Bayesian Inference and Deep Learning

Richard E. Turner (ret26@cam.ac.uk)

Machine Learning Group, University of Cambridge

Robust Deep Learning

fragile (adversarial examples)

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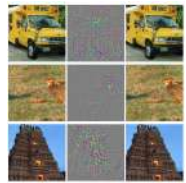
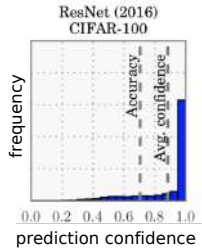
well calibrated uncertainty estimates:
deep learning is often confidently wrong





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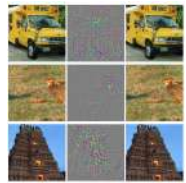

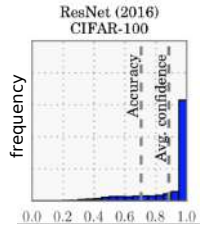

Data-Efficient Deep Learning

small data, big models (few-shot learning
and reinforcement learning)

leverage heterogenous data sources (multi-task learning)

Limitations of Deep Learning

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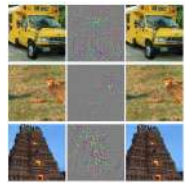

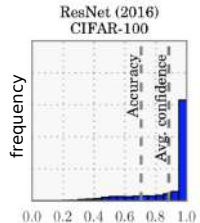

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continual learning (online learning & model building)

active learning (RL exploration-exploitation)

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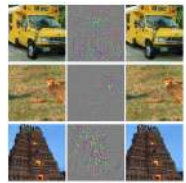

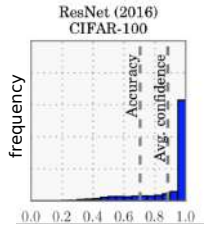

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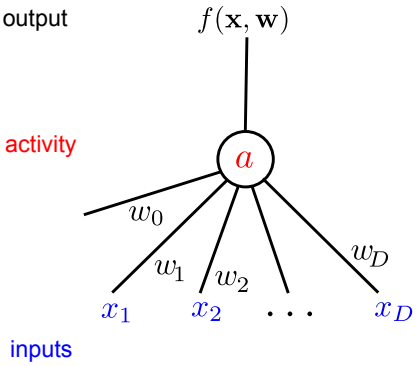
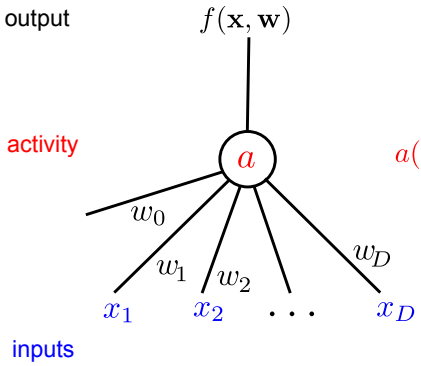
probabilistic modelling
+
probabilistic inference

Logistic Regression as a motivating example

Logistic regression: A single neuron



Logistic regression: A single neuron



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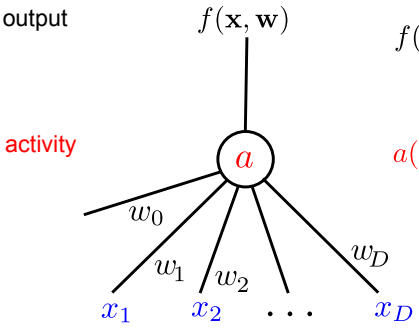

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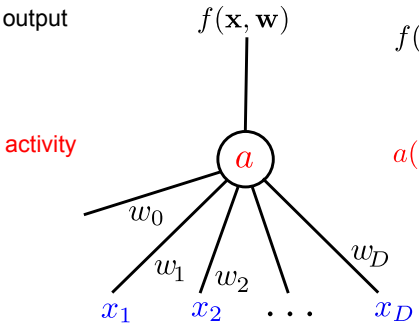

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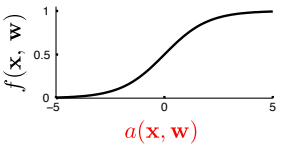

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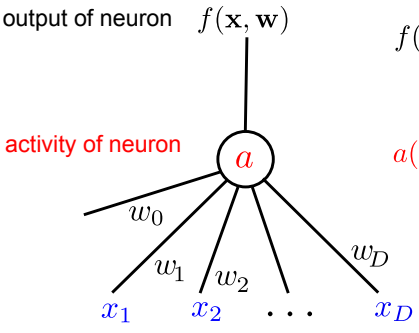

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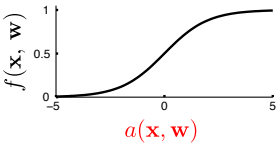

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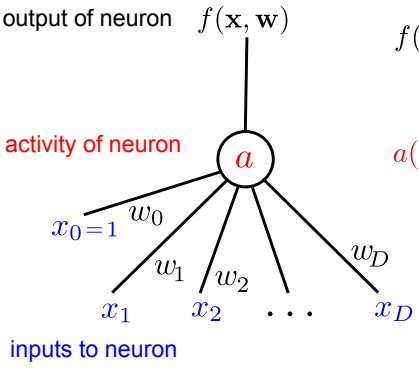

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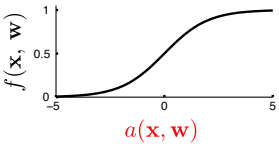

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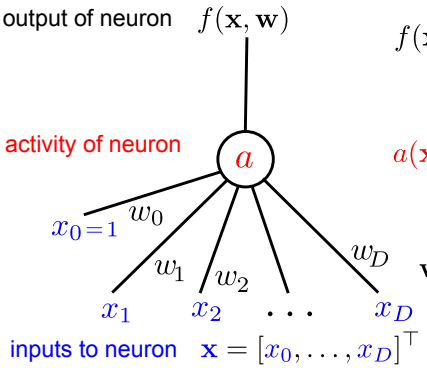

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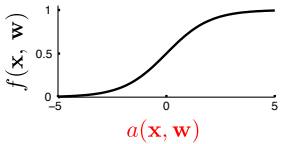

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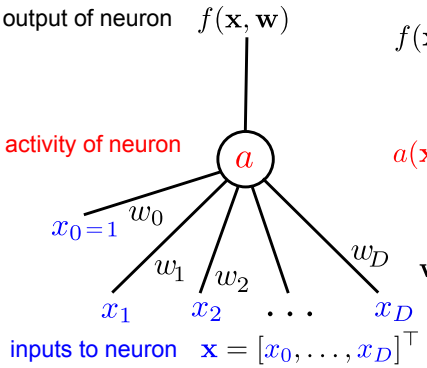

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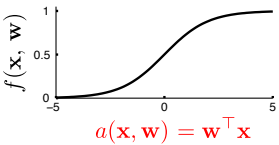

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Logistic regression: Maximum Likelihood Estimation

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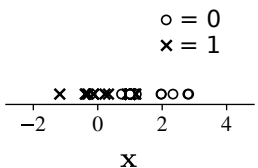

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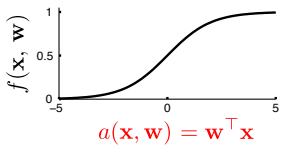

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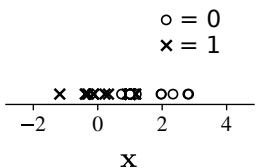

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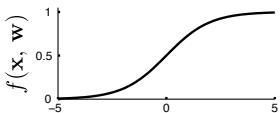

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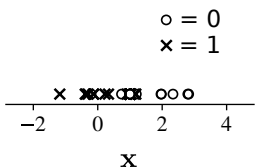

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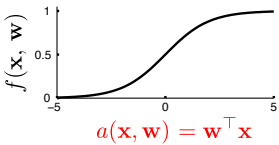

maximum likelihood estimate: parameters that make observed data most probable

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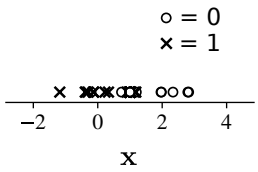

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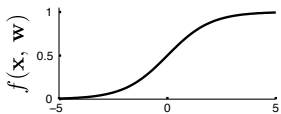

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$$P(\mathcal{Y}|\mathbf{w}, \mathcal{X}) = \prod_{n=1}^{N} P(y_n|\mathbf{w}, \mathbf{x}_n) = \prod_{n=1}^{N} f(\mathbf{w}^\top \mathbf{x}_n)^{y_n} (1 - f(\mathbf{w}^\top \mathbf{x}_n))^{(1-y_n)}$$

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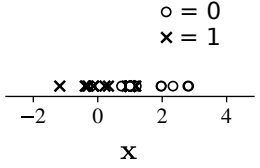

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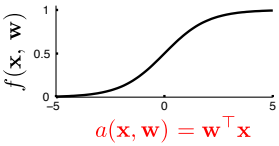

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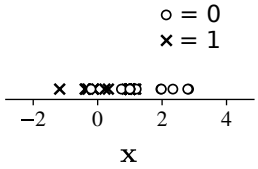

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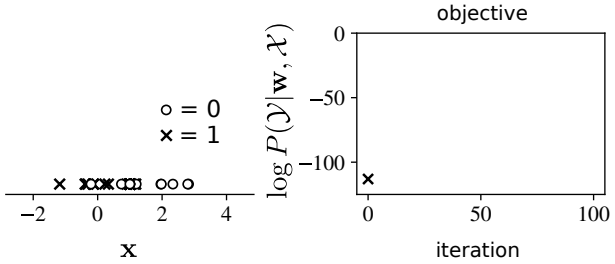

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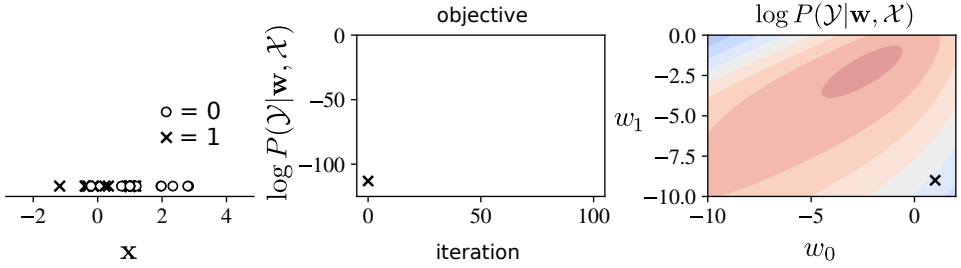

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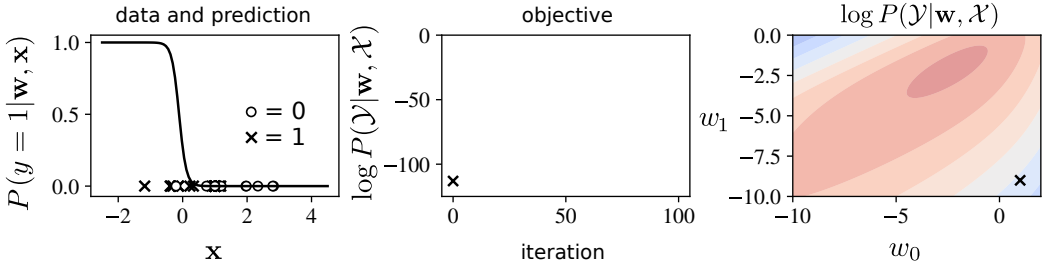
Logistic regression: Maximum Likelihood learning

observe data: estimate parameters



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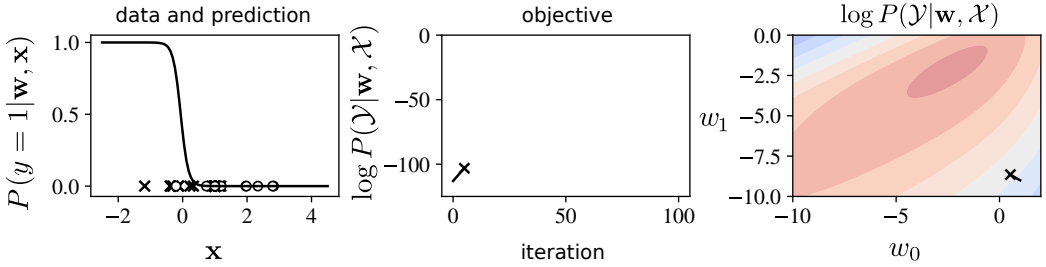

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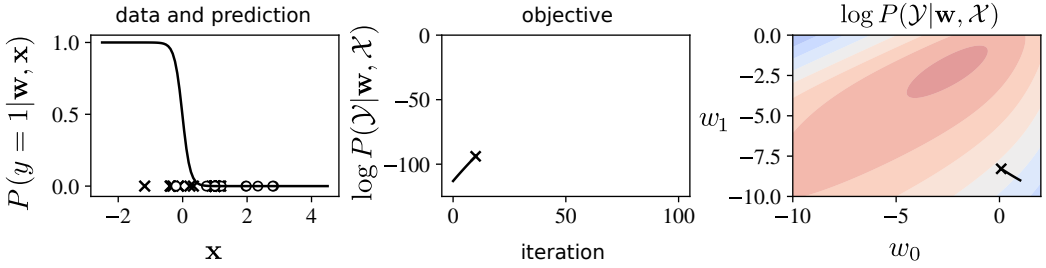

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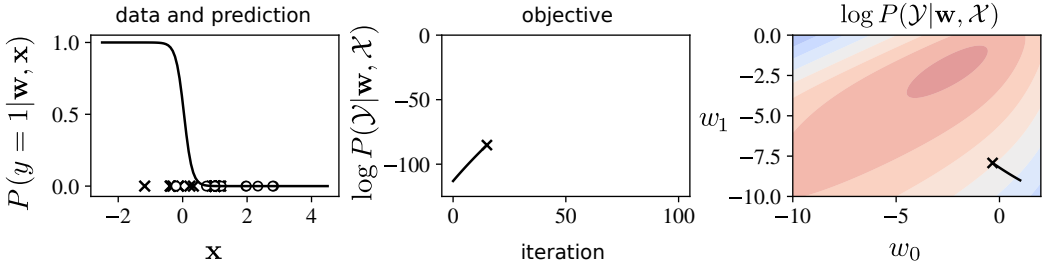

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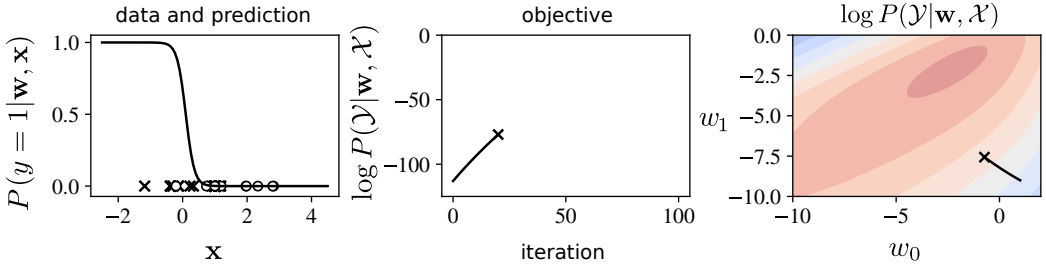

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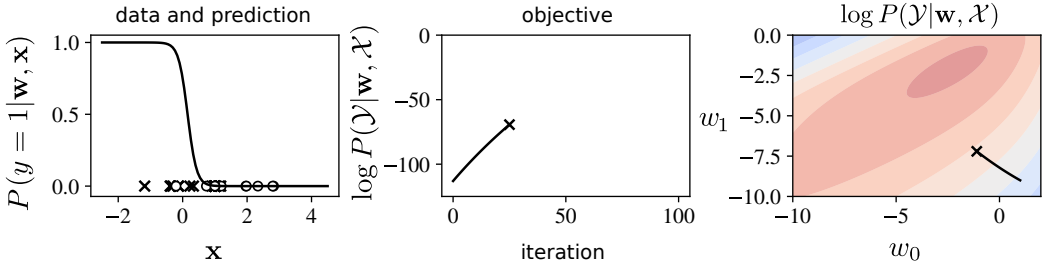

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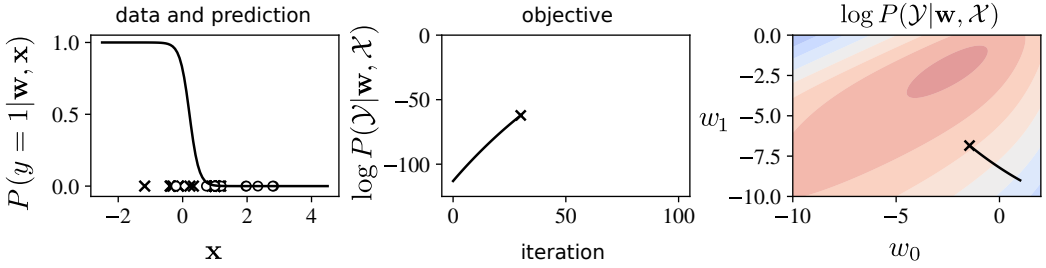

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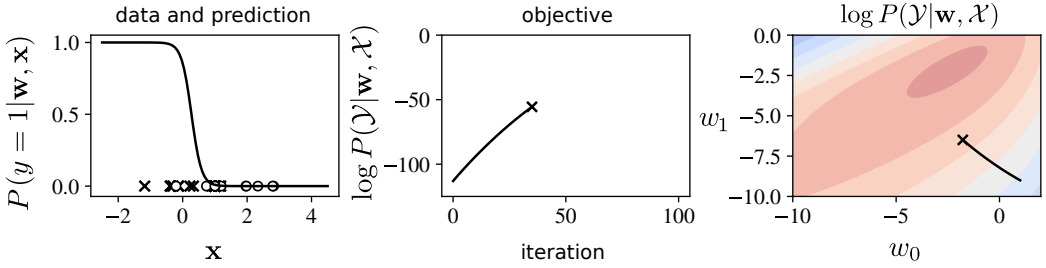

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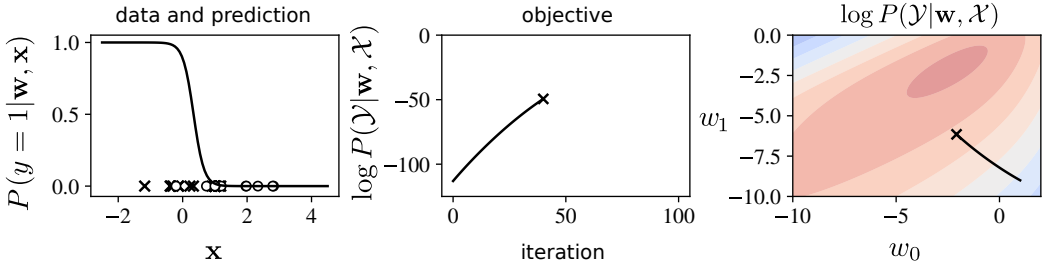

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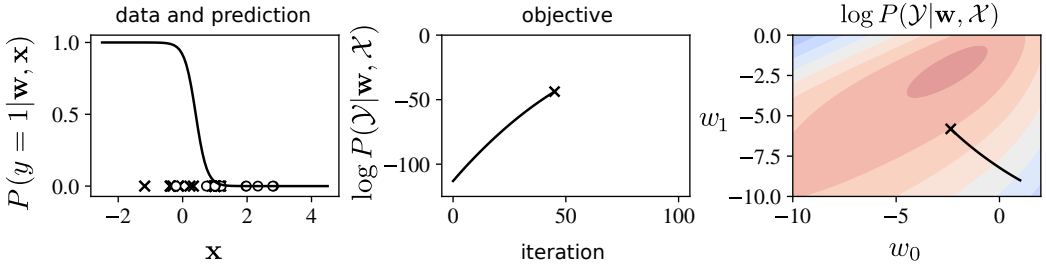

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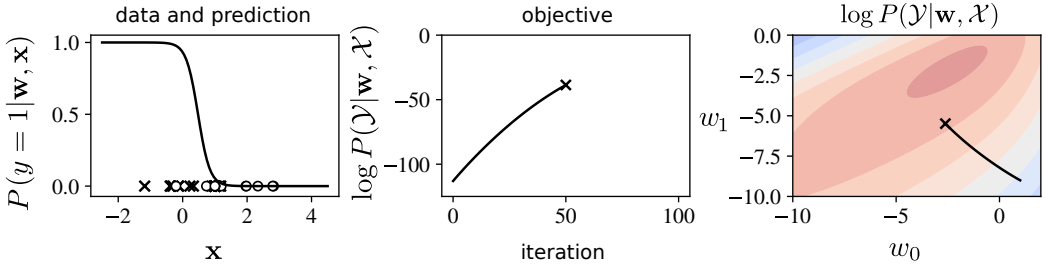

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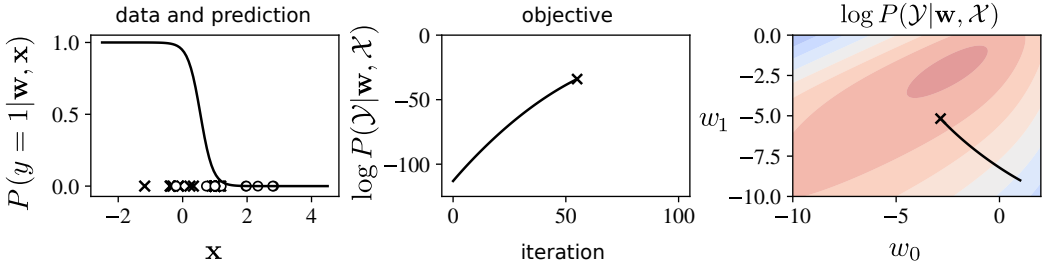

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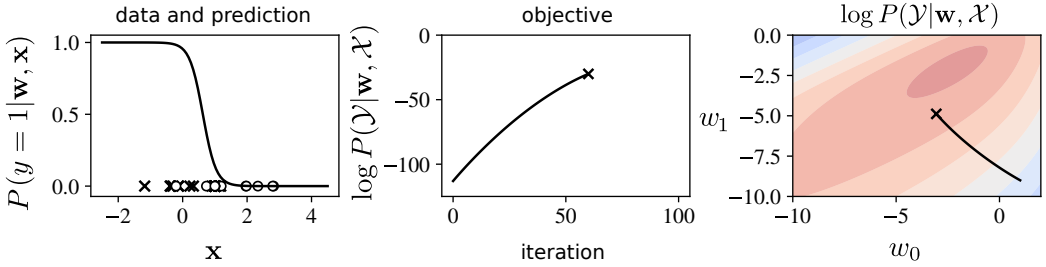

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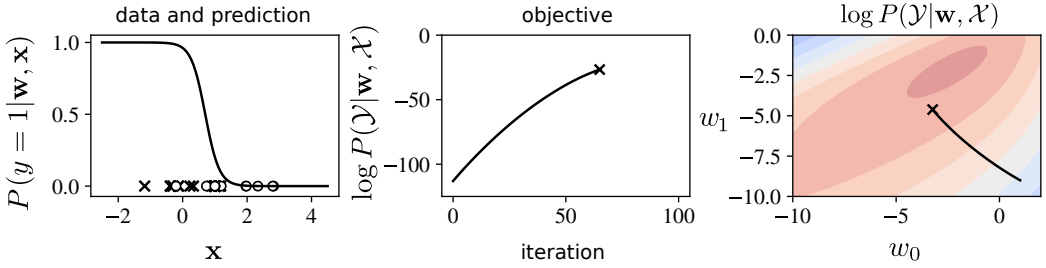

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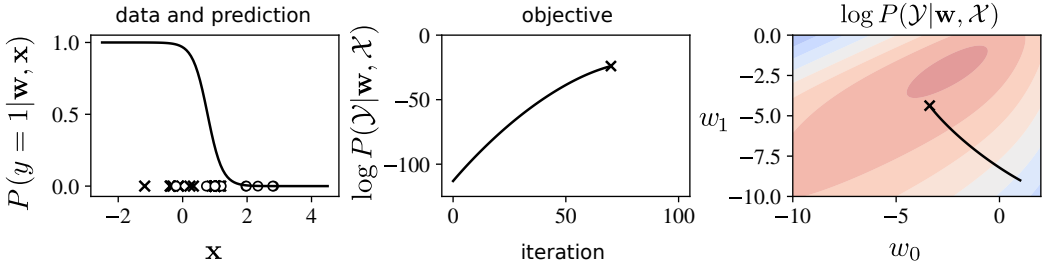

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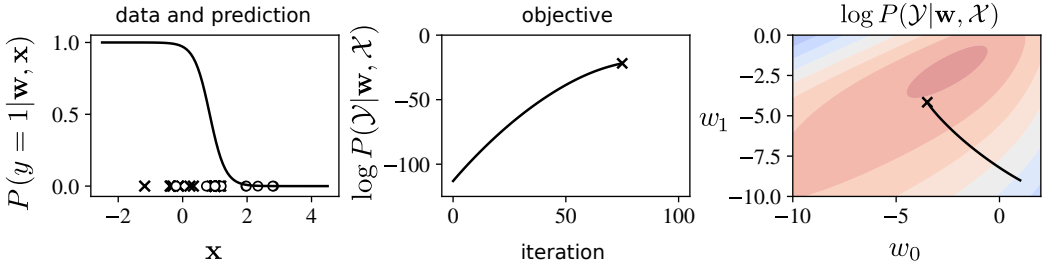

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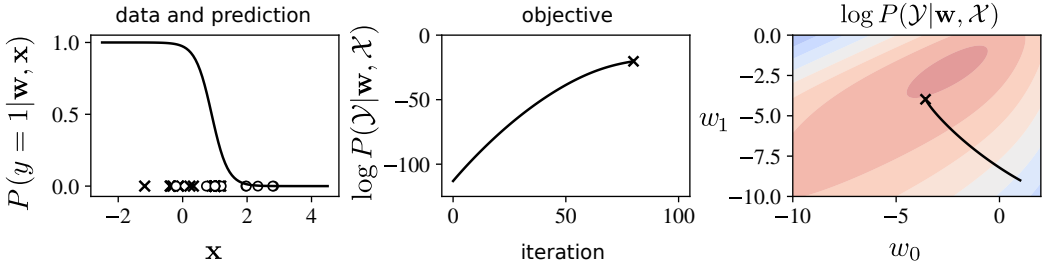

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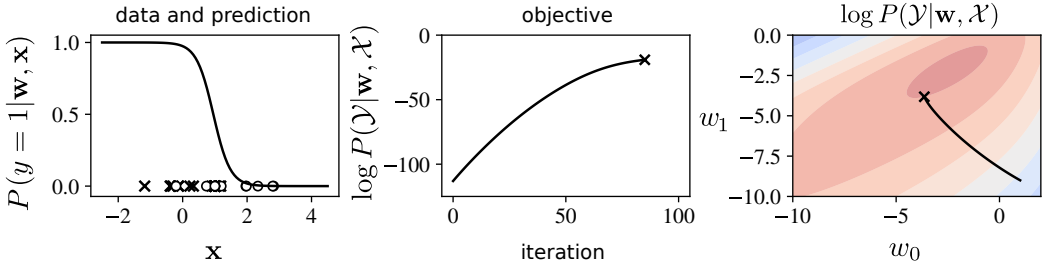

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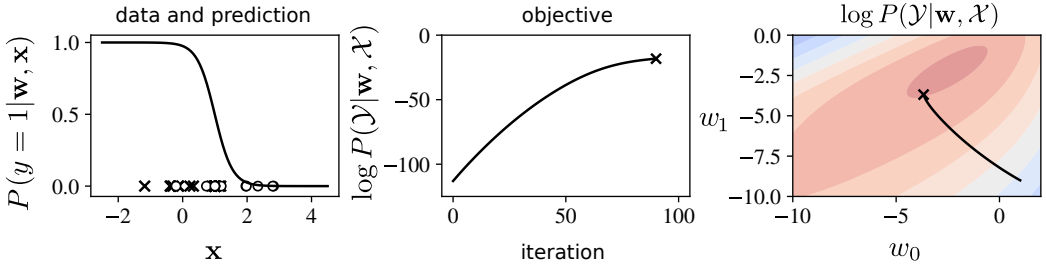

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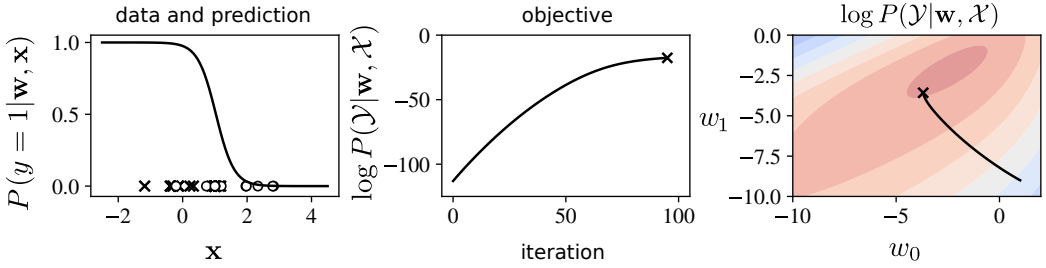

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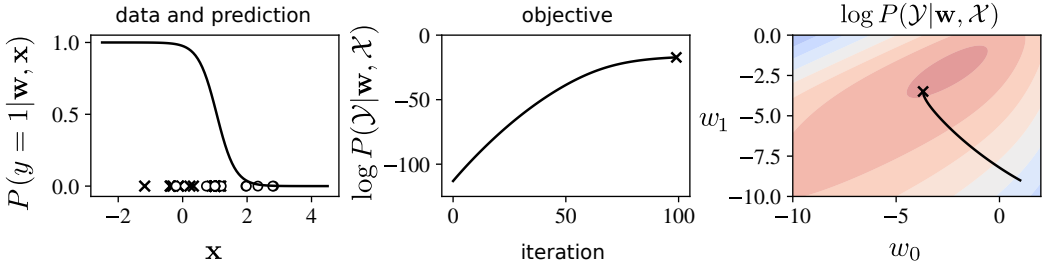
Logistic regression: Maximum Likelihood learning

observe data: estimate parameters



model:

$$f(\mathbf{x},\mathbf{w}) = P\left(y=1|\mathbf{w},\mathbf{x}\right) = \frac{1}{1+\exp(-\mathbf{w}^\top\mathbf{x})} = \frac{1}{1+\exp(-(w_0+w_1x_1))}$$

data:

$$\mathcal{X} = \{\mathbf{x}_n\}_{n=1}^N$$
$$\mathcal{Y} = \{y_n\}_{n=1}^N$$

maximum likelihood estimate: parameters that make observed data most probable

$$\mathbf{w}^{\text{ML}} = \arg\max_{\mathbf{w}} \log P(\mathcal{Y}|\mathbf{w},\mathcal{X}) = \arg\max_{\mathbf{w}} \sum_{n=1}^{N} \left[y_n \log f(\mathbf{w}^\top\mathbf{x}_n) + (1-y_n)\log(1-f(\mathbf{w}^\top\mathbf{x}_n))\right]$$

Bayesian approaches to logistic regression

Probabilistic model

encodes prior assumptions in a recipe for generating datasets

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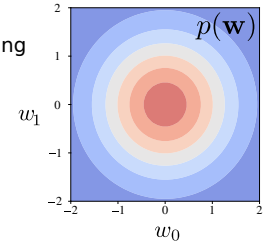

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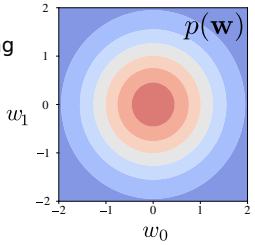

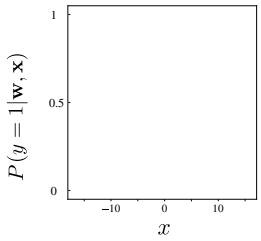

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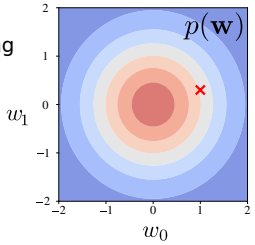

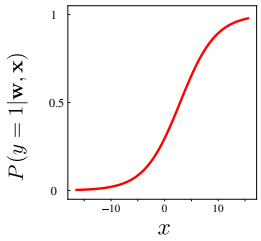

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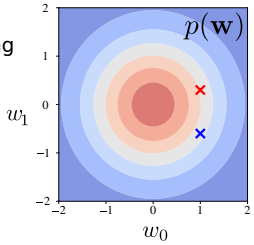

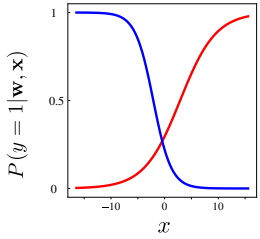

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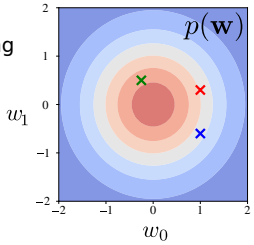

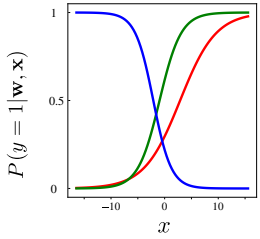

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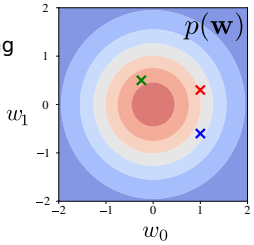

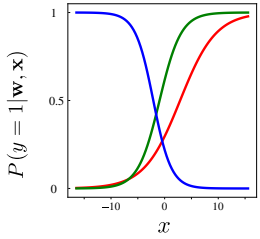

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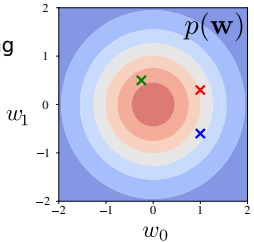

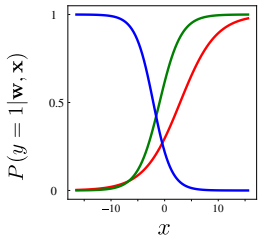

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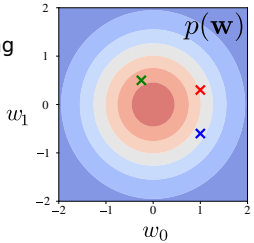

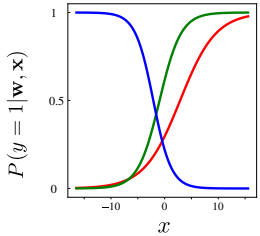

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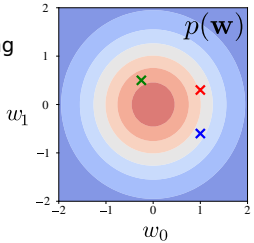

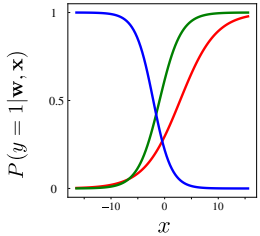

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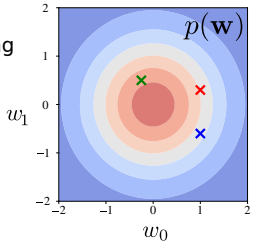

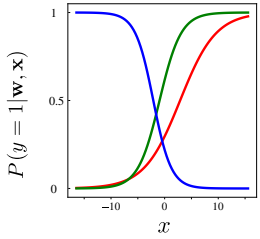

Probabilistic inference

1. "right" answer is a probability distribution over all possible settings of the weights that indicates the plausiblity of that setting given data
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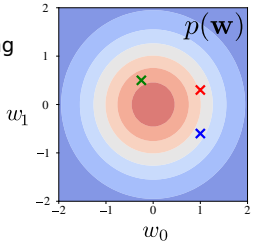

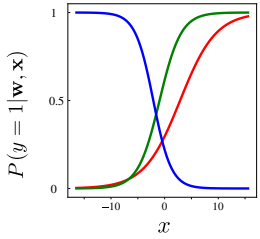

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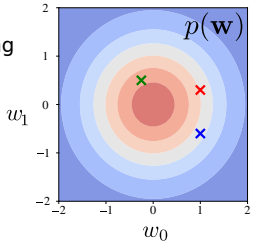

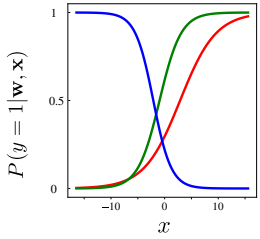

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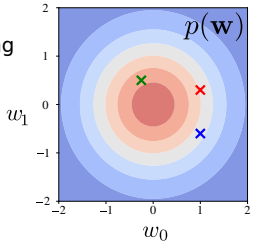

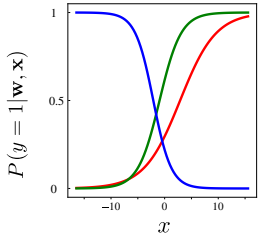

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implies

sum rule: $P(A|C) = \sum_B P(A, B|C)$

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The posterior distribution over weights

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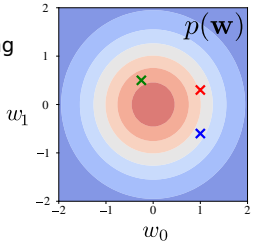

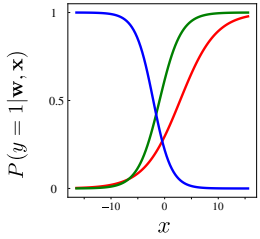

Probabilistic inference

$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X})$$

posterior
what we know
after seeing data

sum rule:	$P(A\|C) = \sum_B P(A, B\|C)$
product rule:	$P(A, B\|C) = P(B\|C)P(A\|B, C) = P(A\|C)P(B\|A, C)$
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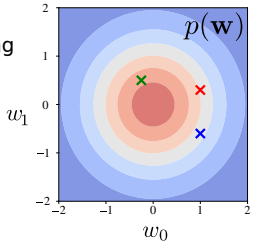

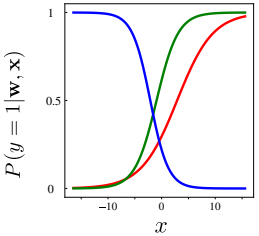

Probabilistic inference

$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}|\mathcal{X})} p(\mathbf{w}|\mathcal{X}) P(\mathcal{Y}|\mathbf{w}, \mathcal{X})$$

$A = \mathbf{w} \quad B = \mathcal{Y} \quad C = \mathcal{X}$

posterior
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sum rule:	$P(A\|C) = \sum_B P(A, B\|C)$
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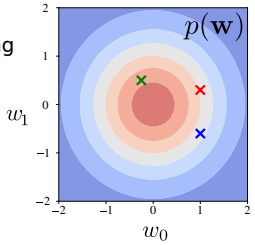

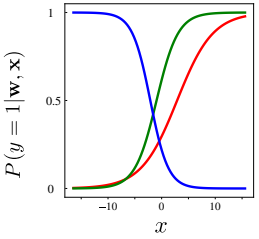

Probabilistic inference

structure of model

$$A = \mathbf{w} \quad B = \mathcal{Y} \quad C = \mathcal{X}$$

$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}|\mathcal{X})}\, p(\mathbf{w})\, P(\mathcal{Y}|\mathbf{w}, \mathcal{X})$$

posterior
what we know
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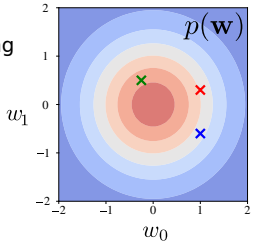
The posterior distribution over weights

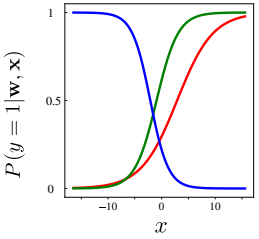
Probabilistic model

encodes prior assumptions in a recipe for generating datasets

$$p(\mathbf{w}) = \mathcal{G}(\mathbf{w}; \mathbf{0}, \sigma_{\mathbf{w}}^2 \mathrm{I})$$

$$P(y = 1|\mathbf{w}, \mathbf{x}) = \frac{1}{1 + \exp(-a(\mathbf{x}, \mathbf{w}))}$$





Probabilistic inference

structure of model

$A = \mathbf{w}$ $B = \mathcal{Y}$ $C = \mathcal{X}$

$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}|\mathcal{X})} p(\mathbf{w}) \; P(\mathcal{Y}|\mathbf{w}, \mathcal{X})$$

posterior ∝ prior × likelihood of $\mathbf{w}$

posterior: what we know after seeing data

prior: what we knew before seeing data

likelihood of $\mathbf{w}$: what the data told us

sum rule:	$P(A\|C) = \sum_B P(A, B\|C)$
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The predictive distribution over class labels

Probabilistic model

encodes prior assumptions in a recipe for generating datasets

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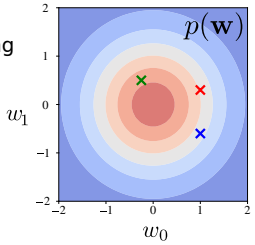

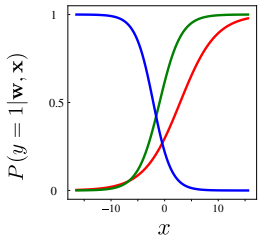

Probabilistic inference

$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}|\mathcal{X})}\, p(\mathbf{w})\; P(\mathcal{Y}|\mathbf{w}, \mathcal{X})$$

$$p(y^*|x^*, \mathcal{Y}, \mathcal{X})$$

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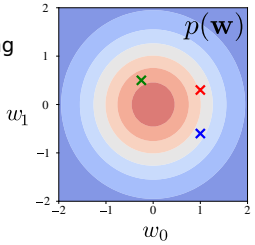

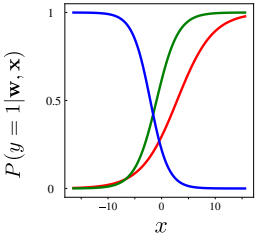

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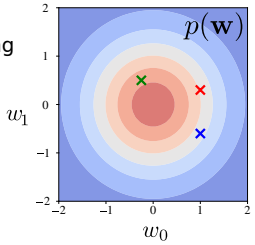

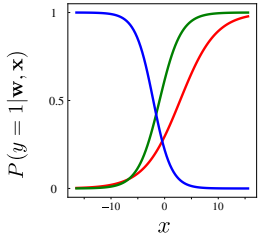

Probabilistic inference

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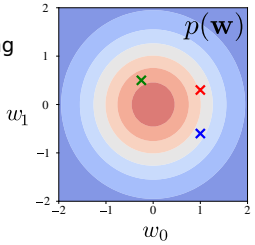

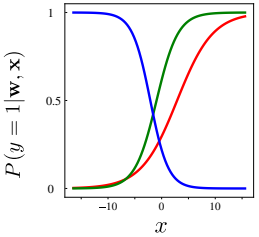

Probabilistic inference

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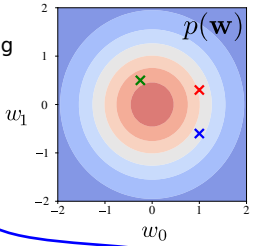

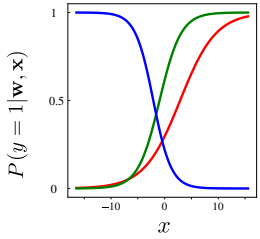

Probabilistic inference

$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}|\mathcal{X})}\, p(\mathbf{w})\; P(\mathcal{Y}|\mathbf{w}, \mathcal{X})$$

structure of model

$$p(y^*|x^*, \mathcal{Y}, \mathcal{X}) = \int p(y^*, \mathbf{w}|x^*, \mathcal{Y}, \mathcal{X}) \mathrm{d}\mathbf{w} = \int p(\mathbf{w}|\mathcal{Y}, \mathcal{X})\, p(y^*|\mathbf{w}, x^*) \mathrm{d}\mathbf{w}$$

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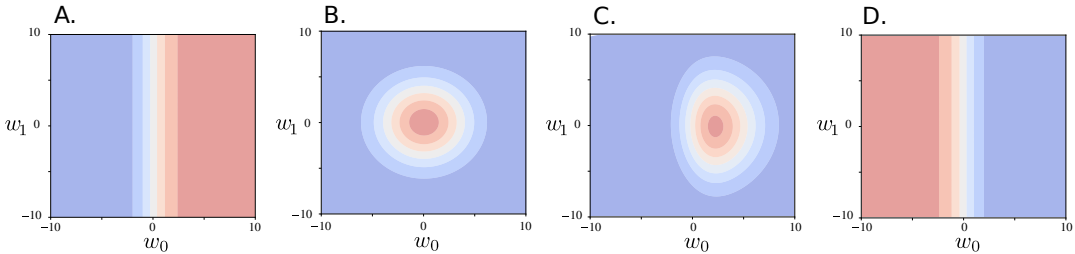
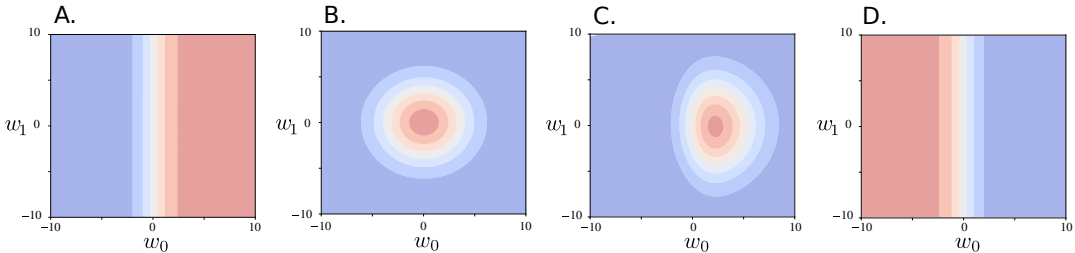
Quiz!



1. $p(\mathbf{w}) = \mathcal{G}(\mathbf{w}; \mathbf{0}, \sigma^2_{\mathbf{w}}\mathrm{I})$
2. $P(y = 1|\mathbf{w}, x = 0)$
3. $P(y = 0|\mathbf{w}, x = 0)$
4. $p(\mathbf{w}|x = 0, y = 1)$

$$P(y = 1|\mathbf{w}, x) = \frac{1}{1+\exp(-(w_1 x + w_0))}$$

Quiz!



B.

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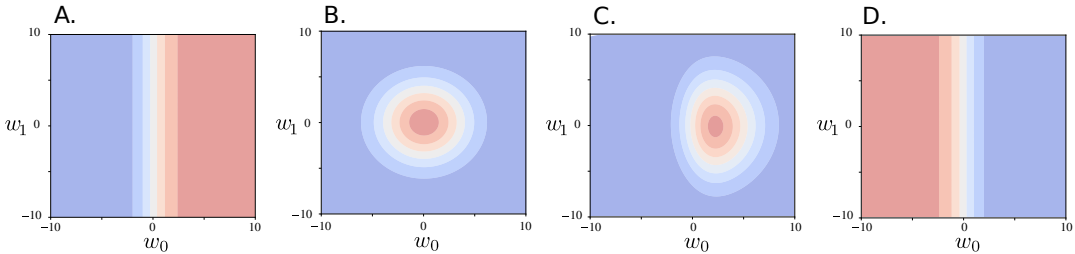

B. 1. $p(\mathbf{w}) = \mathcal{G}(\mathbf{w}; \mathbf{0}, \sigma^2_{\mathbf{w}}\mathrm{I})$

A. 2. $P(y = 1|\mathbf{w}, x = 0)$ $\quad P(y = 1|\mathbf{w}, x) = \frac{1}{1+\exp(-(w_1 x + w_0))}$

3. $P(y = 0|\mathbf{w}, x = 0)$

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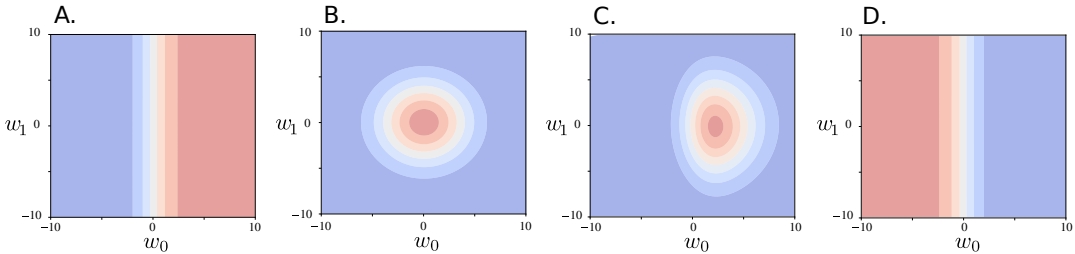
Quiz!



B. 1. $p(\mathbf{w}) = \mathcal{G}(\mathbf{w}; \mathbf{0}, \sigma^2_{\mathbf{w}}\mathrm{I})$

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D. 3. $P(y = 0|\mathbf{w}, x = 0)$

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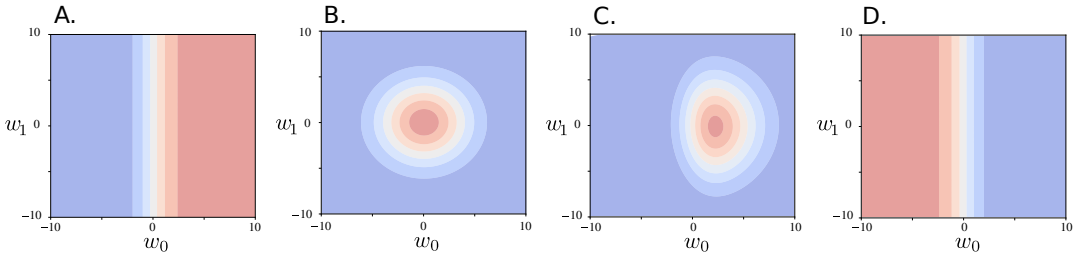
Quiz!



B. 1. $p(\mathbf{w}) = \mathcal{G}(\mathbf{w}; \mathbf{0}, \sigma_{\mathbf{w}}^2 \mathrm{I})$

A. 2. $P(y = 1|\mathbf{w}, x = 0)$ $\qquad P(y = 1|\mathbf{w}, x) = \frac{1}{1+\exp(-(w_1 x + w_0))}$

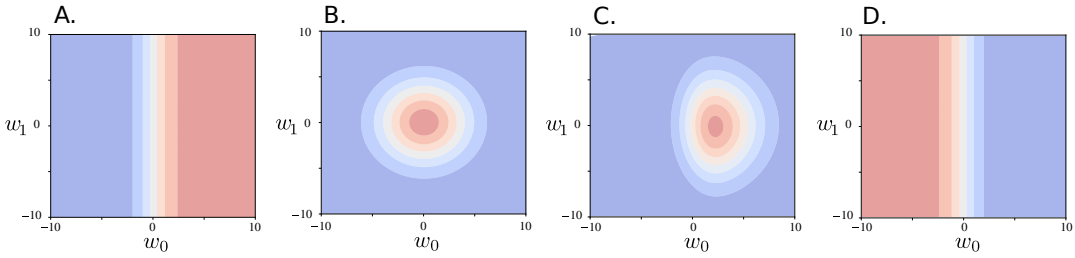
D. 3. $P(y = 0|\mathbf{w}, x = 0)$

4. $p(\mathbf{w}|x = 0, y = 1) \quad \propto \quad p(\mathbf{w}) \quad \times \quad P(y = 1|\mathbf{w}, x = 0)$

posterior — prior — likelihood of $\mathbf{w}$

what we know after seeing data — what we knew before seeing data — what the data told us

Quiz!



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posterior — what we know after seeing data

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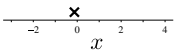
likelihood of $\mathbf{w}$ — what the data told us

Bayesian Inference in Action: 1D Classification Example



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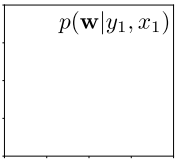

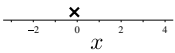

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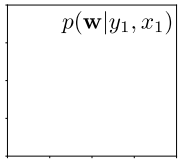

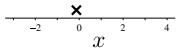

$p(\mathbf{w})$ $\times$ $P(\mathcal{Y}|\mathbf{w}, \mathcal{X})$ $\propto$ $p(\mathbf{w}|\mathcal{Y}, \mathcal{X})$

prior
what we knew
before seeing data

likelihood of $\mathbf{w}$
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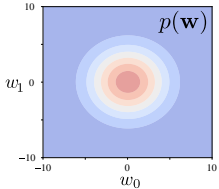
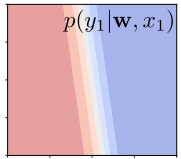


$\times$



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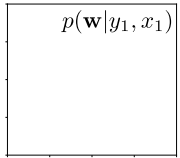

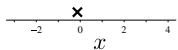

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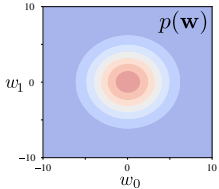

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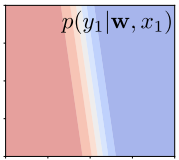

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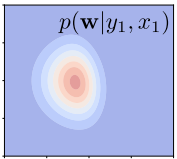

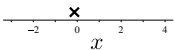

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prior
what we knew
before seeing data

likelihood of $\mathbf{w}$
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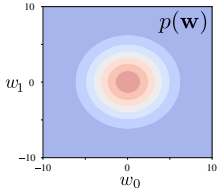
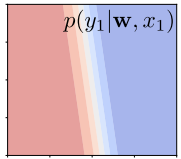
Bayesian Inference in Action: 1D Classification Example



$\times$



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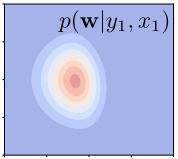

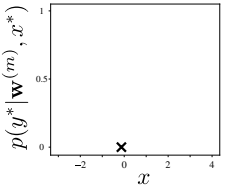

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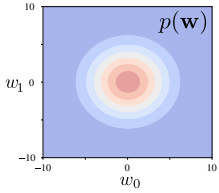

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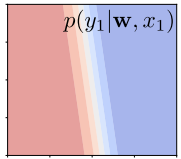

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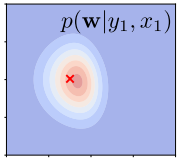

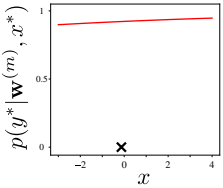

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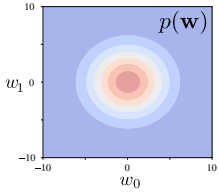

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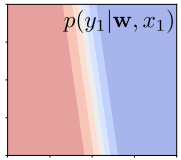

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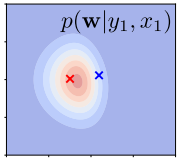

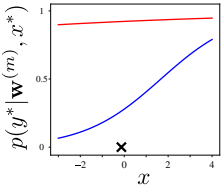

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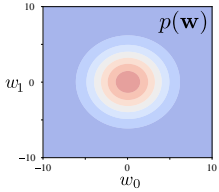

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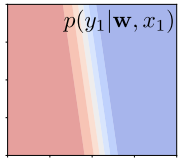

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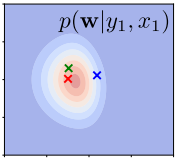

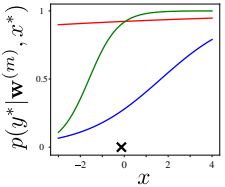

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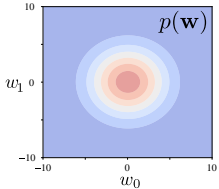
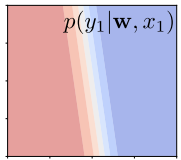
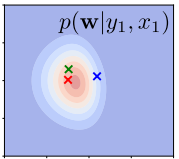
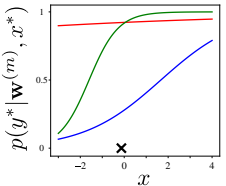
Bayesian Inference in Action: 1D Classification Example

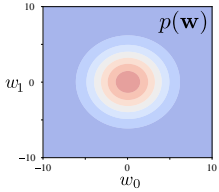


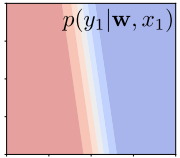


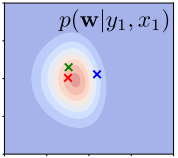


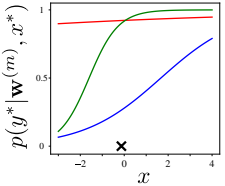


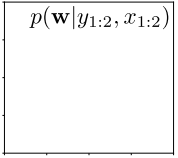
Bayesian Inference in Action: 1D Classification Example

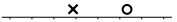




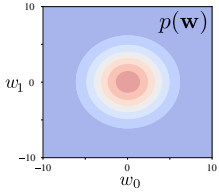




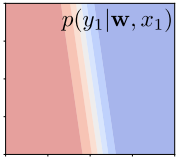




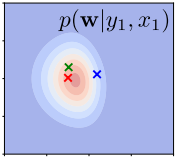
Bayesian Inference in Action: 1D Classification Example

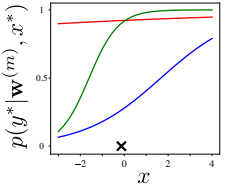


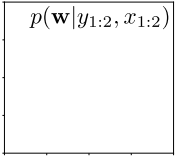
$\times$

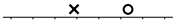


$\propto$









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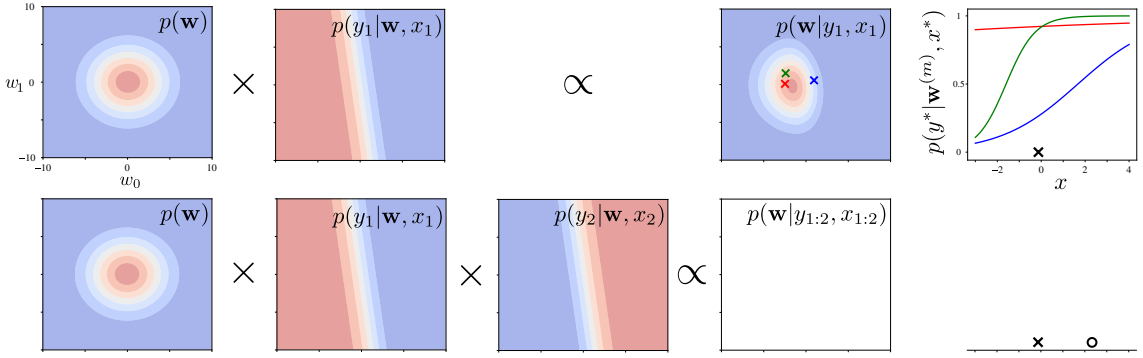
Bayesian Inference in Action: 1D Classification Example



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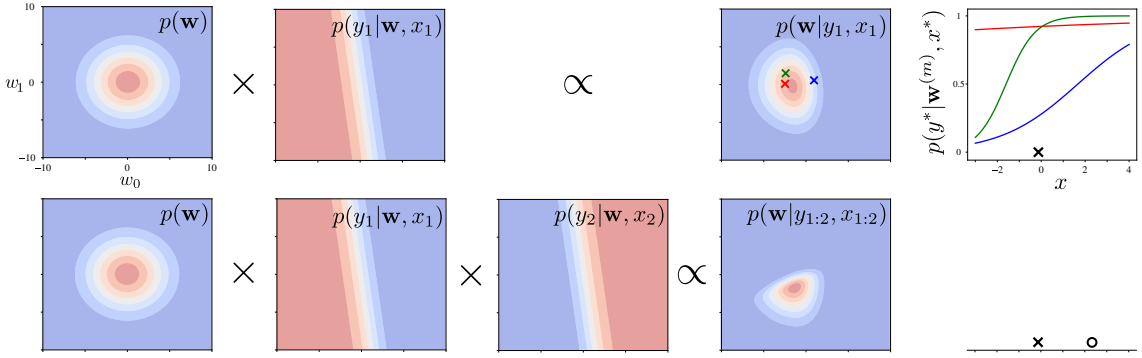

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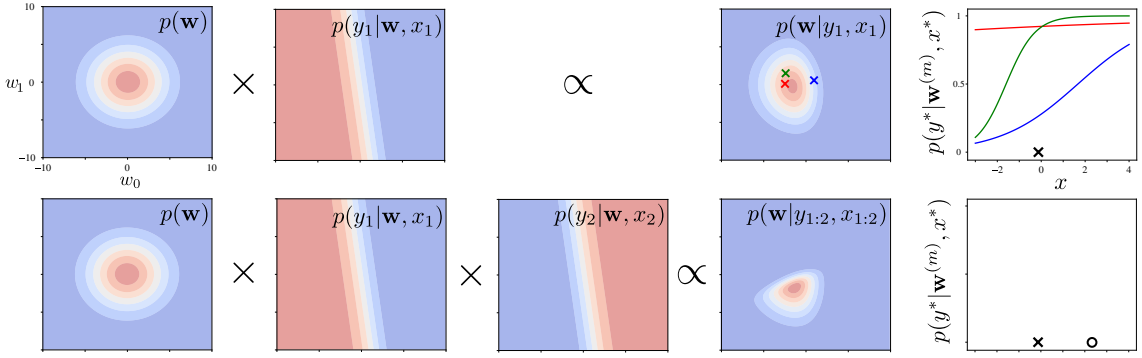

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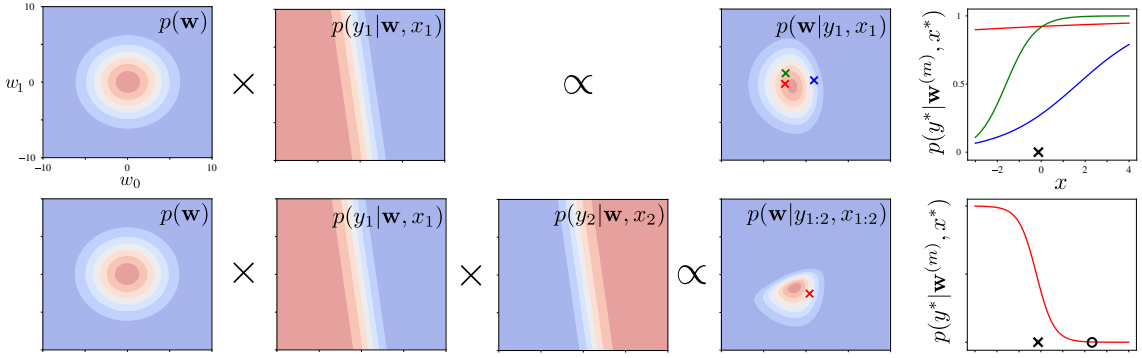

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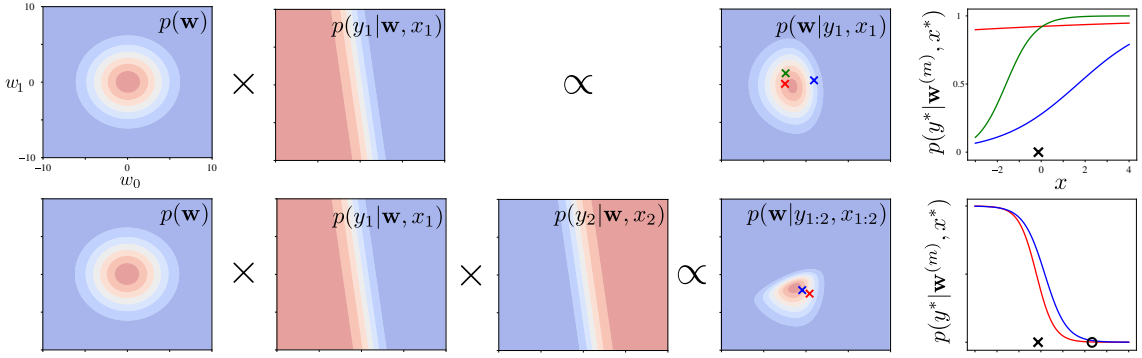

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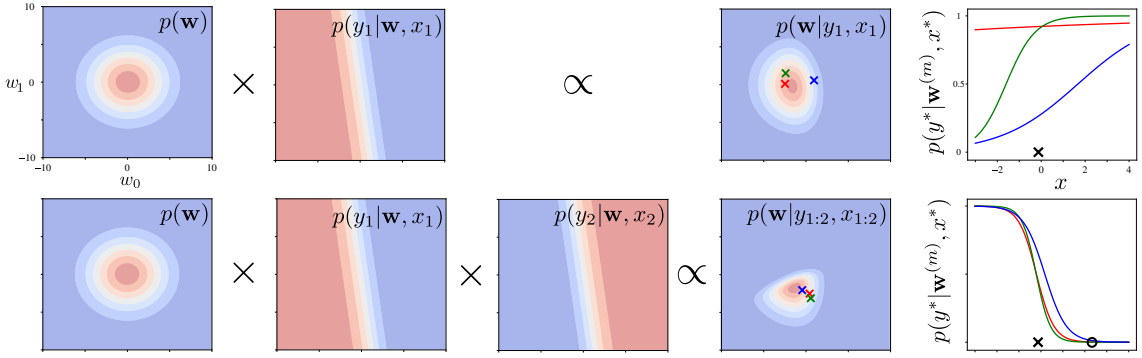

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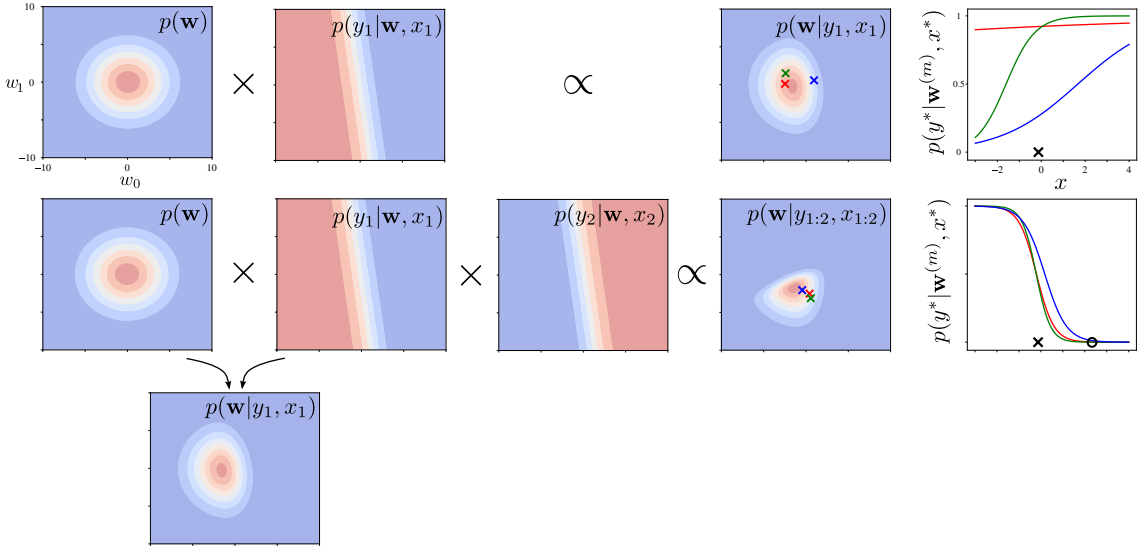

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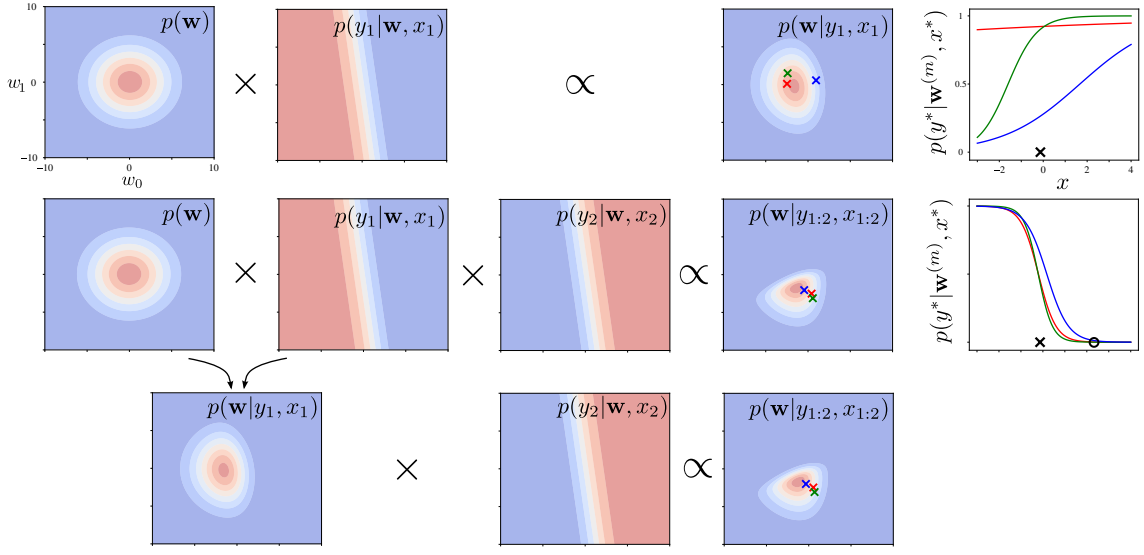
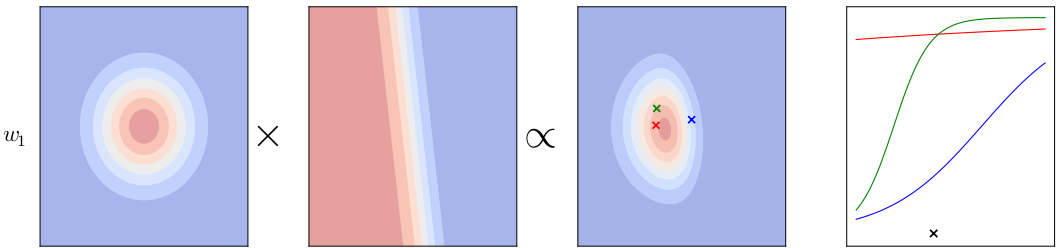


Bayesian Inference in Action: 1D Classification Example

$p(\mathbf{w}|\mathbf{y}_{1:n-1}, \mathbf{x}_{1:n-1})$ $\quad$ $p(y_n|\mathbf{w}, x_n)$ $\quad$ $p(\mathbf{w}|\mathbf{y}_{1:n}, \mathbf{x}_{1:n})$ $\quad$ $p(y^*|\mathbf{w}^{(m)}, x^*)$



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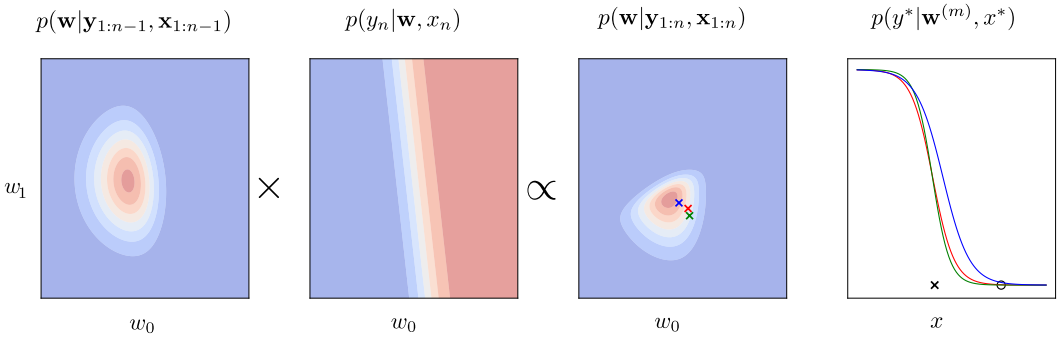

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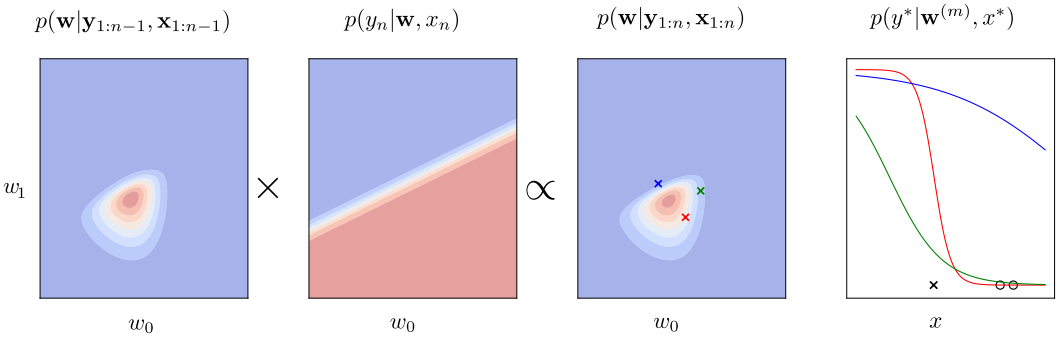
Bayesian Inference in Action: 1D Classification Example



prior

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likelihood of $\mathbf{w}$

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posterior

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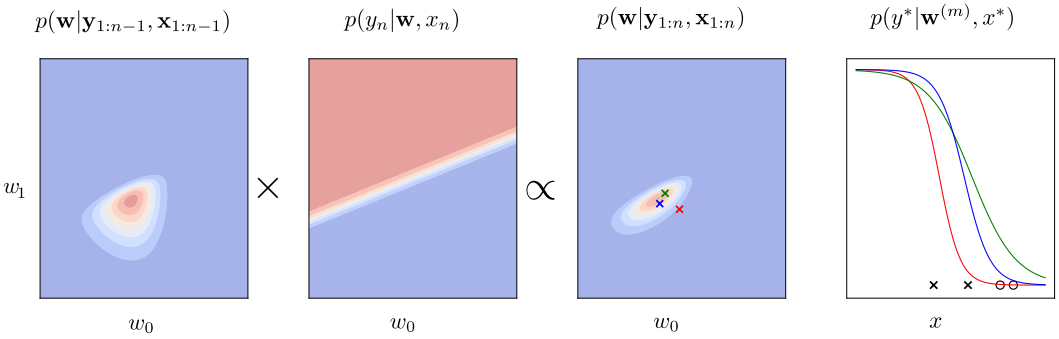

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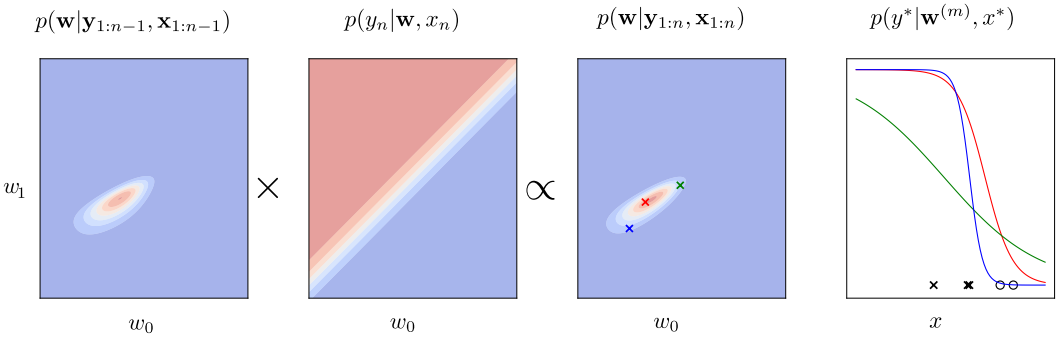

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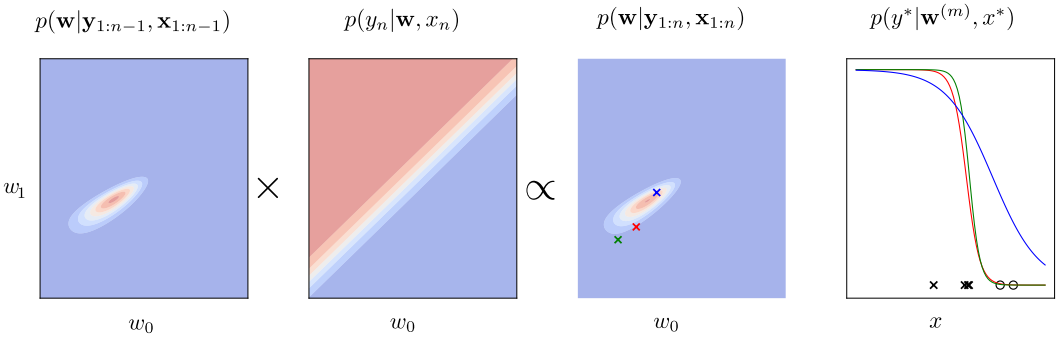

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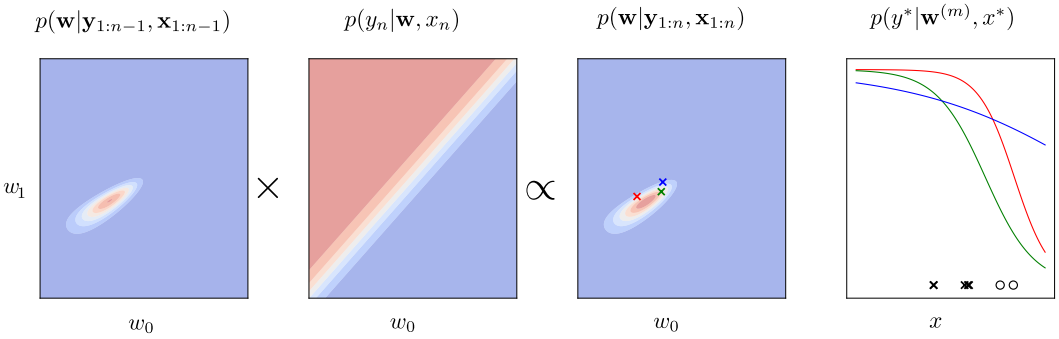

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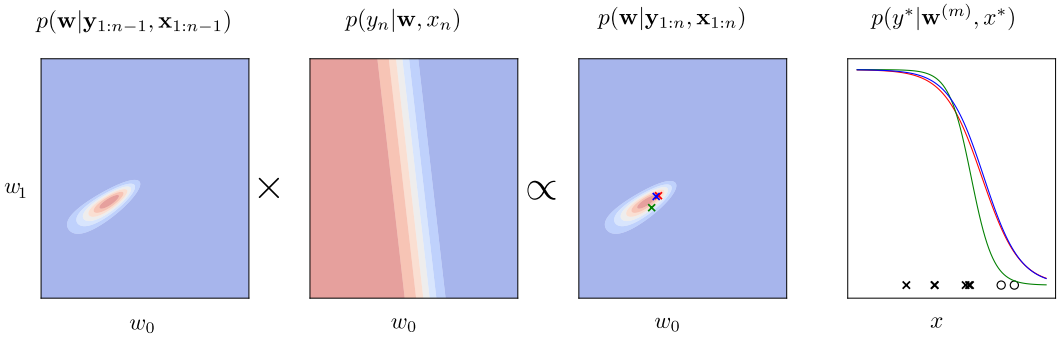

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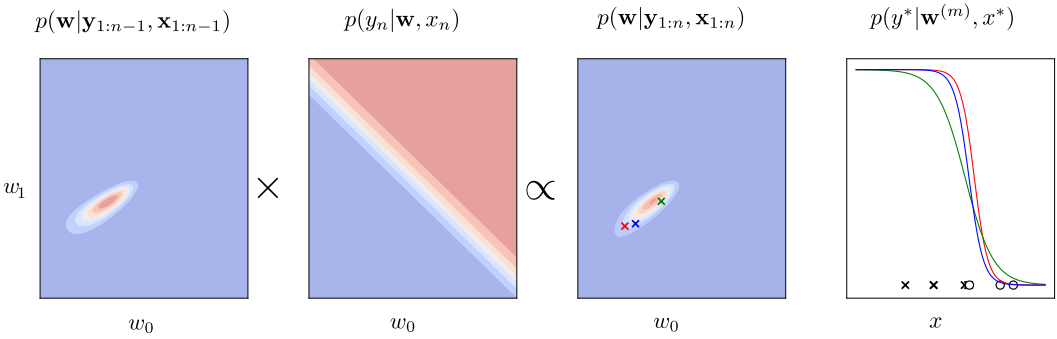

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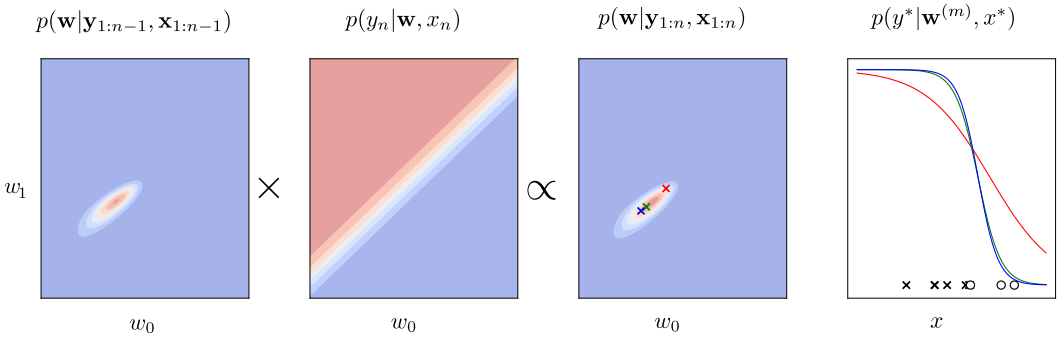

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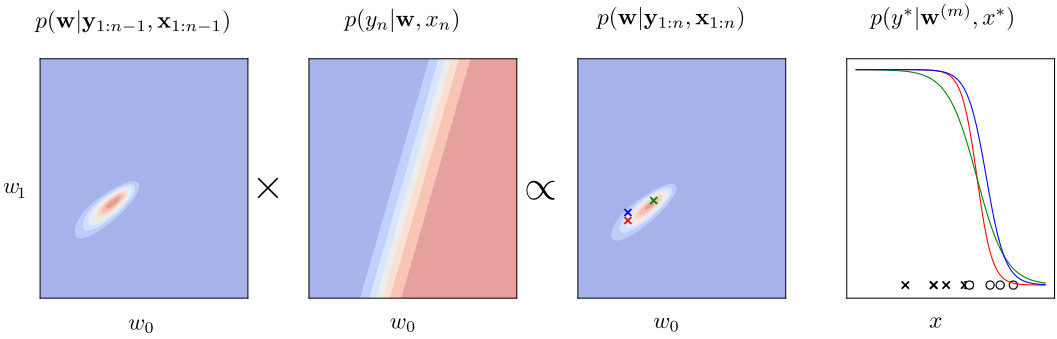

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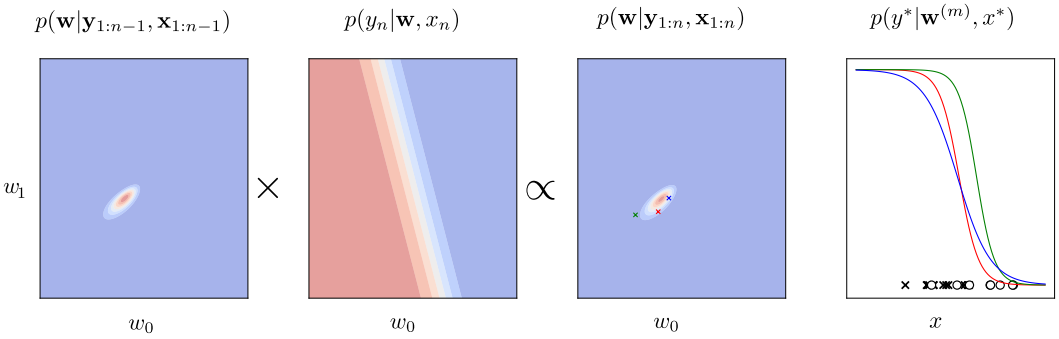

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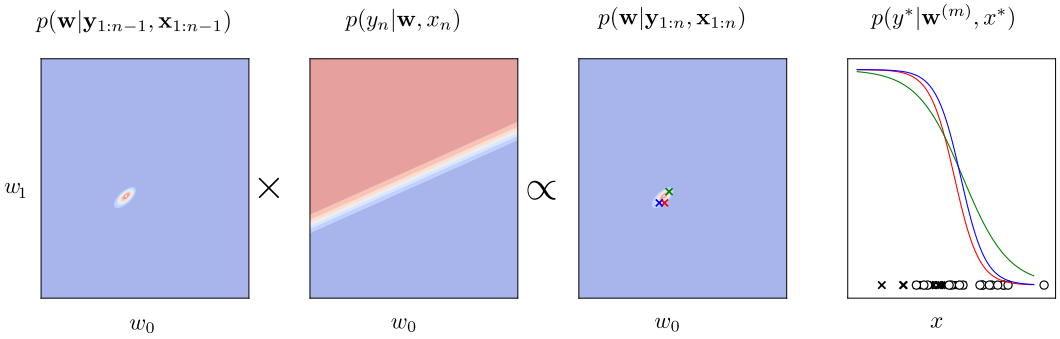

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Why be Bayesian (distributional estimates of weights)?

	maximum-likelihood	Bayes
learning	$\mathbf{w}^{\text{ML}} = \arg\max_{\mathbf{w}} \log P(\mathcal{Y}\|\mathbf{w}, \mathcal{X})$	$p(\mathbf{w}\|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}\|\mathcal{X})}\, p(\mathbf{w})\; P(\mathcal{Y}\|\mathbf{w}, \mathcal{X})$
prediction	$p(y^*\|x^*, \mathcal{Y}, \mathcal{X}) \approx p(y^*\|\mathbf{w}^{\text{ML}}, x^*)$	$p(y^*\|x^*, \mathcal{Y}, \mathcal{X}) = \int p(\mathbf{w}\|\mathcal{Y}, \mathcal{X})\, p(y^*\|\mathbf{w}, x^*)\, \mathrm{d}\mathbf{w}$
	single weight setting	ensemble over weight settings
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Robust Deep Learning

point estimates over-confident, averaging over weight settings less so
Bayesian methods are more robust to adversarial examples (hard to fool ensemble of networks + uncertainty)

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Flexibile Deep Learning

continual learning: use old posterior as prior
active learning: select data that are expected to reduce uncertainty in parameter estimates the most

References: Introduction to the Bayesian approach

Excellent textbook introductions to Bayesian machine learning:

D. J. C. MacKay Information Theory, Inference and Learning Algorithms, 2003

C. Bishop Pattern Recognition and Machine Learning, 2006

K. Murphy Machine Learning: A Probabilistic Perspective, 2012

Beautiful seminal work on Bayesian neural networks:

R. M. Neal Bayesian Learning for Neural Networks, Lecture Notes in Statistics No. 118. New York: Springer-Verlag. 1996

Approximate Bayesian Inference

Laplace's approximation: MacKay 1991 (Saddle point approximation)

$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) \propto p(\mathcal{Y}, \mathbf{w}|\mathcal{X})$

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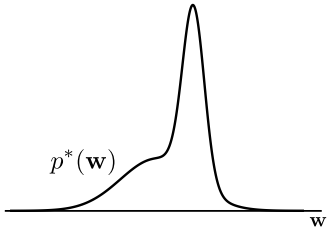
$$p(\mathbf{w}|\mathcal{Y}, \mathcal{X}) \propto p(\mathcal{Y}, \mathbf{w}|\mathcal{X}) = p^*(\mathbf{w})$$



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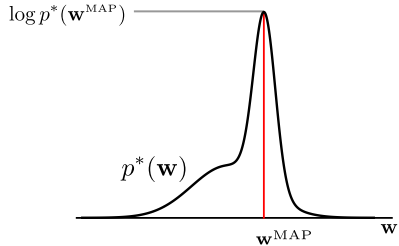

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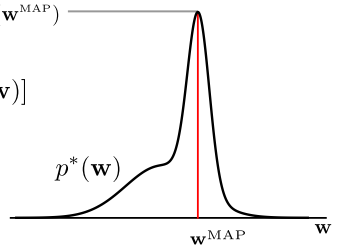

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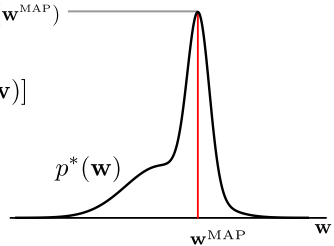

Taylor expand log-prob to 2nd order about MAP est:

$$\log p^*(\mathbf{w}) \approx \log p^*(\mathbf{w}^{\text{MAP}}) + (\mathbf{w}-\mathbf{w}^{\text{MAP}})^\top \frac{\mathrm{d}\log p^*(\mathbf{w})}{\mathrm{d}\mathbf{w}}\Big|_{\mathbf{w}=\mathbf{w}^{\text{MAP}}} + \frac{1}{2}(\mathbf{w}-\mathbf{w}^{\text{MAP}})^\top \frac{\mathrm{d}^2\log p^*(\mathbf{w})}{\mathrm{d}\mathbf{w}\mathrm{d}\mathbf{w}^\top}\Big|_{\mathbf{w}=\mathbf{w}^{\text{MAP}}} (\mathbf{w}-\mathbf{w}^{\text{MAP}})$$

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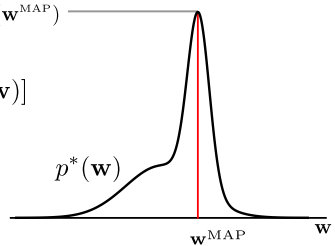

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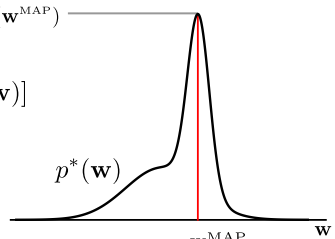

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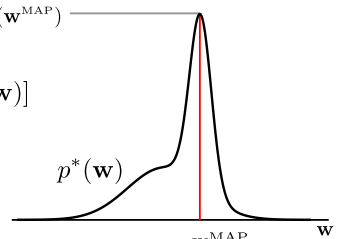

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what's the numerical value of the first order term?

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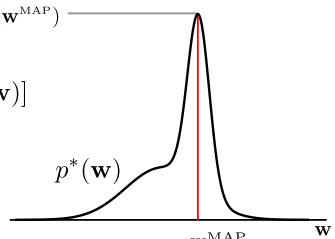

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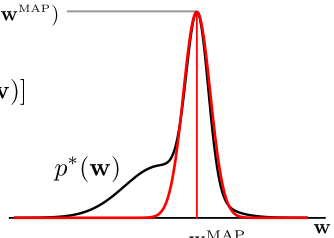

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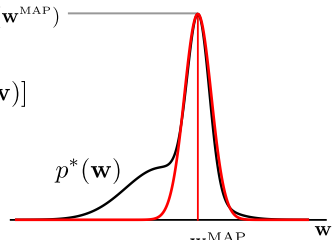

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what's the name of this type of function?

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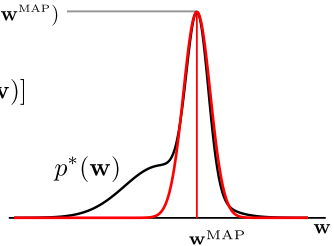

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0

Hessian

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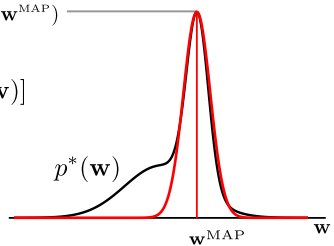

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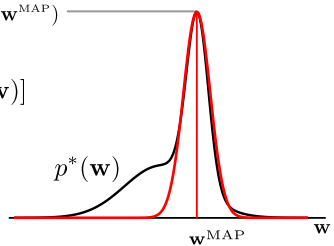

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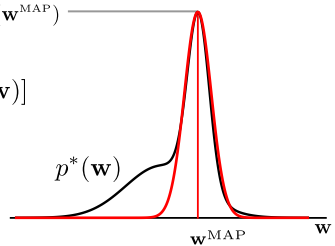

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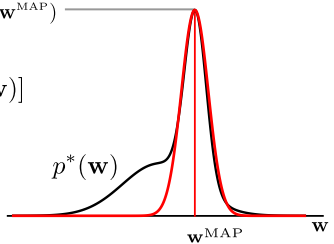

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simple Monte Carlo

$$\mathbf{w}^{(m)} \sim q(\mathbf{w})$$

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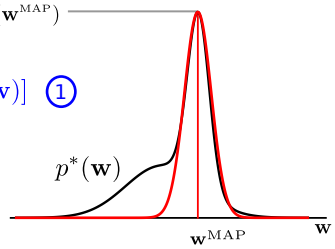

1. optimise to find MAP solution

2. compute Hessian at optimum to form approximate posterior and marginal likelihood

3. predict using samples from approximate posterior and averaging resulting predictions

$$p^*(\mathbf{w}) \approx \underbrace{p^*(\mathbf{w}^{\text{MAP}})\det(2\pi\Sigma)^{1/2}}_{\approx p(\mathcal{Y}|\mathcal{X})} \underbrace{\mathcal{G}\left(\mathbf{w};\mathbf{w}^{\text{MAP}},\Sigma\right)}_{q(\mathbf{w})}$$

(2)

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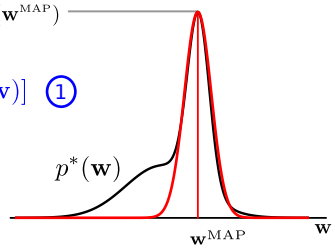

1. optimise to find MAP solution

   might not converge → Hessian not positive definite

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   cubic cost in the number of parameters

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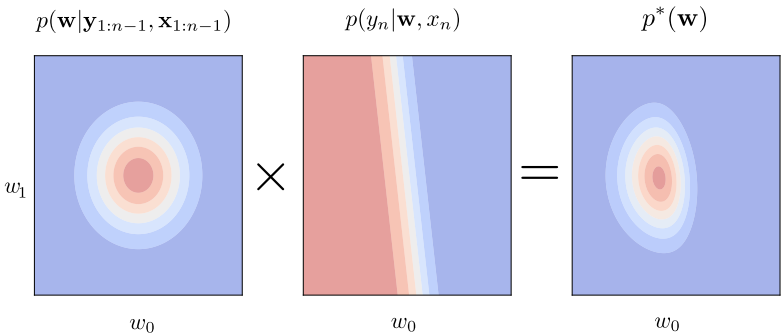
Laplace's approximation: Classification Example



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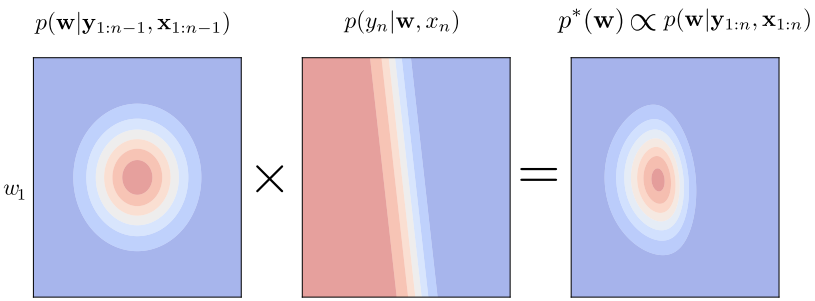

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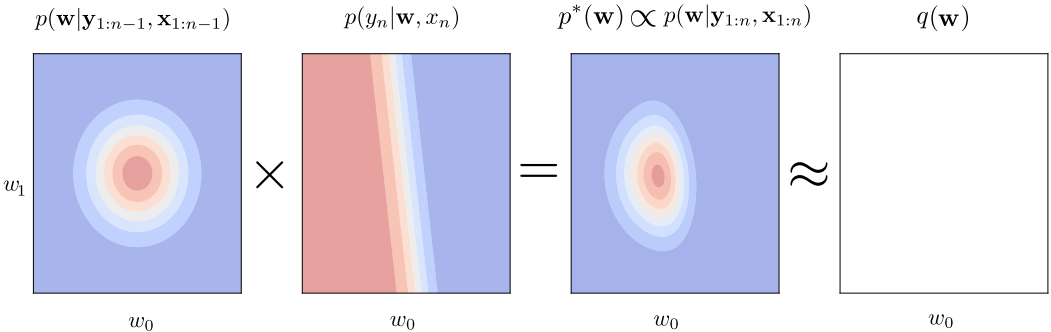

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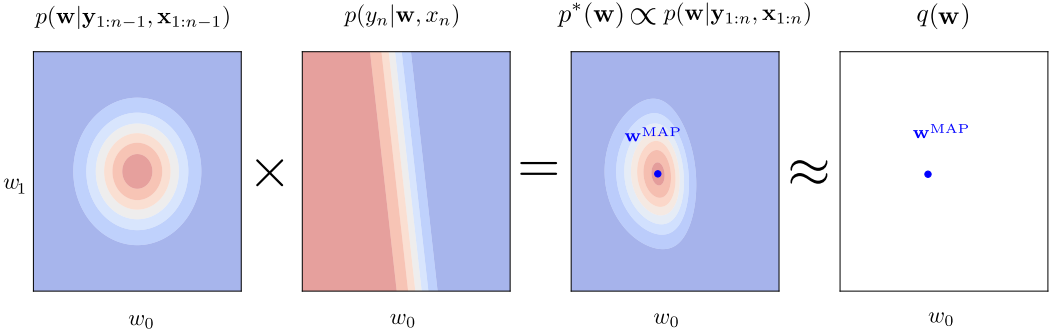

prior
what we knew before seeing data

likelihood of $\mathbf{w}$
what the data told us

$\propto$ posterior
what we know after seeing data

Laplace approx.

1. $\mathbf{w}^{\mathrm{MAP}} = \arg\max_{\mathbf{w}} \log p(\mathbf{w}|\mathcal{Y}, \mathcal{X})$

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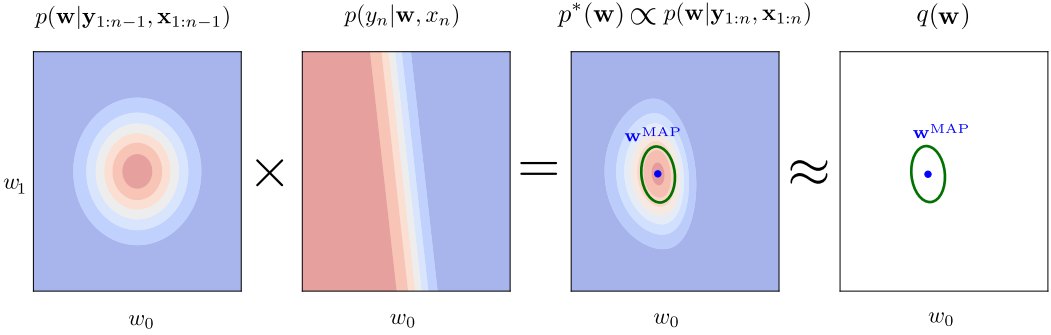

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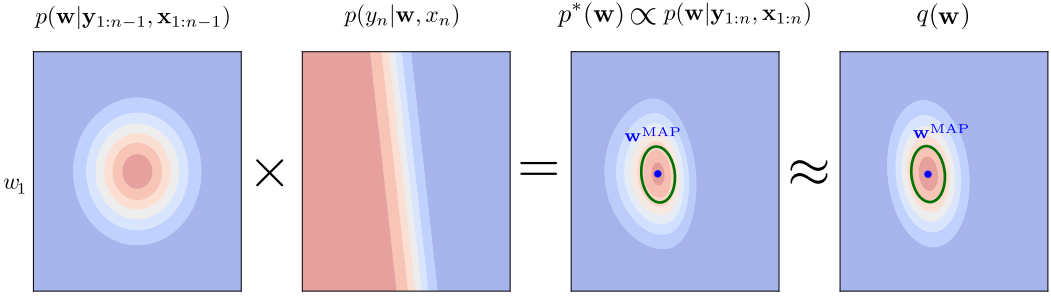

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References: Laplace's method for neural networks

D. J. C. MacKay A practical Bayesian framework for backpropagation networks, Neural Computation 4(3) 448-472, 1992

D. J. C. MacKay Probable networks and plausible predictions – a review of practical Bayesian methods for supervised neural networks , Network: Computation in Neural Systems 6(3), 469-505, 1995

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Variational Inference: the KL Divergence

Kullback–Leibler (KL) divergence

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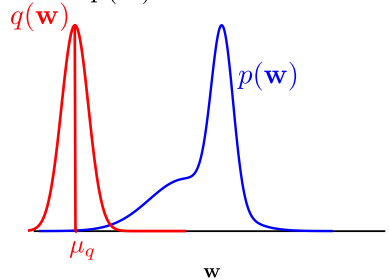

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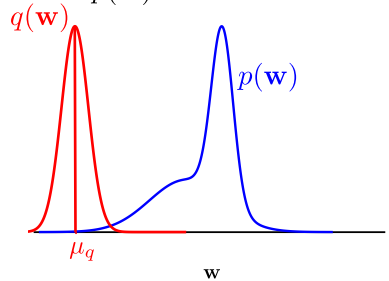

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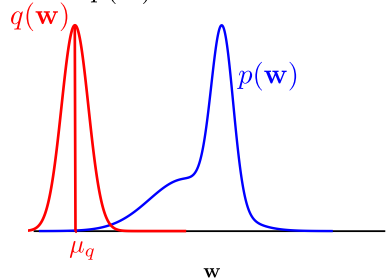

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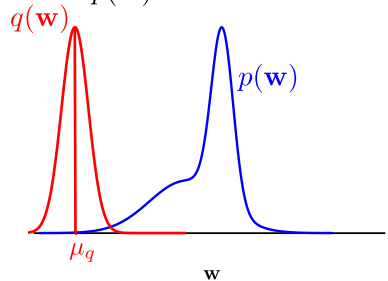

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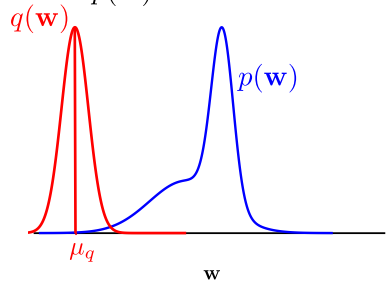

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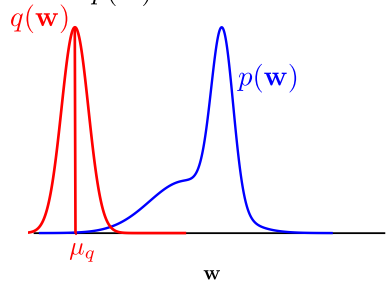

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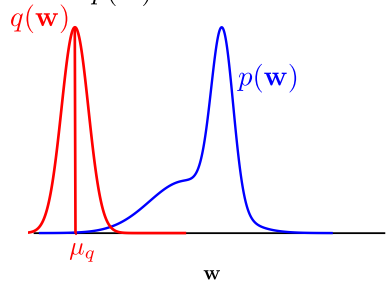

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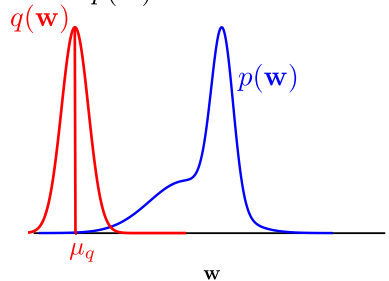

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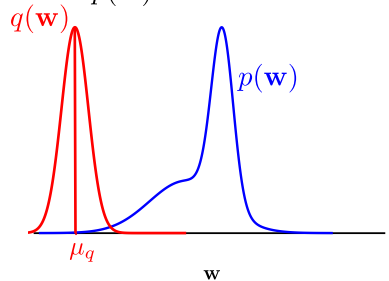

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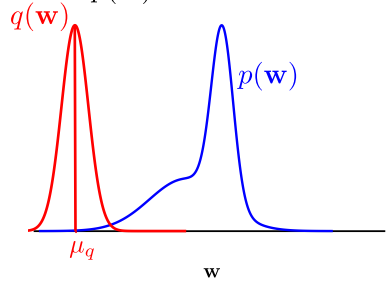

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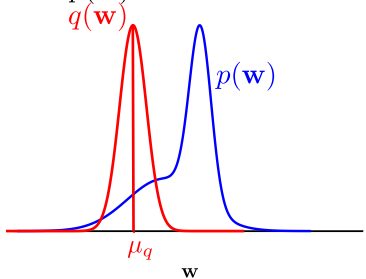

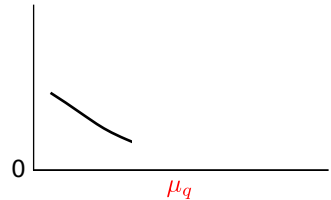

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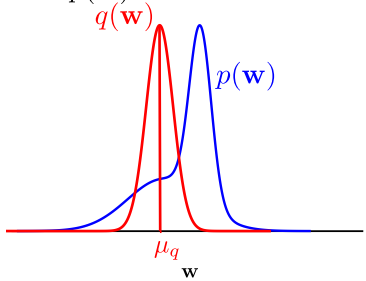

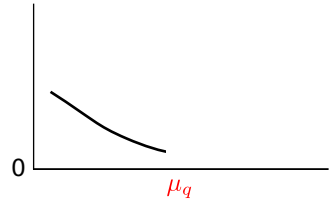

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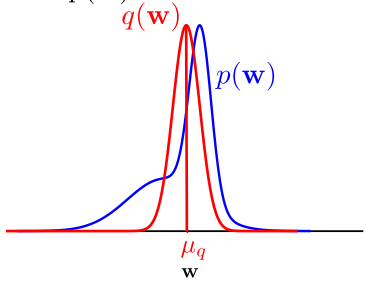

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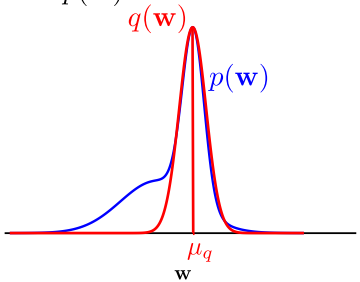

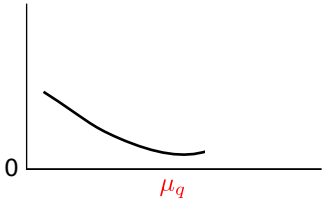

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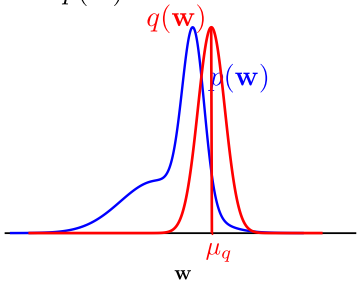

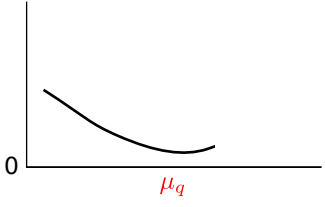

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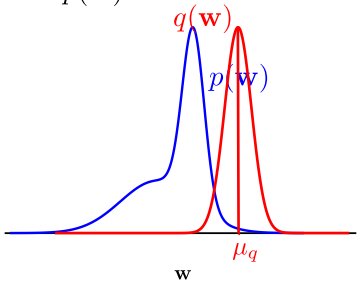

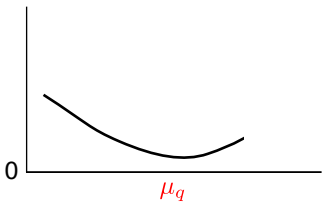

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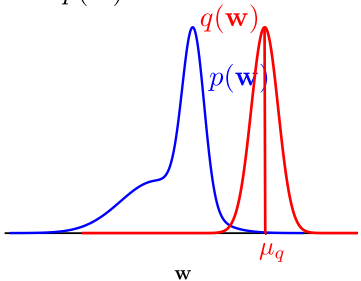

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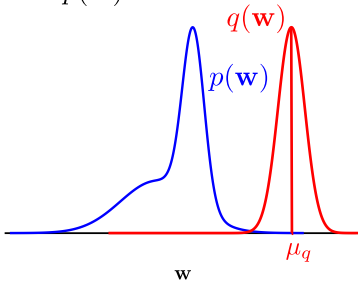

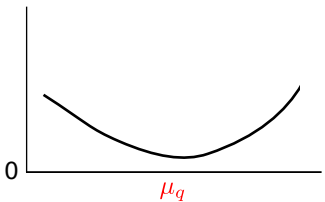

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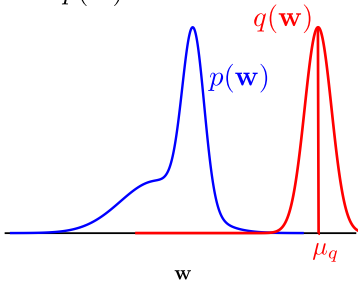

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divergence measures 'distance' between distributions

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3. can be computed up to an additive constant w/o needing normalisation for $p(\mathbf{w})$

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→ suitable for approximate inference

Variational Inference

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Use KL to optimise an approximation $q(\mathbf{w})$ to the true posterior

3. can be computed up to an additive constant w/o needing normalisation for $p(\mathbf{w})$

$$\mathrm{KL}\left(q(\mathbf{w})||\frac{1}{Z}p^*(\mathbf{w})\right) = \int q(\mathbf{w}) \log \frac{q(\mathbf{w})}{\frac{1}{Z}p^*(\mathbf{w})} \mathrm{d}\mathbf{w} = \int q(\mathbf{w}) \log \frac{q(\mathbf{w})Z}{p^*(\mathbf{w})} \mathrm{d}\mathbf{w}$$

$$= \int q(\mathbf{w}) \log \frac{q(\mathbf{w})}{p^*(\mathbf{w})} \mathrm{d}\mathbf{w} + \int q(\mathbf{w}) \log Z \mathrm{d}\mathbf{w} = \int q(\mathbf{w}) \log \frac{q(\mathbf{w})}{p^*(\mathbf{w})} \mathrm{d}\mathbf{w} + \log Z$$

→ suitable for approximate inference

Variational Inference

Kullback–Leibler (KL) divergence $\mathrm{KL}(q(\mathbf{w})||p(\mathbf{w})) = \int q(\mathbf{w}) \log \frac{q(\mathbf{w})}{p(\mathbf{w})} \mathrm{d}\mathbf{w} \overset{1\&2}{\geq} 0$ equality when $q(\mathbf{w}) = p(\mathbf{w})$

Use KL to optimise an approximation $q(\mathbf{w})$ to the true posterior

$$q^{\mathrm{VI}}(\mathbf{w}) \overset{1\&2}{=} \arg\min_{q \in \mathcal{Q}} \mathrm{KL}(q(\mathbf{w})||p(\mathbf{w}|\mathcal{Y}, \mathcal{X}))$$

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Variational Inference

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requires reparameterisation trick

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Variational Inference: Classification Example

$$q^{\mathrm{VI}}(\mathbf{w}) = \arg\min_{q \in \mathcal{Q}} \mathrm{KL}(q(\mathbf{w}) || p(\mathbf{w}|\mathcal{Y}, \mathcal{X}))$$

$$\approx \arg\min_{q \in \mathcal{Q}} \frac{1}{M} \sum_{m=1}^{M} \log \frac{q(\mathbf{w}^{(m)})}{p(\mathbf{w}^{(m)}) p(\mathcal{Y}|\mathbf{w}^{(m)}, \mathcal{X})}$$

approximate family:
factorised Gaussian

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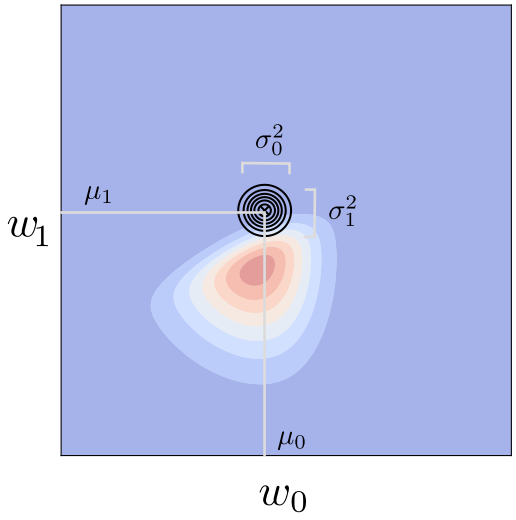
optimise w.r.t. $\mu_0, \sigma_0^2, \mu_1, \sigma_1^2$

VI converts inference into optimization



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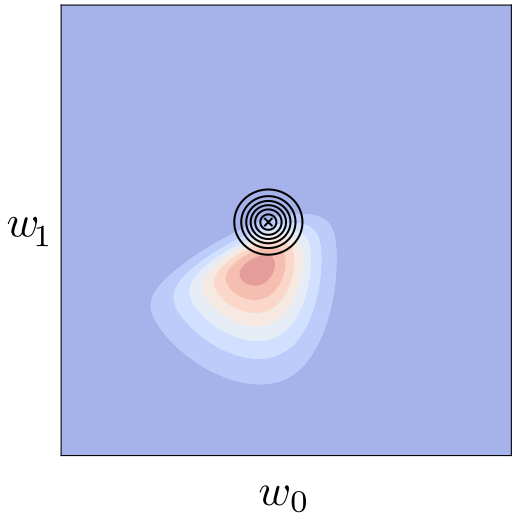

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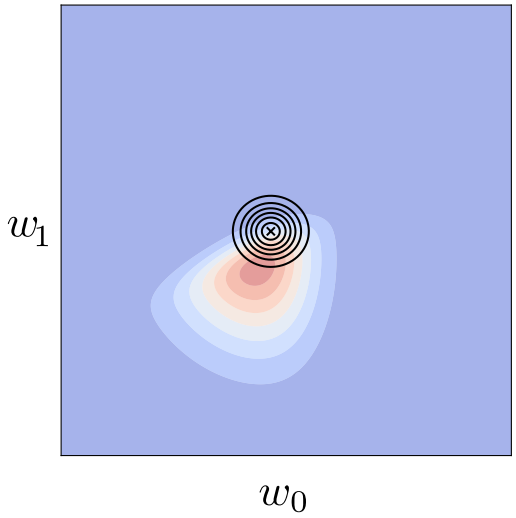

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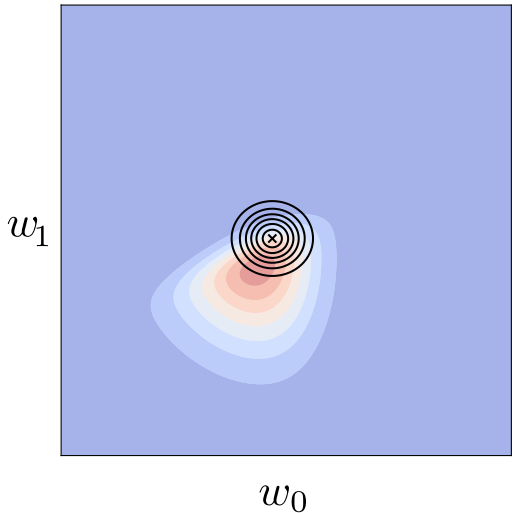

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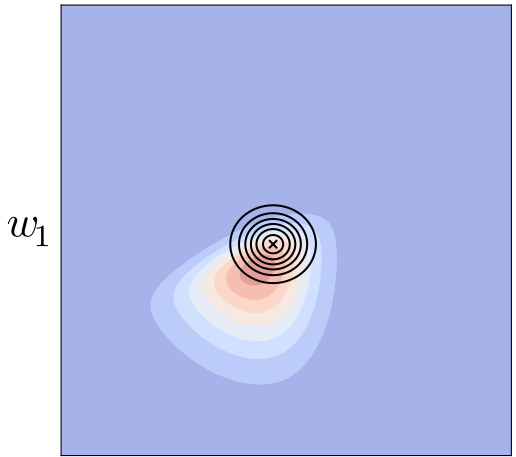

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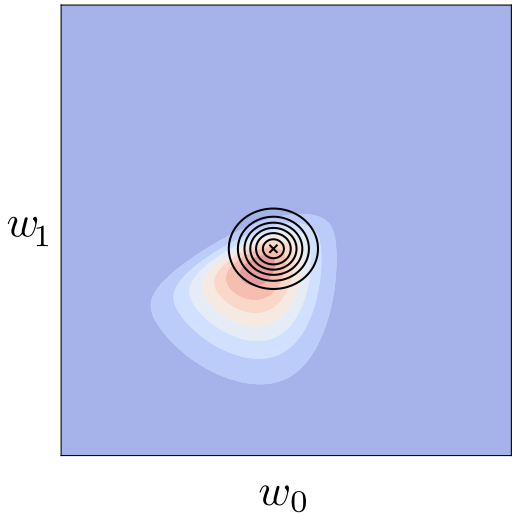

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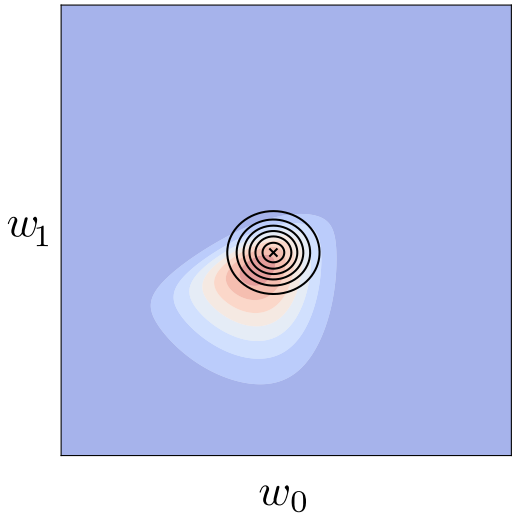

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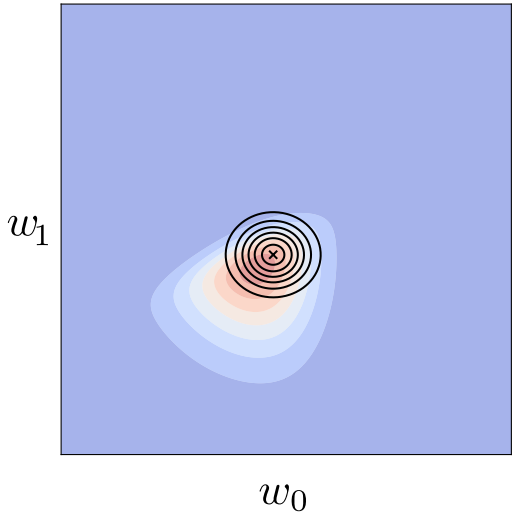

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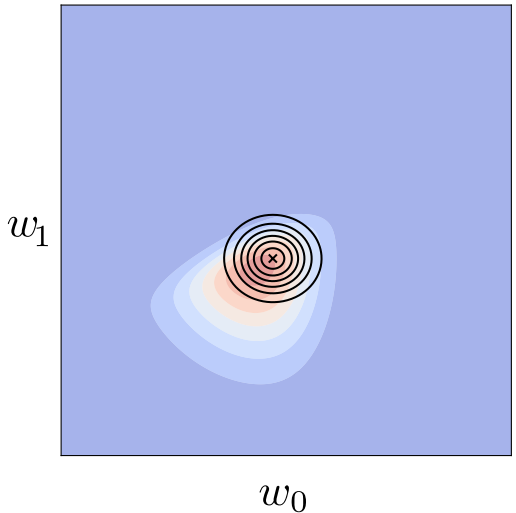

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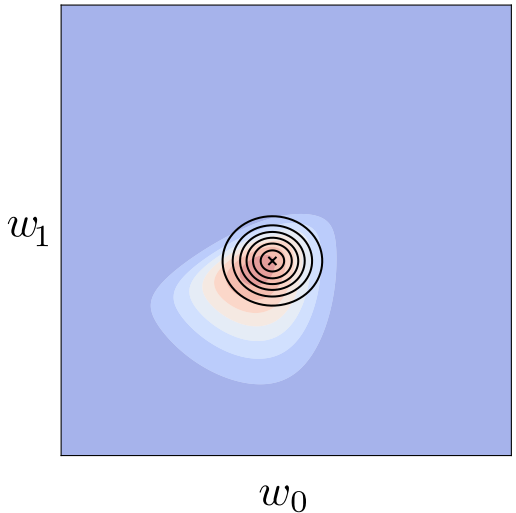

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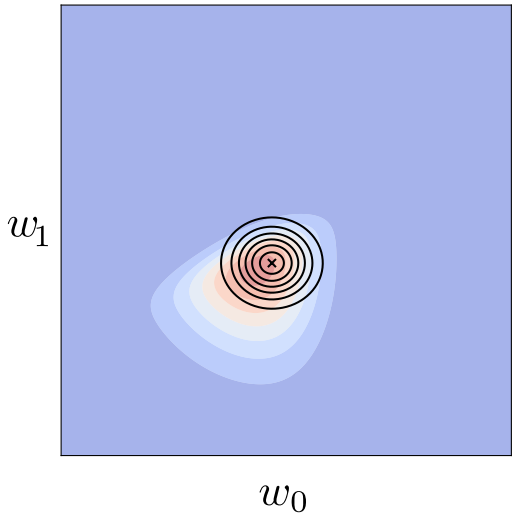

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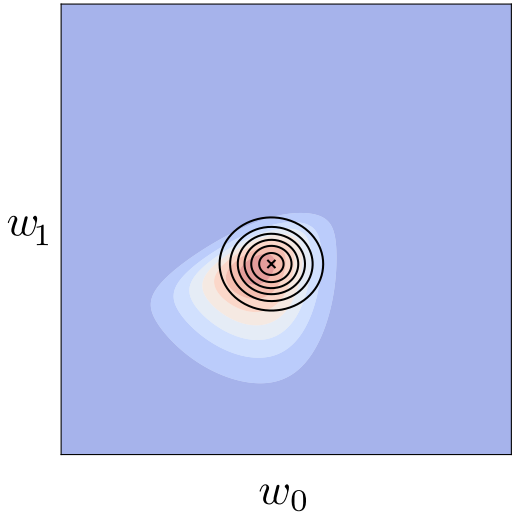

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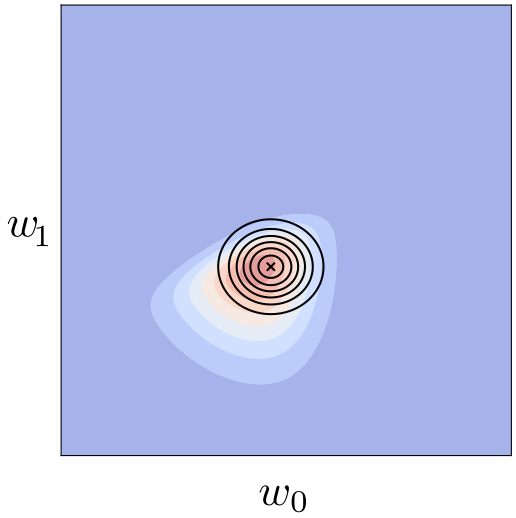

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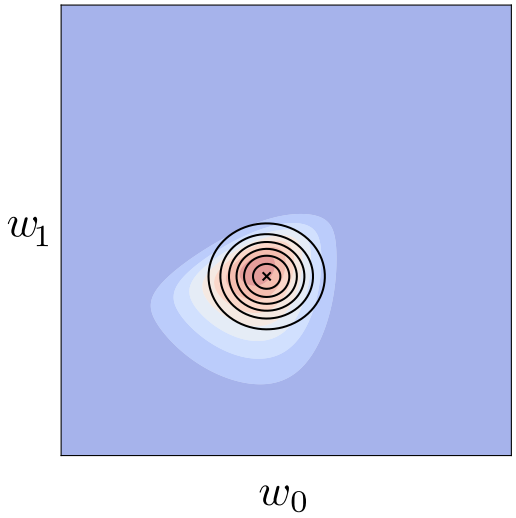

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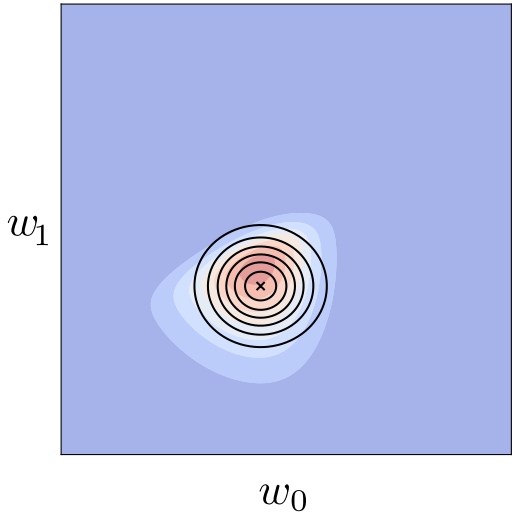

Variational Inference: Classification Example

$$q^{\mathrm{VI}}(\mathbf{w}) = \arg\min_{q \in \mathcal{Q}} \mathrm{KL}(q(\mathbf{w}) || p(\mathbf{w}|\mathcal{Y}, \mathcal{X}))$$

$$\approx \arg\min_{q \in \mathcal{Q}} \frac{1}{M} \sum_{m=1}^{M} \log \frac{q(\mathbf{w}^{(m)})}{p(\mathbf{w}^{(m)}) p(\mathcal{Y}|\mathbf{w}^{(m)}, \mathcal{X})}$$

approximate family:
factorised Gaussian

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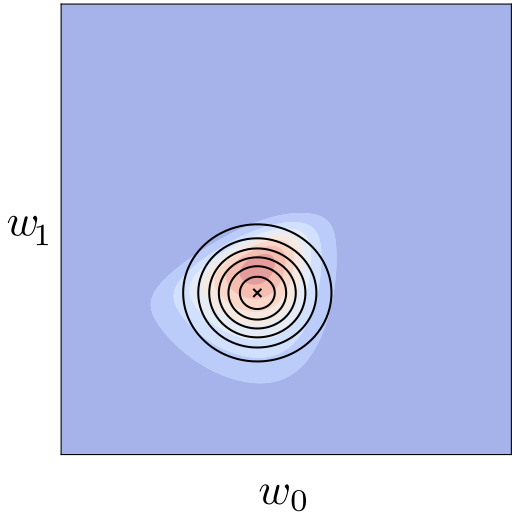

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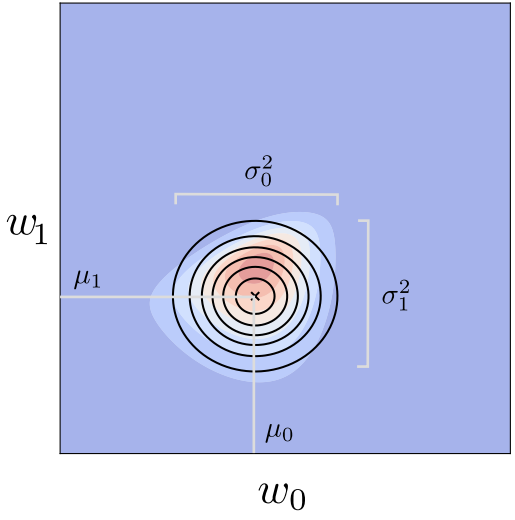
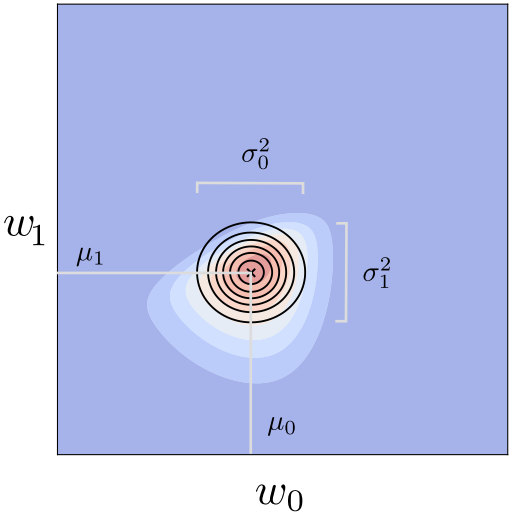
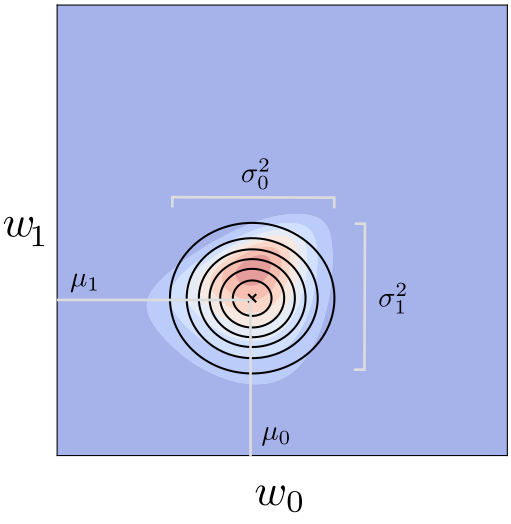


Variational Inference vs Laplace: Classification Example

Laplace Approximation (diagonal Hessian)



Variational Inference (factorised)



References: Variational inference for neural networks

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MCMC for neural networks: Would vanilla Metropolis Hastings work?

samples from posterior can be used for prediction etc. via Monte Carlo

$$\mathbf{w}_t \sim \frac{1}{Z} p^*(\mathbf{w}) \qquad \int f(\mathbf{w}) \frac{1}{Z} p^*(\mathbf{w}) \approx \frac{1}{T} \sum^{T} f(\mathbf{w}_t) \qquad f(\mathbf{w}_t) = p(y|\mathbf{z}, \mathbf{w}_t)$$

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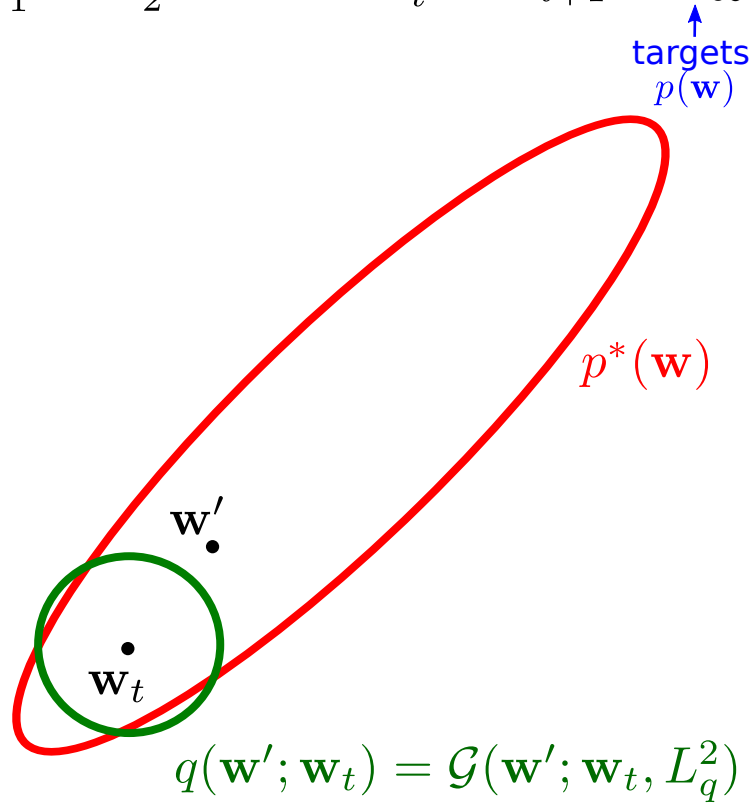

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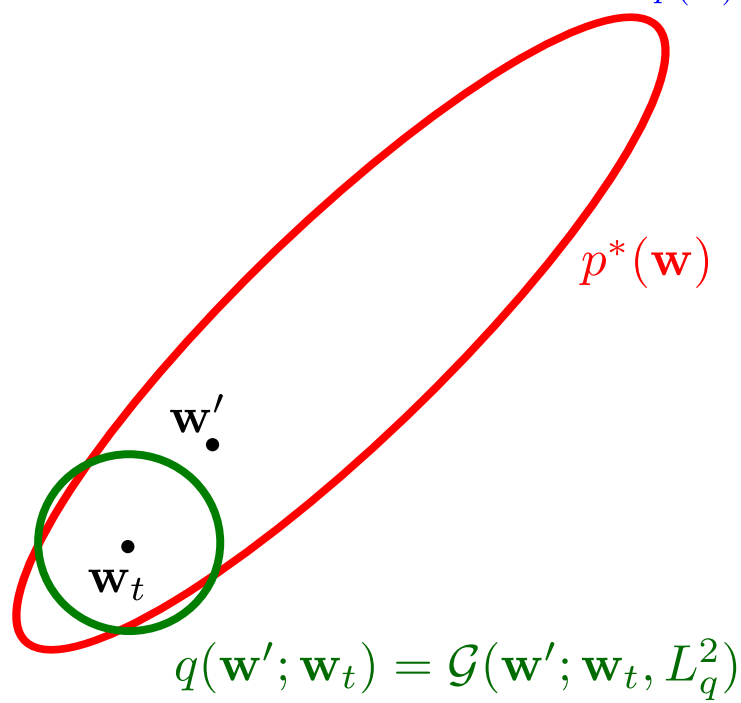

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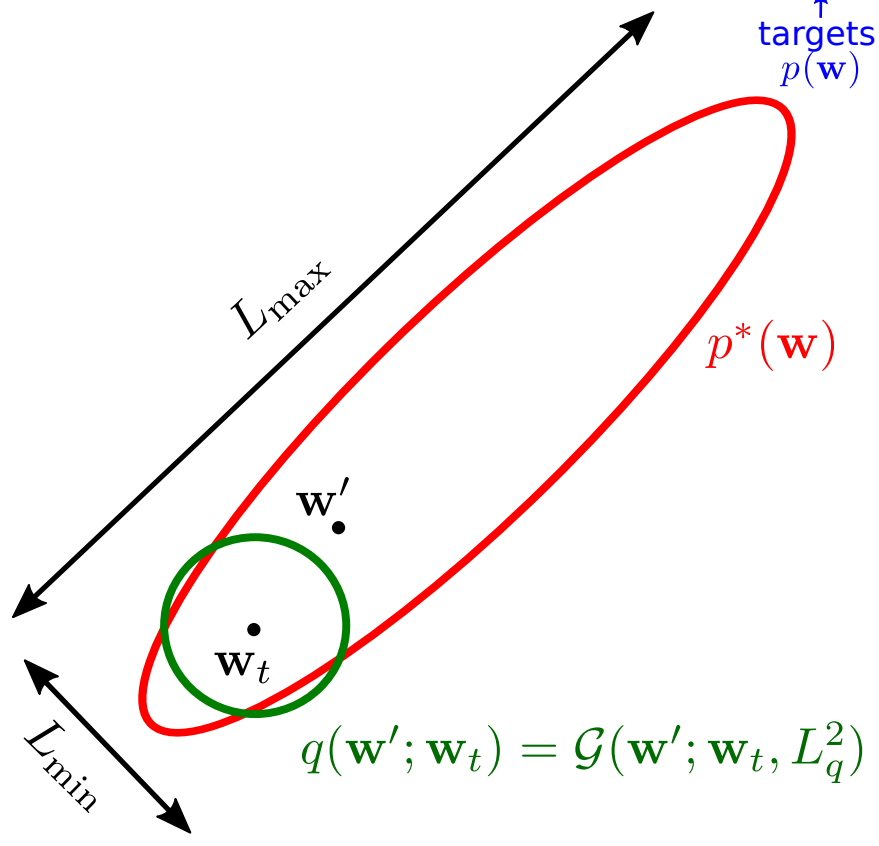

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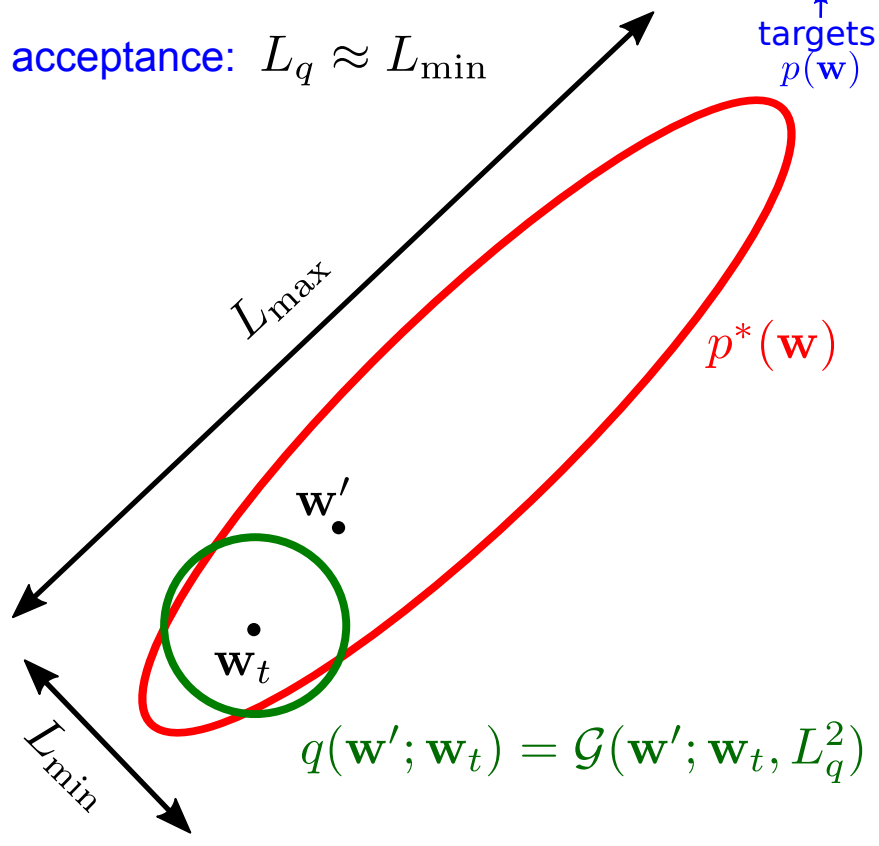

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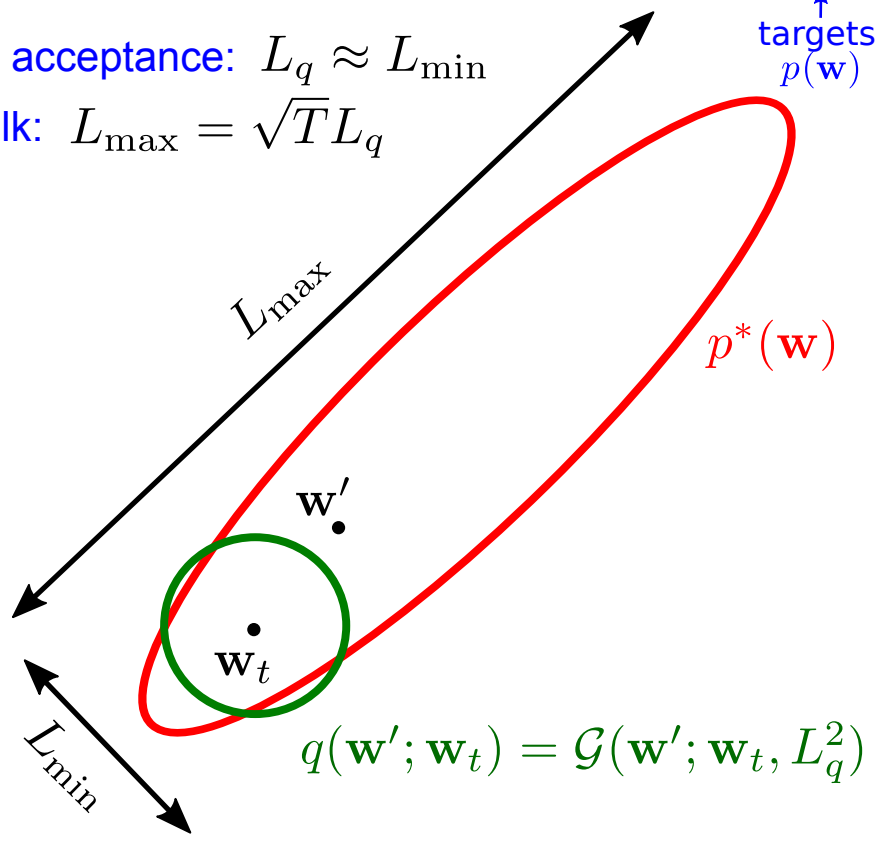

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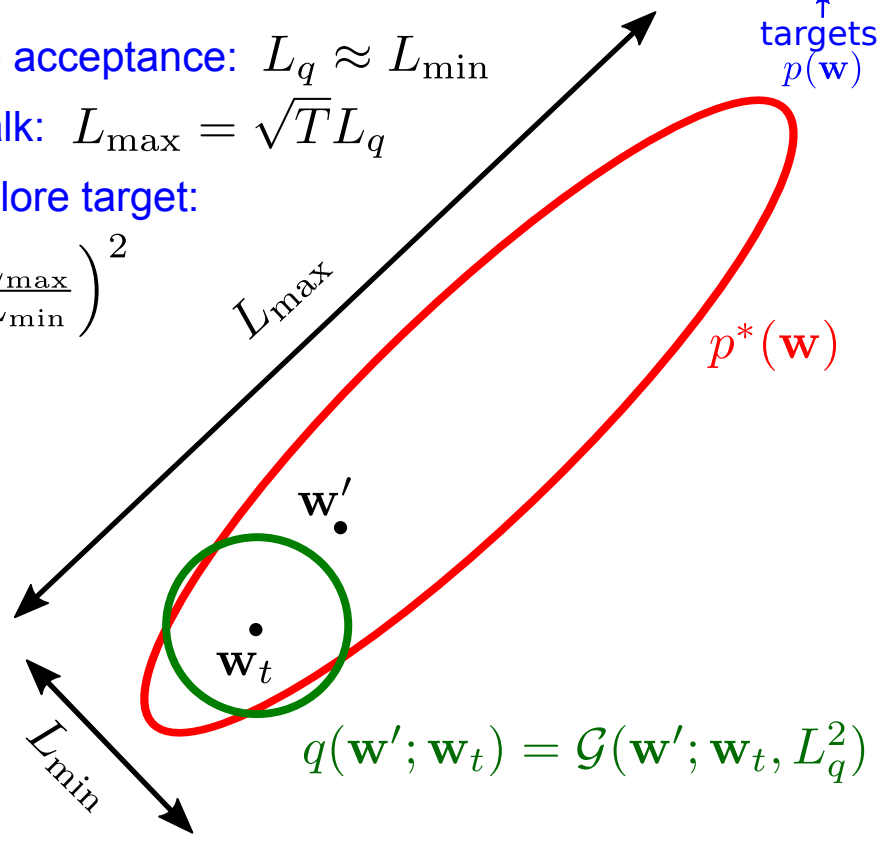

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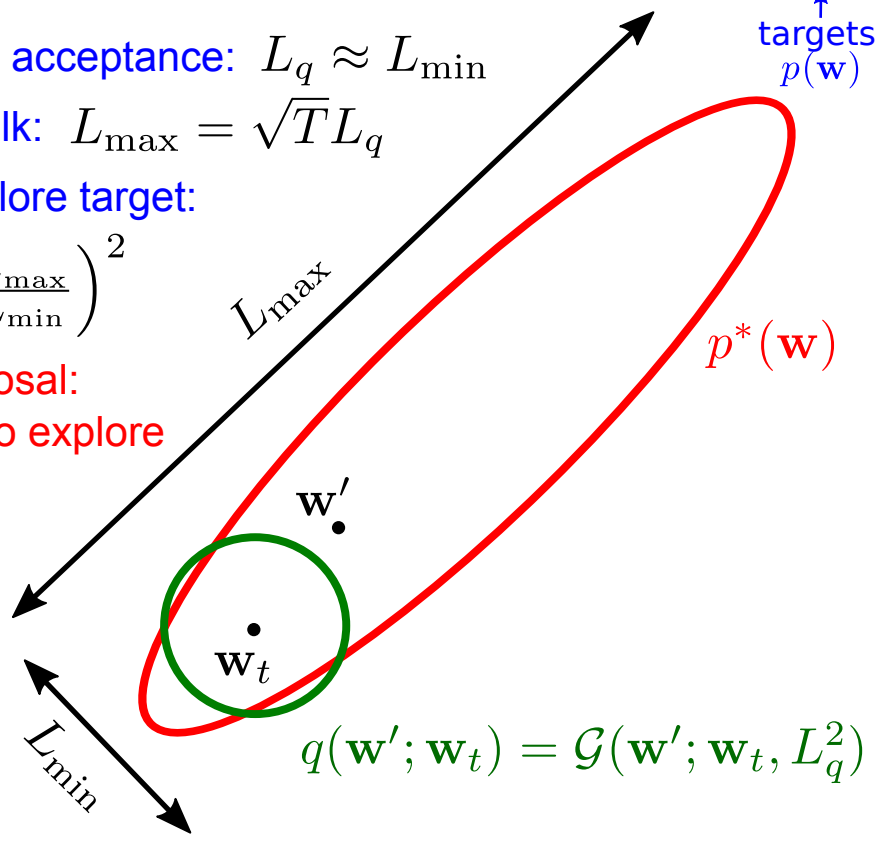
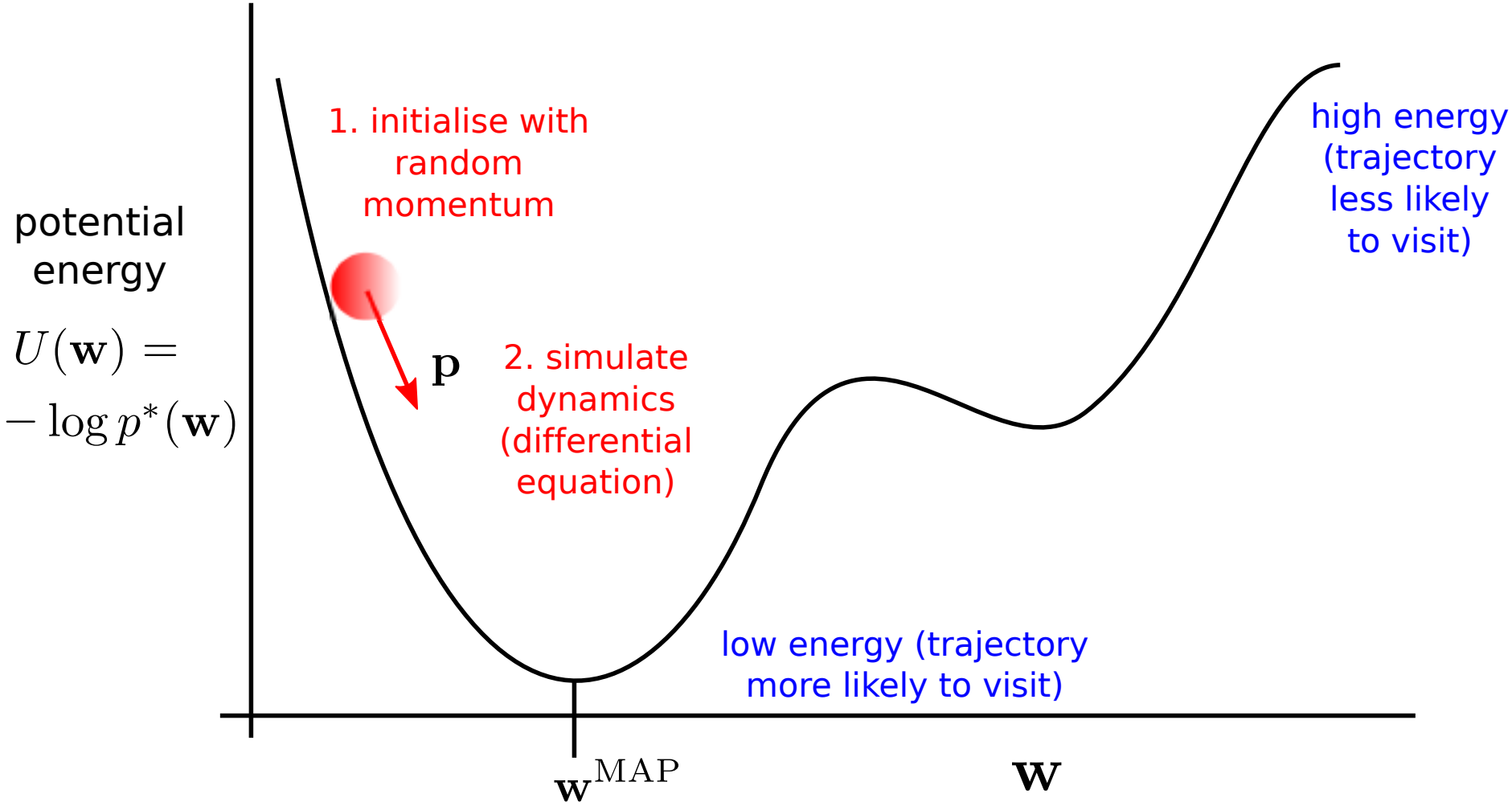
naive proposal:
very slow to explore



Hamiltonian Monte Carlo: Intuition

idea: use gradient information of log-target to shape MH proposal

construct physical system whose dynamics expore weight-space



Hamiltonian Monte Carlo: Neal 1994

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$$H(\mathbf{w}, \mathbf{p}) = U(\mathbf{w}) + \tfrac{1}{2}\mathbf{p}\mathbf{p}^\top$$

weights: spatial variables (potential energy)

$$U(\mathbf{w}) = -\log p^*(\mathbf{w})$$

augment with momentum variables (kinetic energy)

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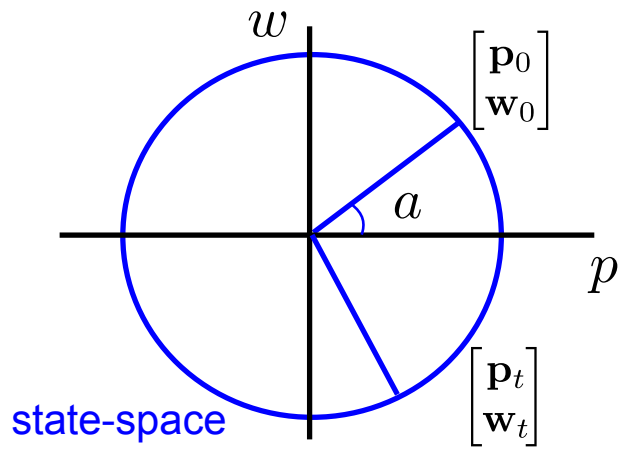

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$$\begin{bmatrix} \mathbf{p}_t \\ \mathbf{w}_t \end{bmatrix} = \mathrm{T}_t\left(\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{w}_0 \end{bmatrix}\right)$$

final state

dynamical mapping

initial state

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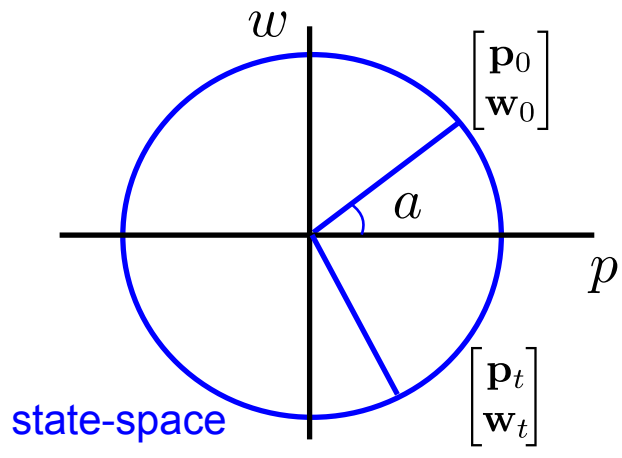

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$$\begin{bmatrix} \mathbf{p}_t \\ \mathbf{w}_t \end{bmatrix} = \mathrm{T}_t \left(\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{w}_0 \end{bmatrix} \right)$$

Reversibility (mapping has an inverse):

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{w} \end{bmatrix} = \mathrm{T}_{-t} \left(\mathrm{T}_t \left(\begin{bmatrix} \mathbf{p} \\ \mathbf{w} \end{bmatrix} \right) \right)$$

used to show detailed balance holds

$$H(w, p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = p \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -w$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} = -w \qquad \frac{\mathrm{d}^2p}{\mathrm{d}t^2} = -p$$

$$w(t) = r\cos(a + t)$$

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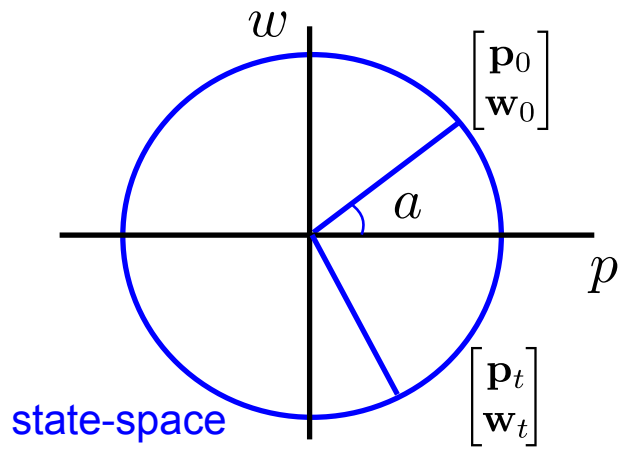

Hamiltonian Monte Carlo: Neal 1994

idea: use gradient information of log-target to shape MH proposal

$$H(\mathbf{w}, \mathbf{p}) = U(\mathbf{w}) + \tfrac{1}{2}\mathbf{p}\mathbf{p}^\top$$

weights: spatial variables (potential energy)

$$U(\mathbf{w}) = -\log p^*(\mathbf{w})$$

augment with momentum variables (kinetic energy)

Hamiltonian dynamics:

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial w_i}$$

Energy conservation (no dissipation)

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \frac{\partial H}{\partial w_i}\frac{\mathrm{d}w_i}{\mathrm{d}t} + \sum_i \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}$$

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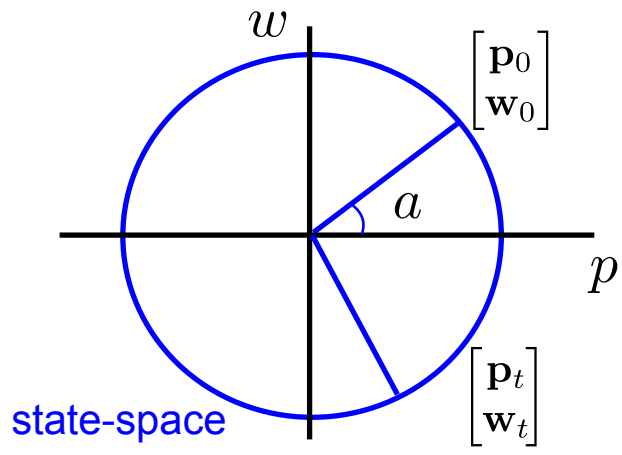

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$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_i \frac{\partial H}{\partial w_i}\frac{\mathrm{d}w_i}{\mathrm{d}t} + \sum_i \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \\ &= \sum_i \frac{\partial H}{\partial w_i}\frac{\partial H}{\partial p_i} - \sum_i \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial w_i} = 0\end{aligned}$$

$$H(w, p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

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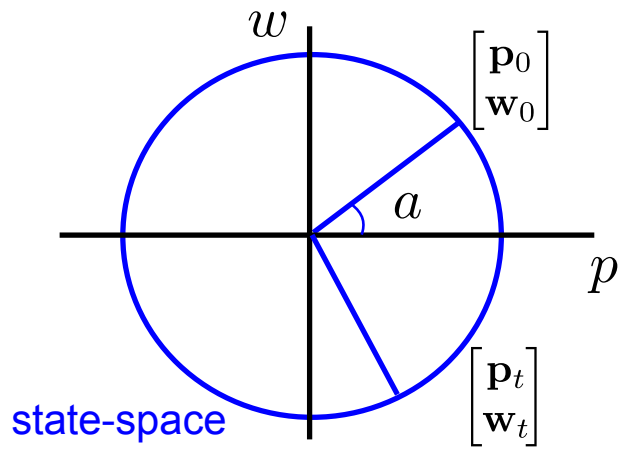

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$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \frac{\partial H}{\partial w_i}\frac{\mathrm{d}w_i}{\mathrm{d}t} + \sum_i \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}$$

$$= \sum_i \frac{\partial H}{\partial w_i}\frac{\partial H}{\partial p_i} - \sum_i \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial w_i} = 0$$

if H = constant, acceptance rate of HMC =1 (numerics)

$$H(w, p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

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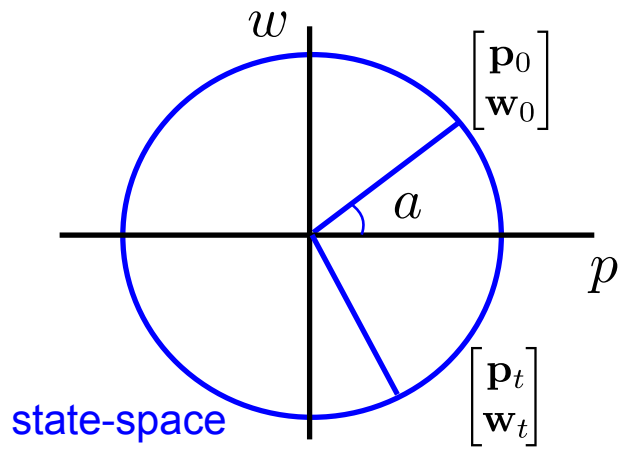

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Hamiltonian dynamics:

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial w_i}$$

Volume preservation (Liouville's theorem)

$$\begin{bmatrix} \mathbf{p}_t \\ \mathbf{w}_t \end{bmatrix} = \mathrm{T}_t\left(\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{w}_0 \end{bmatrix}\right)$$

image of region of state-space has volume indepedent of t
shape of region will change over time
(don't need to account for Jacobians of T in MCMC)

$$H(w, p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = p \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -w$$

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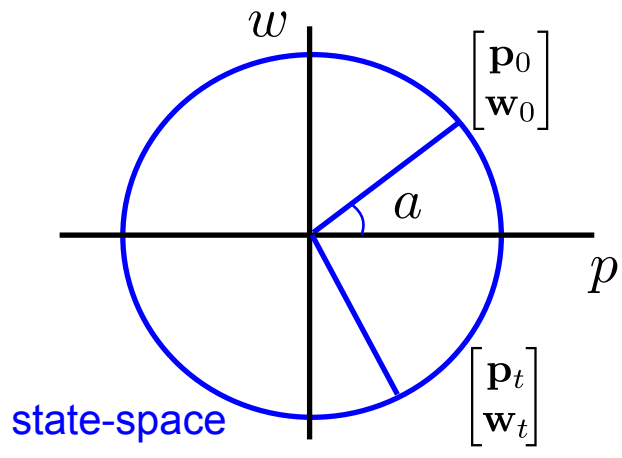

Hamiltonian Monte Carlo: Neal 1994

goal: target joint using MCMC, discard momenta to get samples from $p(\mathbf{w})$

$$H(\mathbf{w},\mathbf{p}) = U(\mathbf{w}) + \tfrac{1}{2}\mathbf{p}\mathbf{p}^\top \qquad p^*(\mathbf{w},\mathbf{p}) = \exp(-H(\mathbf{w},\mathbf{p})) \propto p^*(\mathbf{w})\mathcal{G}(\mathbf{p};0,\mathbf{I})$$

variables independent

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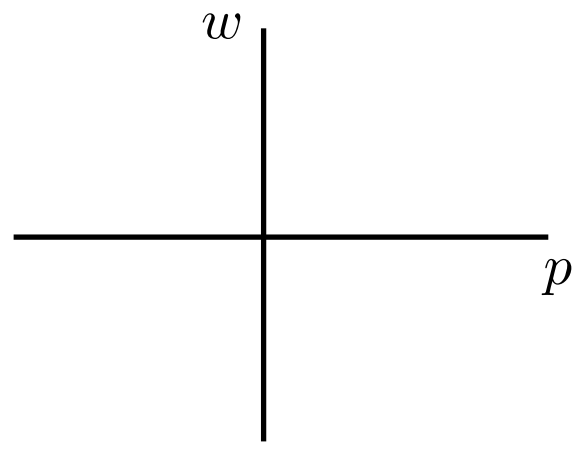

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variables independent

Move 1: resample momenta from marginal (leaves dist invariant)

$$\mathbf{p} \sim \mathcal{G}(\mathbf{p};0,\mathbf{I}) \qquad \{\mathbf{w}_t,\mathbf{p}_t\} \longrightarrow \{\mathbf{w}_t,\mathbf{p}\}$$

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$$w(t) = r\cos(a+t)$$

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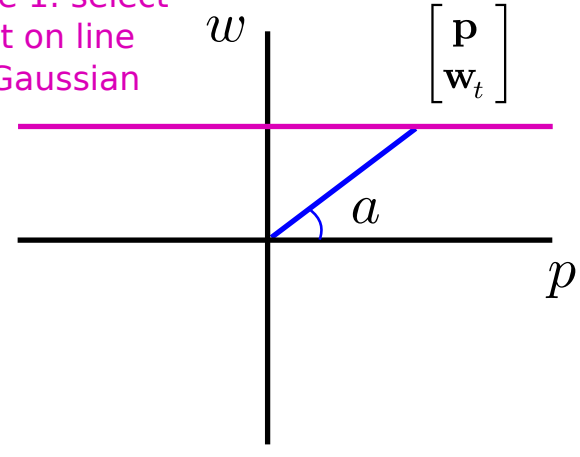
Move 1: select point on line via Gaussian



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Move 2: use Hamiltonian dynamics to make MH update:

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial w_i} \longrightarrow \{\mathbf{p}',\mathbf{w}'\}$$

$$H(w,p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

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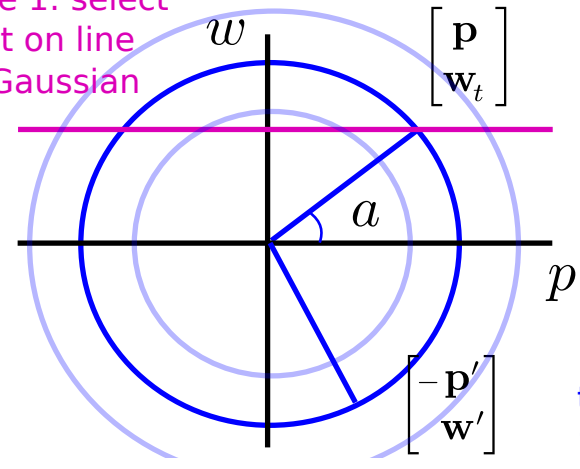

Move 2: simulate trajectory for time s

Hamiltonian Monte Carlo: Neal 1994

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reverse momentum at end (symmetric proposal)

$$H(w,p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

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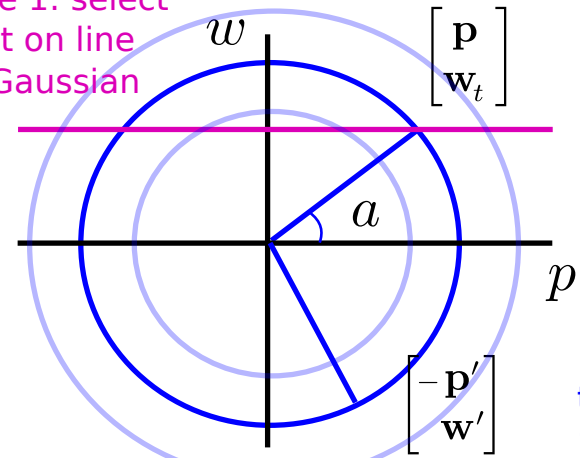

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reverse momentum at end (symmetric proposal)

$$a = \exp(H(\mathbf{w}_t,\mathbf{p}) - H(\mathbf{w}',\mathbf{p}')) = 1$$

if simulation perfect

$$H(w,p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

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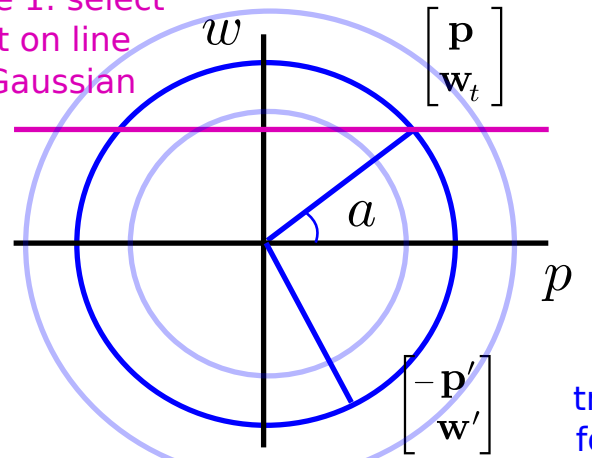

Move 2: simulate trajectory for time s

Hamiltonian Monte Carlo: Neal 1994

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reverse momentum at end (symmetric proposal)

$$a = \exp(H(\mathbf{w}_t,\mathbf{p}) - H(\mathbf{w}',\mathbf{p}')) = 1$$

if simulation perfect

accept if $a \geq 1$ $\{\mathbf{w}_{t+1},\mathbf{p}_{t+1}\} = \{\mathbf{p}',\mathbf{w}'\}$

else

accept with prob a $\{\mathbf{w}_{t+1},\mathbf{p}_{t+1}\} = \{\mathbf{p}',\mathbf{w}'\}$

reject otherwise $\{\mathbf{w}_{t+1},\mathbf{p}_{t+1}\} = \{\mathbf{w}_t,\mathbf{p}\}$

endif

$$H(w,p) = \tfrac{1}{2}w^2 + \tfrac{1}{2}p^2$$

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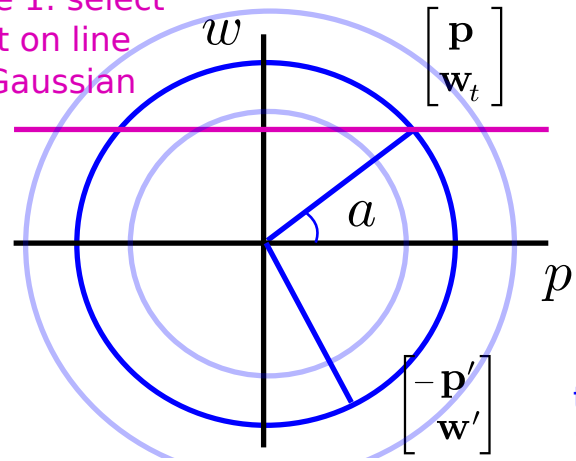


Move 2: simulate trajectory for time s

Hamiltonian Monte Carlo: Neal 1994

```
g = gradE ( x ) ;                  # set gradient using initial x
E = findE ( x ) ;                  # set objective function too

for l = 1:L                        # loop L times
   p = randn ( size(x) ) ;         # initial momentum is Normal(0,1)
   H = p' * p / 2 + E ;            # evaluate H(x,p)

   xnew = x ;  gnew = g ;
   for tau = 1:Tau                 # make Tau 'leapfrog' steps

      p = p - epsilon * gnew / 2 ; # make half-step in p
      xnew = xnew + epsilon * p ;  # make step in x
      gnew = gradE ( xnew ) ;      # find new gradient
      p = p - epsilon * gnew / 2 ; # make half-step in p

   endfor

   Enew = findE ( xnew ) ;         # find new value of H
   Hnew = p' * p / 2 + Enew ;
   dH = Hnew - H ;                 # Decide whether to accept

   if ( dH < 0 )                   accept = 1 ;
   elseif ( rand() < exp(-dH) )    accept = 1 ;
   else                            accept = 0 ;
   endif

   if ( accept )
      g = gnew ;   x = xnew ;    E = Enew ;
   endif
endfor
```

Move 1: sample momentum:

$$\mathbf{p} \sim \mathcal{G}(\mathbf{p}; 0, \mathbf{I})$$

Move 2: Hamiltonian dynamics for MH

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial w_i}$$

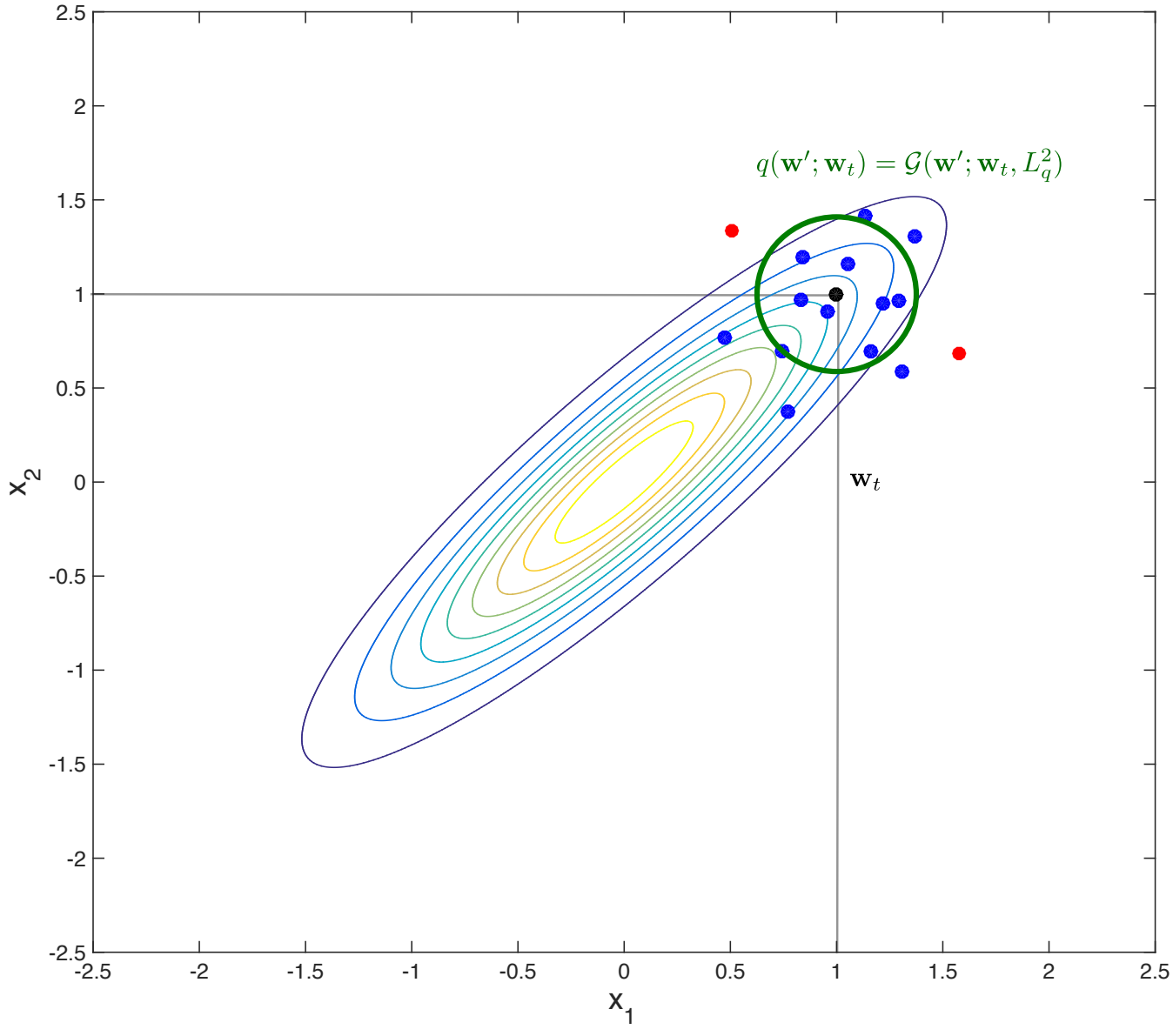
(leap-frog steps preserve volume exactly, but do not exactly preserve energy)

Accept/reject

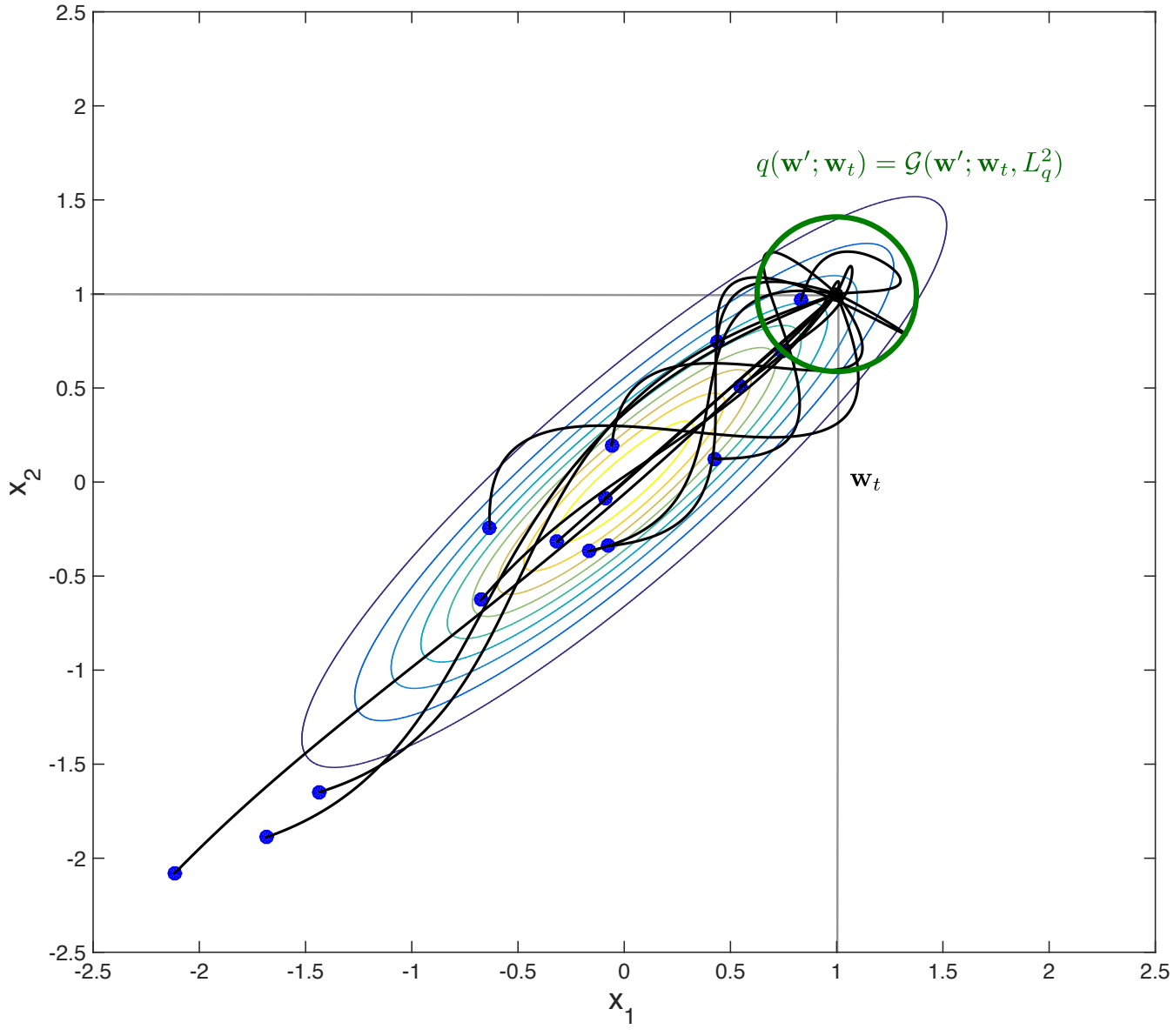
$$a = \exp(H(\mathbf{w}_t, \mathbf{p}) - H(\mathbf{w}', \mathbf{p}'))$$

pseudo-code from Mackay ITILA pg. 388

Isotropic MH proposal: Demo 2D Gaussian 15 proposals from [1,1]



Hamiltonian Monte Carlo: Demo 2D Gaussian 15 proposals from [1,1]



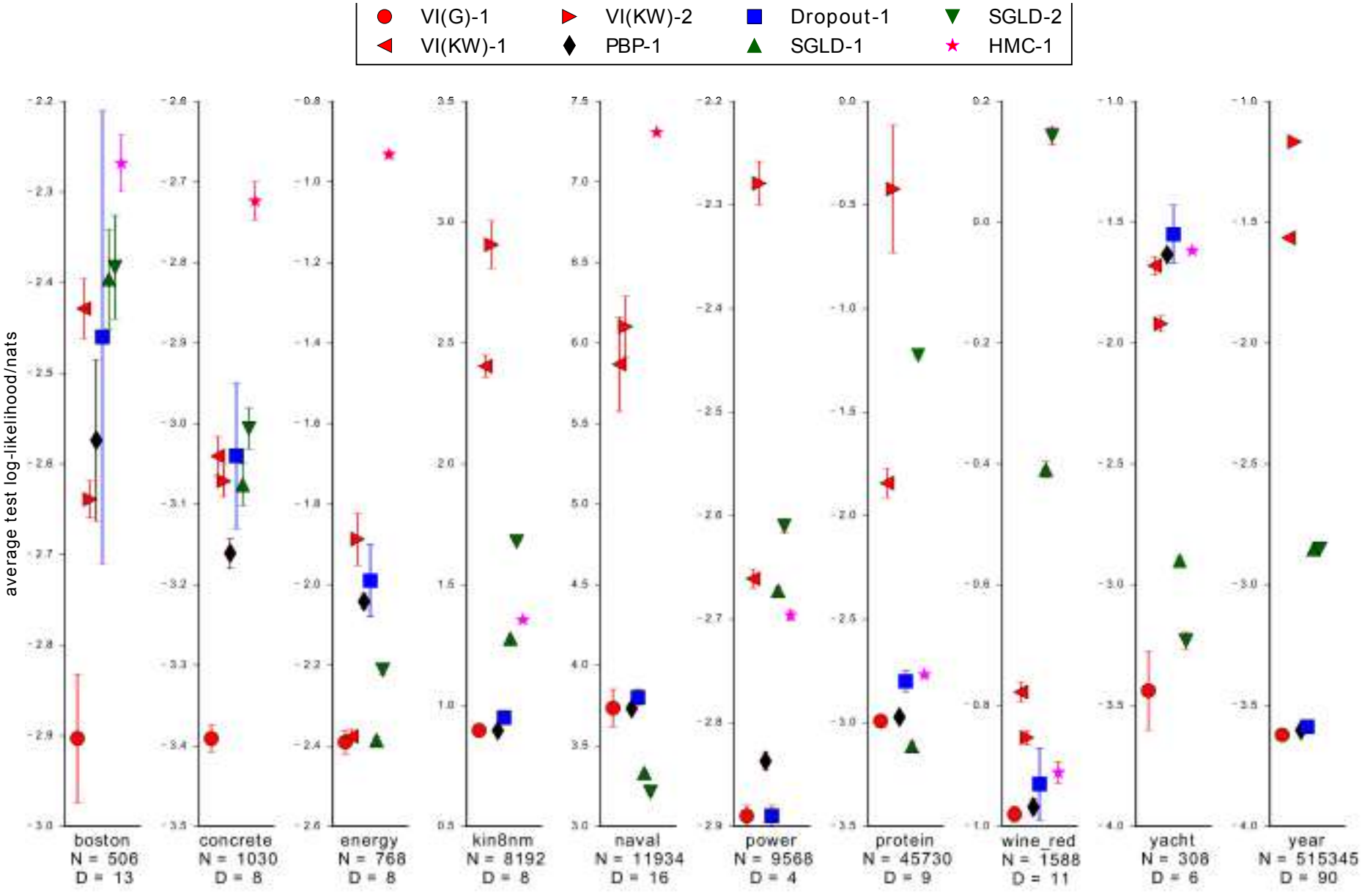
Recent developments

- HMC with one leapfrog step results in Langevin sampling: $q(\mathbf{w}'|\mathbf{w}_t) = \mathcal{G}\left(\mathbf{w}'; \mathbf{w}_t + \epsilon \frac{\mathrm{d} \log p^*(\mathbf{w})}{\mathrm{d}\mathbf{w}}, 2\epsilon\right)$
- Langevin proposal = gradient ascent + Gaussian noise
- Welling and Teh: **stochastic gradient Langevin dynamics** (SGLD):
  - stochastic gradient ascent + Gaussian noise = valid posterior sampler
  - SGA: Robbins-Munro conditions guarantee convergence
  - SGLD: Robbins-Munro conditions applied to ϵ also allow MH step to be dispensed with (nice coincidence)
- SGLD is very slow to mix
- Generalisations to HMC are in development

Bayesian Learning via Stochastic Gradient Langevin Dynamics, Welling and Teh, ICML, 2011

Stochastic Gradient Hamiltonian Monte Carlo, Chen, Fox, Guestrin, NIPS, 2015

Comparison of methods (held out log-likelihood)



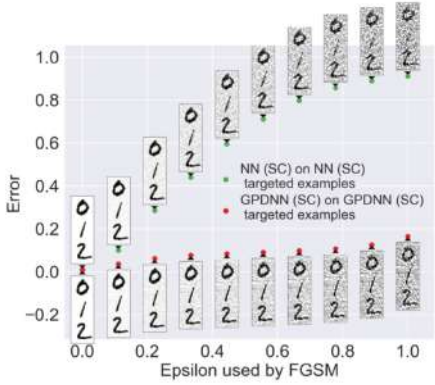
Case study 1. Robust Deep Learning
Uncertainty calibration & adversarial examples

fragile (adversarial examples)

correct diff ostrich

fragile (adversarial examples)

correct diff ostrich

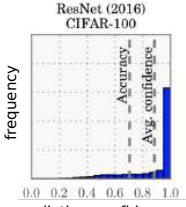


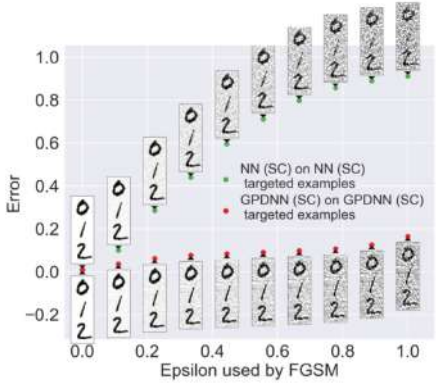
Towards Robust Deep Learning

fragile (adversarial examples)

correct diff ostrich

well calibrated uncertainty estimates: deep learning is often confidently wrong





fragile (adversarial examples)

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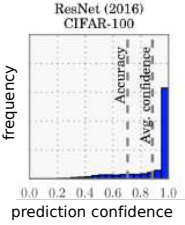
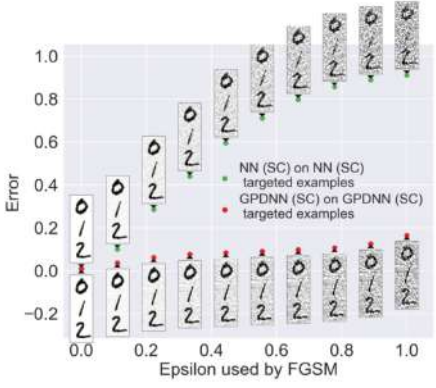
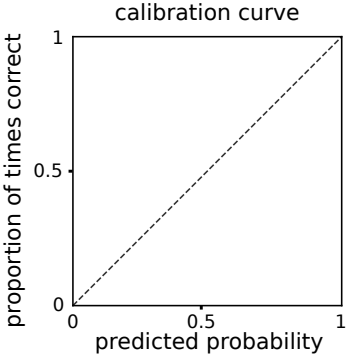
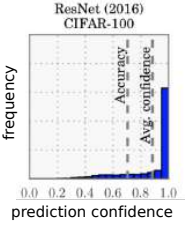


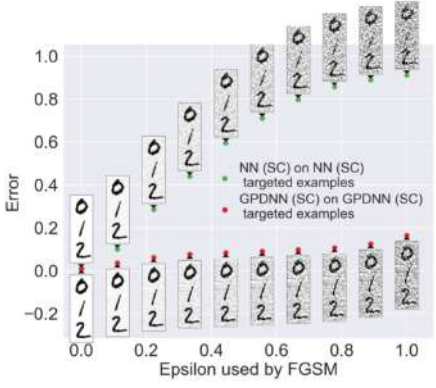


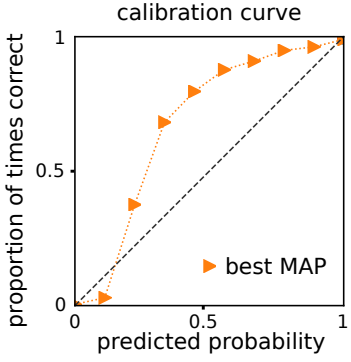


fragile (adversarial examples)

well calibrated uncertainty estimates: deep learning is often confidently wrong



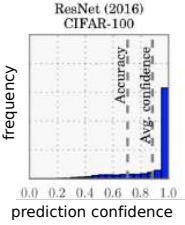


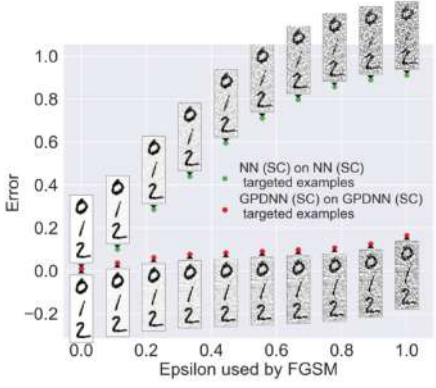


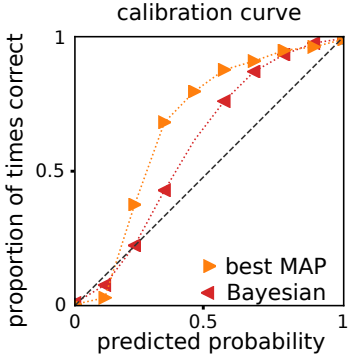
Towards Robust Deep Learning

fragile (adversarial examples)

well calibrated uncertainty estimates: deep learning is often confidently wrong







References: Case Study 1

Bayesian methods yield neural networks that have **better calibrated predictions** and which are **more robust to adversarial examples**

J. Bradshaw et al. Adversarial Examples, Uncertainty, and Transfer Testing Robustness in Gaussian Process Hybrid Deep Networks, arXiv, 2017

M. Bauer et al. Discriminative k-shot learning using probabilistic models, arXiv, 2017

C. Guo et al. On Calibration of Modern Neural Networks, ICML 2017

See also

Y. Li and Y. Gal Dropout Inference in Bayesian Neural Networks with Alpha-divergences, ICML 2017

Case study 2. Flexible Deep Learning
Continual multi-task learning

What is Continual Learning?

"corner cutter" or "line pin"

 ?

 ?

What is Continual Learning?

corner cutter

line pins

"corner cutter" or "line pin"

?

?

collect data set online

predictions

What is Continual Learning?

corner cutter

line pins

"corner cutter" or "line pin"

?

?

collect data set online

predictions

What is Continual Learning?

corner cutter

line pins

"corner cutter" or "line pin"

?

?

collect data set online

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

"corner cutter" or "line pin"

?

?

collect data set online

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

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predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

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?

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

?

?

predictions

What is Continual Learning?

corner cutter

line pins

"corner cutter" or "line pin"

when new examples seen
incrementally update:

new model = f(old model,new data)

collect data set online

predictions

What is Continual Learning?

corner cutter

line pins

collect data set online

"corner cutter" or "line pin"

predictions

What is Continual Learning?

corner cutter

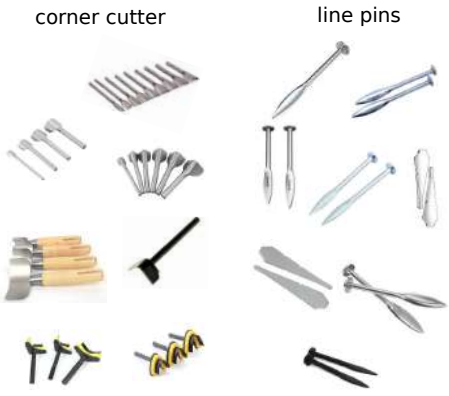
line pins

collect data set online

"corner cutter" or "line pin"

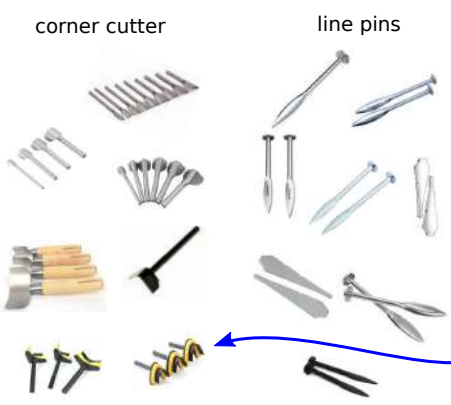
predictions

What is Continual Learning?



"corner cutter" or "line pin"

What is Continual Learning?



statistics of data may change over time: continuously adapt without revisiting all old data

"corner cutter" or "line pin"

What is Continual Learning?

corner cutter

line pins

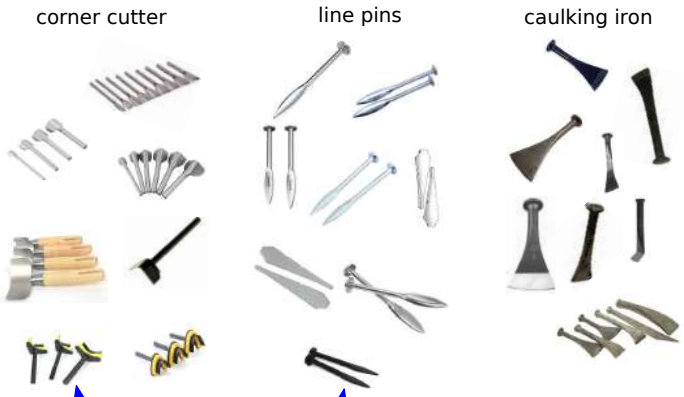
What is Continual Learning?

corner cutter

line pins

caulking iron

What is Continual Learning?



What is Continual Learning?

?

corner cutter

line pins

caulking iron

What is Continual Learning?

corner cutter

line pins

caulking iron

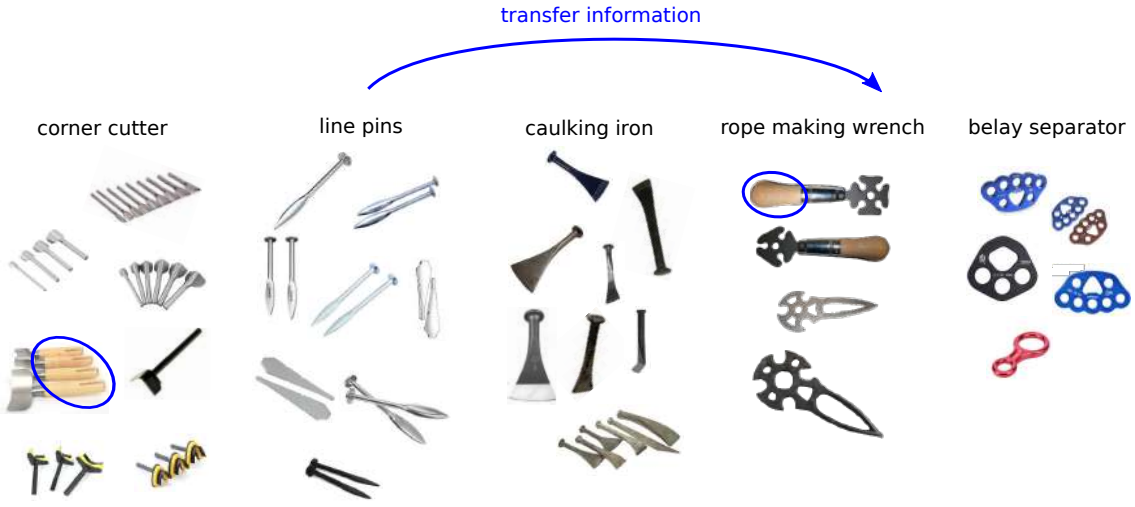
rope making wrench

belay separator

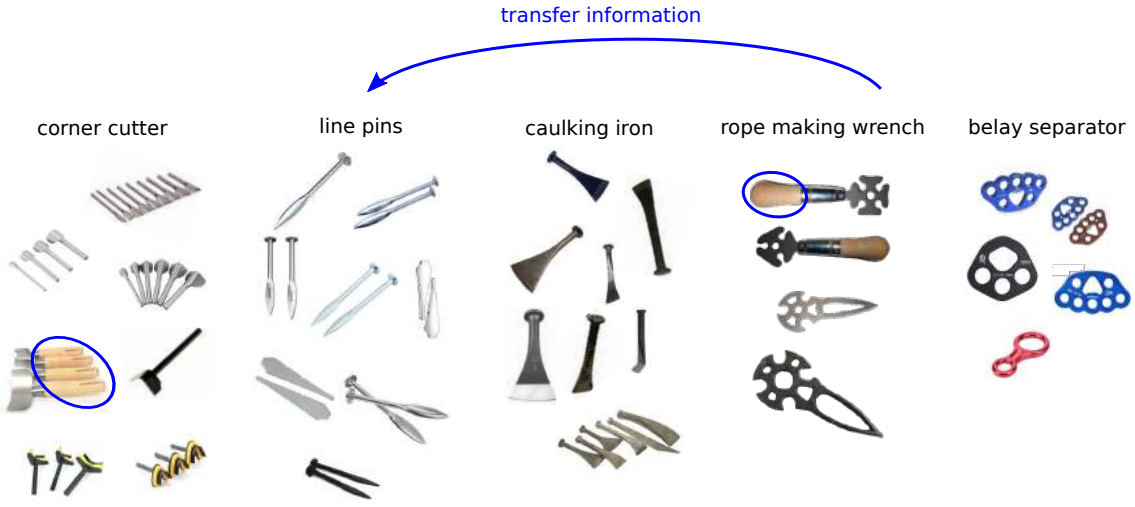
TASK 1

TASK 2

What is Continual Learning?



What is Continual Learning?



What is Continual Learning?

?

corner cutter

line pins

caulking iron

rope making wrench

belay separator

TASK 1

TASK 2

What is Continual Learning?

?

corner cutter

line pins

caulking iron

rope making wrench

belay separator

TASK 1

TASK 2

What is Continual Learning?

avoid forgetting old tasks

?

corner cutter

line pins

caulking iron

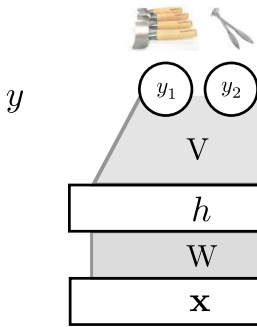
rope making wrench

belay separator

TASK 1

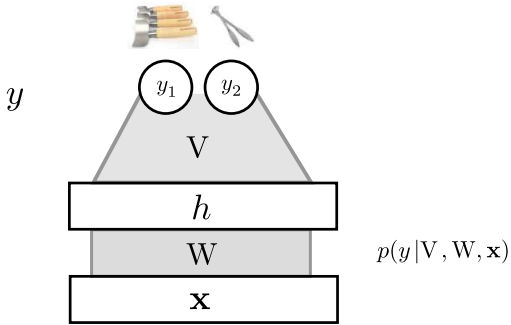
TASK 2

A zoo of discriminative continual learning tasks



$p(y|\mathrm{V}, \mathrm{W}, \mathbf{x})$

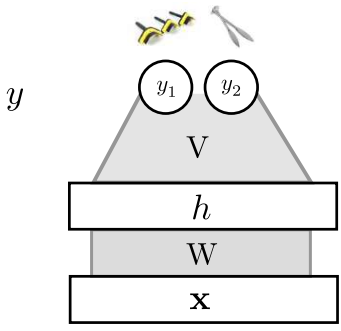
A zoo of discriminative continual learning tasks



1. online iid data (online learning)

$$p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p(\mathbf{x}_n)$$

A zoo of discriminative continual learning tasks



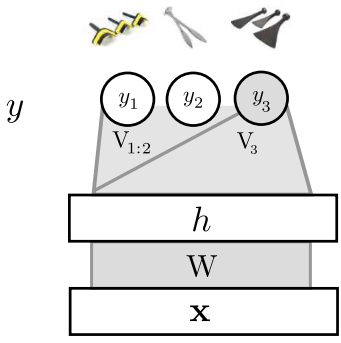
$p(y|\mathrm{V}, \mathrm{W}, \mathbf{x})$

1. online iid data (online learning) $p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p(\mathbf{x}_n)$

2. online non-iid inputs (covariate shift) $p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p_n(\mathbf{x}_n|\mathbf{x}_{1:n-1})$

A zoo of discriminative continual learning tasks

3. new classes
(k-shot learning)



$p(y|\mathrm{V}_{1:2}, \mathrm{V}_3, \mathrm{W}, \mathbf{x})$

1. online iid data
(online learning)

$p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p(\mathbf{x}_n)$

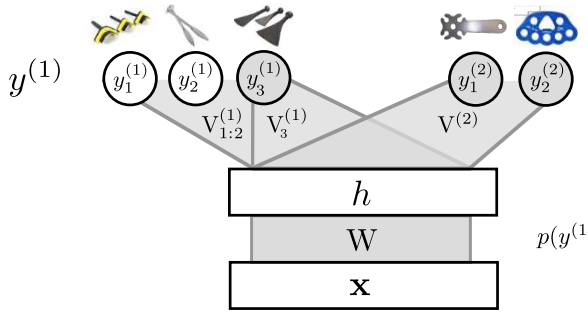
2. online non-iid inputs
(covariate shift)

$p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p_n(\mathbf{x}_n|\mathbf{x}_{1:n-1})$

A zoo of discriminative continual learning tasks

3. new classes
(k-shot learning)

4. new tasks
(online multi-task learning
transfer learning)



$p(y^{(1)}, y^{(2)} | \mathrm{V}^{(1)}, \mathrm{V}^{(2)}, \mathrm{W}, \mathbf{x})$

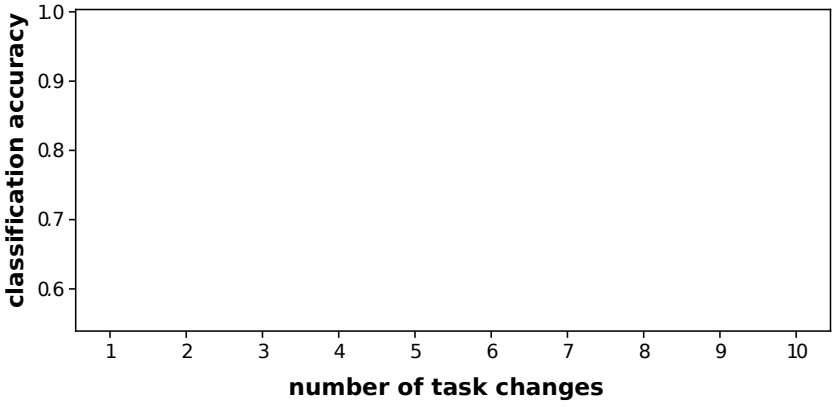
1. online iid data
(online learning)

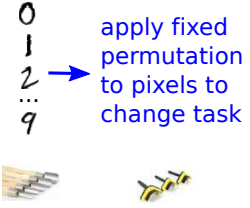
$$p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p(\mathbf{x}_n)$$

2. online non-iid inputs
(covariate shift)

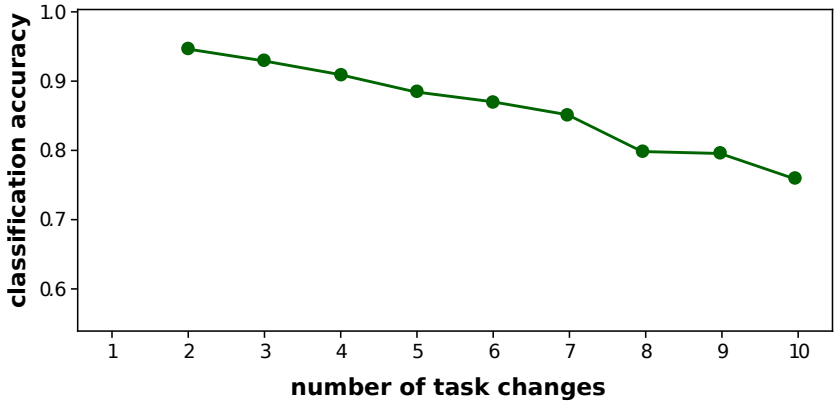
$$p(\mathbf{x}_{1:N}) = \prod_{n=1}^{N} p_n(\mathbf{x}_n | \mathbf{x}_{1:n-1})$$

Continual Learning Test 1: Permuted MNIST (online non-iid inputs, single head)

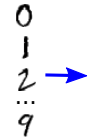




Continual Learning Test 1: Permuted MNIST (online non-iid inputs, single head)

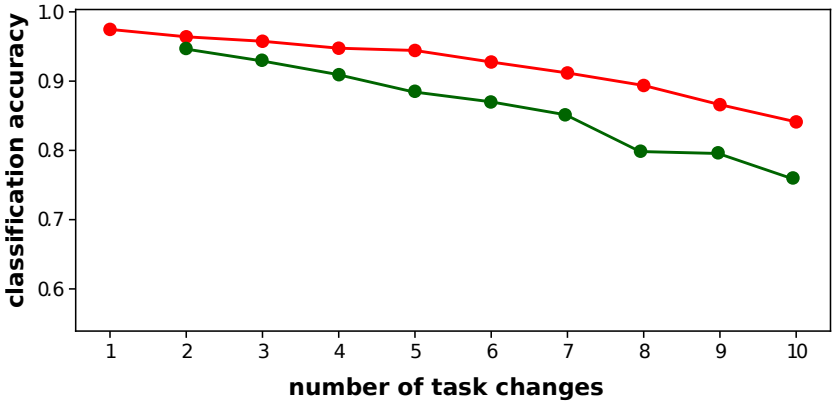


Deep baseline

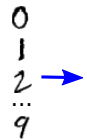


apply fixed permutation to pixels to change task

Continual Learning Test 1: Permuted MNIST (online non-iid inputs, single head)

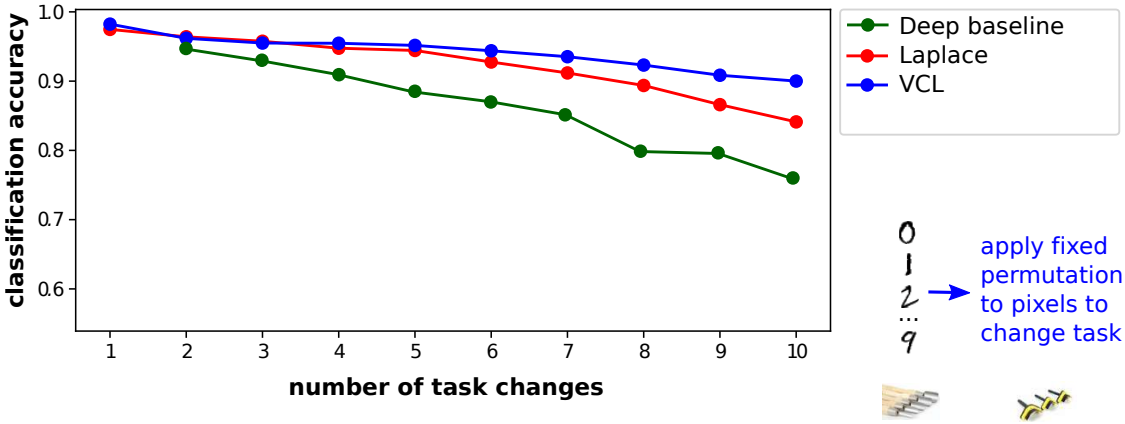


Deep baseline
Laplace

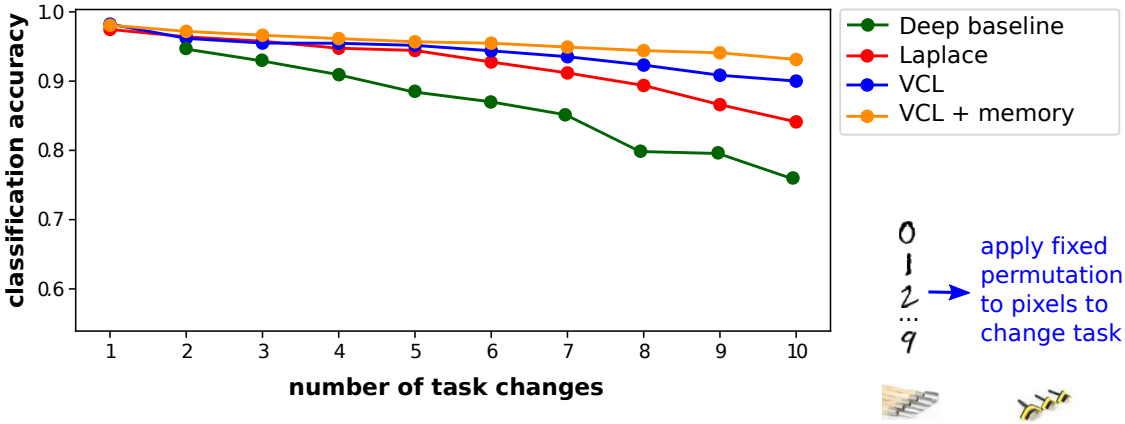


apply fixed permutation to pixels to change task

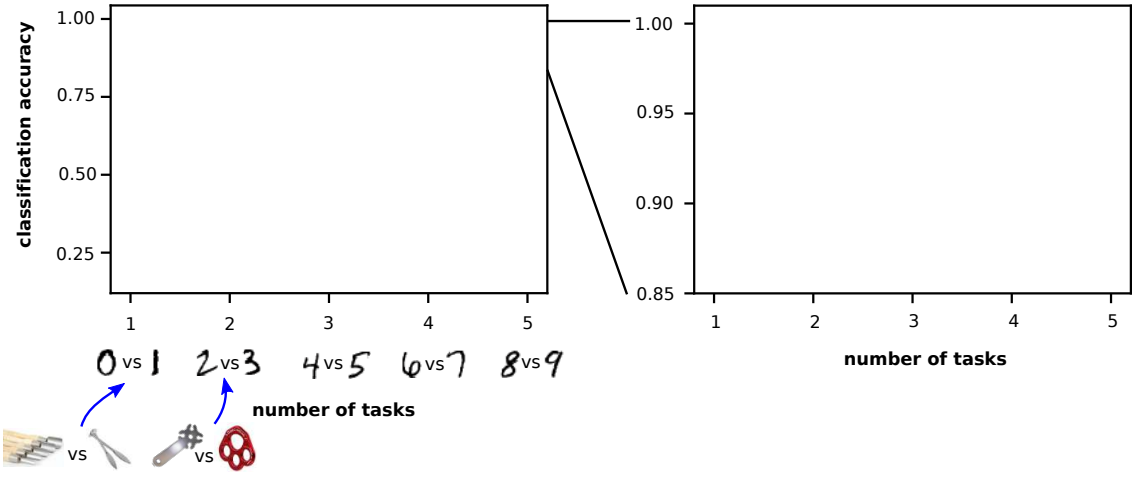
Continual Learning Test 1: Permuted MNIST (online non-iid inputs, single head)



Continual Learning Test 1: Permuted MNIST (online non-iid inputs, single head)



Continual Learning Test 2: Split MNIST (new tasks, multi-head)



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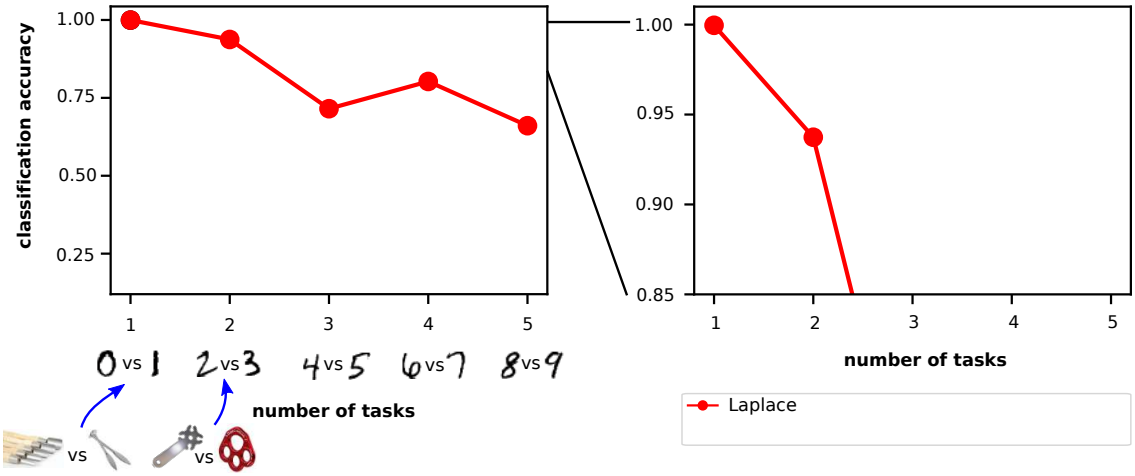
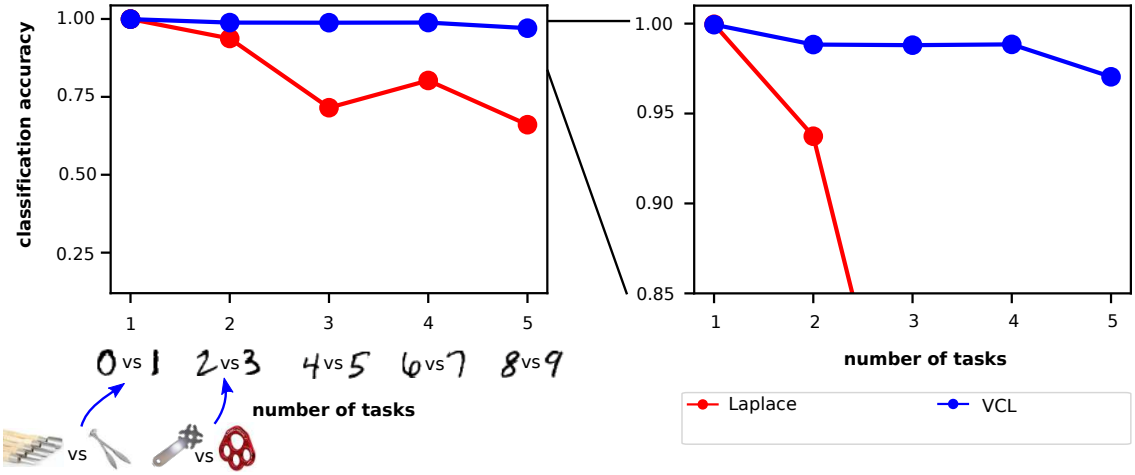


Continual Learning Test 2: Split MNIST (new tasks, multi-head)



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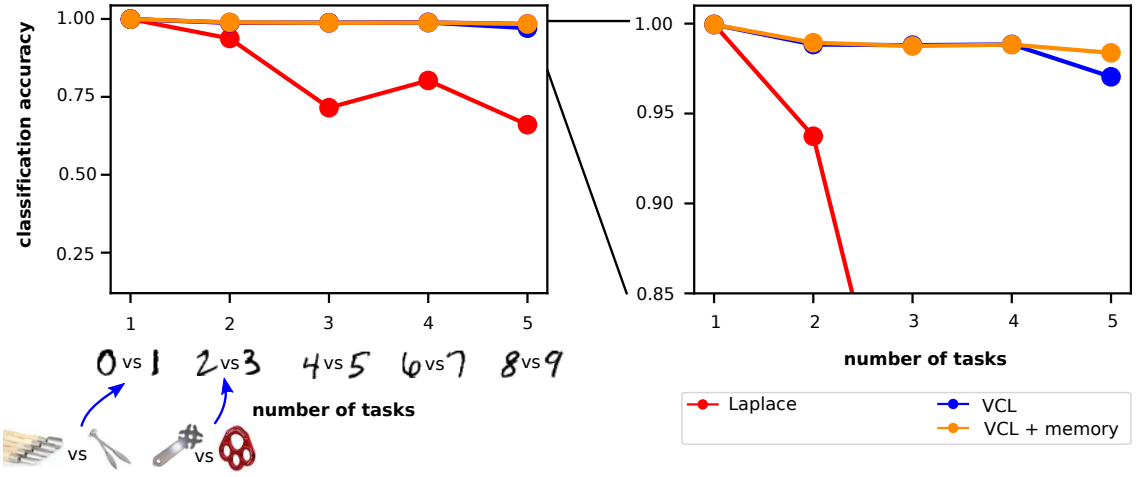


References: Case Study 2

- **Continual learning** is naturally handled by **Bayesian inference**: allows multi-task transfer and avoids catastrophic forgetting
- **Variational Continual Learning** is a **state-of-the-art continual learning method**
- Orthogonal research directions: **complex models** (adapting more than just the head of the network) and **online automatic model building**

J. Kirkpatrick et al. Overcoming catastrophic forgetting in neural networks, PNAS, 2017

C. Nguyen et al. Variational Continual Learning, ICLR 2018

- See also:

F. Huszar On Quadratic Penalties in Elastic Weight Consolidation, PNAS, 2018

F. Zenke et al. Continual Learning Through Synaptic Intelligence, ICML 2017

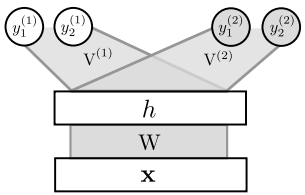
Case study 3: Data Efficient Deep Learning

One-shot learning

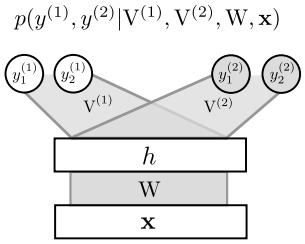
One shot learning using Approximate Bayesian inference

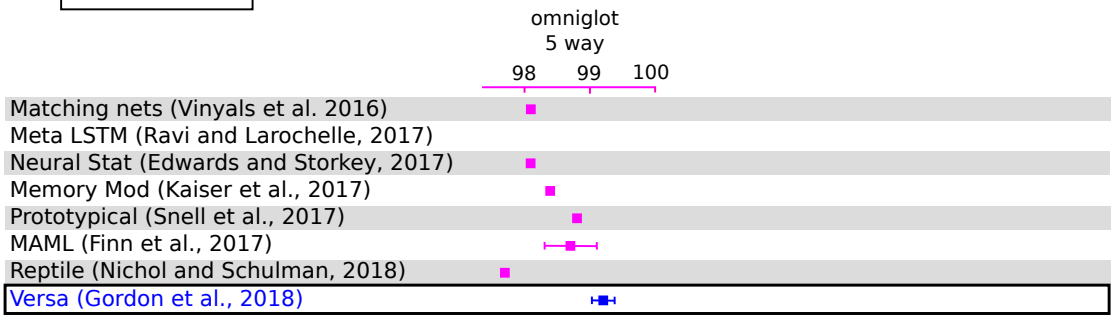
One shot learning using Approximate Bayesian inference

$$p(y^{(1)}, y^{(2)} | \mathrm{V}^{(1)}, \mathrm{V}^{(2)}, \mathrm{W}, \mathbf{x})$$

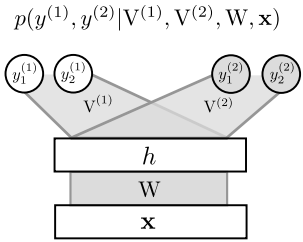


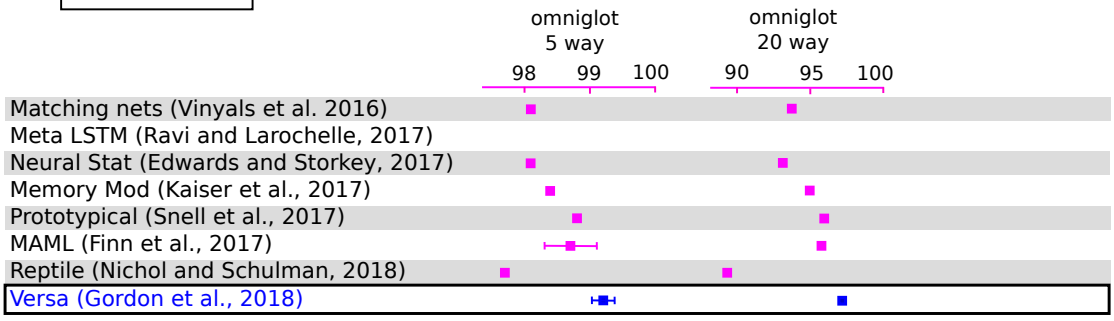
One shot learning using Approximate Bayesian inference



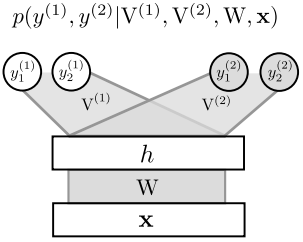


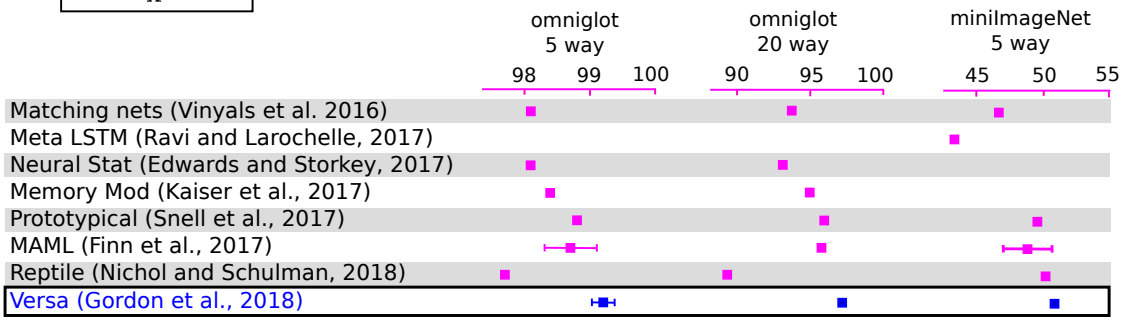
One shot learning using Approximate Bayesian inference





One shot learning using Approximate Bayesian inference





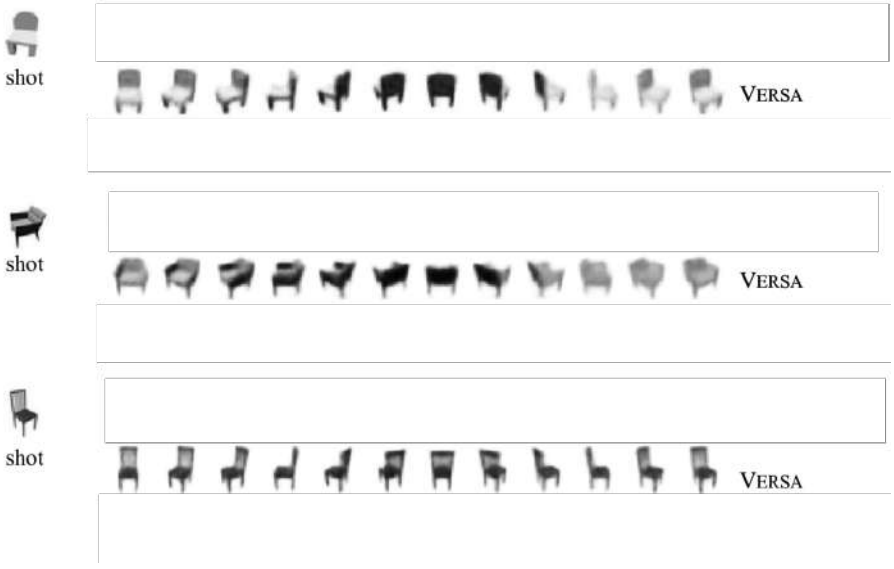
One shot learning using Approximate Bayesian inference


shot

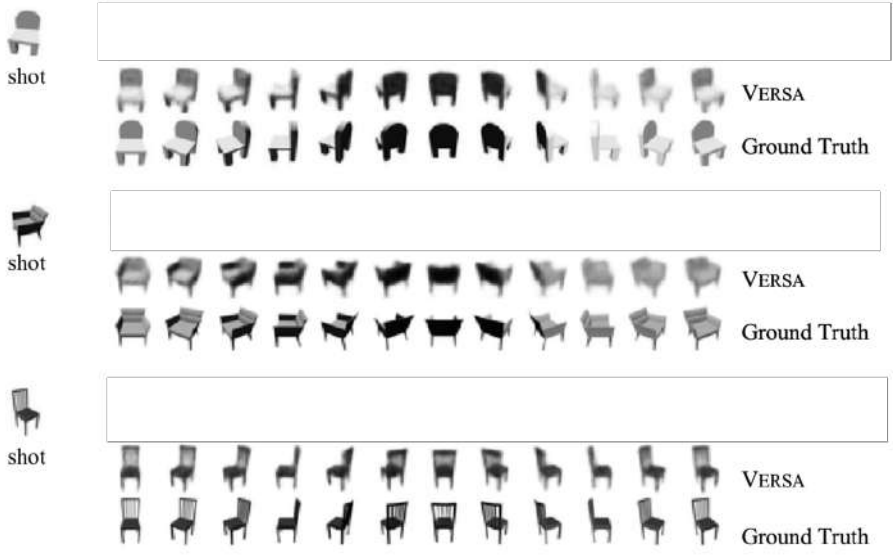

shot


shot

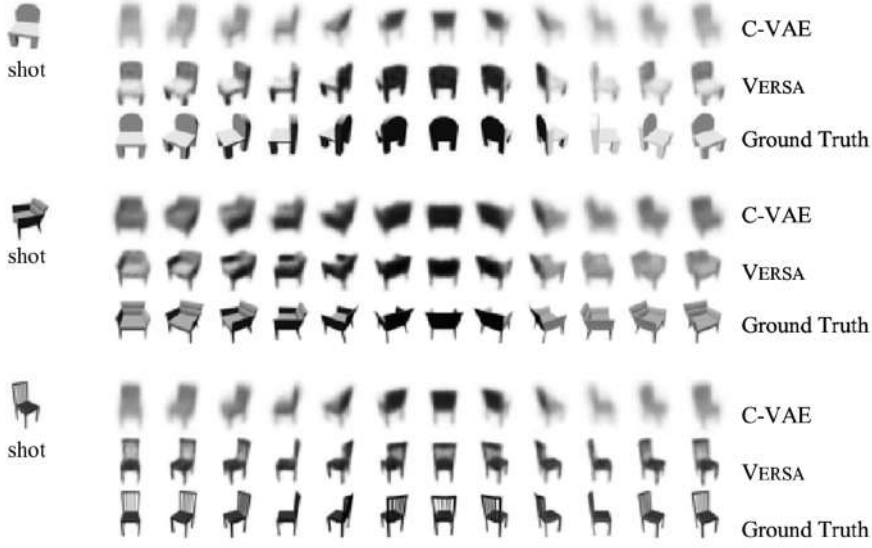
One shot learning using Approximate Bayesian inference



One shot learning using Approximate Bayesian inference



One shot learning using Approximate Bayesian inference



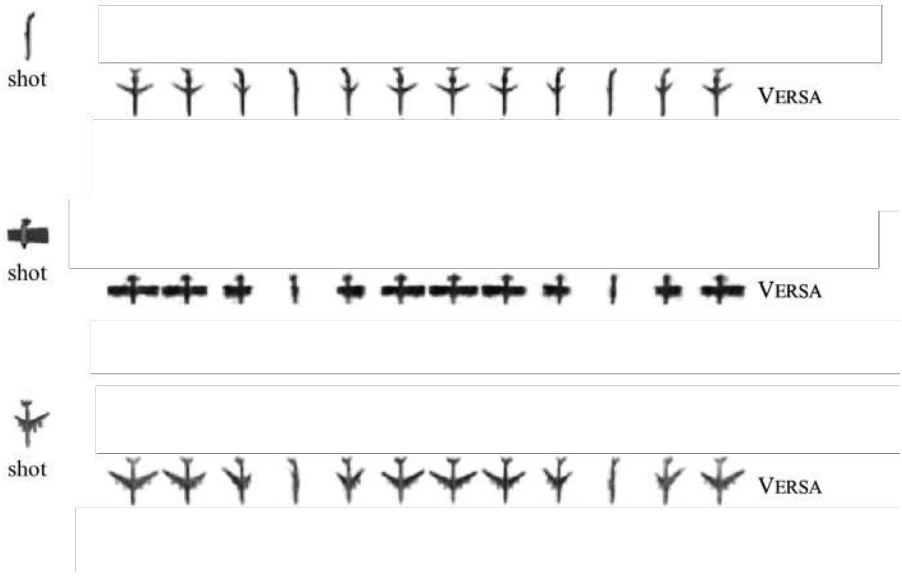
One shot learning using Approximate Bayesian inference


shot

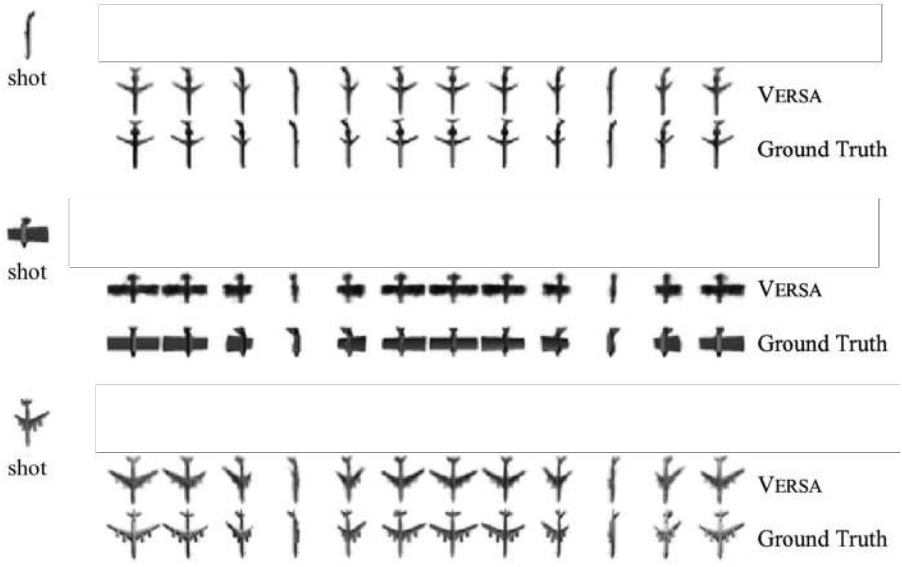

shot


shot

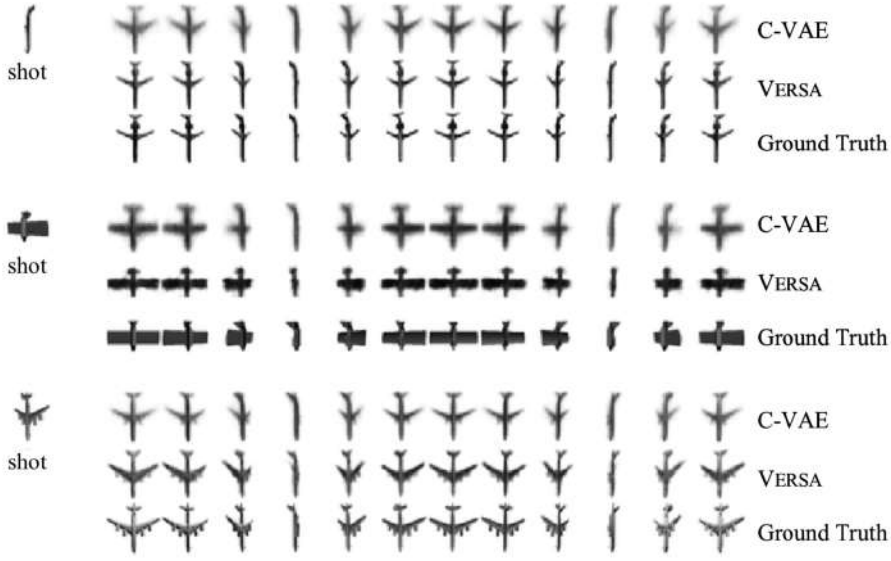
One shot learning using Approximate Bayesian inference



One shot learning using Approximate Bayesian inference



One shot learning using Approximate Bayesian inference



References: Case Study 3

- Bayesian methods enable **neural networks to be accurately trained on small data** and probabilistic modelling allows **statistical information to be shared between many different data sources**

J. Gordon et al. Decision-Theoretic Meta Learning: Versatile and Efficient Amortization of Few-Shot Learning, arXiv, 2018

See also

E. Grant et al. Recasting Gradient Based Meta-Learning as Hierarchical Bayes, ICLR, 2018

C. Finn et al. Probabilistic Model-Agnostic Meta-Learning, arXiv, 2018

T. Kim et al. Bayesian Model-Agnostic Meta-Learning, arXiv, 2018

Why be Bayesian (distributional estimates of weights)?

	maximum-likelihood	Bayes
learning	$\mathbf{w}^{\text{ML}} = \arg\max_{\mathbf{w}} \log P(\mathcal{Y}\|\mathbf{w}, \mathcal{X})$	$p(\mathbf{w}\|\mathcal{Y}, \mathcal{X}) = \frac{1}{P(\mathcal{Y}\|\mathcal{X})}\, p(\mathbf{w})\; P(\mathcal{Y}\|\mathbf{w}, \mathcal{X})$
prediction	$p(y^*\|x^*, \mathcal{Y}, \mathcal{X}) \approx p(y^*\|\mathbf{w}^{\text{ML}}, x^*)$	$p(y^*\|x^*, \mathcal{Y}, \mathcal{X}) = \int p(\mathbf{w}\|\mathcal{Y}, \mathcal{X})\, p(y^*\|\mathbf{w}, x^*) \mathrm{d}\mathbf{w}$
	single weight setting	ensemble over weight settings
		requires approximation

Robust Deep Learning

point estimates over-confident, averaging over weight settings less so
Bayesian methods are more robust to adversarial examples (hard to fool ensemble of networks + uncertainty)

Data-efficient Deep Learning

small data, big model: build models 'the size of a house' & let data prune/learn structure
leverage heterogeneous data sources (multi-task learning) using shared parameters

Flexibile Deep Learning

continual learning: use old posterior as prior
active learning: select data that are expected to reduce uncertainty in parameter estimates the most

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