Supplementary Notes on Homogeneous Coordinates

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1 Homogeneous Coordinates in 2D

1.1 Representation of geometric primitives

- Point $\mathbf{x} = [x, y]$ in cartesian coordinates can be represented as a point in homogeneous coordinates $\tilde{\mathbf{x}}$, where:

$$\tilde{\mathbf{x}} = [x_1, x_2, x_3]^\top = [sx, sy, s]^\top, \tag{1}$$

  where $x_3, s \neq 0$ and $x = \frac{x_1}{x_3}$, $y = \frac{x_2}{x_3}$.

- Points at infinity are represented by homogeneous coordinates with final dimension $x_3 = 0$. A corresponding set of homogenous coordinates is:

$$\{[x_1, x_2, 0]^\top, \text{ s.t. } x_1, x_2 \in R \text{ and } (x_1 \neq 0 \text{ or } x_2 \neq 0)\}, \tag{2}$$

  Point $[0, 0, 0]^\top$ is undefined. Also note that points at infinity do not have mapping to cartesian coordinates.

- Line $ax + by + c = 0$ in cartesian coordinates can be represented as vector $\mathbf{l}$ such that:

$$\mathbf{l}^\top \tilde{\mathbf{x}} = 0, \text{ where } \mathbf{l} = [a, b, c]^\top. \tag{3}$$

- Line at infinity is represented as $[0, 0, c]^\top$ where $c \neq 0$.

- Conic section $ax^2 + bxy + cy^2 + dx + ey + f = 0$ in cartesian coordinates:

$$\tilde{\mathbf{x}}^T \mathrm{C} \tilde{\mathbf{x}} = 0, \text{ where } \mathrm{C} = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}. \tag{4}$$

1.2 Properties

- Point $\tilde{\mathbf{x}}$ at the intersection of two lines $\mathbf{l}_1$ and $\mathbf{l}_2$:

$$\tilde{\mathbf{x}} = \mathbf{l}_1 \times \mathbf{l}_2. \tag{5}$$

  This can be derived as a solution to a system of homogeneous linear equations:

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 = 0, \\ a_2x_1 + b_2x_2 + c_2x_3 = 0. \end{aligned} \tag{6}$$

- Similarly, a line $\mathbf{l}$ going through two points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ is given by:

$$\mathbf{l} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2. \tag{7}$$

- Checking whether 3 points $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$, $\tilde{\mathbf{x}}_3$ all lie on a line:

$$\tilde{\mathbf{x}}_3^\top (\tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2) = 0 \rightarrow \det \left[\begin{array}{ccc} \tilde{\mathbf{x}}_1 & \tilde{\mathbf{x}}_2 & \tilde{\mathbf{x}}_3 \end{array} \right] = 0. \tag{8}$$

- Similarly, checking if 3 lines $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ intersect in a single point:

$$\mathbf{l}_3^\top (\mathbf{l}_1 \times \mathbf{l}_2) = 0 \rightarrow \det \left[\begin{array}{ccc} \mathbf{l}_1 & \mathbf{l}_2 & \mathbf{l}_3 \end{array} \right] = 0. \tag{9}$$

1.3 Planar projective transformation

- Point $\tilde{\mathbf{x}}'$ which is an image of a point $\tilde{\mathbf{x}}$ under planar projective transformation (homography) H is given by:

$$\tilde{\mathbf{x}}' = \mathrm{H}\tilde{\mathbf{x}}. \tag{10}$$

- Line $\mathbf{l}'$ which is an image of a line $\mathbf{l}$ under homography H is:

$$\mathbf{l}' = \mathrm{H}^{-\top}\mathbf{l}. \tag{11}$$

  Note, that: $\mathbf{l}^\top\tilde{\mathbf{x}} = 0 \rightarrow \mathbf{l}^\top \left(\mathrm{H}^{-1}\tilde{\mathbf{x}}'\right) = 0$ (eq. 10) $\rightarrow \left(\mathbf{l}^\top\mathrm{H}^{-1}\right)\tilde{\mathbf{x}}' = 0$.

- Similarly, an image C' of a conic C is:

$$\mathrm{C}' = \mathrm{H}^{-T}\mathrm{C}\mathrm{H}^{-1}. \tag{12}$$

2 Homogeneous coordinates in 3D

2.1 Representation of geometric primitives

- Point $\mathbf{X} = [x, y, z]$ in cartesian coordinates can be represented as a point in homogeneous coordinates $\tilde{\mathbf{X}}$, where:

$$\tilde{\mathbf{X}} = [X_1, X_2, X_3, X_4]^\top = [\lambda x, \lambda y, \lambda z, \lambda]^\top, \tag{13}$$

  where $X_4, \lambda \neq 0$ and $X = \frac{X_1}{X_4}$, $Y = \frac{X_2}{X_4}$, $Z = \frac{X_3}{X_4}$.

- Points at infinity are represented by homogeneous coordinates with final dimension $X_4 = 0$. A corresponding set of homogenous coordinates is:

$$\{[X_1, X_2, X_3, 0]^\top, \text{ s.t. } X_1, X_2, X_3 \in R \text{ and } (X_1 \neq 0 \text{ or } X_2 \neq 0 \text{ or } X_3 \neq 0)\}, \tag{14}$$

  Point $[0, 0, 0, 0]^\top$ is undefined. Also note that points at infinity do not have mapping to cartesian coordinates.

- Plane $aX + bY + cZ = d$ in cartesian coordinates can be represented as vector π such that:

$$\mathbf{\Pi}^\top \tilde{\mathbf{X}} = 0, \text{ where } \mathbf{\Pi} = [\pi_1, \pi_2, \pi_3, \pi_4] = [a, b, c, -d]^\top. \tag{15}$$

- Plane at infinity is represented as $[0, 0, 0, d]^\top$, where $d \neq 0$.

3 Some relevant exam questions.

- **4F12 2003 Q2 (a).** Derive the equation of the vanishing line of parallel planes with normal $\mathbf{n} = (n_x, n_y, n_z)$ when viewed with a pinhole camera under perspective projection.
- **4F12 2005 Q2 (b).** Derive expression for vanishing point in the image plane of lines parallel to X axis.
- **4F12 2017 Q2 (b-iii).** Recover the equation of the horizon of the ground plane (X-Y world plane).
- **4F12 2011 Q2 (b).** Derive algebraic equations of a ray through image point (u, v).