Unifying Review of Linear Gaussian Models

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Contents

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Notice: $P(\mathbf{x_{t+1}}|\mathbf{x_t}) = \mathcal{N}(\mathbf{A}\mathbf{x_t}, \mathbf{Q})|_{\mathbf{x_{t+1}}}$ and $P(\mathbf{y_t}|\mathbf{x_t}) = \mathcal{N}(\mathbf{C}\mathbf{x_t}|\mathbf{R})|_{\mathbf{y_t}}$, so we can calculate an explicit expression for the joint probability of a sequence of τ states and observables:
$P(\{\mathbf{x_1}, \dots, \mathbf{x_\tau}\}, \{\mathbf{y_1}, \dots, \mathbf{y_\tau}\}) = P(\mathbf{x_1}) \prod_{t=1}^{\tau-1} P(\mathbf{x_{t+1}}|\mathbf{x_t}) \prod_{t=1}^{\tau} P(\mathbf{y_t}|\mathbf{x_t})$.
Then, $-2\log P(\{\mathbf{x_1}, \dots, \mathbf{x_\tau}\}, \{\mathbf{y_1}, \dots, \mathbf{y_\tau}\})$ is the cost function used.

Inference and Learning

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$$\mathcal{L}(\theta) = \log P(\mathbf{Y}|\theta) \geq$$

$$\geq \int_{\mathbf{X}} \mathcal{Q}(\mathbf{X}) \log P(\mathbf{X}, \mathbf{Y}|\theta) d\mathbf{X} - \int_{\mathbf{X}} \mathcal{Q}(\mathbf{X}) \log \mathcal{Q}(\mathbf{X}) d\mathbf{X} := \mathcal{F}(\mathcal{Q}, \theta)$$

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E-step: $\mathcal{Q}_{k+1} \leftarrow \arg\max_{\mathcal{Q}} \mathcal{F}(\mathcal{Q}, \theta_k)$.

M-step: $\theta_{k+1} \leftarrow \arg\max_{\theta} \mathcal{F}(\mathcal{Q}_{k+1}, \theta)$.

Continuos-State Linear Gaussian Systems

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- **PCA**: Particular case of SPCA, where α tends to zero.
- **Kalman Filter Models**: Recovering equations (1) because of the time dependency (Linear dynamical systems). We can extend our spatial intuition of the static case to this dynamic model, but now, state-space ball "flows" from time step to time step.

References (I)

- Sam Roweis and Zoubin Ghahramani
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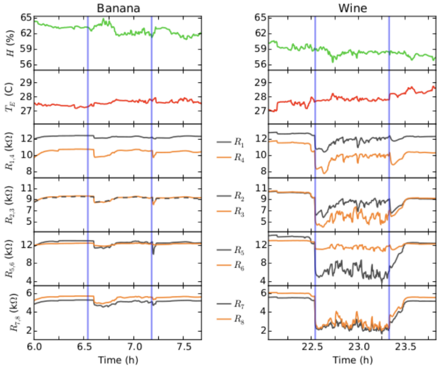
Project description

Monitoring home activity with gas sensors

- Dataset available at UCI Machine Learning Repository
- The original article can be found at arxiv.org/pdf/1608.01719.pdf
- The main goals of the project are:
  - Detect stimulus
  - Classify stimulus

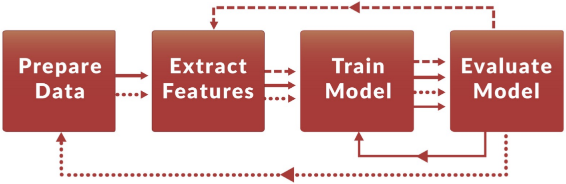


The metadata contains common information for each series

id	date	class	t0	dt
0	07-04-15	banana	13.49	1.64
1	07-05-15	wine	19.61	0.54
2	07-06-15	wine	19.99	0.66
3	07-09-15	banana	6.49	0.72
...				
98	09-16-15	background	14.41	0.71
99	09-17-15	background	11.93	0.68

Iteration process

Common iteration over the different phases of a ML project



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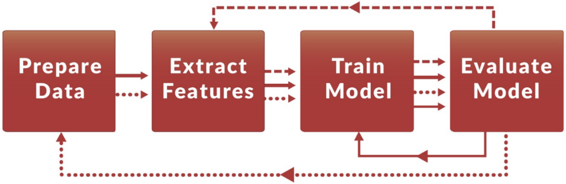

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- Raw data set
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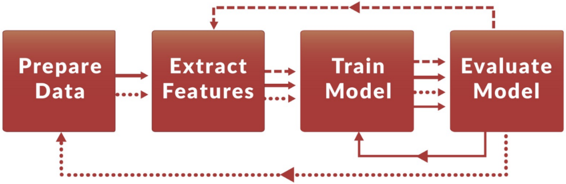

Supervised algorithms used:

- Logistic Regression
- Neural Networks
- Decision Trees
- Support Vector Machines
- Ensembles of the above
- Recurrent Neural Networks

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Best results

Acc.—F1-score	**Raw DB**	**Clean DB**	**Win DB**	**SMOTE DB**
Ensembles NN 3x4 - 0.01	85%—78%	85%—80%	86%—84%	85%—85%
Random Forest	84%—79%	84%—81%	86%—83%	85%—84%
Original paper (SVM)	77%—?	?—?	81%—?	?—?

References (II)

- Online Decorrelation of Humidity and Temperature in Chemical Sensors for Continuous Monitoring

  Ramon Huerta ,Thiago Mosqueiro, Jordi Fonollosa, Nikolai F. Rulkov and Irene Rodriguez-Lujan

- Gas sensors for home activity monitoring
  Machine Learning Repository
  Flavia Huerta, Ramon Huerta

- Smote oversampling for imbalanced classification
  Machine Learning Mastery.
  Jason Brownlee