4f13 Coursework 2: Probablistic Ranking

Alejandro Santorum Varela

TOTAL POINTS

63 / 100

QUESTION 1

1 (a) 15 / 20

✓ **+ 4 pts** **Correctly completed Gibbs sampling code. Gibbs sampling appears to work correctly.**

Contents of the plots

✓ **+ 2 pts** **Trace plots of the skills of at least 3-4 players and with a diverse selection of players to exhibit the behaviour of the sampler for differently skilled players.**

+ 1 pts Plotted the traces of the skills 1-2 players or only a limited selection (e.g. only the top performing players).

Quality of plots

+ 2 pts Nice plots with good labels and captions, figures properly referenced in the main text.

✓ **+ 1 pts** **Reasonable plots with basic labels / captions, some figures may be hard to read or are not referenced in the main text.**

+ 0 pts Basic plots with either missing or hard-to-read labels or captions. Plots may be hard to read or low quality. Figures might not be referenced in the main text.

Discussion on burn-in

✓ **+ 1 pts** **Necessary burn-in is correctly identified to be usually between 10-50 samples, sometimes somewhat larger.**

+ 3 pts Clear discussion of burn-in in terms of the samples converging to an area of high probability under the stationary distribution of the Markov chain / converging to the typical set of the stationary distribution.

✓ **+ 1 pts** **Limited / somewhat incorrect discussion of the burn-in.**

Discussion on autocorrelation time

✓ **+ 1 pts** **Autocorrelation curve plotted and autocorrelation time correctly identified to be usually between 5 and 10 samples.**

+ 3 pts Good discussion of autocorrelation (e.g. using formulae), why samples from the Markov chain are not independent in the first place and autocorrelation time by e.g. discussing the integrated autocorrelation time.

✓ **+ 1 pts** **Limited / somewhat incorrect discussion of autocorrelation time.**

Heuristic on required number of samples for the chain

✓ **+ 1 pts** **Removing the burn-in samples from the chain and thinning based on the autocorrelation time are correctly explained.**

✓ **+ 3 pts** **Good heuristic given for number of samples needed in terms of the number of samples desired, the burn-in and the autocorrelation time, with appropriate justification.**

+ 1 pts Basic / partial heuristic given for determining the minimum chain length for a desired number of samples.

QUESTION 2

2 (b) 12 / 20

✓ **+ 3 pts** **Ran EP implementation, results obtained seem sensible.**

Content of the plots

+ 2 pts Plots of the parameters of the Gaussian approximations of the skills of at least 3-4 players and with a diverse selection of players to exhibit the behaviour of the sampler for differently skilled players.

✓ **+ 1 pts** **Plots of the skills 1-2 players or only a**

limited selection (e.g. only the top performing players).

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+ 2 pts Nice plots with good labels and captions, figures properly referenced in the main text.

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Convergence of Gibbs sampler

+ 5 pts Clear and correct description of convergence of the Gibbs sampler to the stationary distribution of its Markov kernel, the true posterior distribution of the skills. Some discussion on the attractive property of the stationary distribution, e.g. the chain not leaving the typical set, or different chains converging to the typical set regardless of the initialization.

+ 3 pts Somewhat partial and/or mostly correct description of convergence.

✓ **+ 1 pts** **Basic description of convergence, or explanation has major issues.**

Convergence of EP

+ 5 pts Clear and correct description of EP iteratively matching the mean and variance of the approximating Gaussian distribution to the mean and variance of the true marginal distribution of players' skills. Hence in EP, the parameters of the approximating Gaussians are the objects that converge.

✓ **+ 3 pts** **Limited and/or mostly correct description of convergence.**

+ 1 pts Basic description of convergence, or explanation has major issues.

✓ **+ 1 pts** **Good convergence criterion of e.g. thresholding the difference between consecutive iterates for the mean and variance for EP at a reasonable value, e.g. 0.001.**

✓ **+ 2 pts** **The number of message passing steps required correctly identified to be approximately 10 - 100 iterations for a reasonable threshold.**

QUESTION 3

3 (c) 13 / 20

Numerical results

✓ **+ 4 pts** **Correct probabilities calculated for pairs of different players.**

+ 2 pts Partially correct probabilities calculated.

- 2 pts Formulae were applied to pairs of the same player, i.e. "diagonal" entries in the tables. This is wrong, the probability that a player "beats themselves" is 0.

Presentation of results

✓ **+ 2 pts** **Results are presented in a nicely formatted table, they are easy to read.**

+ 1 pts Basic presentation of results. Caption for the table might be missing, or results are presented to unnecessary precision, or the reported precision varies. The player IDs might be displayed instead of the players' names. The tables might not be referenced in the main text.

Calculating the formulae

✓ **+ 3 pts** **The correct formulae $$ P[w_1 > w_2] = \Phi\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma^2_1 + \sigma^2_2}} \right)$$ and $$ P[\text{player 1 wins}] = \Phi\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma^2_1 + \sigma^2_2 + 1}} \right)$$ are neatly presented.**

+ 2 pts The correct formulae are presented, but only either informally, buried in the main text, or only as a code snippet.

+ 1 pts Partially correct formulae presented, or presentation has major flaws.

+ 4 pts Clear and correct derivation for formulae, noting the **unrealistic** independence assumption on the player skills conditioned on the data.

✓ **+ 2 pts** **Limited presentation of the derivations for the formulae**

Explanation of the difference in probabilities

+ **7 pts** Clear explanation of the smoothing effect of the additional noise in the winning probabilities as opposed to the skill differences. In particular, the fact that regardless of the noise the predicted winner does not change, the convergence to equal probabilities as the noise variance tends to infinity is discussed or the noiseless limit are discussed.

+ **4 pts** Limited discussion on the smoothing effect of the noise term.

✓ **+ 2 pts** **Basic / minimal discussion on the smoothing effect of the noise term.**

QUESTION 4

4 (d) **12 / 20**

Formulas/Description of the three approaches to computing players' skills

✓ **+ 4 pts** **Ostensibly correct computations & description**

+ **1 pts** Formulas/Code shown sufficiently

✓ **+ 1 pts** **Bonus: Distribution plots**

+ **2 pts** Partially correct/incomplete computations & description

+ **2 pts** Potentially correct answer; computations not fully verifiable due to lack of code/formulas

Explanation on which method is best

✓ **+ 2 pts** **Choosing either method 2 or 3**

+ **3 pts** Providing a complete explanation

- **1 pts** Not taking into account that method 3 is optimal under certain conditions

- **1 pts** Inaccuracies in the discussion

4x4 skill table

✓ **+ 3 pts** **Table seems correct**

✓ **+ 1 pts** **Used either method 2 or 3 to create table**

+ **1 pts** Brief comparative interpretation wrt. message passing

- **0.5 pts** Numeric values on the diagonal (doesn't make sense)

- **1 pts** Inaccuracies in the discussion/(Partially) missing discussion

- **1 pts** Included incorrect table for one of the methods

Presentation

+ **2 pts** Nice plots/figures with good labels and captions, figures properly referenced in the main text. Text is properly structured and easily readable.

✓ **+ 1 pts** **Reasonable plots/figures with basic labels / captions, some figures may be missing, hard to read or are not referenced in the main text. Text may lack structure.**

+ **0 pts** Basic plots/figures with either missing or hard-to-read labels or captions. Plots may be missing, hard to read or low quality. Figures might not be referenced in the main text. Text may lack structure/be hard to read.

QUESTION 5

5 (e) **11 / 20**

Results

✓ **+ 6 pts** **Plotted results and they seem to be sensible.**

+ **3 pts** Results seem mostly correct

Quality of plots

+ **2 pts** Nice plots with good labels and captions, figures properly referenced in the main text.

✓ **+ 1 pts** **Reasonable plots with basic labels / captions, some figures may be hard to read or are not referenced in the main text.**

+ **0 pts** Basic plots with either missing or hard-to-read labels or captions. Plots may be hard to read or low quality. Figures might not be referenced in the main text.

+ **1 pts** Nice additional results, e.g. plots of comparisons between different ranking methods, e.g. by scatter plotting the different ranking results against each other.

Predictions for ranking

+ **1 pts** Discussion on what "prediction" for ranking means using Gibbs sampling and EP: either use the average probability of a player winning, the average probability of a player having higher skill than others or just sorting the means of the skill posteriors. Some

notes on computational complexity of these methods, or their comparability.

✓ **+ 2 pts Clear description of how rankings were derived.**

Issues with the empirical ranking scheme

+ 4 pts Clear discussion on the drawbacks / unreasonableness of the empirical scheme, mentioning most of the below mentioned issues:

- The rankings are arguably incomparable: some players have played more games than others, e.g. some players drop out really early during tournaments, which might make the estimate of their ranking very noisy / high-variance.
- The skill of the opponent is not taken into account: losing to Djokovic is taken into account as much as losing against a bottom-ranked player.
- On a related note, if a player loses all their games, their they are ranked at the bottom with a predicted 0 probability of winning games
- If a player plays only a single game which they win, they will be ranked first.

+ 2 pts Partial / mostly correct discussion of the above points.

✓ **+ 1 pts Basic discussion of the above points / presentation has major issues.**

Discussion on ranking using Gibbs sampling and EP

+ 4 pts Clear and correct discussion of the advantages of using Gibbs sampling and EP over the empirical scheme, the similarities between the two methods and their differences. In particular, most of the below points are mentioned:

- Both Gibbs sampling and EP take the skill of the opponent into account, hence winning/losing against differently skilled opponents will affect the players' skill differently.
- The above point as well as the general Bayesian set-up guarantee that the players at the bottom of the table are ranked both more reasonably (i.e. all of them are assigned some non-0 probability of winning their next game) and differently based on against whom they lost.
- In a similar vein, if a player who lost all games but only against high skilled players, they still might be ranked higher than a player who won some games, but only against low-ranked players.
- Gibbs sampling and EP produce very similar rankings, hence EP is a produces a good approximation in this case.
- EP is much cheaper to run than Gibbs sampling, hence it is the preferable method for scaling up the method.

+ 2 pts Partial / mostly correct discussion of the above points

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1. Question a)

After completing the code in `gibbsrank.py` (figure 1.1), Gibbs sampling can be executed to sample player skills. Some of them are plotted in figure 1.2.

```python
# Jointly sample skills given performance differences
m = np.zeros((M, 1))
for p in range(M):
    # TODO: COMPLETE THIS LINE
    m[p] = np.dot(t.transpose(), (p == G[:,0]).astype(int) - (p == G[:,1]).astype(int))

iS = np.zeros((M, M))
for g in range(N):
    # TODO: COMPLETE THIS
    iS[G[g,0],G[g,0]] += 1
    iS[G[g,1],G[g,1]] += 1
    iS[G[g,0],G[g,1]] -= 1
    iS[G[g,1],G[g,0]] -= 1
```

Figure 1.1: Completed code in `gibbsrank.py`

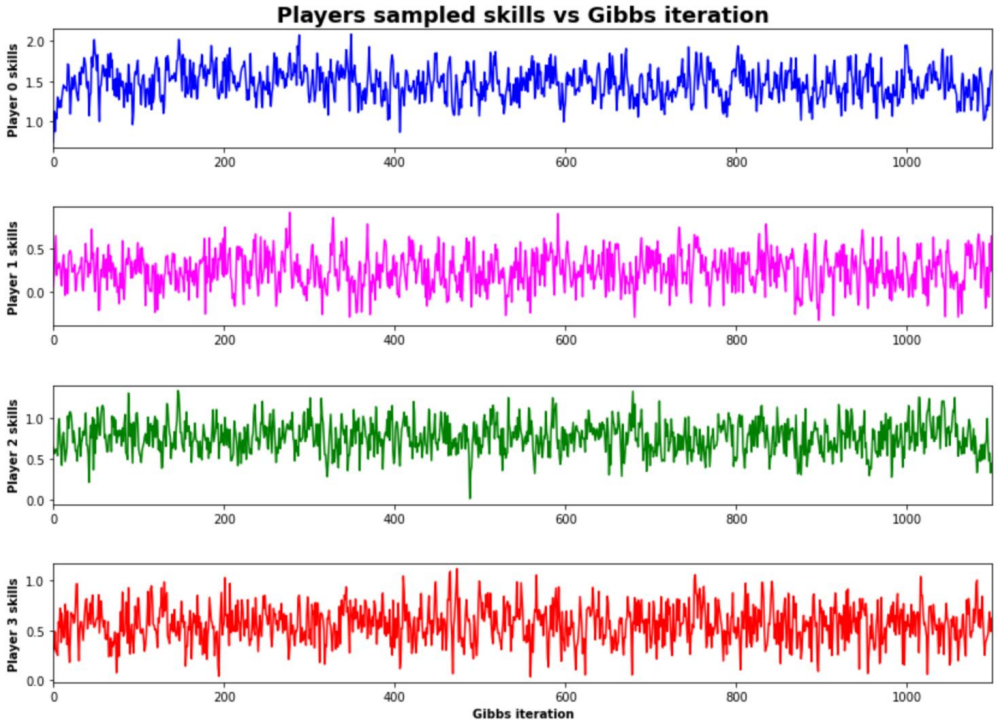
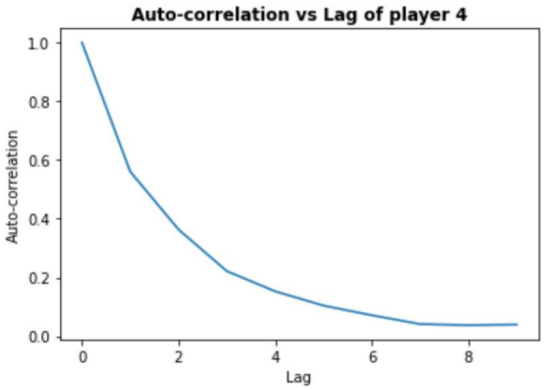
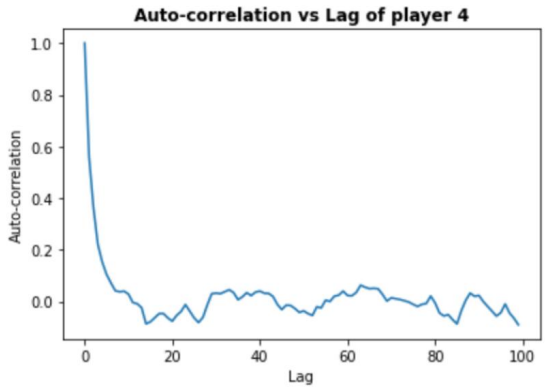


Figure 1.2: Some sampled player skills as a function of Gibbs iteration

Usually when executing Gibbs sampling (GS), the initial sequence of samples is discarded, until the chain has converged to the desired distribution. This is called **burn-in**. In the current problem, skill samples are converging quite fast (check visually the first iterations of figure 1.2), so the burn-in period will be low, around **20-30 iterations**. Question b (section 2) will explain the concept of *convergence*.
Additionally, to get less dependence between samples, GS is often run for a long time, and samples are **thinned** by keeping only every m-th sample.



(a) Auto-correlation plotted against lag



(b) Auto-correlation plotted against lag (greater)

Figure 1.3: Auto-correlation vs Lag for a certain player

After plotting the auto-correlation of samples of a certain player against the lag, shown in figures 1.3, it's reasonable to keep only **10-th sample** to obtain **(pseudo)-independent samples**. If 100 is a reasonable number of samples after burn-in and thinning, then **GS should be executed for about 1100 iterations**.

2. Question b)

Gibbs sampling method tries to obtain a set of independent samples to describe the (intractable) joint distribution. Therefore, convergence is achieved when the majority of samples stay steady for plenty of iterations. Plotting sample mean μ and sample standard deviation σ can confirm that the sample distribution has converged quite quickly, as shown in figure 2.1. So, taking into account *burn-in*, *thinning* and *convergence velocity*, **1100 iterations is still a good amount of iterations for GS**.

In message passing (MP), the mean and variance of each player skill are computed to estimate games outcomes. Every iteration of MP the estimated means and variances are updated, and convergence is achieved when the changes are really small (less than a certain tolerance). In the figure 2.2 the skill means and variances for 3 players are plotted vs EP iteration. The point where the estimated object has changed less than $tol = 10^{-3}$ 10 iterations in a row is represented. As 2.2 shows, **50 iterations** are sufficient to achieve convergence for MP algorithm, although fewer iterations could be used, but since MP is really fast, 50 iterations is a reasonable number.

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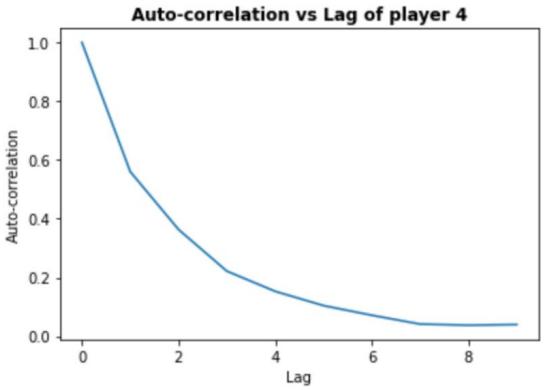

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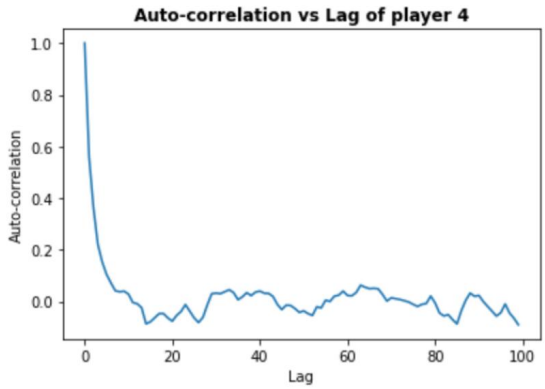

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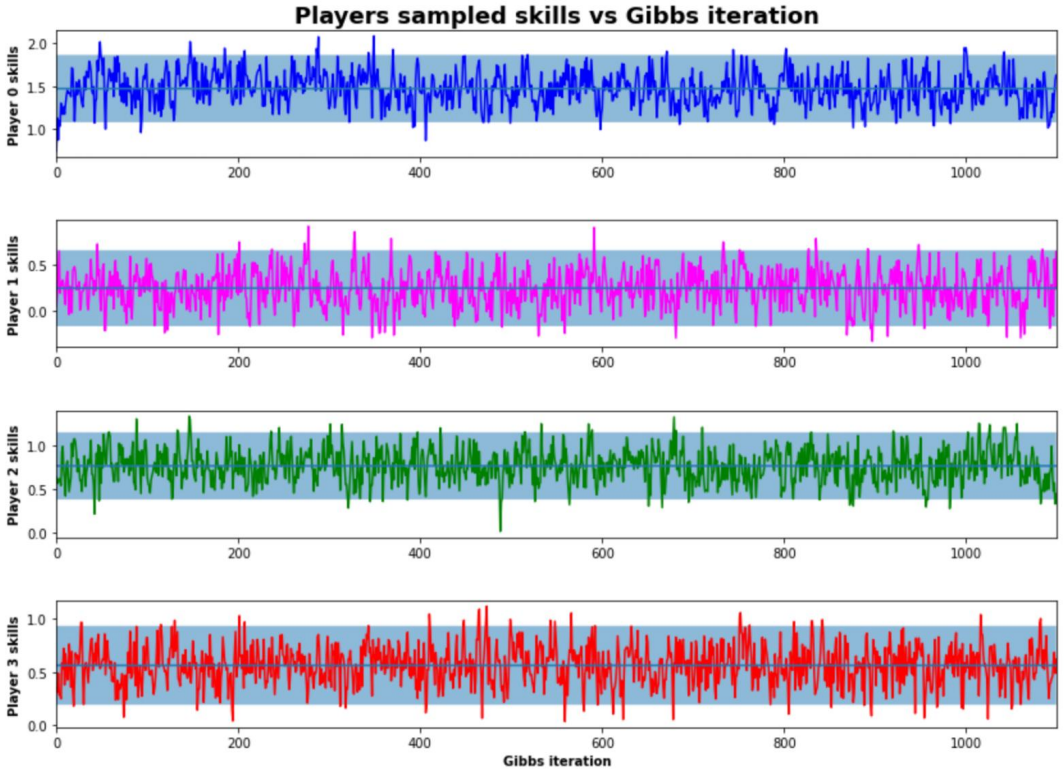

Figure 2.1: Plotting Gibbs samples, its mean μ and the shared area is $\mu \pm 2\sigma$

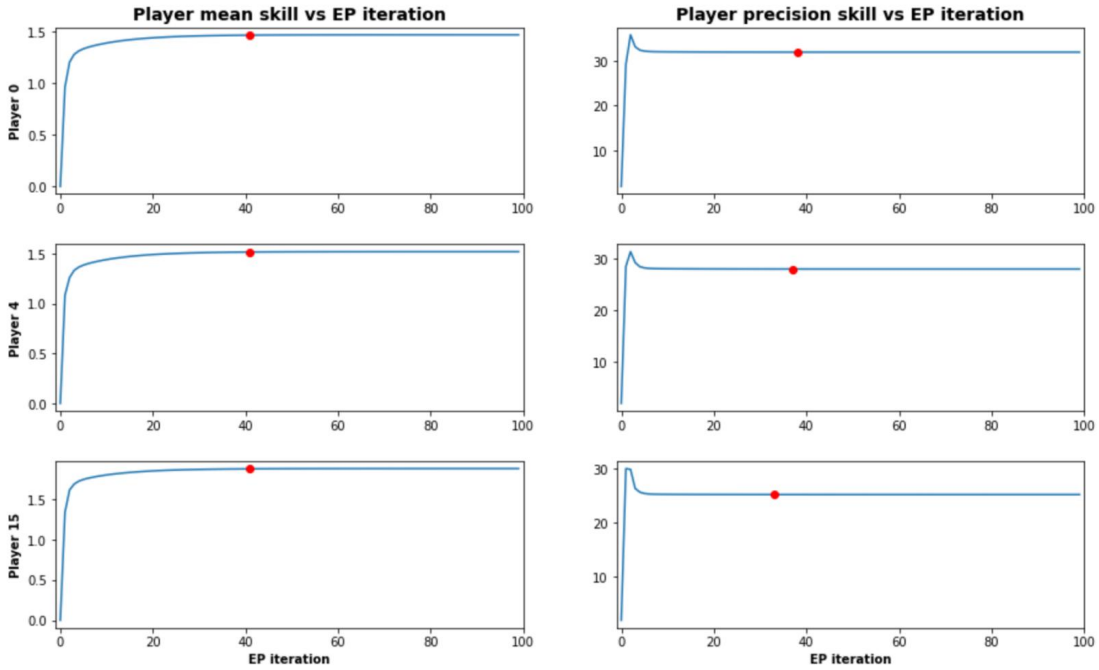

Figure 2.2: Skill mean and variance vs EP iteration for Nadal, Federer and Djokovic

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✓ **+ 3 pts** **Ran EP implementation, results obtained seem sensible.**

Content of the plots

+ 2 pts Plots of the parameters of the Gaussian approximations of the skills of at least 3-4 players and with a diverse selection of players to exhibit the behaviour of the sampler for differently skilled players.

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✓ **+ 1 pts** **Good convergence criterion of e.g. thresholding the difference between consecutive iterates for the mean and variance for EP at a reasonable value, e.g. 0.001.**

✓ **+ 2 pts** **The number of message passing steps required correctly identified to be approximately 10 - 100 iterations for a reasonable threshold.**

3. Question c)

The 4 top players according to the ATP ranking when the data was collected is: Djokovic, Nadal, Federer and Murray.

The table 3.1 contains the probabilities that the skill of one player is higher than the other. To calculate the probability that the player i is greater than that of player j the following reasoning is applied: w_i is skill player i, w_j skill player j, $p(w_i > w_j) = p(w_i - w_j > 0)$. Assuming $w_i \sim N(\mu_i, \sigma_i^2)$ and $w_j \sim N(\mu_j, \sigma_j^2)$, then:

$$p(w_i - w_j > 0) = \int_0^{\infty} N(w_i - w_j; \mu_i - \mu_j, \sigma_i^2 + \sigma_j^2) = 1 - \int_{-\infty}^{0} N(w_i - w_j; \mu_i - \mu_j, \sigma_i^2 + \sigma_j^2) \tag{3.1}$$

In python this can be computed by using ($\star$):

```
mu=meanSkills[i]-meanSkills[j]
var=1/precisionSkills[i]+1/precisionSkills[j]
probSkill(i,j)=1-scipy.stats.norm.cdf(0, mu, np.sqrt(var))
```

	Novak-Djokovic	Rafael-Nadal	Roger-Federer	Andy-Murray
Novak-Djokovic	---	0.9398	0.9089	0.9853
Rafael-Nadal	0.0602	---	0.4272	0.7665
Roger-Federer	0.0911	0.5728	---	0.8108
Andy-Murray	0.0147	0.2335	0.1892	---

Figure 3.1: Probabilities that the skill of one player is higher than the other

The table 3.2 contains the probabilities of one player winning the other one. Since the predicted game outcome depends on the skill difference plus some noise ϵ with mean 0 and variance 1, then the equations are quite similar, we just have to add the noise variance:

$$p(w_i - w_j + \epsilon > 0) = 1 - \int_{-\infty}^{0} N(w_i - w_j + \epsilon; \mu_i - \mu_j, 1 + \sigma_i^2 + \sigma_j^2) \tag{3.2}$$

In python this can be computed as in 1 by just changing:

```
var=1.0+1/precisionSkills[i]+1/precisionSkills[j]
```

The **main difference between theese tables** is that the winning probabilities account for the possible **noise** that a tennis game can have (player motivation, terrain, fan support) and not just the skill difference.

	Novak-Djokovic	Rafael-Nadal	Roger-Federer	Andy-Murray
Novak-Djokovic	---	0.6554	0.638	0.7198
Rafael-Nadal	0.3446	---	0.4816	0.5731
Roger-Federer	0.362	0.5184	---	0.5909
Andy-Murray	0.2802	0.4269	0.4091	---

Figure 3.2: Probabilities of one player winning the other one

4. Question d)

GS algorithm returns a set of sampled player skills. The skills of two players can be compared in different ways:

First, a player skill w_i can be approximated by considering $w_i \sim N(\mu_i, \sigma_i^2)$, where μ_i is the **skill mean** for player i and σ_i^2 is the **skill variance**. Then, the probability that the skill of one player is higher than other can be computed exactly as done in 3.1. In table 4.2a the skills of Nadal and Djokovic are compared after using GS and approximating their marginal skills by Gaussians.

Another method to compare two players skills is to consider their joint distribution:

$$\begin{bmatrix} w_i \\ w_j \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_i \\ \mu_j \end{bmatrix}, \Sigma \right), \tag{4.1}$$

where Σ is the covariance matrix between w_i and w_j. The probability that the skill of one player is higher than other can be computed by calculating the area under the 2D-gaussian pdf limited by the plane $\{x = y, z \in \mathbb{R}\}$, as shown in figure 4.1.

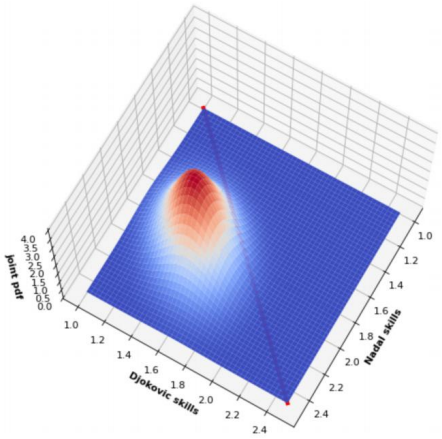

Figure 4.1: Probability that Djokovic skill is higher than Nadal's (around 94%)

3 (c) **13 / 20**

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+ 2 pts Partially correct probabilities calculated.

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+ 7 pts Clear explanation of the smoothing effect of the additional noise in the winning probabilities as opposed to the skill differences. In particular, the fact that regardless of the noise the predicted winner does not change, the convergence to equal probabilities as the noise variance tends to infinity is discussed or the noiseless limit are discussed.

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Rafael-Nadal	0.3446	---	0.4816	0.5731
Roger-Federer	0.362	0.5184	---	0.5909
Andy-Murray	0.2802	0.4269	0.4091	---

Figure 3.2: Probabilities of one player winning the other one

4. Question d)

GS algorithm returns a set of sampled player skills. The skills of two players can be compared in different ways:

First, a player skill w_i can be approximated by considering $w_i \sim N(\mu_i, \sigma_i^2)$, where μ_i is the **skill mean** for player i and σ_i^2 is the **skill variance**. Then, the probability that the skill of one player is higher than other can be computed exactly as done in 3.1. In table 4.2a the skills of Nadal and Djokovic are compared after using GS and approximating their marginal skills by Gaussians.

Another method to compare two players skills is to consider their joint distribution:

$$\begin{bmatrix} w_i \\ w_j \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_i \\ \mu_j \end{bmatrix}, \Sigma \right), \tag{4.1}$$

where Σ is the covariance matrix between w_i and w_j. The probability that the skill of one player is higher than other can be computed by calculating the area under the 2D-gaussian pdf limited by the plane $\{x = y, z \in \mathbb{R}\}$, as shown in figure 4.1.

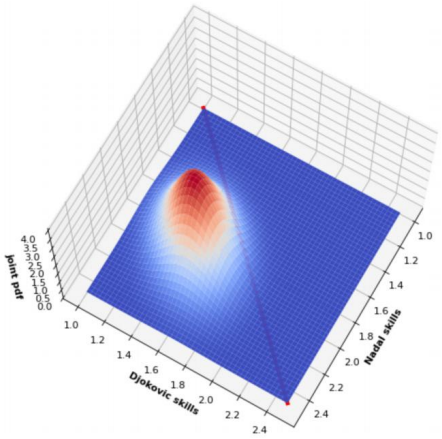

Figure 4.1: Probability that Djokovic skill is higher than Nadal's (around 94%)

	Novak-Djokovic	Rafael-Nadal
Novak-Djokovic	---	0.9192
Rafael-Nadal	0.0808	---

(a) Marginal skills

	Novak-Djokovic	Rafael-Nadal
Novak-Djokovic	---	0.9401
Rafael-Nadal	0.0599	---

(b) Joint skills

	Novak-Djokovic	Rafael-Nadal
Novak-Djokovic	---	0.9383
Rafael-Nadal	0.0617	---

(c) Samples

Figure 4.2: Probability that the skill of one player is higher than other one by approximating their skills with different approaches.

The third technique is **counting** how many **skill samples** of player i are greater than the skill samples of player j and divide it by the total number of skill samples. In python:

```
prob=np.mean(skillSamples[i]>skillSamples[j])
```

To sum up table 4.2, all three methods predict that the **probability of Djokovic having more skill than Nadal is about 91-94%**. However, the method that approximates the skills by a joint Gaussian **takes into account the correlation between the skills** of two players (it isn't *isotropic* like the *marginal* approach). If the number of samples goes to *infinity*, the approximation by a joint Gaussian and the inference using just samples should give the same results, but since the no. samples tends to be considered finite, **the approximation by a joint Gaussian should be the preferred approach**.

	Novak-Djokovic	Rafael-Nadal	Roger-Federer	Andy-Murray
Novak-Djokovic	---	0.9401	0.8834	0.981
Rafael-Nadal	0.0599	---	0.3955	0.7542
Roger-Federer	0.1166	0.6045	---	0.8082
Andy-Murray	0.019	0.2458	0.1918	---

Figure 4.3: Probabilities that the skill of one player is higher than the other using Gibbs sampling.

In figure 4.3 a 4 by 4 table for the skills is shown, and it can be compared to figure 3.1 resulted from using MP algorithm. It's easy to check that the **probabilities are quite similar using both methods**.

5. Question e)

The rankings of players can be compared using different methods of inference. In figure 5.1 **empirical games outcome averages** are plotted. They're computed by **counting how many games a player has won divided by the total played games**. There're players that have no victories, so their predicted outcomes are zero.

4 (d) 12 / 20

Formulas/Description of the three approaches to computing players' skills

✓ **+ 4 pts** **Ostensibly correct computations & description**

+ 1 pts Formulas/Code shown sufficiently

✓ **+ 1 pts** **Bonus: Distribution plots**

+ 2 pts Partially correct/incomplete computations & description

+ 2 pts Potentially correct answer; computations not fully verifiable due to lack of code/formulas

Explanation on which method is best

✓ **+ 2 pts** **Choosing either method 2 or 3**

+ 3 pts Providing a complete explanation

- 1 pts Not taking into account that method 3 is optimal under certain conditions

- 1 pts Inaccuracies in the discussion

4x4 skill table

✓ **+ 3 pts** **Table seems correct**

✓ **+ 1 pts** **Used either method 2 or 3 to create table**

+ 1 pts Brief comparative interpretation wrt. message passing

- 0.5 pts Numeric values on the diagonal (doesn't make sense)

- 1 pts Inaccuracies in the discussion/(Partially) missing discussion

- 1 pts Included incorrect table for one of the methods

Presentation

+ 2 pts Nice plots/figures with good labels and captions, figures properly referenced in the main text. Text is properly structured and easily readable.

✓ **+ 1 pts** **Reasonable plots/figures with basic labels / captions, some figures may be missing, hard to read or are not referenced in the main text. Text may lack structure.**

+ 0 pts Basic plots/figures with either missing or hard-to-read labels or captions. Plots may be missing, hard to read or low quality. Figures might not be referenced in the main text. Text may lack structure/be hard to read.

	Novak-Djokovic	Rafael-Nadal
Novak-Djokovic	---	0.9192
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(a) Marginal skills

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To sum up table 4.2, all three methods predict that the **probability of Djokovic having more skill than Nadal is about 91-94%**. However, the method that approximates the skills by a joint Gaussian **takes into account the correlation between the skills** of two players (it isn't *isotropic* like the *marginal* approach). If the number of samples goes to *infinity*, the approximation by a joint Gaussian and the inference using just samples should give the same results, but since the no. samples tends to be considered finite, **the approximation by a joint Gaussian should be the preferred approach**.

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In figure 4.3 a 4 by 4 table for the skills is shown, and it can be compared to figure 3.1 resulted from using MP algorithm. It's easy to check that the **probabilities are quite similar using both methods**.

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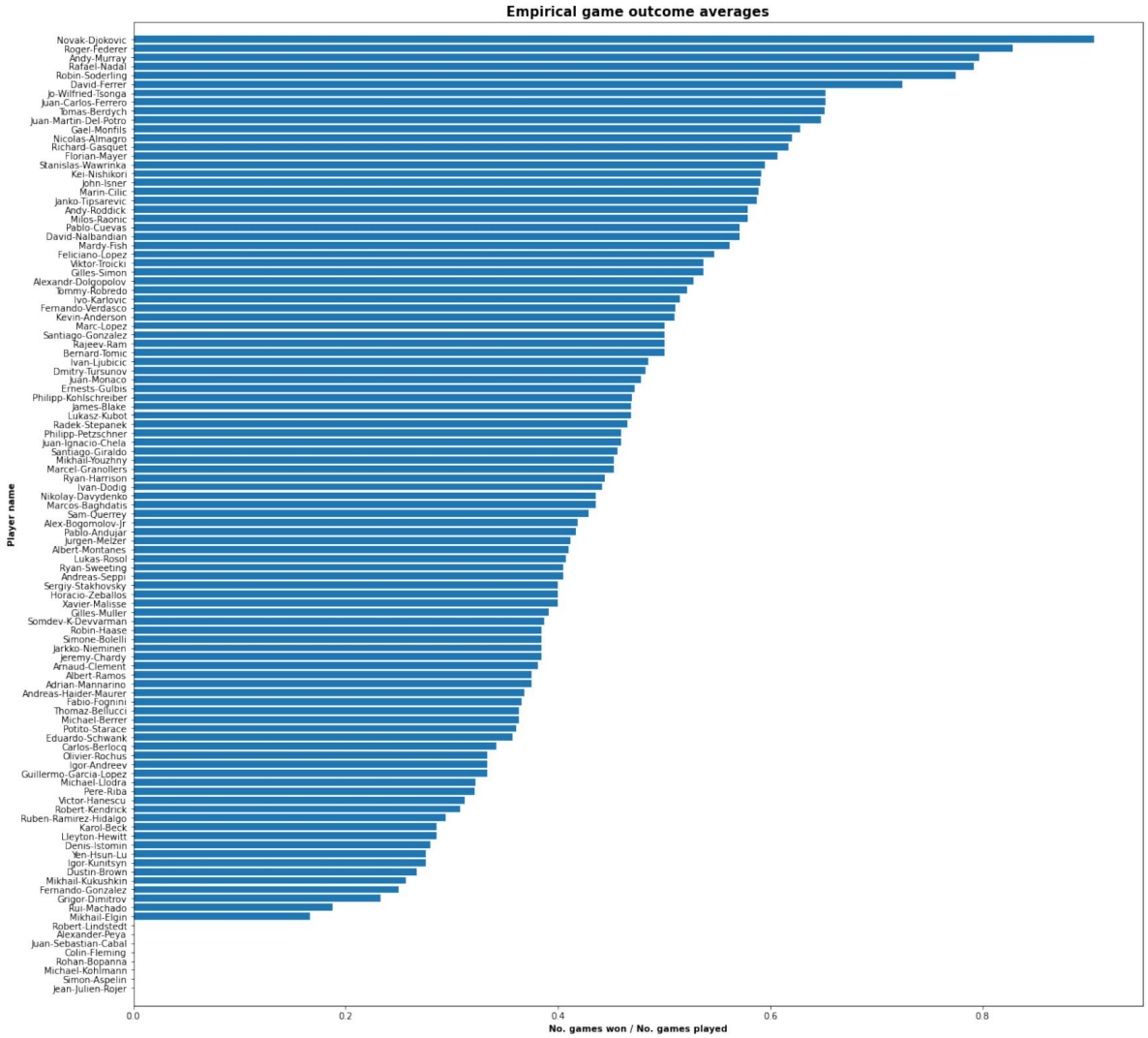

Figure 5.1: Empirical game outcomes averages. There are players with zero probabilities because they haven't won any game.

Gibbs samples can be used to predict game outcomes. This ranking is represented in figure 5.2. The plotted probabilities are the mean probability of winning for a certain player vs the rest. This probabilities have been calculated using equation 3.2 using the Gibbs samples as explained in section 4.

Similar to the previous ranking, **MP algorithm** can be used to get the ranking by computing the probabilities of one certain player winning the rest of players (figure 5.3), using equation 3.2 as done in section 3. The obtained ranking is really close to Gibbs Sampling ranking, that's because both methods address the same goal: approximating and inferring players' skills.

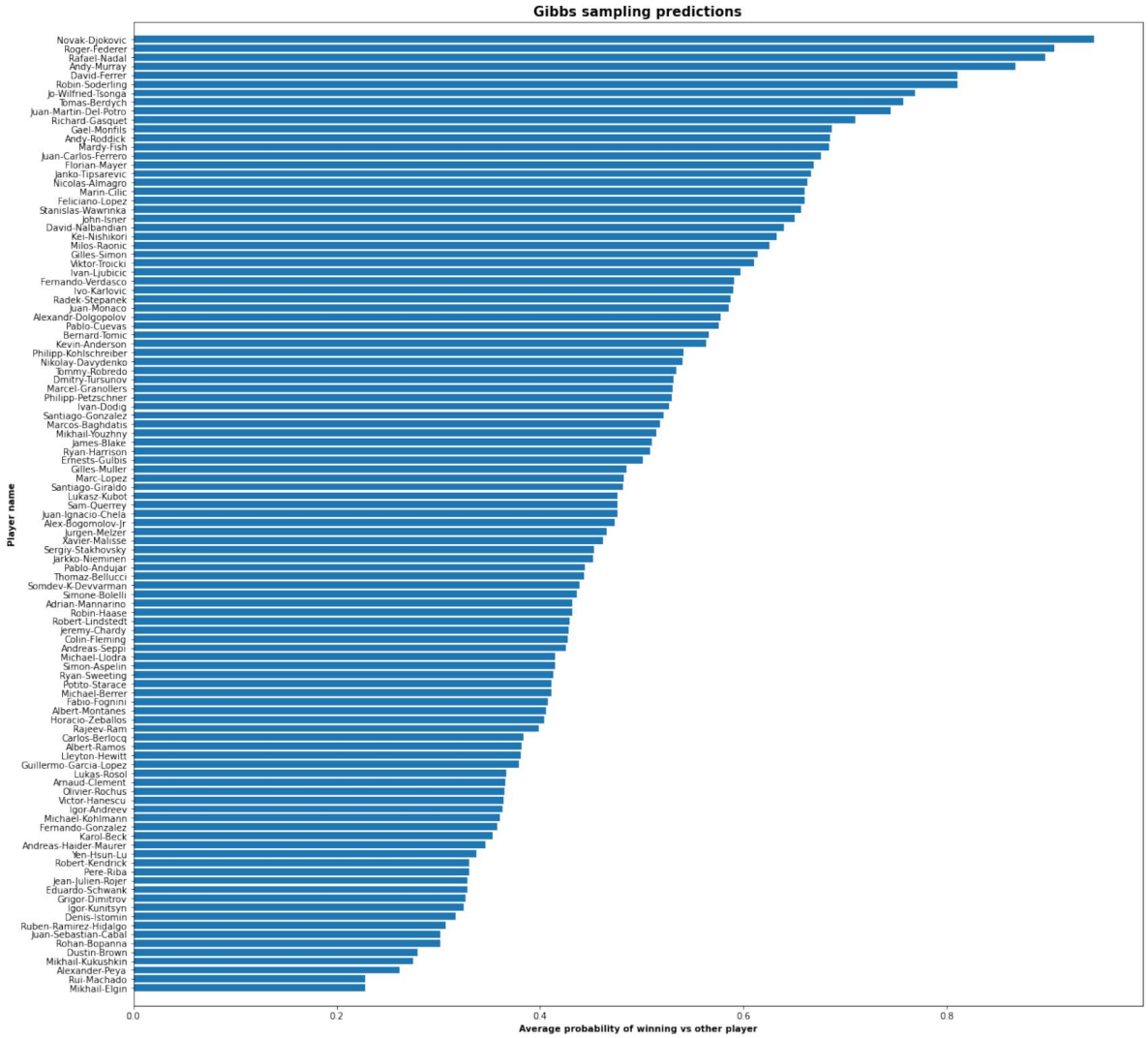

Figure 5.2: Predicted game outcomes using Gibbs sampling. Probabilities for each player come up by averaging the probability of winning the rest of players calculated using Gibbs sampling. There some notorious differences with respect to ranking 5.1 due to the fact that Gibbs sampling is more robust when the number of data points is low.

To sum up, **rankings elaborated with GS and MP are quite similar**: both methods have been proven effective to tackle this problem. On the other hand, ranking using just game outcomes seems worse, since the no. games isn't large enough.

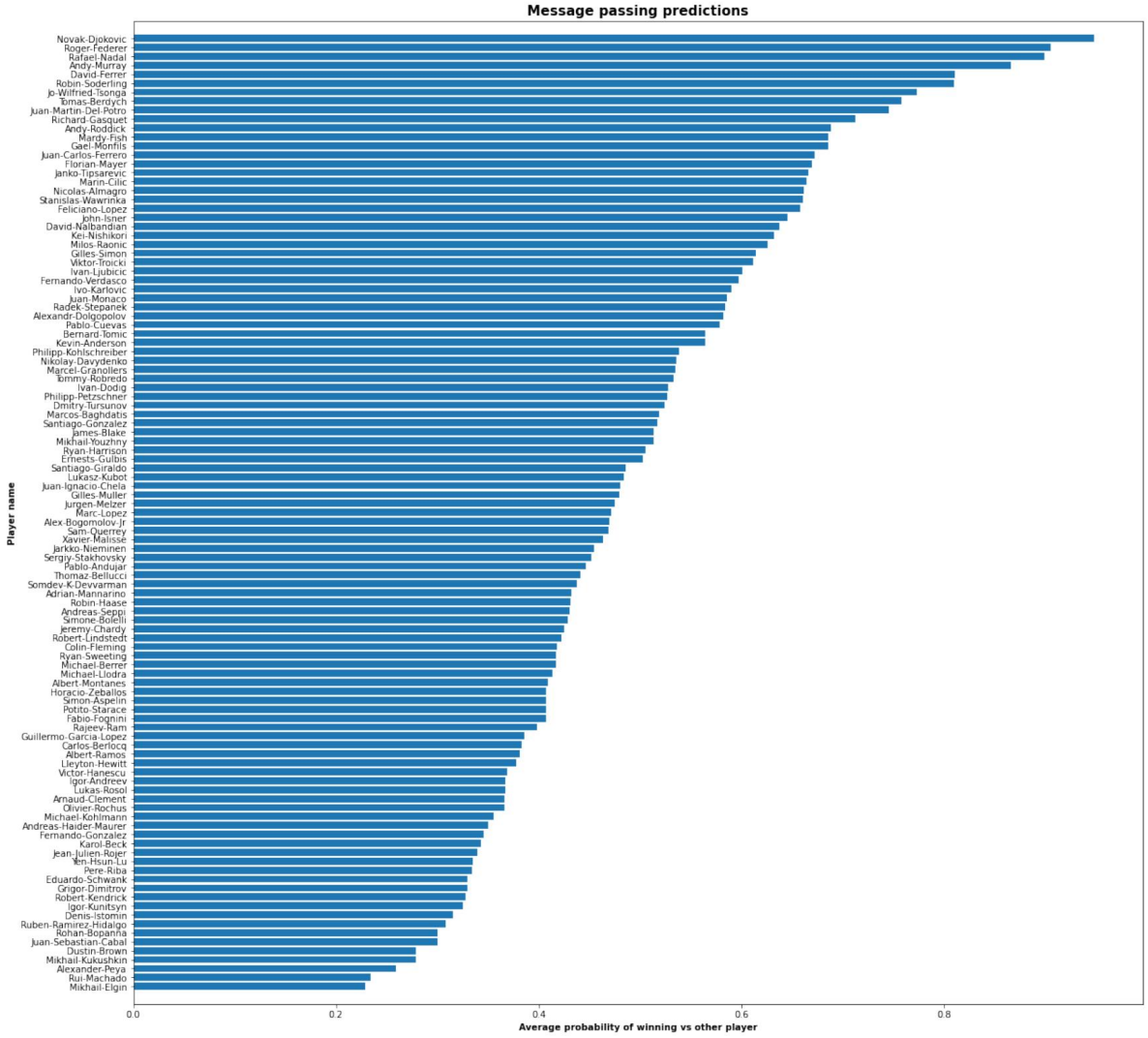

Figure 5.3: Predicted game outcomes using Message Passing. Probabilities for each player come up by averaging the probability of winning the rest of players calculated using Message Passing.

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Results

✓ **+ 6 pts** **Plotted results and they seem to be sensible.**

+ 3 pts Results seem mostly correct

Quality of plots

+ 2 pts Nice plots with good labels and captions, figures properly referenced in the main text.

✓ **+ 1 pts** **Reasonable plots with basic labels / captions, some figures may be hard to read or are not referenced in the main text.**

+ 0 pts Basic plots with either missing or hard-to-read labels or captions. Plots may be hard to read or low quality. Figures might not be referenced in the main text.

+ 1 pts Nice additional results, e.g. plots of comparisons between different ranking methods, e.g. by scatter plotting the different ranking results against each other.

Predictions for ranking

+ 1 pts Discussion on what "prediction" for ranking means using Gibbs sampling and EP: either use the average probability of a player winning, the average probability of a player having higher skill than others or just sorting the means of the skill posteriors. Some notes on computational complexity of these methods, or their comparability.

✓ **+ 2 pts** **Clear description of how rankings were derived.**

Issues with the empirical ranking scheme

+ 4 pts Clear discussion on the drawbacks / unreasonableness of the empirical scheme, mentioning most of the below mentioned issues:

- The rankings are arguably incomparable: some players have played more games than others, e.g. some players drop out really early during tournaments, which might make the estimate of their ranking very noisy / high-variance.
- The skill of the opponent is not taken into account: losing to Djokovic is taken into account as much as losing against a bottom-ranked player.
- On a related note, if a player loses all their games, their they are ranked at the bottom with a predicted 0 probability of winning games
- If a player plays only a single game which they win, they will be ranked first.

+ 2 pts Partial / mostly correct discussion of the above points.

✓ **+ 1 pts** **Basic discussion of the above points / presentation has major issues.**

Discussion on ranking using Gibbs sampling and EP

+ 4 pts Clear and correct discussion of the advantages of using Gibbs sampling and EP over the empirical scheme, the similarities between the two methods and their differences. In particular, most of the below points are mentioned:

- Both Gibbs sampling and EP take the skill of the opponent into account, hence winning/losing against differently skilled opponents will affect the players' skill differently.
- The above point as well as the general Bayesian set-up guarantee that the players at the bottom of the table are ranked both more reasonably (i.e. all of them are assigned some non-0 probability of winning their next game) and differently based on against whom they lost.
- In a similar vein, if a player who lost all games but only against high skilled players, they still might be ranked higher than a player who won some games, but only against low-ranked players.

- Gibbs sampling and EP produce very similar rankings, hence EP is a produces a good approximation in this case.
- EP is much cheaper to run than Gibbs sampling, hence it is the preferable method for scaling up the method.

+ **2 pts** Partial / mostly correct discussion of the above points

✓ **+ 1 pts** **Basic discussion / presentation has major issues**