EJERCICIO 2

1) Sustituimos la parametrización en la ecuación:

$$a^2u^2\cos^2(v) + a^2u^2\operatorname{sen}^2(v) = b^2u^2\tan^2\theta$$

$$a^2u^2 = b^2u^2\tan^2\theta \implies \boxed{\tan\theta = \frac{a}{b}}$$

2) $$\frac{\partial r}{\partial u} = (a\cos(v),\ a\operatorname{sen}(v),\ b)$$

$$\frac{\partial r}{\partial v} = (-au\operatorname{sen}(v),\ au\cos(v),\ 0)$$

$$E = \left\langle \frac{\partial r}{\partial u}, \frac{\partial r}{\partial u} \right\rangle = a^2\cos^2(v) + a^2\operatorname{sen}^2(v) + b^2 = a^2 + b^2$$

$$F = \left\langle \frac{\partial r}{\partial u}, \frac{\partial r}{\partial v} \right\rangle = -a^2u\cancel{\cos(v)}\operatorname{sen}(v) + a^2u\cancel{\cos(v)}\operatorname{sen}(v) + 0 = 0$$

$$G = \left\langle \frac{\partial r}{\partial v}, \frac{\partial r}{\partial v} \right\rangle = a^2u^2\operatorname{sen}^2(v) + a^2u^2\cos^2(v) = a^2u^2$$

Primera forma fundamental: $\boxed{ds^2 = (a^2+b^2)du^2 + a^2u^2\,dv^2}$

Ahora ponemos $dv^2 = v'^2(u)\,du^2$

$$\Rightarrow ds^2 = (a^2+b^2)\,du^2 + a^2u^2\,v'^2(u)\,du^2$$

$$\Rightarrow ds^2 = \left((a^2+b^2) + a^2u^2v'^2(u)\right)du^2$$

Así el funcional es:

(longitud de una curva $(u, v(u))$)

$$L = \int \sqrt{a^2+b^2+a^2u^2v'^2(u)}\,du$$

3) Ahora la v es cíclica, por lo que podemos usar la ecuación de Euler-Lagrange (que nos resulta en conseguir una integral primera)

$$\frac{\partial L}{\partial v'} = \frac{1}{2\sqrt{a^2+b^2+a^2u^2v'^2(u)}} \cdot \left(2a^2u^2 v'(u)\right) = c$$

$$\Rightarrow \frac{\partial L}{\partial v'} = \frac{a^2u^2\, v'(u)}{\sqrt{a^2+b^2+a^2u^2v'^2(u)}} = c$$

4) $a^2u^2v'(u) = c\sqrt{a^2+b^2+a^2u^2v'^2(u)}$

$$a^4u^4v'^2(u) = c^2\left(a^2+b^2+a^2u^2v'^2(u)\right)$$

$$\cancel{v'(u) = \sqrt{\frac{c^2\left(a^2+b^2+a^2u^2v'^2(u)\right)}{a^4v^4}} = \frac{c}{a^2v^2}\sqrt{a^2+b^2+a^2u^2}}$$

$$a^4u^4v'^2(u) = a^2c^2 + b^2c^2 + c^2a^2u^2v'^2(u)$$

$$a^4u^4v'^2(u) - c^2a^2u^2v'^2(u) = a^2c^2 + b^2c^2$$

$$v'^2(u)\left[a^4u^4 - c^2a^2u^2\right] = a^2c^2 + b^2c^2$$

$$v'^2(u) = \frac{a^2c^2+b^2c^2}{a^4u^4 - c^2a^2u^2} \Rightarrow v'(u) = \sqrt{\frac{a^2c^2+b^2c^2}{a^4u^4-c^2a^2u^2}}$$

$$v = \int \sqrt{\frac{a^2c^2+b^2c^2}{a^4u^4-c^2a^2u^2}}\, du = \cdots$$

llegaríamos a las geodésicas del cono (paralelos)