ÁLGEBRA LINEAL

Hoja 1: Matrices y Sistemas Lineales

1. Resuelve los siguientes sistemas mediante el método de eliminación de Gauss.

i) $\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 7 \\ 3x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - 5x_2 + 2x_3 &= 6 \end{aligned}\right\}$ ii) $\left.\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \\ 2x_1 - 2x_3 &= 0 \end{aligned}\right\}$

iii) $\left.\begin{aligned} x + y + z + t &= 0 \\ y - z &= 5 \\ x + z + 2t &= 1 \\ x + 2y &= 0 \end{aligned}\right\}$ iv) $\left.\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 - x_3 &= -2 \\ 3x_1 + 4x_2 + 3x_3 &= 0 \end{aligned}\right\}$

v) $\left.\begin{array}{rcr|r} 2x_1 - 4x_2 & = & 10 & -8 \\ x_1 - 3x_2 + x_4 & = & -4 & -2 \\ x_1 - x_3 + 2x_4 & = & 4 & 9 \\ 3x_1 - 4x_2 + 3x_3 - x_4 & = & -11 & -15 \end{array}\right\}$

vi) $\left.\begin{aligned} x_1 + x_2 &= 5 \\ x_1 - x_2 &= 3 + 6i \end{aligned}\right\}$ vii) $\left.\begin{aligned} x + y + iz + t &= 0 \\ 2x - y + 2z - t &= 1 \\ x + iy - z + it &= 2 \\ x + y + z - t &= 0 \end{aligned}\right\}$

viii) $\left.\begin{array}{rcr|r} 2x_1 - 4x_2 & = & 10 & -8 \\ x_1 - 3x_2 + x_4 & = & -4 & -2 \\ x_1 - x_3 + 2x_4 & = & 4 & 9 \end{array}\right\}$ ix) $\left.\begin{array}{rcr|r} 2x_1 - 4x_2 & = & 10 & -8 \\ x_1 - 3x_2 & = & -4 & -2 \\ x_1 - x_3 & = & 4 & 9 \\ 4x_1 - 7x_2 - x_3 & = & 10 & -15 \end{array}\right\}$

Soluciones: i)$\{x_1 = 2, x_2 = 8, x_3 = 21\}$, ii)$\{x_3 = 0, x_2 = 0, x_1 = 0\}$,
iii) $\{z = -1, y = 4, t = 5, x = -8\}$, iv)$\{x_2 = 0, x_3 = 1, x_1 = -1\}$
v)$\{x_4 = -\frac{101}{13}, x_3 = -\frac{157}{13}, x_1 = \frac{97}{13}, x_2 = \frac{16}{13}\}$ y $\{x_1 = 0, x_3 = -1, x_4 = 4, x_2 = 2\}$
vi)$\{x_1 = 4 + 3i, x_2 = 1 - 3i\}$ vii)$\{x = 1 + i, y = -\frac{7}{10} + \frac{1}{10}i, z = -\frac{7}{5} - \frac{4}{5}i, t = -\frac{11}{10} + \frac{3}{10}i\}$
viii)$\{x_1 = 23 + 2x_4, x_2 = 9 + x_4, x_3 = 19 + 4x_4\}$ y
$\{x_1 = -8 + 2x_4, x_2 = -2 + x_4, x_3 = -17 + 4x_4\}$
ix) $\{x_1 = 23, x_2 = 9, x_3 = 19\}$ y no hay solución respectivamente.

2. Calcula, si existe, la inversa de la matriz A en los siguientes casos

$$\text{i)}A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}, \quad \text{ii)}A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ 3 & 8 & -5 \end{pmatrix}, \quad \text{iii)}A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}$$

i.e. encuentra una matriz $B = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$ tal que $AB = I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solución:

i) $B = \begin{pmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{pmatrix}$ ii) no existe, iii) $B = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

3. Sea A la matriz

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Encuentra el valor de A^n y demuestra el resultado utilizando el método de inducción.

HOJA 1: MATRICES Y SISTEMAS LINEALES

Método recomendado para calcular matrices inversas: Gauss

$$(A \mid I_{n\times n}) \rightsquigarrow \cdots \rightsquigarrow (I \mid A^{-1})$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad ; \quad A^{-1} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

$$A \cdot A^{-1} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3, & a_{11}y_1 + a_{12}y_2 + a_{13}y_3, & a_{11}z_1 + a_{12}z_2 + a_{13}z_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3, & a_{21}y_1 + a_{22}y_2 + a_{23}y_3, & a_{21}z_1 + a_{22}z_2 + a_{23}z_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3, & a_{31}y_1 + a_{32}y_2 + a_{33}y_3, & a_{31}z_1 + a_{32}z_2 + a_{33}z_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; ; \; A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; A\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$A, B \in \mathcal{M}_{n\times n}$ $\qquad$ $A^{-1}B$? $\qquad$ $A\underbrace{\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}}_{A^{-1}\cdot B} = B$

$$(A \mid B) \rightsquigarrow \cdots \rightsquigarrow (I \mid A^{-1}\cdot B)$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad ; \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{pmatrix} \quad ; \quad (A^T)^T = A$$

$$\boxed{(AB)^T = B^T \cdot A^T} \qquad \boxed{\det A = \det(A^T)} \qquad \boxed{(A^{-1})^T = (A^T)^{-1}}$$

$$\boxed{I = (A.A^{-1})^T = (A^{-1})^T . A^T \quad \rightsquigarrow (A^{-1})^T = (A^T)^{-1}}$$

• ¿Cómo calcular $B.A^{-1}$?

$$C = B.A^{-1} \quad ; \quad C^T = (B.A^{-1})^T = (A^{-1})^T . B^T = (A^T)^{-1} . B^T$$

$$\left(A^T \middle| B^T\right) \rightsquigarrow \cdots \rightsquigarrow \left(I \middle| \underbrace{(BA^{-1})^T}\right)$$

$$\rightarrow \left((B.A^{-1})^T\right)^T = B.A^{-1}$$

$\boxed{1.}$ vii)

$$\left.\begin{aligned} x + y + iz + t &= 0 \\ 2x - y + 2z - t &= 1 \\ x + iy - z + it &= 2 \\ x + y + z - t &= 0 \end{aligned}\right\} \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & i & 1 & 0 \\ 2 & -1 & 2 & -1 & 1 \\ 1 & i & -1 & i & 2 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \Leftrightarrow$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & i & 1 & 0 \\ 0 & -3 & 2-2i & -3 & 1 \\ 0 & i-1 & -1-i & i-1 & 2 \\ 0 & 0 & 1-i & -2 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 1 & i & 1 & 0 \\ 0 & 1 & \frac{-2+2i}{3} & 1 & -1/3 \\ 0 & i-1 & -i-1 & -1+i & 2 \\ 0 & 0 & 1-i & -2 & 0 \end{array}\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\begin{array}{cccc|c} 1 & 1 & i & 1 & 0 \\ 0 & 1 & \frac{-2+2i}{3} & 1 & -1/3 \\ 0 & 0 & -1+\frac{7}{3}i & 0 & i-\frac{7}{3} \\ 0 & 0 & 1-i & -2 & 0 \end{array}\right)$$

$$\to (-1 + \tfrac{1}{3}i)z = i - \tfrac{7}{3} \Rightarrow z = \frac{i - 7/3}{-1 + \frac{1}{3}i}$$

$$\to (1-i)z - 2t = 0 \Rightarrow t = \frac{(1-i)z}{2}$$

$$\to y + \frac{-2+2i}{3}z + t = -1/3 \Rightarrow y = \frac{2-2i}{3}z - t - 1/3$$

$$\to x = -y - iz - t$$

1. ix)

$$\left.\begin{array}{l} 2x_1 - 4x^2 = 10 \;;\; -8 \\ x_1 - 3x_2 = -4 \;;\; -2 \\ x_1 - x_3 = 4 \;;\; 9 \\ 4x_1 - 7x_2 - x_3 = 10 \;;\; -15 \end{array}\right\} \iff \left(\begin{array}{ccc|c|c} 2 & -4 & 0 & 10 & -8 \\ 1 & -3 & 0 & -4 & -2 \\ 1 & 0 & -1 & 4 & 9 \\ 4 & -7 & -1 & 10 & -15 \end{array}\right) \iff$$

$$\iff \left(\begin{array}{ccc|c|c} 1 & -2 & 0 & 5 & -4 \\ 0 & -1 & 0 & -9 & 2 \\ 0 & 2 & -1 & -1 & 13 \\ 0 & 1 & -1 & -10 & 1 \end{array}\right) \iff \left(\begin{array}{ccc|c|c} 1 & -2 & 0 & 5 & -4 \\ 0 & -1 & 0 & -9 & 2 \\ 0 & 0 & -1 & -19 & 17 \\ 0 & 0 & -1 & -19 & 3 \end{array}\right)$$

¿$x_3 = 17 \wedge x_3 = 3$? No es posible, segundo sistema sin solución

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 0 & 5 \\ 0 & -1 & 0 & -9 \\ 0 & 0 & -1 & -19 \end{array}\right) \Rightarrow \begin{cases} x_3 = 19 \\ x_2 = 9 \\ x_1 = 5 + 2x_2 = 23 \end{cases}$$

$\boxed{3.}$ Calcular A^n, $n \in \mathbb{N}$ $\qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Si soy capaz de escribir:

diagonalizar $\rightarrow A = PDP^{-1}$, donde D es una matriz diagonal,

entonces es fácil:

$$A^n = \underbrace{PD\overbrace{P^{-1}.P}^{I}D\overbrace{P^{-1}.\ldots.P}^{I}DP^{-1}}_{n \text{ veces}} = P.D^n.P^{-1}$$

Si no se puede diagonilar $\rightarrow$
$\rightarrow$ FORMA DE JORDAN

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix} \quad ; \quad D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix}$$

– SISTEMAS DINÁMICOS

$X_{n+1} = A.X_n$, $n = 1, 2, 3, \ldots$

$X_n = A^n \underbrace{X_0}_{\rightarrow \text{ estado inicial}}$

$\lim\limits_{n \to \infty} A^n$ es interesante

¿Por qué estudiar las potencias de una matriz?

Este es un ejemplo