Modeling the Spruce Budworm

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Abstract

The spruce budworm is one of the most destructive native insects in the forests of Northern America and some parts of Canada. As their population increases, they cause huge damage to the trees the inhabit. These outbreaks have caused the loss of millions of cords of spruce and fir. If we can model the cycles of spruce budworm populations,we can take a step towards developing an effective solution to the problem. We will do this by using the Logistic model.

1. Introduction

The Spruce Budworm (see Figure 1) is one of the most destructive native insect in the northern spruce and fir forests of the Eastern United States and Canada. Majority of the time, the number of budworms remains at a constant low level. However, every few decades, the population of budworms increases to a huge population, depleting the forest and destroying many trees, before dropping back down to its normal population level. Evidence suggests these outbreaks have been repeating itself regularly for hundreds, if not thousands, of years. The first recorded outbreak of the Spruce Budworm in the United States occurred in Maine sometime around 1807. Another outbreak followed in 1878. Since 1909, there have been waves of budworm out breaks throughout the Eastern United States and Canada. The States most often affected are Maine, New Hampshire, New York, Michigan, Minnesota, and Wisconsin. These outbreaks have resulted in the loss of millions of cords of spruce and fir.

Figure 1: Picture of a Spruce Budworm.

2. Spruce and Fir

The fir tree is native to North America, Europe, North Africa and Asia. More than 60 species of the tree exist. This softwood has a wide usage. Its most familiar usage is Christmas trees. Fir lumber is one of the weaker softwood types. Wood from the tree is not considered suitable for general timber use. However, it suits a useful pulp during the processing of plywood. Fir is highly vulnerable to insect attacks and rot. For this reason, it is generally used for indoor applications such as paneling and light frame construction. Red, white and black spruce are cut into lumber, utility poles, pilings, boat building stock, furniture, boxes and crates. These species are also used to make plywood and flake board. Sitka spruce is used to make sounding boards for high-quality pianos, guitar faces, ladders and components of experimental light aircraft. White spruce is also used to manufacture piano sounding boards. Specialty uses for black spruce include distillation of perfume and as the main ingredient in spruce beer.

3. Newly Hatched Budworm

The Spruce Budworm and relatives to it are a group of closely related insects in the genus Choristoneura. Newly hatched budworms are very small and hard to see due to them borrowing themselves into and feeding on needles and expanding buds causing huge damage to the tree.Larvae go through six larval stages in there beginning life. The first larval , about 2 millimeters long, is yellowish green with a light- to medium-brown head. The second larval stage is yellow with a dark brown or black head. During the next four stages, the body of the larva changes from a pale yellow to a dark brown with light-colored spots along the back. In the last larval stage, the larva is about 2.5 centimeters long and the head is dark brown or shiny black. The pupa is pale green at first, later changing to reddish brown as the larva grows.

4. Logistic Model

The model I will use to model the budworm population is the *logistic model equation*. Let t be time and let N be the budworm population size. We model the evolution of the Budworm

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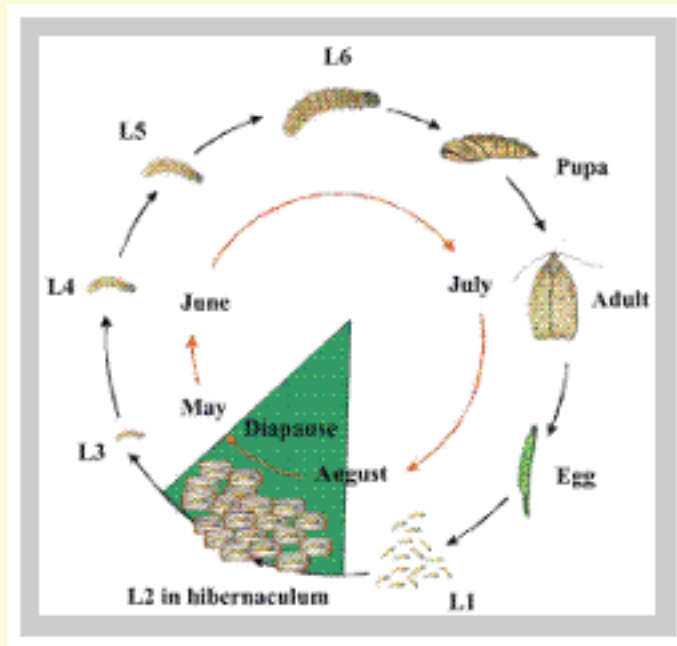
Figure 2: Budworm Larva Lifecycle .

according to the differential equation of the form

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N,$$

where r is the growth rate, N is the population size, and K is the carrying capacity.

4.1. Predator

The rate of predation is highly dependent on the budworm's population. The predator model is modeled by the equation

$$p(N) = \frac{BN^2}{A^2 + N^2},$$

where A and B are constants. If the spruce budworm population is small, the predation rate is low, if not close to zero, since the predators will pursue other prey. The predation rate will grow as the budworms become more numerous. Another important feature of the dependence of predator on prey is that the predation rate cannot grow to become too large. Instead, if the budworm population is very large, the rate of predation reaches some maximum value. This is due to there effects saturates at high prey densities, meaning there is an upper boundary to the death rate of the budworm due to predation.

Now we like to show some prof to our previous statements in our last paragraph. If we take the limit of our equation as N approaches infinite

$$\lim_{N\to\infty} \frac{BN^2}{A^2 + N^2} = \lim_{N\to\infty} \frac{B}{\frac{A^2}{N^2} + 1} = B$$

we will see that B represents the boundary of the death rate of the Budworm due to predation.

By taking the derivative of $p(N)$ we will be able to calculate the rate at which the predation increase. First we use the quotient rule to derive our equation.

$$p'(N) = \frac{(A^2 + N^2)(2BN) - (BN^2)(2N)}{(A^2 + N^2)^2}$$

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This gives us
$$p'(N) = \frac{2A^2BN + 2BN^3 - 2BN^3}{(A^2 + N^2)^2}.$$

We see that $2BN^3 - 2BN^3$ cancels out giving us
$$p'(N) = \frac{2A^2BN}{(A^2 + N^2)^2}.$$

Note that $p'(N)$ is greater then 0 for all values of N, which means the predation rate is increasing. If we integrate for a second time we will be able to solve for N giving us our critical number which represents where our graph shifts concavity from up to down. Just as we did the first time, we use the quotient rule to take the second derivative.

$$p"(N) = \frac{(A^2 + N^2)^2(2A^2B) - (2A^2BN)(2(A^2 + N^2)(2N))}{(A^2 + N^2)^4}$$

After doing a little multiplication we come up with the equation

$$p''(N) = \frac{2A^2B(A^2 + N^2)^2 - 8A^2BN^2(A^2 + N^2)}{(A^2 + N^2)^4}$$
$$= \frac{2A^2B(A^2 + N^2)[(A^2 + N^2) - 4N^2]}{(A^2 + N^2)^4}$$
$$= \frac{2A^2B[A^2 - 3N^2]}{(A^2 + N^2)^3}$$

Now if we set $p''(N) = 0$, we can solve for N and get our critical point.

$$p(N) = 0 \Rightarrow A^2 - 3N^2 = 0 \Rightarrow N^2 = \frac{A^2}{3} \Rightarrow N = \frac{A}{\sqrt{3}}$$

This is our critical point. $A\sqrt{3}$ is equal to the threshold which is the point where our predation graph changes concavity moving from concave up to concave down.

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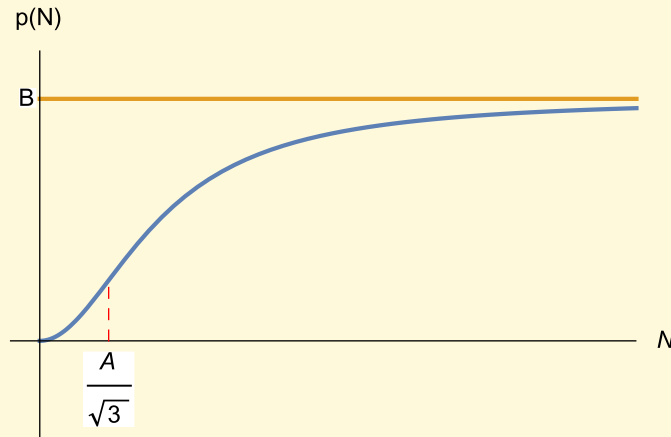
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Figure 3: Predation graph.

4.2. Predator Curve

If we were to graph $p(N)$ and N we would be able to get a better idea of how $p(N)$ is related to N and what happens to the graph as the solution reaches its maximum value or the limit. The limit represents the saturation at high prey densities.(see Figure 3)

5. Final Equation

By adding our predation equation to our logistics model equation we will get our final equation.

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N - p(N)$$

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N - \frac{BN^2}{A^2 + N^2}$$

This represent the rate at which predator eats prey.

6. Eliminating The Parameter

Our equation

$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{K_B}\right)N - \frac{BN^2}{A^2 + N^2} \tag{1}$$

has four parameters, r_B, K_B, B and A, with A and K_B having the same dimensions as N, r_b has dimensions 1/time, and B has dimensions N/time.

6.1. Scaling

By scaling, we can substitute

$$u = \frac{N}{A}, \quad \tau = \frac{Bt}{A}, \quad r = \frac{Ar_B}{B}, \quad q = \frac{K_B}{A},$$

making $N = Au$. So,

$$\frac{dN}{dt} = \frac{dN}{du} \cdot \frac{du}{d\tau} \cdot \frac{d\tau}{dt}$$

$$= A \cdot \frac{du}{d\tau} \cdot \frac{B}{A}$$

$$= B\frac{du}{d\tau}$$

By plugging $B\, du/d\tau$ for $dN/d\tau$ and Au for N into our equation (1) we get

$$B\frac{du}{d\tau} = Ar_B u\left(1 - \frac{Au}{K_B}\right) - \frac{B(Au)^2}{A^2 + (Au)^2}.$$

When we divide both sides by B we get

$$\frac{du}{d\tau} = \frac{Ar_B}{B}u\left(1 - \frac{Au}{K_B}\right) - \frac{(Au)^2}{A^2 + (Au)^2}$$

$$\frac{du}{d\tau} = \frac{Ar_B}{B}u\left(1 - \frac{Au}{K_B}\right) - \frac{A^2 u^2}{A^2 + A^2 u^2}$$

$$\frac{du}{d\tau} = \frac{Ar_B}{B}u\left(1 - \frac{Au}{K_B}\right) - \frac{u^2}{1 + u^2}$$

We will now replace Ar_B/B with r and K_B with qA.

$$\frac{du}{d\tau} = ru\left(1 - \frac{Au}{qA}\right) - \frac{u^2}{1 + u^2}$$

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}$$

By applying the work we have simplified our differential equation significantly.

7. Equilibria

To find the equilibria we have to set the right hand side of our equation equal to zero.

$$ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2} = 0$$

Now we factor out a u.

$$u\left[r\left(1 - \frac{u}{q}\right) - \frac{u}{1 + u^2}\right] = 0$$

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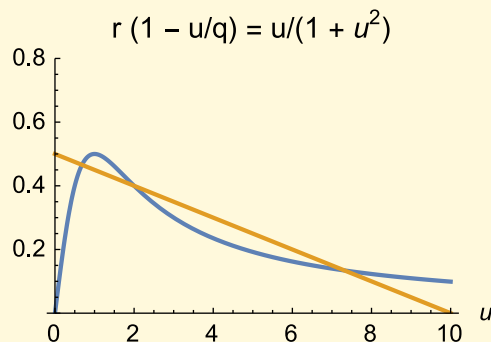
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$$r\,(1 - u/q) = u/(1 + u^2)$$

Figure 4: The intersection of the two graphs represent the equilibrium points.

Thus,

$$u = 0 \quad \text{or} \quad r\left(1 - \frac{u}{q}\right) - \frac{u}{1 + u^2} = 0.$$

So, one of the equilibrium points is $u = 0$. To find the remaining equilibrium points, we need to solve

$$r\left(1 - \frac{u}{q}\right) = \frac{u^2}{1 + u}.$$

The left side of this equation is the per capita growth rate of the scaled variable u. The right hand side represents the per capita death rate due to predation. As you can see in Figure 4, the predator graph is represented by a curve which passes through the origin and asymptotic to the x axis at high densities. The prey graph is represented by a straight line. The equilibria for the budworm variable are defined where the straight growth curve intersects the peaked predation curve.

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7.1. Solving for r

If we take our equation $r(1 - \frac{u}{q}) = \frac{u^2}{1+u^2}$ and solve for r by taking the derivative of the left and right hand side of our equation with respect to u. First let's take the derivative of the left hand side with respect to u.

$$\frac{d}{du} r\left(1 - \frac{u}{q}\right) = r\left(0 - \frac{1}{q}\right)$$
$$= -\frac{r}{q}$$

Now we can derive the right hand side.

$$\frac{d}{du} \frac{u}{1+u^2} = \frac{(u)'(1+u^2) - (1+u^2)'(u)}{(1+u^2)^2}$$
$$= \frac{1+u^2 - 2u^2}{(1+u^2)^2}$$
$$= \frac{1-u^2}{(1+u^2)^2}$$

Now we combine the two equations and get

$$-\frac{r}{q} = \frac{1-u^2}{(1+u^2)^2}.$$

Now we can solve for r by simply multiply both sides of our equation by q.

$$r = \frac{-q(u^2 - 1)}{(1+u^2)^2}$$

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7.2.   Solving For q

We solve for q by plugging $r = -\frac{q(u^2-1)}{(1+u^2)^2}$ into

$$r\left(1 - \frac{u}{q}\right) = \frac{u^2}{1+u}$$

$$-\frac{q(u^2-1)\left(1 - \frac{u}{q}\right)}{(1+u^2)^2} = \frac{u}{1+u^2}$$

$$q(u^2-1)\left(1 - \frac{u}{q}\right) = \frac{u(1+u^2)^2}{1+u^2}$$

$$q(u^2-1)\left(1 - \frac{u}{q}\right) = u(1+u^2)$$

Now we can simplify some thing to make our equation easier to solve. We do this by dividing both sides of the equation by $u^2 - 1$.

$$q\left(1 - \frac{u}{q}\right) = \frac{u(u^2+1)}{u^2-1}$$

This simplies to

$$q - u = \frac{u(u^2+1)}{u^2-1}.$$

Now we find a common denominator giving us

$$q = u + \frac{u(u^2+1)}{u^2-1}$$

$$q = \frac{u(u^2-1)}{u^2-1} + \frac{u(1+u^2)}{u^2-1}$$

$$q = \frac{u(u^2-1) + u(1+u^2)}{u^2-1}.$$

Once we simplify we get

$$q = \frac{2u^3}{u^2 - 1}$$

7.3.   Making r Life Easier

We can finally get our r value with only 1 parameter by plugging q into the equation $r = -q(1 - u^2)/(1 + u^2)^2$.

$$r = \frac{-\dfrac{2u^3}{u^2 - 1}(1 - u^2)}{(1 + u^2)^2}$$

$$r = \frac{2u^3}{(1 + u^2)^2}$$

8.   Bifurcation Diagrams

Significance of our graph is crucial to getting a better understanding to seeing our equilibrium points. The equilibrium points are where the two graphs, predator and prey, intercept one another. As we chose different values for r we have different equilibrium values as well as equilibrium points. These calculated r values gave me a mixture of stable and unstable points.

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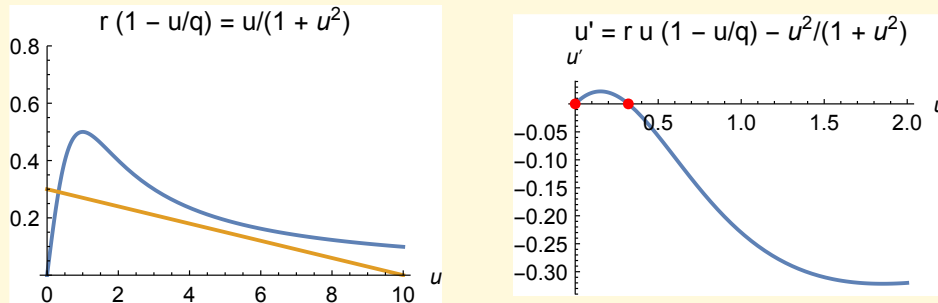Full Screen

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8.1. First Bifurcation Diagram

Figure 5: If r is equal 0.3, then we have two equilibrium points. Those two equilibrium points are (0,0),(.386802,0). The blue lines represent sample points that follow along the vector field.
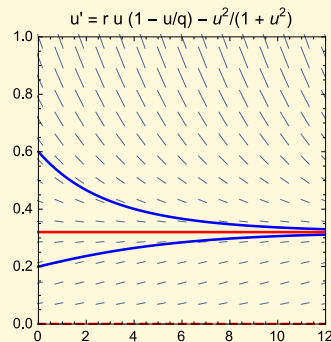

Figure 6: We have two equilibrium points in the above graph. Our point 0 is unstable therefore it is represented by a red dashed line. Our point .386802 is stable, so it is represented by a red solid line.

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8.2. Second Bifurcation Diagram
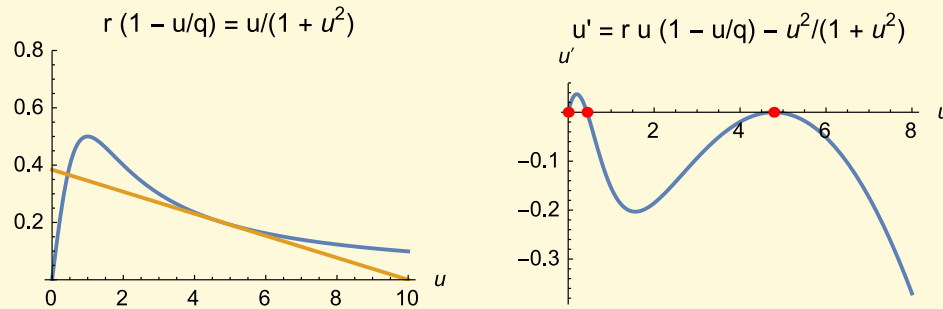

$r\,(1 - u/q) = u/(1 + u^2)$

$u' = r\,u\,(1 - u/q) - u^2/(1 + u^2)$

Figure 7: When r is equal .383971, we have 3 equilibrium points. Those points are $(0,0),(4.78576,0)$, and $(.437433,0)$.

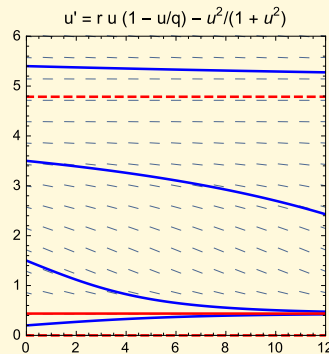$u' = r\,u\,(1 - u/q) - u^2/(1 + u^2)$

Figure 8: We have three calculated equilibrium points. One stable point, solid red line, and 2 unstable points, dashed red lines. Points 4.78576 and 0 are unstable, therefore represented by a dashed red line. Point .437433 is stable, so it is represented by a solid red line.

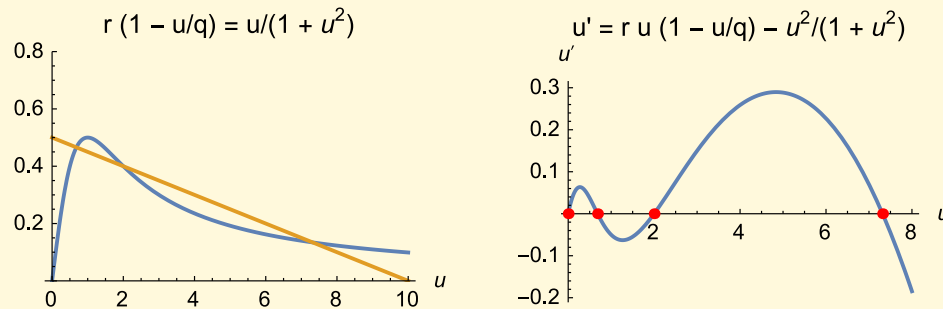## 8.3. Third Bifurcation Diagram

Figure 9: When r is equal .5, we have 4 equilibrium points. Those points are 0 and 0.683375 , 2, and 7.31662.
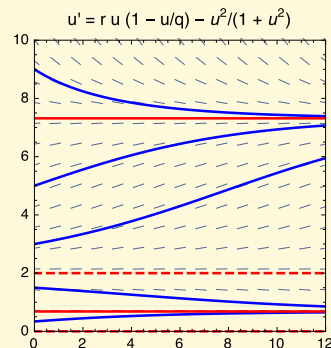

Figure 10: Our above graph represents the equilibrium points we calculated. Our points 0 and 2 are represented by a red dashed line because they are unstable. Our points 7.31662 and .683375 are represented by a solid red line since they are stable.
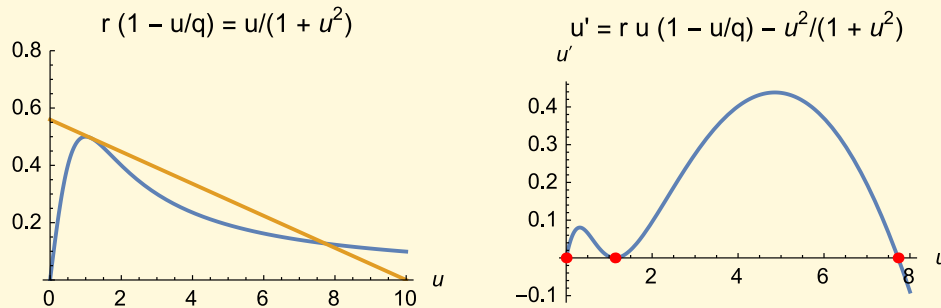
8.4. Fourth Bifurcation Diagram

$r(1 - u/q) = u/(1 + u^2)$

$u' = r\,u\,(1 - u/q) - u^2/(1 + u^2)$

Figure 11: If r is equal .559525, then we have three equilibrium points. Those points are $(0,0)$, $(1.13781,0)$, and $(7.72439,0)$.
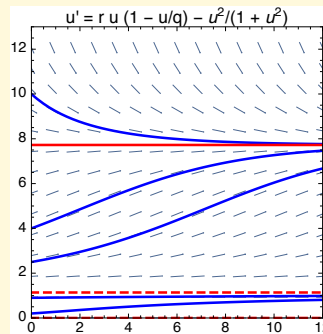

$u' = r\,u\,(1 - u/q) - u^2/(1 + u^2)$

Figure 12: Here we have 3 equilibrium points. 0 and 1.13781 are unstable,red dashed line, and 7.72439 is stable,red solid line .
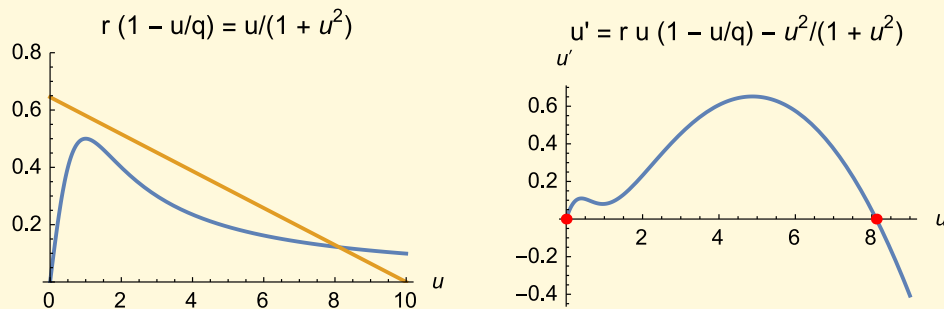
8.5. Fifth Bifurcation Diagram

$$r\,(1 - u/q) = u/(1 + u^2)$$

$$u' = r\,u\,(1 - u/q) - u^2/(1 + u^2)$$

Figure 13: If r is equal .645, then we have 2 equilibrium points. Those 2 equilibrium points are $(0,0)$ and $(8.11894,0)$.

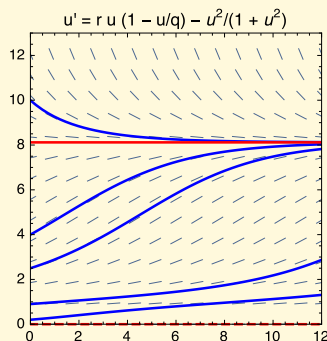$$u' = r\,u\,(1 - u/q) - u^2/(1 + u^2)$$

Figure 14: Here we have 2 equilibrium points. Point 0 is unstable, red dashed line, and 8.11894 is, red solid line.

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9.   Cusp

The graphs in the ru plane can output a number of equilibria solutions, but the number of equilibrium points are between one and three. The stability of these points may vary. As r increases starting at $r = .3$ the solution remains stable until it reaches $r = .383971$. At this point another equilibria solution on the boundary of our graph. If we were to increase our r value even further, we experience another equilibria solution where 2 points are stable and the other isn't (see Figure 15).
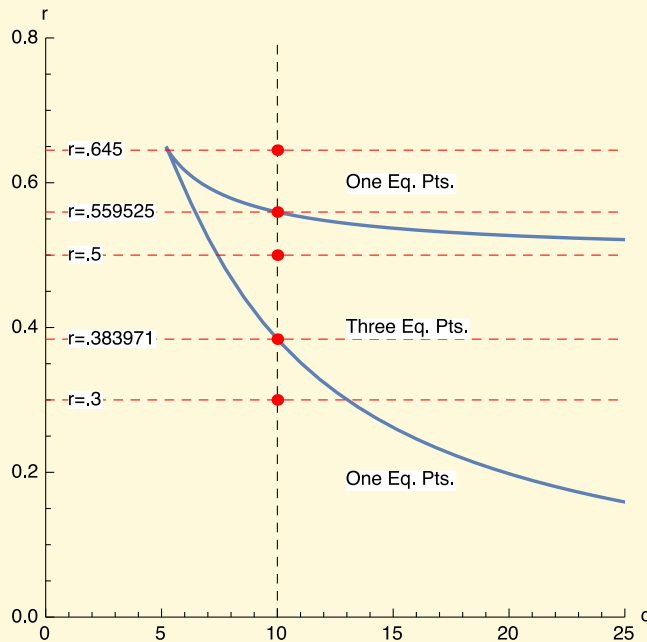

Figure 15: Cusp Graph.

10. Hysteresis

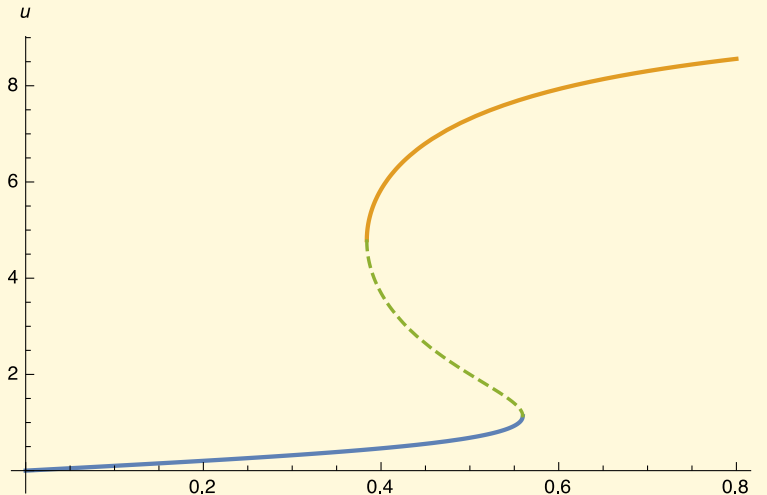An easier model to understand will be the Hysteresis graph (see Figure 16).

Figure 16: The hysteresis effect of the curves on the ru-plane.

This graph models the population changes as r increases. When we follow along the curve starting from $r = 0$ we see that the population increases at a steady rate. This is until we reach around .55 and then the graph undergoes a drastic change. It jumps from the bottom solid line to the top solid line. This represent a sudden increase in the Spruce Budworm population or in other words the Outbreak. Moving further down the curve, we see at $r = .6$ the population starts to become more and more steady. If we were to let nature take care of the outbreak it would take a while before the spruce budworm travels back to its normal small population size. To see this we would simply do what we did before, but instead start on the curve with a high r value. As you follow the graph left to right, you see the population gets smaller and smaller

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until it gets around .56 and then it takes a huge drop straight to the bottom solid line. After the you notice the population suddenly decreases at a steady rate.

11.   Conclusion

By modeling the Spruce Budworm population behavior, we may better get a better understanding of why this occurs. I modeled, with graphs, a huge increase in the population when it reaches a certain level. However, the same cannot be said as the population decreases, as it takes longer for their populations to decrease to its normal low state then it takes to increase. Furthermore, the spruce and fir forests are constantly at risk as the budworm population expand into an almost uncontrollable outbreak.

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[2] item:2 Math Insight: Spruce Budworm Outbreak Model. http://mathinsight.org/spruce_budworm_outbreak_model

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