Ejemplos uso de Gram-Schmidt

Ejemplo 1 $V = \mathbb{R}^3$ con el producto escalar usual.

Aplicamos el algoritmo a la base:

$$B = \{(1,1,0), (0,1,1), (1,0,1)\}$$

$$\boxed{w_1 := \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$$

$\tilde{w}_2 = (0,1,1) - \lambda w_1$; ¿λ? imponemos $\varphi(\tilde{w}_2, w_1) = 0$

$$\varphi(\tilde{w}_2, w_1) = \varphi((0,1,1) - \lambda w_1, w_1) =$$

$$= \varphi((0,1,1), w_1) - \lambda \varphi(w_1, w_1) = \frac{1}{\sqrt{2}} - \lambda = 0$$

$$\Rightarrow \lambda = \frac{1}{\sqrt{2}}$$

$$\tilde{w}_2 = (0,1,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = (0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) =$$

$$= \left(-\tfrac{1}{2}, \tfrac{1}{2}, 1\right)$$ (observa que $w_1 \perp \tilde{w}_2$)

$\|\tilde{w}_2\|^2 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$ / $\boxed{w_2 := \sqrt{\tfrac{2}{3}}\left(-\tfrac{1}{2}, \tfrac{1}{2}, 1\right)}$

$\tilde{w}_3 := (1,0,1) - \alpha w_1 - \beta w_2$ ¿α, β? $\varphi(w_1, \tilde{w}_3) = 0$

$\varphi(w_2, \tilde{w}_3) = 0$

$$\varphi(\tilde{w}_3, w_1) = \varphi((1,0,1) - \alpha w_1 - \beta w_2, w_1) =$$

$$= \varphi((1,0,1), w_1) - \alpha \varphi(w_1, w_1) =$$

$$= \frac{1}{\sqrt{2}} - \alpha \Rightarrow \alpha = \frac{1}{\sqrt{2}};$$

$$\varphi(\tilde{w}_3, w_2) = \varphi((1,0,1) - \alpha w_1 - \beta w_2, w_2) =$$

$$= \varphi((1,0,1), w_2) - \beta \varphi(w_2, w_2) = \frac{1}{2}\sqrt{\tfrac{2}{3}} - \beta = 0$$

$$\tilde{\omega}_3 = (1,0,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) - \frac{1}{2}\sqrt{2/3}\left(\sqrt{2/3}\left(-1/2,1/2,1\right)\right)$$

$$= (1,0,1) - \left(\frac{1}{2},\frac{1}{2},0\right) - \left(-\frac{1}{6},\frac{1}{6},\frac{1}{3}\right) =$$

$$= \left(4/6,-4/6,2/3\right) = \left(2/3,-2/3,2/3\right)$$

Observad que $\omega_1 \perp \tilde{\omega}_3$; $\omega_2 \perp \tilde{\omega}_3$

$$\|\tilde{\omega}_3\|^2 = \frac{4}{9}\cdot 3 = \frac{4}{3}$$

$$\boxed{\omega_3 := \sqrt{3/4}\left(2/3,-2/3,2/3\right) = \sqrt{3}\left(1/3,-1/3,1/3\right)}$$

ahora:

$B' = \{\omega_1, \omega_2, \omega_3\}$ es o.n.

Ejemplo 2 $\mathbb{V} = \{ a_0 + a_1 x + a_3 x^2 ; a_0, a_{1,3} \in \mathbb{R} \}$

Dadas $f, g \in \mathbb{V}$, usamos el producto escalar:

$$\varphi(f, g) = \int_0^1 fg\,dx.$$

Partimos de la base $\{1, x, x^2\}$ y usamos G-S para encontrar una base o.n de $\mathbb{V}$.

$\tilde{w}_1 = 1$

$$\|\tilde{w}_1\|^2 = \int_0^1 dx = 1 \Rightarrow \boxed{w_1 := 1}$$

$\tilde{w}_2 := x - \lambda \cdot 1$ ¿λ? pedimos que $\varphi(1, w_2) = 0$

$$\varphi(1, w_2) = \varphi(1, x - \lambda \cdot 1) = \varphi(1, x) - \varphi(1,1)\lambda =$$

$$= \varphi(1, x) - \lambda$$

$$\varphi(1, x) = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right]_0^1 = 1/2 \Rightarrow \lambda = 1/2$$

$$\tilde{w}_2 = x - \frac{1}{2}, \quad \|w_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}$$

$$\Rightarrow \boxed{w_2 = 2\sqrt{3}\left(x - \frac{1}{2}\right) = 2\sqrt{3}\,x - \sqrt{3}}$$

$\tilde{w}_3 = x^2 - \lambda_1 \cdot 1 - \lambda_2 (2\sqrt{3}x - \sqrt{3})$ ¿λ_1, λ_2?

$$\varphi(\tilde{w}_3, w_1) = 0 \Rightarrow \varphi(x^2, 1) - \lambda_1 = 0$$

$$\Rightarrow \lambda_1 = \int_0^1 x^2\,dx = 1/3$$

$$\lambda_2 = \int_0^1 x^2 \sqrt{3}(2x-1)\,dx = \sqrt{3}\left[2\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 =$$

$$\| \qquad\qquad = \sqrt{3}\left(\frac{1}{2} - \frac{1}{3}\right) = \sqrt{3}\,\frac{1}{6}$$

$$\varphi(x^2, \sqrt{3}(2x-1))$$

$$\tilde{\omega}_3 = x^2 - \frac{1}{3} - \frac{\sqrt{3}}{6}\left(\sqrt{3}(2x-1)\right) =$$

$$= x^2 - \frac{1}{3} - x - \frac{1}{2} = x^2 - x - \frac{5}{6}$$

$$\|\tilde{\omega}_3\|^2 = \int_0^1 \left(x^2 - x - \frac{5}{6}\right)^2 dx = \frac{35}{36}$$ ← ojo: revisar esta última cuenta.

$$\boxed{\omega_3 := \frac{6}{\sqrt{35}}\left(x^2 - x - \frac{5}{6}\right)}$$

Base o.n $= \{\omega_1, \omega_2, \omega_3\} =$

$$= \left\{1, \sqrt{3}(2x-1), \frac{6}{\sqrt{35}}\left(x^2 - x - \frac{5}{6}\right)\right\}$$