HOJA 9

9.

a) Falso.

Si bien existe una biyección:

$$\left\{\begin{array}{c}\text{formas}\\ \text{cuadráticas}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{formas bilineales}\\ \text{simétricas}\end{array}\right\}$$

pueden existir formas bilineales distintas que "generan" la misma forma cuadrática.

P.ej $\varphi : V \times V \to \mathbb{R}$ bilineal no simétrica

$Q(u) = \varphi(u,u)$, Q es una forma cuadrática

$$\varphi_Q(u,v) = \frac{1}{2}\left(\varphi(u,v) + \varphi(v,u)\right) \neq \varphi(u,v)$$

$$\varphi_Q(u,v) = Q(u)$$

Ej:

$$\varphi \leftrightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \varphi' \leftrightarrow \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

$u = (x,y)$

$$Q_\varphi(u) = (x\ \ y)\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 + 4xy$$

$$Q_{\varphi'}(u) = (x\ \ y)\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 + 4xy$$

b) Existe una única forma bilineal $\varphi: \bar{V} \times \bar{V} \to \mathbb{R}$ simétrica tal que $Q(u) = \varphi(u,u)$

Verdadero

Basta probar que $\varphi = \varphi_Q$ $\quad Q(u) = \varphi(u,u)$

$$\varphi_Q(u,v) = \frac{1}{2}\big(Q(u+v) - Q(u) - Q(v)\big) = \frac{1}{2}\big(\varphi(u+v, u+v) - \varphi(u,u) - \varphi(v,v)\big) =$$

$$\underset{\varphi \text{ bilineal}}{=} \frac{1}{2}\big(\cancel{\varphi(u,u)} + \cancel{\varphi(v,v)} + \varphi(u,v) + \varphi(v,u) - \cancel{\varphi(u,u)} - \cancel{\varphi(v,v)}\big) \underset{\varphi \text{ simétr}}{=}$$

$$= \frac{1}{2} \cdot 2\varphi(u,v) = \varphi_Q = \varphi$$

c) Si todos los valores propios de Q son positivos $\Rightarrow$ Q es def. pos

Verdadero

$$Q \xleftrightarrow{1:1} M_B(\varphi) \text{ simétrica}$$

$$\varphi(u,v) = u^T . M_B(\varphi) . v$$

$$Q(u) = \varphi(u,u) = u^T . M_B(\varphi) . u$$

$$\exists B' \text{ o.n } / \ M_{B'}(\varphi) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \lambda_i > 0$$

$$M_B(\varphi) = M_{BB'} . M_{B'}(\varphi) . \underbrace{(M_{BB'})^T}_{P}$$

Verdadero

Existe una matriz simétrica $u = (x_1, \dots, x_n)$ (base o.n).

$Q(u) = u^T . A . u$

$\exists P / \ P^{-1} = P^T$ (cambio de base o.n)

$P^T . A . P = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \lambda_i > 0 \qquad u \to Pu = v$

$$Q(v) = (Pu)^T . A . Pu = u^T . P^T . A . Pu = (x_1 \dots x_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} =$$

$$= \lambda_1 x_1^2 + \dots + \lambda_n x_n^2 > 0 \quad \text{def. pos.}$$

b) $2x - 2x^2 + y^2 + 4xy - 1 = \underbrace{-2x^2 + y^2 + 4xy} + 2x - 1 = 0$

$$(x\ y)\underbrace{\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} + (2,0)\begin{pmatrix} x \\ y \end{pmatrix} - 1 = 0$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & -2 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda+3)(\lambda-2)$$

$\det(A) = (-3).2 = -6 < 0$ Tipo hiperbólico no degenerado

$\rightarrow$ sacar base o.n. vectores propios

$\rightarrow$ cambio

$\rightarrow$ traslación

$\boxed{12.}$ $C := 6x^2 + y^2 = z^2$ y el plano $\Pi := y = 2z + 3$; calcular la ecuación de la canónica $C \cap \Pi$. Resultado: elipse

$$6x^2 + (2z+3)^2 = z^2 \Rightarrow 6x^2 + 4z^2 + 12z + 9 = z^2$$

$$6x^2 + 3z^2 + 12z + 9 = 0$$

$$\underbrace{6x^2}_{x'} + \underbrace{3(z+2)^2}_{z'} = 0 \Rightarrow 2(x')^2 + (z')^2 = 1 \quad \text{elipse}$$

10.

a) $\underbrace{x^2 - 2xy + y^2}_{\text{parte principal}} + \underbrace{4x - 6y}_{(D \;\; E)\begin{pmatrix} x \\ y \end{pmatrix}} + 1 = 0$

$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\qquad D = 4$, $E = -6$

Base o.n. en la que $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ sea diagonal

$|\lambda x - A| = \begin{vmatrix} \lambda-1 & 1 \\ 1 & \lambda-1 \end{vmatrix} = \lambda(\lambda-2)$ $\begin{cases} \lambda = 0 \\ \lambda = 2 \end{cases}$

$\lambda = 0$, $\operatorname{Ker} A = \langle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \rangle$

$\lambda = 2$, $\operatorname{Ker}(A - 2I_2) = \langle (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) \rangle$

$B' = \{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) \}$

$P = M_{BB'} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

$P\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad P^T . A . P = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$(x \;\; y) A \begin{pmatrix} x \\ y \end{pmatrix} + (D \;\; E)\begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \Rightarrow (x' \;\; y') P^T.A.P. \begin{pmatrix} x' \\ y' \end{pmatrix} + (D \;\; E).P.\begin{pmatrix} x' \\ y' \end{pmatrix} + 1 = 0$

$\Rightarrow 2(y')^2 + (4, -6)\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + 1 = 0 \Rightarrow$

$\Rightarrow 2(y')^2 - \sqrt{2}x' + 5\sqrt{2}y' + 1 = 0$

$(\sqrt{2}y' + \frac{5}{2})^2 - \sqrt{2}x' + 1 - \frac{25}{4} = 0 \Rightarrow (\sqrt{2}y' + \frac{5}{2})^2 - \sqrt{2}x' - \frac{21}{4} = 0$

$\Rightarrow \underbrace{(\sqrt{2}y' + \frac{5}{2})^2}_{y''} - \sqrt{2}\underbrace{(x' - \frac{21}{4\sqrt{2}})}_{x''} = 0 \Rightarrow (y'')^2 - \sqrt{2}x'' = 0$

$\Rightarrow (y'')^2 = \sqrt{2}x''$ parábola

$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 21/4\sqrt{2} \\ 5/2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{21}{4\sqrt{2}} \\ 5/2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\boxed{3.}$ $Q(x,y,z) = 3x^2 + y^2 + \alpha z^2 - 2xy - 2yz \qquad \alpha \in \mathbb{R}$ y $\alpha \neq 0$

a) Q es def. positiva $\Leftrightarrow$ los valores propios de $\underbrace{\begin{pmatrix} 3 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & \alpha \end{pmatrix}}_{\overset{\shortparallel}{A}}$ son todos positivos

$$|xI - A| = \begin{vmatrix} x-3 & 1 & 0 \\ 1 & x-1 & 1 \\ 0 & 1 & x-\alpha \end{vmatrix} = x^3 - (4-\alpha)x^2 + (\alpha - 1 + 3\alpha + 3 - 1)x - (2\alpha - 3)$$

$$= x^3 - (4+\alpha)x^2 + (4\alpha+1)x - (2\alpha - 3) =$$

$$= \cdots \quad [\ldots]$$

<u>Método de Gauss</u>: $3x^2 + y^2 + \alpha z^2 - 2y(x+z) =$

$$= 3x^2 + \alpha z^2 + (y - (x+z))^2 - (x+z)^2 =$$

$$= (y-x-z)^2 + 3x^2 + \alpha z^2 - x^2 - 2xz - z^2 =$$

$$= (y-x-z)^2 + 2x^2 - 2xz + (\alpha - 1)z^2 =$$

$$= (y-x-z)^2 + 2x^2 - 2xz + \frac{z^2}{2} + \left(\alpha - \frac{3}{2}\right) \cdot z^2 =$$

$$= \underbrace{(y-x-z)}_{x'}{}^2 + 2\underbrace{\left(x - \frac{z}{2}\right)}_{y'}{}^2 + \left(\alpha - \frac{3}{2}\right) \underset{z'}{\overset{}{z}}{}^2$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = P \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$P = \begin{pmatrix} 0 & 1 & 1/2 \\ 1 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix} = M_{BB'}$ las columnas dan $B' = \{(0,1,0), (1,1,0), (\frac{1}{2}, \frac{3}{2}, 1)\}$

no es o.n.

$$M_{B'}(Q) = P^T \cdot \underbrace{M_B(Q)}_{A} \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \alpha - \frac{3}{2} \end{pmatrix}$$

Q es def. pos $\iff \alpha - \frac{3}{2} > 0 \iff \alpha > \frac{3}{2}$

d) Q y Q' son def. pos, entonces $Q + Q'$ es def. pos.

Verdadero

$$(Q + Q')(u) = \underset{\substack{\vee \\ 0}}{Q(u)} + \underset{\substack{\vee \\ 0}}{Q'(u)} > 0 \qquad \checkmark$$

e) Si Q es indefinida, entonces Q es degenerada

indefinida $\Leftrightarrow$ $\exists u, v \in V /$ $Q(u) < 0$, $Q(v) > 0$ $\Leftrightarrow$ A tiene una valor propio pos y otro neg.

degenerada $\Leftrightarrow$ $\text{Rang}(Q)$ no sea máximo $\Leftrightarrow$ $\lambda = 0$ es un valor propio de A.

Falso

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Longleftrightarrow Q(x,y) = x^2 - y^2$$

(B) Ecuación de la hipérbola $F_1 = (2,-1)$

Asíntotas: $\left.\begin{array}{l} r_1 \Rightarrow x = 0 \\ r_2 \Rightarrow 3x - 4y = 0 \end{array}\right\} \rightarrow$ pasan por el $(0,0)$

como se cortan en el centro $\Rightarrow$ centro $= (0,0)$

$F_1 = (2,-1)$

$F_2 = (-2,1)$

$c^2 = \underbrace{a^2 + b^2}_{\text{distancia foco al centro al cuadrado}}$

$c = \sqrt{5}$

$R' = \{ O = (0,0)\ ;\ u_1 = \frac{1}{\sqrt{5}}(2,-1)\ ;\ u_2 = \frac{1}{\sqrt{5}}(1,2) \}$

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ 2\sqrt{5} \end{pmatrix}$$

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Estas son las direcciones de las asíntotas respecto a $B' = \{u_1, u_2\}$

Entonces: $\frac{b}{a} = \frac{2\sqrt{5}}{\sqrt{5}} = 2 \Rightarrow b = 2a \Rightarrow$

$\Rightarrow c^2 = 5 = a^2 + b^2 = a^2 + 4a^2 \Rightarrow$

$\Rightarrow 5 = 5a^2 \Rightarrow a = 1$

En R': ec. de la hipérbola:

$$\frac{(x')^2}{1} - \frac{(y')^2}{4} = 1 \qquad (x'\ y')\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$$

$\boxed{6.}$ (HECHO POR ANA)

$$Q(x,y,z) = x^2 + 5y^2 - 2xy + 2xz$$

$$Q(x,y,z) = (\underbrace{x-(y-z)}_{x_1})^2 - (\underbrace{y-z}_{z_1})^2 + 5\underbrace{y^2}_{y_1} \Rightarrow \begin{cases} x_1 = x-(y-z) \\ y_1 = y \\ z_1 = y-z \end{cases}$$

$\boxed{5.}$ (ANA)

$$u.v = \frac{(u+v)^2-(u-v)^2}{4}$$

iii) $Q(x,y,z,t) = \underline{xy} + yz + zt$

$$Q(x,y,z,t) = (\underbrace{x+z}_{u})(\underbrace{y+0}_{v}) + zt =$$

$$= \frac{(x+z+y)^2-(x+z-y)^2}{4} + zt = \frac{1}{4}(x+z+y)^2 - \frac{1}{4}(x+z-y)^2 + zt$$

$$= \frac{1}{4}(\underbrace{x+z+y}_{x_1})^2 - \frac{1}{4}(\underbrace{x+z-y}_{y_1})^2 + \frac{1}{4}(\underbrace{z+t}_{z_1})^2 - \frac{1}{4}(\underbrace{z-t}_{t_1})^2$$