EXAMEN ENERO 17-18

1. a) Hallar estimación por momentos $\mathbb{E}_\theta(X) = \overline{X}$

$$\mathbb{E}_\theta(X) = \frac{1}{3} + \frac{2}{3} + 3\,\frac{(1-\theta)}{3} = 1 + 1 - \theta = 2-\theta$$

$\Rightarrow \boxed{EM_\theta = 2-\overline{X}}$ $\quad \overline{x} = 1.27 + 2.23 + 3.40 = 1'93 \quad \Rightarrow \boxed{\hat{\theta} = 0'07}$

b) Hallar estimación por máxima verosimilitud

$$VERO(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{\theta}{3}\right)^{10} \cdot \left(\frac{1}{3}\right)^{27} \cdot \left(\frac{1}{3}\right)^{23} \cdot \left(\frac{1-\theta}{3}\right)^{40}$$

$$\log VERO(\theta) = 10 \log\left(\frac{\theta}{3}\right) + 40 \log\left(\frac{1-\theta}{3}\right) + \text{constante}$$

$$\frac{d}{d\theta} \log VERO(\theta) = 10 \cdot \frac{1/3}{\theta/3} + 40 \cdot \frac{-1/3}{\frac{1-\theta}{3}} = \frac{10}{\theta} - \frac{40}{1-\theta}$$

Igualamos a cero y despejamos θ: $\frac{10(1-\theta) - 40\theta}{(1-\theta)\theta} = 0 \Rightarrow 10(1-\theta) - 40\theta = 0 \Rightarrow$

$\Rightarrow 10 - 10\theta - 40\theta = 0 \Rightarrow \boxed{\hat{\theta} = 1/5}$

2. $\mathbb{X} = (X_1, \dots, X_n) \qquad m = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad V = \begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & \ddots \\ & & & n \end{pmatrix} \qquad V_{i,i} = i \quad i = 1, \dots, n$

a) $\mathbb{Z} = (Z_1, \dots, Z_n) \qquad Z_j = \sum_{i=1}^{j} X_i \qquad \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} X_1 \\ X_1 + X_2 \\ \vdots \\ X_1 + \dots + X_n \end{pmatrix}$

¿$Z \sim N(\vec{m}_2, V_2)$? Calcular m_2 y V_2

$$\begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & 0 \\ 1 & 1 & \\ & & \ddots \\ 1 & & & 1 \end{pmatrix}}_{B} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \qquad det(B) = 1 \neq 0 \Rightarrow \mathbb{Z} \sim N(-,-)$$

$$\vec{m}_2 = B\vec{m} = \vec{m} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad V_2 = BVB^t$$

b) $W_j = \frac{X_j}{\sqrt{j}}$; Sea $A = \frac{1}{n}\sum_{j=1}^{n} W_j$ y $B = \sum_{j=1}^{n} (W_j - A)^2$

No es difícil ver que W_j es X_j tipificada. Por lo tanto A podemos verlo como $\overline{Y}$ donde $Y \sim N(0,1)$, y B como $(n-1)S_Y^2$.

CORREGIR FALLO EN ADELANTE

Entonces: $\mathbb{P}\left(-\frac{5}{\sqrt{n}} \leq \overline{Y} \leq \frac{5}{\sqrt{n}},\ S_Y^2 > \frac{2n-2}{n}\right)$

Por el Teorema de Fisher-Cochran sabemos que $\overline{Y}$ y S_Y^2 son independientes $\Rightarrow \mathbb{P}\left(-\frac{5}{\sqrt{n}} \leq \overline{Y} \leq \frac{5}{\sqrt{n}},\ S_Y^2 > \frac{2n-2}{n}\right) = \underbrace{\mathbb{P}\left(\frac{-5}{\sqrt{n}} \leq \overline{Y} \leq \frac{5}{\sqrt{n}}\right)}_{[1]} \cdot \underbrace{\mathbb{P}\left(S_Y^2 > \frac{2n-2}{n}\right)}_{[2]}$

$$[1] = \mathbb{P}\left(\frac{-5}{\sqrt{n}} \le \bar{Y} \le \frac{5}{\sqrt{n}}\right) = \mathbb{P}\left(\frac{-5\sqrt{n}}{\sqrt{n}} \le \cancel{Y} \le \frac{5\sqrt{n}}{\sqrt{n}}\right) = \Phi(5) - \phi(-5)$$

$$Z \triangleq \frac{\bar{Y}}{1/\sqrt{n}} \sim N(0,1) \qquad Z$$

$$[2] = \mathbb{P}\left(S_Y^2 \ge \frac{2n-2}{n}\right) = \mathbb{P}\left(\frac{(n-1)}{1} S_Y^2 \ge \frac{(2n-2)(n-1)}{n}\right) = 1 - \mathbb{P}\left(\underbrace{(n-1)S_Y^2}_{\chi^2_{n-1}} \le \frac{(2n-2)(n-1)}{n}\right)$$

$$= 1 - F_{\chi^2_{n-1}}\left(\frac{(2n-2)(n-1)}{n}\right)$$

Multiplicando $[1] \cdot [2]$ nos da el resultado buscado.

3.

val	-1	0	1
prob	$\theta/4$	$1-\frac{\theta}{2}$	$\theta/4$

$\theta \in (0,1)$ $\qquad T = 2\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)$

a) ¿T insesgado? $\mathbb{E}_\theta(T) = \frac{2}{n}\mathbb{E}(\sum x_i^2) = 2\mathbb{E}(x^2)$ clones + indep.

$\mathbb{E}(x^2) = \frac{\theta}{4} + \frac{\theta}{4} = \frac{\theta}{2}$; $\quad \mathbb{E}_\theta(T) = \frac{2\theta}{2} = \theta \Rightarrow$ T insesgado

b) Cota Cramer-Rao ¿Mínima varianza?

$Y = \partial_\theta \log f(x_i;\theta)$

$$\begin{cases} \partial_\theta \log f(-1;\theta) = \partial_\theta \log(\theta/4) = \frac{1/4}{\theta/4} = \frac{1}{\theta} \\ \partial_\theta \log f(0;\theta) = \partial_\theta \log\left(\frac{2-\theta}{2}\right) = \frac{-1/2}{\frac{2-\theta}{2}} = \frac{1}{\theta-2} \\ \partial_\theta \log f(1;\theta) = \partial_\theta \log(\theta/4) = \frac{1}{\theta} \end{cases}$$

$\Rightarrow$

val	$1/\theta$	$1/\theta-2$	$1/\theta$
prob	$\theta/4$	$1-\frac{\theta}{2}$	$\theta/4$

$$V(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$$

$$\mathbb{E}(Y) = \frac{1}{\theta}\cdot\frac{\theta}{4} + \frac{1}{\theta-2}\cdot\frac{2-\theta}{2} + \frac{1}{\theta}\cdot\frac{\theta}{4} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\mathbb{E}(Y^2) = \frac{1}{\theta^2}\cdot\frac{\theta}{4} + \frac{1}{(\theta-2)^2}\cdot\frac{2-\theta}{2} + \frac{1}{\theta^2}\cdot\frac{\theta}{4} = \frac{1}{2\theta} - \frac{1}{2(\theta-2)}$$

$$\Rightarrow V_\theta(Y) = I_Y(\theta) = \frac{1}{2\theta} - \frac{1}{2(\theta-2)} = \frac{\theta-2-\theta}{2\theta(\theta-2)} = \frac{1}{\theta(2-\theta)}$$

$$\text{Cota Cramer-Rao} = \frac{1}{nI_Y(\theta)} = \frac{\theta(2-\theta)}{n} \qquad \boxed{V_\theta(T) \ge \frac{\theta(2-\theta)}{n} \quad \forall T} \text{ C-R}$$

$$V_\theta(T) = V\left(\frac{2}{n}\sum_{i=1}^{n} x_i^2\right) = \frac{4}{n^2}\sum V(x_i^2) = \frac{4}{n}V(x^2) \rightarrow = \mathbb{E}(x^4) - \mathbb{E}(x^2)^2$$

$$\mathbb{E}(x^4) = \mathbb{E}(x^2) = \frac{\theta}{2} \; ; \; V(x^2) = \frac{\theta}{2} - \frac{\theta^2}{4} = \frac{2\theta-\theta^2}{4} = \frac{(2-\theta)\theta}{4}$$

$$\Rightarrow V_\theta(T) = \frac{(2-\theta)\theta}{n} \Rightarrow \text{mínima varianza ✓}$$

4. $f(x;\theta) = \frac{1}{2\theta} \cdot e^{-|x|/\theta} \quad \forall x \in \mathbb{R} \quad \theta > 0 \quad T(X_1, \dots, X_n) = \left(\frac{1}{2n} \sum_{i=1}^{n} X_i^2\right)^{1/2}$

$T_n = \left(\frac{1}{2} \overline{X^2}\right)^{1/2}$

$$\mathbb{E}(\overline{X^2}) = \mathbb{E}\left(\frac{1}{n} \sum X_i^2\right) = \mathbb{E}(X^2) = \int_{-\infty}^{\infty} \frac{x^2}{2\theta} e^{-|x|/\theta}\, dx = 2\int_0^{\infty} \frac{x^2}{2\theta} e^{-x/\theta}\, dx =$$

$$= \frac{1}{\theta} \int_0^{\infty} x^2 e^{-x/\theta}\, dx$$ cambio variable: $t = x/\theta \rightarrow x = t\theta \rightarrow dx = \theta dt$

$$= \frac{1}{\theta} \int_0^{\infty} t^2 \theta^2 e^{-t}\, d\cancel{t} = \theta\, \Gamma(3) = 2\theta$$

$$V(\overline{X^2}) = V\left(\frac{1}{n} \sum X_i^2\right) = \frac{1}{n^2} \sum V(X_i^2) = \frac{1}{n} V(X^2) = \frac{1}{n}\left(\mathbb{E}(X^4) - \mathbb{E}(X^2)^2\right)$$

$$\mathbb{E}(X^4) = 2\int_0^{\infty} \frac{x^4}{2\theta} e^{-x/\theta}\, dx = \frac{1}{\theta} \int_0^{\infty} x^4 e^{-x/\theta}\, dx = \frac{1}{\theta} \int_0^{\infty} t^4 \theta^4 e^{-t}\, d\cancel{t} =$$

$$= \theta^3 \Gamma(5) = 24\theta^3$$

$$\Rightarrow V(\overline{X^2}) = \frac{1}{n}\left(24\theta^3 - 4\theta^2\right) = \frac{24\theta^3 - 4\theta^2}{n}$$

$$\sqrt{n}\left(\overline{X^2} - 2\theta\right) \xrightarrow{d} N\left(0, \frac{24\theta^3 - 4\theta^2}{n}\right)$$

Sea ahora $g(x) = \sqrt{\frac{1}{2}x} \longrightarrow g'(x) = \frac{1/2}{2\sqrt{1/2\, x}} = \frac{1}{4\sqrt{\frac{1}{2}x}}$

$g(2\theta) = \sqrt{\theta} \qquad |g'(2\theta)|^2 = \frac{1}{8(\frac{1}{2}2\theta)} = \frac{1}{8\theta}$

$$\Rightarrow \sqrt{n}\left(\sqrt{\tfrac{1}{2}\overline{X^2}} - \sqrt{\theta}\right) \xrightarrow{d} N\left(0, \frac{1}{8\theta} \cdot \frac{24\theta^3 - 4\theta^2}{n}\right)$$

$$\Rightarrow \sqrt{n}\left(\left(\tfrac{1}{2}\overline{X^2}\right)^{1/2} - \sqrt{\theta}\right) \xrightarrow{d} N\left(0, \frac{6\theta^2 - \theta}{2n}\right)$$

(5.) a) Muestra inicial $\begin{cases} 25 \text{ la odian} \\ 5 \text{ la aman} \end{cases}$ estima p con un error menor que 2% (conf. 95%)

Opción conservadora: $Z_{2'5\%}\sqrt{\frac{1/2(1-1/2)}{n}} \leq 0'02 \Rightarrow \sqrt{n} \geq \frac{1'96 \cdot \sqrt{1/4}}{0'02} \Rightarrow$

$\Rightarrow \boxed{n \geq 2401}$

Opción utilizando muestra inicial: $\bar{x} = \frac{5}{30} = \frac{1}{6}$

$Z_{2'5\%}\sqrt{\frac{1/6(1-1/6)}{n}} \leq 0'02 \Rightarrow \sqrt{n} \geq \frac{1'96 \cdot \sqrt{5/36}}{0'02} \Rightarrow \boxed{n \geq 1334}$

b) $n_1 = 100$, $\bar{x} = 30$, $\overline{x^2} = 999$ $\qquad n_2 = 130$, $\bar{y} = 32$, $\overline{y^2} = 1180$

¿Evidencia varianza(Masai Mara(y)) > varianza(Samburu(x))?

$H_0 \equiv$ varianza(x) > varianza(y) (contraria para rechazar)

Es decir, $H_0 \equiv \sigma_2 \geq \sigma_1$ siendo $\begin{cases} \sigma_2 = \text{varianza peso gacelas S.} \\ \sigma_1 = \text{varianza peso gacelas MM} \end{cases}$

suposición (no lo especifica en el b)

Región de rechazo: $\frac{S_1^2}{S_2^2} > F_{\{129, 99, 5\%\}}$

$S_1^2 = 1180 - 32^2 = 156$; $S_2^2 = 99$

¿ $\frac{156}{99} > 1'371$? $\Rightarrow 1'58 > 1'371 \Rightarrow$ rechazamos $\Rightarrow$

$\Rightarrow$ existe evidencia estadística de que la variabilidad en el peso de las gacelas es mayor en MM que en S.

(6.) $X \sim \text{Exp}(\lambda)$ $\lambda > 0$ $\qquad H_0: \lambda < 0'01$ $\qquad \Theta_0 = (0, 0'01)$

Test: $n = 5$ $\qquad$ Si $\min(X_1, \dots, X_5) < 2 \Rightarrow$ rechazamos

Hallar función de potencia del test y su nvl. de significación

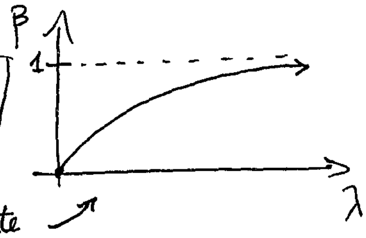
$P_\lambda(\text{rechazar}) = P_\lambda(\min(X_1, \dots, X_5) \leq 2)$

Sabemos que $F_{m_n}(t) = 1 - (1 - F_X(t))^n$ $\qquad F_X(t) = 1 - e^{-\lambda t}$ $\quad t > 0$

$\Rightarrow F_{m_n}(t) = 1 - (1 - 1 + e^{-\lambda t})^n = 1 - e^{-\lambda n t}$

Ahora, $t = 2$ y $n = 5 \Rightarrow P_\lambda(\text{rechazar}) = \boxed{\beta(\lambda) = 1 - e^{-10\lambda}}$

$\beta(0) = 0$
$\beta(\infty) = 1$
monótona creciente



Significación $= \sup_{\lambda \in \Theta_0} \beta(\lambda) = \boxed{\beta(0'01)}$

EXAMEN JUNIO 17-18

1.

val	1	2	3
prob	P_1	P_2	P_3

$P_1 + P_2 + P_3 = 1$ $\quad P_1 \in (0,1)$ $\quad P_2 \in (0,1)$

$P_1 + P_2 < 1$

val	1	2	3
apar.	20	25	55

$n = 100$ Hallar P_1 y P_2 por máxima verosimilitud.

$$\text{VERO}(P_1,P_2\,;X_1,\dots,X_{100}) = \prod_{i=1}^{100} f(x_i;P_1,P_2) = P_1^{20}\cdot P_2^{25}(1-P_1-P_2)^{55}$$

$$\log\text{VERO}(P_1,P_2) = 20\log(P_1) + 25\log(P_2) + 55\log(1-P_1-P_2)$$

$$\frac{\partial}{\partial P_1}(\log\text{VERO}(P_1,P_2)) = \frac{20}{P_1} - \frac{55}{1-P_1-P_2} \; ; \; \frac{\partial}{\partial P_2}(\log\text{VERO}(P_1,P_2)) = \frac{25}{P_2} - \frac{55}{1-P_1-P_2}$$

$$\begin{cases} 20(1-P_1-P_2) - 55P_1 = 0 \\ 25(1-P_1-P_2) - 55P_2 = 0 \end{cases} \Rightarrow 20 - 20P_1 - 20P_2 - 55P_1 = 0 \Rightarrow P_2 = \frac{20-75P_1}{20}$$

$$25 - 25P_1 - 25\cdot\frac{20-75P_1}{20} - 55\cdot\frac{20-75P_1}{20} = 0 \Rightarrow$$

$$\Rightarrow \cancel{25} - 25P_1 - \cancel{25} + \frac{375}{4}P_1 - 55 + \frac{825}{4}P_1 = 0 \Rightarrow 275P_1 = 55 \Rightarrow$$

$$\Rightarrow P_1 = \frac{55}{275} \Rightarrow \boxed{P_1 = 0'2}$$

$$P_2 = \frac{20-75\cdot 0'2}{20} = 0'25 \Rightarrow \boxed{P_2 = 0'25}$$

Máximo debido a que $[0,1]\times[0,1]$ es compacto en $\mathbb{R}^2$ (Teorema Weierstrass)

2. a) $\mathbb{X} = (X_1,X_2,X_3)$ $\quad \vec{m} = (1,1,0)$ $\quad \mathbb{V} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$$\mathbb{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} X_1+X_2 \\ X_1+X_2+X_3 \\ 2X_1+X_2 \end{pmatrix}$$

¿medias de Z_i, varianza de Z_3, $\text{cov}(Z_1,Z_2)$?

$$\mathbb{E}(Z_1) = \mathbb{E}(X_1+X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 2 \; ; \; \mathbb{E}(Z_2) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\mathbb{E}(X_3) = 2$$

$$\mathbb{E}(Z_3) = 2\mathbb{E}(X_1) + \mathbb{E}(X_2) = 3$$

$$V(Z_3) = V(2X_1+X_2) = V(2X_1) + V(X_2) + 2\text{cov}(2X_1,X_2) = \underbrace{4V(X_1)}_{4.1} + \underbrace{V(X_2)}_{2} + \underbrace{4\text{cov}(X_1,X_2)}_{4.1} = 10$$

$$\text{cov}(Z_1,Z_2) = \text{cov}(X_1+X_2\,,X_1+X_2+X_3) = \text{cov}(X_1,X_1) + \text{cov}(X_1,X_2) + \text{cov}(X_1,X_3) +$$

$$+ \text{cov}(X_2,X_1) + \text{cov}(X_2,X_2) + \text{cov}(X_2,X_3) = 1+1+1+1+2+2 = 8$$

b) $n=35$ $\quad N(1,2)$ $\quad ¿\mathbb{P}(\{\bar{X}<1'2\}\cap\{S<1'2\})$?

Por el teorema de Fisher-Cochran sabemos que $\bar{X}$ y S^2 son independientes, por lo que la probabilidad de la intersección es igual al producto de las probabilidades.

$$\mathbb{P}(\{\bar{X}<1'2\}\cap\{S^2<1'44\}) = \mathbb{P}(\bar{X}<1'2)\cdot\mathbb{P}(S^2<1'44) =$$

$$= \mathbb{P}\left(\frac{\bar{X}-1}{\sqrt{2/35}} < \frac{1'2-1}{\sqrt{2/35}}\right)\cdot\mathbb{P}\left(\frac{34.S^2}{2} < \frac{1'44.34}{2}\right) =$$

$$= \phi\left(\frac{0'2}{\sqrt{2/35}}\right)\cdot F_{\chi^2_{34}}\left(\frac{1'44.34}{2}\right)$$

3. $\quad f(x;a)=\frac{2}{a^2}x \quad x\in[0,a] \qquad a>0$

a) $T_1(X_1,\dots,X_n)=\frac{3}{2}\bar{X}$ $\quad$ Comprueba que es insesgado. ¿$V(T_1)$?

$$\mathbb{E}(T_1)=\frac{3}{2}\mathbb{E}(\bar{X})=\frac{3}{2}\mathbb{E}(X)$$

$$\mathbb{E}(X)=\int_0^a x\frac{2}{a^2}x\,dx=\frac{2}{a^2}\int_0^a x^2\,dx=\frac{2}{a^2}\left[x^3/3\right]_0^a=\frac{2a^3}{3a^2}=\frac{2a}{3}$$

$$\Rightarrow \mathbb{E}(T_1)=\frac{3}{2}\cdot\frac{2}{3}\cdot a=a \Rightarrow T_1 \text{ insesgado.}$$

$$V(T_1)=\frac{9}{4}V(\bar{X})=\frac{9}{4n}V(X)=\frac{9}{4n}\left(\mathbb{E}(X^2)-\mathbb{E}(X)^2\right)$$

$$\mathbb{E}(X^2)=\int_0^a x^2\frac{2}{a^2}x\,dx=\frac{2}{a^2}\int_0^a x^3dx=\frac{2}{a^2}\left[\frac{x^4}{4}\right]_0^a=\frac{2a^4}{4a^2}=\frac{a^2}{2}$$

$$\Rightarrow V(T_1)=\frac{9}{4n}\left(\frac{a^2}{2}-\frac{4a^2}{9}\right)=\frac{9}{4n}\left(\frac{2a^2-36a^2}{18}\right)=\frac{-17a^2}{4n}$$

b) $T_2(X_1,\dots,X_n)=\max(X_1,\dots,X_n)$ es sesgado. Calcular su sesgo

$$\mathbb{E}(T_2)=\mathbb{E}(\max(X_1,\dots,X_n))$$

Sabemos que $F_{M_n}(t)=(F_X(t))^n \Rightarrow f_{M_n}(x)=n\,(F_X(x))^{n-1}\cdot f_X(x)$

$$F_X(x)=\begin{cases}0 & x<0\\ \int_0^x\frac{2}{a^2}x\,dx & x\in[0,a]\\ 1 & x>a\end{cases} \Rightarrow F_X(x)=\begin{cases}0 & x<0\\ \frac{x^2}{a^2} & x\in[0,a]\\ 1 & x>a\end{cases}$$

Entonces $f_{M_n}(x) = n\cdot\left(\frac{x^2}{a^2}\right)^{n-1}\cdot\frac{2}{a^2}x = 2n\,\frac{x^{2(n-1)+1}}{a^{2(n-1)+2}}$

$$\Rightarrow \mathbb{E}(T_2) = \int_0^a x\cdot 2n\cdot\frac{x^{2(n-1)+1}}{a^{2(n-1)+2}}\,dx = \frac{2n}{a^{2(n-1)+2}}\int_0^a x^{2(n-1)+2}\,dx = \frac{2n}{a^{2(n-1)+2}}\left[\frac{x^{2(n-1)+3}}{2(n-1)+3}\right]_0^a =$$

$$= \frac{2na}{2(n-1)+3} = \frac{2na}{2n+1}$$

$$\Rightarrow \text{sesgo}(a) = \mathbb{E}(T_2) - a = \frac{2na}{2n+1} - a = \frac{2na-(2n+1)a}{2n+1} = \boxed{\frac{-a}{2n+1}}$$

4. $T(X_1,\dots,X_n) = \left(\frac{\overline{X^2}}{2}\right)^{1/3}$ Resultado normalidad asintótica Método Delta.

DATOS: $\mathbb{E}_a(X) = a^{3/2}$ $\mathbb{E}(X^2) = 2a^3$ $\mathbb{E}(X^3) = 5a^{9/2}$ $\mathbb{E}(X^4) = 15a^6$

$$\mathbb{E}(\overline{X^2}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}(X_i^2) = \mathbb{E}(X^2) = 2a^3$$

$$V(\overline{X^2}) = V\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) = \frac{1}{n^2}\sum_{i=1}^n V(X_i) = \frac{1}{n}V(X) = \frac{1}{n}\left(\mathbb{E}(X^2) - \mathbb{E}(X)^2\right) = \frac{1}{n}(2a^3 - a^3) =$$

$$= \frac{a^3}{n}$$

$$\Rightarrow \sqrt{n}\left(\overline{X^2} - 2a^3\right) \xrightarrow{d} N\left(0, \frac{a^3}{n}\right)$$

Sea $g(x) = \left(\frac{x}{2}\right)^{1/3}$, $g'(x) = \frac{2^{2/3}}{6\,x^{2/3}}$

$$g(2a^3) = \left(\frac{2a^3}{2}\right)^{1/3} = a \qquad |g'(2a^3)|^2 = \left(\frac{2^{2/3}}{6\,(2a^3)^{2/3}}\right)^2 = \frac{1}{6a^2} = \frac{1}{36a^4}$$

$$\Rightarrow \sqrt{n}\left(\left(\frac{\overline{X^2}}{2}\right)^{1/3} - a\right) \xrightarrow{d} N\left(0, \frac{a^3}{36na^4}\right) \Rightarrow$$

$$\Rightarrow \boxed{\sqrt{n}(T_n - a) \xrightarrow{d} N\left(0, \frac{1}{36na}\right)}$$

5. a.1) p = jugadores de Fortnite entre 15 y 20 años

Mínimo n para estimar p con un error $\leq 2\%$ (conf. 95%)

Opción conservadora: $\bar{x} = 1/2$ porque $\sqrt{\bar{x}(1-\bar{x})}$ alcanza un máximo ahí.

Entonces: $Z_{2'5\%}\sqrt{\frac{1/2(1-1/2)}{n}} \leq 0'02 \Rightarrow \sqrt{n} \geq \frac{Z_{2'5\%}\sqrt{1/4}}{0'02} \Rightarrow \boxed{n \geq 2401}$

a.2) $\bar{x} = 0'54$ $\quad n = 3200 \quad$ Confianza = 97'5%

$I = \left(\bar{x} \pm Z_{\alpha/2}\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}\right) \qquad Z_{\alpha/2} = Z_{1'25\%} = 2'241$

Entonces: $I = \left(0'54 \pm 2'241 \cdot \sqrt{\frac{0'54(1-0'54)}{3200}}\right) = \boxed{(0'54 \pm 0'0197)}$

b) Letni-8086 $\begin{cases} n_1 = 100 \\ \bar{x} = 37 \\ s_1^2 = 41 \end{cases}$ $\qquad$ DMA-300 $\begin{cases} n_2 = 130 \\ \bar{y} = 29 \\ \overline{y^2} = 900 \end{cases}$

¿Evidencia estadística de que la variabilidad(LETNI) < variabilidad(DMA)?

Intentaremos rechazar la desigualdad contraria:
variabilidad(DMA) > variabilidad(LETNI)

Llamemos σ_1 = variabilidad(LETNI)
σ_2 = variabilidad(DMA)

$H_0 \equiv \sigma_2 \geq \sigma_1$ $\qquad S_1^2$ = dato = 41

Región de rechazo: $\frac{S_1^2}{S_2^2} > F_{\{n_1-1;\, n_2-1;\, \alpha\}}$ $\qquad S_2^2 = \overline{y^2} - \bar{y}^2 = 900 - 841 = 59$

$\Rightarrow \frac{41}{59} > F_{\{99;\,129;\,5\%\}} = 1'361 \Rightarrow 0'695 \not> 1'361$

$\Rightarrow$ No podemos rechazar $\Rightarrow$ no existe evidencia estadística de que la variabilidad(LETNI) < variabilidad(DMA)

6. $X \sim Geo(p)$ $p \in (0,1)$ Se quiere contrastar $H_0: p > 0'2$

$n=10$ Si sobreviven todas ≥ 3 días $\Rightarrow$ rechazamos H_0

$$\mathbb{P}_p(\text{"rechazar"}) = \mathbb{P}_p(\min(X_1,\dots,X_{10}) \geq 3) = 1 - \underbrace{\mathbb{P}_p(\min(X_1,\dots,X_{10}) \leq 3)}_{F_{m_n}(3)}$$

$$= P_p(X \geq 3)^{10} = (1 - F_X(\leq 2))^{10}$$

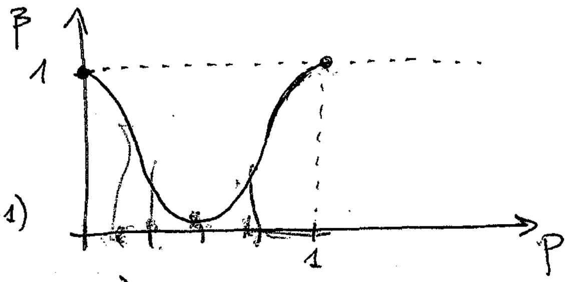
$$F_{m_n}(t) = 1-(1-F_X(t))^n = 1 - (1 - p(1-p)^{t-1})^n$$

Con $n=10$ y $t=3$: $F_{m_n}(3) = 1 - (1-p(1-p)^2)^{10}$

Entonces: $\mathbb{P}_p(\text{rechazar}) = \beta(p) = \cancel{1} - (\cancel{1} - (1-p(1-p)^2)^{10}) \Rightarrow$

$$\Rightarrow \boxed{\beta(p) = (1-p(1-p)^2)^{10}}$$

$\beta(0) = 1$
$\beta(1) = 1$
$\beta(a) < 1$ con $a \in (0,1)$

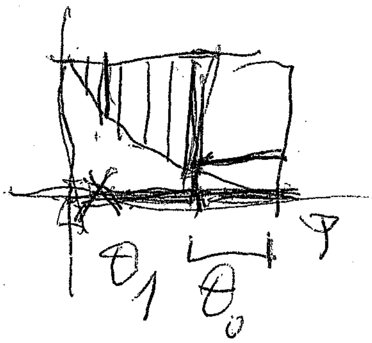


Por lo tanto: (recordar que $\Theta_0 = (0'2, 1)$)

$$\text{Significación} = \sup_{p \in \Theta_0} \beta(p) = \beta(1)$$

$$P_p(X \geq 3) = 1 - P_p(X < 3) = 1 - P_p(X \leq 2)$$
$$= 1 - P_p(X=1) - P_p(X=2)$$
$$= 1 - p - (1-p)p$$
$$= (1-p)^2$$

$$P_p(\text{"rechazar"}) = (1-p)^{20}$$



EJERCICIO 1

a)

valores	0	1
prob	$\frac{1}{2}-\theta$	$\frac{1}{2}+\theta$

$\theta \in (0, 1/2)$

$\bar{x} \in (1/2, 1)$

Momentos

$$\mathbb{E}_\theta(X) = \bar{x} \Rightarrow \bar{x} = 0\cdot\left(\tfrac{1}{2}-\theta\right) + 1\cdot\left(\tfrac{1}{2}+\theta\right) = \tfrac{1}{2}+\theta \Rightarrow \boxed{M_\theta = \bar{X} - \tfrac{1}{2}}$$

ESTIMADOR POR MOMENTOS

Máx Vero

$$\text{VERO}(\theta; x_1,\dots,x_n) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\tfrac{1}{2}+\theta\right)^{n\bar{x}} \cdot \left(\tfrac{1}{2}-\theta\right)^{n(1-\bar{x})}$$

$$\log \text{VERO}(\theta; x_1,\dots,x_n) = n\bar{x}\log\left(\tfrac{1}{2}+\theta\right) + n(1-\bar{x})\log\left(\tfrac{1}{2}-\theta\right)$$

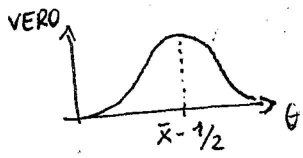
$$\frac{d}{d\theta}\log\text{VERO}(\theta; x_1,\dots,x_n) = \frac{n\bar{x}}{\frac{1}{2}+\theta} - \frac{n(1-\bar{x})}{\frac{1}{2}-\theta}$$

Igualamos a cero y despejamos θ: $n\bar{x}\left(\tfrac{1}{2}-\theta\right) - n(1-\bar{x})\left(\tfrac{1}{2}+\theta\right) = 0 \iff$

$$\iff \frac{n\bar{x}}{2} - n\bar{x}\theta - n\left(\tfrac{1}{2}+\theta-\tfrac{\bar{x}}{2}-\bar{x}\theta\right) = 0 \iff \frac{n\bar{x}}{2} - \cancel{n\bar{x}\theta} - \tfrac{1}{2}n - n\theta + \frac{\bar{x}n}{2} + \cancel{n\bar{x}\theta} = 0$$

$$\iff n\bar{x} - \frac{n}{2} - n\theta = 0 \iff n\theta = n\left(\bar{x}-\tfrac{1}{2}\right) \iff \theta = \bar{x} - \tfrac{1}{2}$$

$$\Rightarrow \boxed{EMV_\theta = \bar{X} - 1/2}$$ ESTIMADOR MÁXIMA VEROSIMILITUD



b)

valores	0	θ
prob	$\frac{1}{2}-\theta$	$\frac{1}{2}+\theta$

$\theta \in (0, 1/2)$

$\bar{x} \in (0, 1/2)$

Momentos

$$\mathbb{E}_\theta(X) = \bar{x} \Rightarrow \bar{x} = 0\cdot\left(\tfrac{1}{2}-\theta\right) + \theta\left(\tfrac{1}{2}+\theta\right) \Rightarrow \bar{x} = \tfrac{1}{2}\theta + \theta^2 \Rightarrow \theta^2 + \tfrac{1}{2}\theta - \bar{x} = 0 \Rightarrow$$

$$\Rightarrow \theta = \frac{-1/2 \pm \sqrt{1/4 - 4\cdot(-\bar{x})}}{2} = \frac{-1/2 \pm \sqrt{\frac{1+16\bar{x}}{4}}}{2} \Rightarrow \theta \begin{cases} = -\frac{1}{4} + \frac{1}{4}\sqrt{1+16\bar{x}} \in (0, 1/2) & \bar{x}\in(0,1/2) \\ = -\frac{1}{4} - \frac{1}{4}\sqrt{1+16\bar{x}} \in \left(-\frac{1}{2}, -1\right) & \text{no válido } \theta\in(0,1/2) \end{cases}$$

$$\Rightarrow \boxed{M_\theta = \frac{1}{4}\left(\sqrt{1+16\bar{X}} - 1\right)}$$ ESTIMADOR POR MOMENTOS

$n\theta \to$ todo θ's en la muestra
$n\bar{x} \to$ suma de θ's en la muestra
$/\theta \to$ para obtener el número de ceros
} número de ceros muestra

MáxVero

$$\text{VERO}(\theta; x_1,\dots,x_n) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\tfrac{1}{2}-\theta\right)^{\frac{n\theta - n\bar{x}}{\theta}} \cdot \left(\tfrac{1}{2}+\theta\right)^{n - \frac{n\theta - n\bar{x}}{\theta}}$$

$n - \frac{n\theta - n\bar{x}}{\theta} \to = \frac{n\bar{x}}{\theta}$ después de pensar

$$\log\text{VERO}(\theta; x_1,\dots,x_n) = \frac{n\theta - n\bar{x}}{\theta}\cdot\log\left(\tfrac{1}{2}-\theta\right) + \frac{n\bar{x}}{\theta}\log\left(\tfrac{1}{2}+\theta\right)$$

$$\frac{d}{d\theta}\log\text{VERO}(\theta; x_1,\dots,x_n) = \frac{n\bar{x} - n\theta}{\theta(1/2-\theta)} + \frac{n\bar{x}}{\theta(1/2+\theta)}$$

Igualamos a cero y despejamos θ: $\frac{n\bar{x}-n\theta}{\theta(1/2-\theta)} + \frac{n\bar{x}}{\theta(1/2+\theta)} = 0 \iff$

$\iff (n\bar{x}-n\theta)(\frac{1}{2}+\theta) + n\bar{x}(\frac{1}{2}-\theta) = 0 \iff \frac{n\bar{x}}{2} + \cancel{n\bar{x}\theta} - \frac{n\theta}{2} - n\theta^2 + \frac{n\bar{x}}{2} - \cancel{n\bar{x}\theta} = 0$

$\iff n\bar{x} - \frac{n\theta}{2} - n\theta^2 = 0 \iff n\theta^2 + \frac{n\theta}{2} - n\bar{x} = 0 \Rightarrow \theta = \frac{-n/2 \pm \sqrt{\frac{n^2}{4}+4n^2\bar{x}}}{2n}$

$\Rightarrow \theta = \frac{-1/2 \pm \sqrt{\frac{1}{4}+4\bar{x}}}{2} \Rightarrow \theta$

$\to = -\frac{1}{4} + \frac{1}{4}\sqrt{1+16\bar{x}} \in (0, 1/2) \leftarrow \bar{x} \in (0, 1/2)$

$\to = -\frac{1}{4} - \frac{1}{4}\sqrt{1+16\bar{x}} \in (-\frac{1}{2}, -1)$ no válido

$\Rightarrow \boxed{EMV_\theta = \frac{1}{4}\left(\sqrt{1+16\bar{x}} - 1\right)}$ ESTIMADOR POR MÁXIMA VEROSIMILITUD.

EJERCICIO 2

a) $\alpha = 5\% \Rightarrow Z_{\alpha/2} = 1'96$

Nos colocamos en el peor caso posible: $\bar{x}_1 = \bar{x}_2 = 1/2$

$Z_{\alpha/2}\sqrt{\frac{1/2(1-1/2)}{n} + \frac{1/2(1-1/2)}{n}} \leq 1'96\% \Rightarrow 1'96\sqrt{\frac{1/2}{n}} \leq 1'96\%$

$\Rightarrow \frac{0'5}{n} \leq \frac{1}{10000} \Rightarrow \boxed{n \geq 5000} \longrightarrow n = 5000$

b) $n=100$ $\qquad H_0: p < 1/4 \qquad X \sim Ber(p)$

b.1) Región de rechazo: $\bar{x} > \frac{1}{4} + 1'645\frac{\sqrt{3}}{4\sqrt{100}} = 0'321$

Rechazaremos a partir de 33 unos (significación 5%).

b.2) 65 ceros $\to$ 35 unos $\to \bar{x} = 0'35$

$\Rightarrow 0'35 - 0'25 = Z_\alpha \frac{\sqrt{3}}{40} \Rightarrow Z_\alpha = 2'31$

$\Rightarrow \boxed{\alpha = \text{p-valor} \simeq 1\%}$

EJERCICIO 3

a) $T(x_1,\dots,x_n) = \ln\left(\frac{1}{6n}\sum_{i=1}^{n} x_i^2\right) = \ln\left(\frac{1}{6}\overline{X^2}\right)$

TCL: $\sqrt{n}\left(\overline{X^2} - \mathbb{E}(X^2)\right) \xrightarrow{d} N(0, V(X^2))$

$V(X^2) = \mathbb{E}(X^4) - \mathbb{E}(X^2)^2 = 150e^{2\theta} - 36e^{2\theta} = 144e^{2\theta}$

$\Rightarrow$ TCL: $\sqrt{n}\left(\overline{X^2} - 6e^{\theta}\right) \xrightarrow{d} (0, 144e^{2\theta})$

$g(x) = \ln\left(\frac{1}{6}x\right) \qquad g'(x) = \frac{1}{x} \quad . \quad |g'(6e^{\theta})|^2 = \frac{1}{36e^{2\theta}}$

Método delta

$$\boxed{\sqrt{n}\left(\ln\left(\tfrac{1}{6}\overline{x^2}\right) - e^{\theta}\right) \xrightarrow{d} N(0,4)}$$

b) $\hat{t}_n = \ln\left(\frac{1}{6}\cdot 8'2\right) = 0'31 \qquad \alpha = 5\% \longrightarrow z_{\alpha/2} = 1'96$

Intervalo: $\left(0'31 - \frac{\sqrt{4}}{\sqrt{100}}\cdot 1'96\ ,\ 0'31 + \frac{\sqrt{4}}{\sqrt{100}}\, 1'96\right)$ ✓

$\Rightarrow$ $\boxed{\text{Intervalo: } (-0'08\ ,\ 0'7)}$

EJERCICIO 4

a) $H_0: N \leq 6 \qquad N \geq 1 \qquad X \sim \chi^2_N \qquad \sum_{i=1}^{10} \chi^2_N = \chi^2_{10N}$

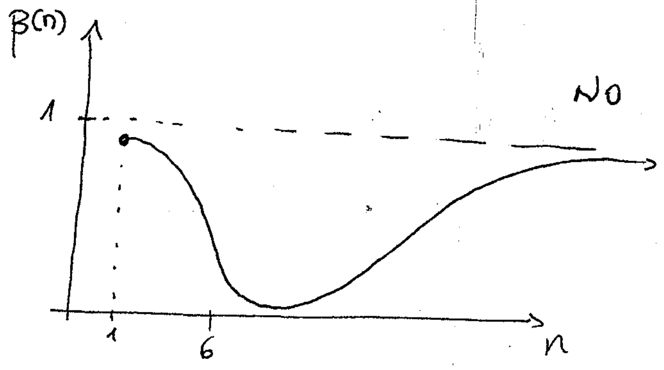
Función de potencia $= \beta(N) = \mathbb{P}(\text{rechazar}) = \mathbb{P}(\chi^2_{10n} \geq 80) =$

$= 1 - F_{\chi^2_{10n}}(80) \Rightarrow \boxed{\beta(n) = 1 - F_{\chi^2_{10n}}(80)}$

$\Theta = \{1, 2, 3, \dots\}$

b) $\Theta = [1, \infty)$

$\Theta_0 = [1, 6]$



NO

<u>significación</u> $\equiv \sup_{n \in \Theta_0} \beta(n)$

$$\boxed{\sup_{n \in \Theta_0} \beta(n) = \beta(1)}$$