CÁLCULO 1: 1^0 del grado de Matemáticas y doble grado MAT-IngINF

Hoja 7: Polinomios de Taylor.

1.- Hallar el polinomio de Taylor de grado 4 de las siguientes funciones:

(a) $f(x) = \cos x$ en $a = \frac{\pi}{4}$ (b) $f(x) = \log x$ en $a = 1$ (c) $f(x) = x^{\frac{1}{2}}$ en $a = 1$

(d) $f(x) = \dfrac{1}{1+x^2}$ en $a = 0$ (e) $f(x) = \dfrac{1}{1+x}$ en $a = 0$ (f) $f(x) = \arctan x$ en $a = 0$

(g) $f(x) = x^5$ en $a = 3$ (h) $f(x) = \dfrac{e^x}{1+x^2}$ en $a = 0$ (i) $f(x) = \log(1+x)$ en $a = 0$

(j) $f(x) = 3 + (x-1) + 2\,(x-1)^2 + 5\,(x-1)^3$ en $a = 0$

2.- Calcular los siguientes límites utilizando el polinomio de Taylor:

$$\lim_{x\to 0} \frac{(x - \operatorname{sen} x)^4}{(\log(1+x) - x)^6}, \qquad \lim_{x\to 0} \frac{e^{-x} - 1 + x}{\cos(2\,x) - 1}.$$

3.- Probar que para $x > 0$ se cumple

$$1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}.$$

4.- Probar que para $x > 0$ se cumple

$$x - \frac{x^2}{2} < \log(1+x) < x.$$

5.- Probar que para $x > 0$ se cumple

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} \le e^{-x} \le 1 - x + \frac{x^2}{2}.$$

6.- Sea f una función 4 veces derivable en un intervalo alrededor del 0. Supongamos que

$$\lim_{x\to 0} \frac{f(x) - 1 + 3\,x - 5\,x^2}{x^3} = 0.$$

Calcular $f(0)$, $f'(0)$, $f''(0)$ y $f'''(0)$.

7.- Usando la función $y = \arctan x$, calcular π con un error menor que 10^{-3}.

8.- Calcular $\cos(1)$ con un error menor que 10^{-3}.

$\boxed{\text{HOJA 7}}$

$\boxed{1.}$

a) P_4 de $f(x) = \cos x$, $a = \frac{\pi}{4}$

$f'(x) = -\operatorname{sen} x$

$f''(x) = -\cos x$

$f'''(x) = \operatorname{sen} x$

$f^{IV}(x) = \cos x$

$$P_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f^{IV}(x)\,(x-a)^4}{4!}$$

$$P_4\left(x, \frac{\pi}{2}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^2}{2} + $$

$$+ \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^3}{6} + \frac{\sqrt{2}}{2}\frac{(x-\pi/4)^4}{4!}$$

$\boxed{2.}$

a) $\lim\limits_{x\to 0} \frac{(x - \operatorname{sen} x)^4}{(\log(1+x) - x)^6}$

$f(x) = \operatorname{sen} x$

$f'(x) = \cos x$

$f''(x) = -\operatorname{sen} x$

$f'''(x) = -\cos x$

$f^{IV}(x) = \operatorname{sen} x$

$f^{V}(x) = \cos x$

$$f(x) = f(0) + f'(0)x + \cancel{\frac{f''(0)x^2}{2}}^{\,0} + \frac{f'''(0)x^3}{3!} + \cancel{\frac{f^{IV}(0)x^4}{4!}}^{\,0} + \frac{f^{V}(0)x^5}{5!} =$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$g(x) = \log(1+x)$

$g'(x) = \frac{1}{1+x}$

$g''(x) = -\frac{1}{(1+x)^2}$

$g'''(x) = \frac{2}{(1+x)^3}$

$$g(x) = x - \frac{x^2}{2} + \frac{2x^3}{3!} = x - \frac{x^2}{2} + \frac{x^3}{3}$$

O GRANDE

$$\lim_{x\to 0} \frac{(x-\operatorname{sen} x)^4}{(\log(1-x)-x)^6} = \lim_{x\to 0} \frac{\left(x - \left(x - \frac{x^3}{3!} + O(x^5)\right)\right)^4}{\left(x - \frac{x^2}{2} + O(x^3) - x\right)^6} =$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^3}{3!} - O(x^5)\right)^4}{\left(-\frac{x^2}{2} + O(x^3)\right)^6} = \lim_{x\to 0} \frac{\cancel{x^{12}}\left(\frac{1}{6} - \overbrace{O(x^2)}^{\text{tiende a cero}}\right)^4}{\cancel{x^{12}}\left(-\frac{1}{2} + \underbrace{O(x^1)}_{\text{tiende a cero}}\right)^6} =$$

$$= \frac{1/6^4}{1/2^6} = \frac{2^6}{6^4}$$

$\boxed{3.}$ $f(x) = \sqrt{1+x}$ $\forall x > 0$

$$f'(x) = \frac{1}{2\sqrt{1+x}} = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$$

$$f(x) = f(0) + f'(0)x + R_1(x,0) = 1 + \frac{x}{2} + \frac{f''(c)\,x^2}{2!}$$ para c cercano a 0. $c > 0$

$$R_1(x,0) = \frac{-1}{2!\,4}(1+c)^{-3/2}x^2 < 0$$ para c cercano a 0.

$$R_1(x,0) > \frac{-1}{2!\,4}x^2$$

$\boxed{8.}$ a) Calcular $\cos(1)$ con un error $< 10^{-3}$.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + R_n(x,0)$$

$$R_n(x,0) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$, c cercano a 0.

$$\cos(1) = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \dots + R_n(1,0).$$

$$|R_n(1,0)| < 10^{-3}$$

$$\parallel$$

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| < \frac{1}{(n+1)!} < \frac{1}{10^3}$$

HOJA 7

$\boxed{2.}$ a) $\lim_{x \to 0} \frac{(x - \operatorname{sen} x)^4}{(\log(1+x) - x)^6}$

Taylor de $f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = \frac{1}{(1+x)^2} = -(1+x)^{-2}$

$f'''(x) = 2(1+x)^{-3}$

$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}$

$\log(1+x) = 0 + 1x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3 \cdot 2}{4!}x^4 + $

$+ \frac{4!}{5!}x^5 + \dots + \frac{(-1)^{n+1}(n-1)!}{n!}x^n + (-1)^{n+2} \cdot \frac{n!}{(n+1)!}(1+c)^{-(}$

Entonces:

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots +$

$+ \frac{(-1)^{n+1} x^n}{n} + \frac{(-1)^{n+2}}{(n+1)} \cdot \frac{x^{n+1}}{(1+c)^{n+1}}$

Vamos a intentar acotar el resto:

$$\left| \frac{(-1)^{n+2}}{(n+1)} \cdot \frac{x^{n+1}}{(1+c)^{n+1}} \right| = \left| \frac{1}{(n+1)} \cdot \frac{x^{n+1}}{(1+c)^{n+1}} \right|$$

- Si $|x|<1 \longrightarrow \left|\frac{x}{1+c}\right| < 1 \longrightarrow$ Resto tiende a 0.

$$\log(1+x) = \underbrace{\sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k}}_{An} + \underbrace{\frac{(-1)^{n+2}}{(n+1)} \cdot \frac{x^{n+1}}{(1+c)^{n+1}}}_{Bn}$$

- Si $0<x<1 \longrightarrow \lim_{x \to \infty} Bn = 0$, es decir:

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

- para $x>1$ todo lo anterior no vale. La serie diverge.

$$\lim_{x\to 0}\frac{(x-\operatorname{sen}x)'}{(\log(1+x)-x)^6}$$

$$f(x) - T_{n,a}(x) = o((x-a)^n)$$

↑ o pequeña

$$\lim_{x\to a}\frac{o((x-a)^n)}{(x-a)^n}=0$$

$$\operatorname{sen}x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

$$\operatorname{sen}x = x - \frac{x^3}{3!} + o(x^3)$$

$$\operatorname{sen}x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$x - \operatorname{sen}x = x - \left(x - \frac{x^3}{3!} + o(x^3)\right) = \frac{x^3}{3!} - o(x^3)$$

$$\underbrace{\log(1+x)}_{} - x = \underbrace{x - \frac{x^2}{2} + o(x^2)}_{} - x = \frac{-x^2}{2} + o(x^2)$$

$$\frac{(x-\operatorname{sen}x)^4}{(\log(1+x)-x)^6} = \frac{\left(\frac{x^3}{3!} - o(x^3)\right)^4}{\left(-\frac{x^2}{2}+o(x^2)\right)^6} = \frac{\left[x^3\left(\frac{1}{3!} - \frac{o(x^3)}{x^3}\right)\right]^4}{\left(x^2\left[-\frac{1}{2} + \frac{o(x^2)}{x^2}\right]\right)^6} = \frac{\left(\frac{1}{3!} - \frac{o(x^3)}{x^3}\right)^4}{\left(-\frac{1}{2} + \frac{o(x^2)}{x^2}\right)^6}$$

$$\Rightarrow \text{Cuando } x\to 0 \quad \frac{(1/3!)^4}{(-1/2)^6}$$

Sea f derivable 4 veces en un entorno alrededor de 0.

Se sabe que $\displaystyle\lim_{x\to 0}\frac{f(x) - 1 + 3x - 5x^2 + 0.x^3}{x^3} = 0$

¿$f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$?

$$\lim_{x\to 0}\frac{f(x) - (1 - 3x + 5x^2 - 0.x)}{x^3} = 0 \Rightarrow 1 - 3x + 5x^2 - 0.x^3 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$\boxed{f(0) = 1 \;;\; f'(0) = -3 \;;\; f''(0) = 2!.5 = 10 \;;\; f'''(0) = 0}$$

$x>0$

$$1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$$

$$f(x) = \sqrt{1+x} = (1+x)^{1/2}$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$$

$$f''(c) = -\frac{1}{4}(1+c)^{-3/2}$$

$$f'''(c) = \frac{3}{8}(1+c)^{-5/2}$$

$$f(0) = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}(1+c)^{-3/2} \le 1 + \frac{1}{2}x$$

$$f'(0) = \frac{1}{2} \;;\; f''(0) = -\frac{1}{4}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4.2!}x^2 + \underbrace{\frac{3}{8}(1+c)^{-5/2}}_{>0} \Rightarrow$$

$$\Rightarrow \sqrt{1+x} \ge 1 + \frac{1}{2}x - \frac{1}{8}x^2$$