$\boxed{7.}$ DESIGUALDADES RECURRENTES

$$\boxed{\begin{array}{ll} r \geq 1 & \dfrac{UR - P}{R - 1} \\[2ex] r < 1 & \dfrac{P - UR}{1 - R} \end{array}}$$

$\boxed{63.}$

4) $\displaystyle\sum_{0}^{K-1} 2^{j+2}$ $\qquad N = 8^K$

$$\sum_{0}^{K-1} 2^2 \cdot 2^j = 4 \sum_{0}^{K-1} 2^j = 4 \cdot \frac{2^{K-1} \cdot 2 - 1}{2 - 1} = 4(2^K - 1)$$

$$N = 8^K = (2^3)^K = (2^K)^3 \Rightarrow 2^K = N^{1/3}$$

Entonces: $\displaystyle\sum_{0}^{K-1} 2^{j+2} = 4(N^{1/3} - 1)$

5) $\displaystyle\sum_{0}^{K-1} \frac{3^{j-2}}{4^{j+2}}$ $\qquad N = 12^K$

$$\sum_{0}^{K-1} \frac{1}{3^2 \cdot 4^2} \cdot \left(\frac{3}{4}\right)^j = \frac{1}{9.16} \sum_{0}^{K-1} \left(\frac{3}{4}\right)^j = \frac{1}{9.16}\left(\frac{1 - (3/4)^K}{1/4}\right) =$$

$$= \frac{1}{9.4}\left(1 - \left(\frac{3}{4}\right)^K\right)$$

Como $N = 12^K$ queremos: $\left(\underbrace{\left(\frac{3}{4}\right)^{algo}}_{12}\right)^K$ $\qquad algo = \log_{3/4} 12$

65. $T(1) = 0$

① Caso particular $N = 2^K$

a) $T(N) \leq 2T(\lfloor N/2 \rfloor) + N^3$

Con el caso particular $N = 2^K$:

$$T(N) \leq 2T\left(\frac{N}{2}\right) + N^3 \Rightarrow T(N) \leq N^3 + 2\left(\left(\frac{N}{2}\right)^3 + 2T\left(\frac{N}{2^2}\right)\right) =$$

$$= N^3 + \frac{2}{2^3}N^3 + 2^2 T\left(\frac{N}{2^2}\right) \leq$$

$$\leq N^3\left(1 + \frac{2}{2^3}\right) + 2^2\left(\left(\frac{N}{2^2}\right)^3 + 2T\left(\frac{N}{2^3}\right)\right) =$$

$$= N^3\left(1 + \frac{2}{2^3} + \frac{2^2}{(2^2)^3}\right) + 2^3 T\left(\frac{N}{2^3}\right) =$$

$$= N^3\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2\right) + 2^3 T\left(\frac{N}{2^3}\right) \leq$$

$$\leq N^3 \sum_{0}^{K-1} \left(\frac{1}{4}\right)^j + 2^K T\Big(\underbrace{\frac{N}{2^K}}_{=1}\Big) \to 0$$

$$\Rightarrow T(N) \leq N^3 \cdot \sum_{0}^{K-1} \left(\frac{1}{4}\right)^j = N^3\left(\frac{1 - \frac{1}{4}^K}{1 - \frac{1}{4}}\right) = \frac{4}{3}N^3\left(1 - \frac{1}{4^K}\right)$$

Sabemos que $2^K = N$, entonces $4^K = (2^2)^K = (2^K)^2 = N^2$

$$\Rightarrow \boxed{T(N) \leq \frac{4}{3}(N^3 - N)}$$

Ahora: CASO GENERAL POR INDUCCIÓN: H.I. $T(N') \leq \frac{4}{3}(N'^3 - N')\ \forall N' < N$

Metodología:

$$T(N) \overset{DR}{\leq} N^3 + 2T(\overbrace{\lfloor \tfrac{N}{2} \rfloor}^{<N}) \leq N^3 + 2\left(\frac{4}{3}\left(\lfloor \tfrac{N}{2} \rfloor^3 - \lfloor \tfrac{N}{2} \rfloor\right)\right) \overset{\text{función creciente}}{\leq}$$

$$\leq N^3 + \frac{8}{3}\left(\frac{N^3}{2^3} - \frac{N}{2}\right) = N^3 + \frac{N^3}{3} - \frac{4}{3}N = \left(1 + \frac{1}{3}\right)N^3 - \frac{4}{3}N =$$

$$= \frac{4}{3}N^3 - \frac{4}{3}N = \frac{4}{3}(N^3 - N) \quad \checkmark$$

b) $T(N) \leq 1 + 2T(N-1) \leq 1 + 2\left(1 + 2T(N-2)\right) =$

$= 1 + 2 + 2^2 T(N-2) \leq 1 + 2 + 2^2\left(1 + 2T(N-3)\right) =$

$= 1 + 2 + 2^2 + 2^3 T(N-3) \leq \cdots \leq 2^0 + 2^1 + \cdots + 2^{N-1} T(N-(N-1)) =$

$= \sum_{j=0}^{N-2} 2^j + 2^{N-1} \underbrace{T(1)}_{0} = \dfrac{2^{N-2} \cdot 2 - 1}{2-1} = 2^{N-1} - 1$

$\boxed{67.}$ $T(N) \leq N + 2T\left(\lfloor \frac{N}{2} \rfloor\right)$

$T(1) = 0$

① $N = 2^K$

$T(N) \leq N + 2\left(\frac{N}{2} + 2T\left(\frac{N}{2^2}\right)\right) = 2N + 2^2 T\left(\frac{N}{2^2}\right) \leq$

$\leq 2N + 2^2\left(\frac{N}{2^2} + 2T\left(\frac{N}{2^3}\right)\right) = 3N + 2^3 T\left(\frac{N}{2^3}\right) \leq \cdots \leq$

$\leq KN + 2^K \cdot \underbrace{T\left(\frac{N}{2^K}\right)}_{T(1)=0} = K.N = N \log_2 N$

② Inducción. caso general $T(N) \leq N \log_2 N$

caso base $N=1$ $0 = T(1) \leq 1 \log 1 = 0$ ✓

caso general: $T(N) \overset{\text{desi. recurrente}}{\leq} N + 2T\left(\lfloor \frac{N}{2} \rfloor\right) \overset{\text{H.I.}}{\leq} N + 2 \lfloor \frac{N}{2} \rfloor \log \lfloor \frac{N}{2} \rfloor \overset{\text{creciente}}{\leq}$

$\lfloor \frac{N}{2} \rfloor < N$

$\leq N + 2 \cdot \frac{N}{2} \log \frac{N}{2} = \cancel{N} + N \log N - \cancel{N} \underbrace{\log 2}_{=1} = N \log N$ ✓

$\boxed{66.}$ $T(N) \leq \sqrt{N} + 4T\left(\left\lfloor \frac{N}{4} \right\rfloor\right)$

$T(1) = 0$

① $N = 4^K$

$$T(N) \leq \sqrt{N} + 4\left(\sqrt{\frac{N}{4}} + 4T\left(\frac{N}{4^2}\right)\right) = \sqrt{N} + 2\sqrt{N} + 4^2 \cdot T\left(\frac{N}{4^2}\right) \leq$$

$$\leq \sqrt{N} + 2\sqrt{N} + 4^2\left(\sqrt{\frac{N}{4^2}} + 4T\left(\frac{N}{4^3}\right)\right) = \sqrt{N} + 2\sqrt{N} + 4\sqrt{N} + 4^3 T\left(\frac{N}{4^3}\right)$$

$$\leq \cdots \leq \sqrt{N}\sum_{j=0}^{j=K-1} 2^j + \underbrace{\cancel{4^K T\left(\frac{N}{4^K}\right)}}_{0} = \sqrt{N}\,\frac{2^K - 1}{2-1} = \sqrt{N}\left(2^K - 1\right)$$

Como $N = 4^K = (2^2)^K = (2^K)^2 \longrightarrow 2^K = \sqrt{N}$

$$\sqrt{N}\left(2^K - 1\right) = \sqrt{N}\left(\sqrt{N} - 1\right) = N - \sqrt{N}$$

② Inducción. $T(N) \leq N - \sqrt{N}$

C.B: $N = 1$ $\quad 0 = T(1) \leq 1 - \sqrt{1} = 0$ ✓

C.G $\quad T(N) \overset{\text{des. recurrente}}{\leq} \sqrt{N} + 4T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) \leq \sqrt{N} + 4\left(\left\lfloor \frac{N}{4} \right\rfloor - \sqrt{\left\lfloor \frac{N}{4} \right\rfloor}\right) \overset{\text{creciente}}{\leq}$

$$\leq \sqrt{N} + 4\left(\frac{N}{4} - \frac{\sqrt{N}}{2}\right) = \sqrt{N} + N - 2\sqrt{N} = N - \sqrt{N} \quad ✓$$

$\boxed{64.}$ $T(N) \leq \log N + T\left(\lfloor \frac{N}{2} \rfloor\right)$ $\qquad T(1)=0$

① CASO PARTICULAR: $N = 2^K$

$$T(N) \leq \log N + \left(\log \frac{N}{2} + T\left(\frac{N}{2^2}\right)\right) = 2\log N - 1 + T\left(\frac{N}{2^2}\right) \leq$$

$$\leq 2\log N - 1 + \left(\log \frac{N}{2^2} + T\left(\frac{N}{2^3}\right)\right) =$$

$$= 3\log N - 1 - 2 + T\left(\frac{N}{2^3}\right) \leq 4\log N - 1 - 2 - 3 + T\left(\frac{N}{2^4}\right) \leq$$

$$\leq \dots \leq K\log N - \sum_{j=1}^{K-1} j + \underbrace{T\big(\underbrace{N/2^K}_{1}\big)}_{0} = K.\log N - \frac{K(K-1)}{2} \leq$$

$K = \log N$

$$\leq \log^2 N$$

② Caso general INDUCCIÓN

H.I. $T(N') < (\log N')^2 \quad \forall N' < N$

$0 = T(1) \leq \log 1^2 = 0$ ✓

caso general: $T(N) \leq \log N + T\left(\lfloor \frac{N}{2} \rfloor\right) \overset{H.I.}{\leq} \log N + \left(\log \lfloor \frac{N}{2} \rfloor\right)^2 \leq$ f crec.

$\lfloor \frac{N}{2} \rfloor < N$

$$\leq \log N + \log^2\left(\frac{N}{2}\right) = \log N + \log^2(N-1) = \log N + \log^2 N - 2\log N + 1$$

$$= \log^2 N \underbrace{- \log N + 1}_{\leq 0} \leq \log^2 N$$

70.

C.B $\quad m(1) = 1$

C.G $\quad m(N) = \underbrace{1}_{\text{tr. explícito}} + \underbrace{2m(N-1)}_{\text{tr. implícito}}$

$$m(n) = 1 + 2m(N-1) = 1 + 2\big(1 + 2m(N-2)\big) = 1 + 2 + 2^2 m(N-2) =$$

$$= 1 + 2 + 2^2\big(1 + 2m(N-3)\big) = 1 + 2 + 2^2 + 2^3 m(N-3) = \cdots =$$

$$= \sum_{j=0}^{N-2} 2^j + 2^{N-1} m(N-N+1) = \sum_{j=0}^{N-2} 2^j + 2^{N-1} \underbrace{m(1)}_{\overset{\shortparallel}{1}} =$$

$$= 2^N - 1$$

76.

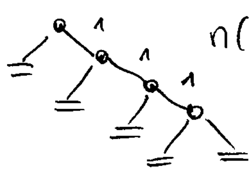
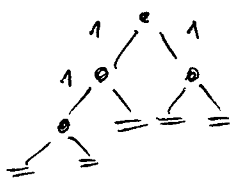
$n(T) = \#$ sumas

$n(\emptyset) = 0$
$n(\bullet) = 0$ $\Big\}$ $n(0) = n(1) = 0$

$n\left(\overset{\bullet}{T_i \;\; T_d}\right) = 1 + n(T_i) + n(T_d)$

$$\underset{0 \le K \le N-1}{n(N) = 1 + n(K) + n(N-K-1)} \le 1 + \max_{0 \le K \le N-1} \{n(K) + n(N-K-1)\}$$

iii) Pensamos tanteando el peor caso

→ para árboles con 4 nodos  $n(4) \le 3$ 

Inducción:

C.B. pensando

C.G. si T' tiene $N' < N$ nodos $\quad n(N') \le N' - 1$

$$n(N) \le 1 + \max_{1 \le K \le N-1} \{ \underset{\hat{N}}{n(K)} + \underset{\hat{N}}{n(N-K-1)} \} \overset{H.I.}{\le} 1 + \max\{\cancel{K}-1 + N - \cancel{K} - 2\} \le$$

$$\le 1 + N - 3 = N-2 \le N-1$$

$$n(N) \le 1 + \max\{N-2, \max_{1 \le K \le N-1}\{n(K) + n(N-K-1)\}\} \le$$

mejoras (nos hemos pasado de listos) $\quad \le 1 + \max\{N-2, N-3\} = N-$

68. $T(N) \leq T\left(\lfloor \frac{N}{2} \rfloor\right) + N\log(N)$ $\qquad T(1)=0$

① CASO PARTICULAR: $N=2^K$

$$T(N) \leq N\log N + \left(\frac{N}{2}\log\frac{N}{2} + T\left(N/2^2\right)\right) = N\log N + \frac{N}{2}\log N - \frac{N}{2}\cdot 1 + T\left(\frac{N}{2^2}\right)$$

$$\leq N\log N + \frac{N}{2}\log N - \frac{N}{2}\cdot 1 + \frac{N}{2^2}\log\frac{N}{2^2} + T\left(\frac{N}{2^3}\right) =$$

$$= N\log N\left(1 + \frac{1}{2} + \frac{1}{2^2}\right) - \frac{N}{2}\cdot 1 - \frac{N}{2^2}\cdot 2 + T\left(N/2^3\right) \leq \cdots \leq$$

$$\leq N\log N\left(\sum_{j=0}^{K-1}\frac{1}{2^j}\right) - N\sum_{1}^{K-1}\frac{j}{2^j} + \underbrace{T\Big(\underbrace{\frac{N}{2^K}}_{1}\Big)}_{0} \leq$$

$$\leq N\log N\,\frac{1-\frac{1}{2^K}}{1-\frac{1}{2}} = 2N.\log N\left(1-\frac{1}{2^K}\right) = 2N\log N\left(1-\frac{1}{N}\right) =$$

$$= 2N\log N - 2\log N \leq 2N\log N$$

② CASO GENERAL: INDUCCIÓN

C.B. $0 = T(1) \leq 2.1.\log 1 = 0$ ✓

C.G. $T(N) \leq N\log N + T\left(\lfloor \frac{N}{2} \rfloor\right) \overset{H.I}{\leq} N.\log N + 2\lfloor \frac{N}{2} \rfloor \log\lfloor \frac{N}{2} \rfloor \overset{f\ creciente}{\leq}$

$$\leq N\log N + N\log\frac{N}{2} = N\log N + N\log N - N = 2N\log N - N \leq 2N\log N$$

8. Análisis de algoritmos recurrentes

71.

$$n(T) = n(T_i) + n(T_d) + 1 \qquad n(1) = 0$$

$$n(N) = 1 + n\left(\left\lceil \tfrac{N}{2} \right\rceil\right) + n\left(\left\lfloor \tfrac{N}{2} \right\rfloor\right)$$

CASO PARTICULAR $\quad n(N) \leq N-1 \longrightarrow$ ¿$n(N) = N-1$?

CASO GENERAL

C.B $\quad 0 = n(1) = N-1 = 0$ ✓

C.G. $\quad n(N') = N'-1 \quad \forall N < N$

$$n(N) \overset{\text{I.R.}}{=} 1 + n\left(\left\lceil \tfrac{N}{2} \right\rceil\right) + n\left(\left\lfloor \tfrac{N}{2} \right\rfloor\right) \overset{\text{H.I.}}{=} 1 + \left\lceil \tfrac{N}{2} \right\rceil - 1 + \left\lfloor \tfrac{N}{2} \right\rfloor - 1 =$$

$$= \underbrace{\left\lceil \tfrac{N}{2} \right\rceil + \left\lfloor \tfrac{N}{2} \right\rfloor}_{\overset{?}{=} N} - 1 = N - 1$$

? → para N par sí

→ para N impar sí

77.

árbol no completo

$$n(\emptyset) = 0$$

$$n(0) = 0$$

$$n_{PO}(T) = 2 + n(T_i) + n(T_d)$$

$$n_{PO}(N) = 2 + n(K) + n(N-1-K) \qquad 0 \leq K \leq N-1$$

$$n_{PO}(N) \leq 2 + \max_{1 \leq j \leq N-1} \{ \underbrace{n(j)}_{\wedge\!\backslash\; 2j} + n(N-1-j) \}$$

C.G. $\quad n(N') \leq 2N' \quad \forall N' < N$

$$n(N) \leq 2 + \max \{ \underset{\hat{N}}{n(j)} + n(\underset{\hat{N}}{N-1}-j) \} \overset{\text{H.I.}}{\leq} 2 + \max_{0 \leq j \leq N-1} \{ 2j + 2N - 2 - 2j \} = 2N \checkmark$$

$\boxed{77.}$ árbol completo

$$n_{PO}(T) = 2 + n(T_i) + n(T_d)$$

$$n(N) = 2 + 2n\left(\frac{N-1}{2}\right) \qquad \frac{N-1}{2} = \frac{2^h-2}{2} = 2^{h-1}-1$$

$N = 2^h - 1$

$$n(2^h-1) = 2 + 2n(2^{h-1}-1)$$

$$\phi(h) = 2 + 2\phi(h-1) = 2 + 2(2+2\phi(h-2)) = 2+2^2+2^2\phi(h-2) =$$

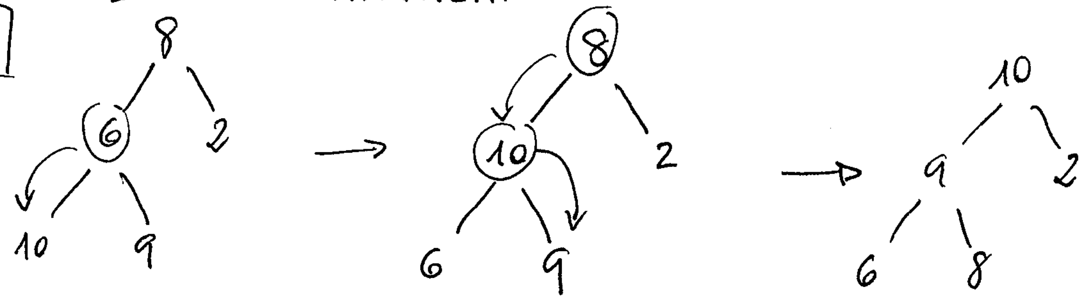
$$= 2(1+2^1) + 2^2\phi(h-2) = \dots = 2\sum_{j=0}^{h-2} 2j + 2^{h-1}\phi(h-(h-1)) =$$

$$= 2(2^{h-1}-1) + 2^h = 2^h - 2 + 2^h = 2(2^h-1) = 2N$$

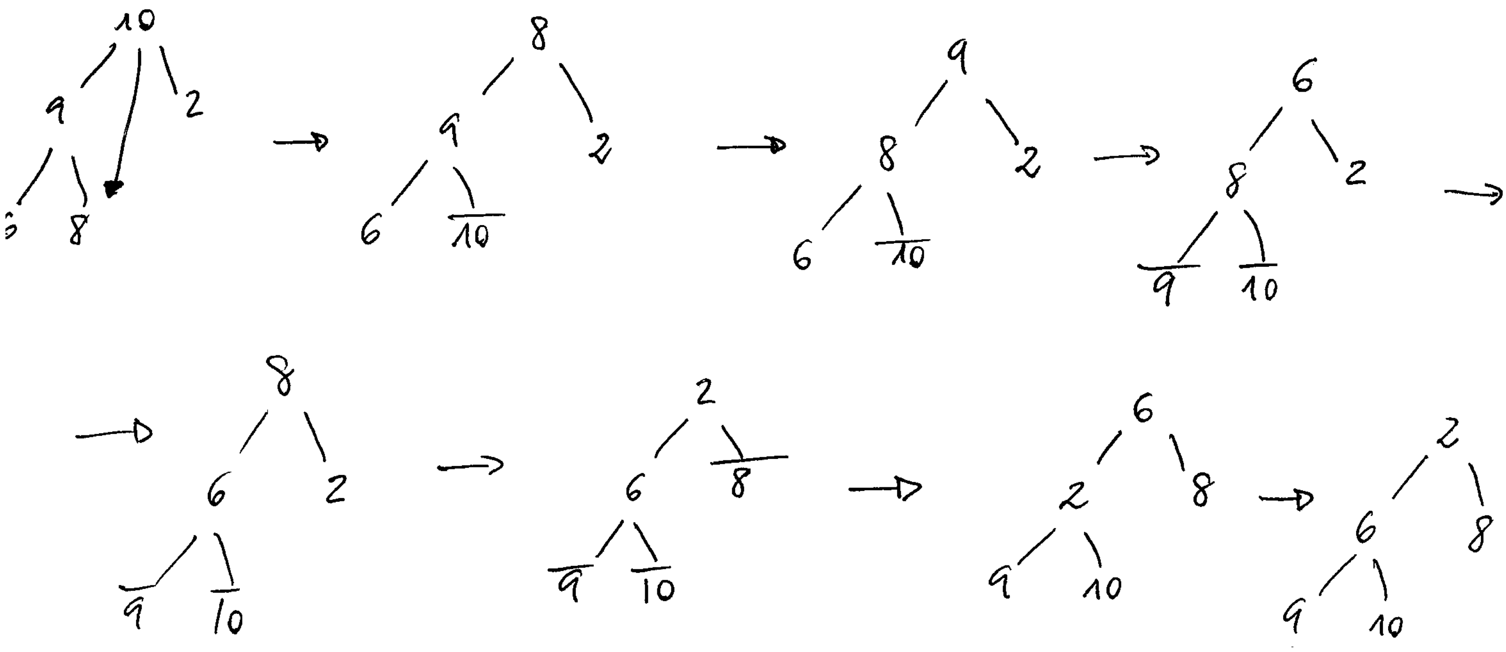
9. HEAPSORT

80.

1. CREAR MAXHEAP

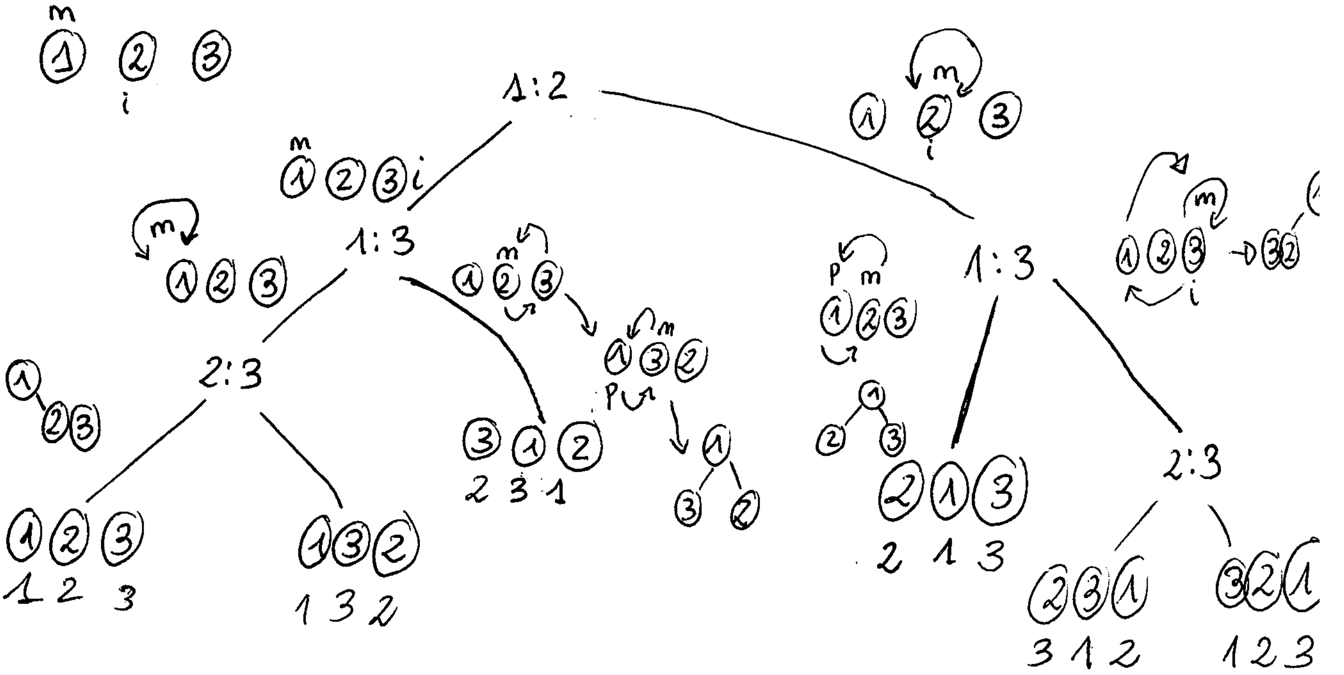


2. Ordenar MAXHEAP

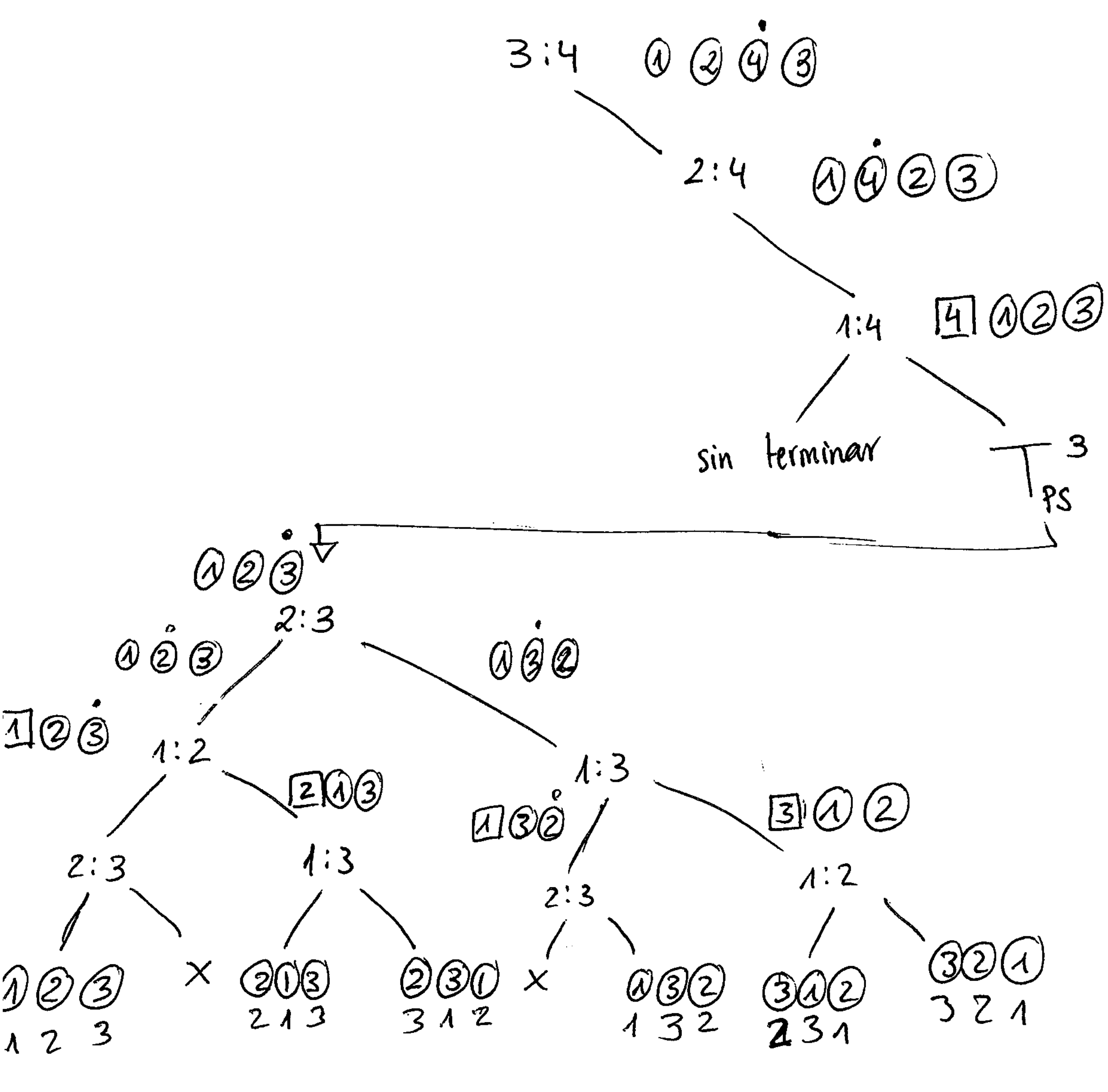


10. ÁRBOLES DE DECISIÓN

91.



99.



EJERCICIOS EXAMEN FINAL

Evolución $[4\ 3\ 5\ 6\ 2\ 7\ 1]$ partir

m	i	TABLA
2	2	$[4\ \overset{m}{3}\ 5\ 6\ 2\ 7\ 1]$
2	3	$[4\ \overset{m}{3}\ 5\ 6_i$
2	4	$[4\ 3\ \overset{m}{5}\ 6\ \underset{i}{2}$
3	5	$[4\ 3\ 2\ 6\ 5\ 7$
3	6	$[4\ 3\ 2\ \overset{m}{6}\ 5\ 7\ \underset{i}{1}$
4	7	$[4\ 3\ 2\ \overset{m}{1}\ 5\ 7\ 6$
		$[1\ 3\ 2\ 4\ 5\ 7\ 6]$

Desigualdad recurrente $T(N) \leq N^{1/3} + T\left(\left\lfloor \frac{N}{8} \right\rfloor\right)$; $T(1)=0$

(1) CASO PARTICULAR : $N = 8^k \longrightarrow N = (2^k)^3$

$$T(N) \leq N^{1/3} + \left(\left(\frac{N}{8}\right)^{1/3} + T\left(\frac{N}{8^2}\right)\right) = N^{1/3} + N^{1/3} + T\left(\frac{N}{8^2}\right) \leq$$

$$\leq N^{1/3} + N^{1/3}\cdot\frac{1}{2} + \left(\left(\frac{N}{8^2}\right)^{1/3} + T\left(\frac{N}{8^3}\right)\right) = N^{1/3} + N^{1/3}\cdot\frac{1}{2} + N^{1/3}\cdot\frac{1}{2^2} +$$

$$+ T\left(\frac{N}{8^3}\right) = N^{1/3}\left(1 + \frac{1}{2} + \frac{1}{2^2}\right) + T\left(\frac{N}{8^3}\right) \leq \cdots \leq$$

$$\leq N^{1/3}\sum_{j=0}^{k-1}\frac{1}{2^j} + T\Big(\underbrace{\frac{N}{8^k}}_{1}\Big) = N^{1/3}\left(\frac{P + UR}{1-R}\right) = N^{1/3}\left(\frac{1 - 1/2^k}{1-\frac{1}{2}}\right) =$$

$$\underline{\ \ 1\ \ }$$
$$0$$

$$= 2N^{1/3}\left(1 - \frac{1}{2^k}\right) = 2N^{1/3}\left(1 - \frac{1}{N^{1/3}}\right) = 2\left(N^{1/3} - 1\right)$$

② CASO GENERAL : Inducción

$$T(N') \le 2(N')^{1/3} \quad \forall N' < N$$

$$T(1) = 0 \le 2 \cdot 1 = 2$$

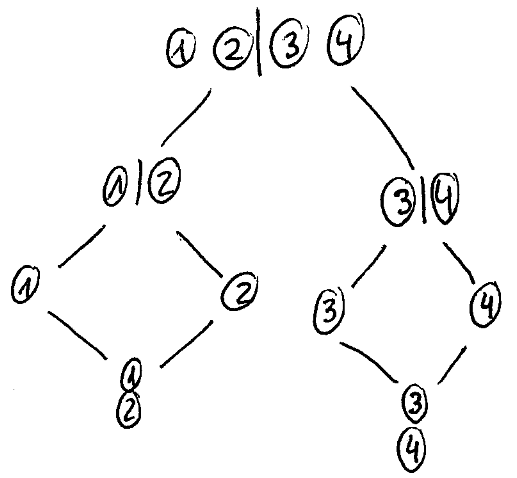
$$T(N) \overset{D.R}{\le} N^{1/3} + T\Big(\underset{\hat{N}}{\left\lfloor \tfrac{N}{8} \right\rfloor}\Big) \overset{H.I}{\le} N^{1/3} + 2\left\lfloor \tfrac{N}{8} \right\rfloor^{1/3} \overset{\text{creciente}}{\le}$$

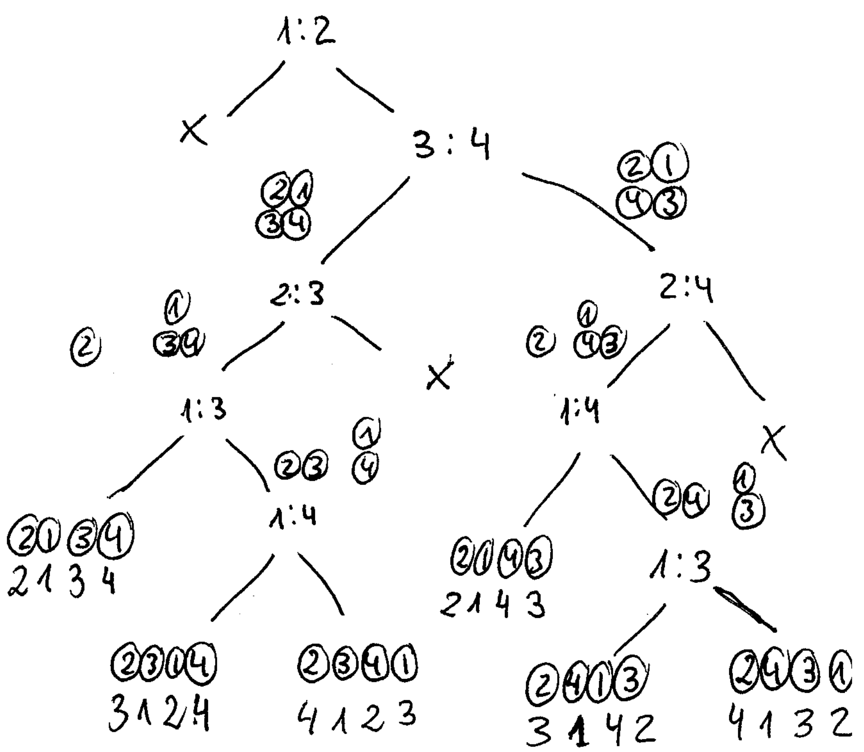
$$\le N^{1/3} + 2 \cdot \frac{N^{1/3}}{(2^3)^{1/3}} = 2N^{1/3}$$

Mergesort $N=4$

$\sigma(2) \equiv 1$

① ② ③ ④ (② = 1)





PR (x , N)

$N \equiv 0 \rightarrow 1$

$N \equiv 1 \rightarrow x$

else:

y = PR(x, N//2)

si $N \% 2 \equiv 0$

return y*y

si $N \% 2 \equiv 1$

return y*y*x

$$n(0) = n(1) = 0$$

$$n(N) \leq 2 + n\left(\left\lfloor \frac{N}{2} \right\rfloor\right)$$

- C.P. $N = 2^K$

$$n(N) \leq 2 + \left(2 + n\left(\frac{N}{2^2}\right)\right) = 2 + 2 + n\left(\frac{N}{2^2}\right) \leq$$

$$\leq 2 + 2 + 2 + n\left(\frac{N}{3^3}\right) = 2 \cdot 3 + n\left(\frac{N}{2^3}\right) \leq \cdots \leq$$

$$\leq 2K + n\left(\frac{N}{2^K}\right) = 2 \cdot \log N$$

- CASO GENERAL: Inducción

[...]