$\boxed{1.}$ $y' = \dfrac{2x^3}{1+x^2+y^2}$ a) Existencia y unicidad en $x \in \mathbb{R}$.

$y(0) = 0$

ⓐ

i) f es continua en $\mathbb{R}^2$

(Razón: f es cociente de dos polinomios con denominador que nunca se anula ≥ 1).

ii) $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{-4x^3y}{(1+x^2+y^2)^2}$, continua en $\mathbb{R}^2$ por la misma razón que en i.

$2|x|\dfrac{x^2}{1+x^2+y^2} \leq 2|x|$ dos opcio

HASTA AQUÍ EXISTENCIA Y UNICIDAD LOCAL ——

iii) $\left|\dfrac{\partial f}{\partial y}\right| = \dfrac{4|x|^3|y|}{(1+x^2+y^2)^2} = 4|x|^3\dfrac{|y|}{(1+x^2+y^2)^2} \underset{(*)}{\leq} 4|x|^3\dfrac{1}{2(1+x^2+y^2)} \leq 2|x$

$$(*)\begin{cases} |y| \leq \dfrac{1+y^2}{2} & \left(0 \leq (|y|-1)^2 = |y|^2+1-2|y|,\ \text{e.d., } 2|y| \leq y^2+1\right) \\ |y| \leq \dfrac{1+x^2+y^2}{2} & \longrightarrow \dfrac{|y|}{1+x^2+y^2} \leq \dfrac{1}{2} \end{cases}$$

$\Longrightarrow \left|\dfrac{\partial f}{\partial y}\right| \leq 2|x|^3 \leq 2M^3$ si $|x| \leq M$

e.d., $\dfrac{\partial f}{\partial y}$ está acotada en $[-M,M]\times\mathbb{R}$ $\Longrightarrow$

$\underset{\text{T}^{\text{ma}}\text{ global}}{\Longrightarrow} \exists!$ en $[-M,M]$ Como M puede ser cualquiera, $\exists!$ en $\mathbb{R}$.

) $|y(x)| \leq x^2 \qquad \forall x \in \mathbb{R}$

$$y'(x) = \frac{2x^3}{1+x^2+y^2}$$

- $\underline{x \geq 0}$

$$0 \leq y'(x) = \frac{2x^3}{1+x^2+y^2} \underset{x\geq 0}{\leq} 2x$$

$$\frac{x^2}{1+x^2+y^2}$$

$$\underbrace{\int_0^x 0\, ds}_{\overset{\shortparallel}{0}} \leq \underbrace{\int_0^x y'(s)\, ds}_{y(x)-y(0)} \leq \int_0^x 2s\, ds = x^2 \implies$$

Importante

$$\Rightarrow y(x) \geq 0 \implies |y(x)| = y(x) \leq x^2$$

- $\underline{x \leq 0}$

$$2x \leq y'(x) = \frac{2x^3}{1+x^2+y^2} \leq 0$$

Hacemos lo mismo

TRA POSIBILIDAD PARA HACERLO:

$$\bar{y}(x) \overset{def.}{=} y(-x) \implies \bar{y}'(x) = -y'(-x) = \frac{-2(-x)^3\, y(-x)}{1+(-x)^2+(y(-x))^2}$$

e.d. $\bar{y}'(x) = \dfrac{2x^3}{1+x^2+\bar{y}^2} \xrightarrow{\text{unic.}} \bar{y}(x) = y(x)$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{y(-x)}$

$\bar{y}(0) = 0$

$\boxed{2.}$ $X' = AX$

$$X(t) = e^{2t}\begin{pmatrix}1\\2\\1\end{pmatrix} \qquad Z(t) = \underbrace{e^{(1+i)t}}_{e^t \cdot \underbrace{e^{it}}_{\hookrightarrow = \cos t + i \operatorname{sen} t}}\begin{pmatrix}1+i\\1-i\\0\end{pmatrix}$$

a) 3 soluciones reales indep.

$$X_2(t) = \operatorname{Re}(Z(t)) = e^t\begin{pmatrix}\cos t - \operatorname{sen} t\\ \cos t + \operatorname{sen} t\\ 0\end{pmatrix}$$

$$X_3(t) = \operatorname{Im}(Z(t)) = e^t\begin{pmatrix}\cos t + \operatorname{sen} t\\ -\cos t + \operatorname{sen} t\\ 0\end{pmatrix}$$

$$\det\begin{pmatrix}\uparrow & \uparrow & \uparrow\\ X_1(0) & X_2(0) & X_3(0)\\ \downarrow & \downarrow & \downarrow\end{pmatrix} = \det\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 1 & 0 & 0\end{pmatrix} \neq 0$$

$$\overset{\text{wros.}}{\Longrightarrow} \det(X_1(t), X_2(t), X_3(t)) \neq 0 \quad \forall t$$

b) $\mathbb{F}$ tal que $\mathbb{F}(0) = Id.$

$\Phi(t)$ fundam., $\Phi(0) = Id$

i) $\mathbb{F}(t)B$ fundamental si $\det(B) \neq 0$ B constante

$B = \mathbb{F}(0)^{-1}$

$\Phi(t) = \mathbb{F}(t) \cdot \mathbb{F}(0)^{-1}$

c) $\lambda_1 = 2$, $\lambda_2 = 1+i$, $\lambda_3 = 1-i$

$\Phi(t) = e^{At}$ explicar esto.

$$e^{t}(\cos t + i \,\mathrm{sen}\, t)\cdot\left(\begin{pmatrix}1\\1\\0\end{pmatrix} + i\begin{pmatrix}1\\-1\\0\end{pmatrix}\right)$$

$$e^{t}\left\{\underbrace{\cos t\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\text{REAL}} + \underbrace{i\cos t\begin{pmatrix}1\\-1\\0\end{pmatrix}}_{\mathbb{C}} + \underbrace{i\,\mathrm{sen}\, t\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\mathbb{C}} + \underbrace{i\,\mathrm{sen}\, t\begin{pmatrix}1\\-1\\0\end{pmatrix}}_{\text{Real}}\right\}$$

$$i\,\boxed{e^{t}\left(\ \mathbb{C}\ \right)} + \boxed{e^{t}\left(\ \mathbb{R}\ \right)}$$

$\text{Im } z(t)$ $\qquad\qquad$ $\text{Re } z(t)$

$$z(t) = a + bi$$