$\boxed{48.}$ $\quad \Omega = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 \neq -1 \}$

$f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ cuyas componentes son:

$$f_1(x) = \frac{x_i}{1 + x_1 + x_2 + x_3} \qquad x \in \Omega$$

1. Demostrar que f es inyectiva en Ω

$$f(x) = f(y) \Rightarrow \frac{x_i}{1+x_1+x_2+x_3} = \frac{y_i}{1+y_1+y_2+y_3} \qquad i = 1, 2, 3$$

Sumo en i:
$$\frac{x_1+x_2+x_3}{1+x_1+x_2+x_3} = \frac{y_1+y_2+y_3}{1+y_1+y_2+y_3}$$

$$\underbrace{1 - \left(\quad '' \quad \right)}_{\displaystyle \frac{1}{1+x_1+x_2+x_3}} = \underbrace{1 - \left(\quad '' \quad \right)}_{\displaystyle \frac{1}{1+y_1+y_2+y_3}}$$

$$\frac{1}{1+x_1+x_2+x_3} = \frac{1}{1+y_1+y_2+y_3} \Rightarrow 1+x_1+x_2+x_3 \overset{(*)}{=} 1+y_1+y_2+y_3$$

Como $\displaystyle \frac{x_i}{\underbrace{1+x_1+x_2+x_3}} = \frac{y_i}{\underbrace{1+y_1+y_2+y_3}} \Rightarrow x_i = y_i$

son iguales por $(*)$

2) $Jf(x)$ $\qquad \dfrac{\partial x_i}{\partial x_j} = 1$ cuando $i = j$ ó 0 cuando $i \neq j$

$$\frac{\partial f}{\partial x_i} = \frac{\delta_{ij}(1+x_1+x_2+x_3) - x_i \cdot 1}{(1+x_1+x_2+x_3)^2}$$

$$Jf(x) = \begin{pmatrix} \dfrac{1+x_2+x_3}{(1+x_1+x_2+x_3)^2} & \dfrac{-x_1}{(\ )^2} & \dfrac{-x_1}{(\ )^2} \\[2ex] \dfrac{-x_2}{(\ )^2} & \dfrac{1+x_1+x_3}{(\ )^2} & \dfrac{-x_2}{(\ )^2} \\[2ex] \dfrac{-x_3}{(\ )^2} & \dfrac{-x_3}{(\ )^2} & \dfrac{1+x_1+x_2}{(\ )^2} \end{pmatrix}$$

$\hookrightarrow$ calcular det $= \cdots = \dfrac{1}{\ }$

3. Calcular $f(\Omega)$ y la expresión de ...

Supongamos $(y_1, y_2, y_3) \in f(\Omega)$ $\qquad y_i = \dfrac{x_i}{1 + \sum_{j=1}^{3} x_j}$

$$y_1 + y_2 + y_3 = \frac{\sum_{j=1}^{3} x_j}{1 + \sum_{j=1}^{3} x_j}$$

$f(\Omega)$ no contiene puntos (y_1, y_2, y_3) con $y_1 + y_2 + y_3 = 1$

Parece que $f(\Omega) = \mathbb{R}^3 \setminus \{y_1 + y_2 + y_3 = 1\}$

$$\Rightarrow 1 - \sum_{1}^{3} y_j = 1 - \frac{\sum x_j}{1 + \sum x_j} = \frac{1 + \sum x_j - \sum x_j}{1 + \sum x_j} = \frac{1}{1 + \sum x_j} \Rightarrow$$

$$\Rightarrow 1 + \sum_{j=1}^{3} x_j = \frac{1}{1 + \sum_{j=1}^{3} y_j} \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{3} x_j = \frac{1}{1 - \sum y_j} - 1 = \frac{\sum y_j}{1 - \sum y_j}$$

$$\Rightarrow \begin{cases} x_1 = \dfrac{y_1}{1 - \sum y_j} \\ x_2 = \dfrac{y_2}{1 - \sum y_j} \\ x_3 = \dfrac{y_3}{1 - \sum y_j} \end{cases}$$

Lo demuestro:

$$f(x_1, x_2, x_3) = f\left(\frac{y_1}{1 - \Sigma}, \ldots, \ldots\right) = (y_1, y_2, y_3)$$

$$f^{-1}(y_1, \ldots, y_3) = \left(\frac{y_1}{1 - \Sigma}, \ldots, \frac{y_3}{1 - \Sigma}\right) \Rightarrow \text{Dom} f^{-1} = f(\Omega)$$

49.1 Coordenadas polares

$C = \{(r,\theta) \mid r>0,\ \theta\in(-\pi,\pi)\}$

$\Omega = \mathbb{R}^2 \setminus \{(x_1,0) / x_1 \le 0\}$

$f: C \longrightarrow \Omega, \quad f(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta)$

1. f biyectiva de f en Ω.

1.1. Inyectiva

$f(r,\theta) = f(\bar r,\bar\theta) \qquad \begin{matrix}(r,\theta)\\ (\bar r,\bar\theta)\end{matrix} \in C$

$r^2 = (r\cos\theta)^2 + (r\operatorname{sen}\theta)^2 = (\bar r\cos\bar\theta)^2 + (\bar r\operatorname{sen}\bar\theta)^2 = \bar r^2 \Rightarrow r = \pm\bar r$

pero como en C r y $\bar r > 0 \Rightarrow \boxed{r=\bar r}$

$\left.\begin{matrix} r\cos\theta = \bar r\cos\bar\theta \\ r\operatorname{sen}\theta = \bar r\operatorname{sen}\bar\theta \end{matrix}\right\} \begin{matrix}\cos\theta = \cos\bar\theta \\ \operatorname{sen}\theta = \operatorname{sen}\bar\theta\end{matrix}$ ya que $r=\bar r>0$

Pero $-\pi<\theta,\bar\theta<\pi$ como el intervalo es de longitud $<2\pi$, un ángulo queda determinado de forma unívoca por su coseno y su seno $\Rightarrow \theta=\bar\theta$.

1.2. Sobreyectiva

Sea $(x_1,x_2)\in\Omega$. Necesito r,θ tal que $\begin{matrix}x_1 = r\cos\theta\\ x_2 = r\operatorname{sen}\theta\end{matrix}$

Con $r>0$, $-\pi<\theta<\pi$.

$\Rightarrow$ Necesito $r=\sqrt{x_1^2+x_2^2}$, y además $r>0$ y como $(x_1,x_2)\in\Omega$ y $(0,0)\notin\Omega \Rightarrow r>0$.

Ahora necesito θ.

$\left(\frac{x_1}{r}\right)^2 + \left(\frac{x_2}{r}\right)^2 = 1 \Rightarrow \exists\theta\in(-\pi,\pi]$ tal que $\cos\theta = \frac{x_1}{r}$, $\operatorname{sen}\theta = \frac{x_2}{r}$

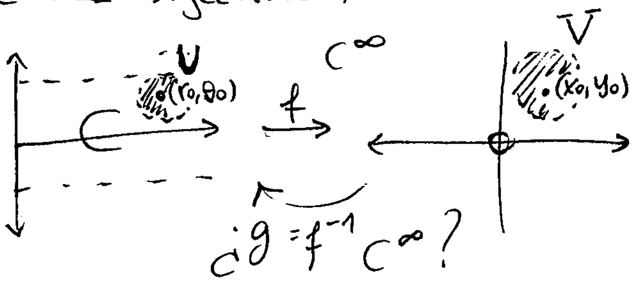
Si $\theta=\pi$, $\left.\begin{matrix}\frac{x_1}{r} = \cos\theta = -1 \Rightarrow x_1<0 \\ \frac{x_2}{r} = \operatorname{sen}\theta = 0 \Rightarrow x_2 = 0\end{matrix}\right\} \Rightarrow (x_1,x_2)\notin\Omega \Rightarrow$

$\Rightarrow \theta\neq\pi$ siempre

$\Rightarrow \theta\in(-\pi,\pi)$

2. $g : \Omega \longrightarrow C$, $g = f^{-1}$ ($f : C \to \Omega$ biyectiva)

2.1. g es diferenciable



f es C^∞

$$Df(r,\theta) = \begin{pmatrix} \cos\theta & -r\,\text{sen}\theta \\ \text{sen}\theta & r\cos\theta \end{pmatrix}$$

$$Jf(r,\theta) = |Df(r,\theta)| = r\cos^2\theta + r\,\text{sen}^2\theta = r > 0$$

Sea $(x_0,y_0) \in \Omega$ donde voy a ver g diferenciable

$(x_0,y_0) = f(r_0,\theta_0) \Rightarrow$ Como $Jf(r_0,\theta_0) = r_0 > 0$

El TFInversa dice que $\exists$abierto $U_{(r_0,\theta_0)}$, abierto $V_{(x_0,y_0)}$

tal que $f|_U : U \longrightarrow V$ biyectiva, diferenciable, con inversa $(f|_U)^{-1} : V \longrightarrow U$ diferenciable.

Pero $f^{-1} = g$ es global, e.d., existe en todo Ω

$$(f|_U)^{-1} = g|_{f(V)} = g|_U$$

2.2. Calcula $Dg(x,y)$

$f \circ g = Id_\Omega$; $D(f\circ g)(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$$D(f\circ g)(x,y) = Df(g(x,y)) \cdot Dg(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Dg(x,y) = Df(g(x,y))^{-1}$$

recordar
$x = r\cos\theta$
$y = r\,\text{sen}\theta$

$\underset{(r,\theta)}{Df(\underline{g(x,y)})}$; $g(x,y) = (r,\theta)$ $x = r\cos\theta$
$f(r,\theta) = (x,y)$ $y = r\,\text{sen}\theta$

$$Df^{-1}(g(x,y)) = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\,\text{sen}\theta \\ -\text{sen}\theta & \cos\theta \end{pmatrix} =$$

$$= \frac{1}{r}\begin{pmatrix} x & y \\ -y/r & x/r \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

$r = \sqrt{x^2+y^2}$

3. $g(x) = \left(\|x\|, \; 2 \arctan \dfrac{x_2}{\|x\| + x_1} \right)$

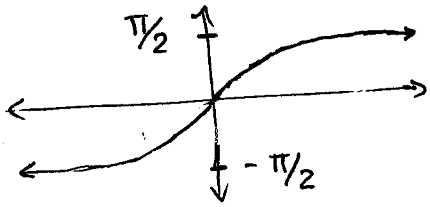
$g(x_1, x_2) = (r, \theta)$ tal $\quad x_1 = r\cos\theta$

$x_2 = r \operatorname{sen}\theta$

$r > 0, \quad \theta \in (-\pi, \pi)$

$$\begin{array}{c} r > 0 \\ \shortparallel \\ \sqrt{x_1^2 + x_2^2} = \|x\| > 0 \end{array}$$

$$\arctan(\varphi) = \arctan\left(\frac{x_2}{\|x\| + x_1}\right) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$



$$\theta = 2\arctan\left(\frac{x_2}{\|x\| + x_1}\right) \in (-\pi, \pi)$$

Recordar:

$$\cos(2\varphi) = \frac{1 - \tan^2(\varphi)}{1 + \tan^2(\varphi)} \qquad \operatorname{sen}(2\varphi) = \frac{2\tan(\varphi)}{1 + \tan^2(\varphi)}$$

$$x_1 = r \cos\Big(2 \overbrace{\arctan\left(\frac{x_2}{\|x\| + x_1}\right)}^{\varphi'}\Big)$$

$$x_2 = r \operatorname{sen}\Big(2 \underbrace{\arctan\left(\frac{x_2}{\|x\| + x_1}\right)}_{\varphi'}\Big)$$

$\Rightarrow \cdots \Rightarrow$

$$\cos 2\varphi' = \frac{x_1}{\|x\|}$$

$$\operatorname{sen} 2\varphi' = \frac{x_2}{\|x\|}$$

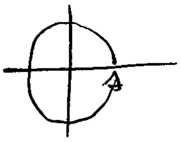
50.1 $F: \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2$

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

1) Ver que F es C^1 en $\mathbb{R}^2 \setminus \{0\}$ $\quad$ ¿ $\frac{\partial}{\partial x_1} F_2 = \frac{\partial}{\partial x_2} F_1$?

2) $\int_C F ds$ $\quad$ C = circunferencia unidad $\circlearrowleft$

C : $c(t) = (\cos t, \operatorname{sen} t)$ $\quad t \in [0, 2\pi]$



$$\int_C F ds = \int_0^{2\pi} F(c(t)) \cdot c'(t) dt = \int_0^{2\pi} \langle F(c(t)), c'(t) \rangle dt =$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi$$

3) Demostrar que no existe $U: \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}$ C^1 tal que $F = \nabla U$.

Supongamos que sí $\quad F = \nabla U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right)$

$$\int_C F ds = \int_C \nabla U = \int_0^{2\pi} \nabla U(c(t)) \cdot c'(t) dt = \int_0^{2\pi} \frac{d}{dt}\left(U(c(t)) \right) =$$

$$= U(c(2\pi)) - U(c(0)) = U((1,0)) - U(1,0) = 0 \quad (\text{antes} = 2\pi)$$

contradicción

$F: \Omega = \mathbb{R}^2 \setminus \{(x_1, 0), x_1 \leq 0\}$ $\quad \theta$ es C^1 en Ω

$$\tilde{C} = \{r > 0, -\pi < \theta < \pi\} \xrightarrow{f} \underset{\Omega}{(r\cos\theta, r\operatorname{sen}\theta)}$$

problema 49

$$(r(x,y), \theta(x,y)) \xleftarrow{g} (x,y) \in \Omega$$

$$\boxed{\text{p. 49} \quad \theta = \arctan\left(\frac{x_2}{\|x\| + x_1} \right)}$$

Ver que $F(x) = \nabla\theta(x)$

$$\nabla\theta(x) = \left(\frac{\partial\theta}{\partial x_1}, \frac{\partial\theta}{\partial x_2} \right)$$

$$Dg(x_1, x_2) = \begin{pmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{pmatrix} \xrightarrow{\text{p. 49}} \left(\frac{\partial\theta}{\partial x_1}, \frac{\partial\theta}{\partial x_2} \right) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

$\boxed{51}$ $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f(x_1, x_2) = (x_1^2 - x_2^2 \, , \; 2x_1x_2)$$

1. f no es inyectiva en $\mathbb{R}^2$.

$f(1,1) = f(-1,-1) \Longrightarrow$ no es inyectiva

f es inyectiva en $\Omega = \{x_1 > 0\}$

$f(x_1, x_2) = f(y_1, y_2)$ con $x_1, y_1 > 0$

$$\left.\begin{array}{l} x_1^2 - x_2^2 = y_1^2 - y_2^2 \\ \cancel{2}x_1x_2 = \cancel{2}y_1y_2 \end{array}\right\} \quad \frac{x_2}{y_1} = \frac{y_2}{x_1} = \lambda \qquad \begin{array}{l} x_2 = \lambda y_1 \\ y_2 = \lambda x_1 \end{array}$$

$$x_1^2 - \lambda^2 y_1^2 = y_1^2 - \lambda^2 x_1^2$$

$$x_1^2(1+\lambda^2) = y_1^2 \underbrace{(1+\lambda^2)}_{\neq 0}$$

$$x_1^2 = y_1^2 \Longrightarrow \underset{\underset{x_1}{\parallel}}{|x_1|} = \underset{\underset{y_1}{\parallel}}{|y_1|}$$

$$\cancel{x_1}\, x_2 = \cancel{y_1}\, y_2 \Longrightarrow x_2 = y_2 \quad \text{porque } x_1 = y_1 > 0$$

Calcular $f(\Omega)$: $f(\Omega) = f(\{x_1 > 0\})$

$f(x_1, x_2) = (x, y)$

$$\left.\begin{array}{l} x = x_1^2 - x_2^2 \\ y = 2x_1x_2 \Longrightarrow x_2 = \frac{1}{2}\frac{y}{x_1} \end{array}\right\} \Longrightarrow \quad x = x_1^2 - \frac{1}{4}\frac{y^2}{x_1^2} \Longrightarrow x_1^2 x = x_1^4 - \frac{1}{4}y$$

$$\Longrightarrow x_1^4 - x_1^2 x - \frac{1}{4}y^2 = 0$$

$$\Longrightarrow x_1^2 = \frac{x \pm \sqrt{x^2+y^2}}{2}$$

$$x_1^2 = \frac{x - \sqrt{\cdots}}{2} \leq 0 \quad (\text{no váli}$$

$$\Longrightarrow x_1 = \sqrt{\frac{x + \sqrt{x^2+y^2}}{2}}$$

$$\Longrightarrow x_2 = \frac{1}{2}\frac{y}{x_1} = \frac{1}{2} \cdot \frac{y}{\sqrt{\frac{x+\sqrt{\cdots}}{2}}}$$

Tenemos que ver que $\dfrac{\quad}{2} > 0$

$x+\sqrt{x^2+y^2}$ → si $y>0$: $x+\sqrt{x^2+y^2} > 0$

$x+\sqrt{x^2+y^2}$ → si $y=0$: $x+\sqrt{x^2+y^2} = x+|x| = 0$ si $x \leq 0$

$$\Longrightarrow f(\Omega) = \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\}$$

$$(y_1, y_2) = \left(\frac{\operatorname{sen} x_1}{\cos x_2}, \frac{\operatorname{sen} x_2}{\cos x_1}\right)$$

$$y_1 = \frac{\operatorname{sen} x_1}{\cos x_2}$$

$\cancel{x_1}$ $\operatorname{sen} x_1 = y_1 \cos x_2$

$$x_1 = \operatorname{arcsen}(y_1 \cdot \cos x_2)$$

~~$x_1 = \operatorname{arcsen}$~~

$$\operatorname{sen} x_1 = y_1 \cdot \cos\left(\arccos\left(\text{to' la pesca}\right)\right)$$

$$\operatorname{sen} x_1 = y_1 \cdot \text{to' la pesca}$$

$$\boxed{x_1 = \operatorname{arcsen}\left(y_1 \cdot \sqrt{\frac{y_2^2 - 1}{y_2^2 y_1^2 - 1}}\right)}$$

$$y_2 = \frac{\operatorname{sen} x_2}{\cos x_1} = \frac{\operatorname{sen} x_2}{\sqrt{1 - \operatorname{sen}^2 x_1}}$$

$$\frac{\operatorname{sen} x_2}{\sqrt{1 - \operatorname{sen}^2(\operatorname{arcsen}(y_1 \cdot \cos x_2))}} =$$

$$= \frac{\operatorname{sen} x_2}{\sqrt{1 - (y_1^2 \cdot \cos^2 x_2)}} =$$

$$\Rightarrow y_2^2 = \frac{\operatorname{sen}^2 x_2}{1 - y_1^2 \cos^2 x_2}$$

$$y_2^2 = \frac{1 - \cos^2 x_2}{1 - y_1^2 \cos^2 x_2}$$

$$y_2^2 - y_2^2 y_1^2 \cos^2 x_2 = 1 - \cos^2 x_2$$

$$y_2^2 - 1 = \cos^2 x_2 (y_2^2 y_1^2 - 1)$$

$$\cos^2 x_2 = \frac{y_2^2 - 1}{y_2^2 y_1^2 - 1}$$

$$\Rightarrow \boxed{x_2 = \arccos\left(\sqrt{\frac{y_2^2 - 1}{y_2^2 y_1^2 - 1}}\right)}$$

$\cancel{g(x) = sen}$ $\quad g(x,y) = \dfrac{\operatorname{sen} x}{\cos y}$

$\cos \quad y$

$-(\cos y^{-2}) - se$

$= \dfrac{\operatorname{sen} y}{\cos^2 y}$

$$\nabla f = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right) = \left(\frac{\cos x}{\cos y} \quad \frac{\operatorname{sen} x \operatorname{sen} y}{\cos^2 y}\right)$$

$$\nabla f = (0,0) \iff \begin{cases} \dfrac{\cos x}{\cos y} = 0 \iff \boxed{x = \dfrac{\pi}{2}} \\ \dfrac{\operatorname{sen} x \operatorname{sen} y}{\cos^2 y} = 0 \iff \end{cases}$$

$$\begin{cases} 1 = \cos^2 x_1 + y_1^2 \cos^2 x_2 \\ 1 = \cos^2 x_2 + y_2^2 \cos^2 x_1 \end{cases}$$

$$\begin{cases} 1 = \cos^2 x_1 + \dfrac{\operatorname{sen}^2 x_1}{\cancel{\cos^2 x_2}} \cdot \cancel{\cos^2 x_2} \\ 1 = \cos^2 x_2 + \dfrac{\operatorname{sen}^2 x_2}{\cancel{\cos^2 x_1}} \cdot \cancel{\cos^2 x_1} \end{cases}$$

$$\cancel{1 = \cos^2 x_2 \left(1 + \frac{\operatorname{sen}^2 x_2}{\cos^2 x_1}\right)}$$

INÚTIL

$1 =$

Calcular: $\quad 1 -$

$\cancel{1 - co}$