HOJA 1

$\boxed{1.}$

$\boxed{\Longrightarrow}$ A def. pos. $\Rightarrow$ todos sus autovalores son reales

Como es simétrica, podemos diagonalizar A:

$A = O^T \Lambda O$ con O matriz ortogonal con los autovectores de A, Λ matriz diagonal con los $\underbrace{\text{autovalores}}_{\text{(positivos)}}$ de A.

$$A = O^T \Lambda O \underset{\substack{\uparrow \\ \Lambda \text{ diagonal} \\ A \text{ def. pos.}}}{=} O^T \Lambda^{1/2} \Lambda^{1/2} O = \underbrace{O^T \Lambda^{1/2} O}_{R^T} \underbrace{O^T \Lambda^{1/2} O}_{R} = R^T R$$

Como $\det(O^T \Lambda^{1/2} O) = \det(\Lambda^{1/2}) > 0 \Longrightarrow R$ invertible.

$\boxed{\Longleftarrow}$ Queremos ver que $\forall x \in \mathbb{R}^n \setminus \{\vec{0}\}$ $x^T A x > 0$ y que $x^T A x = 0 \Longleftrightarrow \vec{x} = \vec{0}$.

$$x^T A x = x^T R^T R x = (Rx)^T (Rx) = ⊛$$

Notemos que $Rx \in \mathcal{M}_{n\times 1}$ y que $(Rx)^T \in \mathcal{M}_{1\times n} \Longrightarrow ⊛ \in \mathcal{M}_{1\times 1} \equiv \mathbb{R}$

$$Rx = \begin{pmatrix} r_{11}x_1 + r_{12}x_2 + \cdots + r_{1n}x_n \\ \vdots \\ r_{n1}x_1 + r_{n2}x_2 + \cdots + r_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ | \\ b_n \end{pmatrix} \Longrightarrow (Rx)^T(Rx) = b_1^2 + \text{—} + b_n^2 > 0$$

y $b_1^2 + \text{—} + b_n^2 = 0 \Longleftrightarrow b_1 = \text{—} = b_n = 0 \Longleftrightarrow \sum_{j=1}^{n} r_{ij} x_j = 0 \quad \begin{matrix} \forall i = 1, \dots, n \\ \forall j = 1, \dots, n \end{matrix}$

y esto solo pasa cuando $x_1 = \text{—} = x_n = 0$, e.d., $\vec{x} = \vec{0}$, ya que R es invertible. ∎

$\boxed{2.}$

$\boxed{3.}$ $a^T V a = 0 \iff Va = 0$

$\boxed{\Leftarrow}$ Obvio, trivial

$\boxed{\Rightarrow}$ $0 \leq p(\lambda) = (a+\lambda b)^T V (a+\lambda b) =$

$= a^T V a + 2\lambda a^T V b + \lambda^2 b^T V b$

para $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^n$

???

El discriminante es: $4(a^T V b)^2 - 4(a^T V a)(b^T V b) \leq 0$

$\Rightarrow \begin{cases} a^T V b = 0 \\ b^T V a = 0 \end{cases} \forall b \in \mathbb{R}^n \Rightarrow Va = 0.$ ∎

???

$\boxed{5.}$ $\mathbb{Y} = (Y_1, Y_2, Y_3)^T \sim N(\vec{m}, V)$ con $\vec{m} = \vec{0}$

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

a) ¿ distrib. $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} Y_1 + Y_3 \\ Y_2 + Y_3 \end{pmatrix}$?

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{\substack{\shortparallel \\ A}} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \qquad A V A^T = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$$

$$\Rightarrow \mathbb{X} \sim N(A\vec{m}, AVA^T) = N\left(\vec{0}, \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}\right)$$

b) $\alpha Y_1 + \beta Y_2 + \gamma Y_3 = Z$

Como $\mathbb{Y}, \mathbb{X}$ son normales, la independencia $\Leftrightarrow$ covarianza nula.

Buscamos $\operatorname{cov}(Z, X_1) = 0$: linealidad covarianza

$$\operatorname{cov}(\alpha Y_1 + \beta Y_2 + \gamma Y_3, Y_1 + Y_3) = \alpha \underbrace{\operatorname{cov}(Y_1, Y_1)}_{=1} + \alpha \underbrace{\operatorname{cov}(Y_1, Y_3)}_{=0} +$$

$$+ \beta \underbrace{\operatorname{cov}(Y_2, Y_1)}_{=0} + \beta \underbrace{\operatorname{cov}(Y_2, Y_3)}_{=1} + \gamma \underbrace{\operatorname{cov}(Y_3, Y_1)}_{=0} + \gamma \underbrace{\operatorname{cov}(Y_3, Y_3)}_{=2} =$$

$$= \alpha + \beta + 2\gamma$$

$$\Rightarrow \operatorname{cov}(Z, X_1) = 0 \Leftrightarrow \boxed{\alpha + \beta + 2\gamma = 0}$$

$\boxed{6.}$ $\mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N(\vec{m}, V)$ con $\vec{m} = \vec{0}$, $V = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

¿Qué parejas son independientes?

i) X_1 y X_2:

¿ $\mathrm{cov}(X_1, X_2) = 0$? Sí $\Longrightarrow$ independientes

ii) $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ y X_2:

Reordenamos $\mathbb{X}$: $\hat{\mathbb{X}} = \begin{pmatrix} X_1 \\ X_3 \\ X_2 \end{pmatrix} \Longrightarrow \hat{\vec{m}} = \vec{0}$

$$\hat{V} = \left(\begin{array}{cc|c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hline 0 & 0 & 5 \end{array}\right)$$

$\hat{V}_{1,1}$, $\hat{V}_{1,2}$, $\hat{V}_{2,2}$, $\hat{V}_{2,1}$

Usamos punto 8 (mi resumen) o diapositiva 15 (punto 1)

$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ y X_2 indep. $\Longleftrightarrow \hat{V}_{1,2} = V_{2,1}^{T} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓ $\Longrightarrow$

$\Longrightarrow \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ y X_2 sí son indep.

iii) X_1 y $X_1 + 3X_2 - 2X_3$

Llamamos $Y_1 = X_1$
$Y_2 = X_1 + 3X_2 - 2X_3$ $\Longrightarrow \mathbb{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$\Longrightarrow \mathbb{Y} \sim N(A\vec{m}, AVA^T) = N\left(\vec{0}, \begin{pmatrix} 4 & 6 \\ 6 & 61 \end{pmatrix}\right)$

Y_1 y Y_2 indep $\Longleftrightarrow AVA^T$ es diagonal

Como no lo es $\Longrightarrow Y_1, Y_2$ no son indep.

$\boxed{7.}$ $\mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N(\vec{m}, V)$ con $\vec{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ y $V = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

¿Distr. de $X_1 + X_2$ condicionada al valor $X_1 - X_2$?

Sea $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$ $\Longrightarrow \mathbb{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$\Longrightarrow \mathbb{Y} \sim N(A\vec{m}, AVA^T) = N\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 1 & 3 \end{pmatrix} \right)$$

$$\Longrightarrow Y_1 | Y_2 = a \sim N(\tilde{m}_1, \tilde{V}_1) \text{ con } \begin{cases} \tilde{m}_1 = 2 + 1 \cdot \frac{1}{3}(a-0) = 2 + \frac{a}{3} \\ \tilde{V}_1 = 7 - 1 \cdot \frac{1}{3} \cdot 1 = \frac{20}{3} \end{cases}$$

punto ⑧ (mi resumen)
diap. 15 (punto 2)

$\boxed{8.}$ $\mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N(\vec{m}, V)$ con $\vec{m} = \vec{0}$ y $V = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 4 \end{pmatrix}$

Definimos $\mathbb{Y} = \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$$\Longrightarrow \mathbb{Y} \sim N(A\vec{m}, AVA^T) = N\left(\vec{0}, \left(\begin{array}{c|cc} \overbrace{22}^{V_{11}} & \multicolumn{2}{c}{\overbrace{4 \quad -2}^{V_{12}}} \\ \hline 4 & 3 & -2 \\ \underbrace{-2}_{V_{21}} & \multicolumn{2}{c}{\underbrace{-2 \quad 2}_{V_{22}}} \end{array}\right) \right)$$

$$\Longrightarrow Y_3 \,\Big|\, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{a} \sim N(\tilde{m}_3, \tilde{V}_3) \text{ con:}$$

$$\tilde{m}_3 = 0 + (4 \quad -2) \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}^{-1} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = 1$$

$$\tilde{V}_3 = 22 - (4 \quad -2) \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 16$$

$\boxed{9.}$ $\mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N(\vec{m}, V)$ con $\vec{m} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ y $V = \begin{pmatrix} 3 & a & 1/2 \\ a & 2 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$

$a \in \mathbb{R}$

a) Valores de a para que V sea def. pos.

Usamos el criterio de Sylvester:

$3 > 0$ ✓

$\begin{vmatrix} 3 & a \\ a & 2 \end{vmatrix} = 6 - a^2 > 0 \iff a^2 < 6$

$\begin{vmatrix} 3 & a & 0'5 \\ a & 2 & 0 \\ 0'5 & 0 & 1 \end{vmatrix} = \frac{11}{2} - a^2 > 0 \iff a^2 < \frac{11}{2}$

$\Longrightarrow -\sqrt{11/2} < a < \sqrt{11/2} \approx 2'34$

b) $\mathbb{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 + 2X_2 \\ X_1 - X_2 \end{pmatrix}$ Buscamos a para que Y_1 e Y_2 sean independientes:

$\mathbb{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \Rightarrow \mathbb{Y} \sim N(A\vec{m}, AVA^T) =$

$= N\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \hat{V}\right)$

Y_1, Y_2 indep $\iff \hat{V}$ es diagonal $\iff \operatorname{cov}(Y_1, Y_2) = \operatorname{cov}(Y_2, Y_1) = 0$

$\operatorname{cov}(Y_1, Y_2) = \operatorname{cov}(X_1 + 2X_2, X_1 - X_2) = \operatorname{cov}(X_1, X_1) - \operatorname{cov}(X_1, X_2) +$

$+ 2\operatorname{cov}(X_2, X_1) - 2\operatorname{cov}(X_2, X_2) =$

$= 3 - a + 2a - 4 = a - 1 = 0$

$\iff \boxed{a = 1}$

c) $a = 2$ ahora.

$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Big| X_3 = \frac{1}{2} \sim N(\tilde{m}_1, \tilde{V}_1)$ con $\begin{cases} \tilde{m}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \cdot 1 \left(1/2 - 0\right) = \begin{pmatrix} 5/4 \\ 1 \end{pmatrix} \\ \tilde{V}_1 = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} 1 \begin{pmatrix} 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 11/4 & 2 \\ 2 & 2 \end{pmatrix} \end{cases}$

$\boxed{10.}$ $\mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N(\vec{m}, V)$ con $\vec{m} = \vec{0}$ y $V = \begin{pmatrix} 10 & 1 & 2 & 3 \\ 1 & 9 & 4 & 5 \\ 2 & 4 & 8 & 6 \\ 3 & 5 & 6 & 7 \end{pmatrix}$

a) Calcular la prob. $\mathbb{P}(Y = 2X_1 - 3X_4 \leq A)$:

$Y = 2X_1 - 3X_4$

$\mathbb{E}(Y) = \mathbb{E}(2X_1 - 3X_4) = 2\,\cancelto{0}{\mathbb{E}(X_1)} - 3\,\cancelto{0}{\mathbb{E}(X_4)} = 0$

$V(Y) = V(2X_1 - 3X_4) = V(2X_1) + V(-3X_4) - 2\operatorname{cov}(2X_1, -3X_4) =$

$= 4\underbrace{V(X_1)}_{=10} + 9\underbrace{V(X_4)}_{=7} - 12\underbrace{\operatorname{cov}(X_1, X_4)}_{=3} = 67$

Observaciones:

1) $V(X+Y) = V(X) + V(Y) - 2\operatorname{cov}(X,Y)$

$V(aX) = a^2 V(X)$

$\Rightarrow Y \sim N(0, 67) \Rightarrow Y = \sqrt{67}\, Z$ con $Z \sim N(0,1)$

$\Rightarrow \mathbb{P}(Y \leq A) = \mathbb{P}(\sqrt{67}\, Z \leq A) = \mathbb{P}(Z \leq A/\sqrt{67}) = \Phi(A/\sqrt{67})$

$\boxed{11.}$ Ejercicio inmediato con el Corolario 2 tema 1.

Solo tenemos que comprobar que B es idempotente y $\mu^T B \mu = 0$:

$$B = \begin{pmatrix} 1/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 2/3 \end{pmatrix} \; ; \quad B^2 = \begin{pmatrix} 1/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 2/3 \end{pmatrix} = B \Rightarrow B \text{ idempotente } \checkmark$$

$$\mu^T B \mu = \left(1, \tfrac{-\sqrt{2}}{2}\right) \begin{pmatrix} 1/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{3} \end{pmatrix} = 0 \; \checkmark$$

$$\Rightarrow \frac{1}{3} \mathbb{X}^T . B . \mathbb{X} \underset{\substack{\uparrow \\ \text{CORO } 2}}{\sim} \chi^2_{\text{traza}(B)} = \chi^2_1$$