HOJA 4

1.

X_1	X_2	Y
3	7	0
2	4	0
4	7	0
6	9	1
5	7	1
4	8	1

$\xrightarrow[\bar{X}_1=(5,8)]{\text{restamos } \bar{X}_0=(3,6)}$

X_1	X_2	Y
0	1	0
−1	−2	0
1	1	0
1	1	1
0	−1	1
−1	0	1

a) $n_0 = 3 \quad , \quad n_1 = 3$

$$S_0 = \frac{1}{n_0-1}\sum_{i=1}^{n_0}(X_i-\bar{X}_0)(X_i-\bar{X}_0)^T = \frac{1}{2}\left[\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1) + \begin{pmatrix}-1\\-2\end{pmatrix}(-1\ \ -2) + \begin{pmatrix}1\\1\end{pmatrix}(1\ \ 1)\right] =$$

$$= \frac{1}{2}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix} + \begin{pmatrix}1&2\\2&4\end{pmatrix} + \begin{pmatrix}1&1\\1&1\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}2&3\\3&6\end{pmatrix}$$

$$S_1 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i-\bar{X}_1)(X_i-\bar{X}_1)^T = \frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix}(1\ \ 1) + \begin{pmatrix}0\\-1\end{pmatrix}(0\ \ -1) + \begin{pmatrix}-1\\0\end{pmatrix}(-1\ \ 0)\right] =$$

$$= \frac{1}{2}\left[\begin{pmatrix}1&1\\1&1\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}2&1\\1&2\end{pmatrix}$$

$$S_p = \frac{(n_0-1)S_0 + (n_1-1)S_1}{n_0+n_1-2} = \frac{1}{4}\left[\begin{pmatrix}2&3\\3&6\end{pmatrix} + \begin{pmatrix}2&1\\1&2\end{pmatrix}\right] = \begin{pmatrix}1&1\\1&2\end{pmatrix} \Longrightarrow S_p^{-1} = \begin{pmatrix}2&-1\\-1&1\end{pmatrix}$$

$$\Longrightarrow \hat{a}_m = S_p^{-1}(\bar{X}_0-\bar{X}_1) = \begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}-2\\-2\end{pmatrix} = \begin{pmatrix}-2\\0\end{pmatrix}$$

b) Nuevo dato $x = (2,7)^T$

$$\left.\begin{aligned}\hat{a}_m^T x &= (-2\ \ 0)\begin{pmatrix}2\\7\end{pmatrix} = -4\\ \hat{a}_m^T\left(\frac{\bar{X}_0+\bar{X}_1}{2}\right) &= (-2\ \ 0)\begin{pmatrix}4\\7\end{pmatrix} = -8\end{aligned}\right\} \text{Como } \hat{a}_m^T x > \hat{a}_m^T\left(\frac{\bar{X}_0+\bar{X}_1}{2}\right) \Longrightarrow \text{clase } 1\ \Pi_1$$

OJO! En la teoría se dice que este '>' indica clase 1, pero NO siempre es así, depende de dónde esté $\bar{x}_0$ y $\bar{x}_1$.

FORMA CORRECTA:

Proyectamos dato: $\hat{a}_m^T . x = -4$

Proyectamos $\bar{x}_0$: $\hat{a}_m^T . \bar{x}_0 = (-2\ \ 0)\begin{pmatrix}3\\6\end{pmatrix} = -6$

Proyectamos $\bar{x}_1$: $\hat{a}_m^T . \bar{x}_1 = (-2\ \ 0)\begin{pmatrix}5\\8\end{pmatrix} = -10$

$\hat{a}_m^T . x$ más cerca de $\hat{a}_m^T . \bar{x}_0$ que de $\hat{a}_m^T . \bar{x}_1$ $\Longrightarrow \boxed{x \text{ clase } 0}$

2.

Recuerdo: teoría

Nueva observación $x = (x_1, \dots, x_K)^T \Longrightarrow \tilde{x} = (1, x_1, \dots, x_K)^T$

Parámetros $\beta = (\beta_0, \beta_1, \dots, \beta_K)^T$

$$\mathbb{P}(Y=1|x) = h(\beta^T \tilde{x}) \xrightarrow{\text{LOGIT}} h(z) = \frac{1}{1+e^{-z}} \quad \text{(sigmoide)}$$

$$\mathbb{P}(Y=1|x) = h(\beta^T \tilde{x}) \xrightarrow[\text{PROBIT}]{} h(z) = \Phi(z) \quad (\mathcal{N}(0,1))$$

a) TRUCO: Usar la expresión de odds: $O(x) = \dfrac{p(x)}{1-p(x)}$

$$\mathbb{P}(Y=1|x) = p(x) = \frac{1}{1+e^{-\beta^T\tilde{x}}}$$

$$O(x) = \frac{p(x)}{1-p(x)} = \frac{1/1+e^{-\beta^T\tilde{x}}}{1-1/1+e^{-\beta^T\tilde{x}}} = e^{\beta^T\tilde{x}} = e^{\beta_0+\beta_1x_1+\cdots+\beta_Kx_K}$$

$$\frac{O(x^{(j)})}{O(x)} = \frac{e^{\beta_0+\beta_1x_1+\cdots+\beta_j(x_j+1)+\cdots+\beta_Kx_K}}{e^{\beta_0+\beta_1x_1+\cdots+\beta_Kx_K}} = \frac{e^{\beta^T\tilde{x}}\cdot e^{\beta_j}}{e^{\beta^T\tilde{x}}} = e^{\beta_j}$$

$$\Longrightarrow \beta_j = \ln\left(\frac{O(x^{(j)})}{O(x)}\right) = \ln\left(\frac{p(x^{(j)})/1-p(x^{(j)})}{p(x)/1-p(x)}\right)$$

b) DATOS: $\beta_j = 2$, $p(x) = 0'3 \Longrightarrow$ despejamos $p(x^{(j)})$

$$e^2 = \frac{p(x^{(j)})/1-p(x^{(j)})}{0'3/1-0'3} \Longrightarrow 3'167 = \frac{p}{1-p} \Longrightarrow 4'167p = 3'167$$

$$\Longrightarrow p(x^{(j)}) = \frac{3'167}{4'167} = 76\%$$

c)

C.a

$$p(x) = \mathbb{P}(Y=1|x) = \Phi(\beta^T \tilde{x}) = \Phi(\beta_0 + \beta_1 x_1 + \cdots + \beta_K x_K) \Longrightarrow$$

$$\Longrightarrow \beta^T \tilde{x} = \Phi^{-1}(p(x))$$

Análogamente, $\beta_0 + \beta_1 x_1 + \cdots + \beta_j(x_j+1) + \cdots + \beta_K x_K = \Phi^{-1}(p(x^{(j)})) \Longrightarrow$

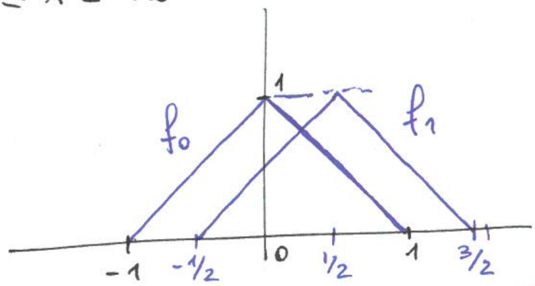
$$\Longrightarrow \underbrace{\beta^T \tilde{x}}_{\Phi^{-1}(p(x))} + \beta_j = \Phi^{-1}(p(x^{(j)})) \Longrightarrow \boxed{\beta_j = \Phi^{-1}(p(x^{(j)})) - \Phi^{-1}(p(x))}$$

C.b $\beta_j = 2, \quad p(x) = 0'3, \quad ¿\, p(x^{(j)})\,?$

$$\overset{\text{excel}}{\Longrightarrow} \Phi^{-1}(0'3) = -0'524$$

$$\Longrightarrow 2 - 0'524 = \Phi^{-1}(p(x^{(j)})) \Longrightarrow p(x^{(j)}) = \Phi(1'476) = 93\%$$

$\boxed{3.}$ a) $X \begin{cases} \xrightarrow{\pi_0} f_0(x) = 1-|x|, \quad |x| \leq 1 \\ \xrightarrow{\pi_1} f_1(x) = 1 - |x - \frac{1}{2}|, \quad -\frac{1}{2} \leq x \leq \frac{3}{2} \end{cases}$



$P_1 = P_0 = 50\%$

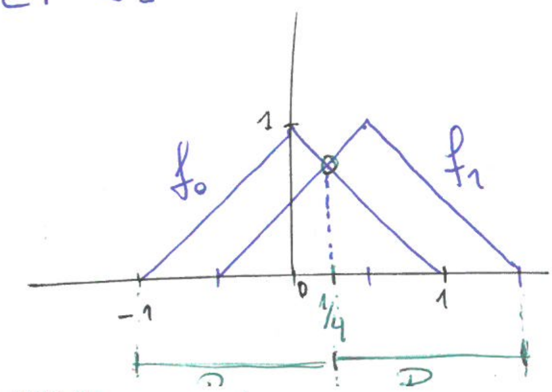
$$R_1 = \{x \in \mathbb{R} : f_1(x) P_1 \geq f_0(x) P_0\}$$

$$f_1 P_1 \geq f_0 P_0 \iff f_1 \geq f_0$$

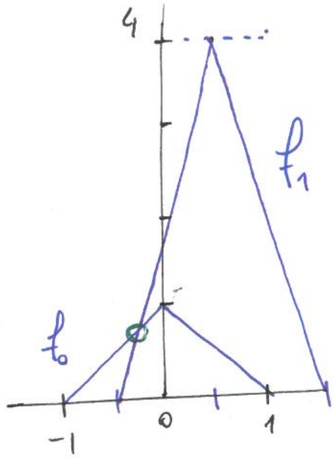
$$f_1 = f_0 \iff 1 - (-(x - \tfrac{1}{2})) = 1 + x - \tfrac{1}{2} = 1 - x \iff 2x = \tfrac{1}{2} \iff \boxed{x = \tfrac{1}{4}}$$

escogemos signo $|\cdot|$ mirando la pendiente correspondiente en el dibujo

Entonces: $R_0 = [-1, \frac{1}{4})$, $R_1 = [\frac{1}{4}, \frac{3}{2}]$



$P_0 = 20\%$, $P_1 = 80\%$



$$R_1 = \{x \in \mathbb{R} : f_1(x)\, P_1 \geq f_0(x)\, P_0\}$$

$$0'8\, f_1 \geq 0'2\, f_0 \iff 4 f_1 \geq f_0$$

$$4f_1 = f_0 \iff 4\left(1 + x - \tfrac{1}{2}\right) = 1 + x$$

el signo del |·| se escoge observando la gráfica
En el punto de corte hay pendiente positiva.

$$\iff 4 + 4x - 2 = 1 + x$$

$$\iff 3x = -1 \iff \boxed{x = \frac{-1}{3}}$$

$$R_0 = \left[-1, -\tfrac{1}{3}\right) \quad , \quad R_1 = \left[\tfrac{-1}{3}, \tfrac{3}{2}\right]$$

b) $P_0 = P_1 = 0'5$, ¿ prob. mala clasificación?

$$\mathbb{P}\binom{\text{mala}}{\text{clasif.}} = P_1 \int_{R_0} f_1(x)\,dx + P_0 \int_{R_1} f_0(x)\,dx$$

$$\int_{R_0} f_1(x)\,dx = \int_{-1/2}^{1/4} \left(1 + x - \tfrac{1}{2}\right) dx = \left[\frac{x^2}{2} + \frac{x}{2}\right]_{-1/2}^{1/4} = \frac{1}{32} + \frac{1}{8} - \frac{1}{8} + \frac{1}{4} = \frac{9}{32}$$

$$\int_{R_1} f_0(x)\,dx = \int_{1/4}^{1} (1 - x)\,dx = \left[x - \frac{x^2}{2}\right]_{1/4}^{1} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{32} = \frac{9}{32}$$

$$\Rightarrow \mathbb{P}\binom{\text{mala}}{\text{clasif.}} = \frac{1}{2} \cdot \frac{9}{32} + \frac{1}{2} \cdot \frac{9}{32} = \boxed{\frac{9}{32} = 0'28125}$$

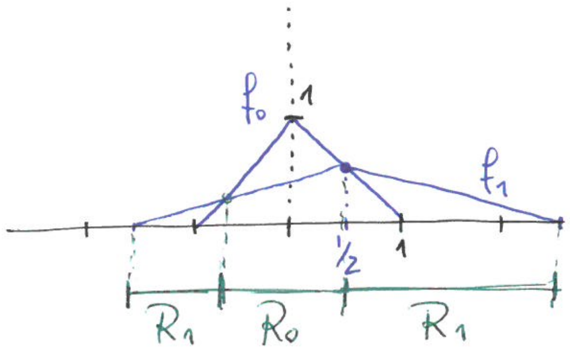
c) $P_0 = P_1 = \frac{1}{2}$

$$X \begin{cases} \xrightarrow{\Pi_0} 1 - |x| \ , \ |x| \leq 1 \\ \xrightarrow[\Pi_1]{} \frac{1}{4}\left(2 - \left|x - \frac{1}{2}\right|\right) \ , \ \frac{-3}{2} \leq x \leq \frac{5}{2} \end{cases}$$

$f_1\left(\frac{-3}{2}\right) = 0$, $f_1\left(\frac{5}{2}\right) = 0$

Buscamos máximo: $\cancel{\frac{1}{4}}\left(\cancel{2} - x + \frac{1}{2}\right) = \cancel{\frac{1}{4}}\left(\cancel{2} + x - \frac{1}{2}\right) \iff 2x = 1 \iff x = \frac{1}{2}$

$f_1(1/2) = 1/2$. Por tanto, $f_1(1/2) = f_0(1/2) = 1/2$



Buscamos 2º pto. de corte:

$f_0 = f_1 \iff 1 + x = \frac{1}{4}\left(2 + x - \frac{1}{2}\right) \iff$

$\iff 4 + 4x = 2 + x - 0'5 \iff 3x = -\frac{5}{2} \iff$

$\iff x = \frac{-5}{6} \approx 0'8\widehat{3}$

$R_0 = \left[\frac{-5}{6}, \frac{1}{2}\right]$, $R_1 = \left[\frac{-3}{2}, \frac{-5}{6}\right) \cup \left(\frac{1}{2}, \frac{5}{2}\right]$

$\boxed{4.}$ $\mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ $\xrightarrow{\Pi_0}$ $\mathbb{X} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\xrightarrow{\Pi_1}$ UNIF(S) , donde $S :=$ rectángulo centrado origen, lado vertical $= 2$, lado horiz $= e\pi$

$P_0 = P_1 \Rightarrow R_1 = \{(x,y) \in \mathbb{R}^2 : f_1(x,y) \geq f_0(x,y)\}$

$$f_1(x,y) \geq f_0(x,y) \iff \underbrace{\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}}_{\text{función densidad normal bidimensional}} \geq \underbrace{\frac{1}{2e\pi}}_{\frac{1}{\text{área}(S)}} \cdot \underbrace{\mathbb{1}_S(x,y)}_{\text{función indicatriz } S} \iff$$

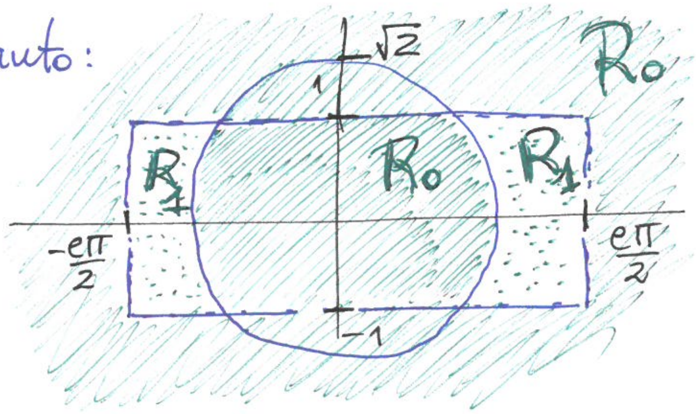
$$\iff e^{-\frac{1}{2}(x^2+y^2)} \geq e^{-1} \cdot \mathbb{1}_S(x,y)$$

- En el rectángulo S: $\mathbb{1}_S(x,y) = 1$

$$-\frac{1}{2}(x^2+y^2) \geq -1 \iff \boxed{x^2 + y^2 \geq 2}$$

- Fuera del rectángulo S: $\mathbb{1}_S(x,y) = 0$ y $\exp(-) \geq 0$

Por tanto:

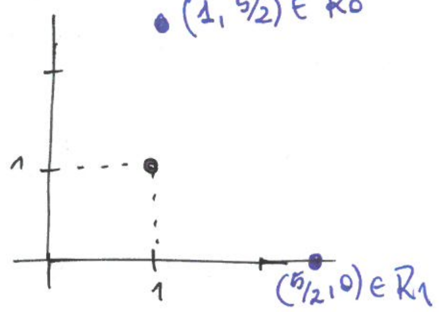


$\boxed{5.}$ $\Pi_0 \longrightarrow X \sim N\left(\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\Pi_1 \longrightarrow X \sim N\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$\left(\frac{5}{2}, 0\right) \in R_1$

$\left(1, \frac{5}{2}\right) \in R_0$



$R_1 = \{(x,y) \in \mathbb{R}^2 : f_1(x) \geq f_0(x)\}$

$R_1 \equiv f_1 \geq f_0 \iff \frac{1}{2\pi} e^{-\frac{1}{2}((x-1)^2+(y-1)^2)} \geq$

$\geq \frac{1}{2\pi} e^{-\frac{1}{2}((x-a)^2+(y-a)^2)} \iff$

$\iff -\cancel{\frac{1}{2}}((x-1)^2+(y-1)^2) \geq -\cancel{\frac{1}{2}}((x-a)^2+(y-a)^2) \iff$

OJO

$\iff (x-1)^2+(y-1)^2 \leq (x-a)^2+(y-a)^2 \iff$

$\iff \cancel{x^2}-2x+1+\cancel{y^2}-2y+1 \leq \cancel{x^2}-2ax+a^2+$
$+\cancel{y^2}-2ay+a^2 \iff$

Viendo el dibujo podemos asumir que $a > 1$, ya que la frontera $R_0 = R_1$ está "alejada" del $(1,1)$

$\iff -x-y+1 \leq -ax-ay+a^2 \iff (a-1)x+(a-1)y \leq a^2-1 \iff$

$\iff \cancel{(a-1)}(x+y) \leq \cancel{(a-1)}(a+1) \iff x+y \leq a+1 \iff \boxed{y \leq -x+a+1}$

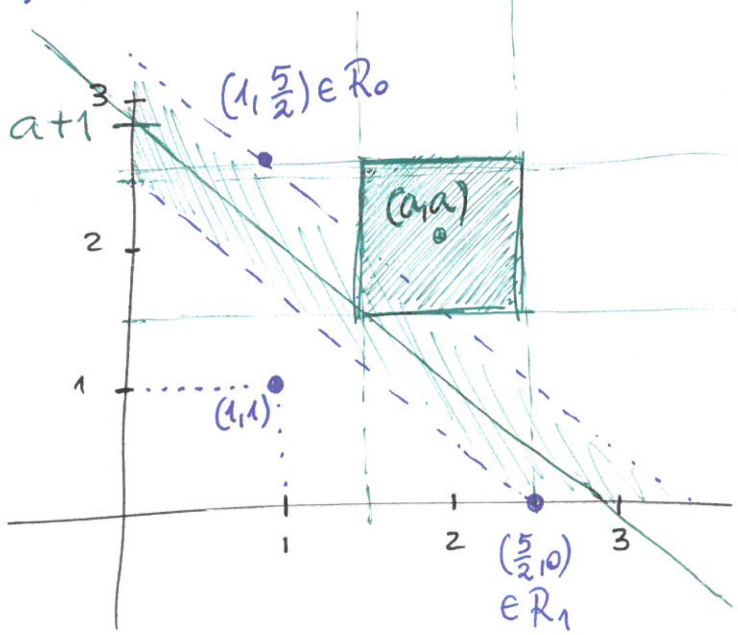
$a > 1$
$a-1 > 0$

recta (la frontera) de pendiente -1

$\left(\frac{5}{2}, 0\right) \in R_1 \implies 0 \leq -2'5+a+1 \implies a \geq 1'5$

$\left(1, \frac{5}{2}\right) \in R_0 \implies 2'5 \geq -1+a+1 \implies a \leq 2'5$

$\boxed{a \in \left[\frac{3}{2}, \frac{5}{2}\right]}$



recta $y = -x+a+1$

$\boxed{6.}$ $f_0(x,y) = e^{-x-y}$, $x,y>0$ $\quad P_1 = P_0$

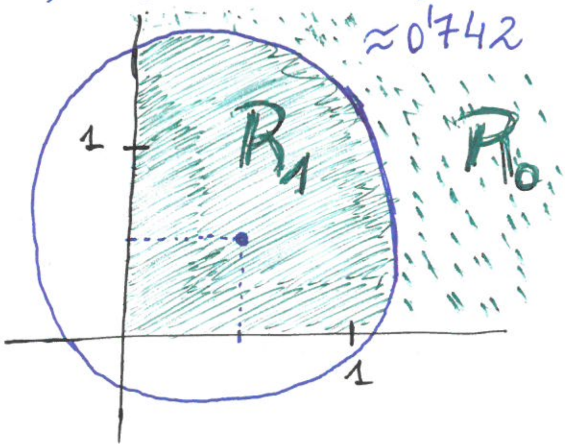
$f_1(x,y) = \frac{4}{\pi} e^{-(x^2+y^2)}$, $x,y>0$

$R_1 = \{(x,y) \in \mathbb{R}^2_+ : f_1(x,y) \geq f_0(x,y)\}$

$f_1 \geq f_0 \iff \frac{4}{\pi} e^{-x^2-y^2} \geq e^{-x-y} \iff \ln\left(\frac{4}{\pi}\right) - x^2 - y^2 \geq -x-y$

$\iff x^2+y^2-x-y \leq \ln\left(\frac{4}{\pi}\right) \underset{\substack{\uparrow \\ \text{completar cuadrados}}}{\iff} \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \leq \underbrace{\ln\left(\frac{4}{\pi}\right)+\frac{1}{2}}_{\approx 0'742}$

La frontera es una circunferencia de radio $\sqrt{0'742}$ centrada en $\left(\frac{1}{2},\frac{1}{2}\right)$ restringida al primer cuadrante de $\mathbb{R}^2$.



$\boxed{7.}$ $\mathbb{X} = (X_1, X_2, X_3)^T$ en Π_0 y Π_1.

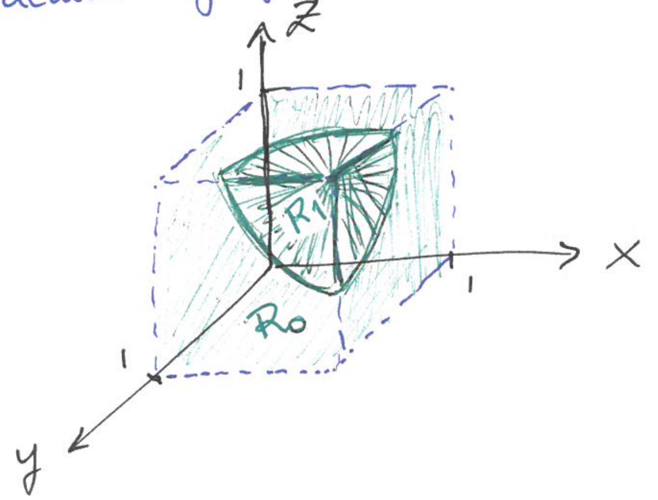
$\Pi_0 \longrightarrow f_0(x,y,z) = \mathbb{1}_C(x,y,z)$, con $C := [0,1]^3$ (cubo unidad)

$\Pi_1 \longrightarrow f_1(x,y,z) = 12x^2yz$, $0 \leq x,y,z \leq 1$

$R_1 = \{(x,y,z) \in [0,1]^3 : f_1(x,y,z) \geq f_0(x,y,z)\} =$

$= \{(x,y,z) \in [0,1]^3 : 12x^2yz \geq 1\}$

Podemos graficar dicha región usando algún software.

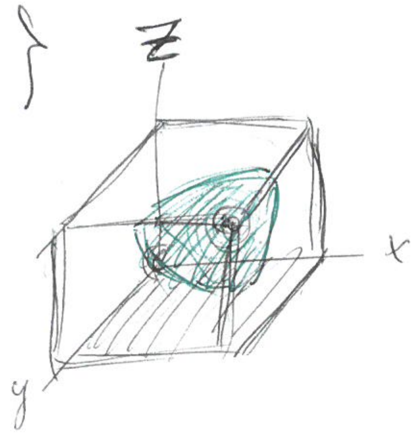


$\boxed{7.}$ $\quad \mathbb{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

En $\Pi_0 \longrightarrow f_0(x,y,z)$ es la indicadora del cubo unidad

En $\Pi_1 \longrightarrow f_1(x,y,z) = 12x^2yz \qquad 0 \leq x,y,z \leq 1$

$R_1 \longrightarrow \{ \underset{0}{\overset{\vee\!\shortmid}{x,y,z}} \leq 1 : \; 12x^2yz \geq 1 \}$



A partir de ahora los ejercicios son más teóricos pero "más interesantes".

$\boxed{8.}$ $\quad \mathbb{X} = \begin{pmatrix} X_1 \\ | \\ X_K \end{pmatrix}$ → en $\Pi_0 \longrightarrow N(\mu_0, \Sigma)$; → en $\Pi_1 \longrightarrow N(\mu_1, \Sigma)$

$P_0 = P_1 = \frac{1}{2}$

Llamamos $Y = (\mu_1 - \mu_0)^T \Sigma^{-1} \mathbb{X}$, Y es normal

$\mathbb{E}(Y) =$ → $(\mu_1 - \mu_0)^T \Sigma^{-1} \mu_0$ en Π_0 ; → $(\mu_1 - \mu_0)^T \Sigma^{-1} \mu_1$ en Π_1

$$V(Y) = (\mu_1 - \mu_0)^T \Sigma^{-1} \Sigma \Sigma^{-1} (\mu_1 - \mu_0) = \overbrace{(\mu_1 - \mu_0)^T \Sigma^{-1} (\mu_1 - \mu_0)}^{\Delta^2}$$

$$\mathbb{P}\begin{pmatrix}\text{mala}\\ \text{clasif.}\end{pmatrix} = \underbrace{\frac{1}{2}\int_{R_0} f_1(\vec{x})\,d\vec{x}}_{\|} + \frac{1}{2}\int_{R_1} f_0(\vec{x})\,d\vec{x}$$

$$\mathbb{P}\left(Y < \frac{1}{2}(\mu_1-\mu_0)^T\Sigma^{-1}(\mu_1+\mu_0)\right)$$

con media la de Π_1

$$\|$$

$$\mathbb{P}\left(\frac{Y-(\mu_1-\mu_0)^T\Sigma^{-1}\mu_1}{\Delta} < \frac{1}{\Delta}\overbrace{\left[(\mu_1-\mu_0)^T\Sigma^{-1}\frac{1}{2}(\mu_0-\mu_1)\right]}^{-\frac{1}{2}\Delta^2}\right)$$

$$\|$$

$$\mathbb{P}\left(N(0,1) \leq -\frac{\Delta}{2}\right) = \Phi\left(-\frac{\Delta}{2}\right) = 1-\Phi\left(\frac{\Delta}{2}\right)$$

$\boxed{9.}$ $\mathbb{X} = \begin{pmatrix}X_1\\ \vdots\\ X_K\end{pmatrix}$ $\qquad \begin{matrix}\Pi_1 & f_1(\vec{x})\\ \vdots & \vdots\\ \Pi_m & f_m(\vec{x})\end{matrix}$

$$g \xleftarrow{\text{regla}} \longrightarrow \Omega \subset \mathbb{R}^K = \bigcup_{i=1}^{m} R_i^{(g)}$$

$$\mathbb{P}\begin{pmatrix}\text{mala}\\ \text{clasif.}\end{pmatrix} = \sum_{i=1}^{m} \underbrace{\mathbb{P}\left(\begin{matrix}\text{mala}\\ \text{clasif.}\end{matrix}\,\middle|\,\Pi_i\right)}_{\| \atop \sum_{j\neq i}\int_{R_j} f_i(\vec{x})\,d\vec{x}} \underbrace{\mathbb{P}(\Pi_i)}_{P_i}$$

<u>Simplificación</u> : Argumento bayesiano

Tenemos observación x : $\mathbb{P}(\Pi_i|x) = \underbrace{\mathbb{P}(x|\Pi_i)}_{f_i} \dfrac{\overbrace{\mathbb{P}(\Pi_i)}^{P_i}}{\mathbb{P}(x)}$

Clasificamos x en Π_i si $P_i f_i(\vec{x}) \geq P_j f_j(\vec{x})$ $\quad j \neq i$

Si Normal:

$$\ln\left(P_i f_i(\vec{x})\right) = \ln P_i - \frac{1}{2}\ln|\Sigma_i| - \frac{K}{2}\ln(2\pi) -$$

$$- \frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i) \qquad i=1,\dots,m$$