+v^2}\begin{pmatrix} 1 & 0 \\ 0 & 1+v^2 \end{pmatrix}\begin{pmatrix} 0 & v & 0 \\ -v & 0 & 0 \end{pmatrix} \qquad K = \frac{-\lambda_{vv}}{\lambda} = \frac{-1}{(1+v^2)^{3/2}}\cdot\frac{1}{(1+v^2)^{1/2}} = \frac{-1}{(1+v^2)^2}$$

Utilizamos que $K$ se conserva para sacar información:

$$\bar{K}(\bar{u},\bar{v}) = \frac{-1}{(1+\bar{v}^2)^2} = K(u,v) = \frac{-1}{(1+v^2)^2} \implies \bar{v} = \varepsilon v \quad \text{con} \quad \varepsilon = \pm 1$$

$$\begin{cases} u = u(\bar{u},\bar{v}) \\ v = \varepsilon\bar{v} \end{cases}$$

Deben conservar la 1ª F F. y tratamos de sacar $u$.

$$\bar{\mathbb{X}}(\bar{u},\bar{v}) = \mathbb{X}(u(\bar{u},\bar{v}),\ v(\bar{u},\bar{v}))$$

$$\mathbb{X}_{\bar{u}} = \frac{\partial u}{\partial \bar{u}}\mathbb{X}_u + \cancel{\frac{\partial v}{\partial \bar{u}}}\mathbb{X}_u \implies \underbrace{\|\mathbb{X}_{\bar{u}}\|^2}_{1+\bar{v}^2} = \left|\frac{\partial u}{\partial \bar{u}}\right|^2 \|\mathbb{X}_u\|^2 = \left|\frac{\partial u}{\partial \bar{u}}\right|^2 (1+v^2)$$

$$\frac{\partial u}{\partial \bar{u}} = \delta = \pm 1$$

$$u = \delta\bar{u} + a(\bar{v}) \qquad \frac{\partial u}{\partial \bar{v}} = a(\bar{v})$$

$$\mathbb{X}_{\bar{v}} = \frac{\partial u}{\partial \bar{v}}\mathbb{X}_u + \frac{\partial v}{\partial \bar{v}}\mathbb{X}_v = \frac{\partial u}{\partial \bar{v}}\mathbb{X}_u + \varepsilon\mathbb{X}_v = a'(\bar{v})\mathbb{X}_u + \varepsilon\mathbb{X}_v$$

$$\langle \mathbb{X}_{\bar{u}}, \mathbb{X}_{\bar{v}} \rangle = \langle \delta\mathbb{X}_u,\ a'(\bar{v})\mathbb{X}_u + \varepsilon\mathbb{X}_v \rangle = \delta a'(\bar{v})(1+v^2) = 0$$

$$a'(\bar{v}) = 0 \implies a(\bar{v}) = a$$

isometrías del helicoide en sí mismo

$$\begin{cases} u = \delta \bar{u} + a \\ v = \varepsilon \bar{v} \end{cases}$$

$$(v\cos u, v \operatorname{sen} u, u) = (\varepsilon \bar{v} \cos(\delta \bar{u} + a), \varepsilon \bar{v} \operatorname{sen}(\delta \bar{u} + a), \delta \bar{u} + a) =$$

$$= \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} + \begin{pmatrix} \cos a & -\operatorname{sen} a & 0 \\ \operatorname{sen} a & \cos a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon \bar{v} \cos(\delta \bar{u}) \\ \varepsilon \bar{v} \operatorname{sen}(\delta \bar{u}) \\ \delta \bar{u} \end{pmatrix}$$

Movimientos rígidos de $\mathbb{R}^3$. (giros, simetrías y traslaciones)

Recuerdo: las isometrías de la esfera son las aplicaciones ortogonales. $\longrightarrow$ giros y simetrías.

$\boxed{H9 - \#EJ1}$ $$K(p) = \lim_{r\to 0} \frac{12\pi r - A}{\pi r^4}$$

$$\lambda(\theta,r)\, d\theta^2 + dr^2 \qquad \sqrt{EG-F^2} = \lambda$$

$$\lambda(\theta,0) = 0 \qquad \lambda_r(\theta,0) = 1 \qquad \lambda_{rr}(\theta,0) = 0$$

$$\lambda_{rr}(\theta,r) = -K(\theta,r)\lambda(\theta,r)$$

$$\lambda_{rrr}(\theta,r) = -K_r(\theta,r)\lambda(\theta,r) - K(\theta,r)\lambda_r(\theta,r)$$

$$\lambda_{rrr}(\theta,r) = -K(\theta,r) = -K(p)$$

$$\lambda(\theta,r) = r - \frac{Kr^3}{6} + \varphi(\theta,r) \qquad \frac{\varphi(\theta,r)}{r^3} \xrightarrow{r\to 0} 0$$

$$A_r = \iint_{D_r} \lambda\, d\theta\, dr = \int_0^r \int_0^{2\pi} \lambda(\theta,r)\, d\theta\, dr = \int_0^r \int_0^{2\pi} r - \frac{K(p)r^3}{6} + \varphi(\theta,r)\, d\theta\, dr =$$

$$= 2\pi\left(\frac{r^2}{2} - \frac{K(p)r^4}{24}\right) + \varphi(\theta,r) = \qquad \frac{\psi(\theta,r)}{r^4} \xrightarrow[r\to 0]{} 0$$

$$= \pi r^2 - \frac{K(p)r^4}{12}\pi + \psi$$

$$\pi K(p)\frac{r^4}{12} = \pi r^2 - A_r + \psi \implies K(p) = \frac{12}{\pi}\frac{(\pi r^2 - A_r)}{r^4} + \frac{12\psi}{\pi r^4}$$

$$\xrightarrow[r\to 0]{} \frac{12}{\pi}\left(\frac{\pi r^2 - A_r}{r^4}\right) = K(p)$$

17- EJ.3

$u>0 \quad v>1$

$$ds^2 = (du)^2 + \frac{2}{v}\,du\,dv + e^u (dv)^2$$

a) $\alpha(t) = \mathbb{X}(t, v_0) \quad \|\alpha'(t)\| = 1$ geodésicas

b) $\mathbb{X}(1,e) \quad \mathbb{X}(h(v), v) = \beta(v)$

c) $(\bar{u},\bar{v}) \qquad d\bar{s}^2 = (d\bar{u})^2 + c(\bar{u},\bar{v})(d\bar{v})^2$

$$\begin{cases} u(t) = t & v(t) = v_0 \\ u' = 1 & v' = 0 \\ u'' = 0 & v'' = 0 \end{cases}$$

$$\boxed{\begin{cases} u''(t) + \Gamma_{uu}^{u}\, u'(t)^2 + 2\Gamma_{uv}^{u}\, u'(t)v'(t) + \Gamma_{vv}^{u}\, v'(t)^2 = 0 \\ v''(t) + \Gamma_{uu}^{v}\, u'(t)^2 + 2\Gamma_{uv}^{v}\, u'(t)v'(t) + \Gamma_{vv}^{v}\, v'(t)^2 = 0 \end{cases}}$$ ECUACIONES GEODÉSICAS

$$\Gamma_{uu}^{u} = 0 \qquad \Gamma_{uu}^{v} = 0$$

$$\begin{pmatrix} \Gamma_{uu}^{u} & \Gamma_{uv}^{u} & \Gamma_{vv}^{u} \\ \Gamma_{uu}^{v} & \Gamma_{uv}^{v} & \Gamma_{uv}^{v} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_v \end{pmatrix}$$

$E = 1 \implies E_u = 0 \qquad E_v = 0$

$F = \frac{2}{v} \implies F_v = 0$

$\alpha'(t) = \mathbb{X}_u(t, v_0) \qquad \|\alpha'(t)\| = \sqrt{E} = 1$

$\beta(t) = \mathbb{X}(u(t), v(t)) \qquad \beta'(t) = \mathbb{X}(u(t), v(t)) \qquad \beta(0) = \mathbb{X}(1,e)$

$u(0) = 1 \qquad v(0) = e$

$\beta'(t) = u'\mathbb{X}_u + v'\mathbb{X}_v \qquad \langle \beta'(t), \mathbb{X}_u(\beta(t))\rangle = 0 = Eu' + Fv' = u' + \frac{v'}{v} = 0$

$u + \ln v = \text{cte.}$

$u = 2 - \log v = \log \frac{e^2}{v} = h(v)$

$\implies 1 + \log e = c \qquad c = 2$

...

**4.** Demostrar que no existe una superficie $\mathbb{X}(u,v)$ tal que $E=G=1$, $F=0$ y $e=1$, $f=0$, $g=-1$

$$K = \frac{-1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right] = 0 \quad \left(\begin{array}{l}\text{usando esta} \\ \text{fórmula de } F=0\end{array}\right)$$

$$K = \frac{eg-f^2}{EG-F^2} = \frac{-1}{1} = -1$$

contradicción

**5.** Decidir si existe una superficie $\mathbb{X}(u,v)$ tal que $E=1$, $F=0$, $G=\cos^2 u$ y $e=\cos^2 u$, $f=0$, $g=1$.

$$K = (1^{\underline{a}} \text{ fórmula}) = 1$$

$$K = \frac{eg-f^2}{EG-F^2} = 1$$

no hemos encontrado

Opción 1: se han equivocado y $g=-1$ $\left(\begin{array}{l}\text{fácil llegar} \\ \text{a una contr.}\end{array}\right)$

Opción 2: utilizar las ecuaciones de compatibilidad

$\longrightarrow$ que solo sabemos que existen.

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## FÓRMULA IMPORTANTE:

$$K = \frac{1}{EG-F^2}\left[F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} - \begin{pmatrix}\frac{1}{2}E_u & F_u - \frac{1}{2}E_v\end{pmatrix}\begin{pmatrix}E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix}F_v - \frac{1}{2}G_v \\ \frac{1}{2}G_v\end{pmatrix} + \begin{pmatrix}\frac{1}{2}E_v & \frac{1}{2}G_u\end{pmatrix}\begin{pmatrix}E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{2}E_v \\ \frac{1}{2}G_u\end{pmatrix}\right]$$

H9-E3

$E_\Phi = \frac{1}{2u}$ $\quad G_\Phi = u^2$ $\quad a = \frac{1}{\sqrt{2u}}$ $\quad b = u$

$b_u = 1$ $\quad (\sqrt{2u})_u = \frac{2}{2\sqrt{2u}} = \frac{1}{\sqrt{2u}}$

$$K_\Phi = \frac{-1}{\frac{1}{\sqrt{2u}}\cdot u}\left[\cancel{\left(\frac{a_v}{b}\right)_v}^{0} + \left(\frac{b_u}{a}\right)_u\right] =$$

$$= \frac{-1}{\frac{1}{\sqrt{2u}}\cdot u}\left(\frac{1}{\sqrt{2u}}\right) = \frac{-1}{u}$$

---

$E_\Psi = \frac{1}{2u}$ $\quad G_\Psi = \frac{1}{2u}$ $\quad a = \frac{1}{\sqrt{2u}}$ $\quad b = \frac{1}{\sqrt{2u}}$

$a_v = 0$

$$b_u = -\frac{2}{2\sqrt{2u}} = \frac{\frac{-1}{\sqrt{2u}}}{2u} = \frac{-1}{2u\sqrt{2u}}$$

$$\frac{b_u}{a} = \frac{\frac{-1}{2u\sqrt{2u}}}{\frac{1}{\sqrt{2u}}} = \frac{-1}{2u} \xrightarrow{du} \frac{+1\cdot 2}{4u^2} = \frac{1}{2u^2}$$

$$K_\Psi = \frac{-1}{\frac{1}{2u}}\left[\frac{1}{2u^2}\right] = \frac{-2u}{2u^2} = \frac{-1}{u}$$

$u = u(\bar u, \bar v)$
$v = v(\bar u, \bar v)$

---

$\rightarrow \boxed{u = \bar u}$

$dv = V_{\bar u}\bar u' + V_{\bar v}\cdot \bar v'$

$$\frac{1}{2u}(du)^2 + u^2(dv)^2$$

$$\Downarrow$$

$$\frac{1}{2\bar u}(d\bar u)^2 + \bar u^2\left(V_{\bar u}\bar u' + V_{\bar v}\bar v'\right)^2 = \frac{1}{2\bar u}(d\bar u)^2 + \bar u^2\Big(V_{\bar u}^2\bar u'^2 + V_{\bar v}^2\bar v'^2 +$$

$$+ 2V_{\bar u}\bar u' V_{\bar v}\bar v'\Big) = \frac{1}{2\bar u}(d\bar u)^2 + \bar u^2 V_{\bar u}^2 d\bar u^2 + \bar u^2 V_{\bar v}^2 d\bar v^2 +$$

$$+ 2V_{\bar u}V_{\bar v}\,d\bar u\,d\bar v = \boxed{\left(\frac{1}{2\bar u} + \bar u^2 V_{\bar u}^2\right)} d\bar u^2 + 2V_{\bar u}V_{\bar v}\,d\bar u\,d\bar v +$$

$$+ \bar u^2 V_{\bar v}^2 d\bar v^2 = \boxed{\frac{1}{2\bar u}} d\bar u^2 + \frac{1}{2\bar u}d\bar v^2 \rightarrow (*)$$

$\bar u^2 V_{\bar u}^2 = 0 \rightarrow V_{\bar u}^2 = 0 \rightarrow V$ no depende de $\bar u$

contradicción

$(*)$ $\bar u^2 V_{\bar v}^2 = \frac{1}{2\bar u} \Rightarrow V_{\bar v}^2 = \frac{1}{2\bar u^3} \Rightarrow V_{\bar v} = \sqrt{\frac{1}{2\bar u^3}} \Rightarrow V$ depende de $u$