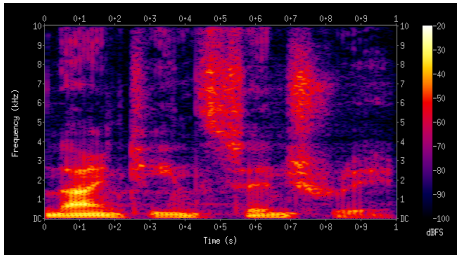
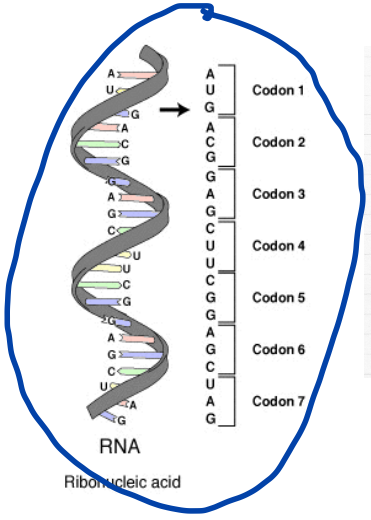
Sequence Modelling

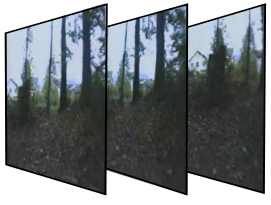
Rich Turner

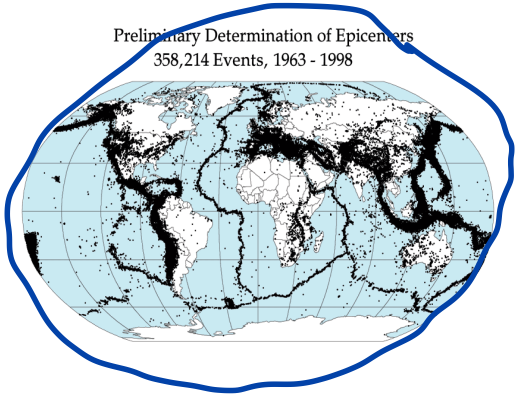
Sequence data

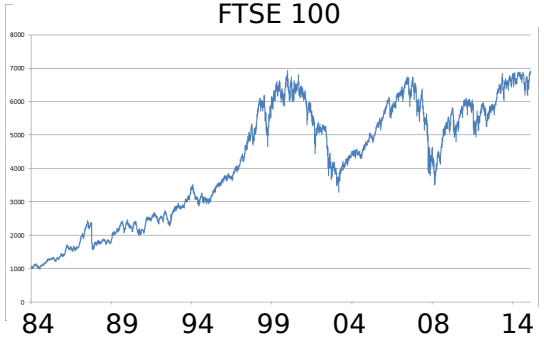


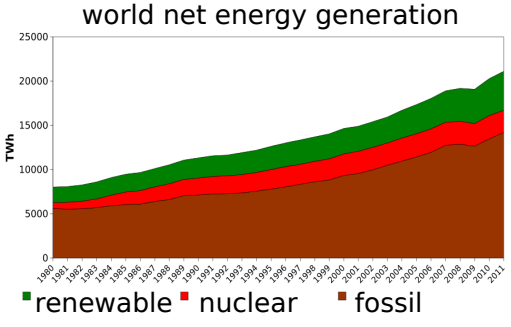


Good King Wenceslas looked out,
On the feast of Stephen,
When the snow lay round about,
Deep and crisp and even;
Brightly shone the moon that night,
Though the frost was cruel,
When a poor man came in sight,
Gathering winter fuel.









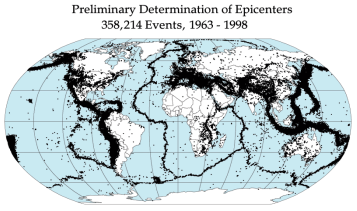
I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted. A. Turing

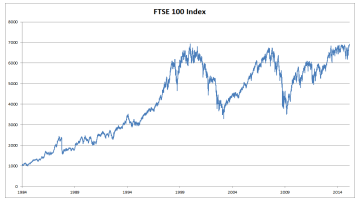
Some images taken from wikipedia

Goals of sequence modelling

Predict future items in sequence

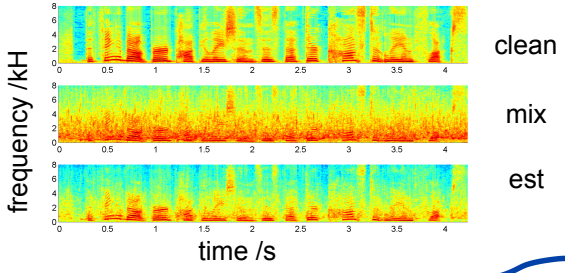
$$p(y_t|y_1,\ldots,y_{t-1})$$





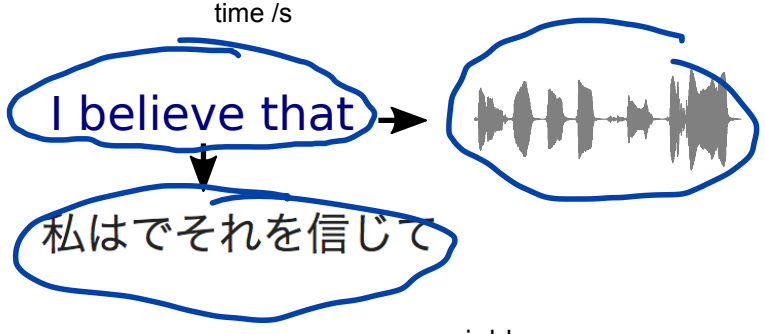
Remove noise from a sequence

$$p(y'_1,\ldots,y'_t|y_1,\ldots,y_t)$$



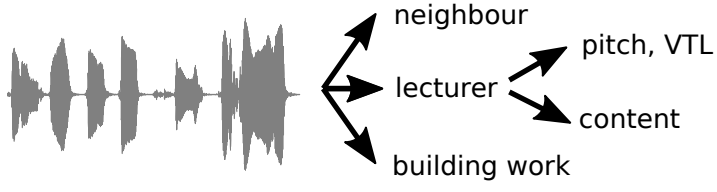
Predict one sequence from another

$$p(y'_1,\ldots,y'_t|y_1,\ldots,y_t)$$



Discover underlying latent variables

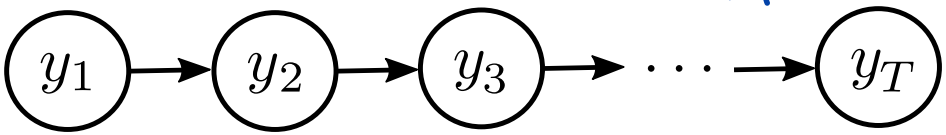
$$p(x_1,\ldots,x_t|y_1,\ldots,y_t)$$



Markov models

First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2) \ldots p(y_T|y_{T-1})$$



$$p(y_{1:T}) = \mathcal{N}(y_{1:T};\, \mu_{1:T}, \Sigma_{1:T,1:T})$$

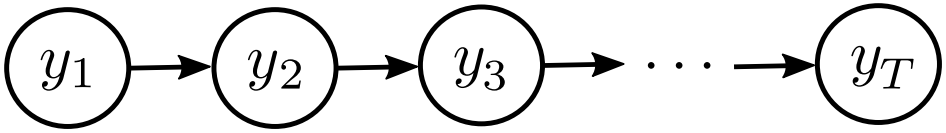
Markov models

parameters tied

∞ number of variables
finite number of
parameters

First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



Markov models

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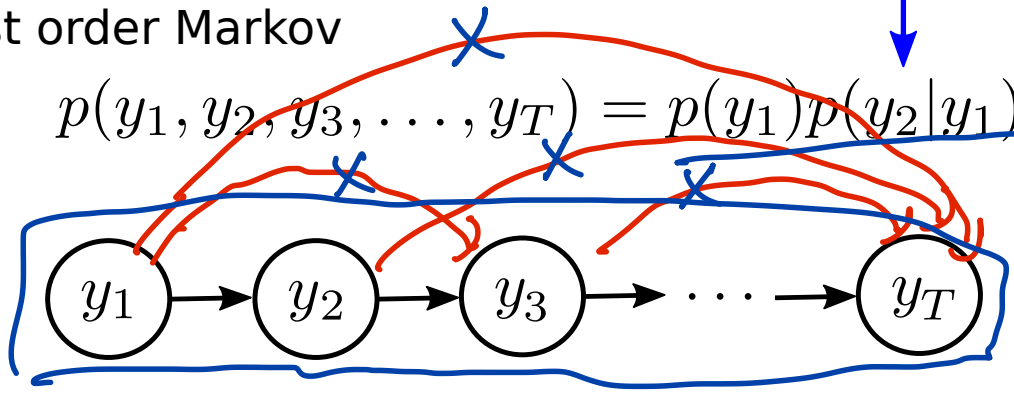
First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



Markov model = conditional independence relationship + product rule

independent from

independent of past

future → $y_{t+1} \perp y_{1:t-1} | y_t$ ← given present

$$p(y_{1:T}) = \prod_{t=1}^{T} p(y_t|y_{1:t-1})$$

$p(y_{1:T}) = p(y_1)p(y_2|y_1)p(y_3|y_1,y_2) \cdots p(y_T|y_{1:T-1})$

$= p(y_3|y_2)$

$= p(y_T|y_{T-1})$ $\quad y_{1:T-2}$

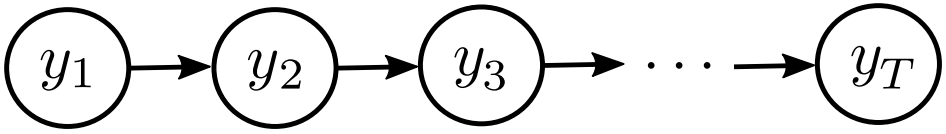
Markov models

parameters tied

∞ number of variables
finite number of parameters

First order Markov

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



Markov model = conditional independence relationship + product rule

future

independent of past

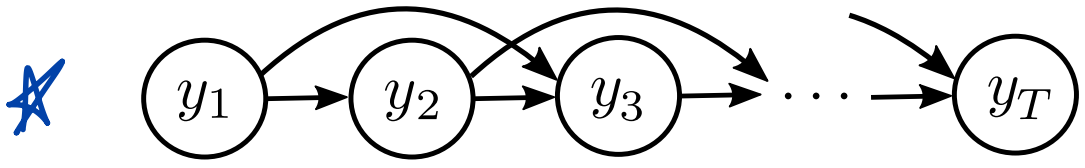
$$y_{t+1} \perp y_{1:t-1} | y_t$$

given present

$$p(y_{1:T}) = \prod_{t=1}^{T} p(y_t|y_{1:t-1})$$

Second order Markov

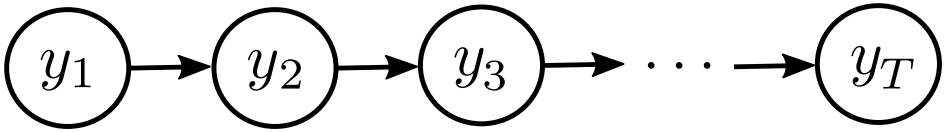
$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$



Markov models for discrete data: n-gram models

First order Markov (bi-gram)

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



$y_t \in \{1, ..., K\}$ discrete states

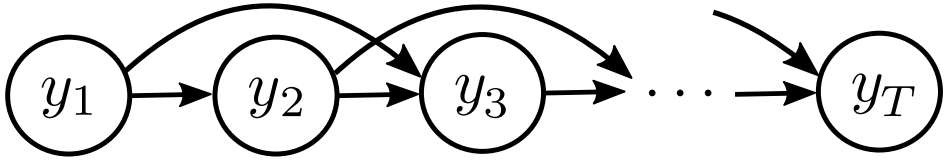
$p(y_1 = k) = \pi_k^0$ initial state probabilities

$p(y_t = k|y_{t-1} = l) = T_{k,l}$ transition probabilities (stochastic matrix)

$\sum_{k=1}^{K} T_{k,l} = 1$

Second order Markov (tri-gram)

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$



$p(y_t = k|y_{t-1} = l, y_{t-2} = m) = T_{k,l,m}$

n-grams require large multidimensional arrays

100 x 100 x 100

Some questions about n-gram models

First order Markov (bi-gram)

$$y_t \in \{1, ..., K\} \quad p(y_1 = k) = \pi_k^0 \quad p(y_t = k | y_{t-1} = l) = T_{k,l}$$

Q1. How can we compute the marginal distribution over the second state?

$$p(y_2 = k|\theta) = \sum_{l=1}^{K} p(y_2 = k, y_1 = l | \theta) \quad \text{sum rule}$$

$$= \sum_{l=1}^{K} p(y_2 = k | y_1 = l, \theta) p(y_1 = l | \theta) \quad \text{product rule}$$

$$= \sum_{l=1}^{K} T_{kl} \pi_l^0$$

$$= [T \pi^0]_k$$

Some questions about n-gram models

First order Markov (bi-gram)

$$y_t \in \{1, ..., K\} \quad p(y_1 = k) = \pi_k^0 \qquad p(y_t = k | y_{t-1} = l) = T_{k,l}$$

Q1. How can we compute the marginal distribution over the second state?

$$p(y_2 = k) = \sum_{l=1}^{K} p(y_2 = k | y_1 = l) p(y_1 = l) = \sum_{l=1}^{K} T_{k,l} \pi_l^0$$

Some questions about n-gram models

First order Markov (bi-gram)

$$y_t \in \{1, ..., K\} \quad p(y_1 = k) = \pi_k^0 \quad p(y_t = k | y_{t-1} = l) = T_{k,l}$$

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$$\Rightarrow \quad p(y_2 = k) = \sum_{l=1}^{K} p(y_2 = k | y_1 = l) p(y_1 = l) = \sum_{l=1}^{K} T_{k,l} \pi_l^0$$

Q2. How can we compute the stationary distribution for the Markov chain?

$$\lim_{t \to \infty} \quad p(y_t = k | \theta) = \sum_{l} T_{k,l} \, p(y_{t-1} = l | \theta)$$

(invariant)

$$\| \qquad\qquad \|$$

$$\pi_k^\infty = \sum_{l=1}^{K} T_{k,l} \, \pi_l^\infty$$

$$\| \qquad\qquad \|$$

$$1 \times \underline{\pi}^\infty = \underline{\underline{T}} \, \underline{\pi}^\infty$$

$$\underline{\underline{M}} \, \underline{e}_\mu = \lambda_\mu \underline{e}_\mu$$

Some questions about n-gram models

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Some questions about n-gram models

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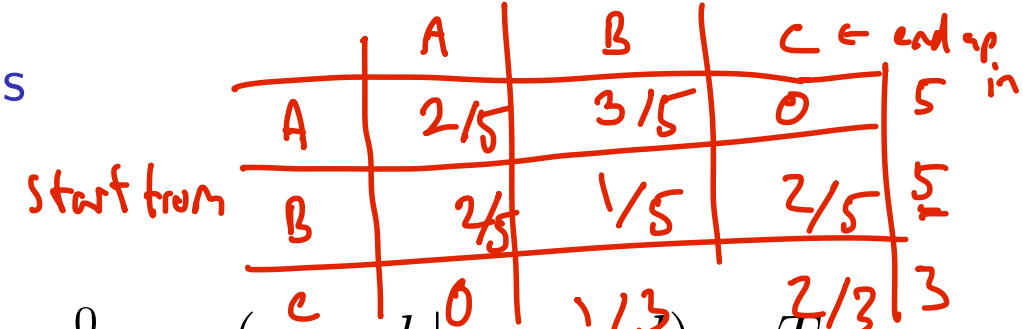
$$\pi_k^\infty = \sum_{l=1}^{K} T_{k,l} \pi_l^\infty$$

eigenvectors of transition matrix with eigenvalue = 1

Some questions about n-gram models



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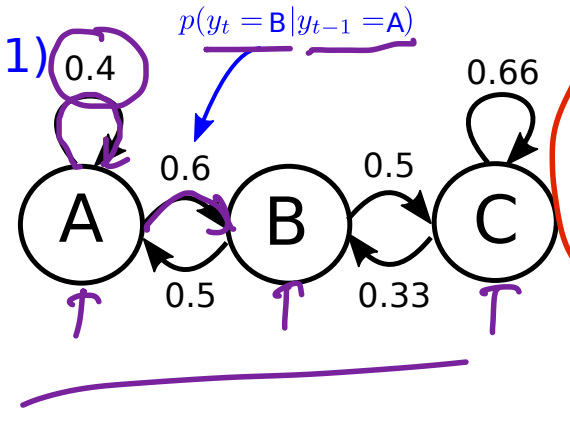
$$\pi_k^\infty = \sum_{l=1}^{K} T_{k,l} \pi_l^\infty$$

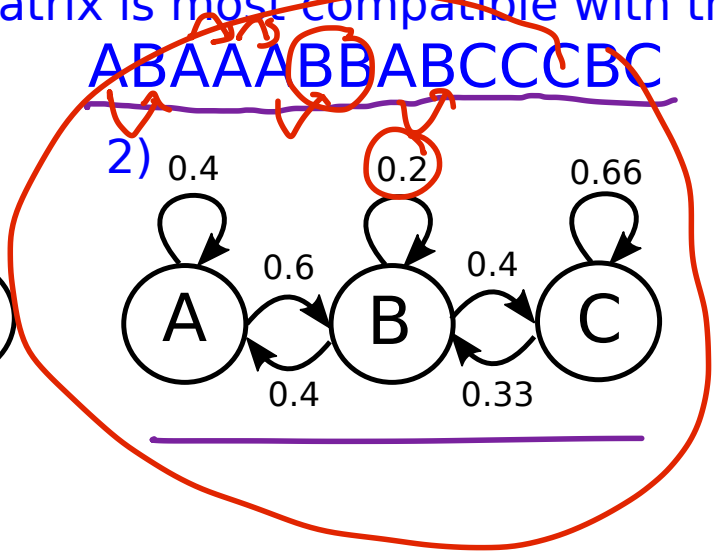
eigenvectors of transition matrix with eigenvalue = 1

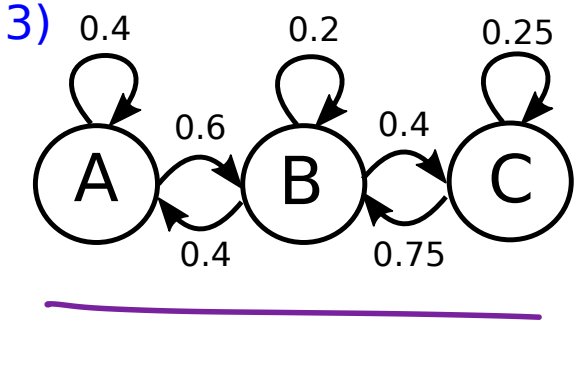
Q3. Which transition matrix is most compatible with the following sequence?

ABAAABBABCCCBC

'State Transition Diagrams'







Some questions about n-gram models

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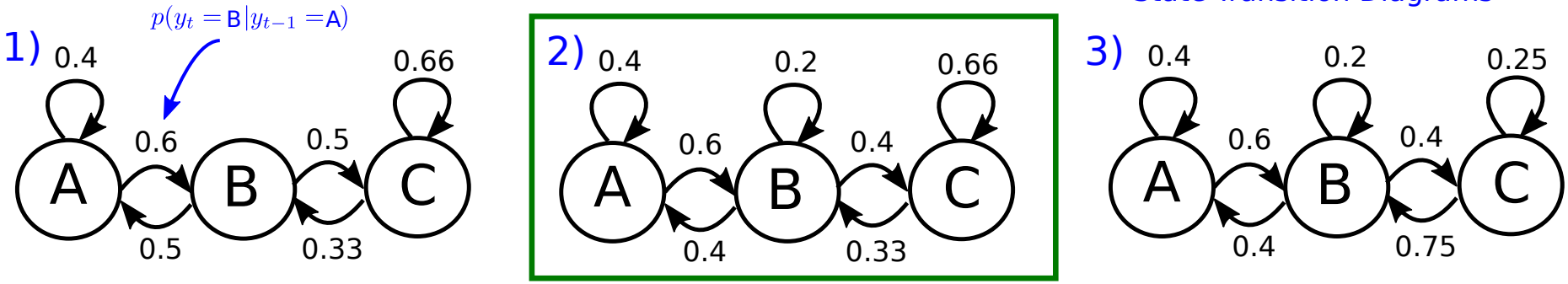
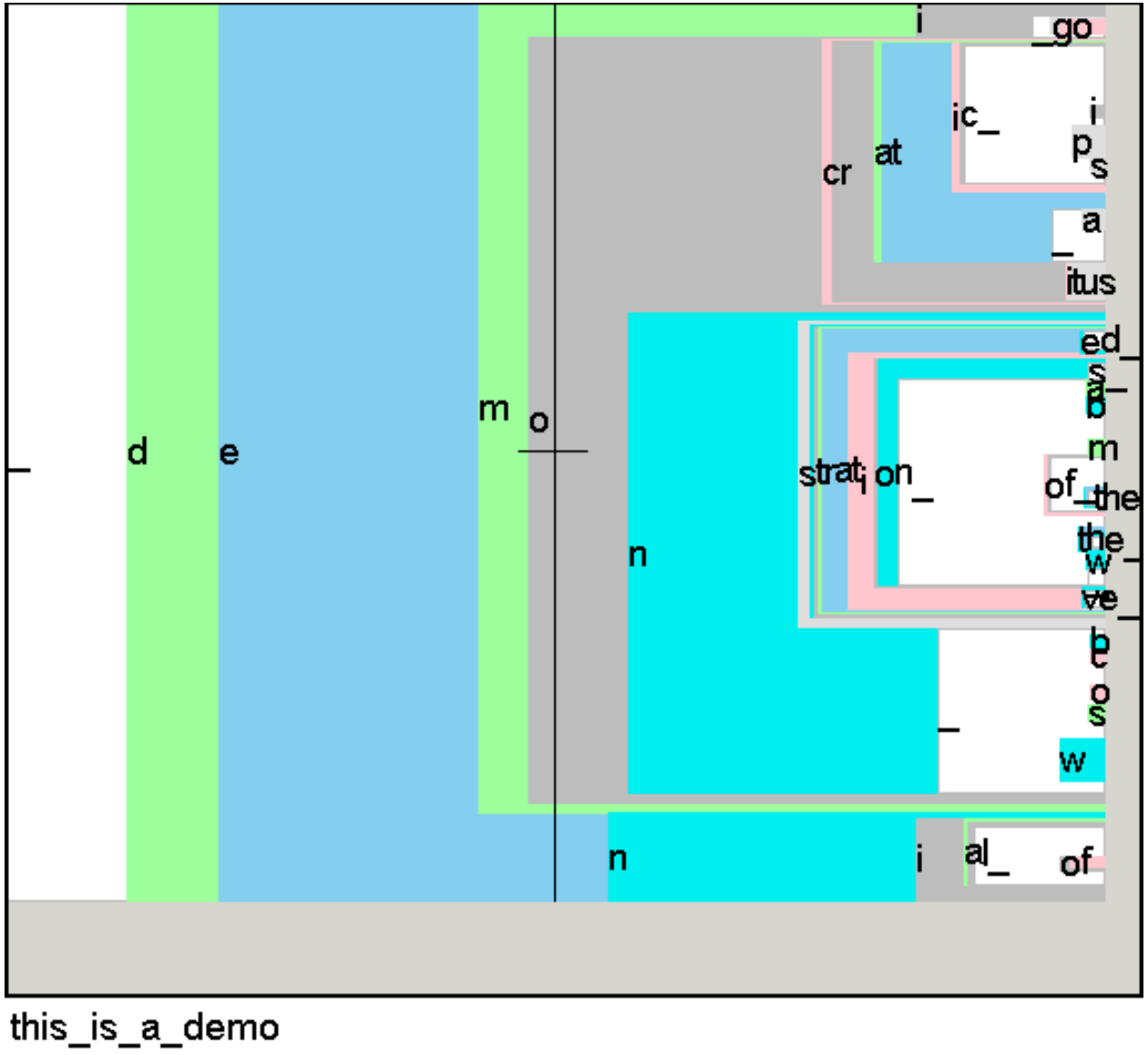


Example application of n-grams: text modelling for dasher



http://www.inference.phy.cam.ac.uk/dasher/ https://www.youtube.com/watch?v=nr3s4613DX8

Markov models for discrete data: n-gram models

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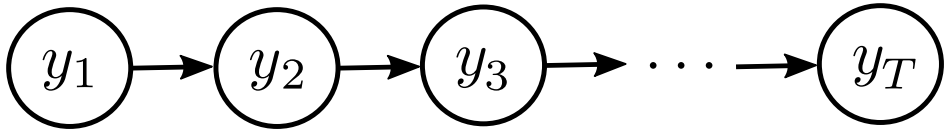

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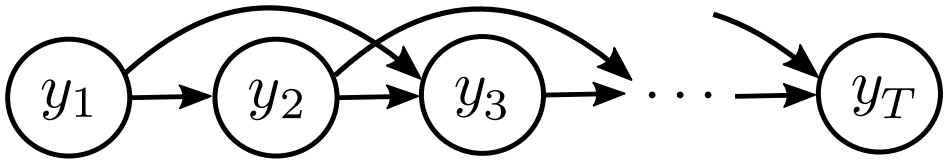


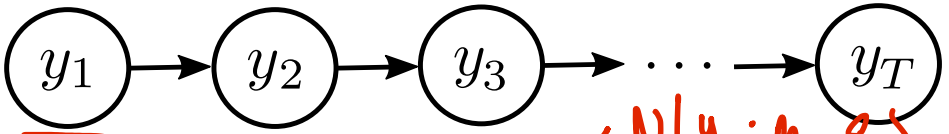
$p(y_t = k|y_{t-1} = l, y_{t-2} = m) = T_{k,l,m}$

n-grams require large multidimensional arrays

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2)\ldots p(y_T|y_{T-1})$$



$y_t \in \mathbb{R}^D$ — continuous vector states

$p(y_1) = \mathcal{G}(y_1; \mu_0, \Sigma_0)$ — initial state density

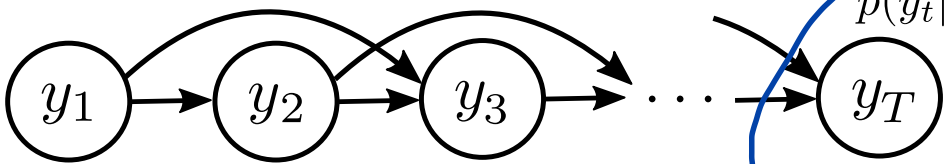
$p(y_t|y_{t-1}) = \mathcal{G}(y_t; \Lambda y_{t-1}, \Sigma)$ — transition density

$$\mathcal{G}(y; \mu, \Sigma) = \frac{1}{(2\pi)^{D/2}\det(\Sigma)^{1/2}} \exp\left\{-\frac{1}{2}(y-\mu)^T \Sigma^{-1}(y-\mu)\right\}$$

Second order Markov (AR(2))

$$p(y_1, y_2, y_3, \ldots, y_T) = p(y_1)p(y_2|y_1)p(y_3|y_2, y_1)\ldots p(y_T|y_{T-1}, y_{T-2})$$

$$p(y_t|y_{t-1}, y_{t-2}) = \mathcal{G}(y_t; \Lambda_1 y_{t-1} + \Lambda_2 y_{t-2}, \Sigma)$$



joint distribution over all variables is always multivariate Gaussian

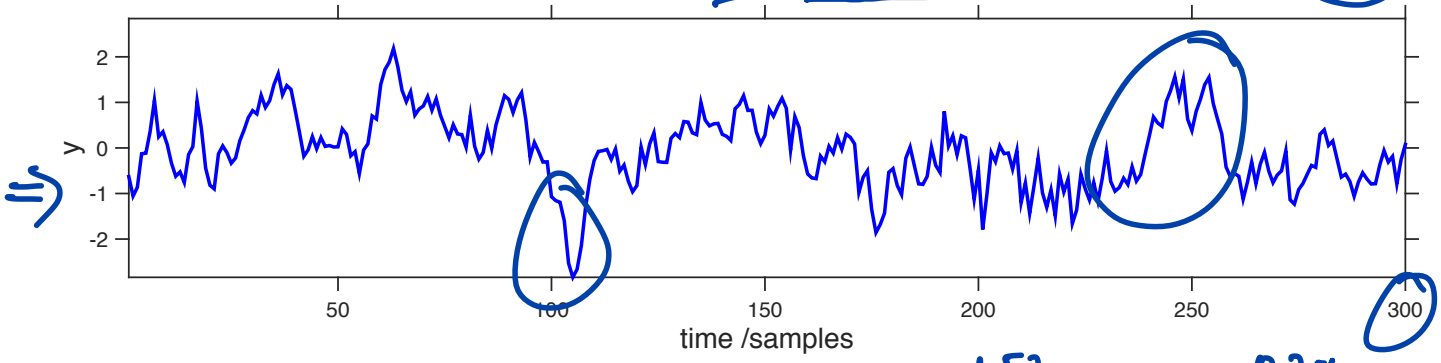
$$p(y_{1:T}|\theta) = \mathcal{G}(y_{1:T}; \mu_{1:T}, \Sigma_{1:T,1:T})$$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

$y_t = 0.9 \times y_{t-1} + \epsilon_t \sqrt{0.01} \quad \epsilon_t \sim \mathcal{N}(0,1)$

First order Markov (AR(1))

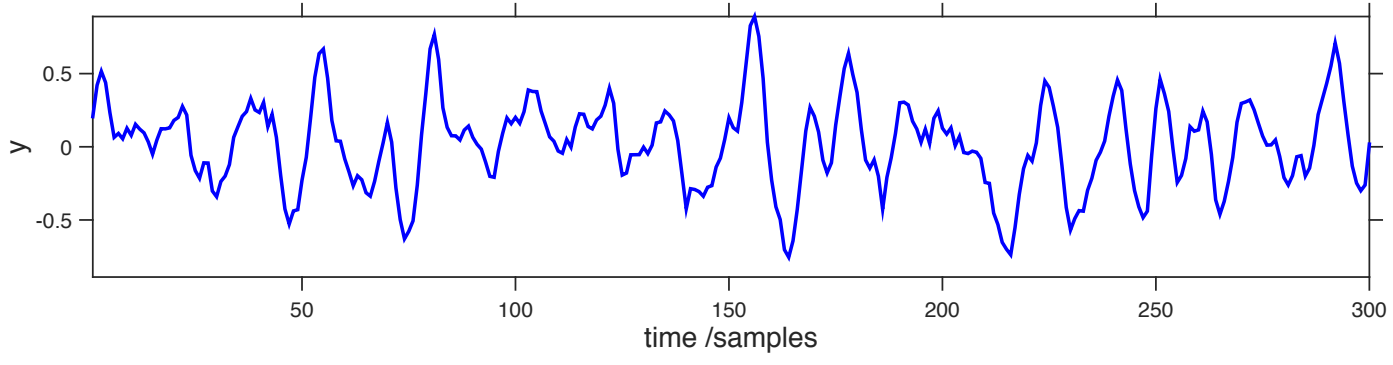
$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \lambda y_{t-1}, \sigma^2) \qquad \lambda = 0.9 \quad \sigma^2 = 0.01$$



Second order Markov (AR(2))

$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}, y_{t-2}) = \mathcal{G}(y_t; \lambda_1 y_{t-1} + \lambda_2 y_{t-2}, \sigma^2)$$

$$[\lambda_1, \lambda_2] = [1.57, -0.78] \qquad \sigma^2 = 0.01$$



Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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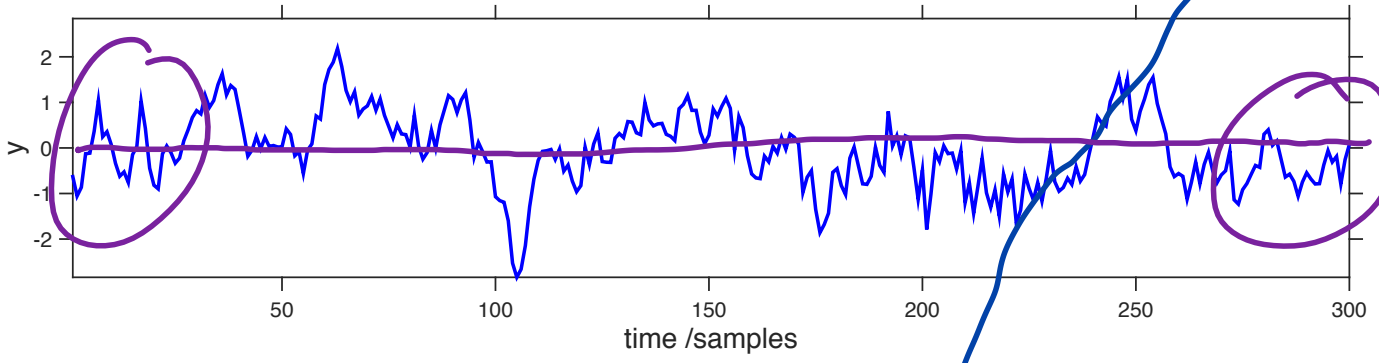


What is the stationary distribution of this process?

$$y_t = \lambda y_{t-1} + \varepsilon_t \sigma \qquad \varepsilon_t \sim \mathcal{G}(0,1)$$

$$p(y_\infty) = \mathcal{G}(y_\infty | \mu_\infty, \Sigma_\infty)$$

$$= \mathbb{E}(y_t) = \mathbb{E}(\lambda y_{t-1} + \varepsilon_t \sigma)$$

$p(y_{t-1}, \varepsilon_t)$

$$= \mu_t = \lambda \mu_{t-1} + \sigma \mathbb{E}(\varepsilon_t) \Rightarrow \mu_t = \lambda \mu_{t-1}$$

$\mathbb{E}(\varepsilon_t) \to 0$

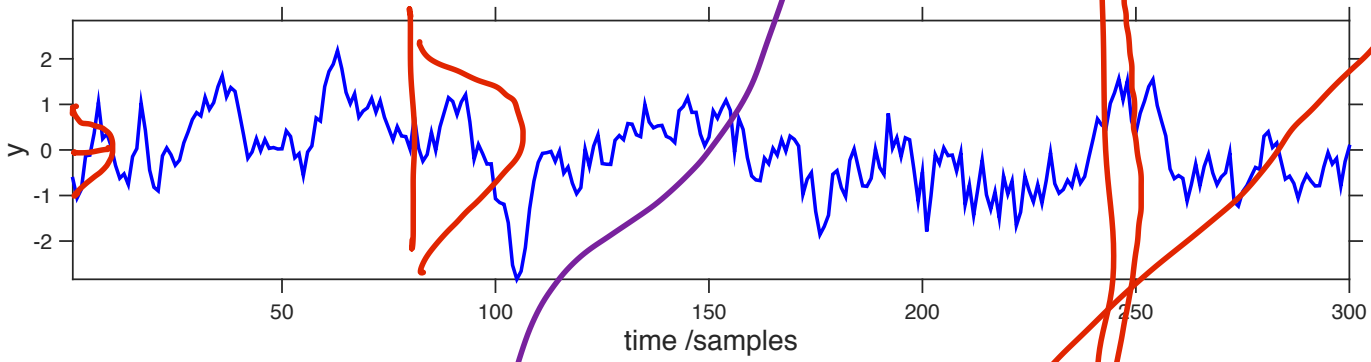
$\lambda = 0.9$

$\mu_\infty = \lambda \mu_\infty \Rightarrow \mu_\infty = 0$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \lambda y_{t-1}, \sigma^2) \qquad \lambda = 0.9 \quad \sigma^2 = 0.01$$



$y_t = \lambda y_{t-1} + \varepsilon_t \sigma \qquad \varepsilon_t \sim G(0,1)$

$y_t = y_{t-1} + \varepsilon_t \sigma = y_{t-2} + \varepsilon_{t-1}\sigma + \varepsilon_t \sigma$

What is the stationary distribution of this process? $p(y_\infty) = ?$

Everything is linear Gaussian => must be Gaussian $p(y_\infty) = \mathcal{G}(y_\infty; \mu_\infty, \sigma_\infty^2)$

$$\mathbb{E}(y_t^2) = \mathbb{E}\left((\lambda y_{t-1} + \varepsilon_t \sigma)^2\right) = \lambda^2 \mathbb{E}(y_{t-1}^2) + 2\mathbb{E}(y_{t-1}\varepsilon_t)\sigma\lambda + \mathbb{E}(\varepsilon_t^2)\sigma^2$$

$p(y_{t-1}, \varepsilon_t) = p(y_{t-1})p(\varepsilon_t)$

$\mathbb{E}_{p(y_{t-1})}(y_{t-1}) \cdot \mathbb{E}_{p(\varepsilon_t)}(\varepsilon_t) = 0$

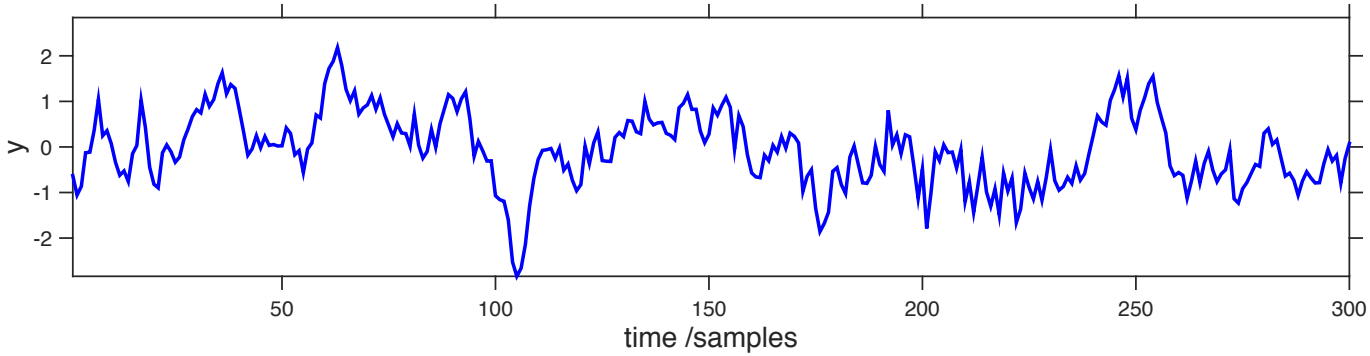
$1 - 0.9^2 = 0.19$

$$\sigma_\infty^2 = \lambda^2 \sigma_\infty^2 + \sigma^2 \quad \Rightarrow \quad \sigma_\infty^2 = \frac{\sigma^2}{1-\lambda^2} = \frac{0.01}{1-0.9^2} = \frac{0.01}{0.19}$$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

$$y_t \in \mathbb{R}^1 \qquad p(y_t|y_{t-1}) = \mathcal{G}(y_t; \lambda y_{t-1}, \sigma^2) \qquad \lambda = 0.9 \quad \sigma^2 = 0.01$$



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$$y_t = \lambda y_{t-1} + \sigma \epsilon_t \;\; \epsilon_t \sim \mathcal{G}(0, 1)$$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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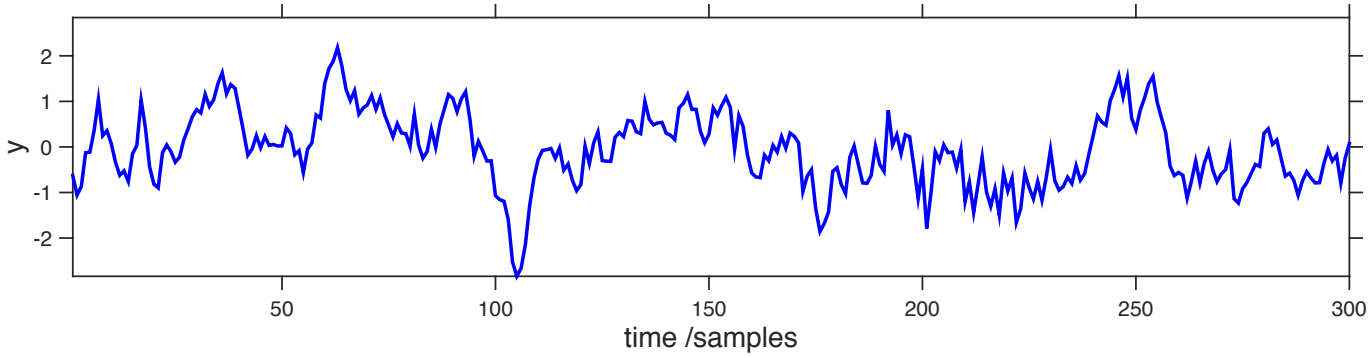

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Mean: $\langle y_t \rangle$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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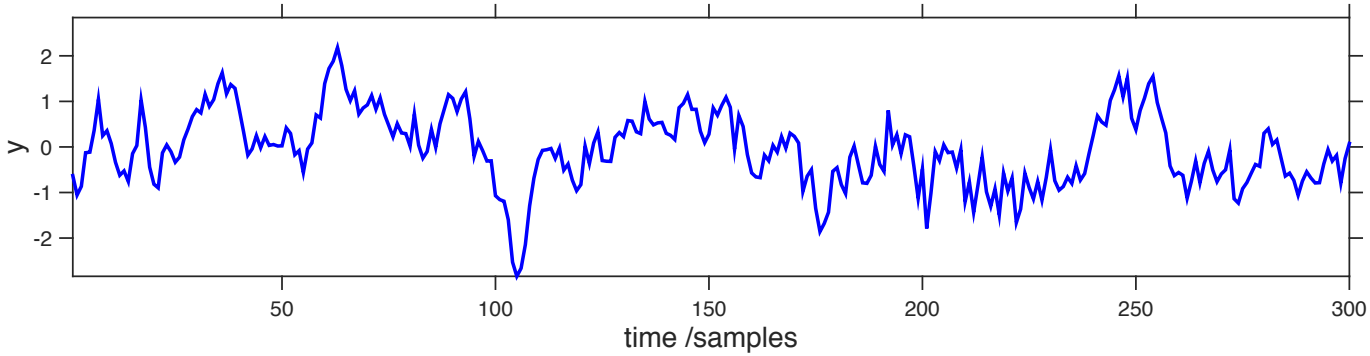

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Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

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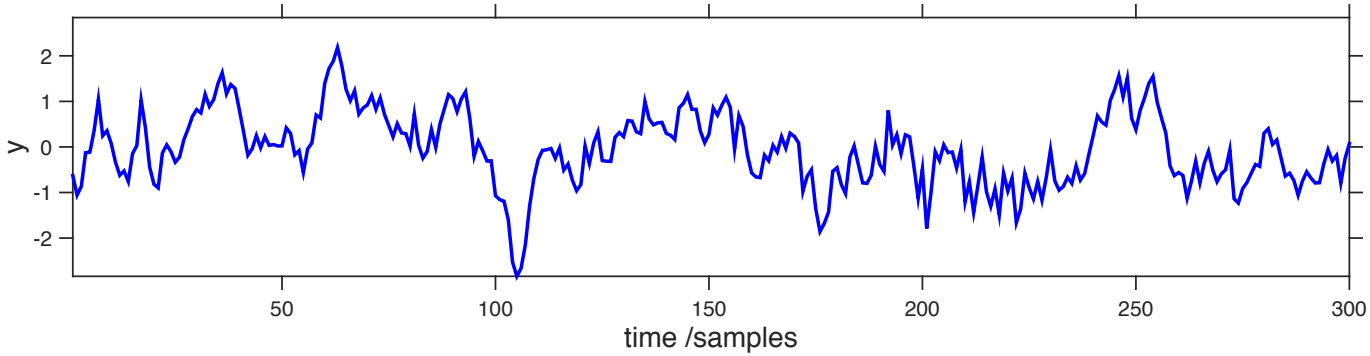

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Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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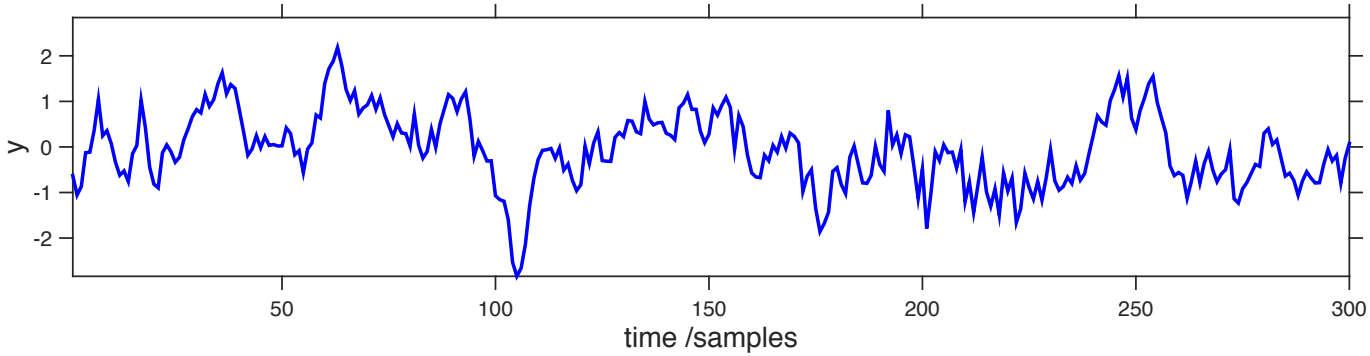

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Mean: $\langle y_t \rangle = \lambda \langle y_{t-1} \rangle + \sigma \langle \epsilon_t \rangle = 0 \qquad \mu_\infty = 0$

Variance: $\langle y_t^2 \rangle$

Markov models for continuous data: Auto-Regressive (AR) Gaussian models

First order Markov (AR(1))

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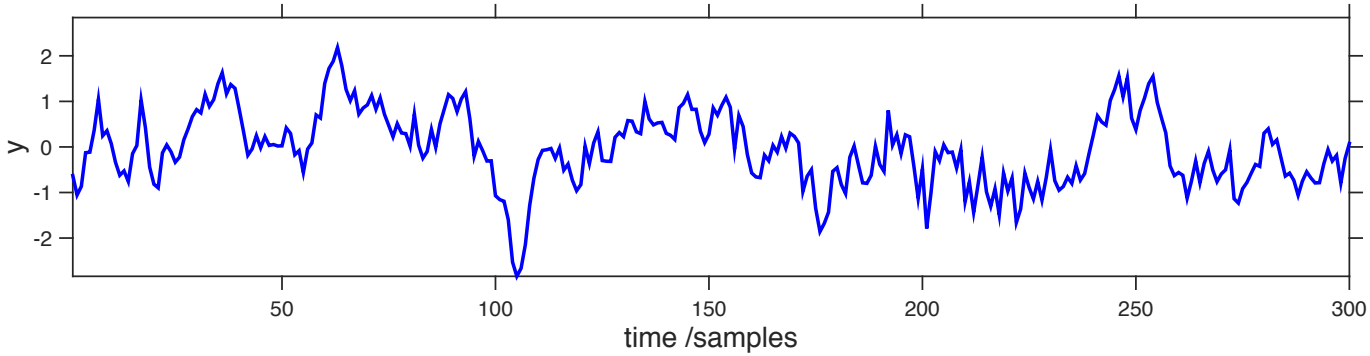

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Markov models for continuous data: Auto-Regressive (AR) Gaussian models

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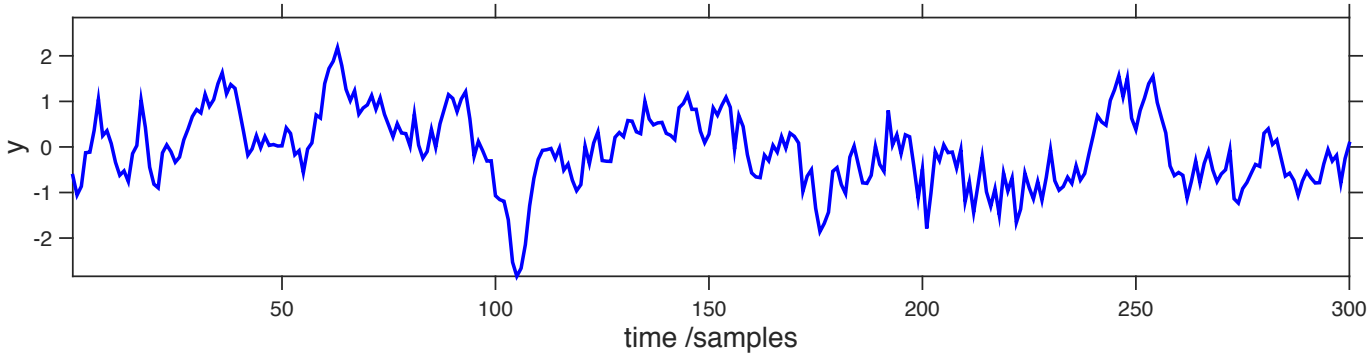

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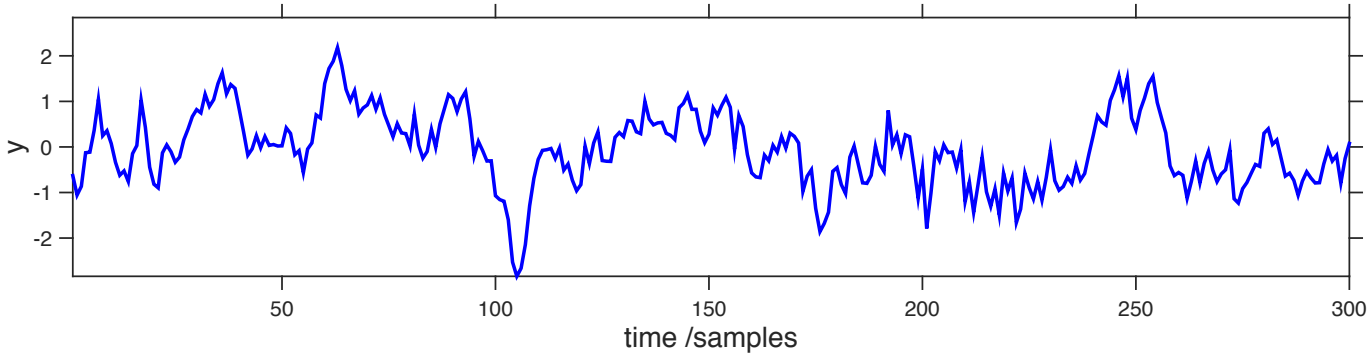

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$$\langle y_t^2 \rangle = \lambda^2 \langle y_{t-1}^2 \rangle + \sigma^2$$

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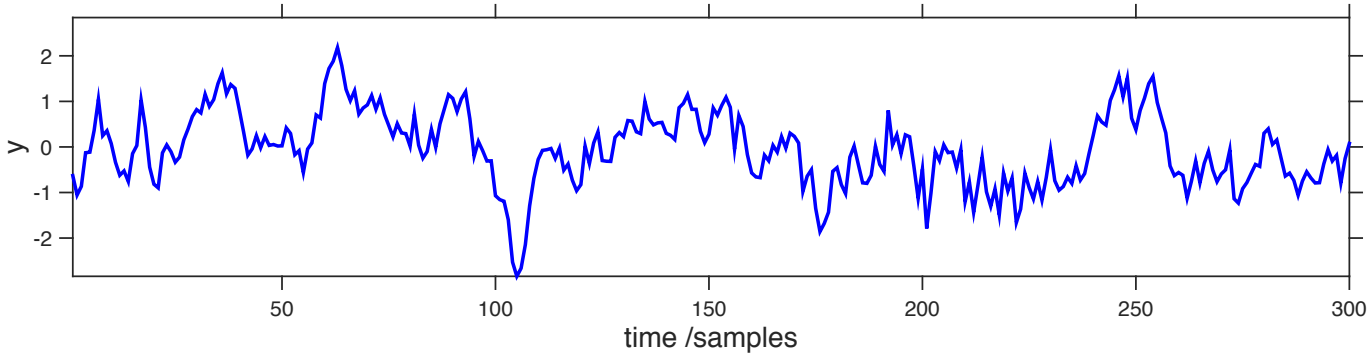

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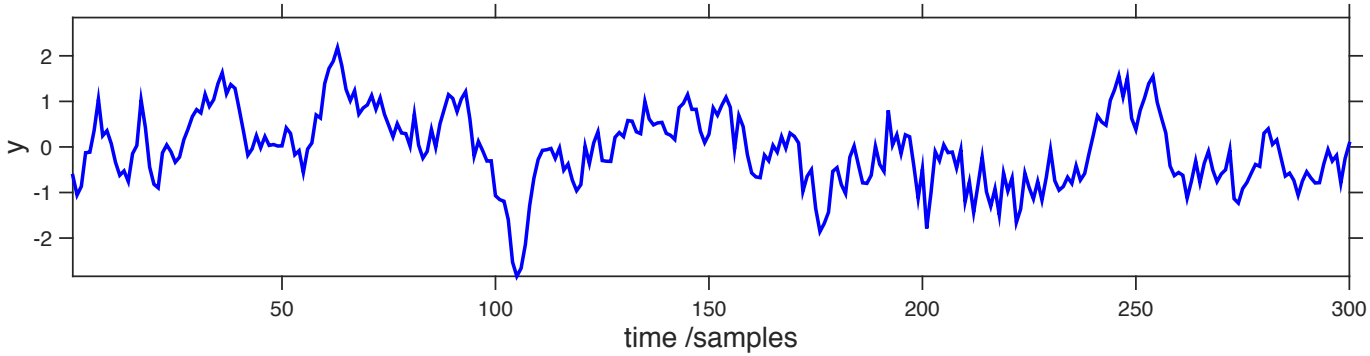

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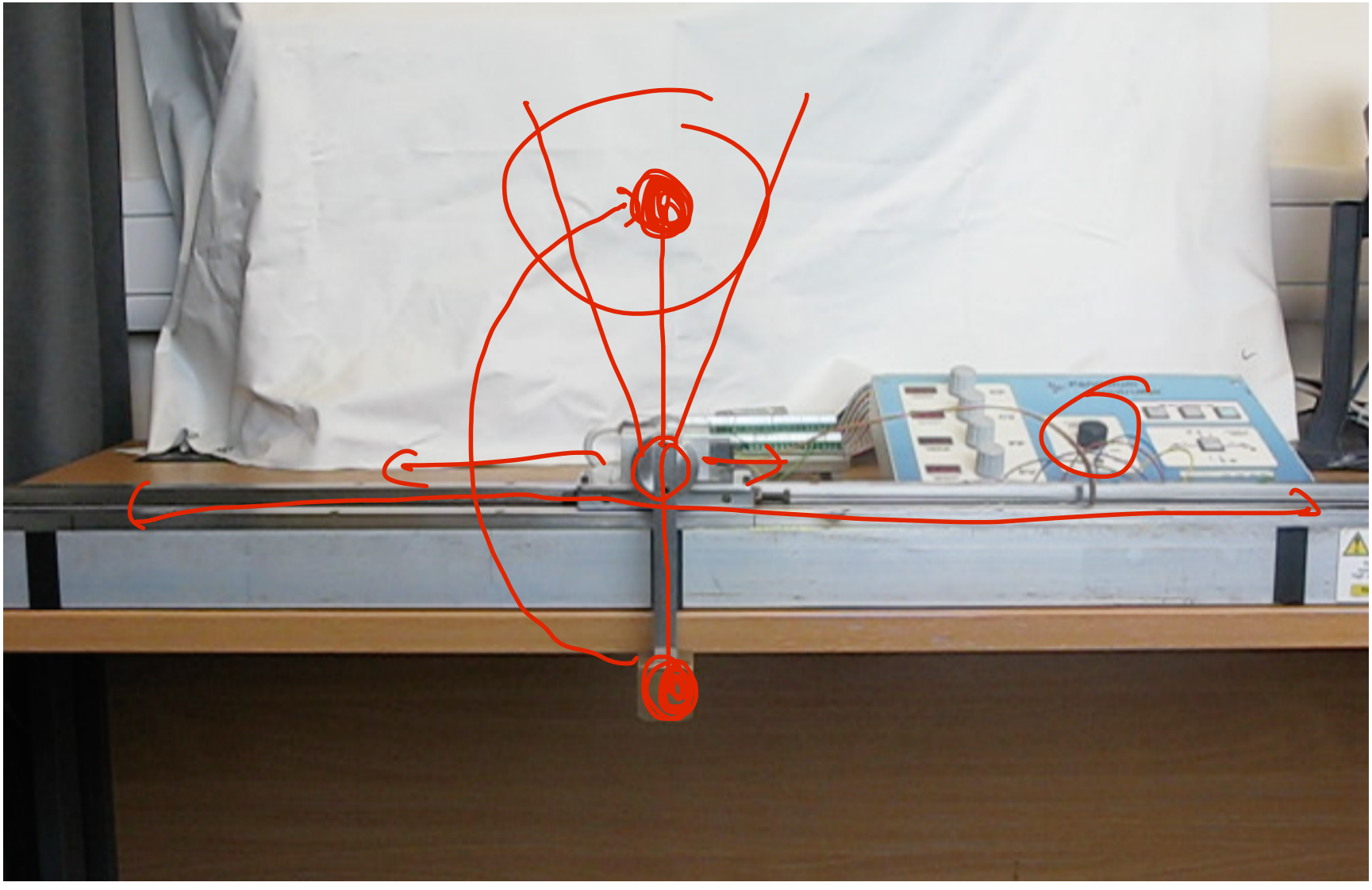
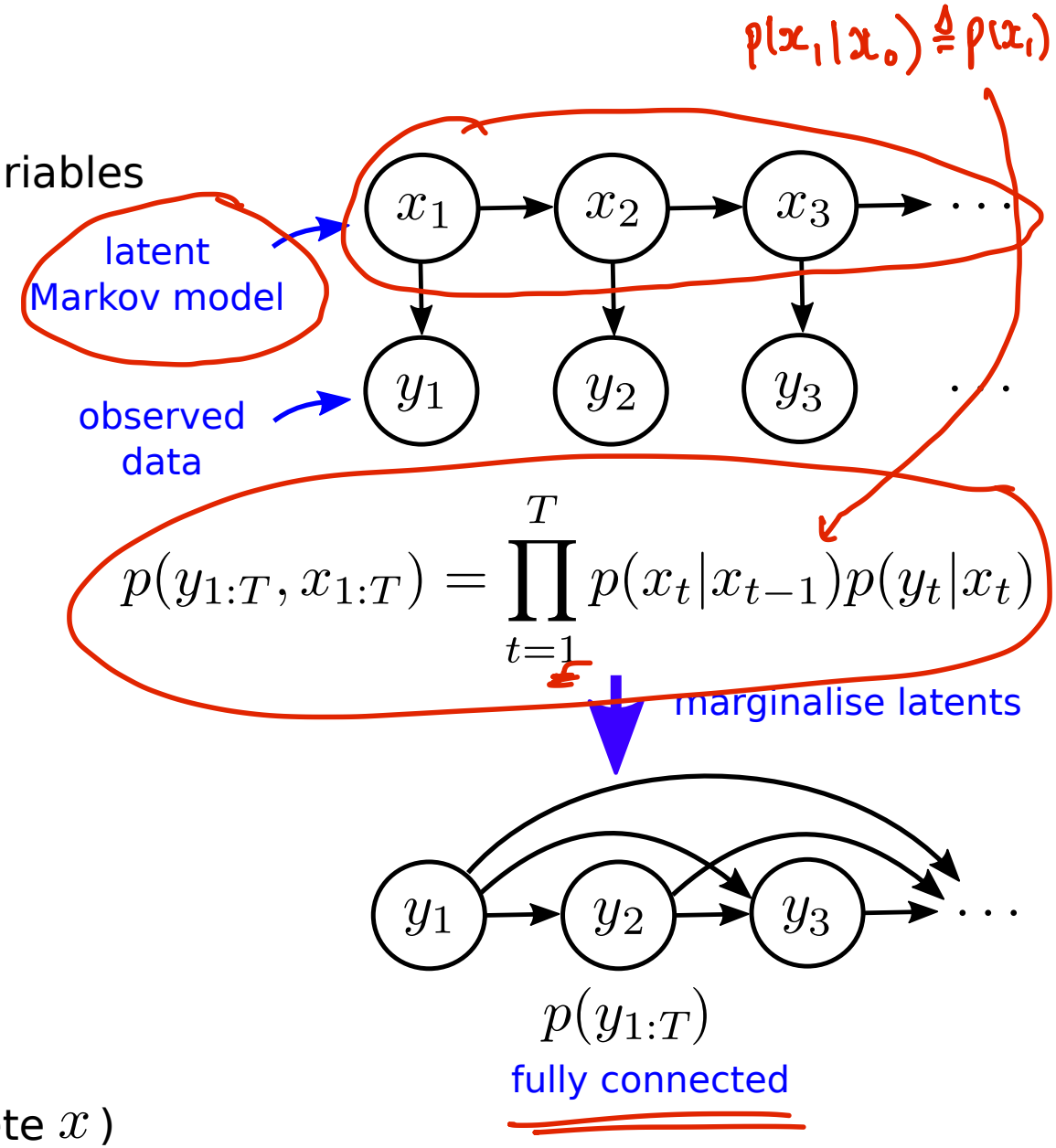
$$\langle y_t^2 \rangle = \lambda^2 \langle y_{t-1}^2 \rangle + \sigma^2 \quad \sigma^2_\infty = \lambda^2 \sigma^2_\infty + \sigma^2 \qquad \sigma^2_\infty = \frac{\sigma^2}{1-\lambda^2}$$

Example application of Markov Models: pendulum swing up control problem



Hidden Markov models

Real data depend on latent variables



ASR

- x phonemes/words
- y waveform/feature

Computer Vision

- x objects, pose, lighting
- y image pixel intensities

Natural Language Processing

- x topics
- y words

Two prevelant Examples:

Hidden Markov Models (discrete x)
Linear Gaussian State Space Models (Gaussian x and y)

Hidden Markov models: discrete hidden state

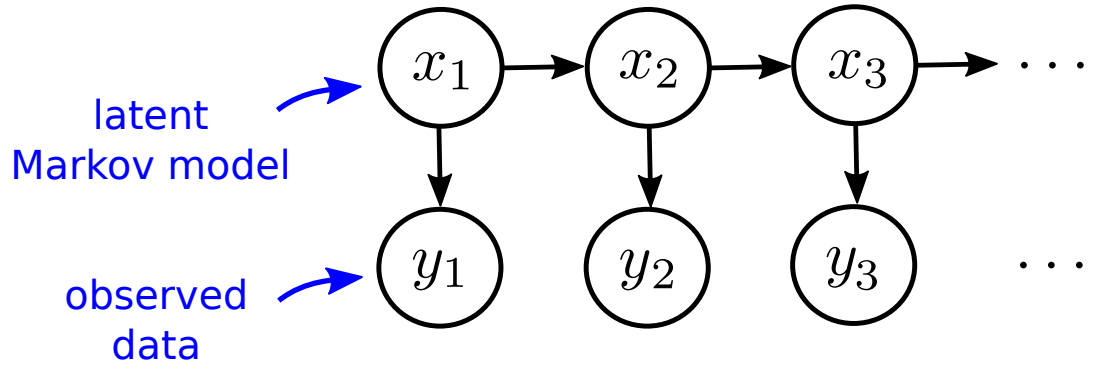
Discrete Hidden State

$$x_t \in \{1, ..., K\}$$

$$p(x_t = k | x_{t-1} = l) = T_{k,l}$$

E.g. in examples below $K = 2$

$$\Rightarrow T = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

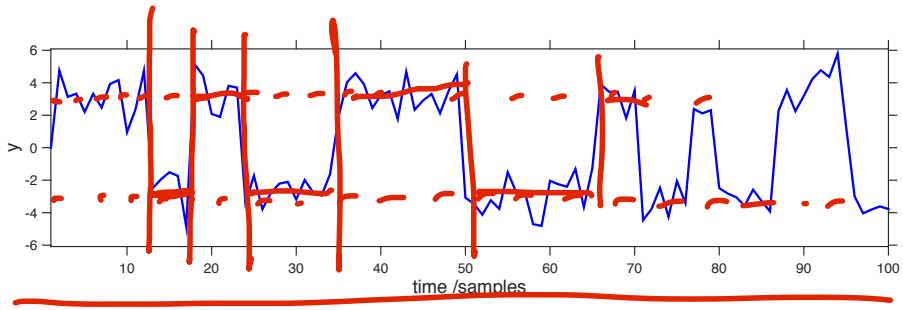


$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t | x_{t-1}) p(y_t | x_t)$$

Continuous Observed State

$$\Rightarrow p(y_t | x_t = k) = \mathcal{G}(y_t; \mu_k, \Sigma_k)$$

$\mu_1 = 3 \quad \mu_2 = -3 \quad \sigma_1^2 = \sigma_2^2 = 1$



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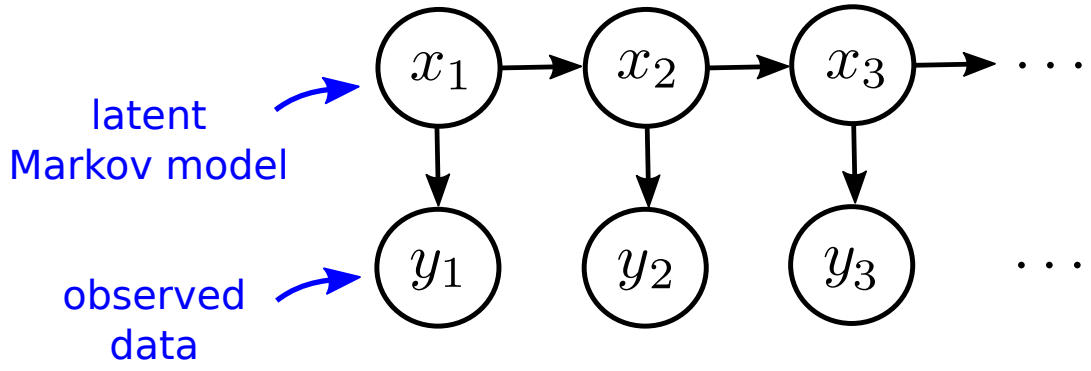

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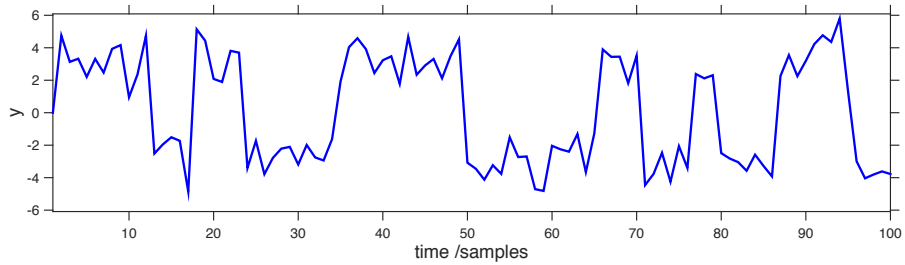

Discrete Observed State

$$p(y_t = l | x_t = k) = S_{l,k}$$

$$S = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

ABBBBAAABAAACCCCCBBBBBBCCCCCCCCCCCC
AAABBBBAABAAABBCCCCCCCCCCCCCCCCBBA
AACCCCCCBABCCCCCCCAABBAABABCCCCC

Sequence Modelling Recap...

Markov Models

1st order $p(y_{1:T}) = p(y_1)\,p(y_2|y_1)\,p(y_3|y_2)\cdots p(y_T|y_{T-1})$

⇒ Discrete y ⇒ bigram models

$p(y_1 = k) = \pi_k^0$

$p(y_t = k | y_{t-1} = l) = T_{kl}$

Continuous y ⇒ Autoregressive models

$p(\underline{y}_1) = \mathcal{G}(\underline{y}_1; \underline{\mu}_0, \underline{\underline{\Sigma}}_0)$

$p(\underline{y}_t|\underline{y}_{t-1}) = \mathcal{G}(\underline{y}_t; \underline{\underline{A}}\,\underline{y}_{t-1}, \underline{\underline{\Sigma}})$

Hidden Markov Models

observed $\underline{y}_{1:T}$, hidden / latent $\underline{x}_{1:T}$

$$p(\underline{y}_{1:T}, \underline{x}_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})\,p(\underline{y}_t|x_t)$$ where $p(x_1|x_0) \triangleq p(x_1)$

discrete hidden state HMM: $p(x_t = k | x_{t-1} = l) = T_{kl}$ ←

$p(\underline{y}_t | x_t = k) \overset{e.g.}{=} \mathcal{G}(\underline{y}_t; \underline{\mu}_k, \underline{\underline{\Sigma}}_k)$ ⇐

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

$$x_t \in \{1, ..., K\}$$

$$p(x_1 = k) = \pi_k^0$$

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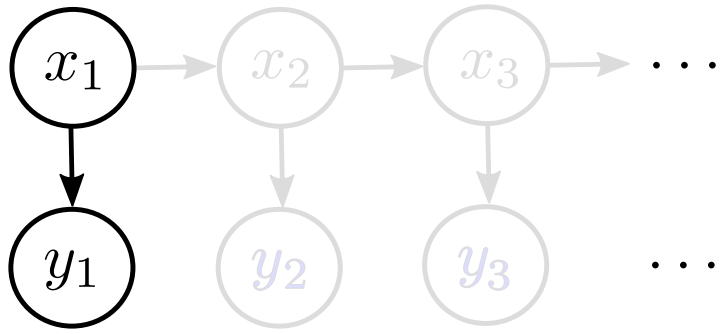

Consider T = 1

Q1: What type of distribution is $p(y_1)$?

Hidden Markov models: discrete hidden state

Discrete Hidden State, Continuous Observed State

$$x_t \in \{1, ..., K\}$$

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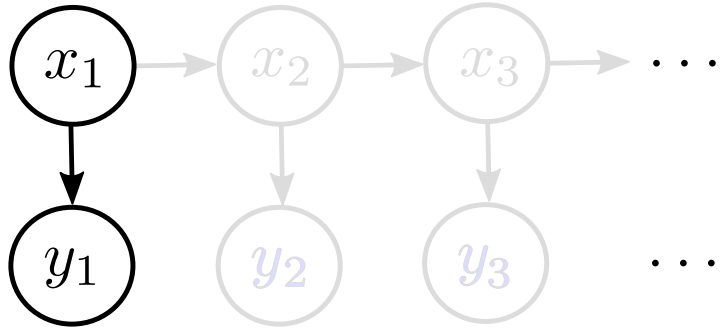

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$$p(y_1) = \sum_k p(y_1 | x_1 = k) p(x_1 = k)$$

$p(y_1, x_1) = p(y_1 | x_1) p(x_1)$

← sum rule + product rule

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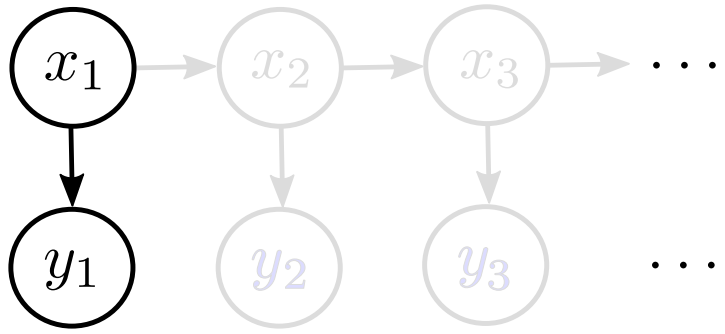

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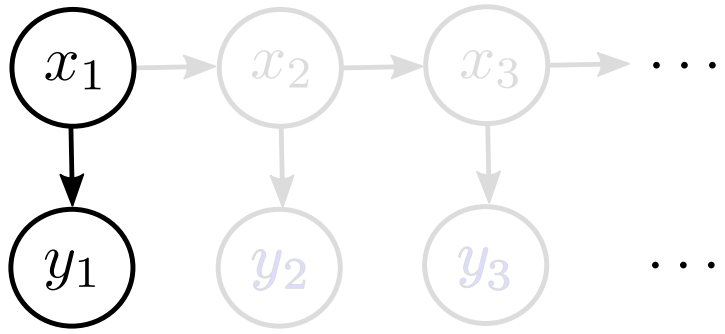

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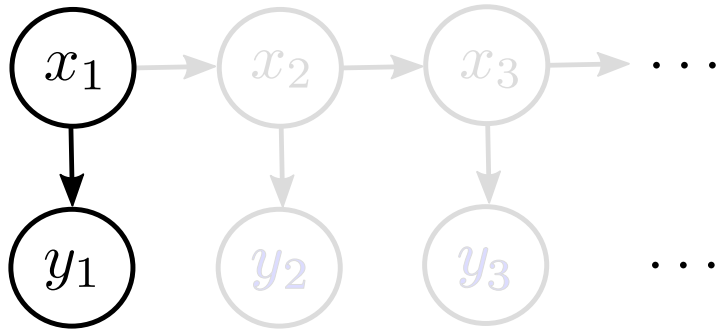

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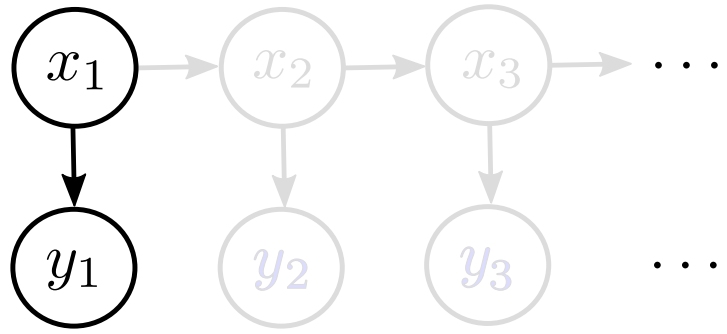

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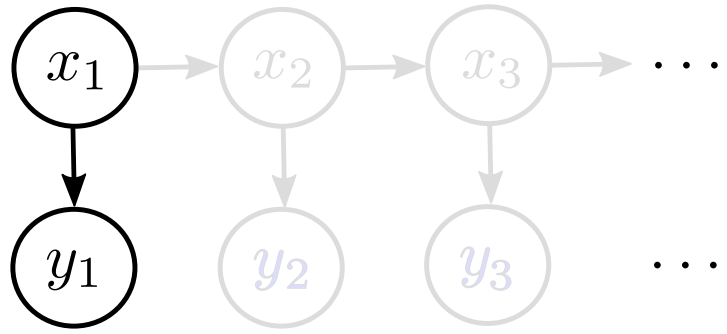

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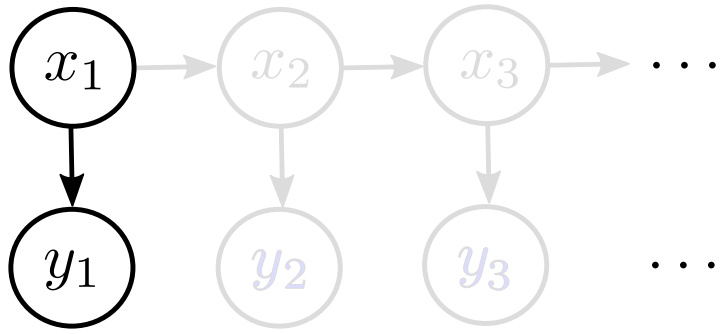

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this HMM = Mixture of Gaussian Models with dynamic cluster assignments

Hidden Markov models: continuous hidden state (LGSSMs)

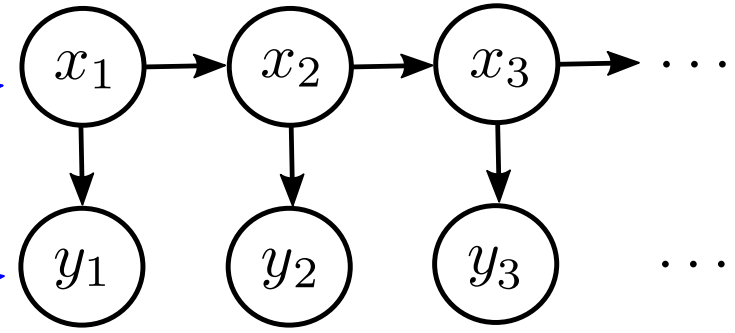
Continuous Hidden State

$x_t \in \mathbb{R}^K$

$p(x_t|x_{t-1}) = \mathcal{G}(x_t; Ax_{t-1}, Q)$

latent Markov model

observed data



Continuous Observed State

$y_t \in \mathbb{R}^D$

$p(y_t|x_t) = \mathcal{G}(y_t; Cx_t, R)$

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

E.g. simple example $K = 2$ $D = 1$

$$A = \lambda \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

rotation matrix

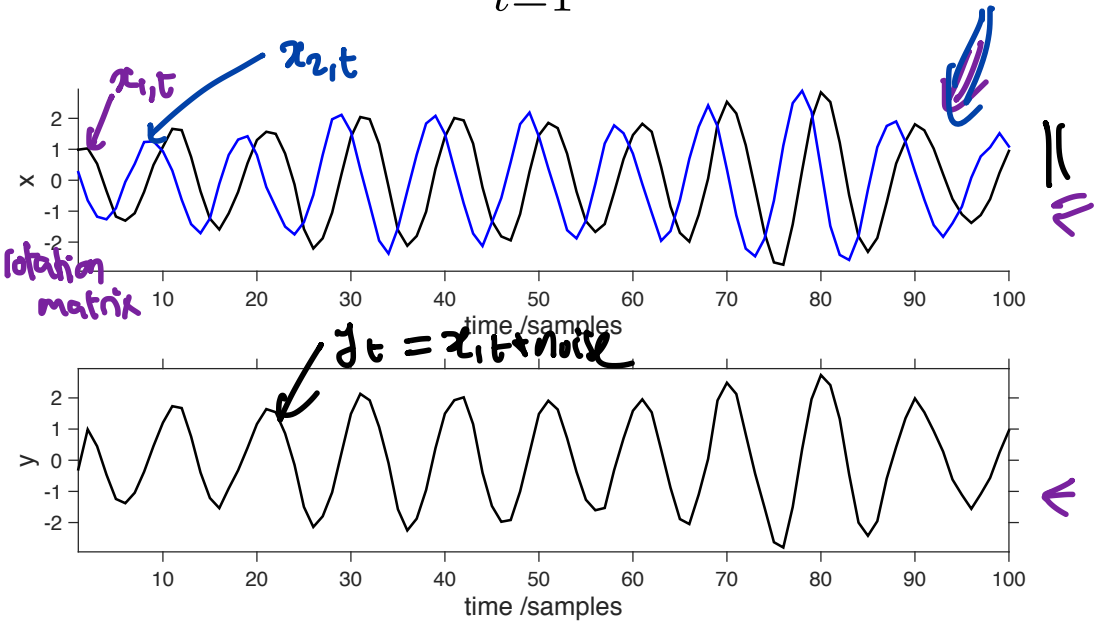
$\lambda = 0.99$ $\theta = 2\pi/10$

$Q = (1-\lambda^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C = [1, 0]$ $R = 0.01$

dynamics model

obs. model

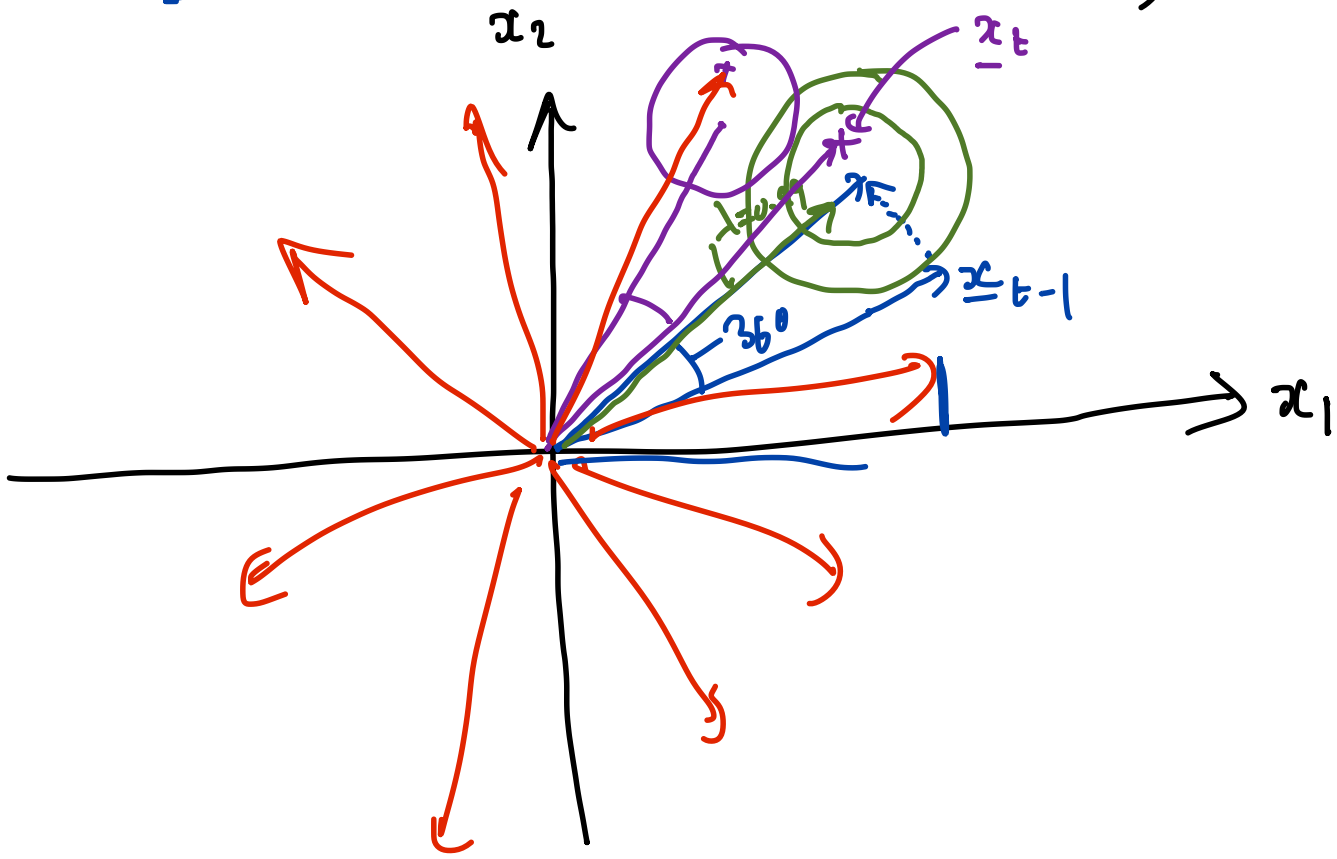


$$\underline{x}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \lambda \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + (1-\lambda^2)^{1/2} \underline{\varepsilon}_t$$

λ: 0.99 $\quad \theta$: $2\pi/10$

$$\underline{\varepsilon}_t \sim G(\underline{0}, \underline{\underline{I}})$$

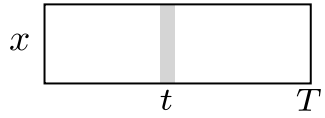
$$y_t = x_{1t} + (0.01)^{1/2} \eta_t \qquad \eta_t \sim G(0,1)$$

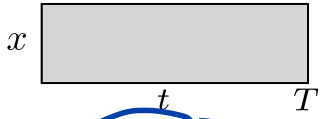


Varieties of Inference

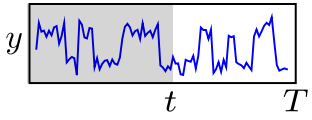
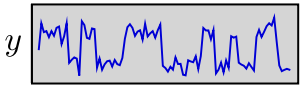
Distributional estimates

infer single state or sequence?





future data available?

	marginal	joint
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Varieties of Inference

Distributional estimates

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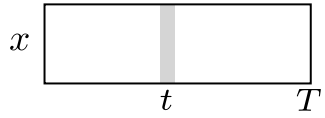

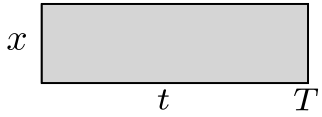

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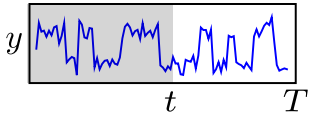
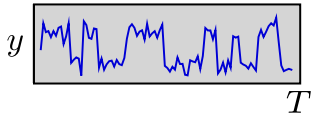
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Point estimates

most probable state @ t

$$x_t^* = \arg\max_{x_t} p(x_t \mid y_{1:T})$$

most probable sequence

$$x'_{1:T} = \arg\max_{x_{1:T}} p(x_{1:T} \mid y_{1:T})$$

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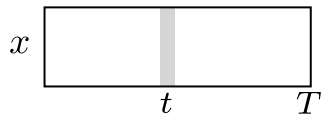

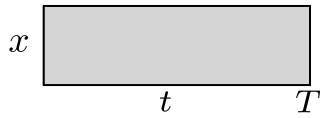

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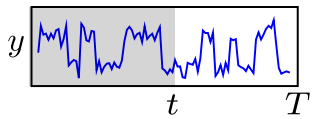
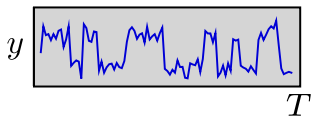
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

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Question: are these estimates the same $x^*_{1:T} \stackrel{?}{=} x'_{1:T}$ for

1. Linear Gaussian State Spate Models?
2. Discrete Hidden State HMMs?

$T=2, K=2$

$p(x_{n2} \mid y_{1:T})$ ✓

sequences	probability
0 0	0
0 1	0.3
1 0	0.4
1 1	0.3

$x^*_{1:2} = [1]$

$x'_{1:T} = (1,0)$

$$p(x_{1:T} \mid y_{1:T}) = \mathcal{N}(x_{1:T}; \mu_{1:T}, \Sigma_{1:T,1:T})$$

$$x^*_{1:T} = \mu_{1:T}$$

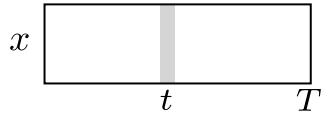
$$p(x_t \mid y_{1:T}) = \int dx_{1:T \neq t}\, \mathcal{N}(x_{1:T}; \mu_{1:T}, \Sigma_{1:T,1:T}) = \mathcal{N}(x_t; \mu_t, \Sigma_{t,t})$$

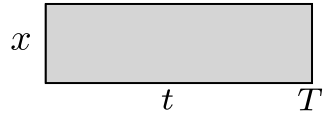
$$x^*_t = \mu_t \Leftarrow \quad x^*_{1:T} = \mu_{1:T}$$

Varieties of Inference

Distributional estimates

infer single state or sequence?





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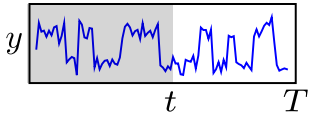
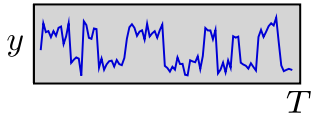
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

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Question: are these estimates the same $x^*_{1:T} \stackrel{?}{=} x'_{1:T}$ for

1. Linear Gaussian State Spate Models? $x^*_{1:T} = x'_{1:T}$ (Gaussian)

2. Discrete Hidden State HMMs?

Varieties of Inference

Distributional estimates

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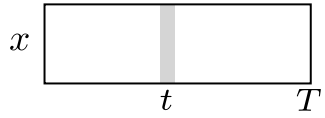
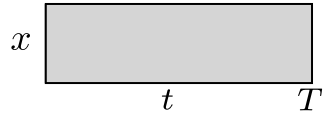
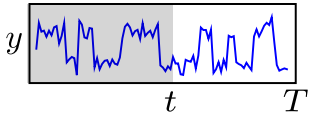
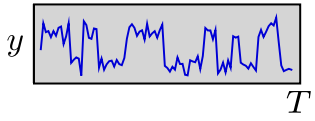




future data available?

	marginal	joint
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Point estimates

most probable state @ t

$$x_t^* = \arg\max_{x_t} p(x_t \mid y_{1:T})$$

most probable sequence

$$x'_{1:T} = \arg\max_{x_{1:T}} p(x_{1:T} \mid y_{1:T})$$

Question: are these estimates the same $x^*_{1:T} \stackrel{?}{=} x'_{1:T}$ for

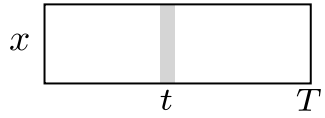
1. Linear Gaussian State Spate Models? $x^*_{1:T} = x'_{1:T}$ (Gaussian)

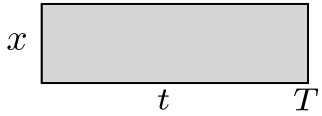
2. Discrete Hidden State HMMs? $x^*_{1:T} \neq x'_{1:T}$

Varieties of Inference

Distributional estimates

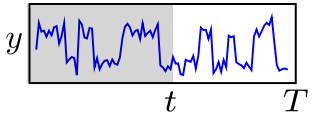
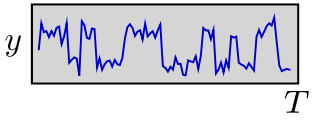
infer single state or sequence?





look at this next

future data available?

	marginal	joint
 filter	$p(x_t \mid y_{1:t})$	$p(x_{1:t} \mid y_{1:t})$
 smoother	$p(x_t \mid y_{1:T})$	$p(x_{1:T} \mid y_{1:T})$

Point estimates

most probable state @ t

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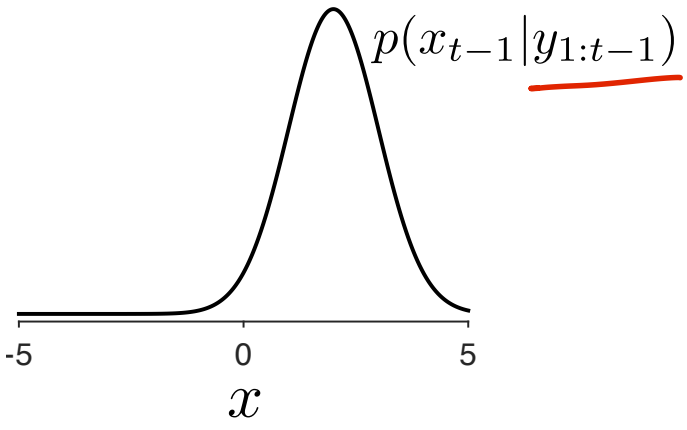
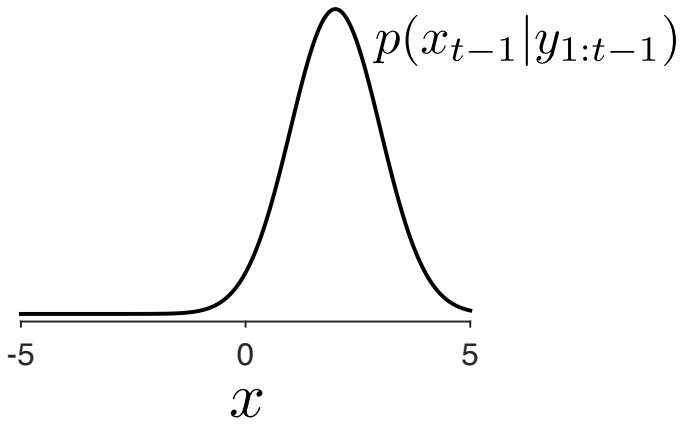
Inference: Kalman Filter



$$p(x_{t-1}|y_{1:t-1})$$

Inference: Kalman Filter



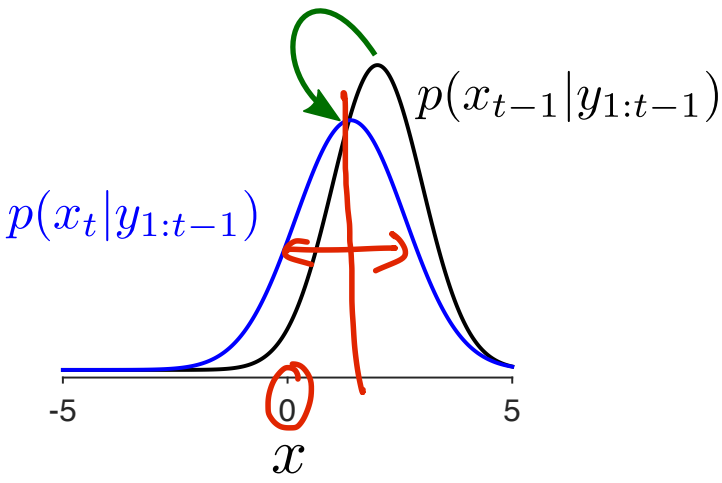
$$p(x_{t-1}|y_{1:t-1})$$

diffuse via dynamics

sum for discrete hidden state

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

Inference: Kalman Filter



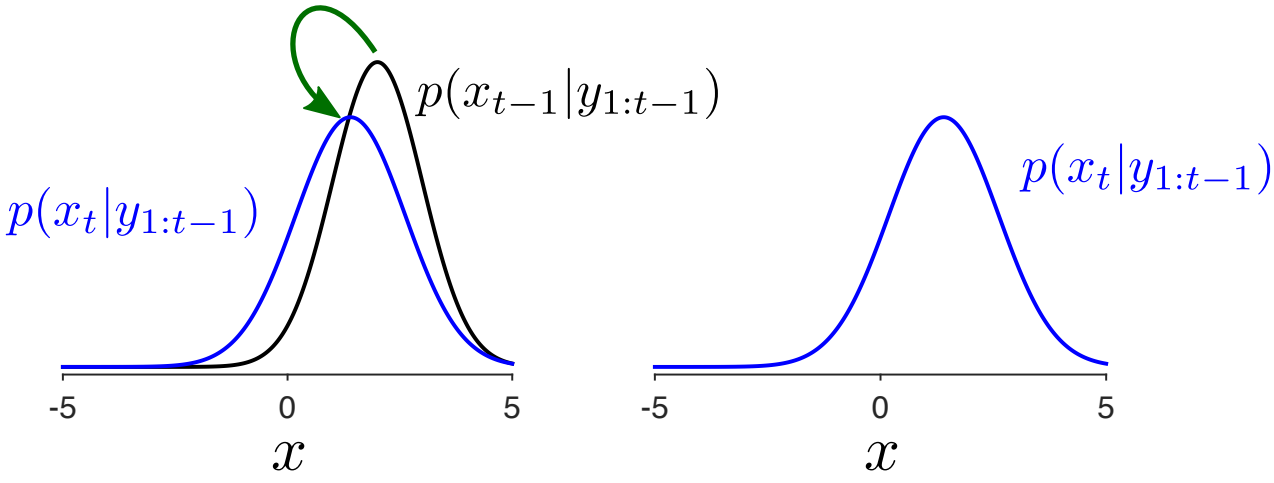
diffuse via dynamics

$$p(x_{t-1}|y_{1:t-1})$$

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

sum for discrete hidden state

Inference: Kalman Filter



$p(x_{t-1}|y_{1:t-1})$

diffuse via dynamics

sum for discrete hidden state

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

combine with likelihood

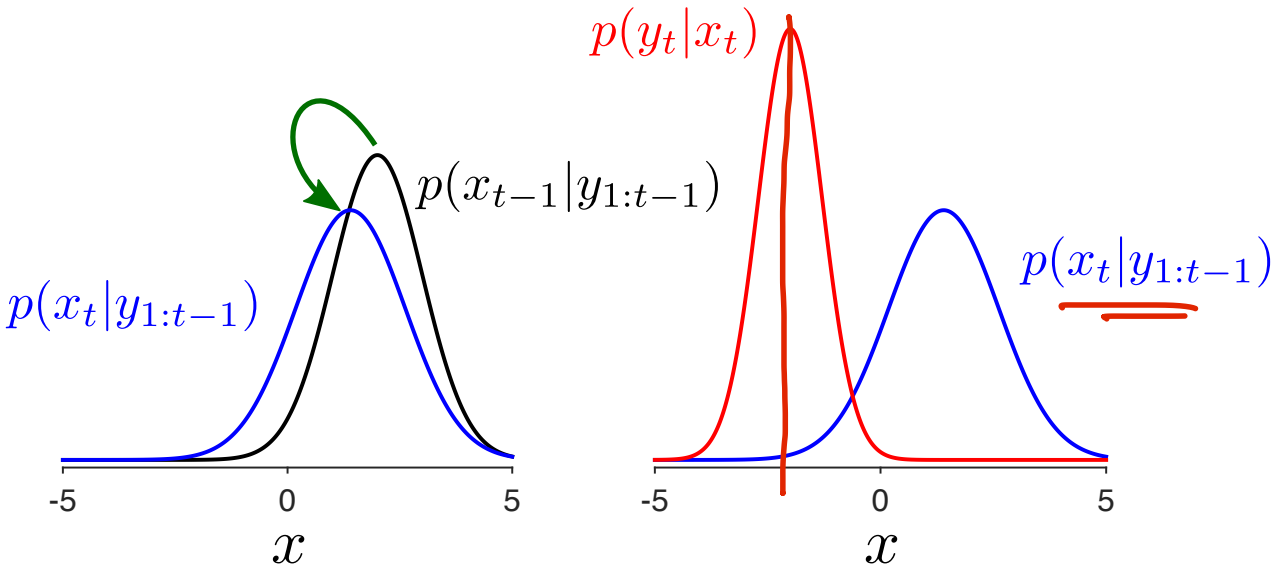
$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior

likelihood

Bayes' Rule

Inference: Kalman Filter



$$p(x_{t-1}|y_{1:t-1})$$

diffuse via dynamics

sum for discrete hidden state

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

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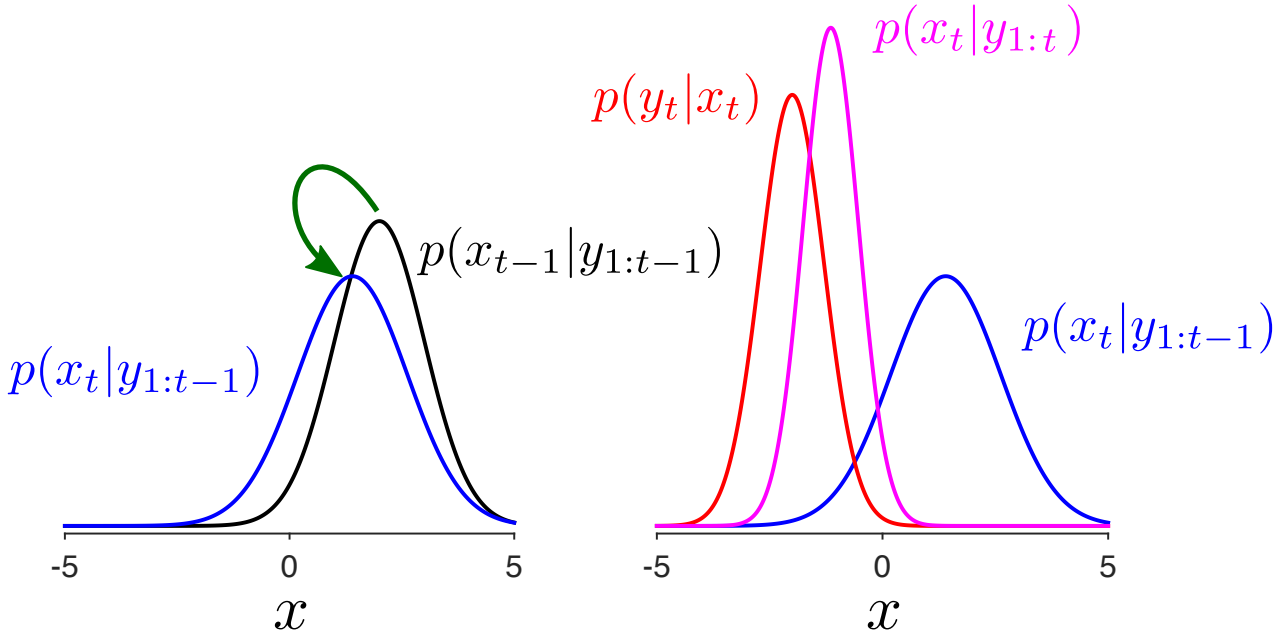
Bayes' Rule

prior likelihood

Inference: Kalman Filter



$$p(x_{t-1}|y_{1:t-1})$$

diffuse via dynamics

sum for discrete hidden state

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

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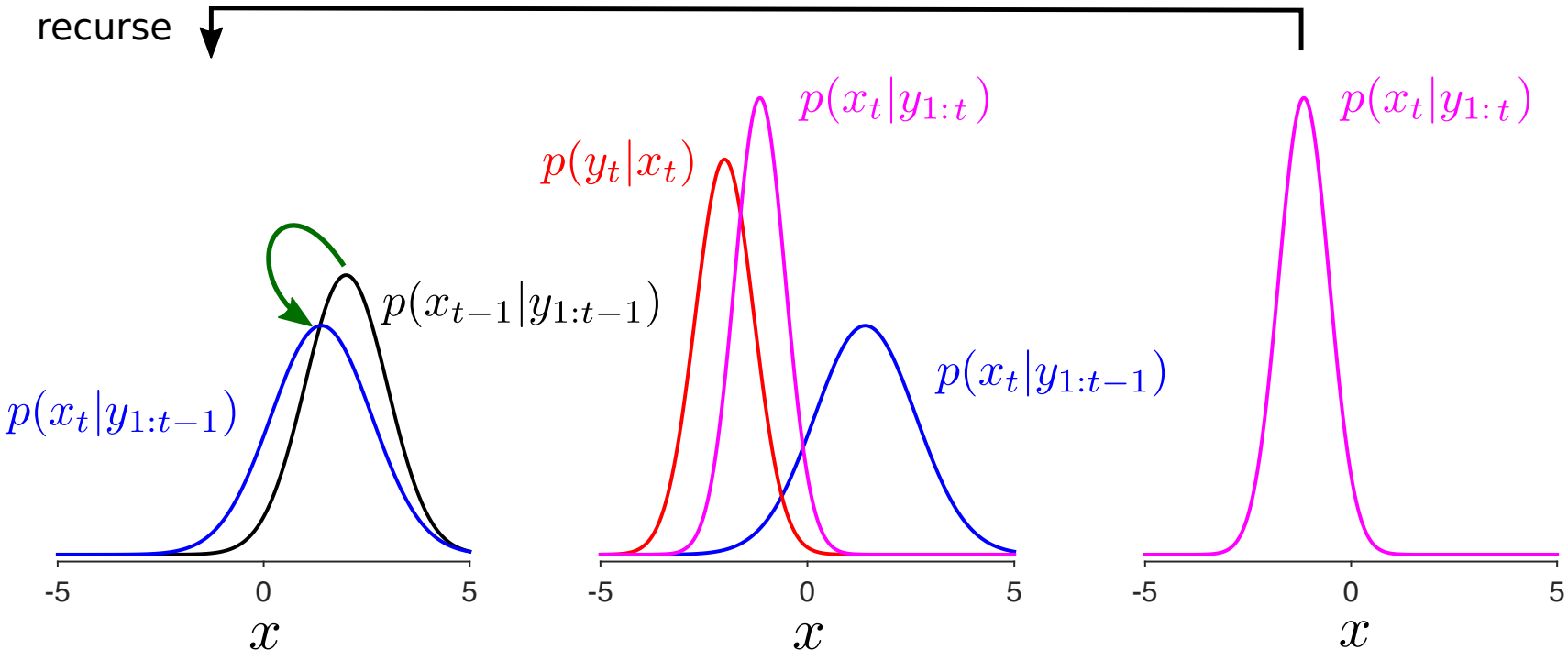
Bayes' Rule

prior

likelihood

Inference: Kalman Filter



$$p(x_{t-1}|y_{1:t-1})$$

diffuse via dynamics

sum for discrete hidden state

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

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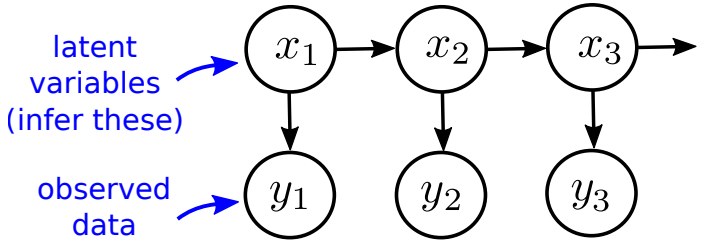
Bayes' Rule

prior likelihood

Inference: Derivation of General Filtering Equations

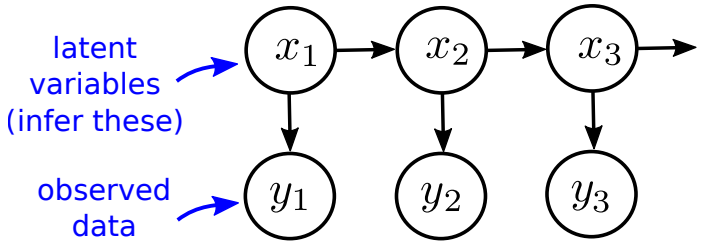
Model



$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

Inference: Derivation of General Filtering Equations

Model



+

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

Rules of probability

product rule

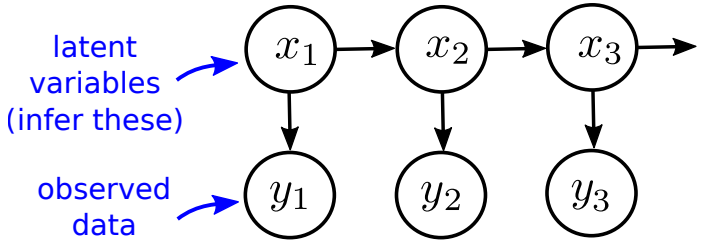
$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

sum rule

$$p(A|C) = \sum_B p(A,B|C)$$

Inference: Derivation of General Filtering Equations

Model



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+

Rules of probability

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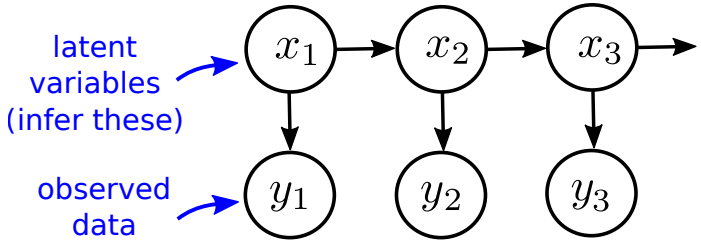
Inference

$$= \ ?$$

Inference: Derivation of General Filtering Equations

Model



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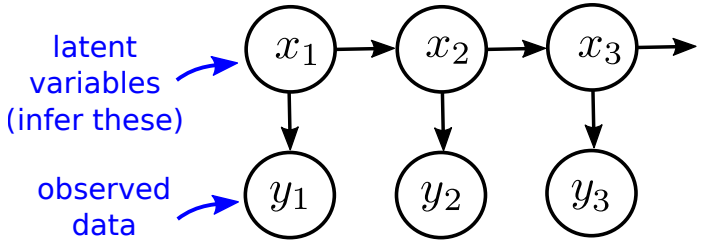
Inference

$$p(x_t|y_{1:t}) = ?$$

Inference: Derivation of General Filtering Equations

Model



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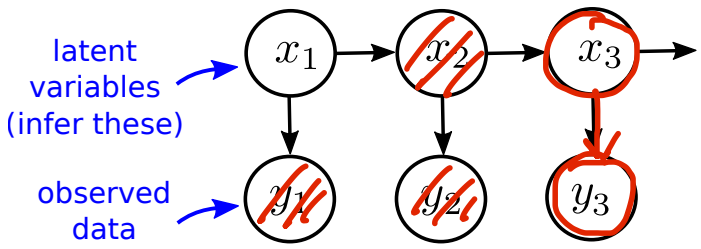
Inference

$$p(x_t|y_{1:t}) = p(x_t|y_t, y_{1:t-1})$$

Inference: Derivation of General Filtering Equations

Model



Rules of probability

+

product rule

$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

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Inference

= ?

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

$$\begin{aligned} p(x_t|y_{1:t}) &= p(x_t|y_t, y_{1:t-1}) \\ &= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t, y_{1:t-1})\, p(x_t|y_{1:t-1}) \end{aligned}$$

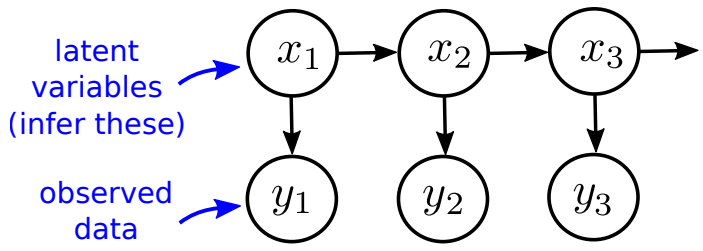
product rule

$A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

$= p(y_t|x_t)$

Inference: Derivation of General Filtering Equations

Model



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+

Rules of probability

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Inference

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product rule

$A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

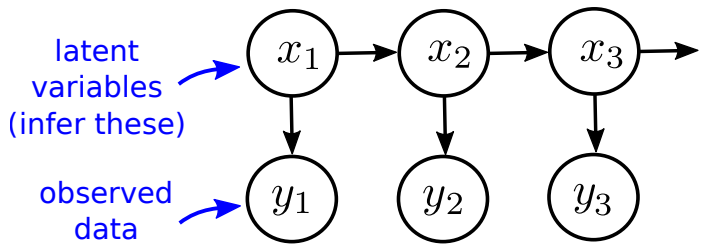
conditional independence from model

$y_t \perp y_{1:t-1}|x_t$

$$p(y_{1:T}|\theta) = \prod_{t=1}^{T} p(y_t|y_{1:t-1},\theta)$$

Inference: Derivation of General Filtering Equations

Model



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Rules of probability

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Inference

= ?

$$\begin{aligned}
p(x_t|y_{1:t}) &= p(x_t|y_t, y_{1:t-1}) \\
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\end{aligned}$$

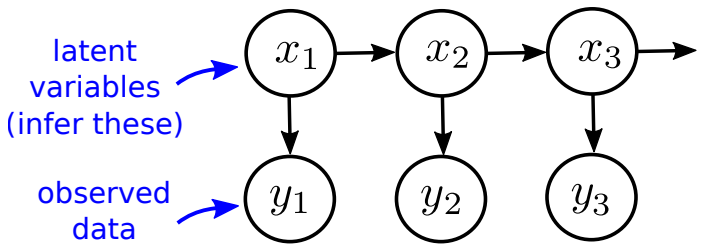
product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

Inference: Derivation of General Filtering Equations

Model



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Rules of probability

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Inference

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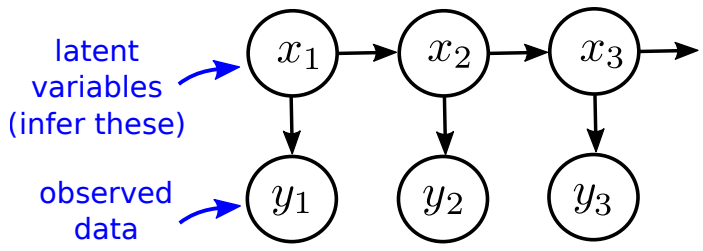
$p(x_t|y_{1:t-1})$

Inference: Derivation of General Filtering Equations

Model



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Rules of probability

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sum rule

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Inference

$= ?$

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\end{aligned}$$

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$A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

conditional independence from model
$y_t \perp y_{1:t-1}|x_t$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

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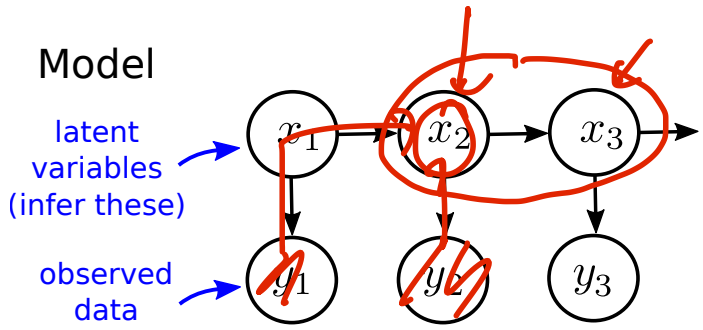
sum rule
$A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

Inference: Derivation of General Filtering Equations

Model



+

Rules of probability

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Inference

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product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

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constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$\begin{aligned}
p(x_t|y_{1:t-1}) &= \int p(x_t, x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1} \\
&= \int p(x_t|x_{t-1}, y_{1:t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}
\end{aligned}$$

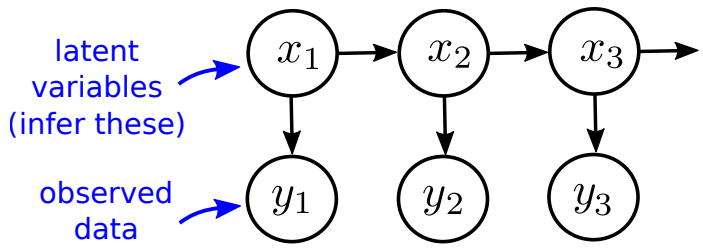
sum rule $A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

product rule

$= p(x_t|x_{t-1})$

Inference: Derivation of General Filtering Equations

Model



+

Rules of probability

product rule

$$p(A|B,C) = \frac{1}{p(B|C)}\, p(B|A,C)p(A|C)$$

sum rule

$$p(A|C) = \sum_B p(A,B|C)$$

Inference

= ?

$$p(y_{1:T}, x_{1:T}) = \prod_{t=1}^{T} p(x_t|x_{t-1})p(y_t|x_t)$$

$$\begin{aligned}
p(x_t|y_{1:t}) &= p(x_t|y_t, y_{1:t-1}) \\
&= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t, y_{1:t-1})p(x_t|y_{1:t-1}) \\
&= \frac{1}{p(y_t|y_{1:t-1})}\, p(y_t|x_t)p(x_t|y_{1:t-1}) \\
&\propto p(y_t|x_t)p(x_t|y_{1:t-1})
\end{aligned}$$

product rule $A = x_t \;\; B = y_t \;\; C = y_{1:t-1}$

conditional independence from model $y_t \perp y_{1:t-1}|x_t$

constant of proportionality $p(y_t|y_{1:t-1})$ (see learning)

$$\begin{aligned}
p(x_t|y_{1:t-1}) &= \int p(x_t, x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1} \\
&= \int p(x_t|x_{t-1}, y_{1:t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1} \\
&= \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}
\end{aligned}$$

sum rule $A = x_t \;\; B = x_{t-1} \;\; C = y_{1:t-1}$

product rule

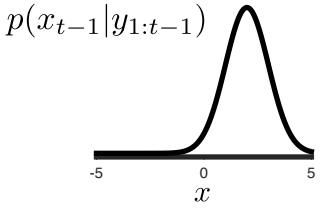
conditional independence from model

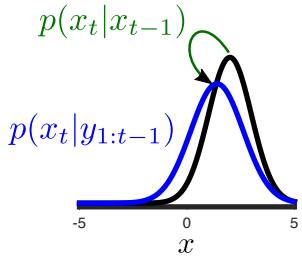
Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1})$$

diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

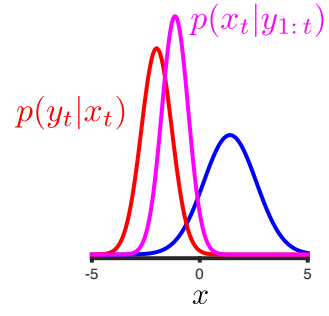




combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior likelihood

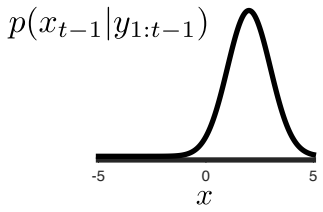


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

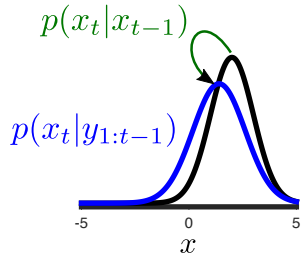
most recent data used in prediction

variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

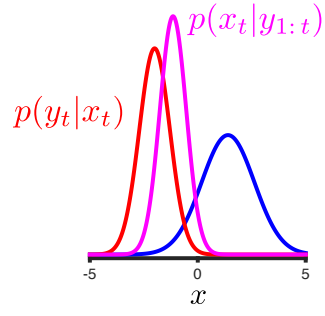


combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior

likelihood

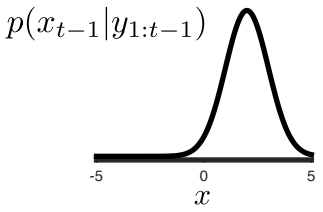


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

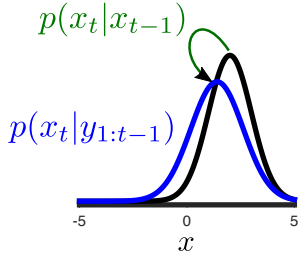
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1})$$

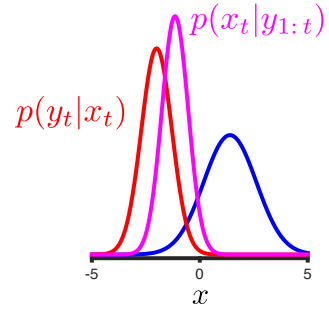


combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior

likelihood

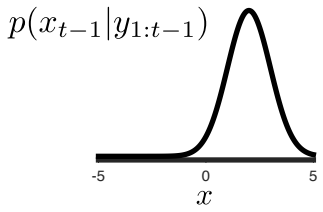


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

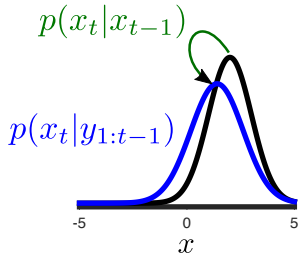
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

diffuses toward 0



$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1})$$

$$\mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

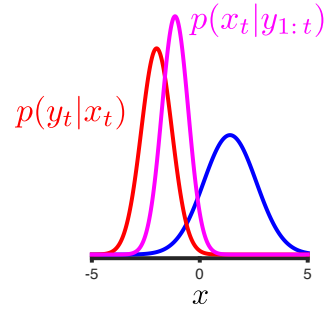
variance inflates

combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1}) p(y_t|x_t)$$

prior

likelihood

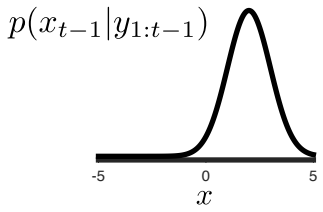


Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

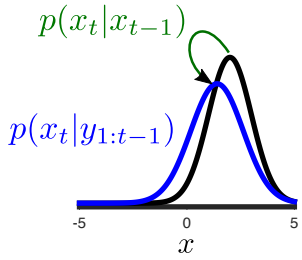
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}$$

diffuses toward 0



$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1}) \quad \mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

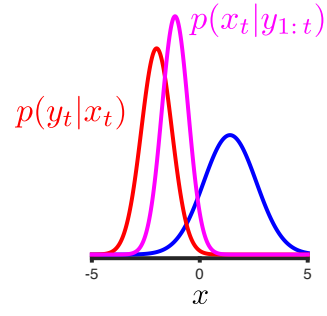
variance inflates

combine with likelihood

$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1}) p(y_t|x_t)$$

prior

likelihood



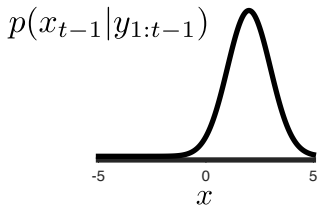
$$p(x_t|y_{1:t}) = \mathcal{G}(x_t; \mu_t^t, V_t^t)$$

Inference: Kalman Filter

$$p(x_{t-1}|y_{1:t-1}) = \mathcal{G}(x_{t-1}; \mu_{t-1}^{t-1}, V_{t-1}^{t-1})$$

most recent data used in prediction

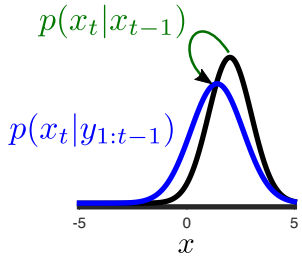
variable being predicted



diffuse via dynamics

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1})\mathrm{d}x_{t-1}$$

diffuses toward 0

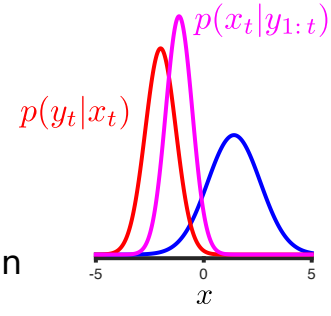


$$p(x_t|y_{1:t-1}) = \mathcal{G}(x_t; \mu_t^{t-1}, V_t^{t-1}) \quad \mu_t^{t-1} = A\mu_{t-1}^{t-1}$$

$$V_t^{t-1} = AV_{t-1}^{t-1}A^\top + Q$$

variance inflates

combine with likelihood



$$p(x_t|y_{1:t}) \propto p(x_t|y_{1:t-1})p(y_t|x_t)$$

prior likelihood

prediction correction

$$p(x_t|y_{1:t}) = \mathcal{G}(x_t; \mu_t^t, V_t^t)$$

$$\mu_t^t = \mu_t^{t-1} + K_t(y_t - C\mu_t^{t-1})$$

$$V_t^t = V_t^{t-1} - K_t C V_t^{t-1}$$

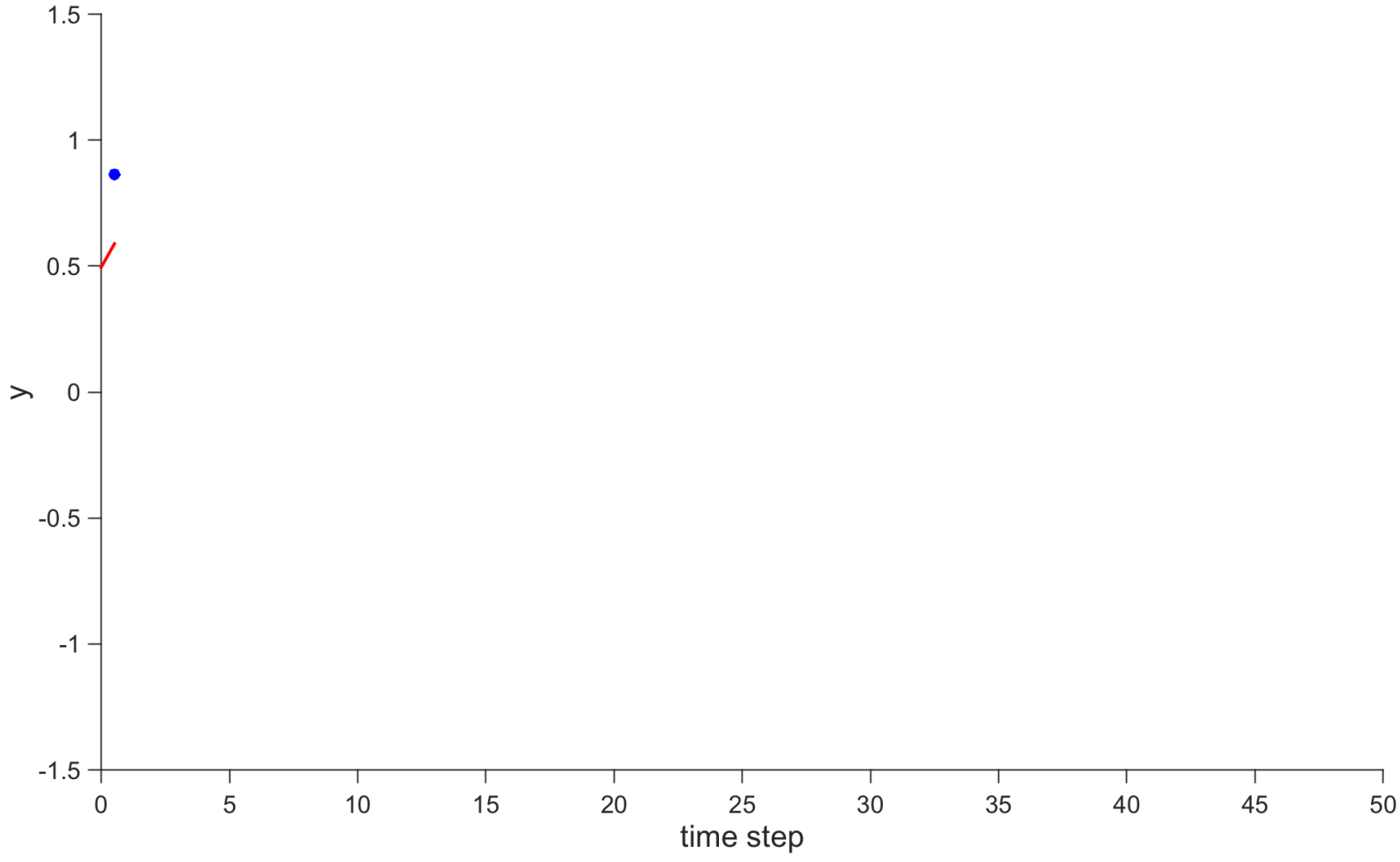
Kalman gain $K_t = V_t^{t-1}C^\top(CV_t^{t-1}C^\top + R)^{-1}$

Kalman Filter Demo

- data: $y_t = \sin(\omega t) + \sigma_y \epsilon_t$ where $\sigma_y^2 = 0.1$
- model: $x_t = \lambda x_{t-1} + \sigma\eta$ and $y_t = x_t + \sigma_y \eta'_t$
  where $\lambda = 0.99$ and $\sigma^2 = 1 - \lambda^2$
- demo shows how the Kalman filter processes the data to form estimates of the hidden state at each time point $p(x_t | y_{1:t})$

Kalman Filter Demo

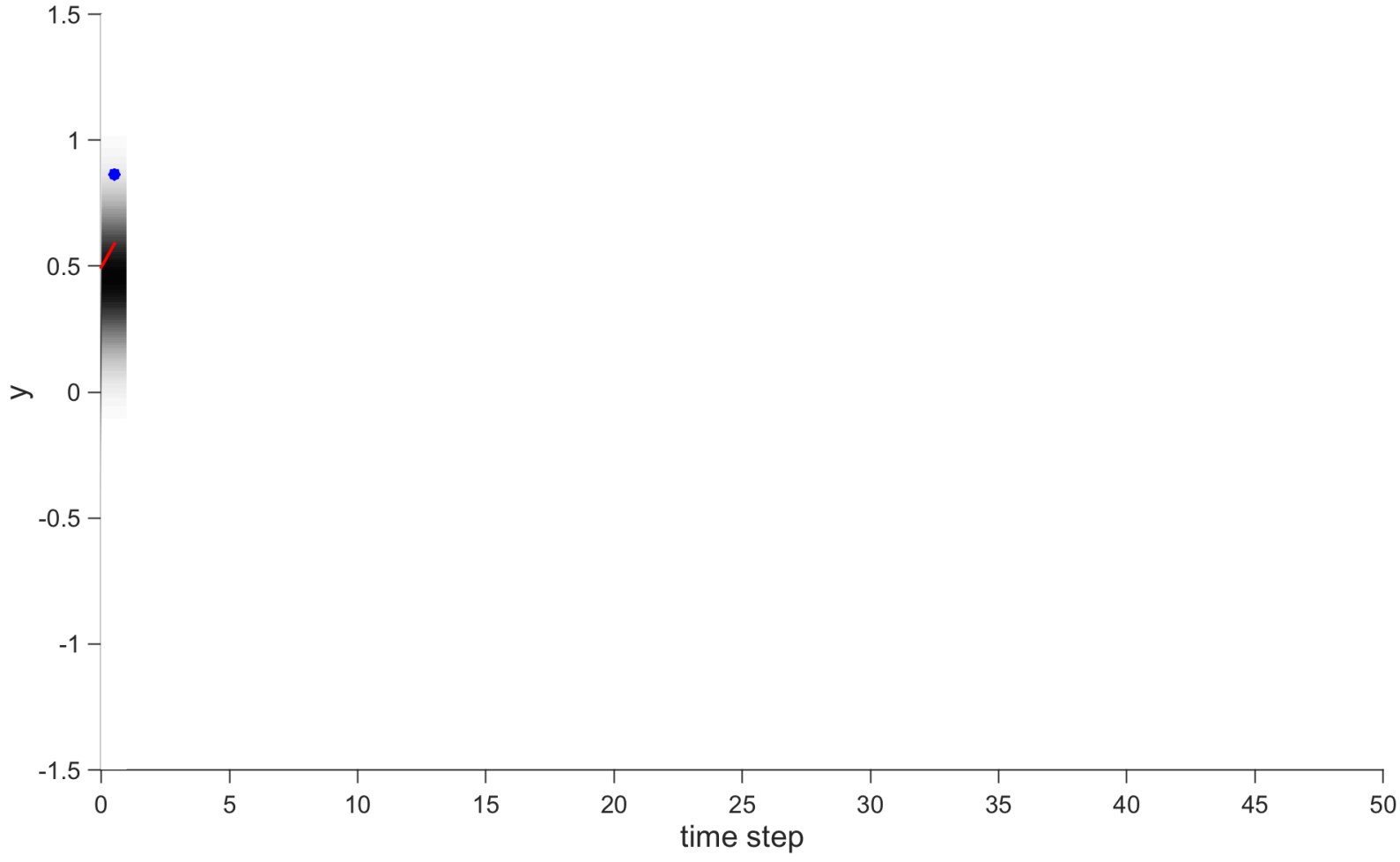
observed noisy data y_t, ground truth sinusoid



observe first data point y_1

Kalman Filter Demo

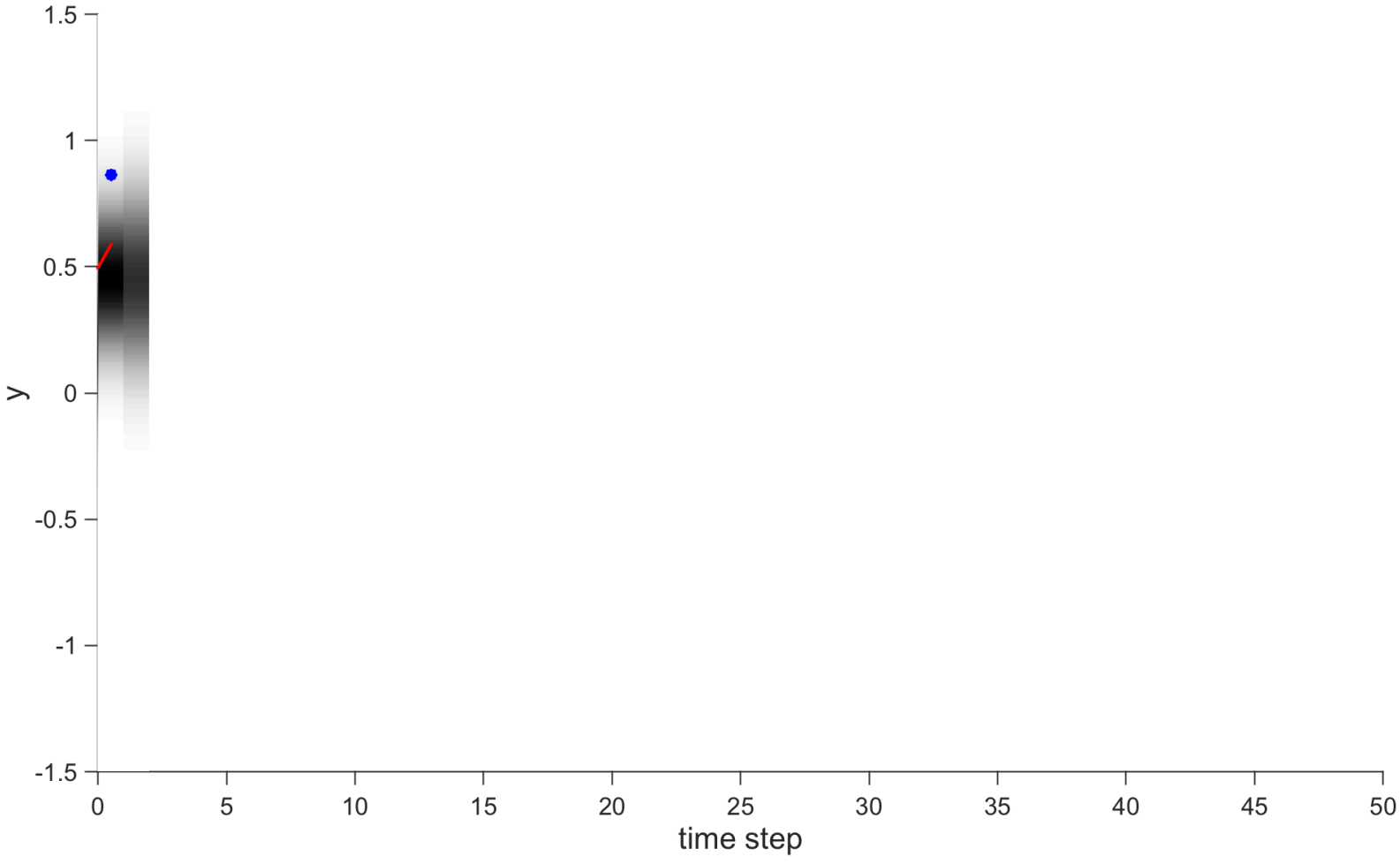
observed noisy data y_t, ground truth sinusoid



posterior over first latent variable $p(x_1|y_1)$

Kalman Filter Demo

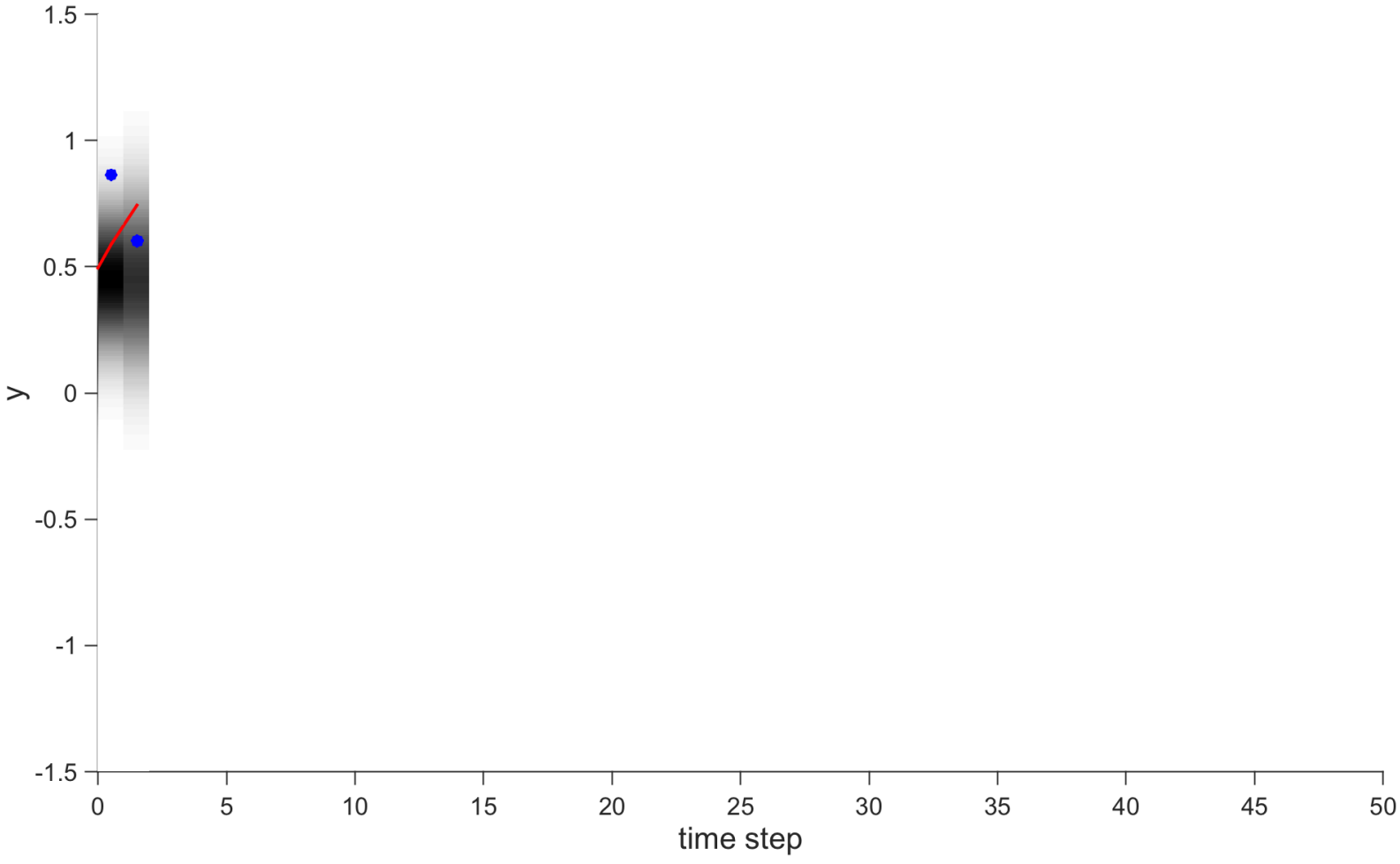
observed noisy data y_t, ground truth sinusoid



prediction for second latent variable $p(x_2|y_1)$

Kalman Filter Demo

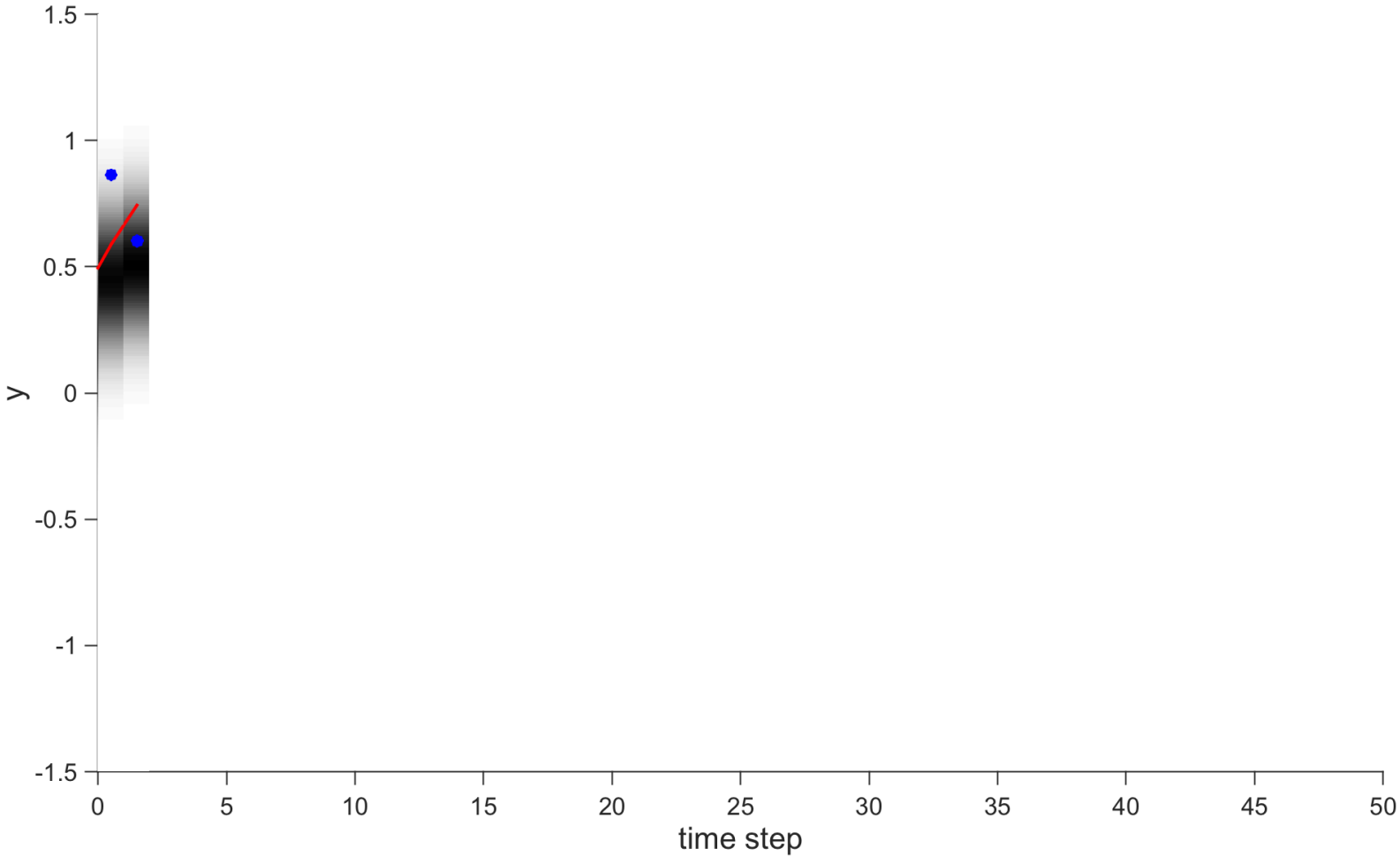
observed noisy data y_t, ground truth sinusoid



observe next data point y_2

Kalman Filter Demo

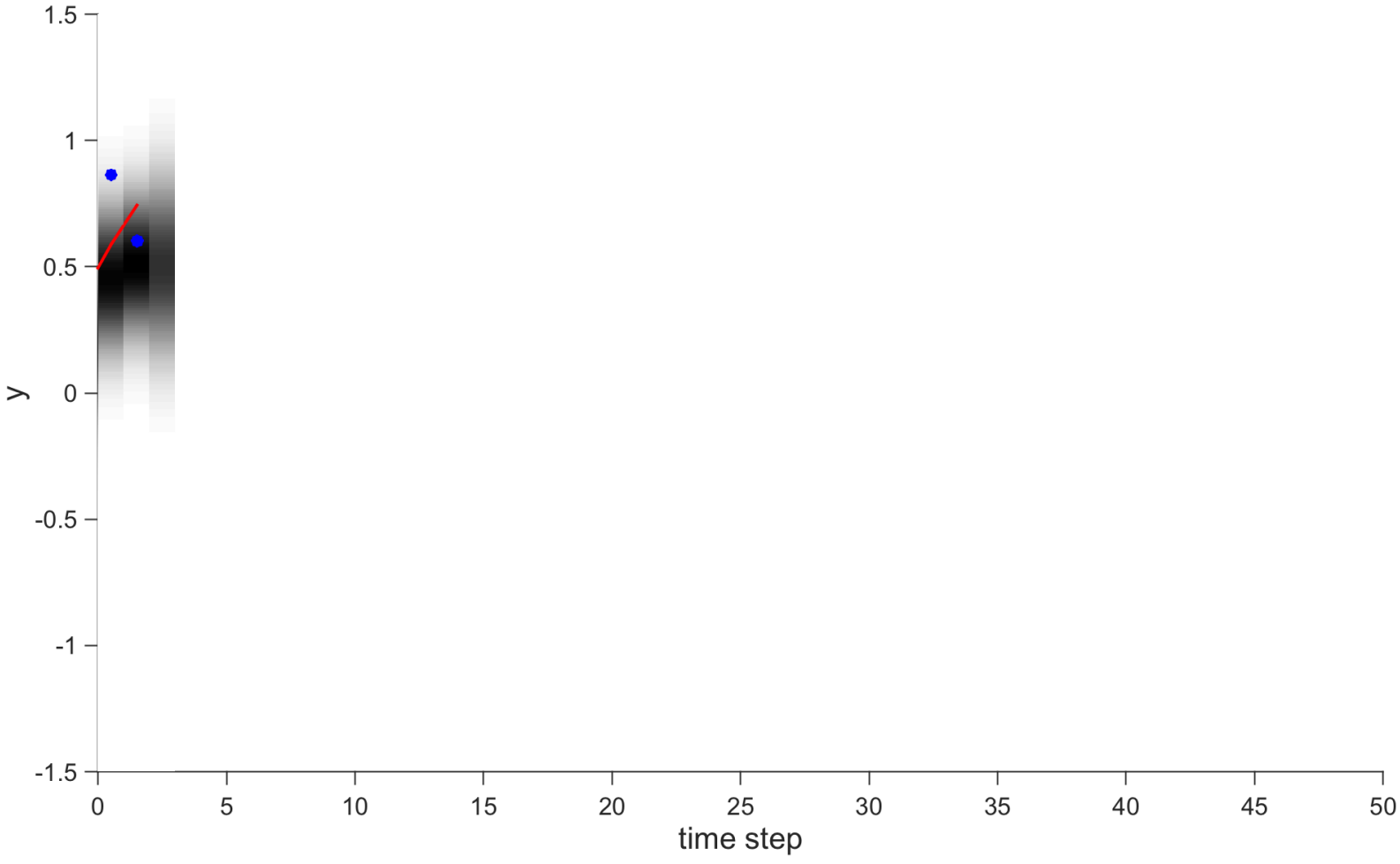
observed noisy data y_t, ground truth sinusoid



form posterior over second latent variable $p(x_2|y_1, y_2)$

Kalman Filter Demo

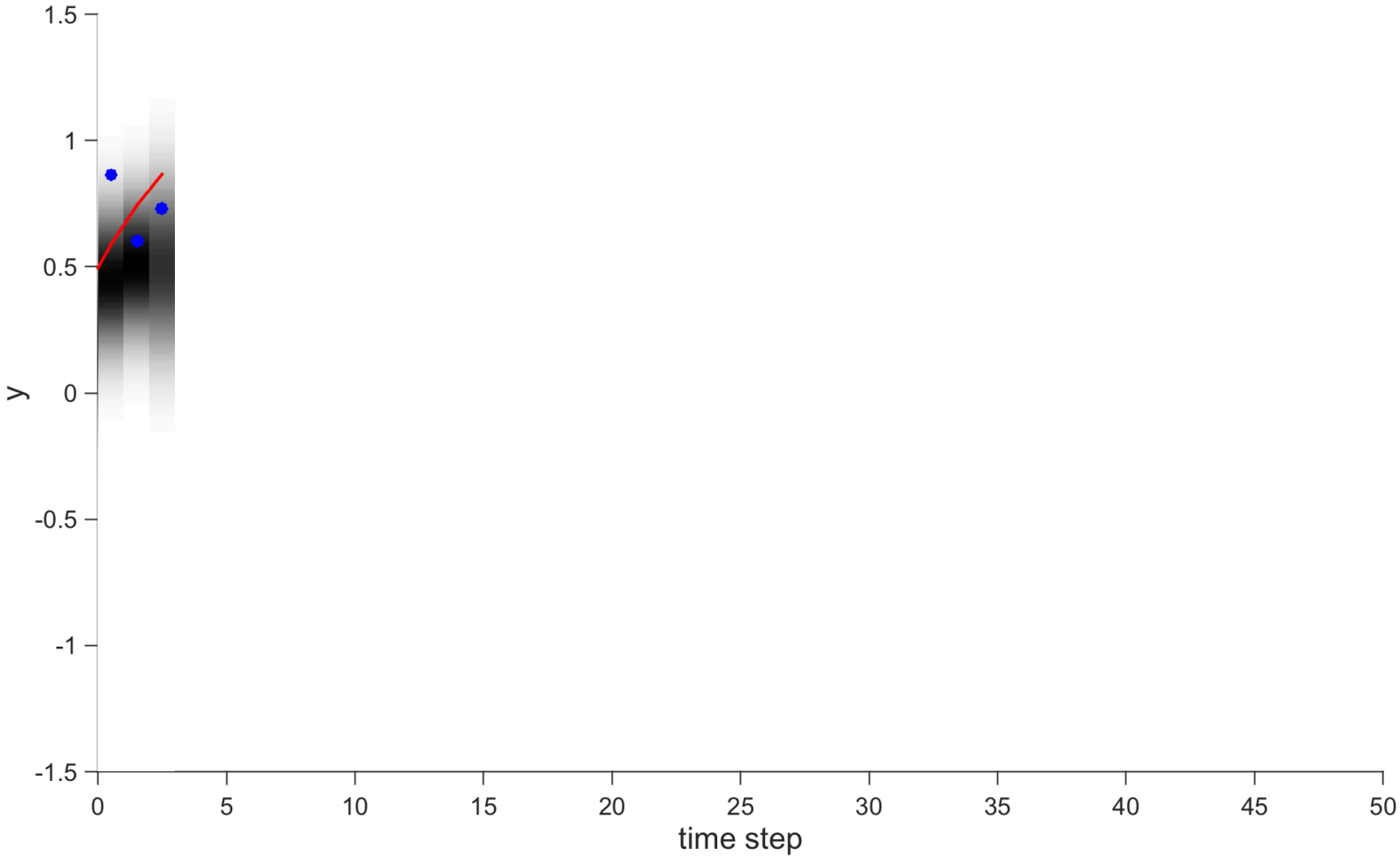
observed noisy data y_t, ground truth sinusoid



prediction for third latent variable $p(x_3|y_1, y_2)$

Kalman Filter Demo

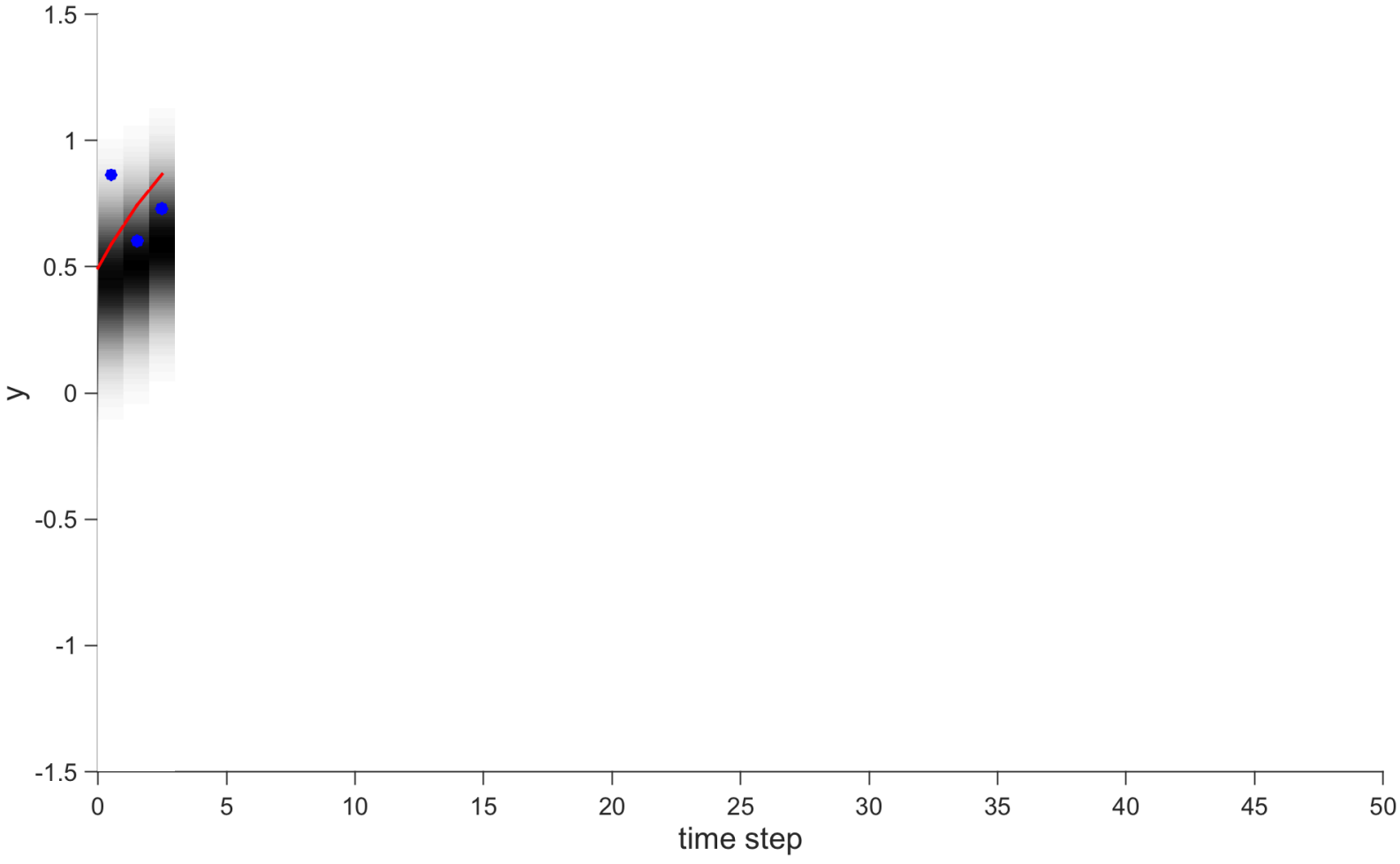
observed noisy data y_t, ground truth sinusoid



observe next data point y_3

Kalman Filter Demo

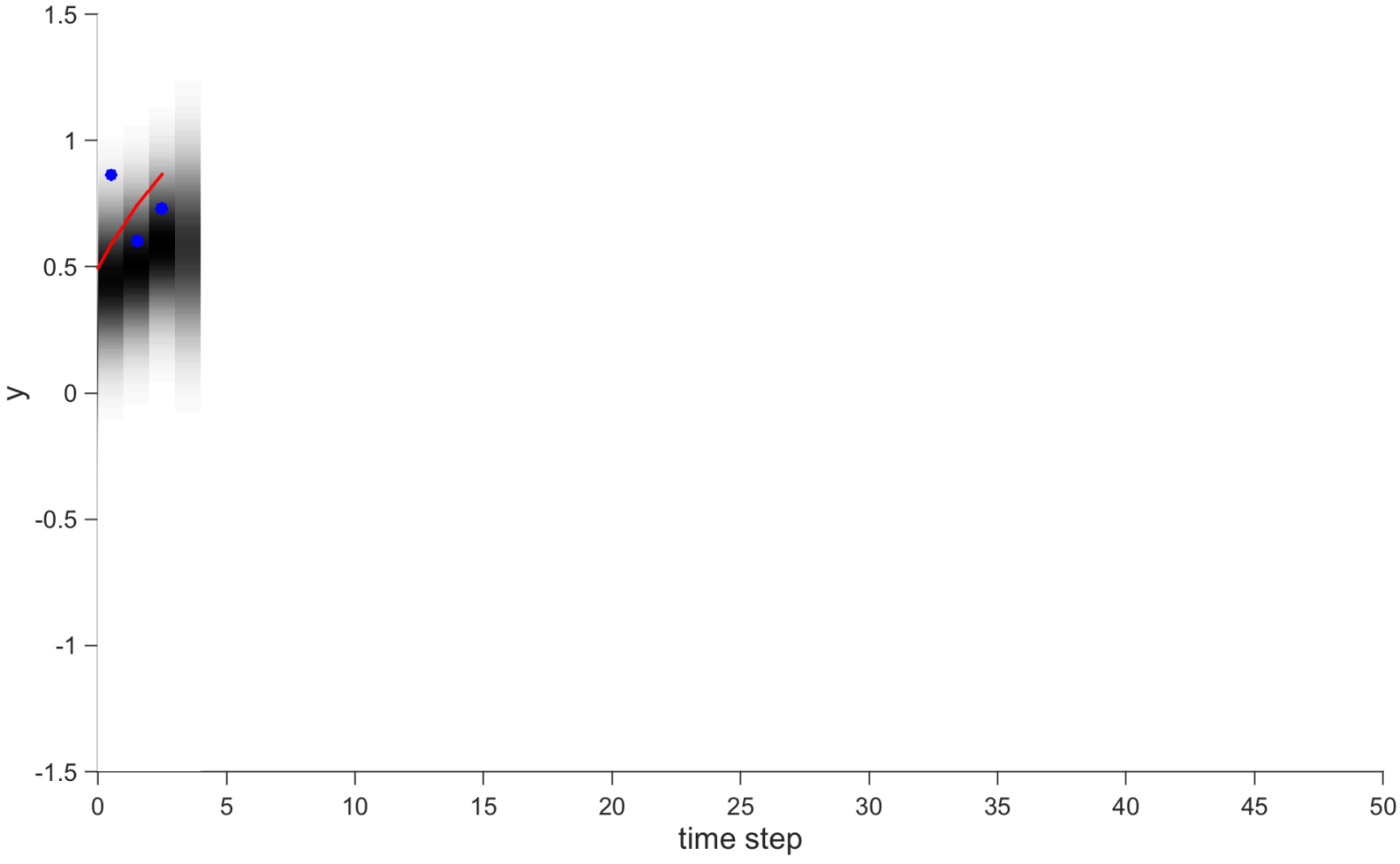
observed noisy data y_t, ground truth sinusoid



form posterior over third latent variable $p(x_3|y_1, y_2, y_3)$

Kalman Filter Demo

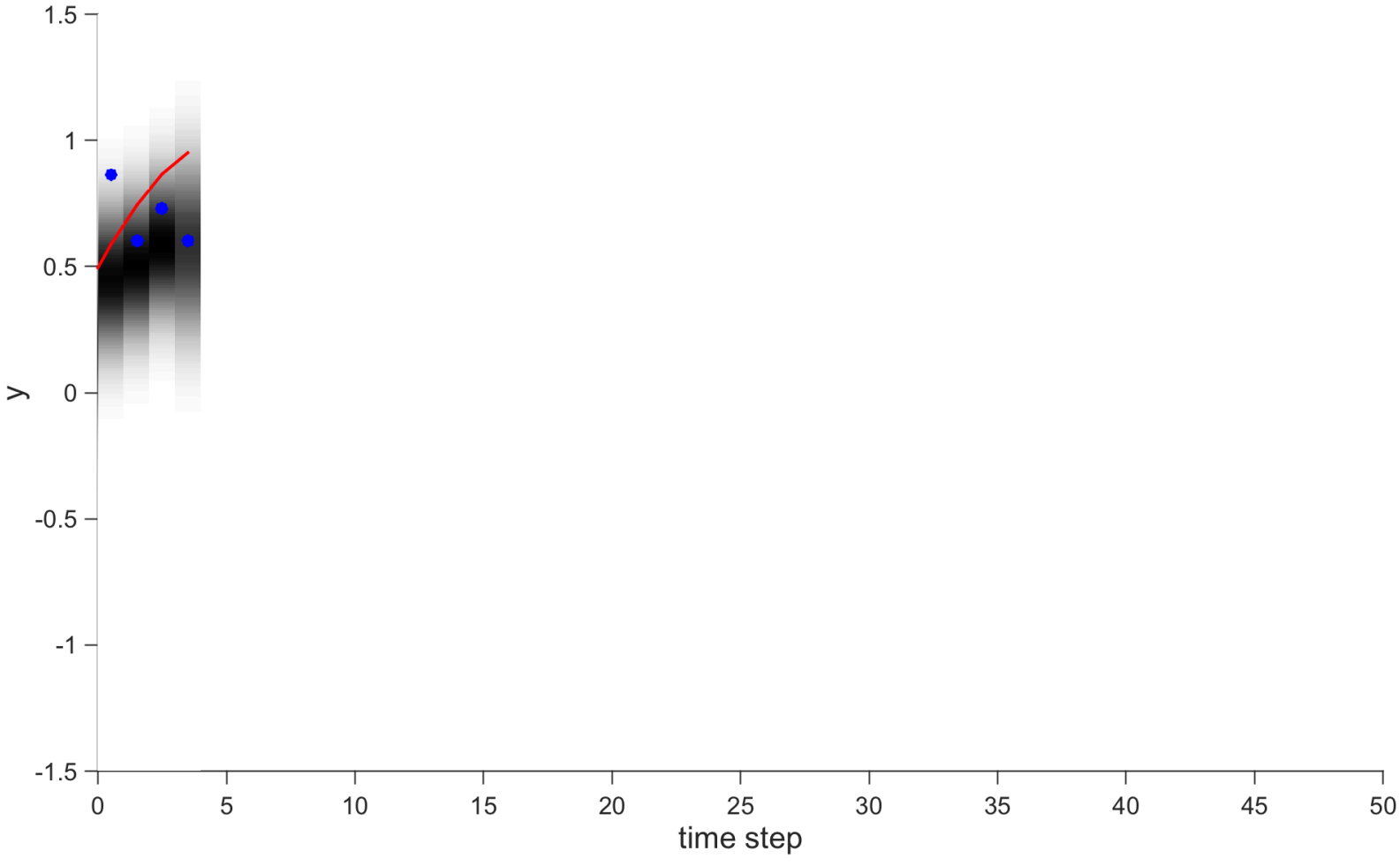
observed noisy data y_t, ground truth sinusoid



prediction for fourth latent variable $p(x_4|y_{1:3})$

Kalman Filter Demo

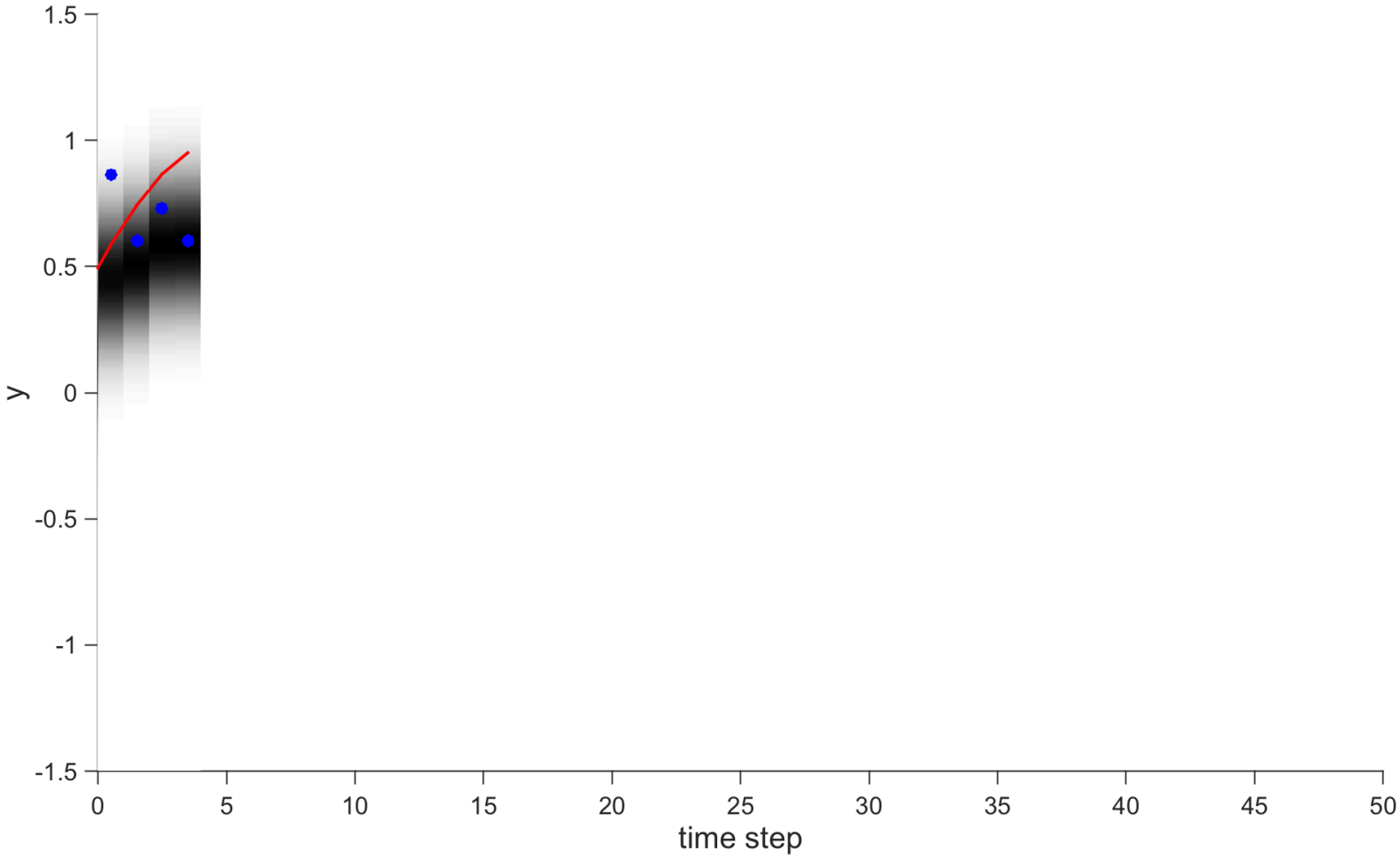
observed noisy data y_t, ground truth sinusoid



observe next data point y_4

Kalman Filter Demo

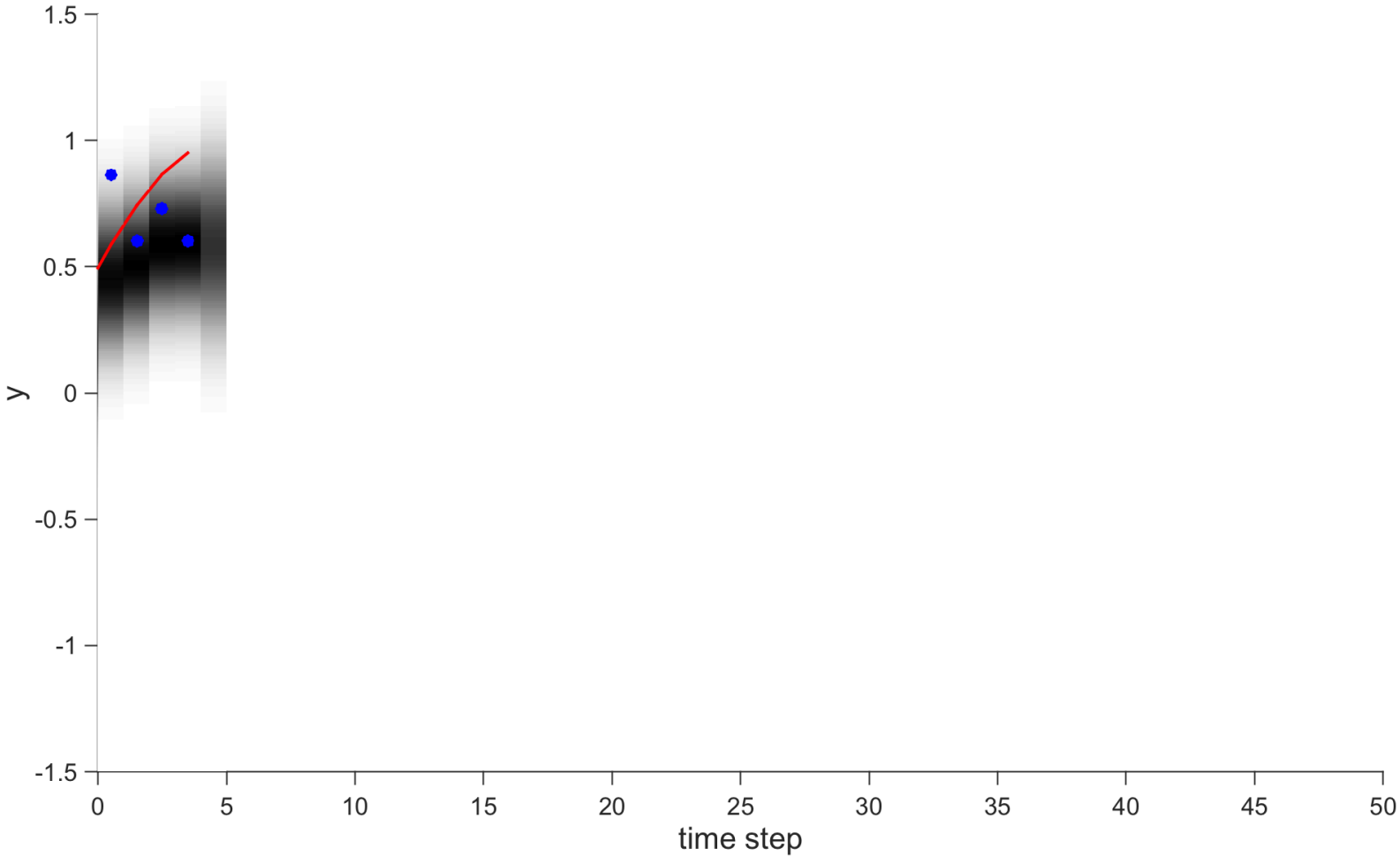
observed noisy data y_t, ground truth sinusoid



form posterior over fourth latent variable $p(x_4|y_{1:4})$

Kalman Filter Demo

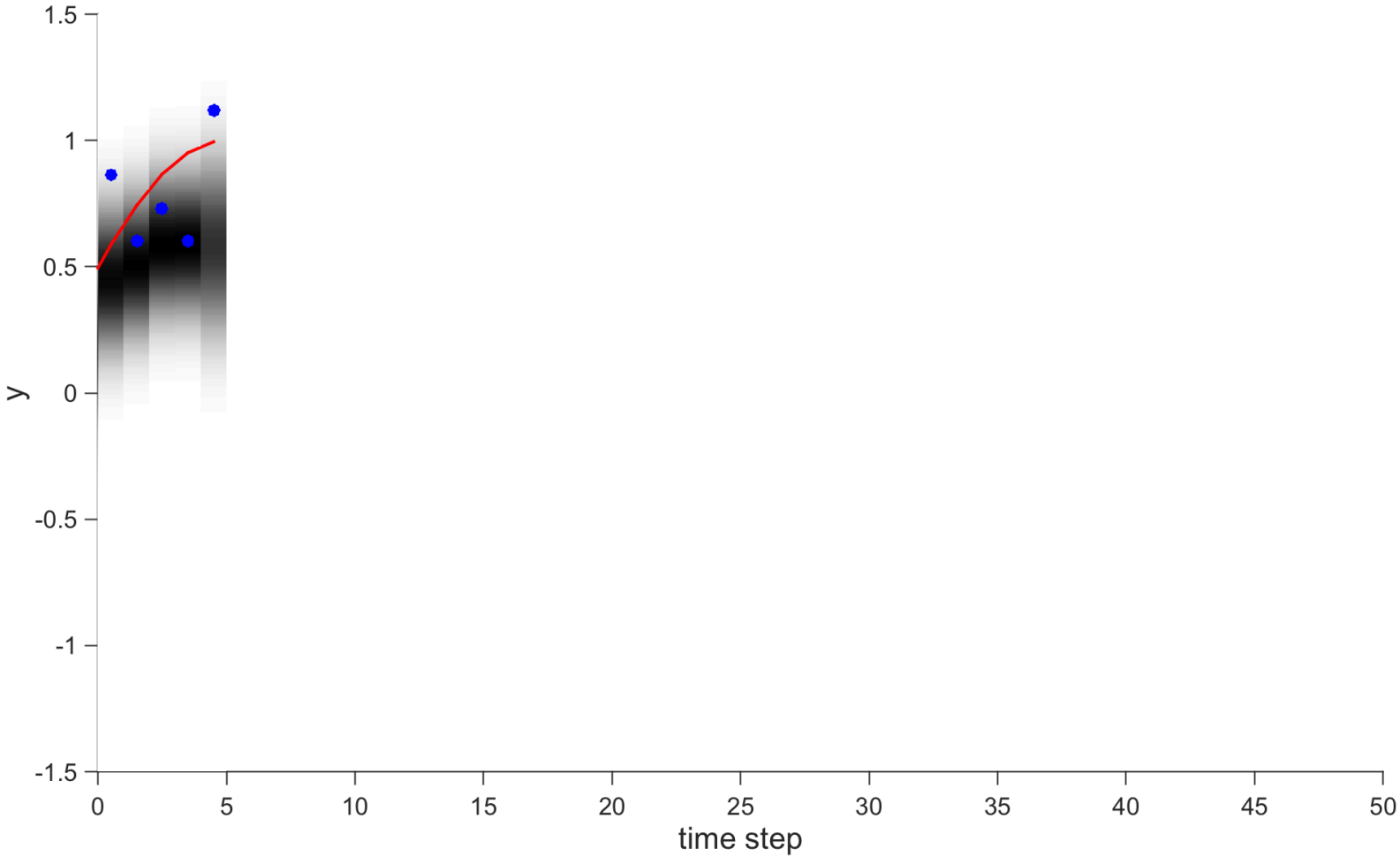
observed noisy data y_t, ground truth sinusoid



prediction for fifth latent variable $p(x_5|y_{1:4})$

Kalman Filter Demo

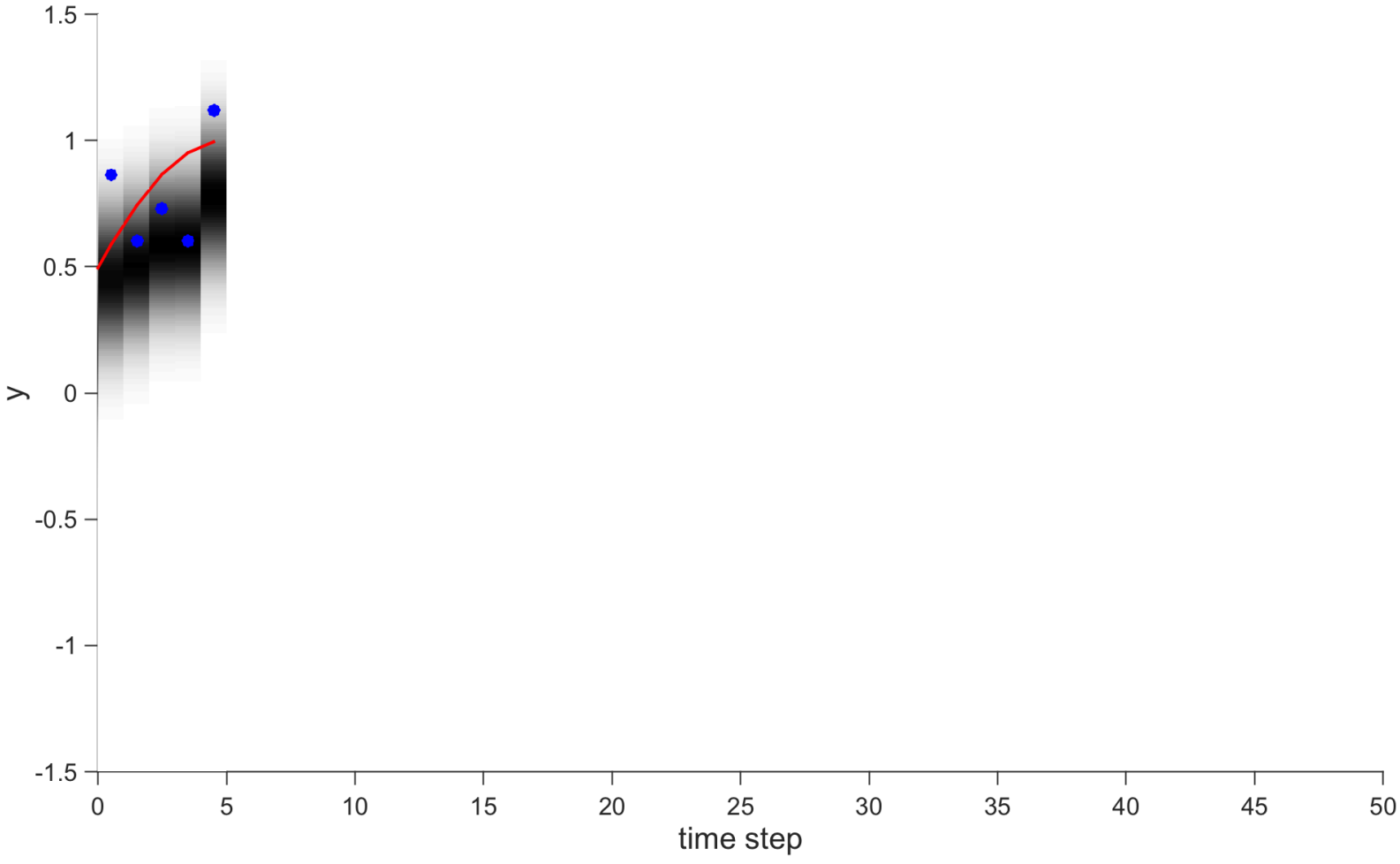
observed noisy data y_t, ground truth sinusoid



observe next data point y_5

Kalman Filter Demo

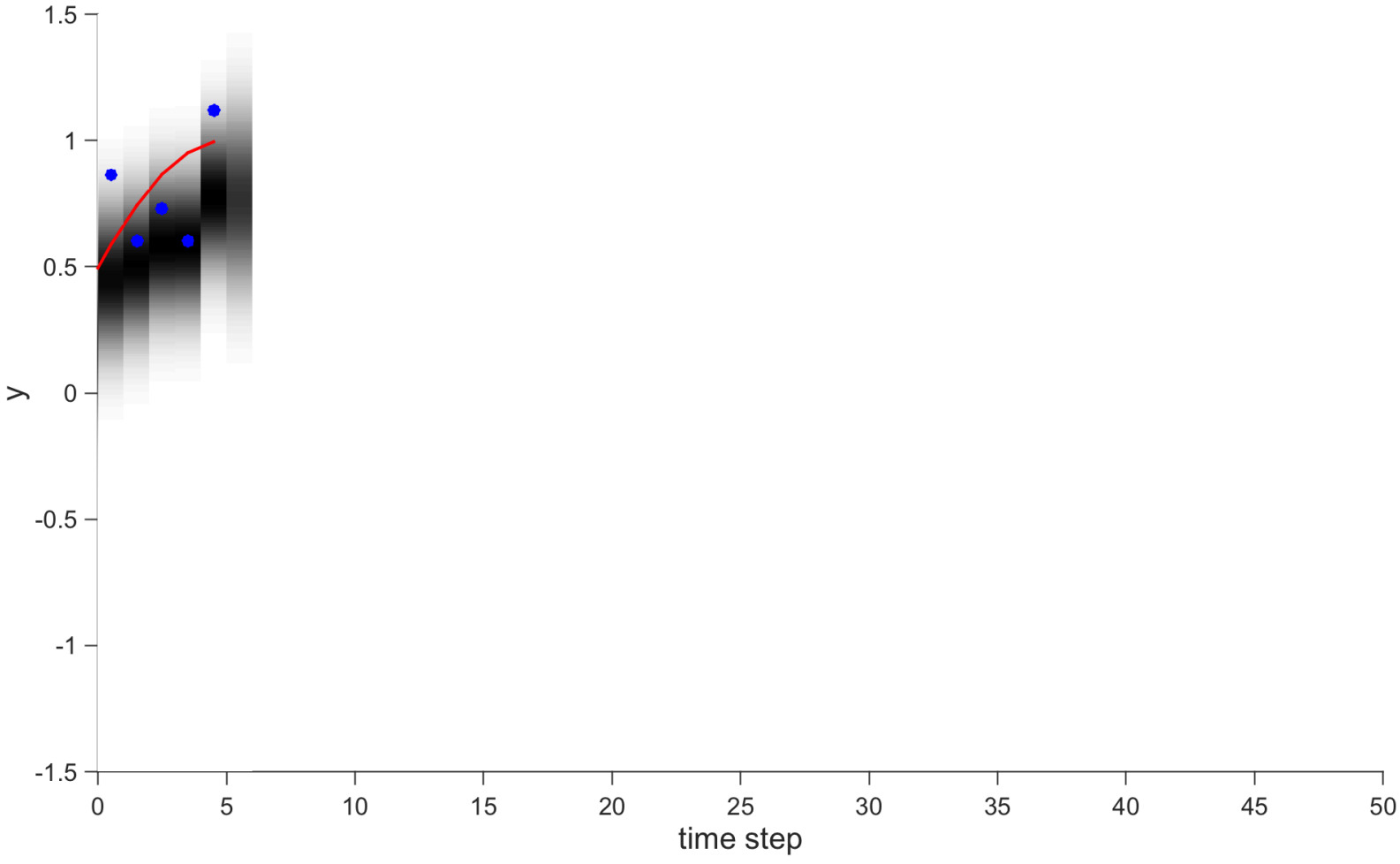
observed noisy data y_t, ground truth sinusoid



form posterior over fifth latent variable $p(x_5 | y_{1:5})$

Kalman Filter Demo

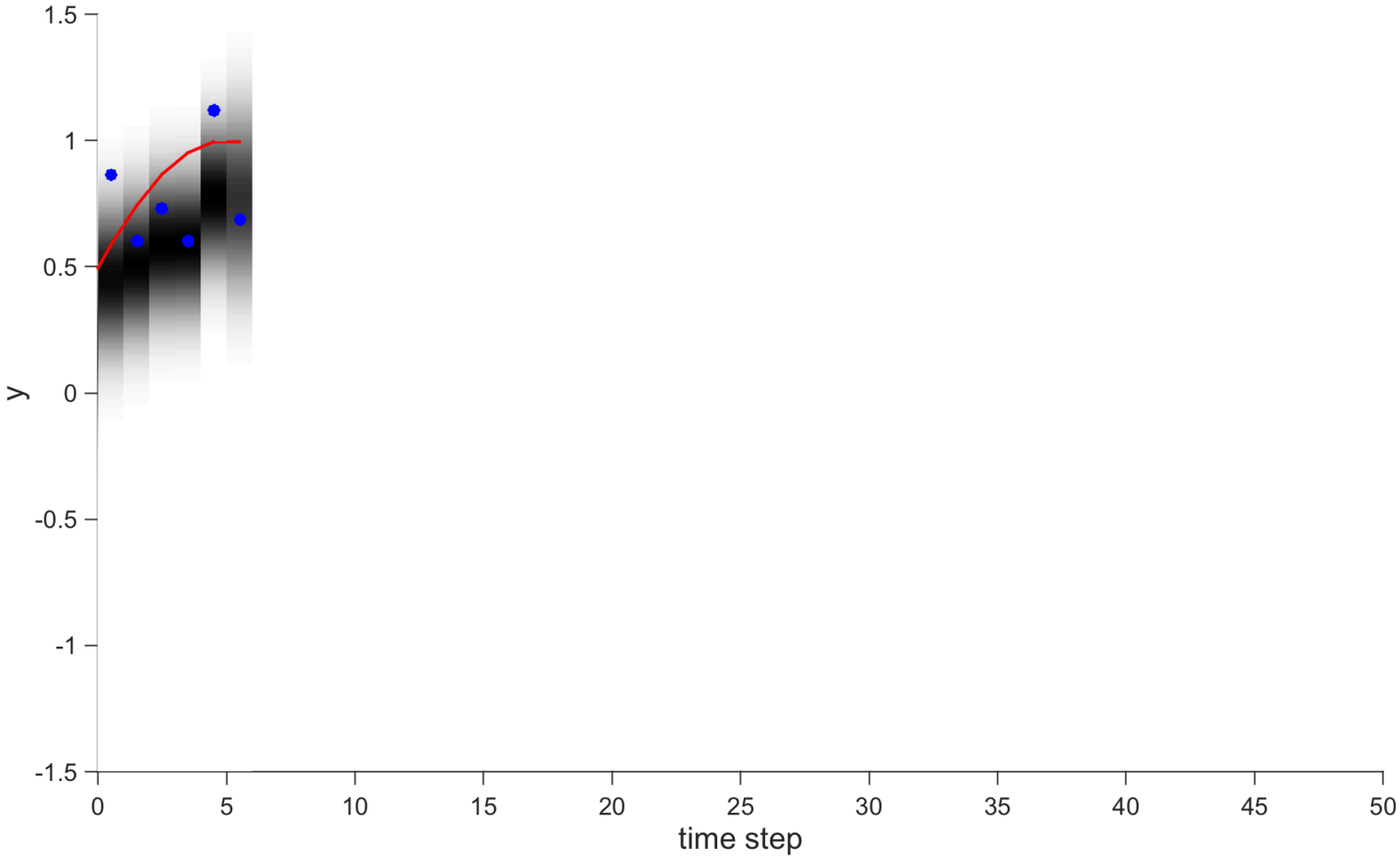
observed noisy data y_t, ground truth sinusoid



prediction for sixth latent variable $p(x_6|y_{1:5})$

Kalman Filter Demo

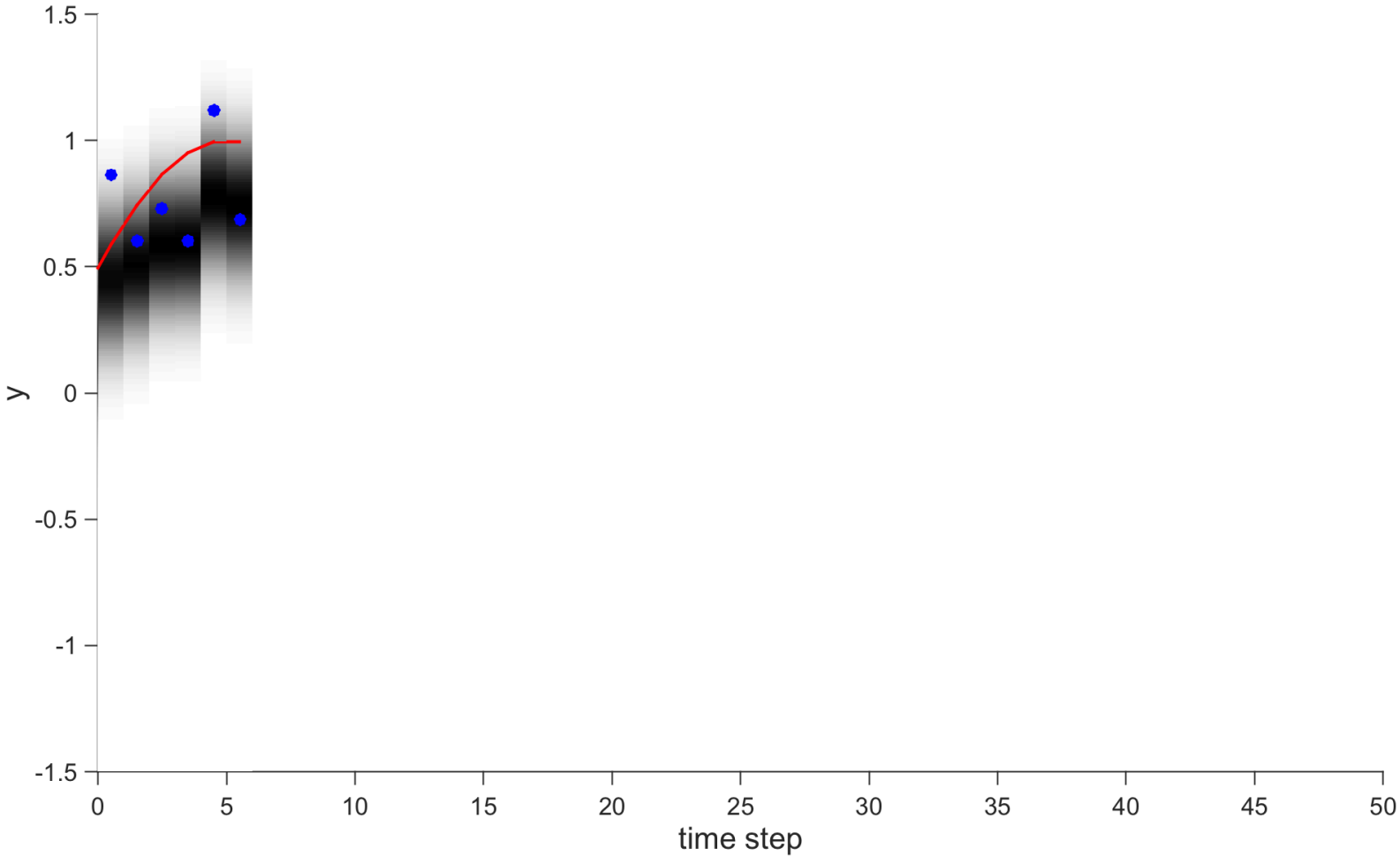
observed noisy data y_t, ground truth sinusoid



observe next data point y_6

Kalman Filter Demo

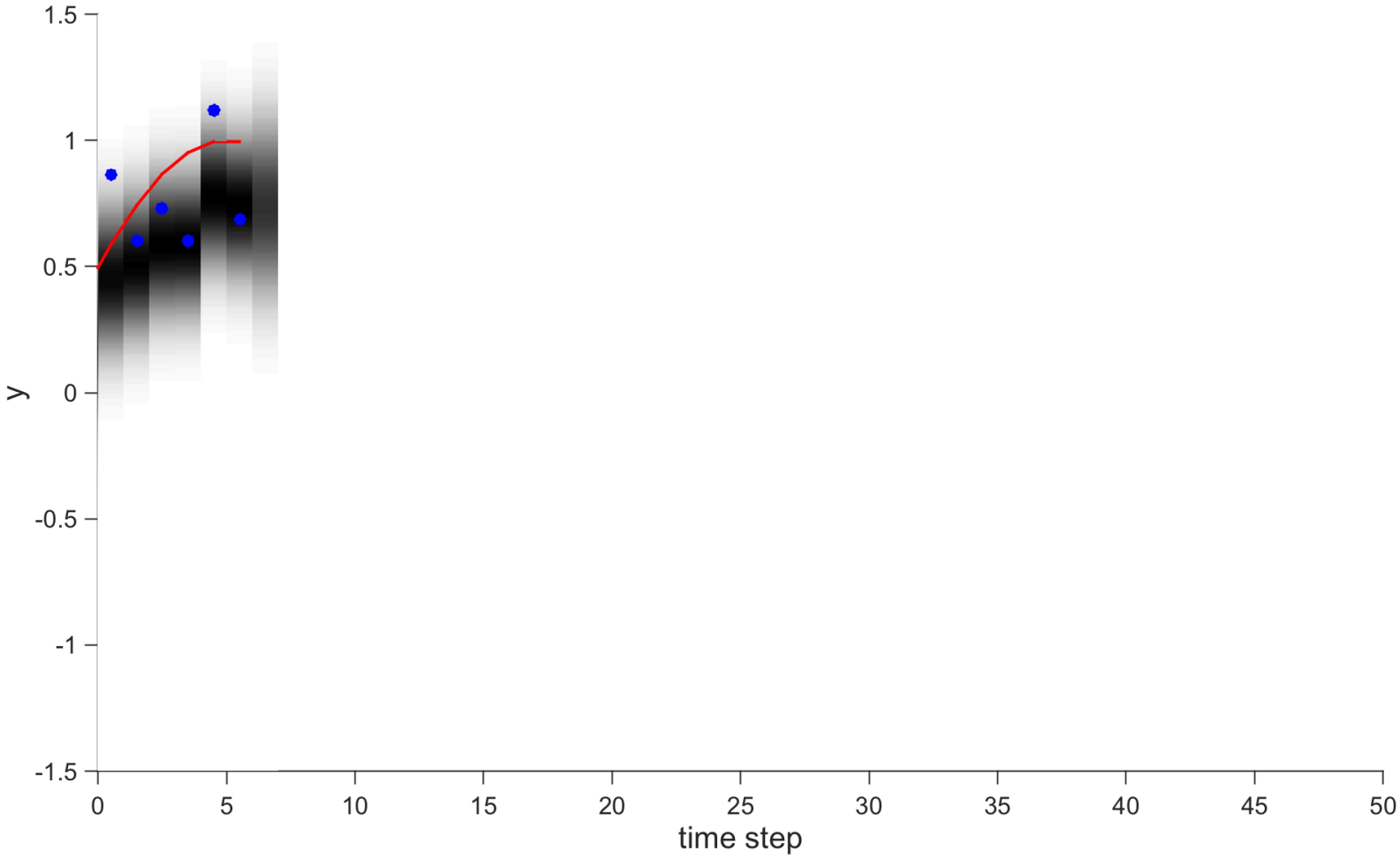
observed noisy data y_t, ground truth sinusoid



form posterior over sixth latent variable $p(x_6|y_{1:6})$

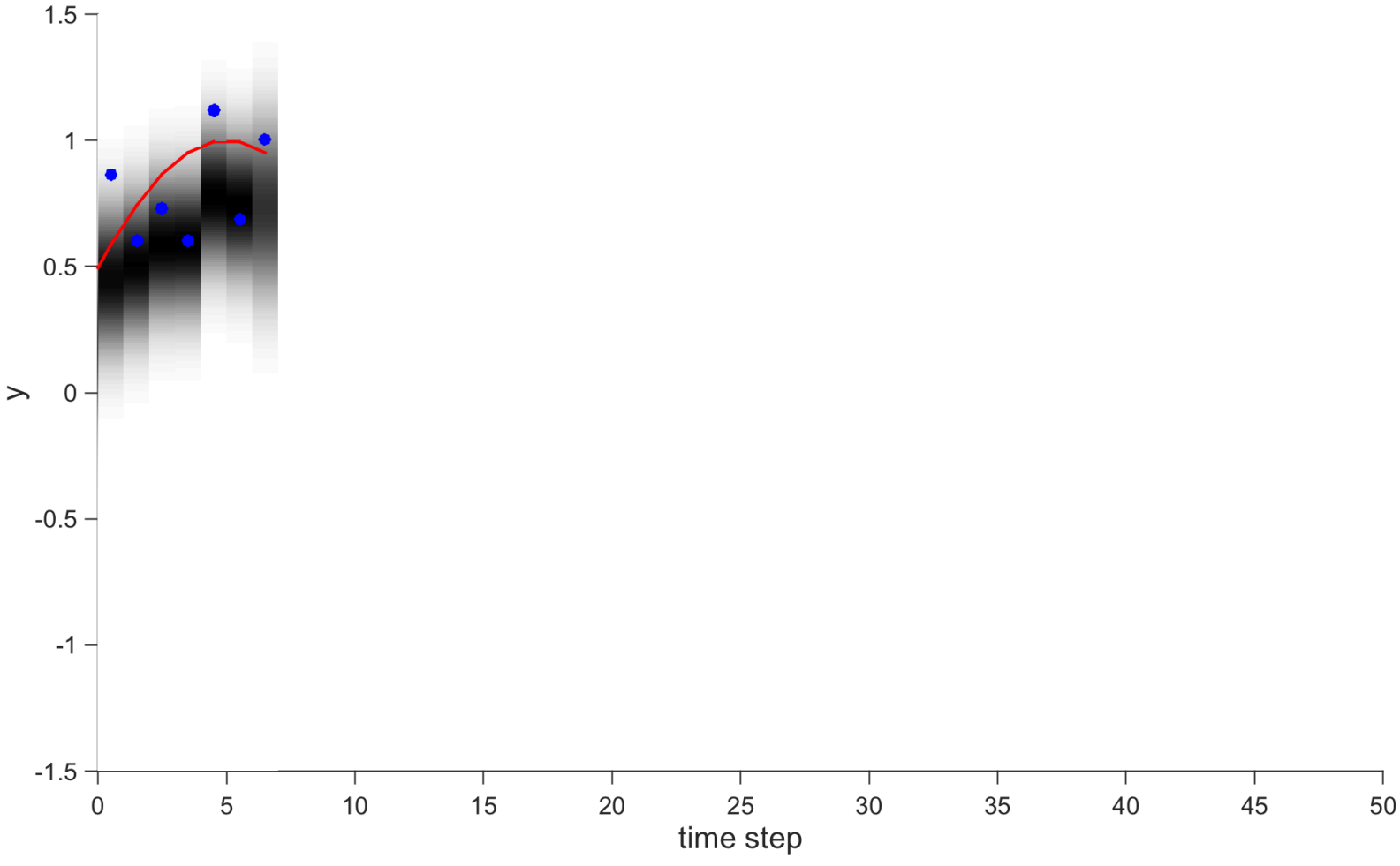
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



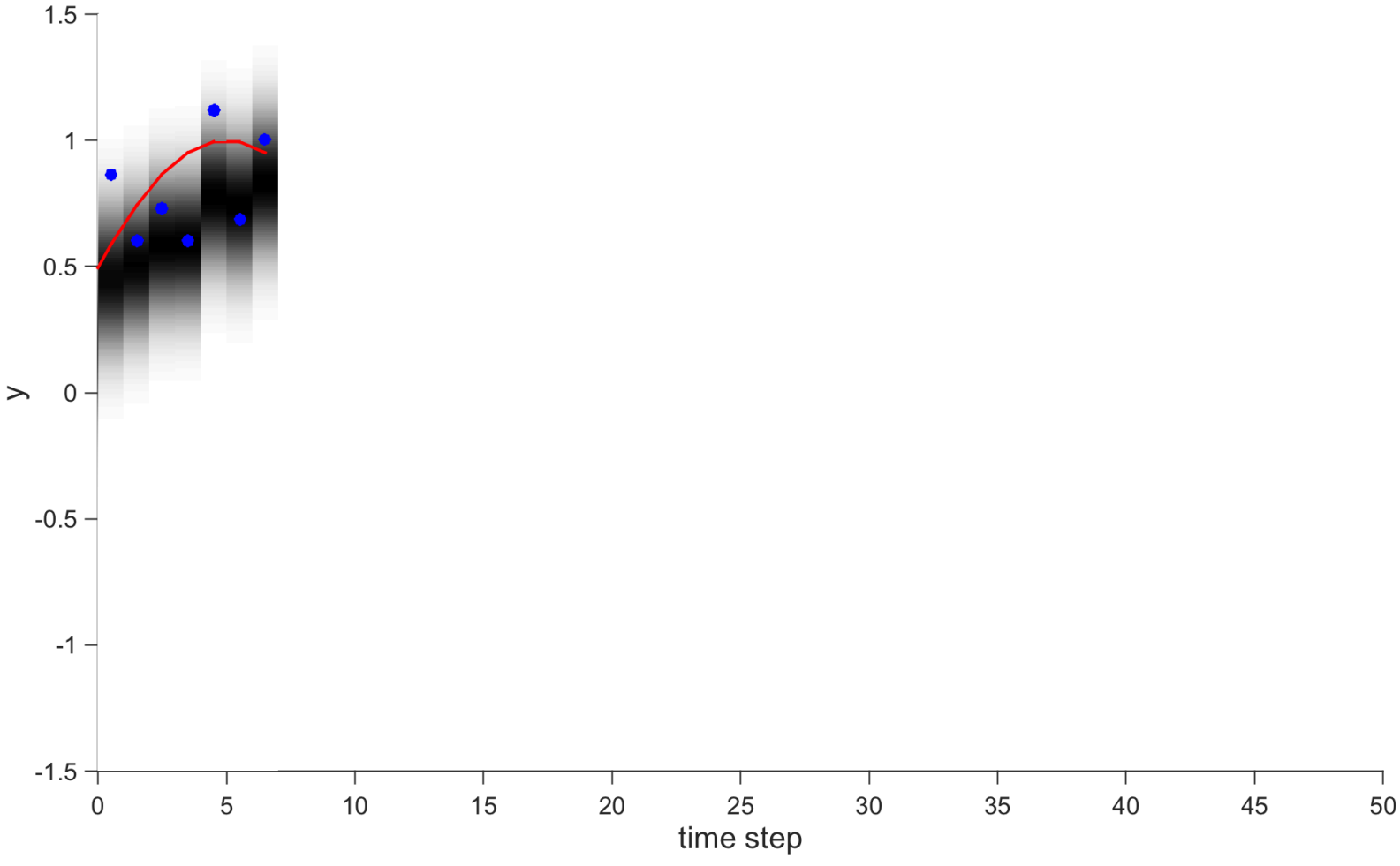
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



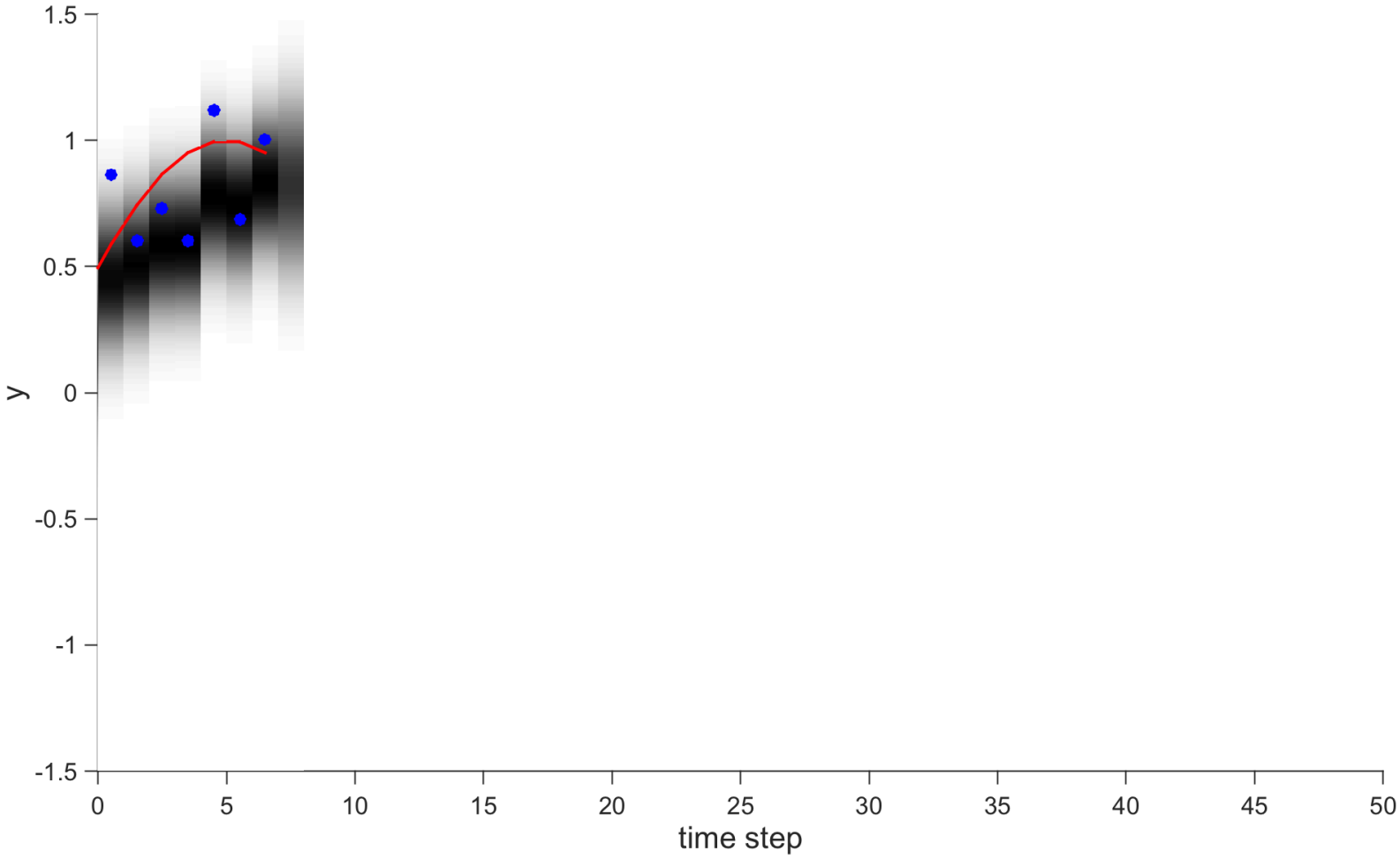
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



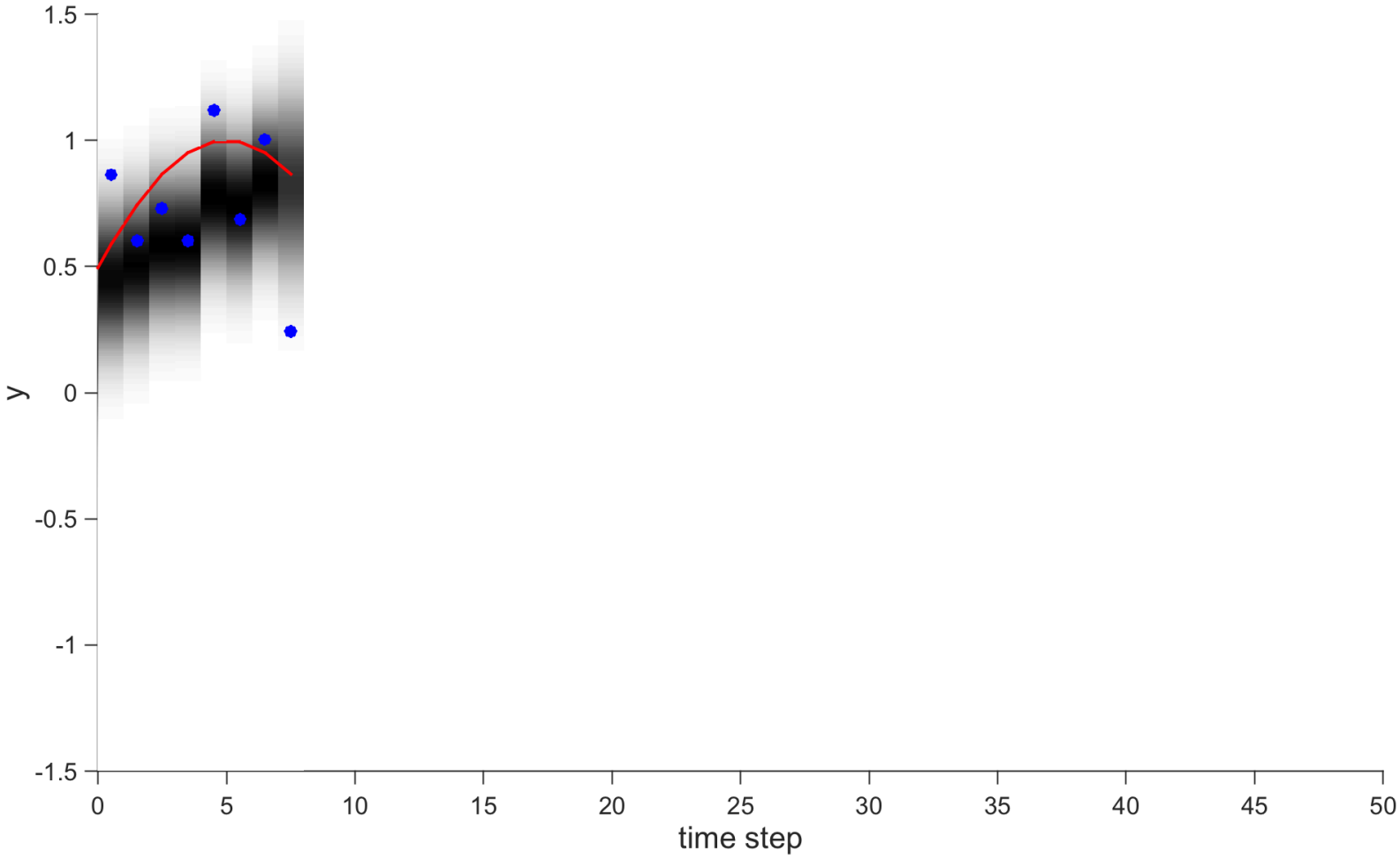
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



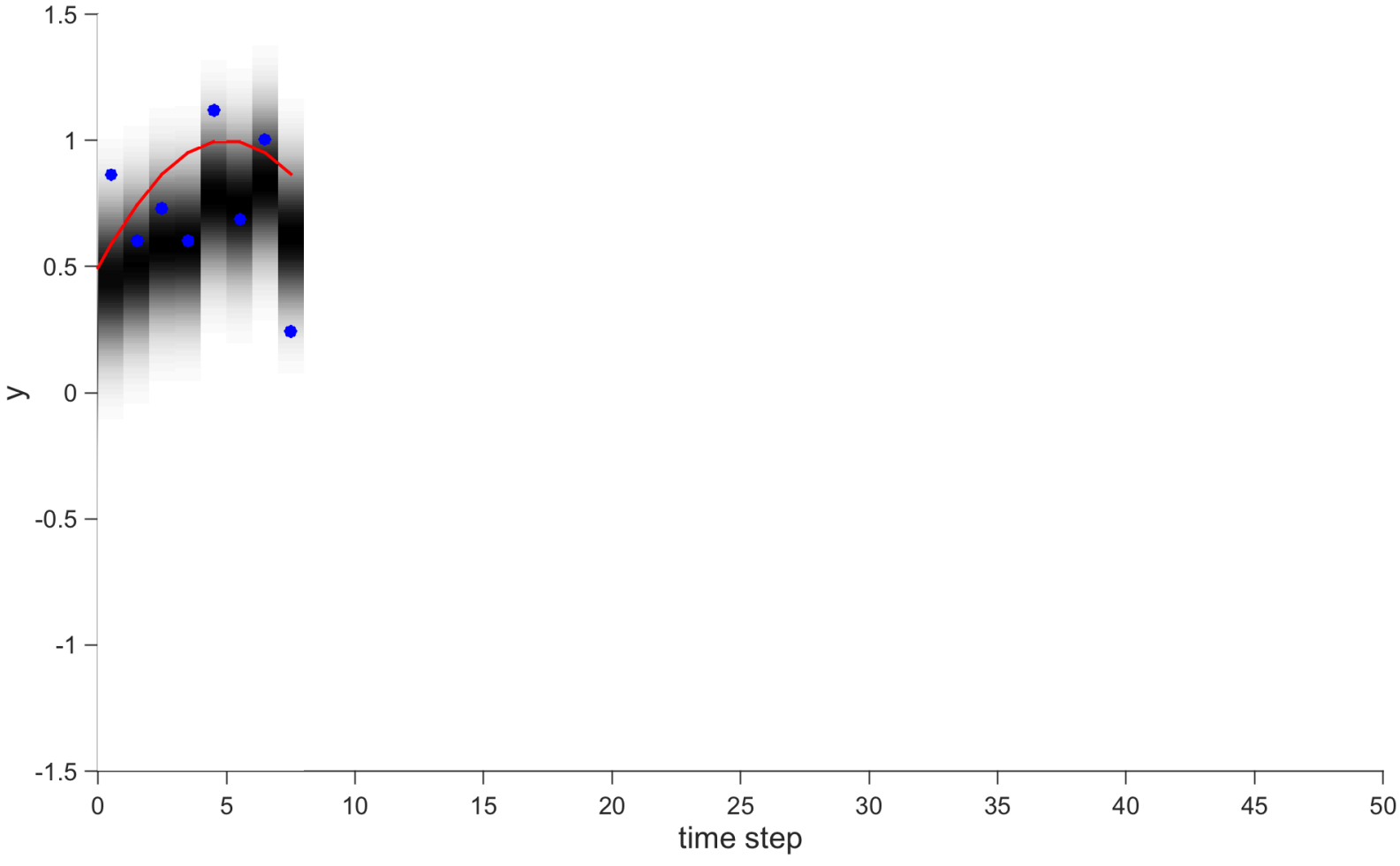
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



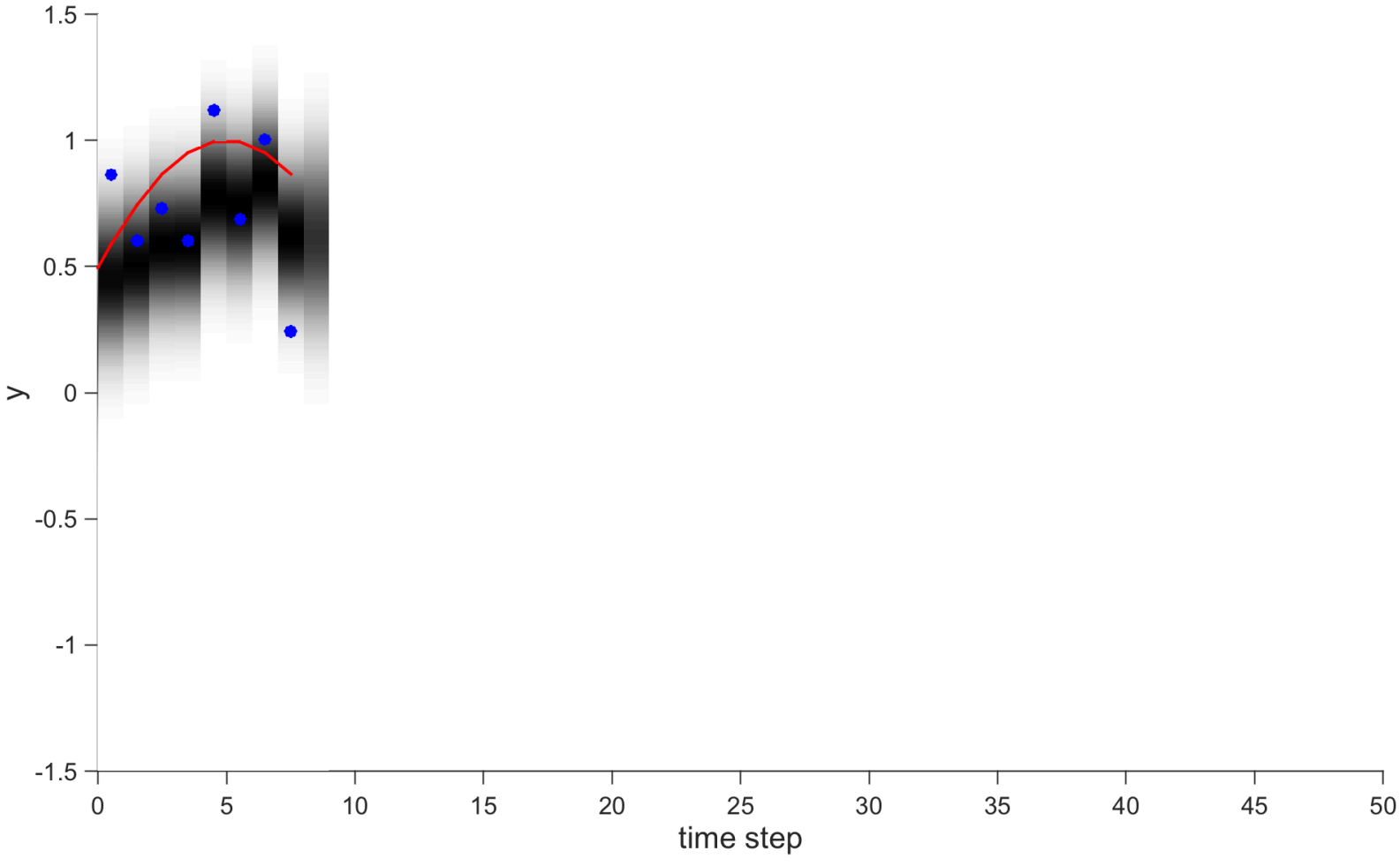
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



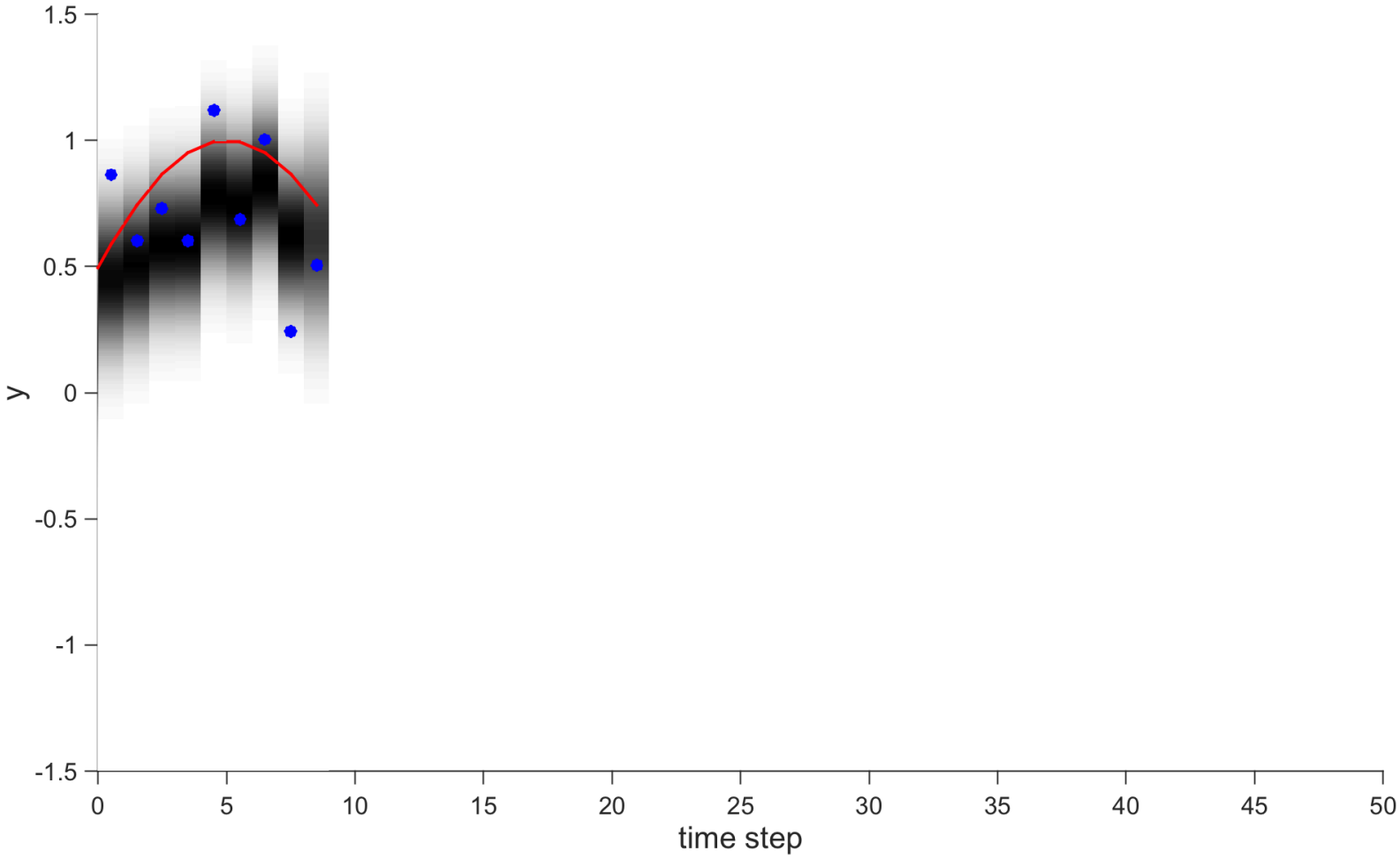
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



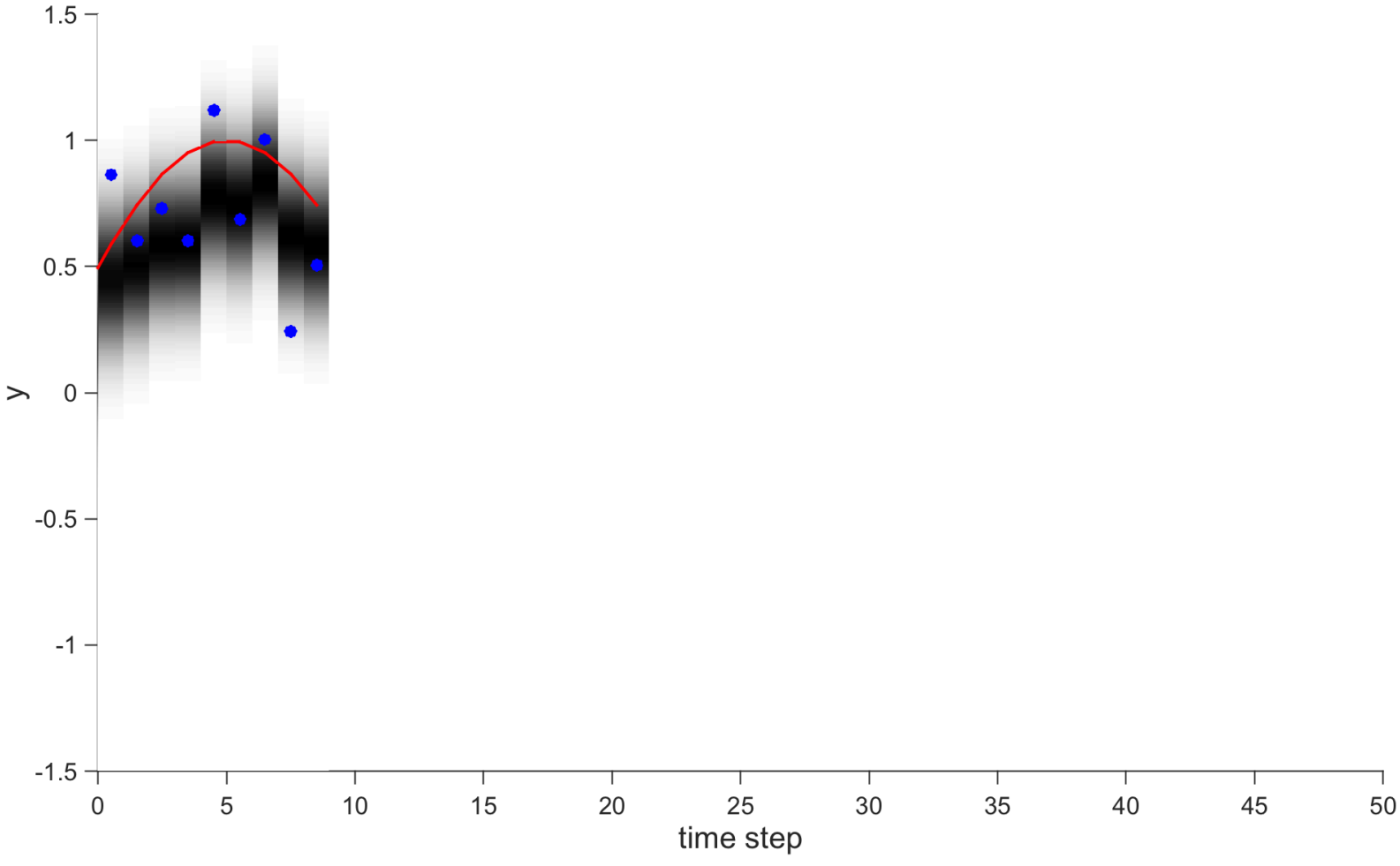
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



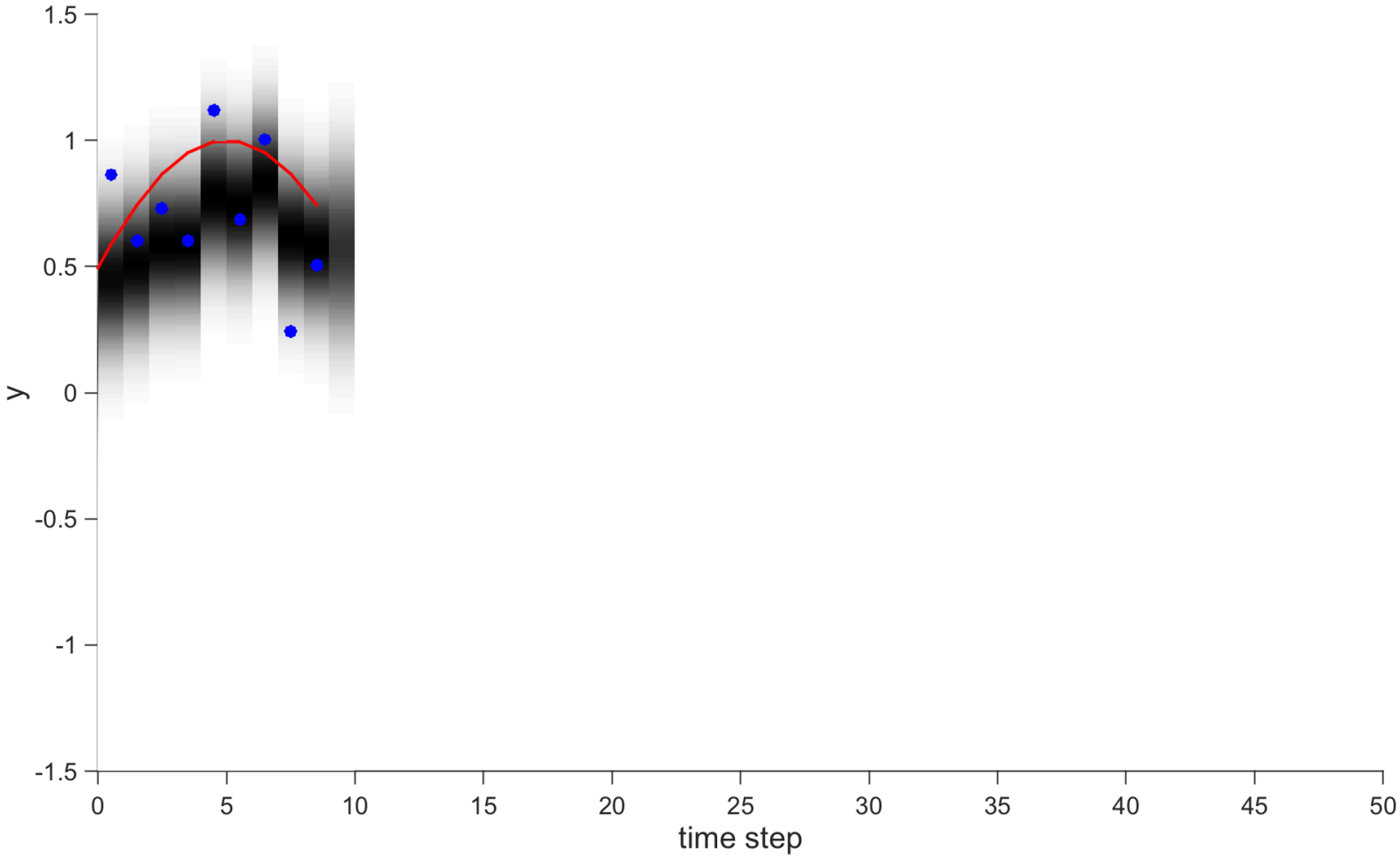
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



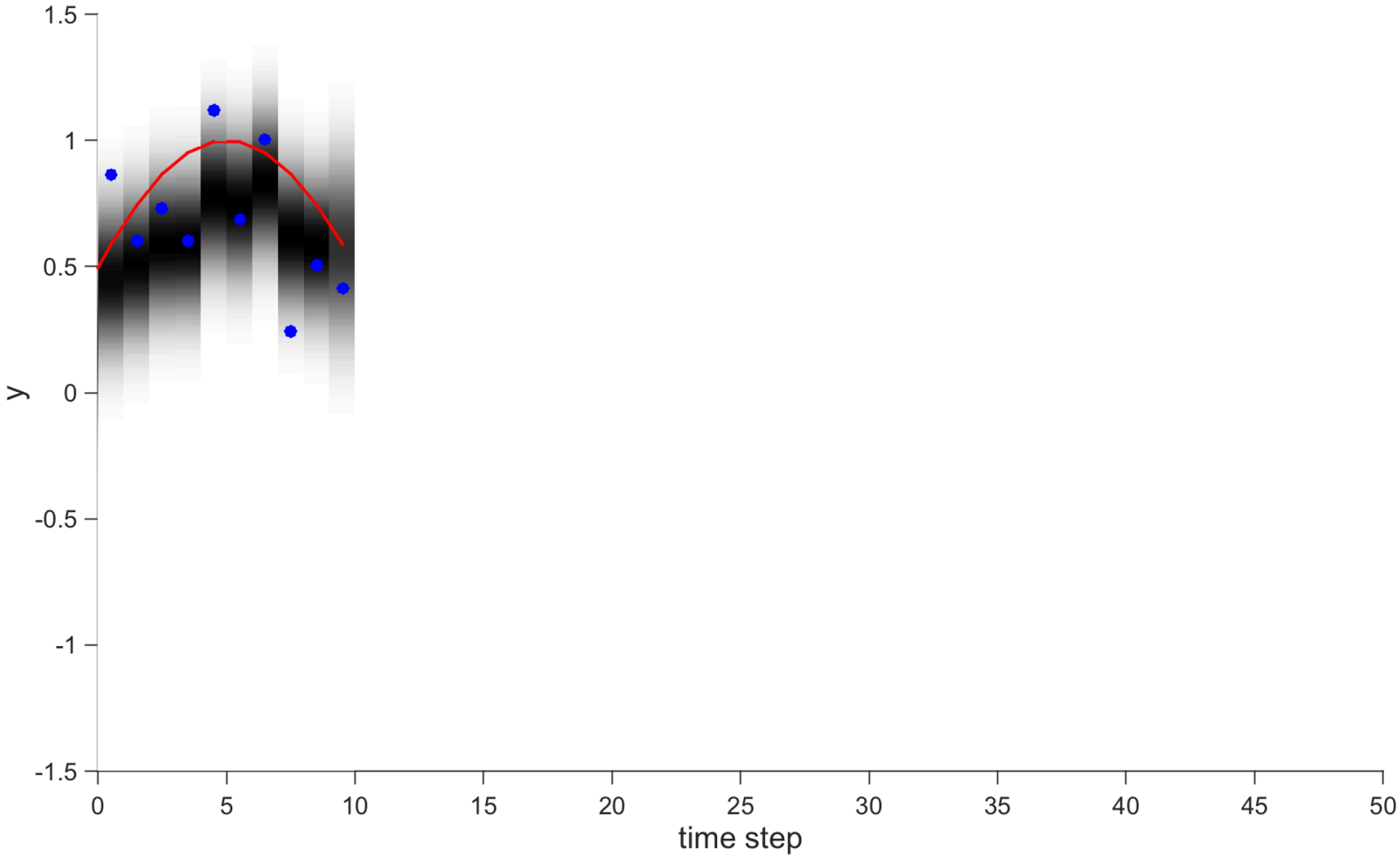
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



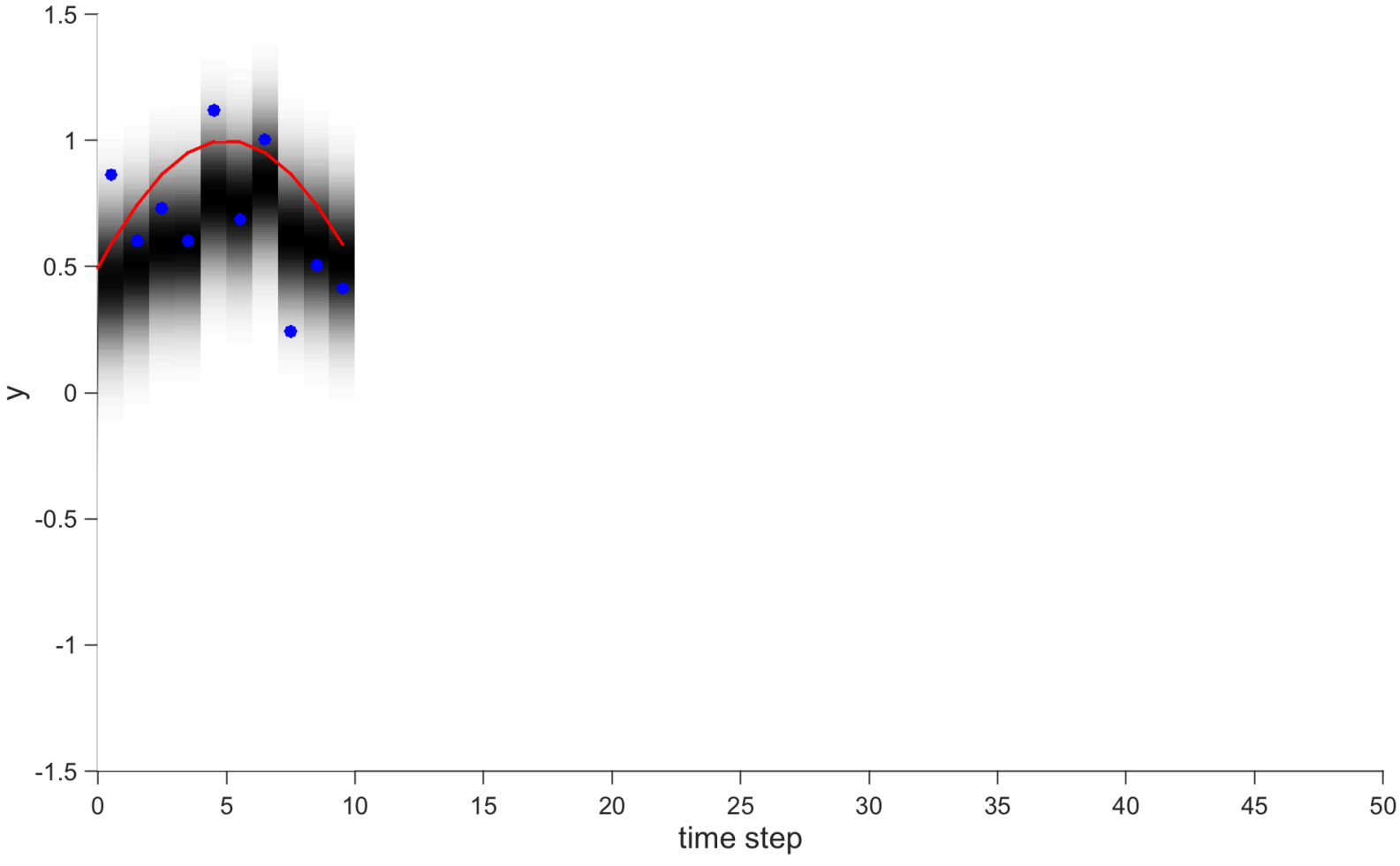
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



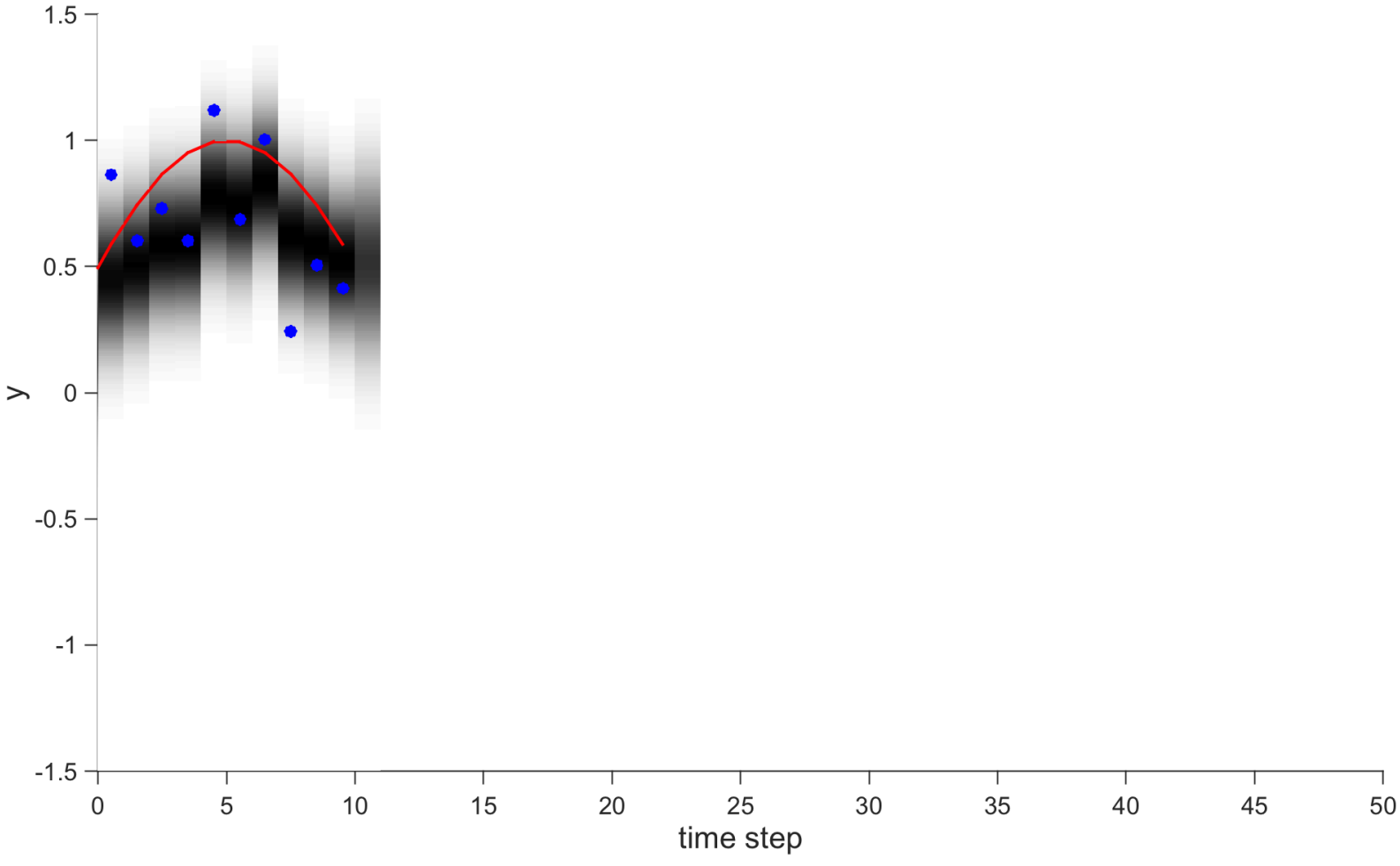
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



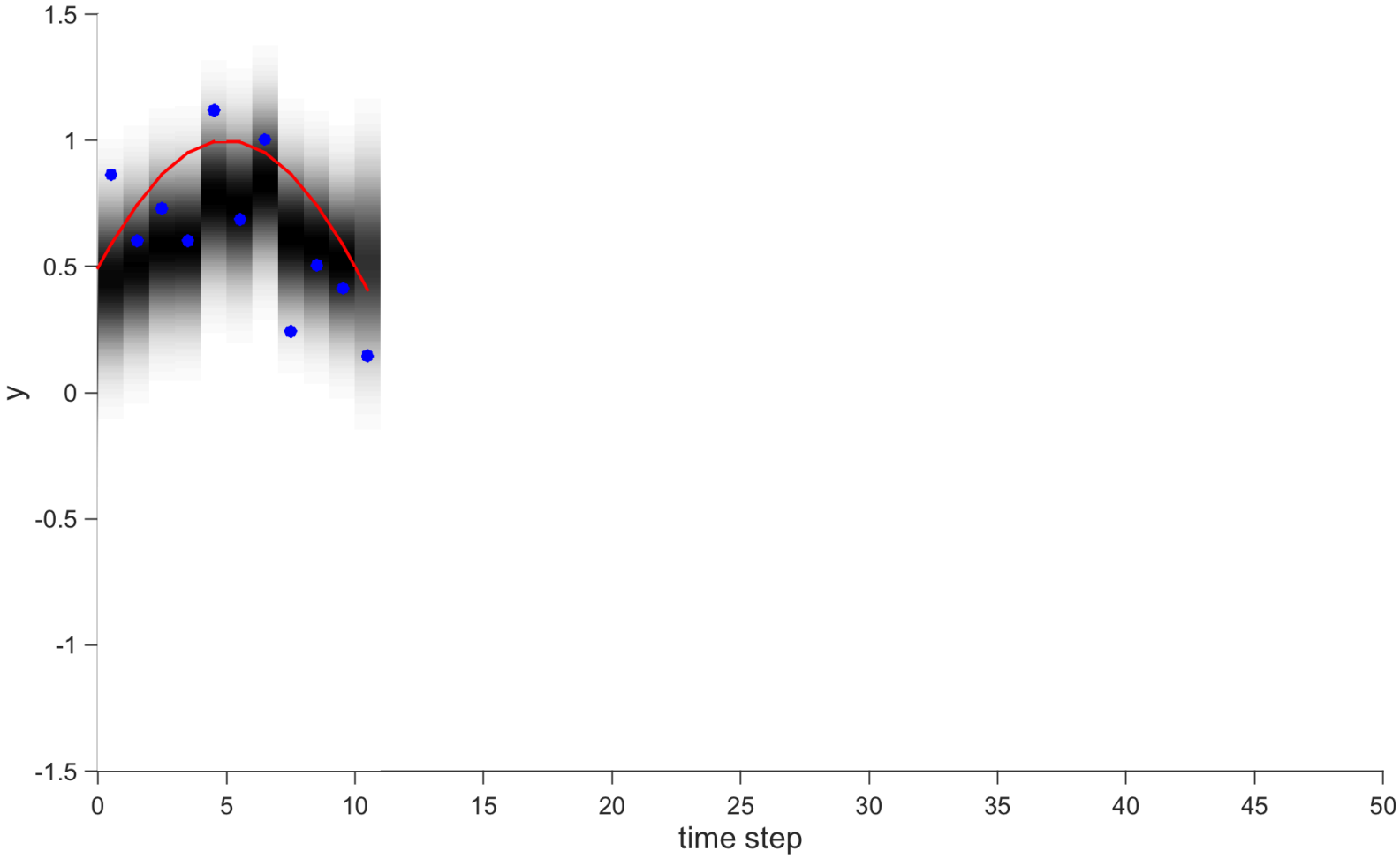
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



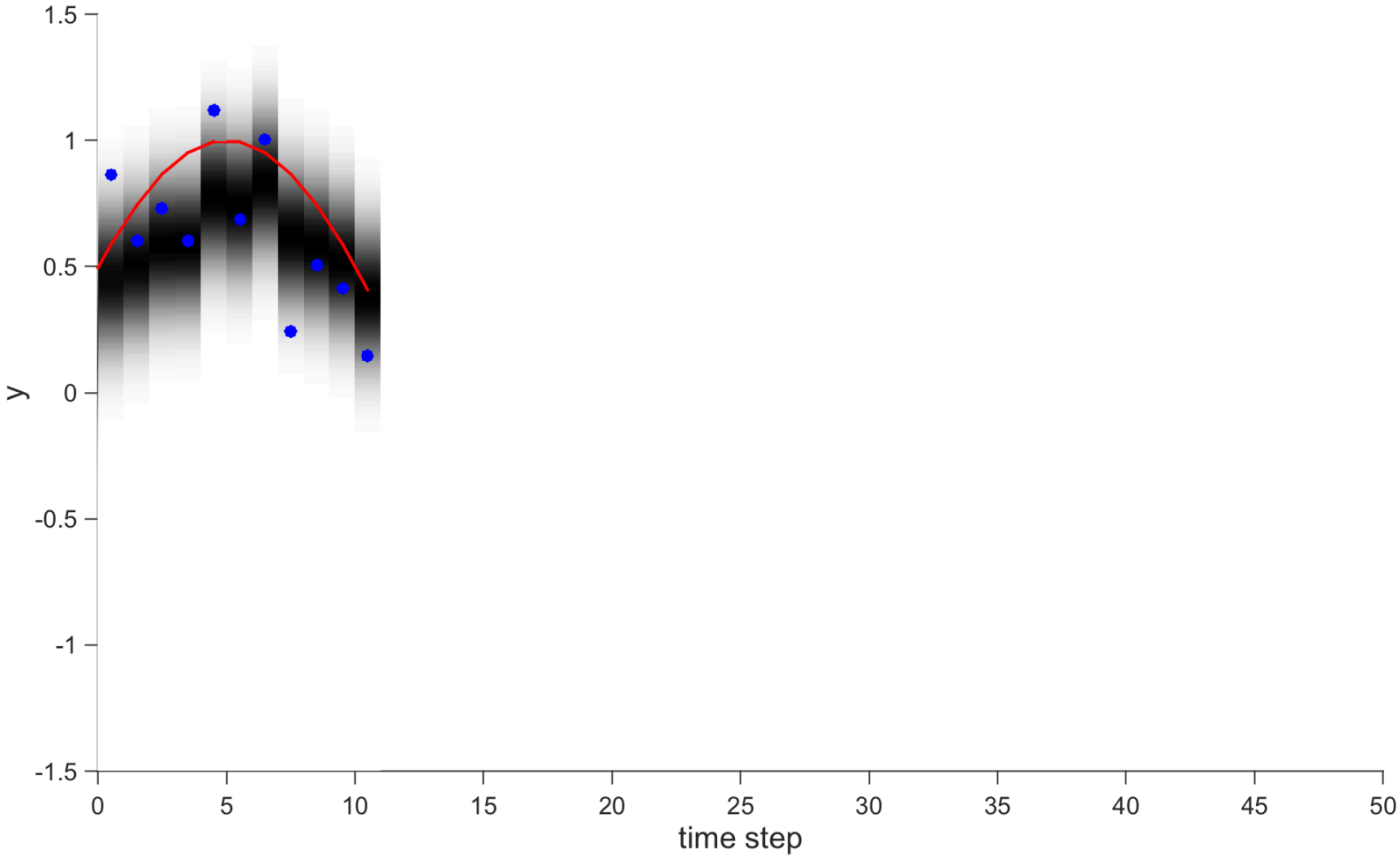
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



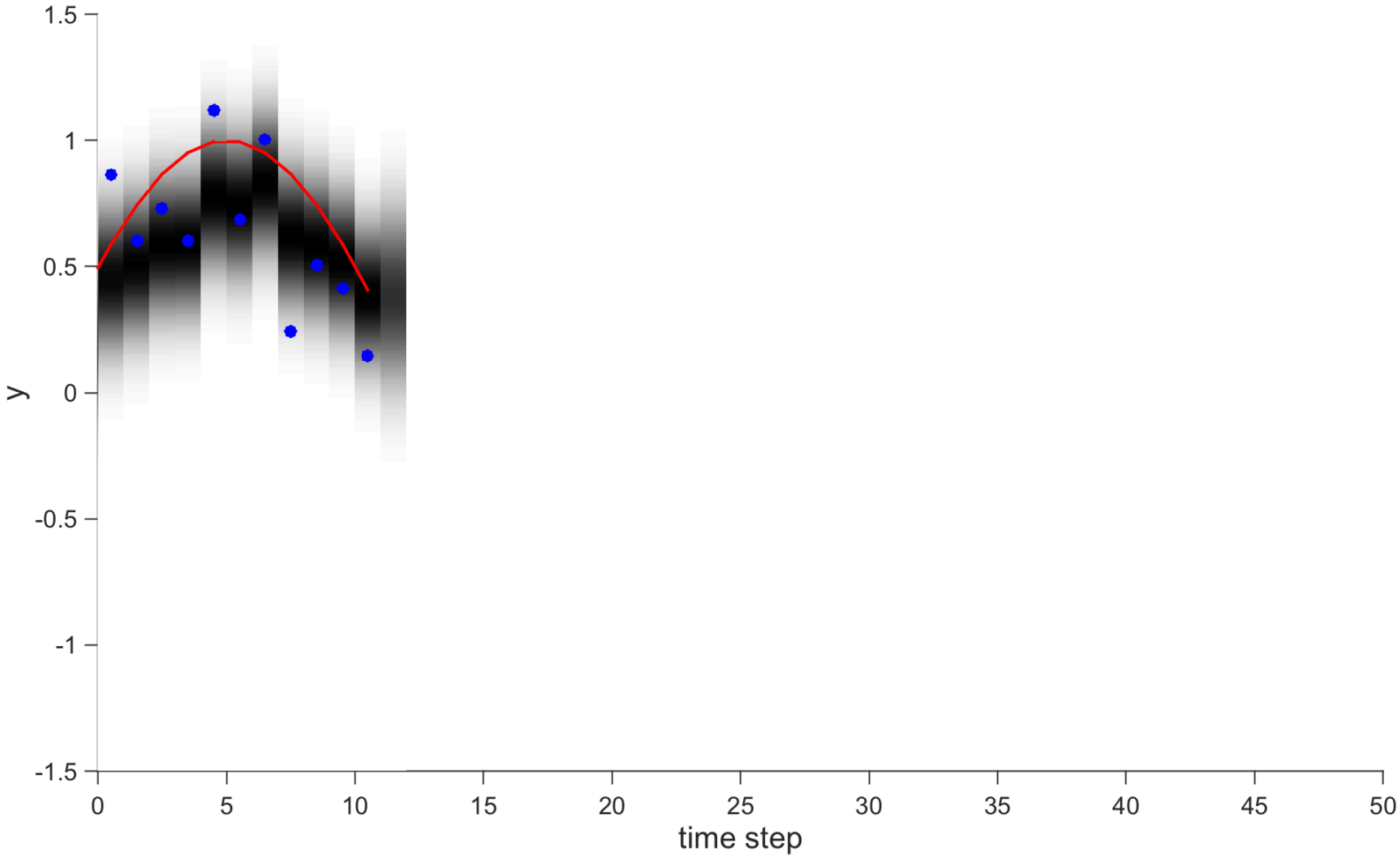
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



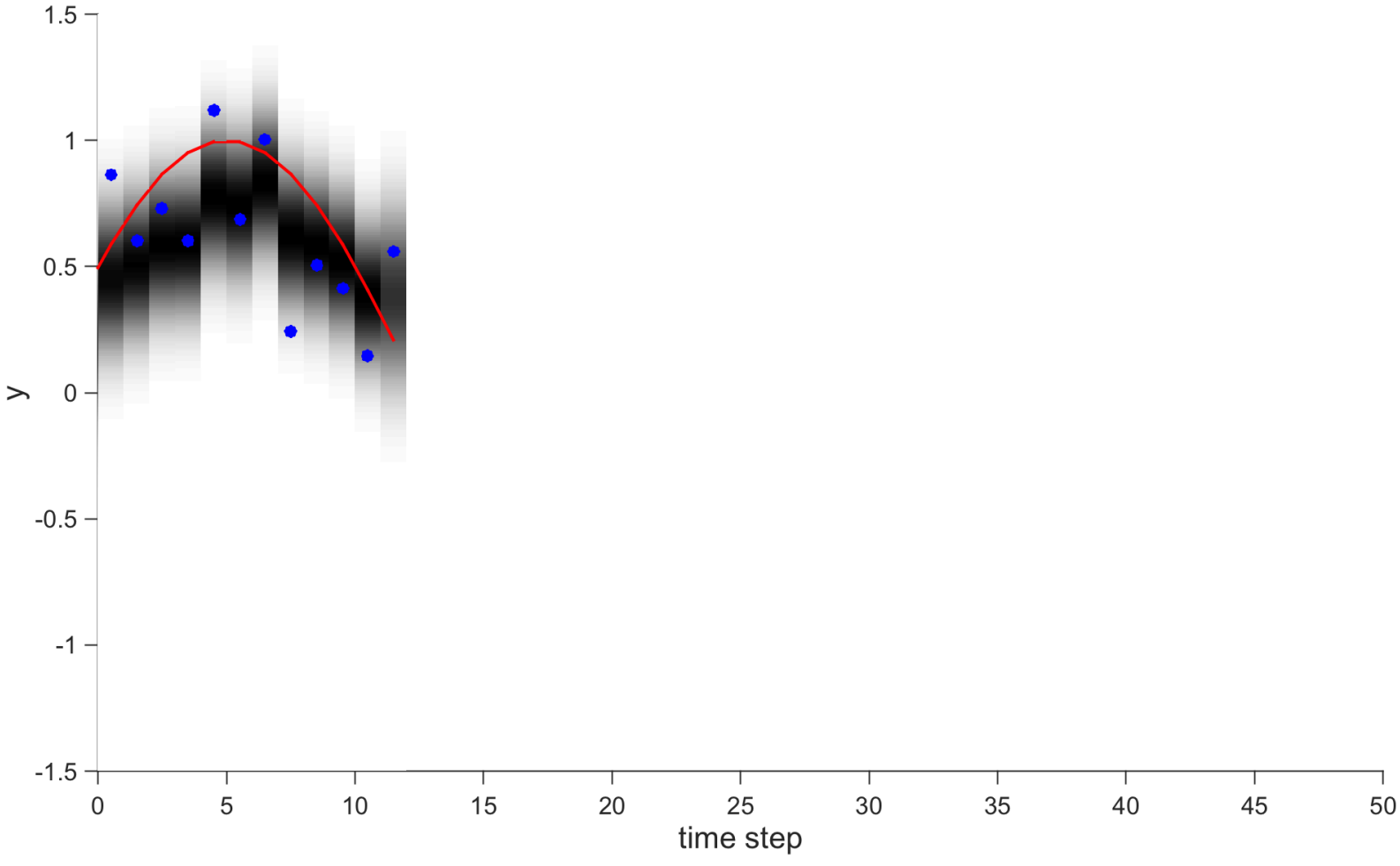
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



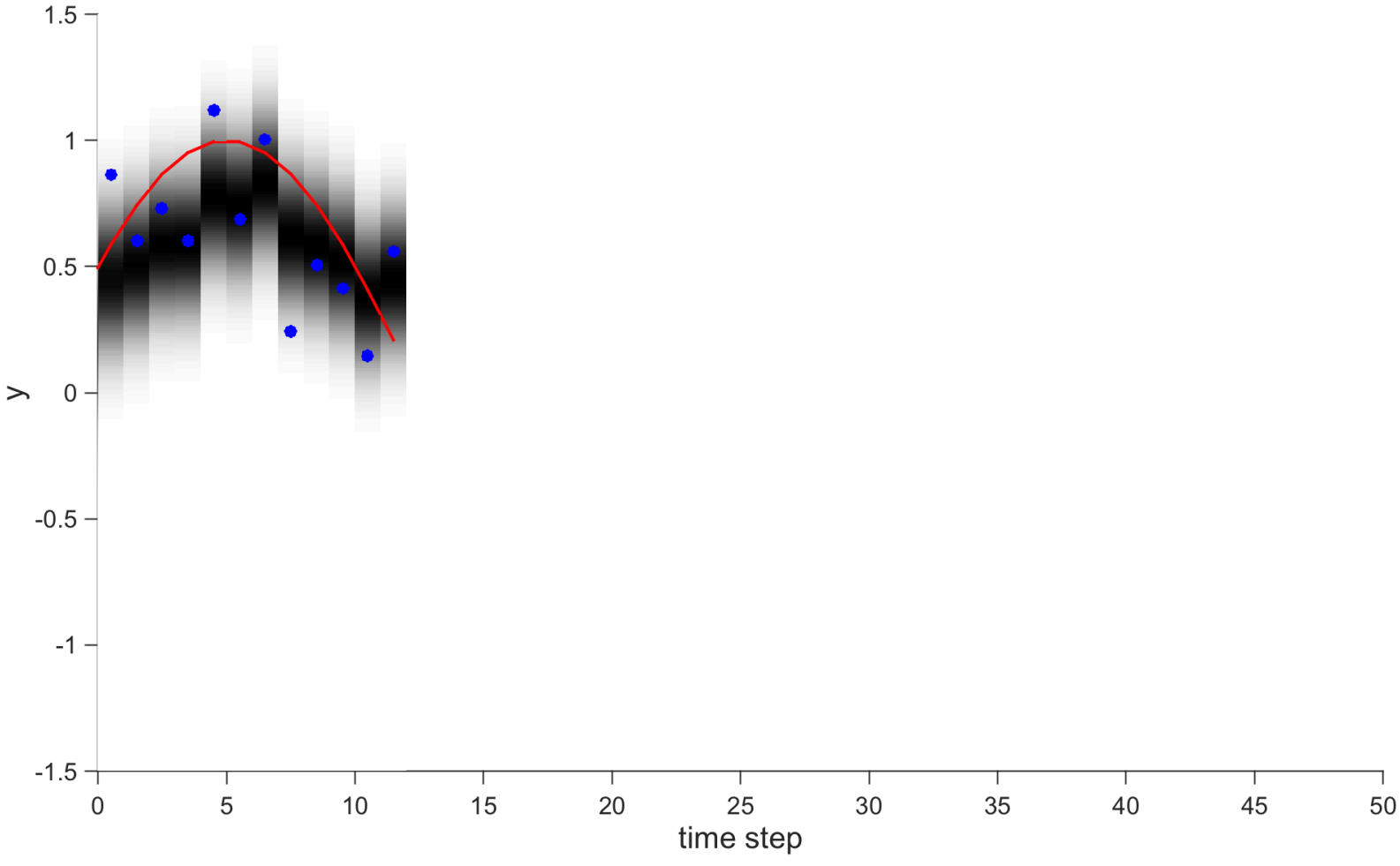
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



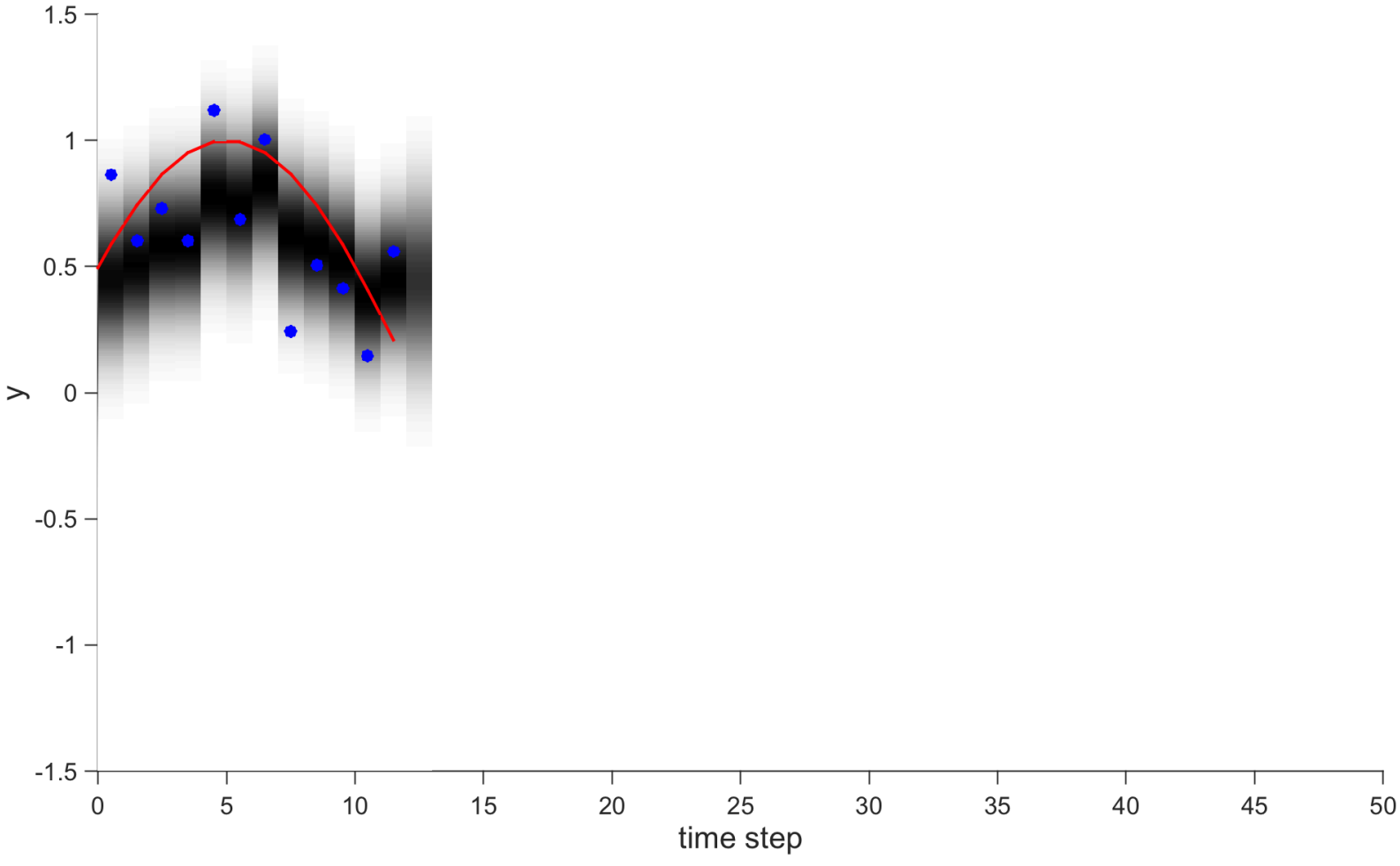
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



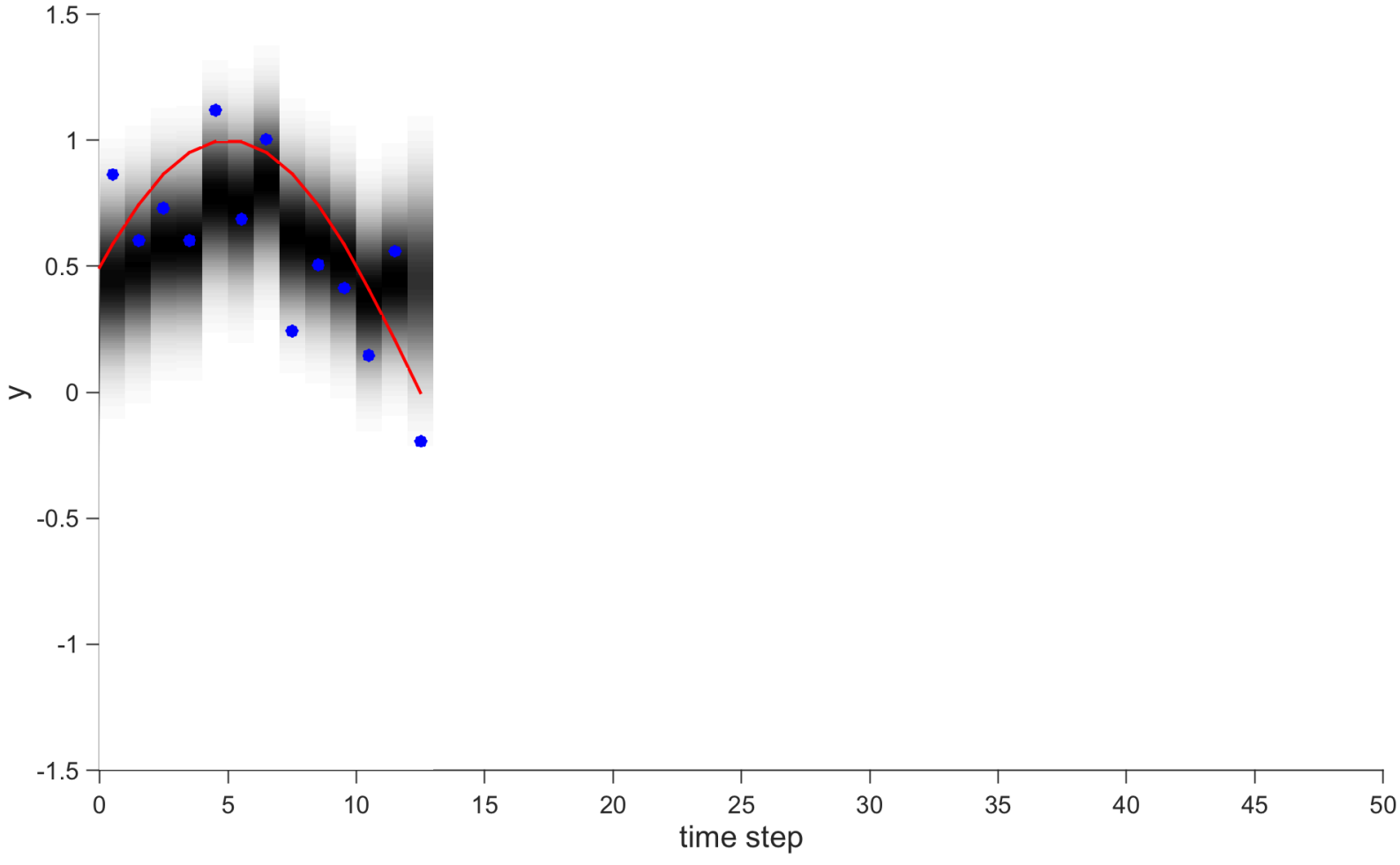
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



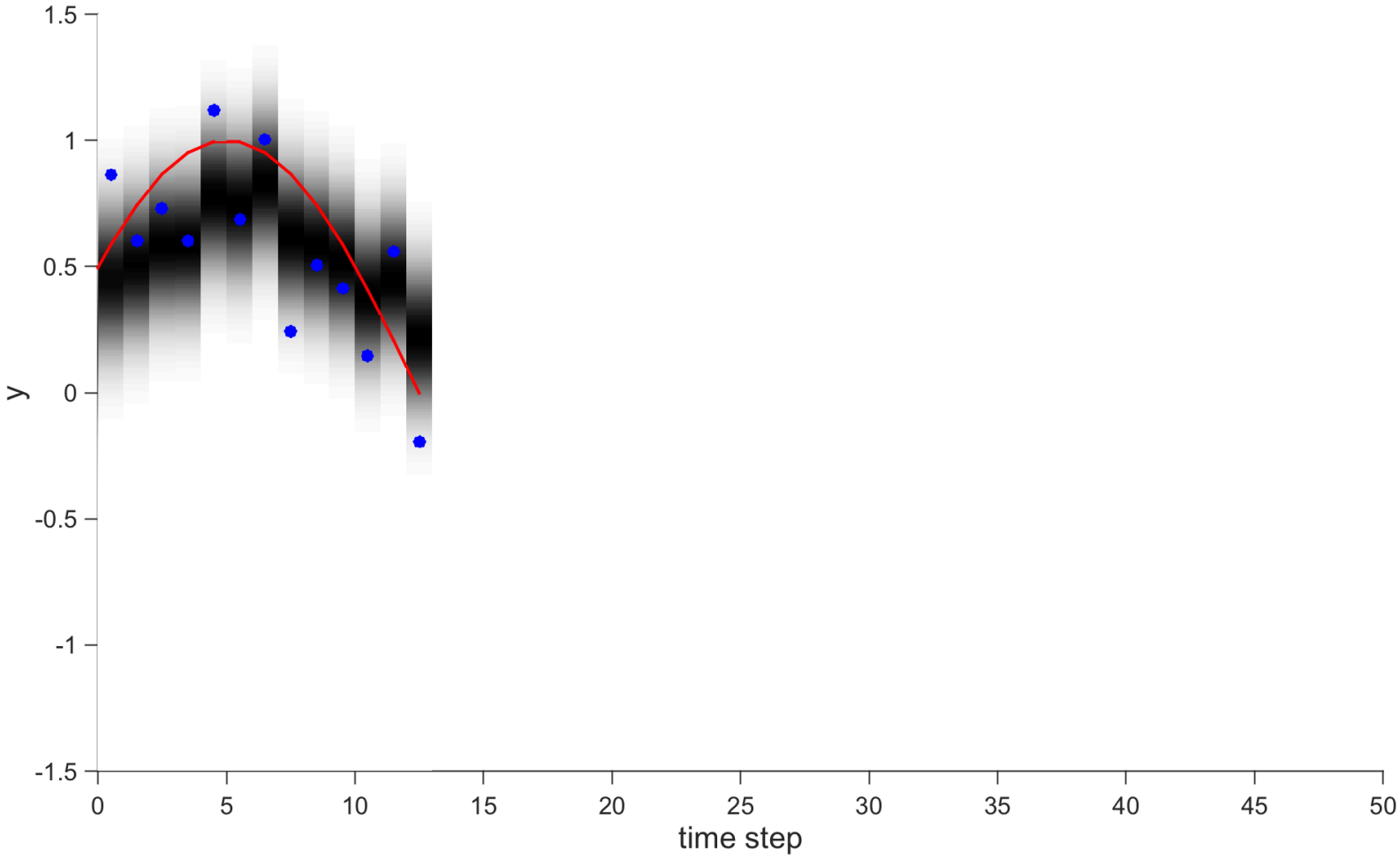
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



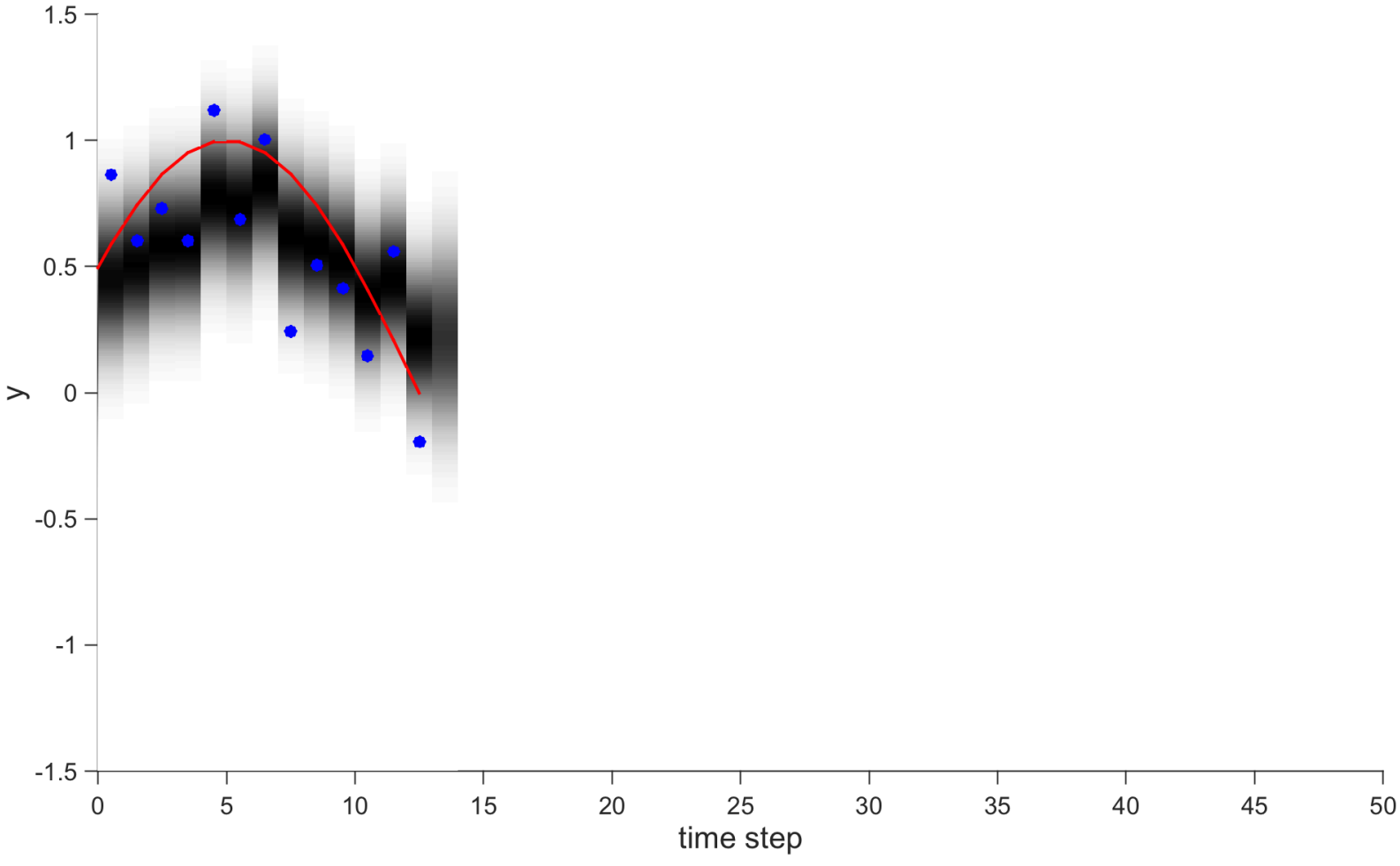
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



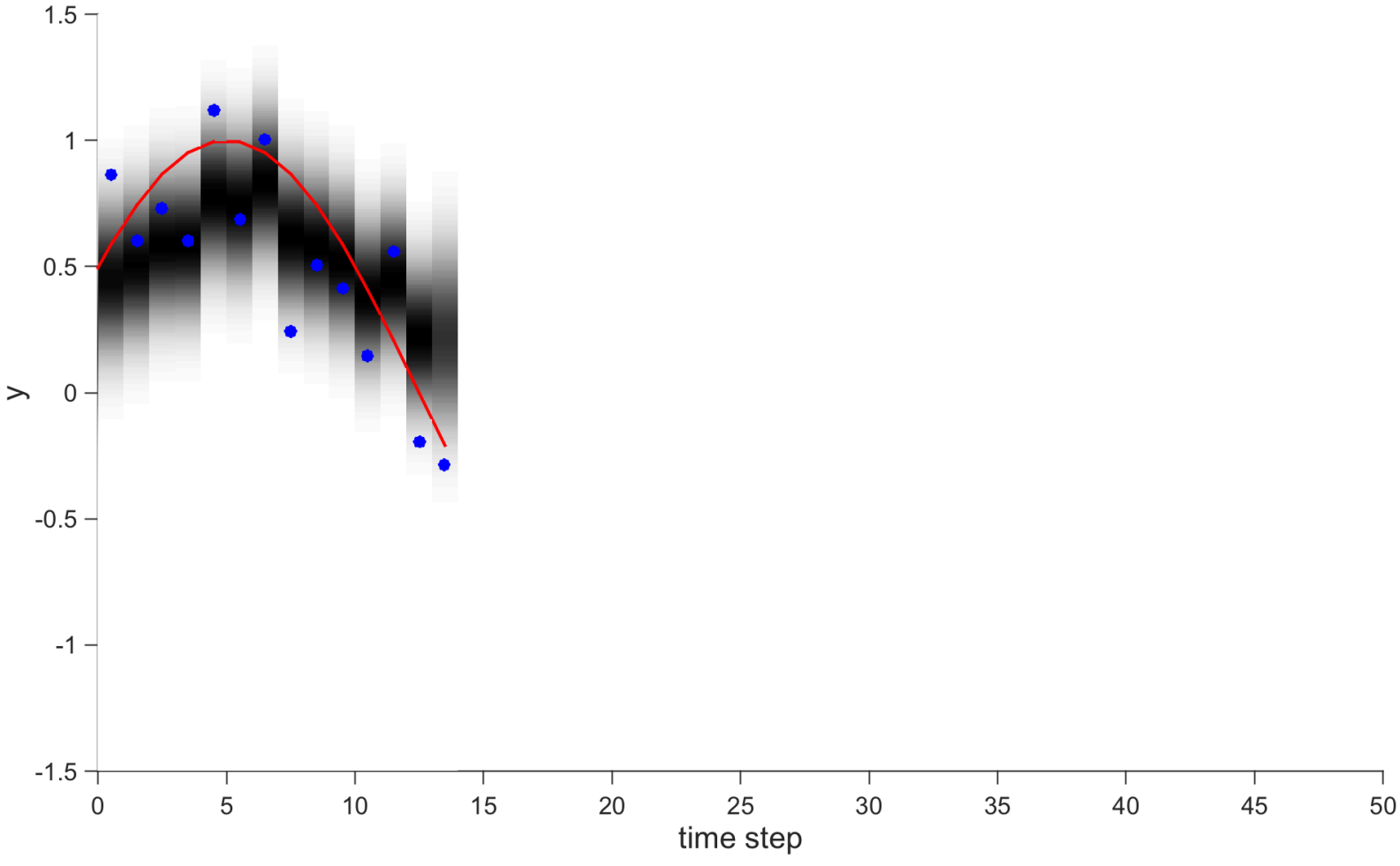
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



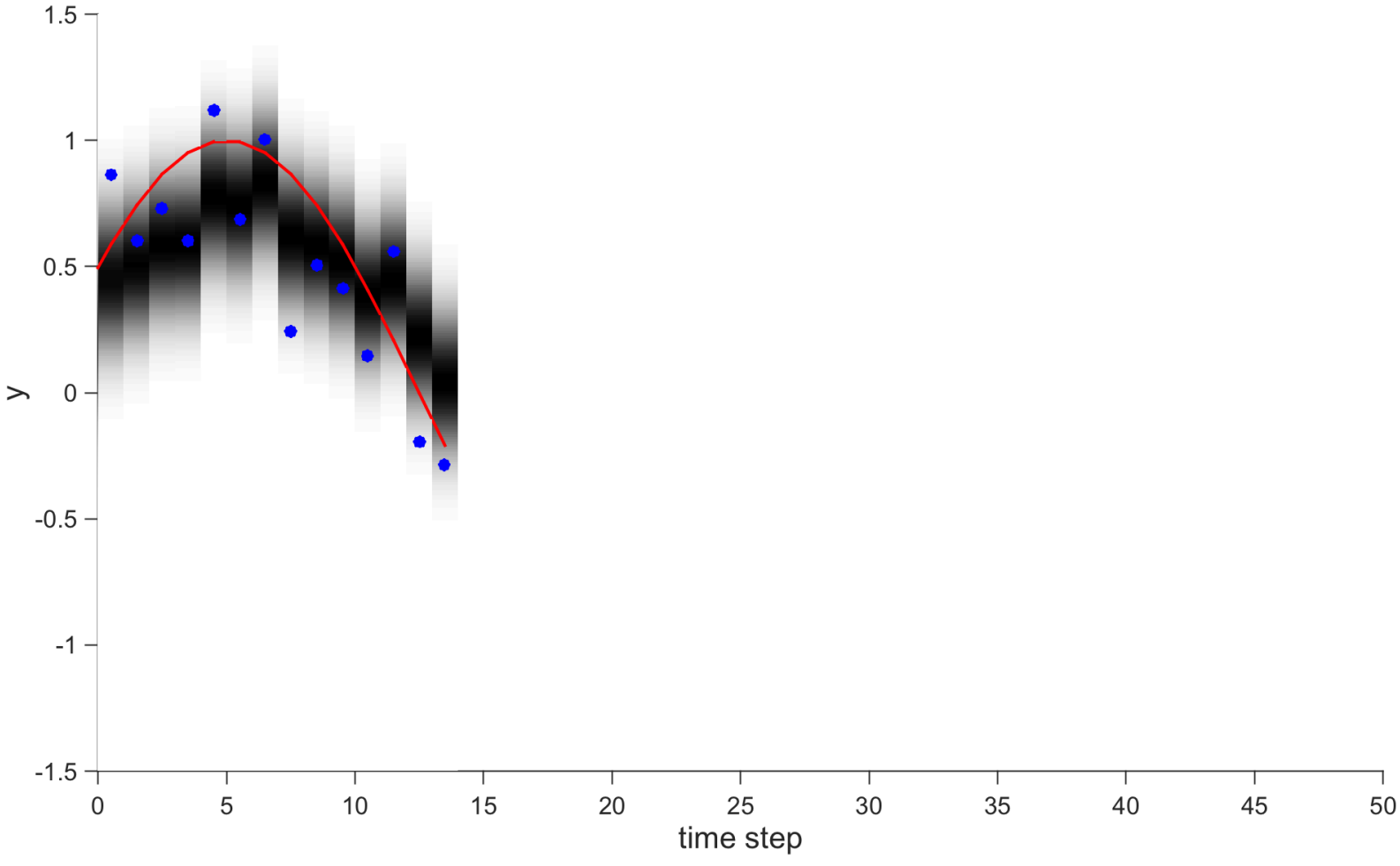
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



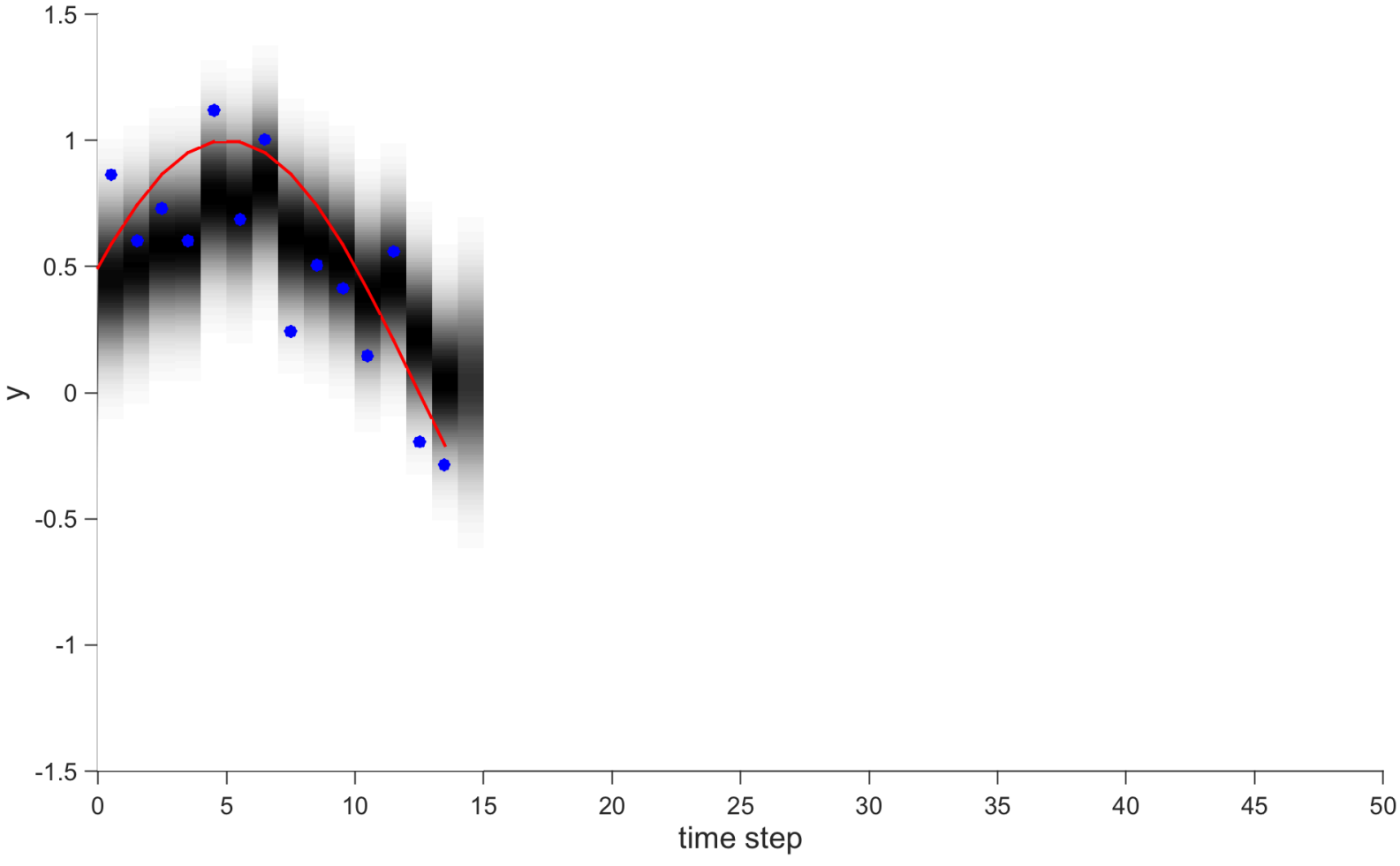
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



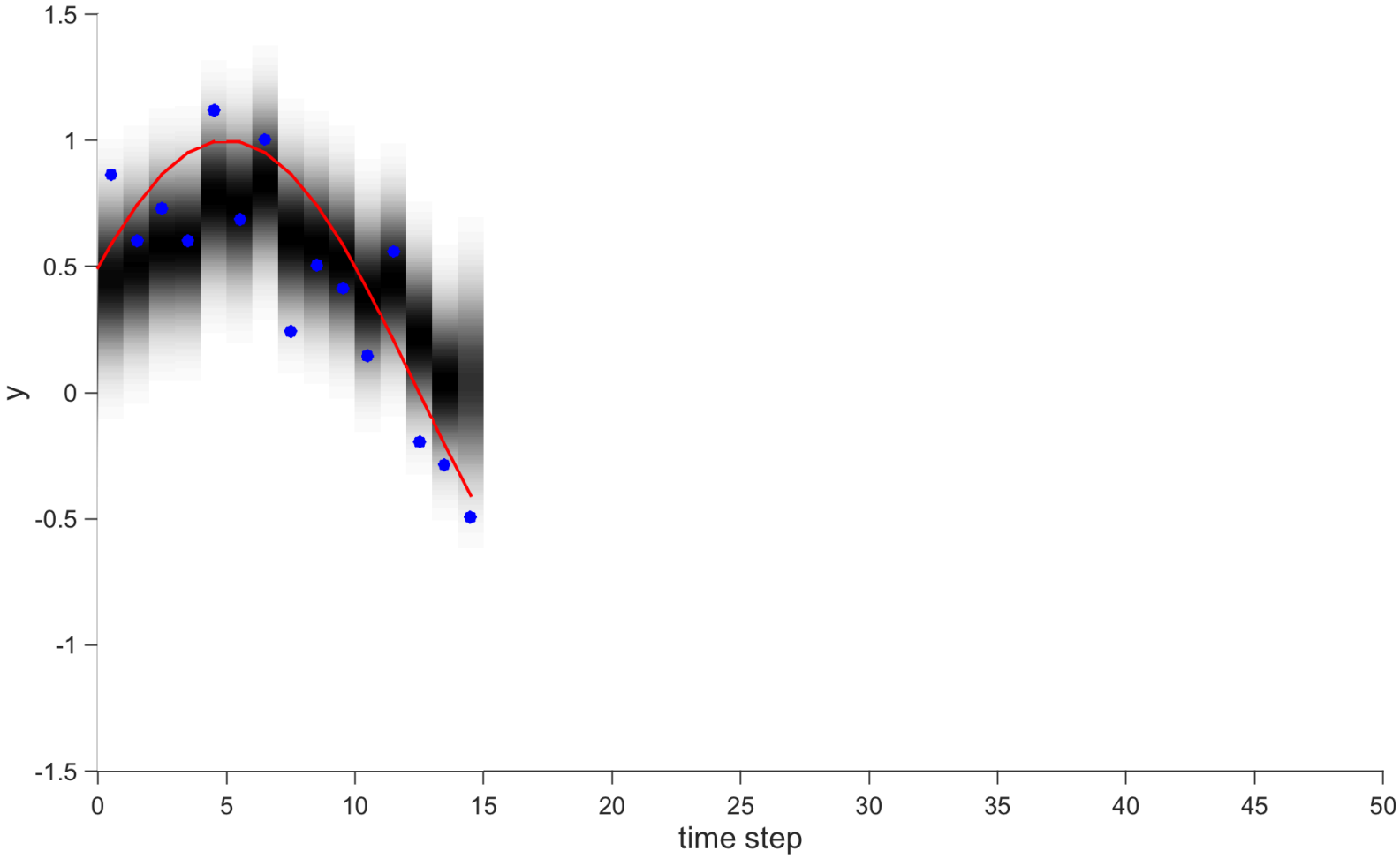
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



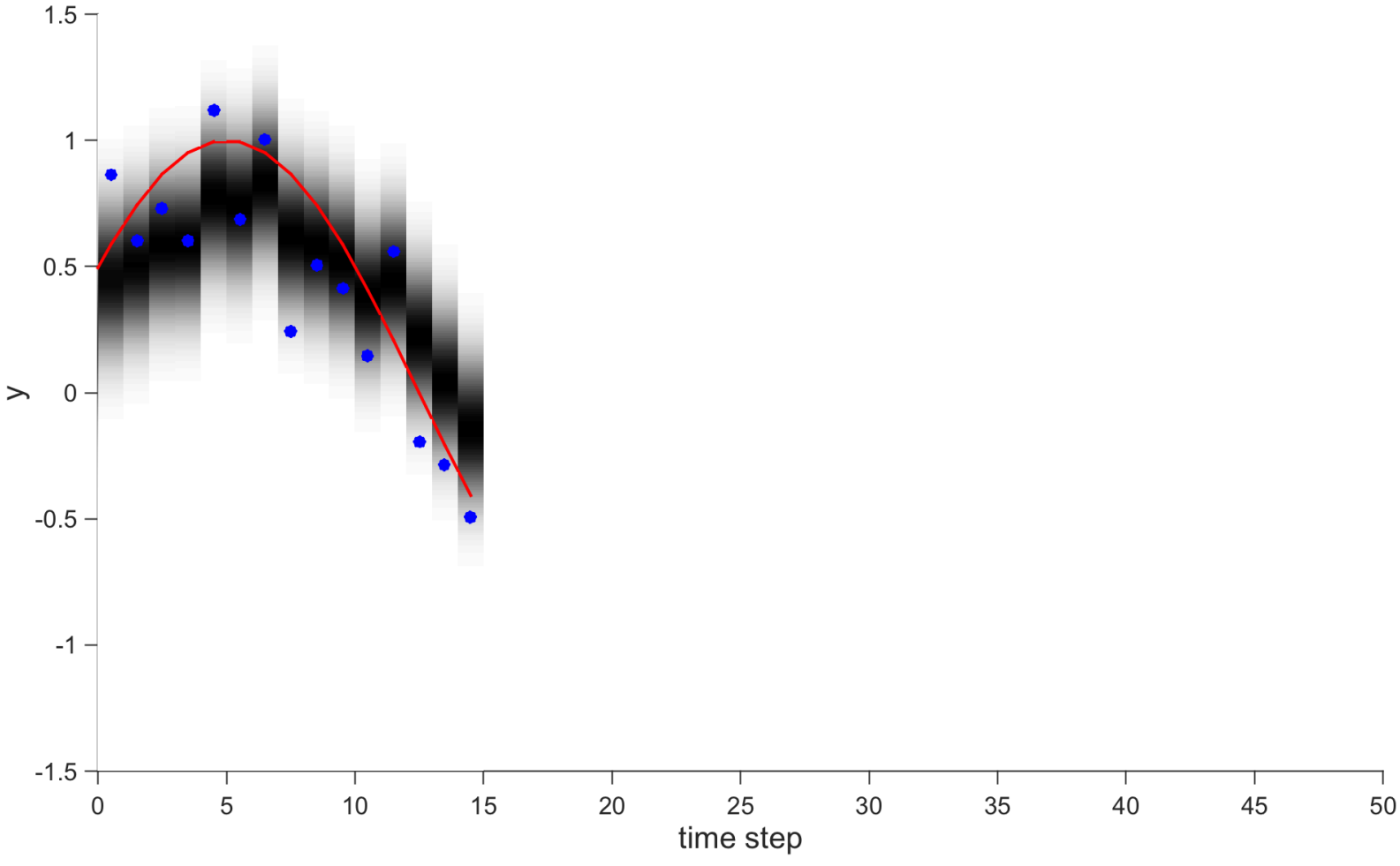
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



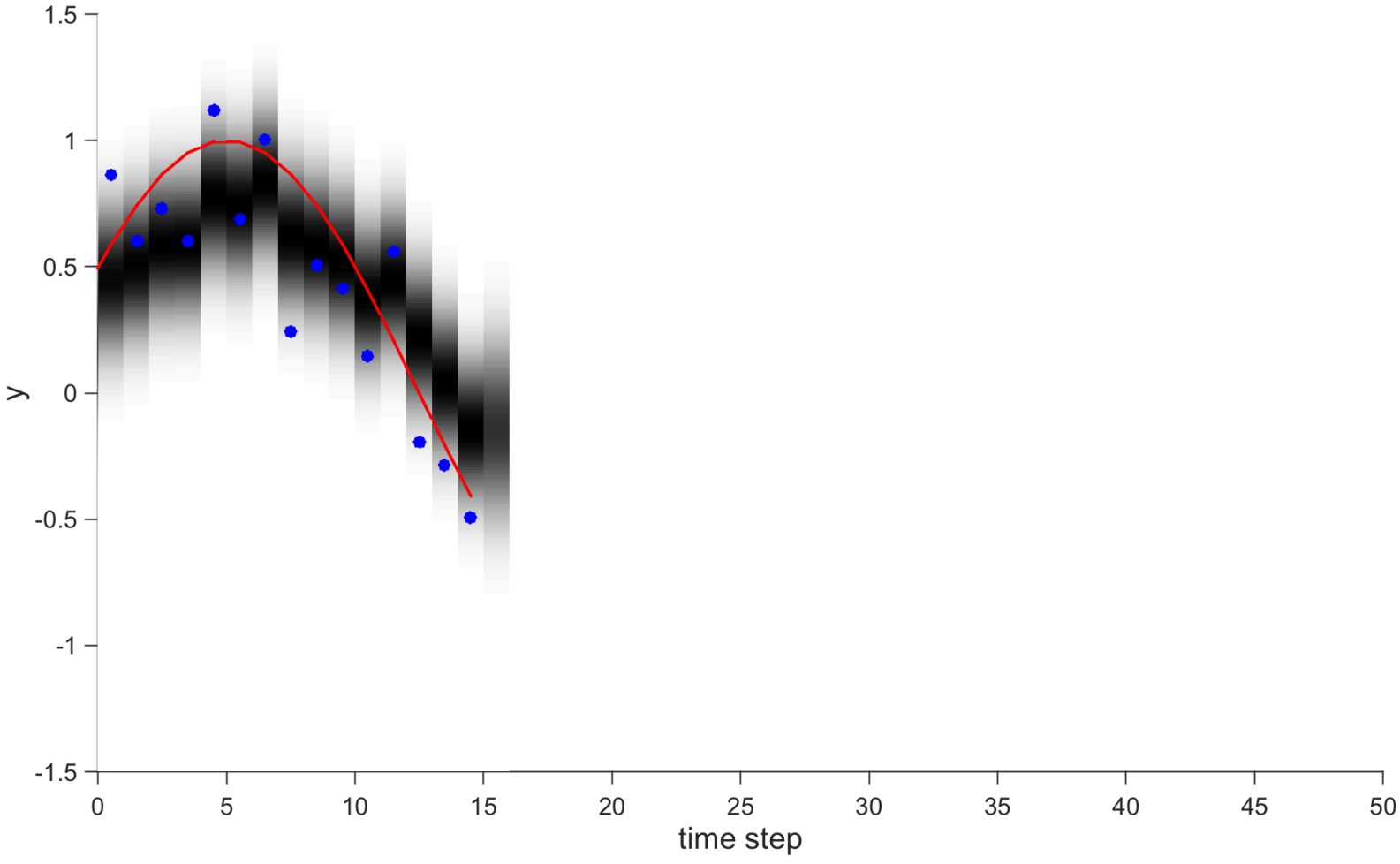
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



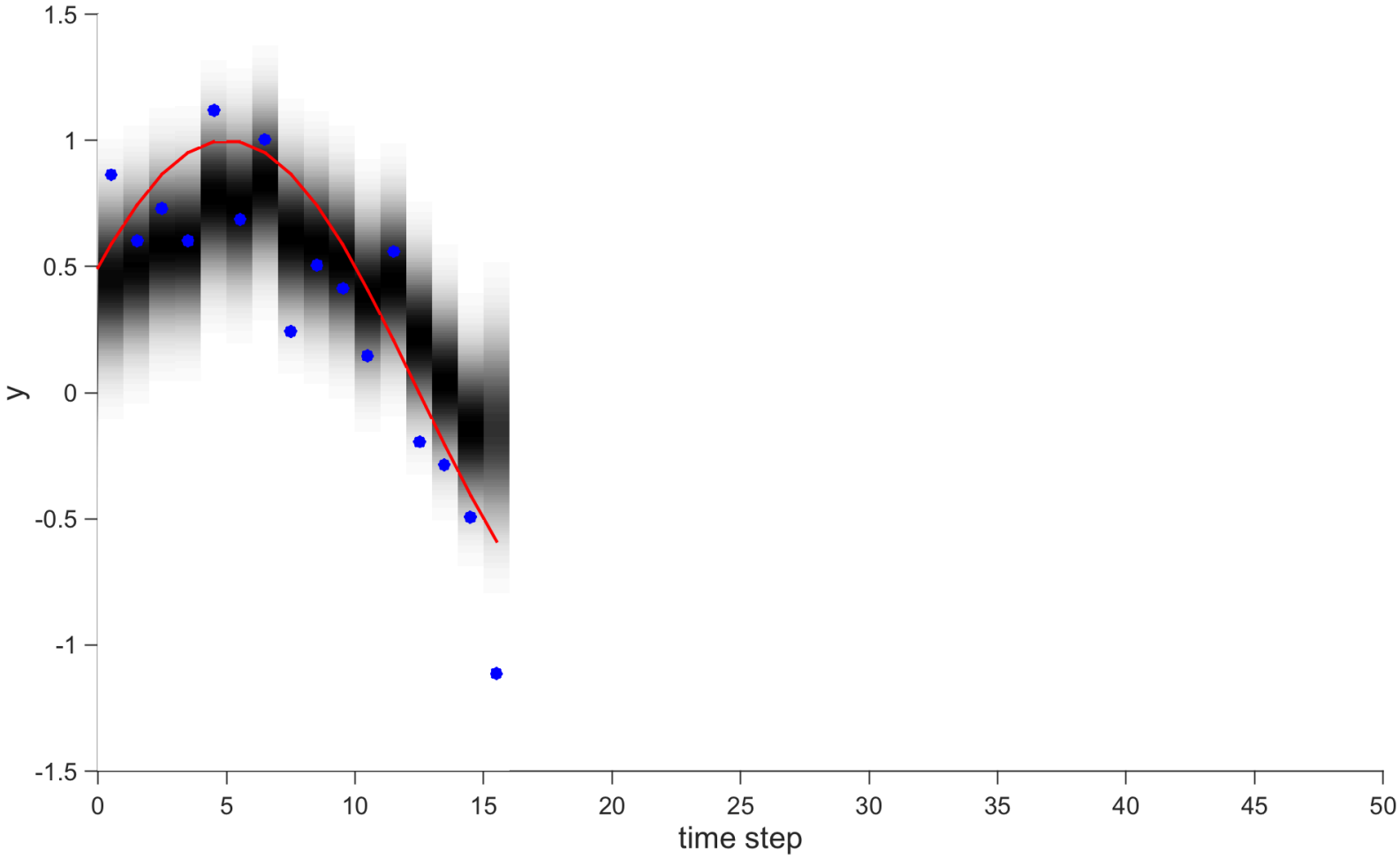
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



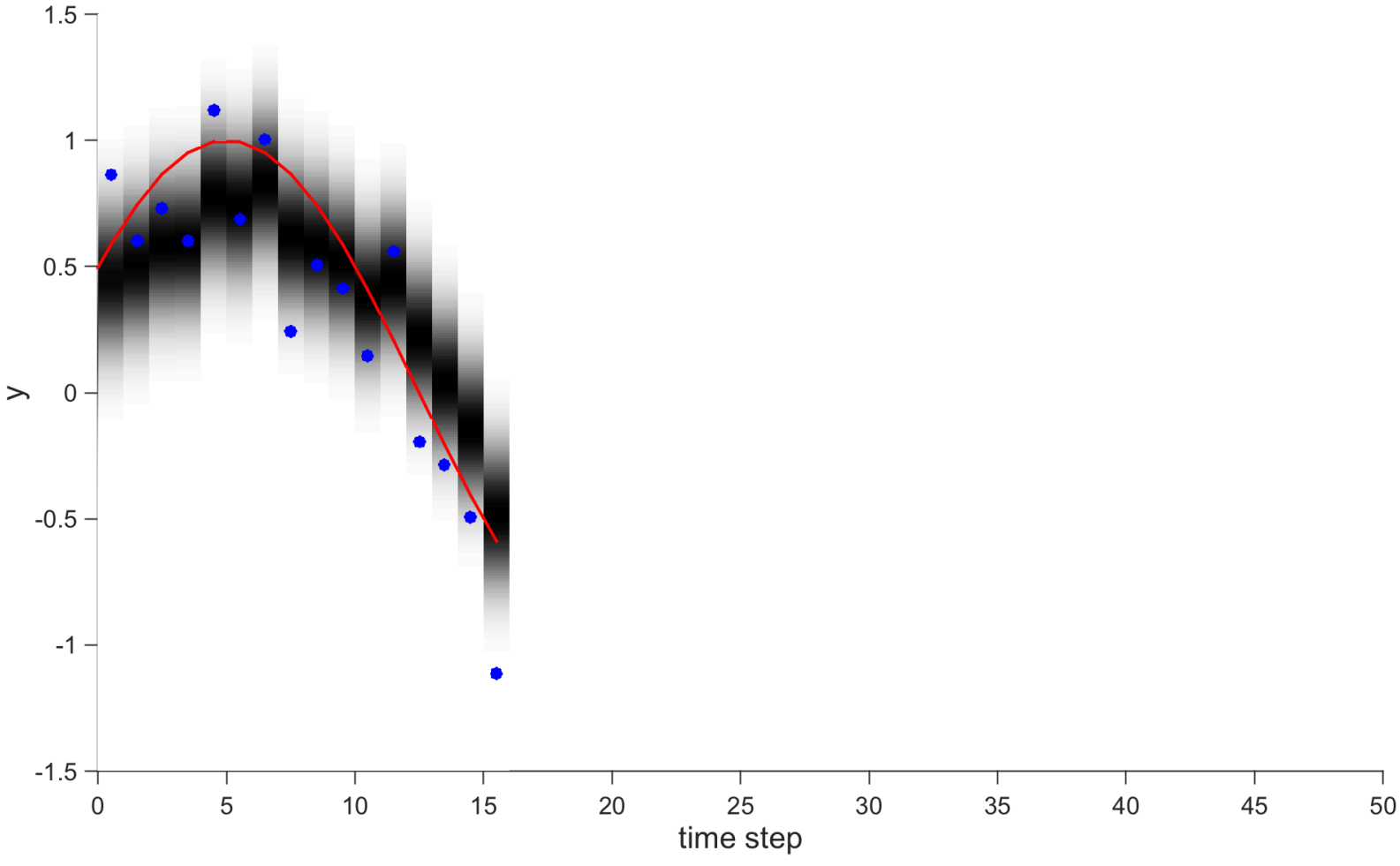
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



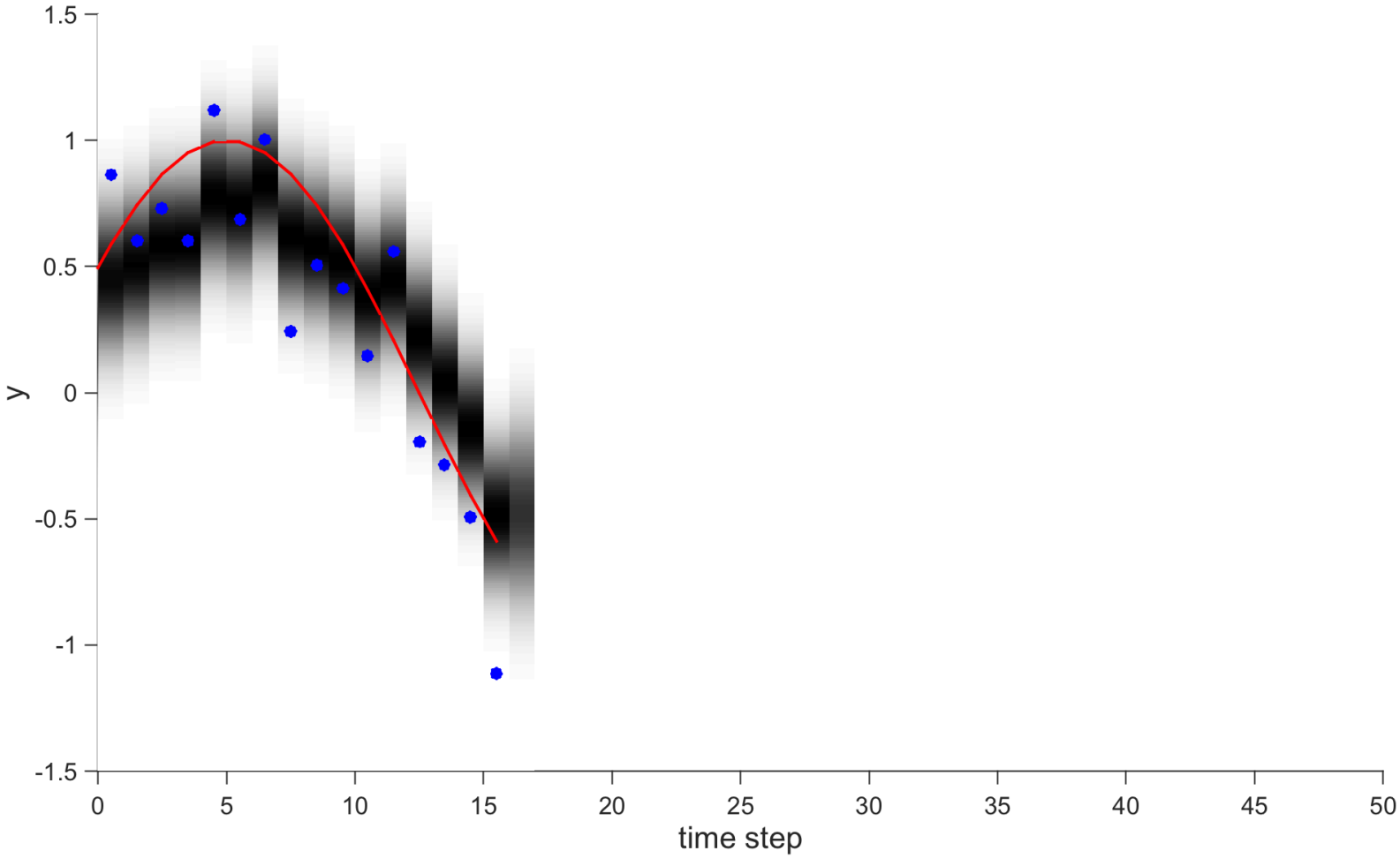
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



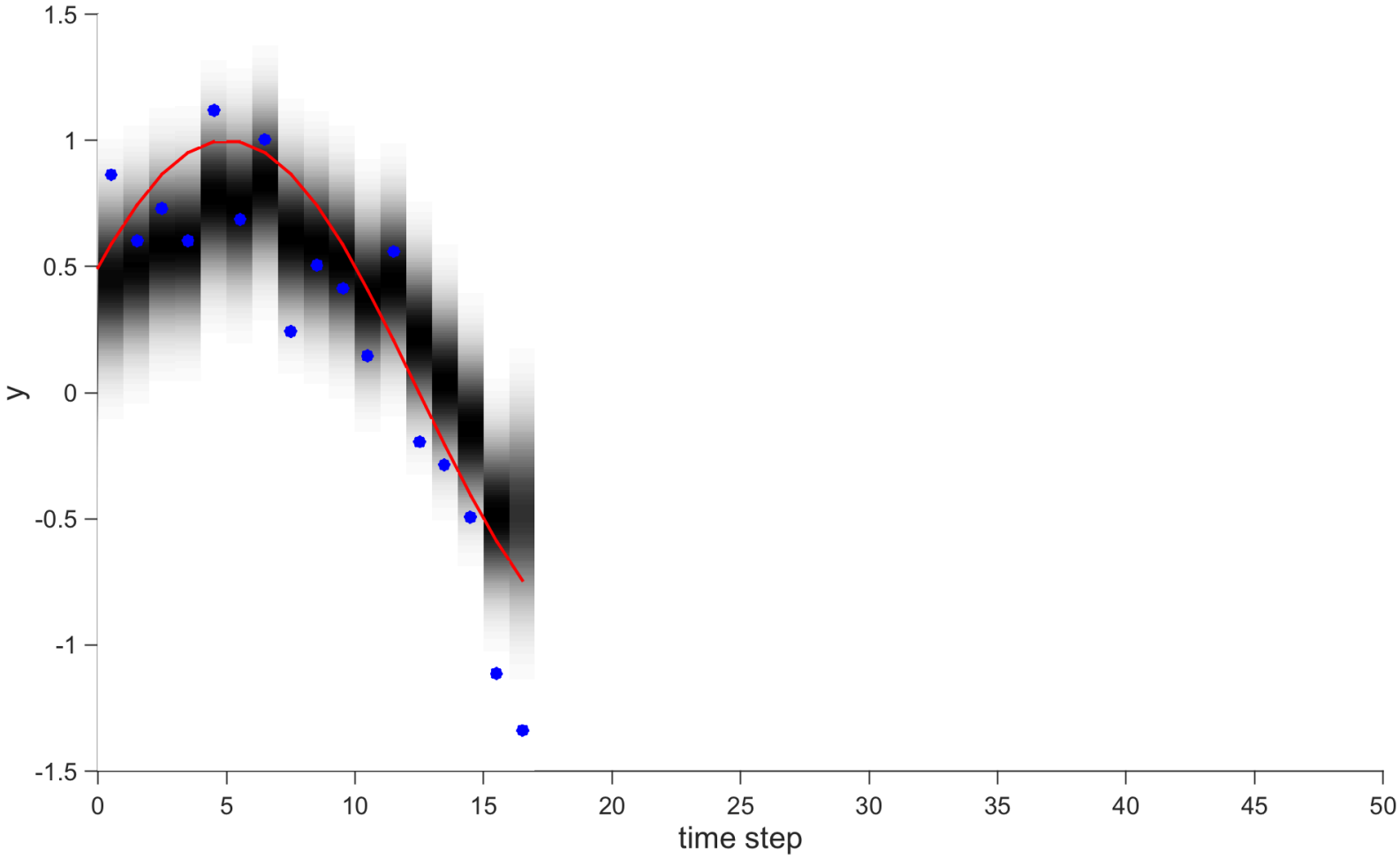
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



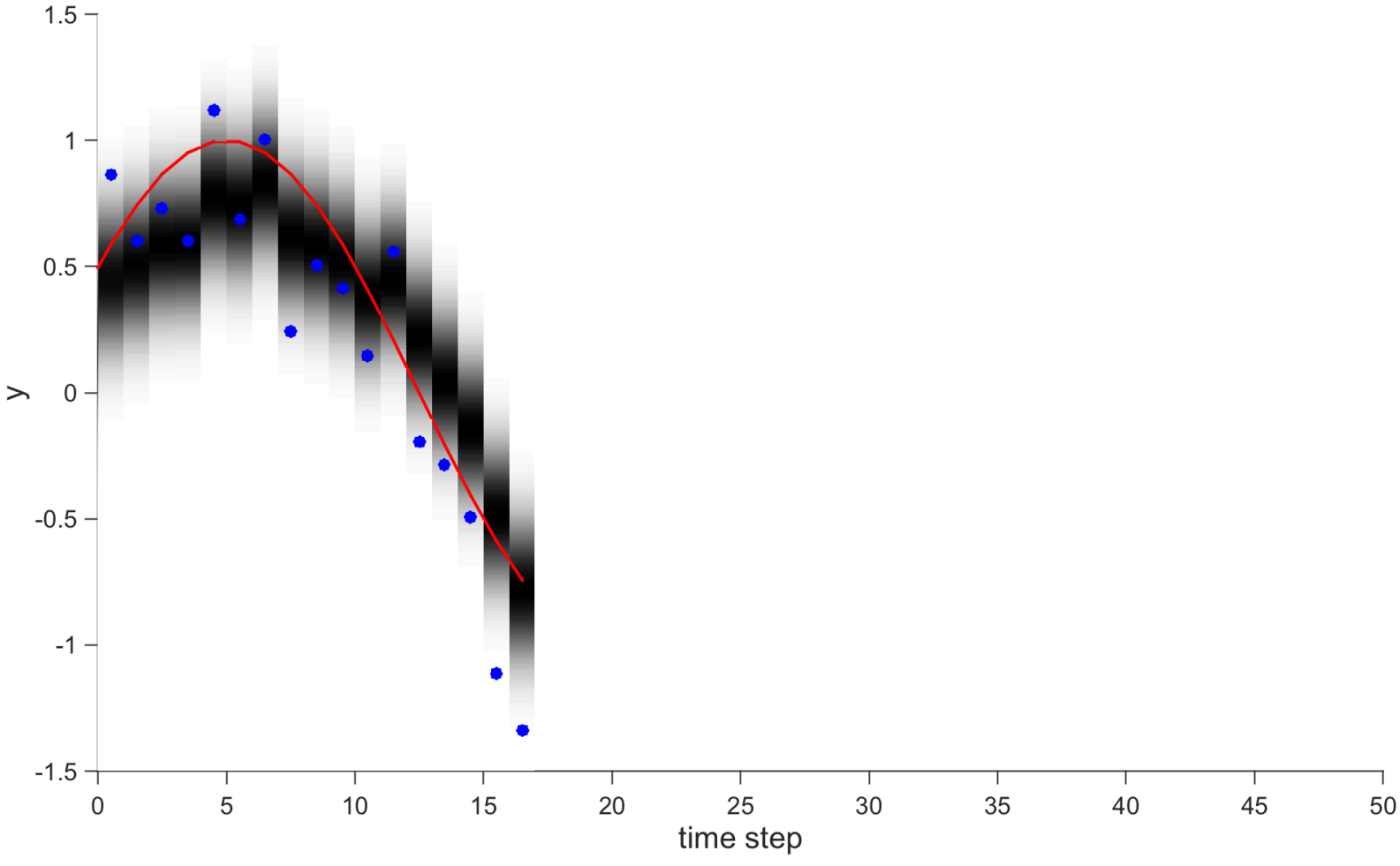
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



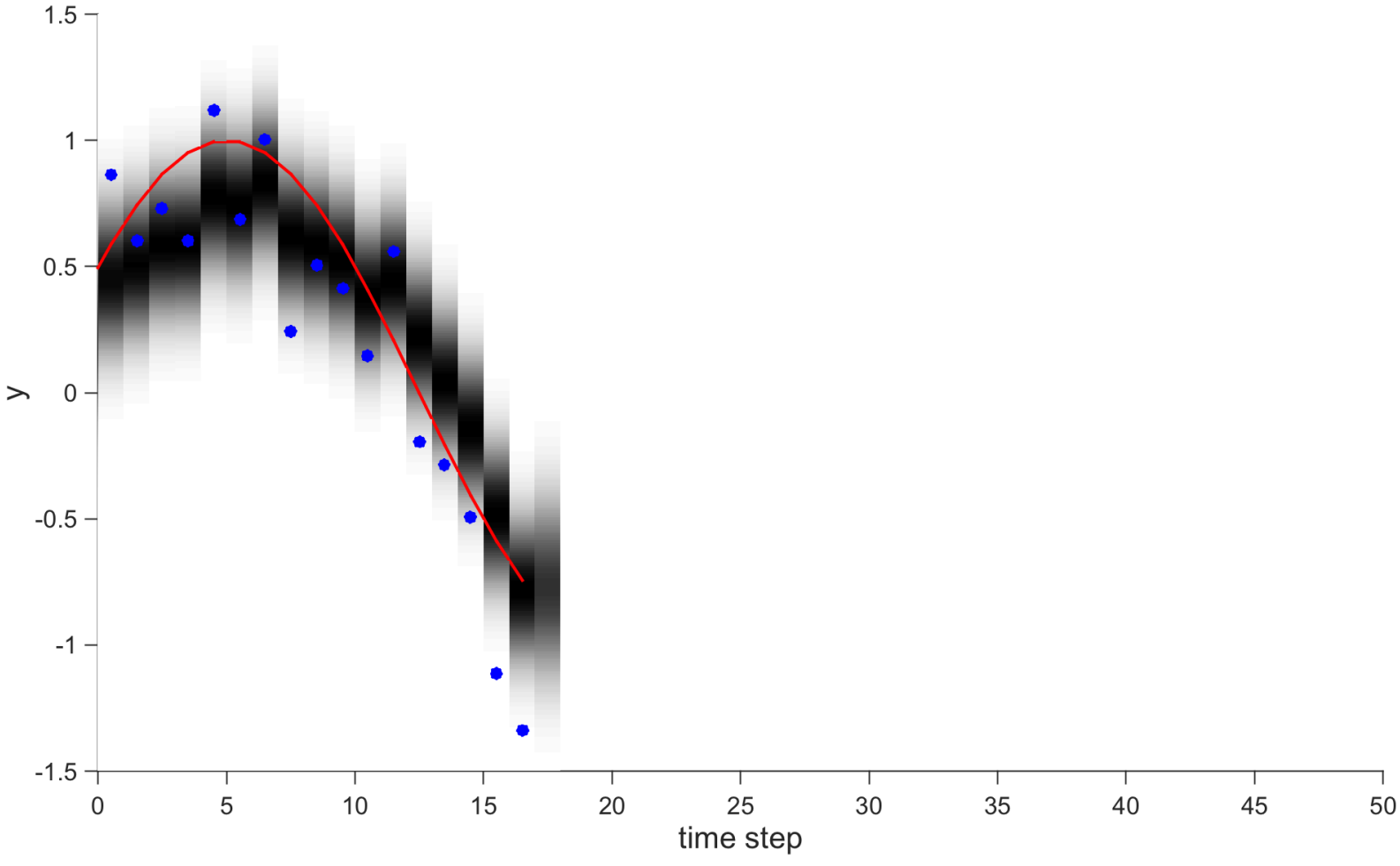
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



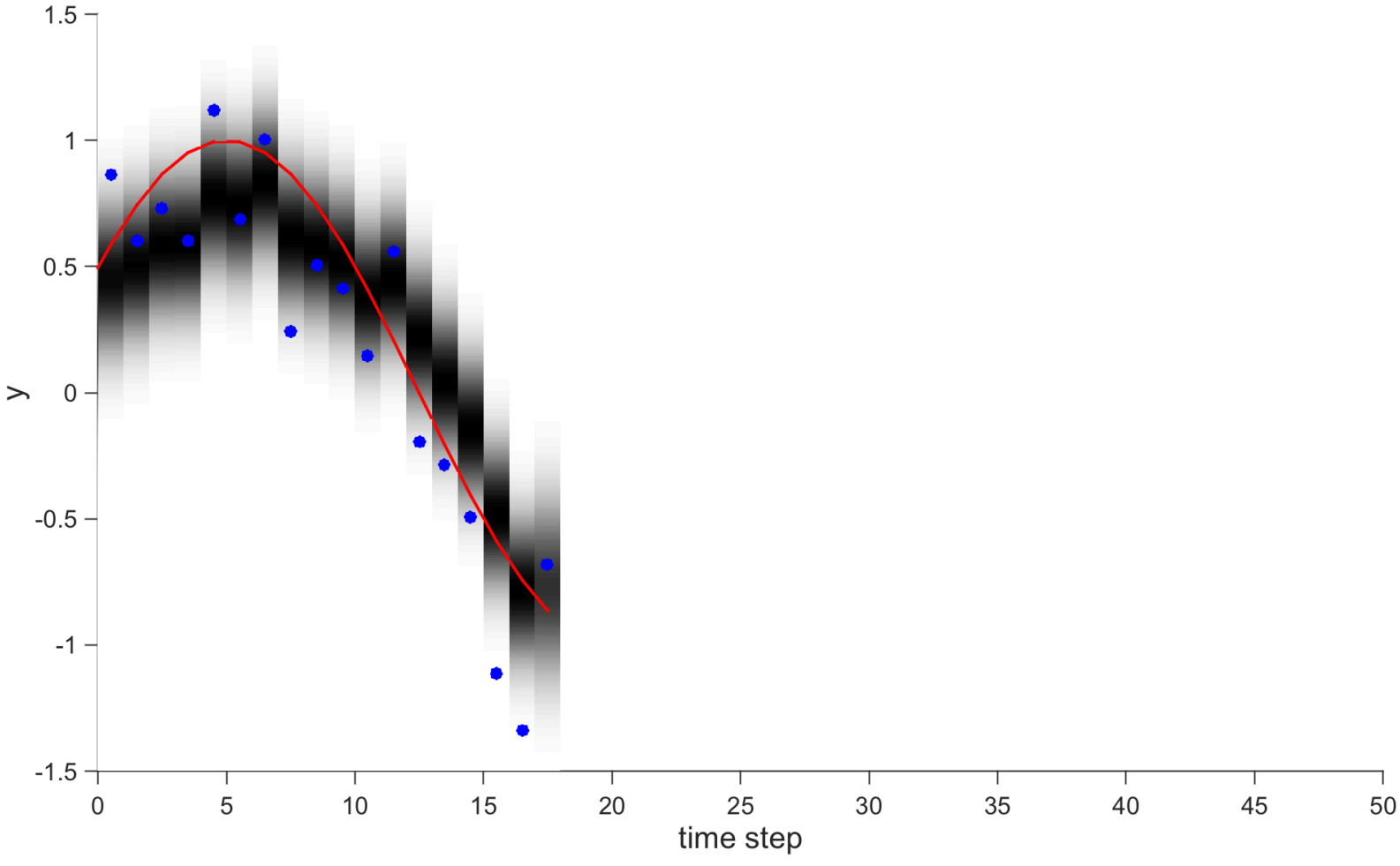
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



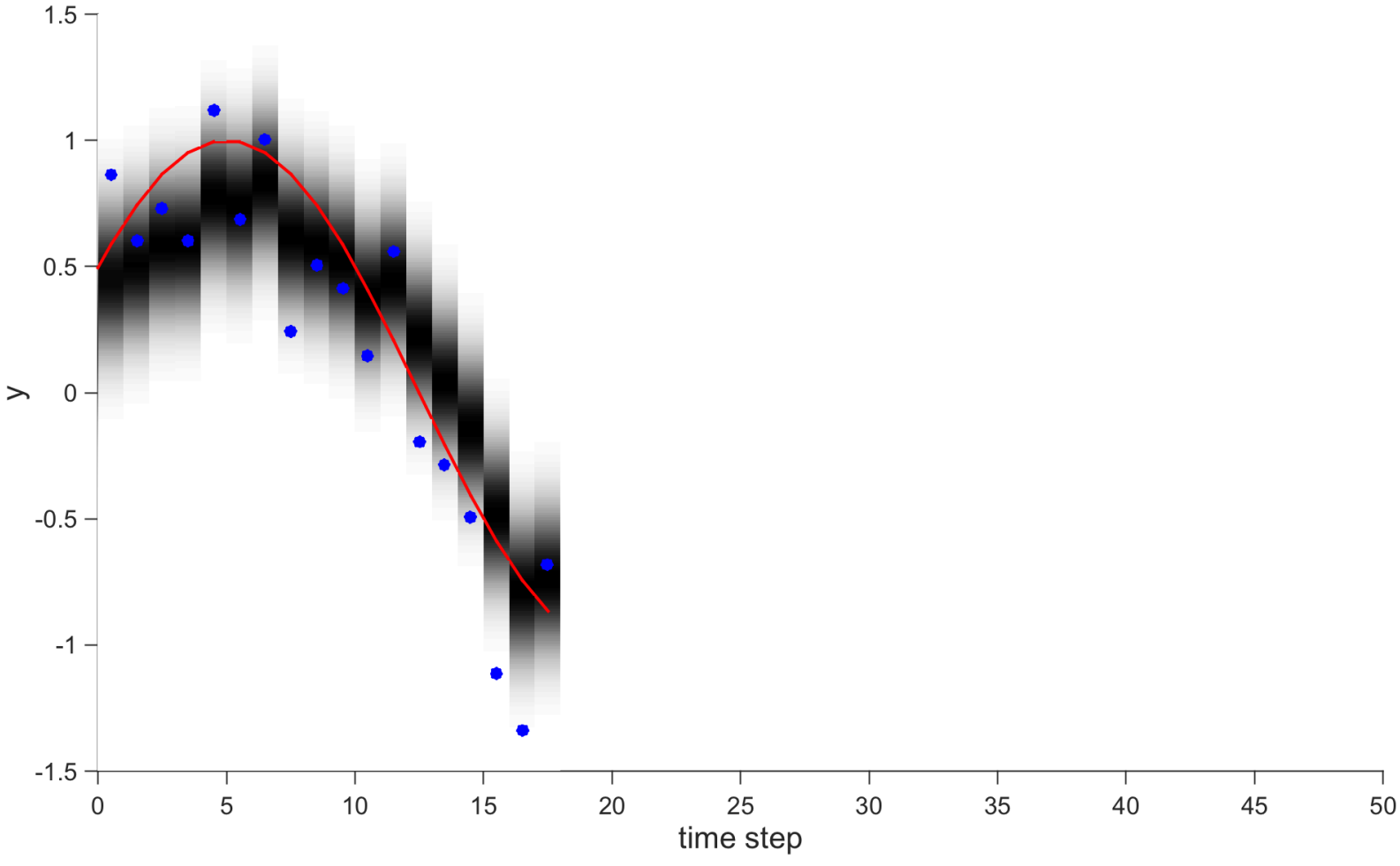
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



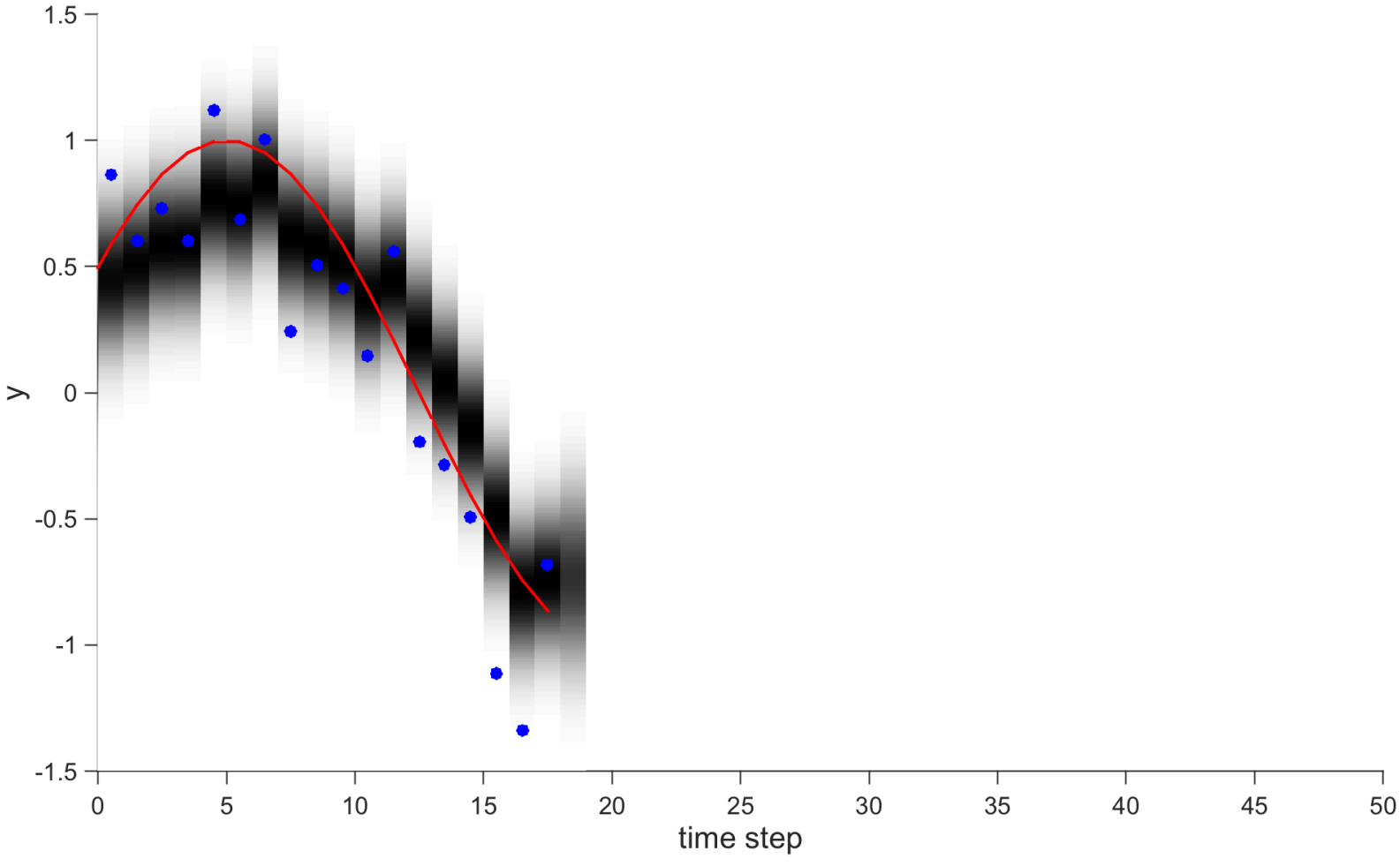
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



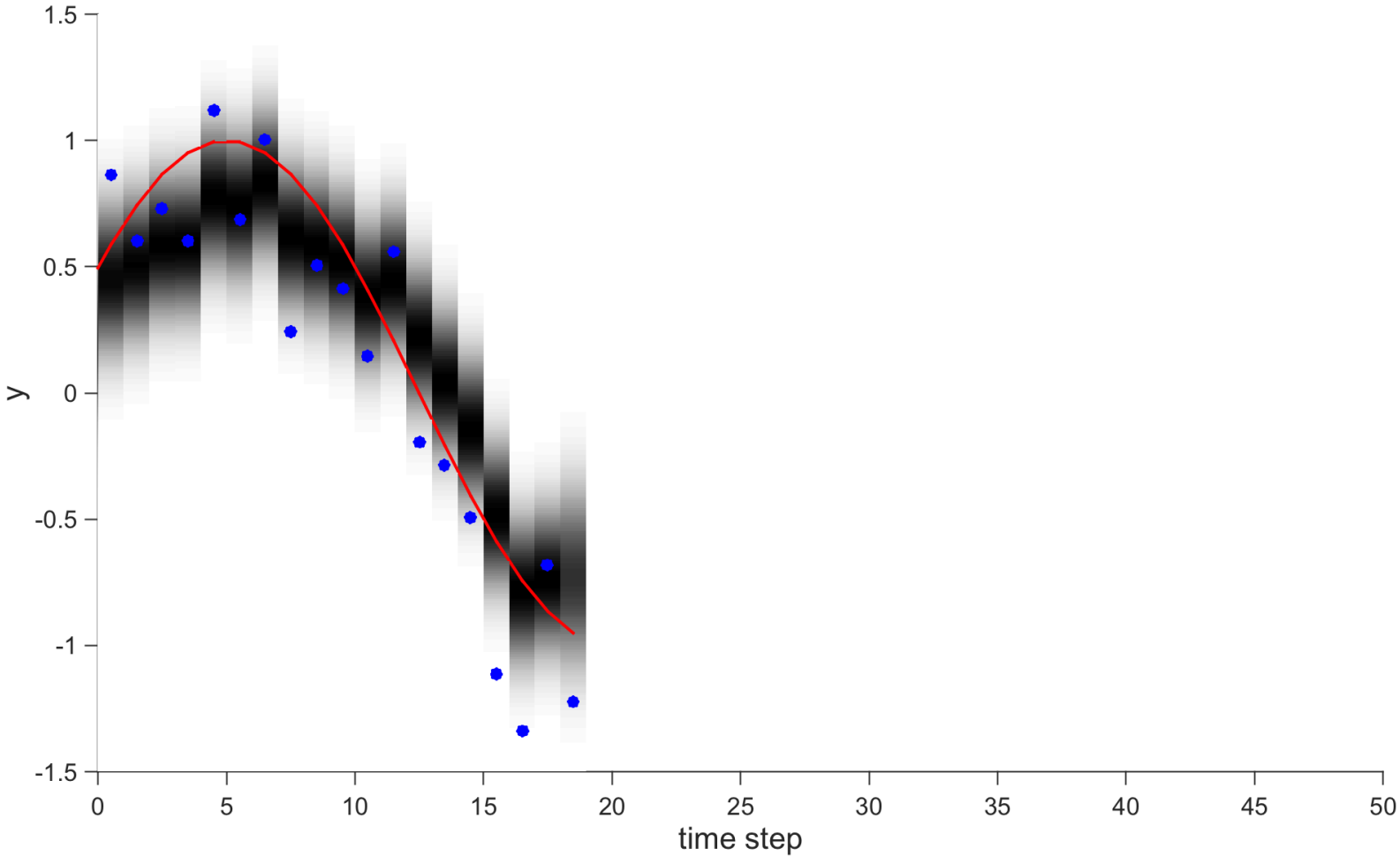
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



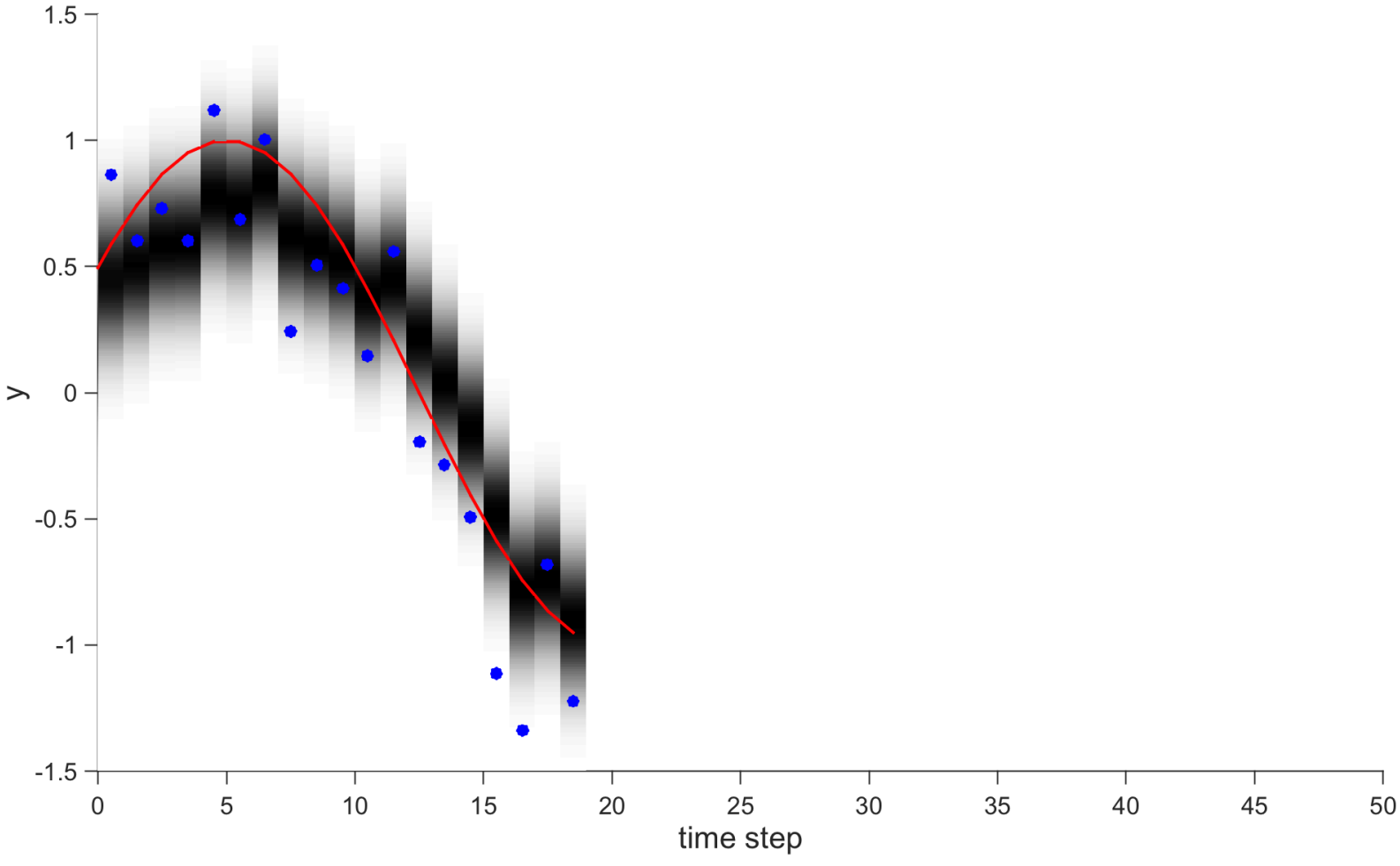
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



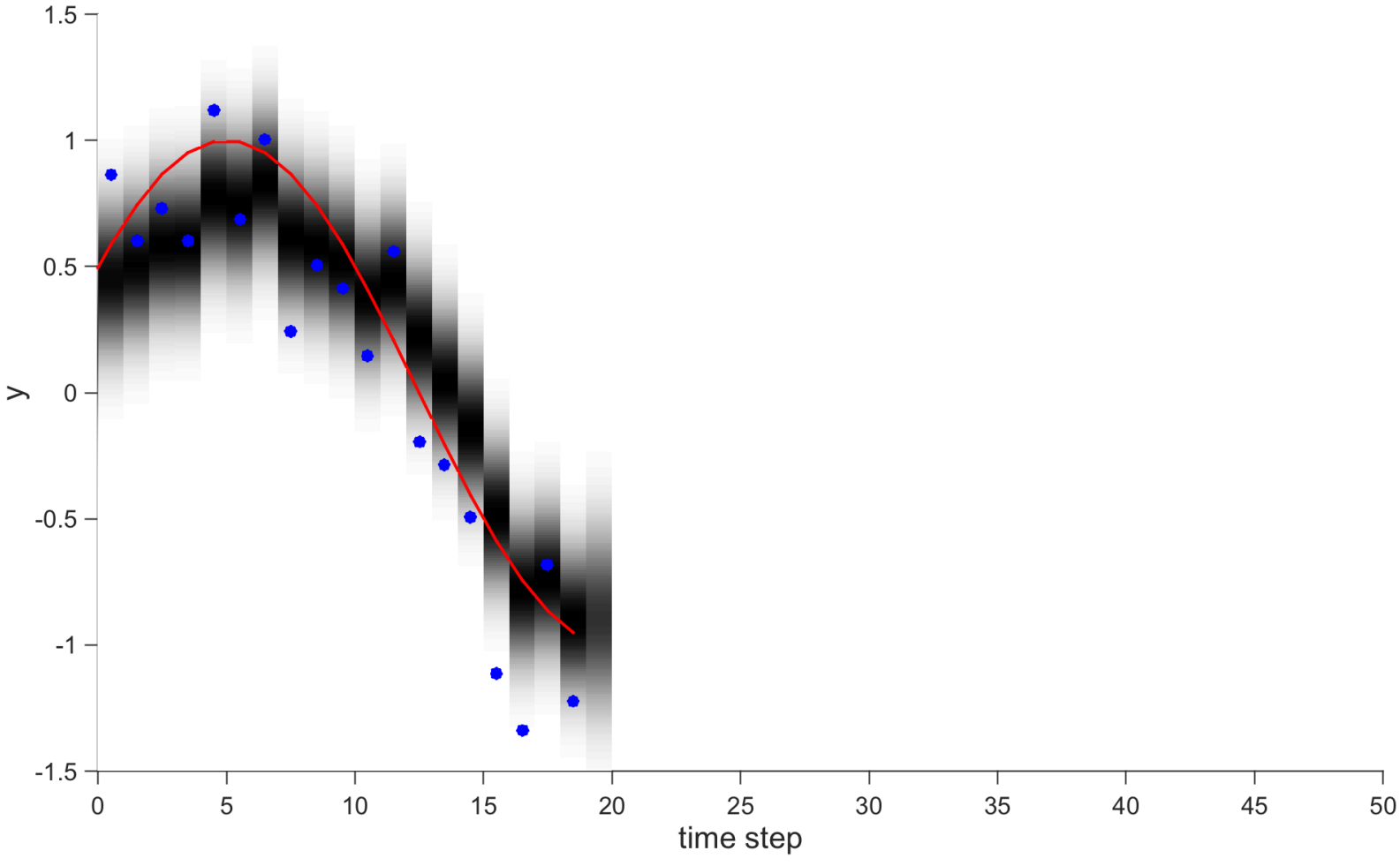
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



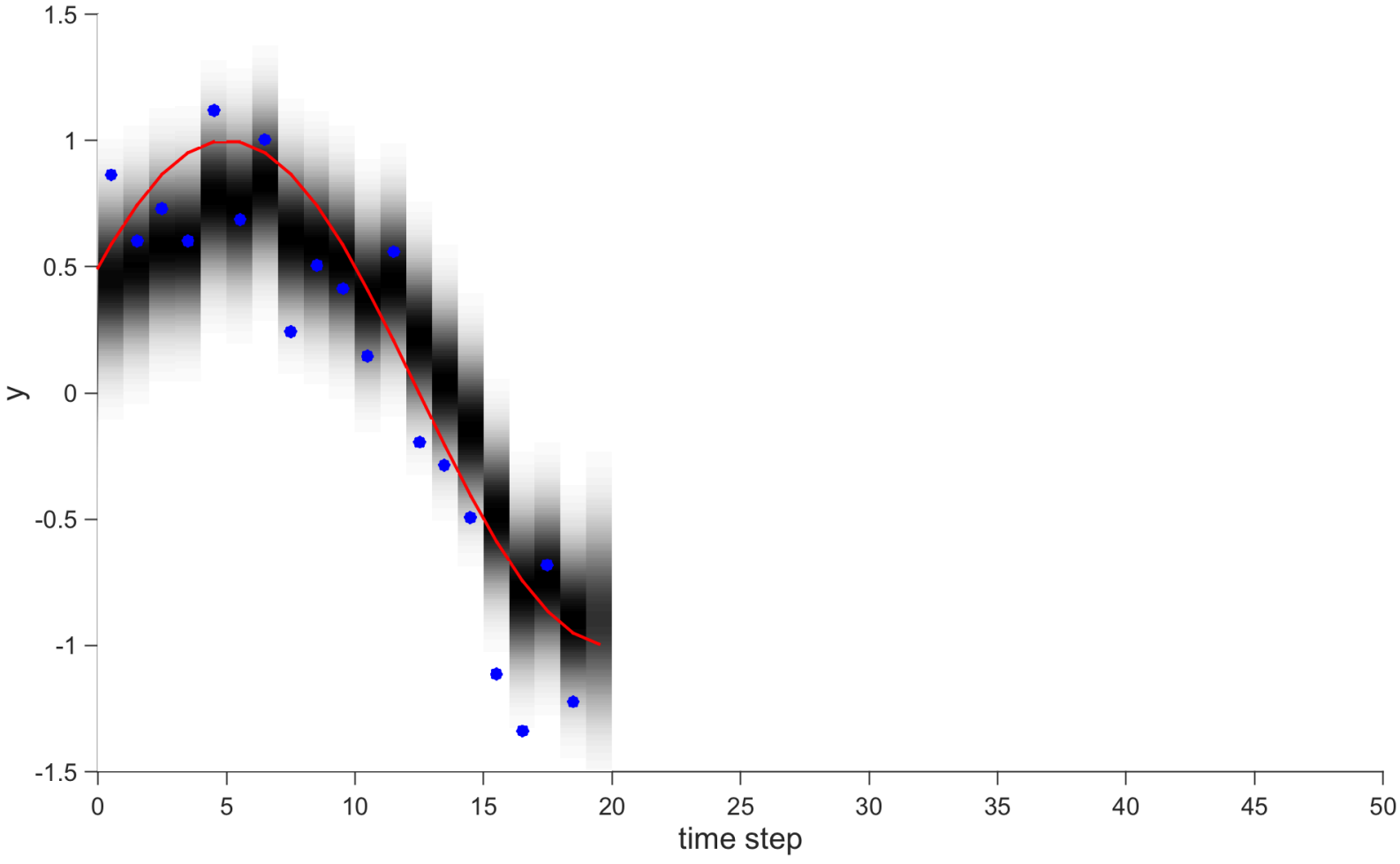
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



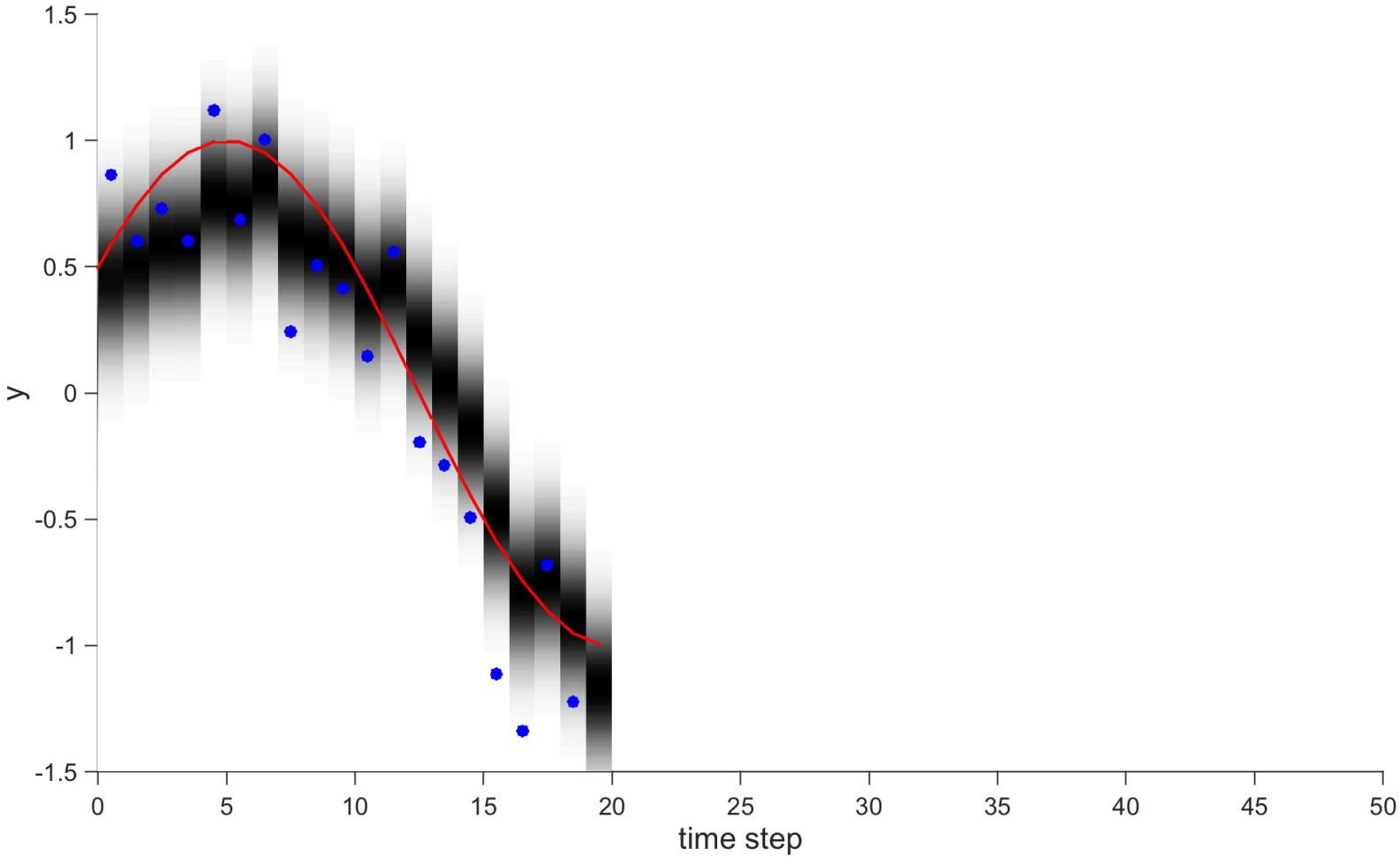
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



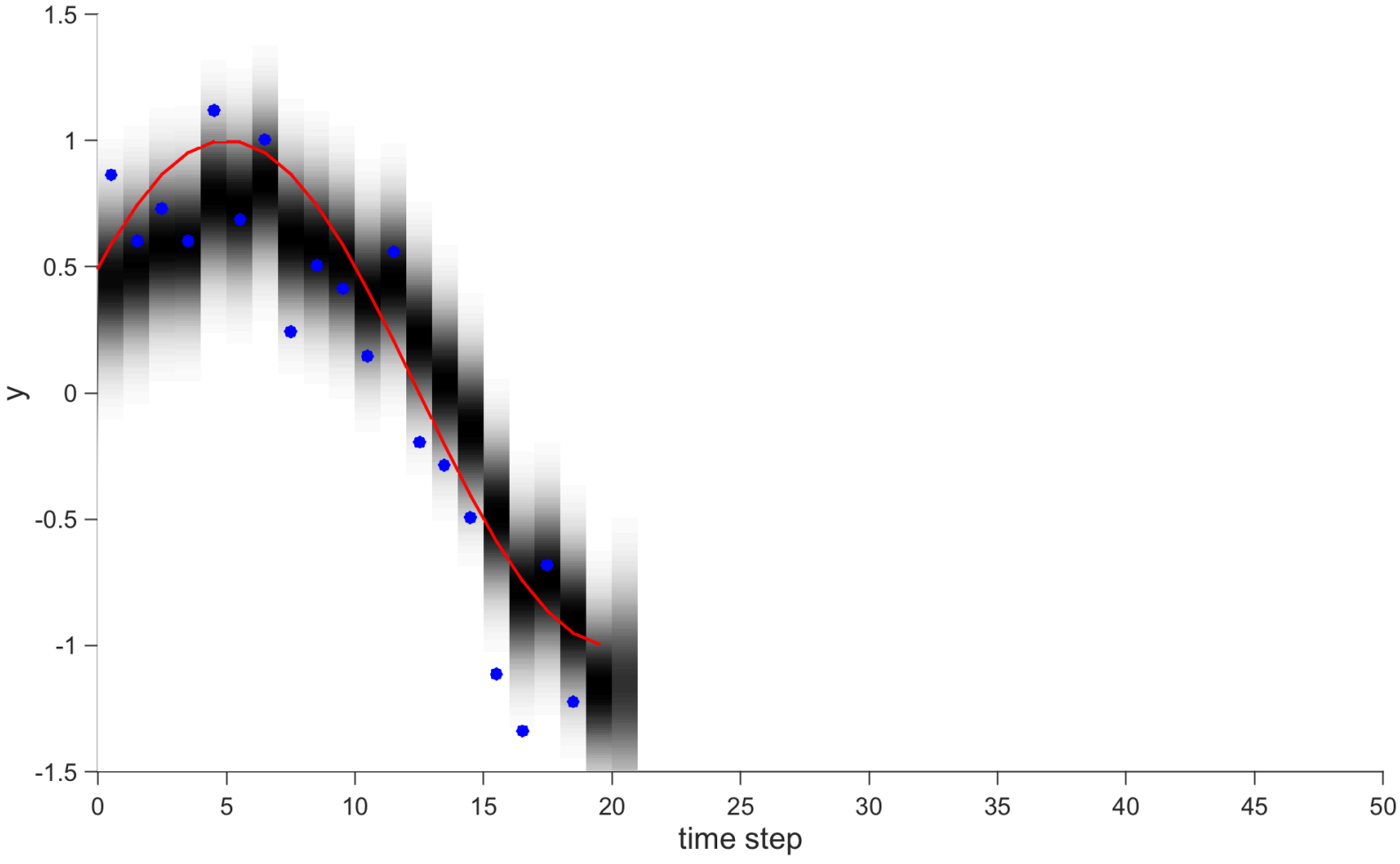
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



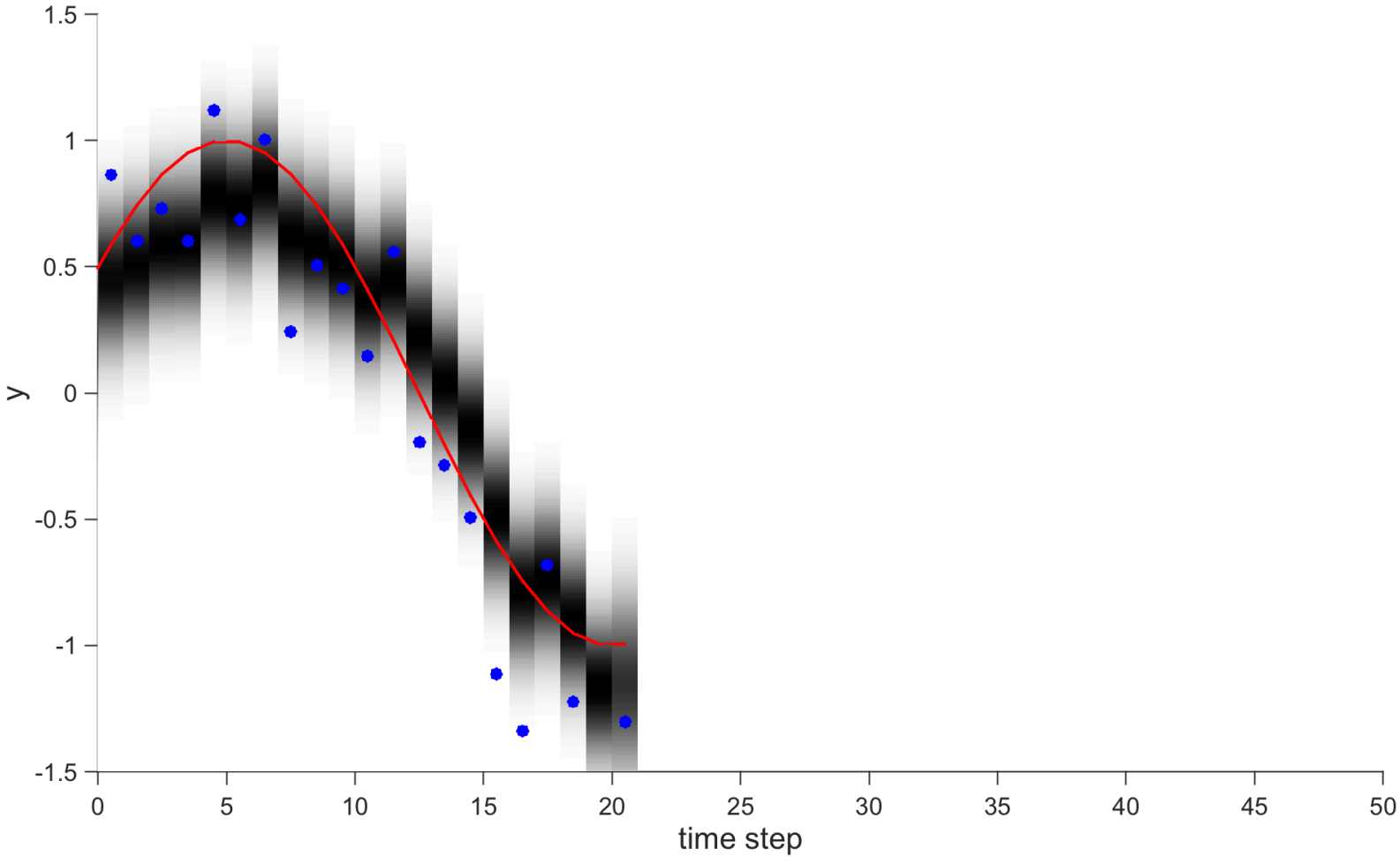
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



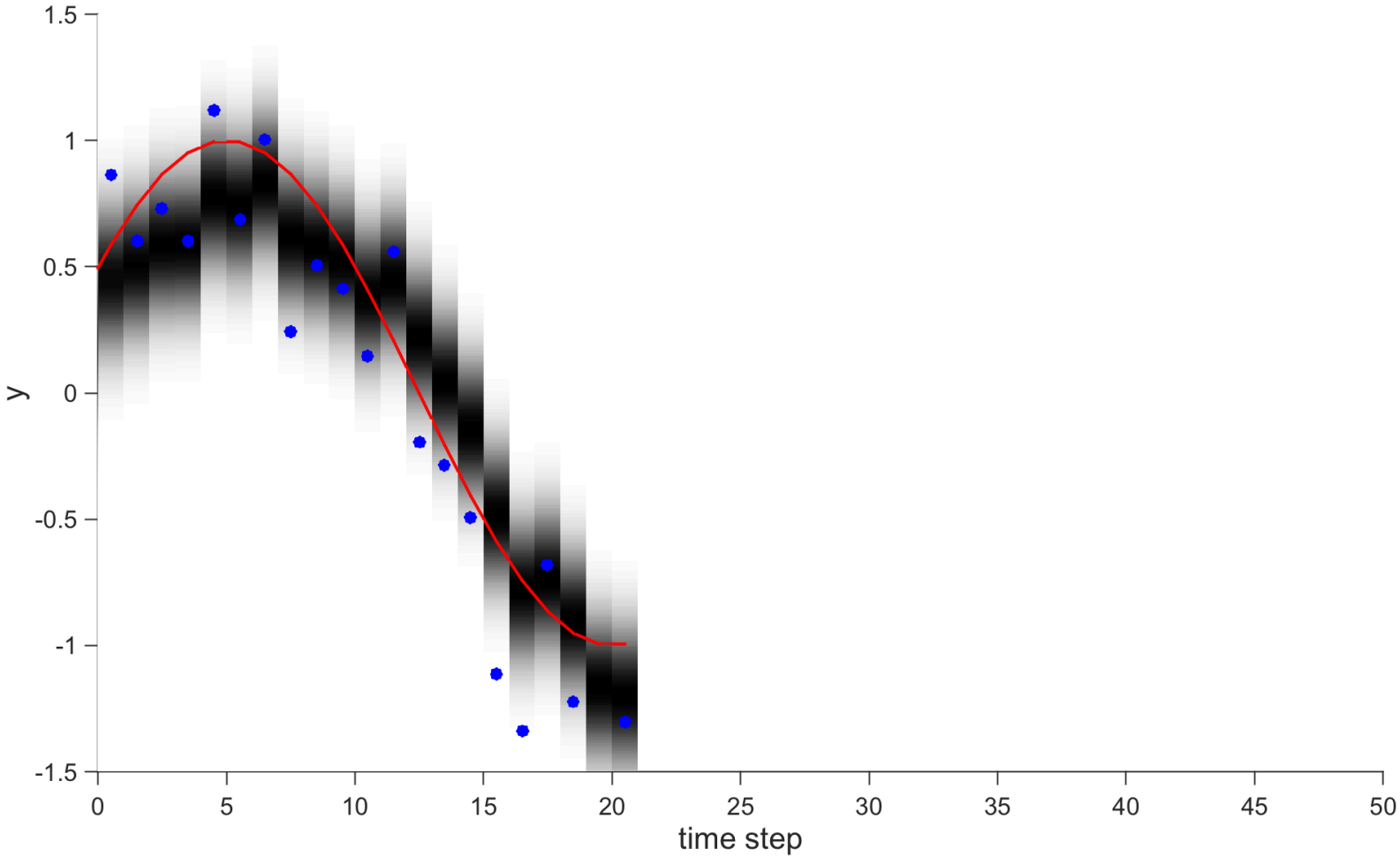
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



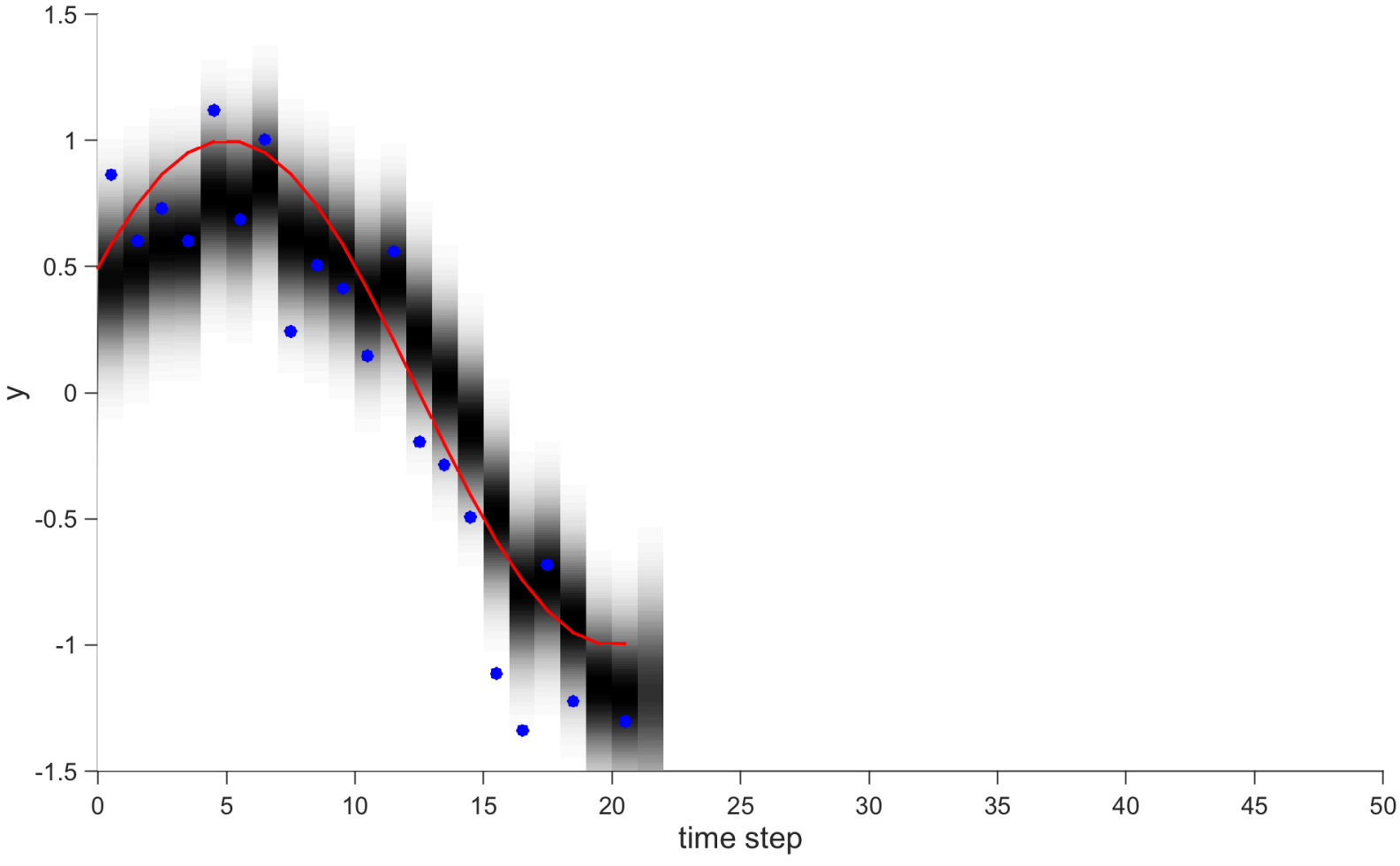
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



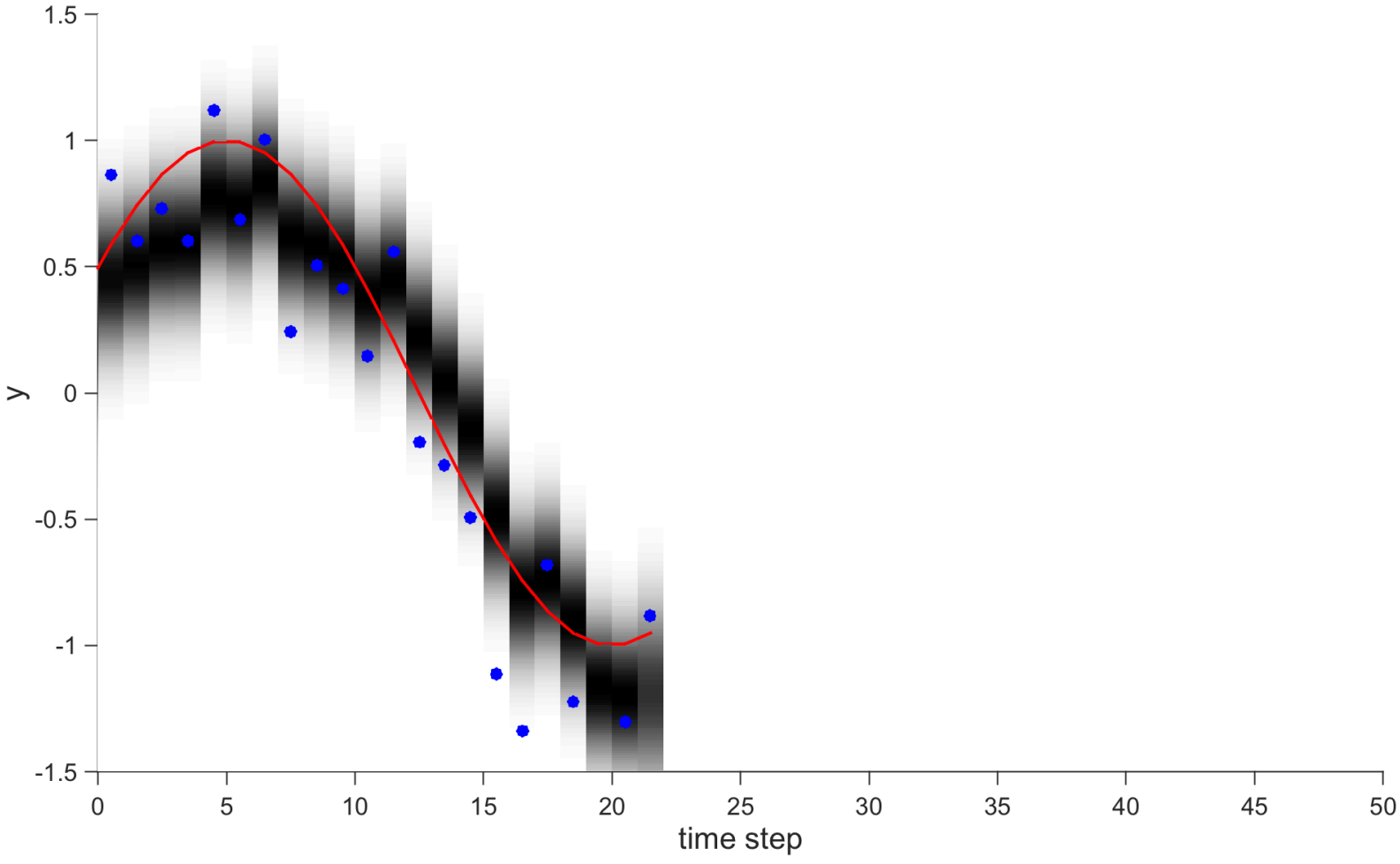
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



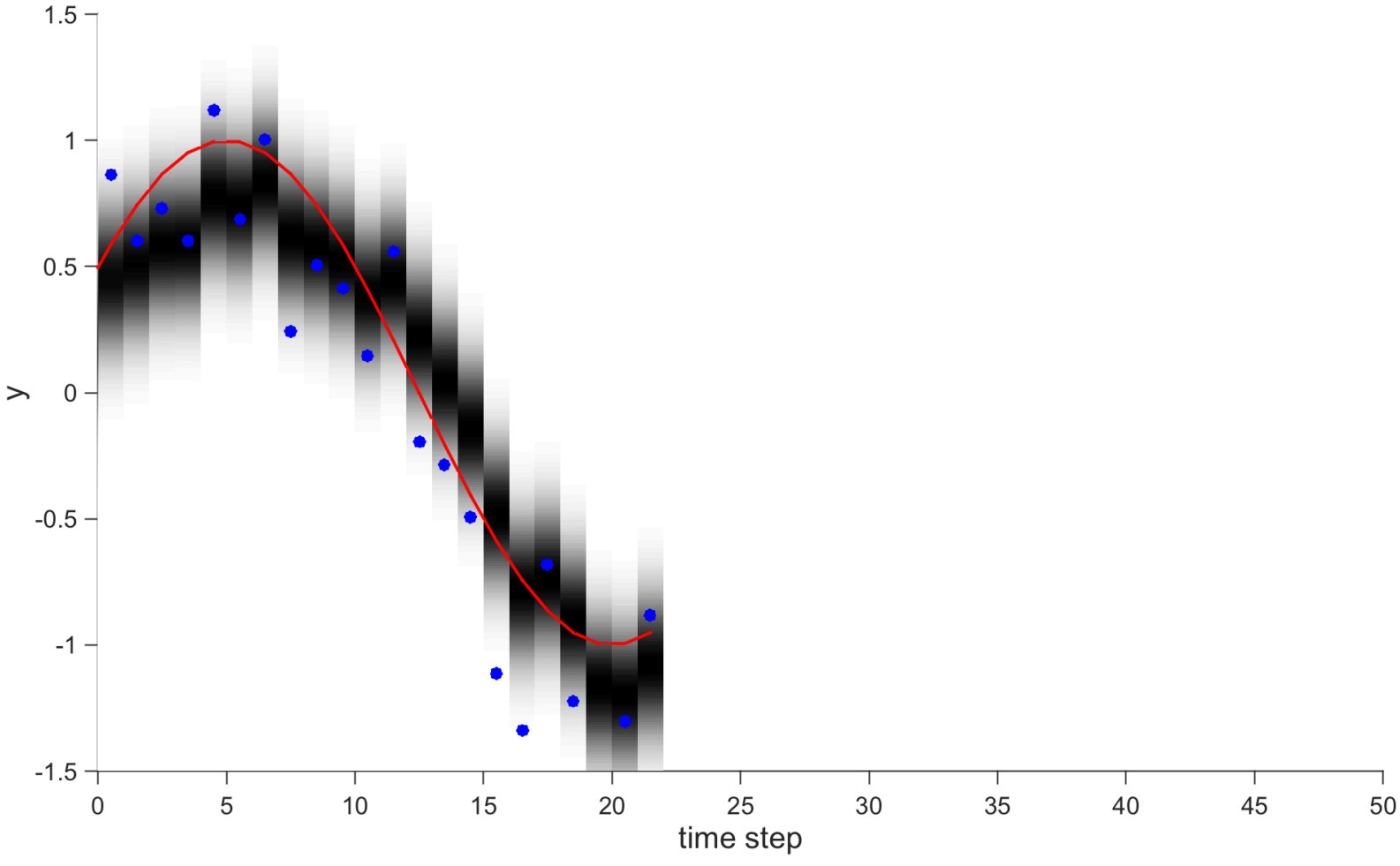
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



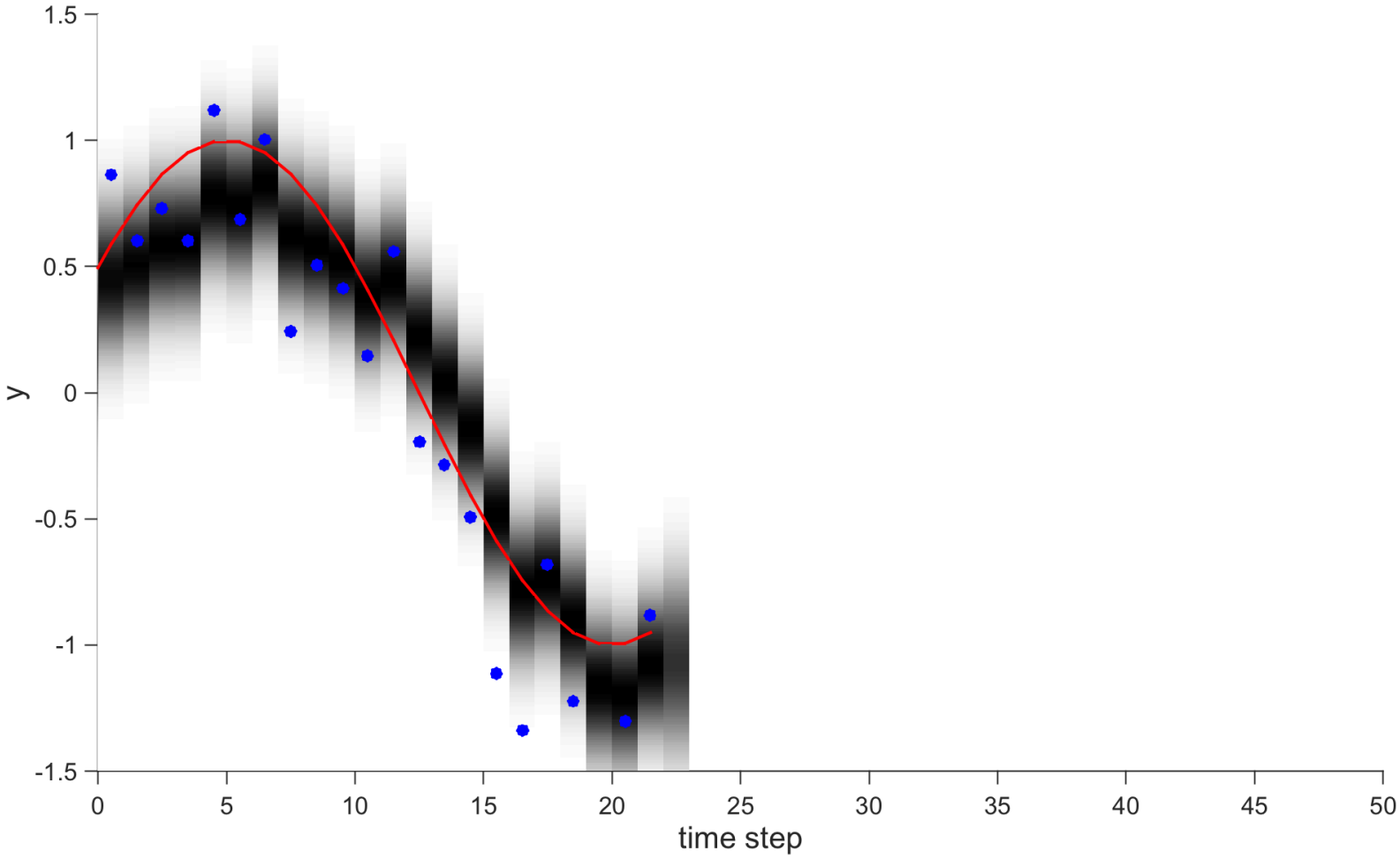
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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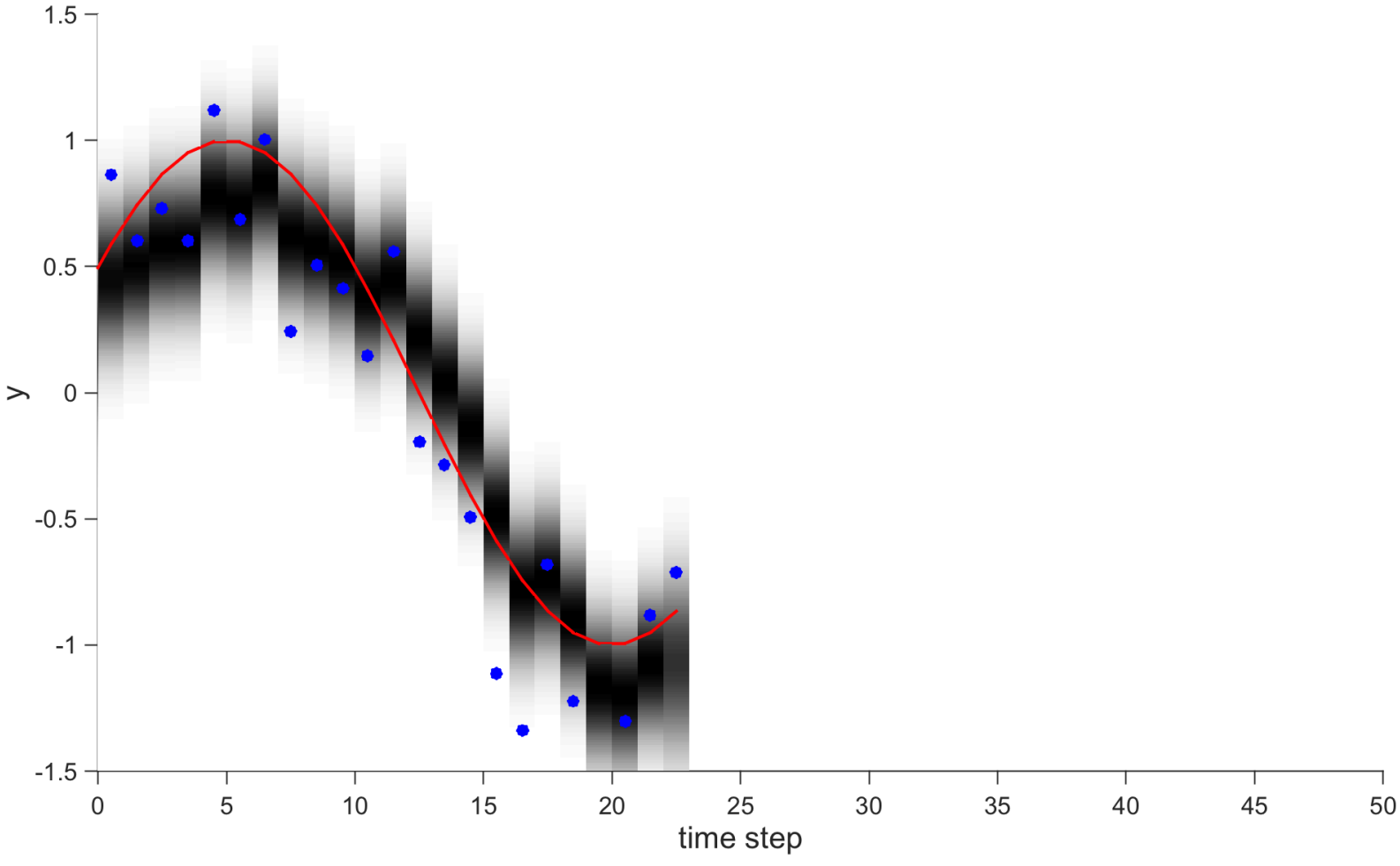
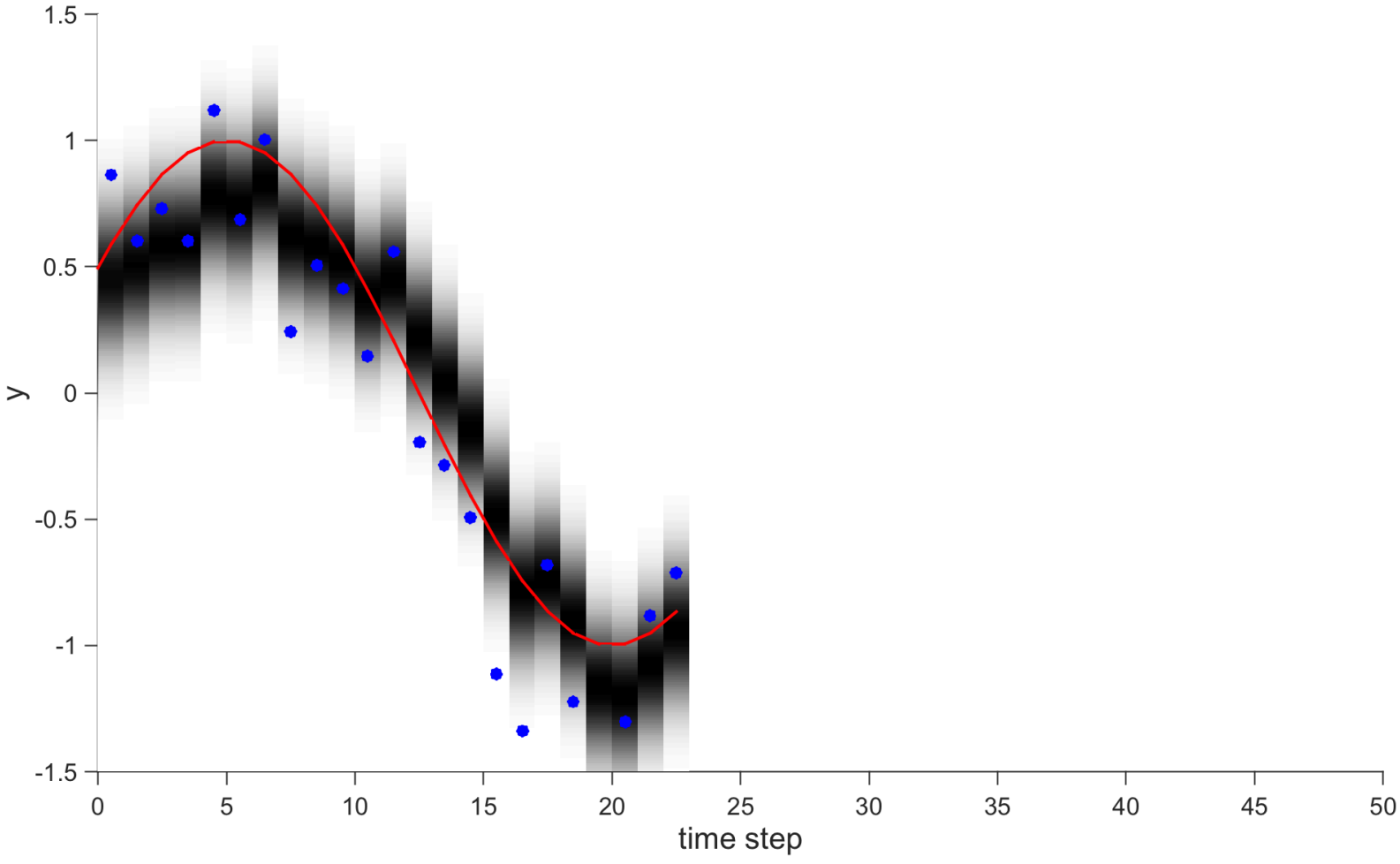


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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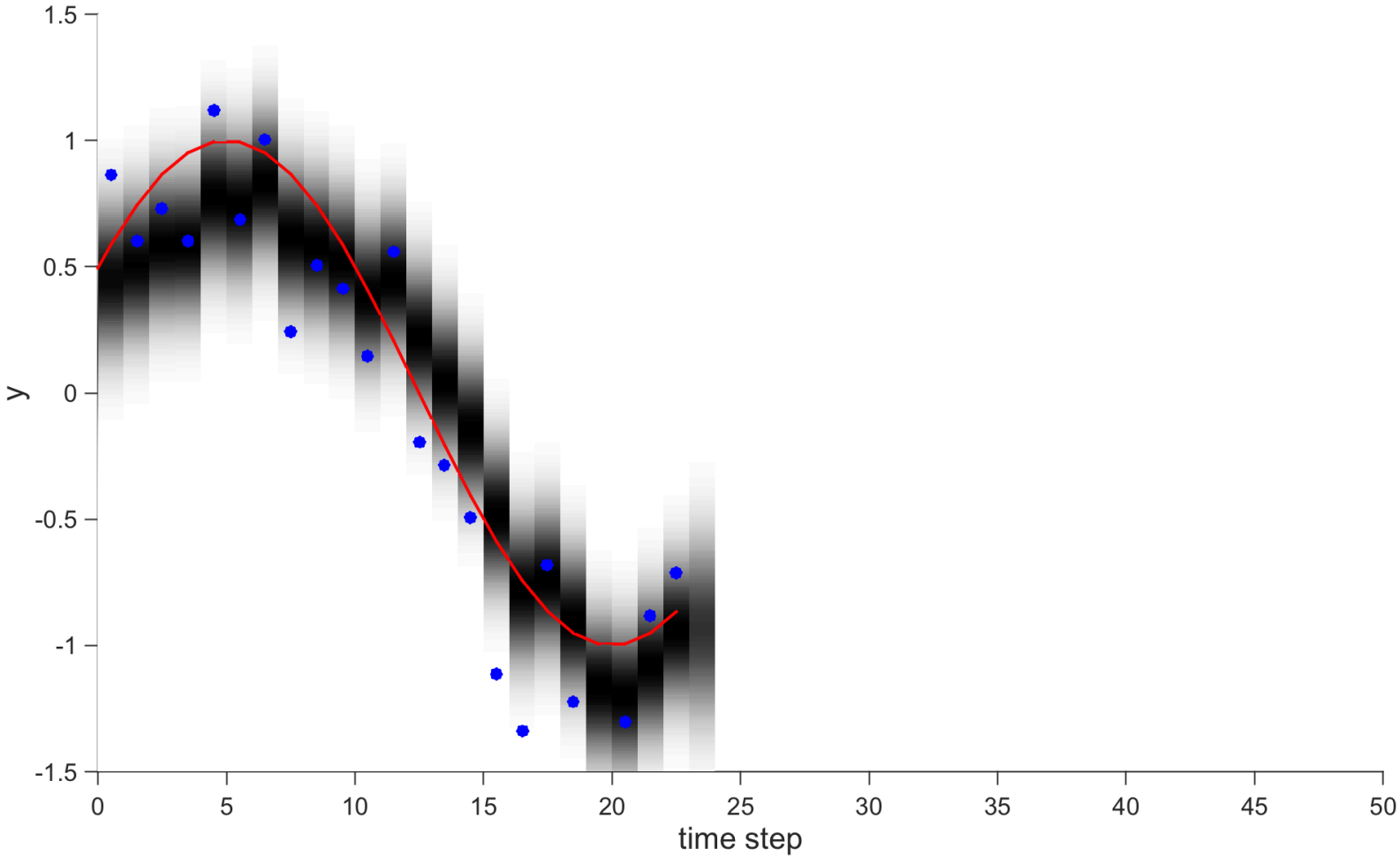

Kalman Filter Demo

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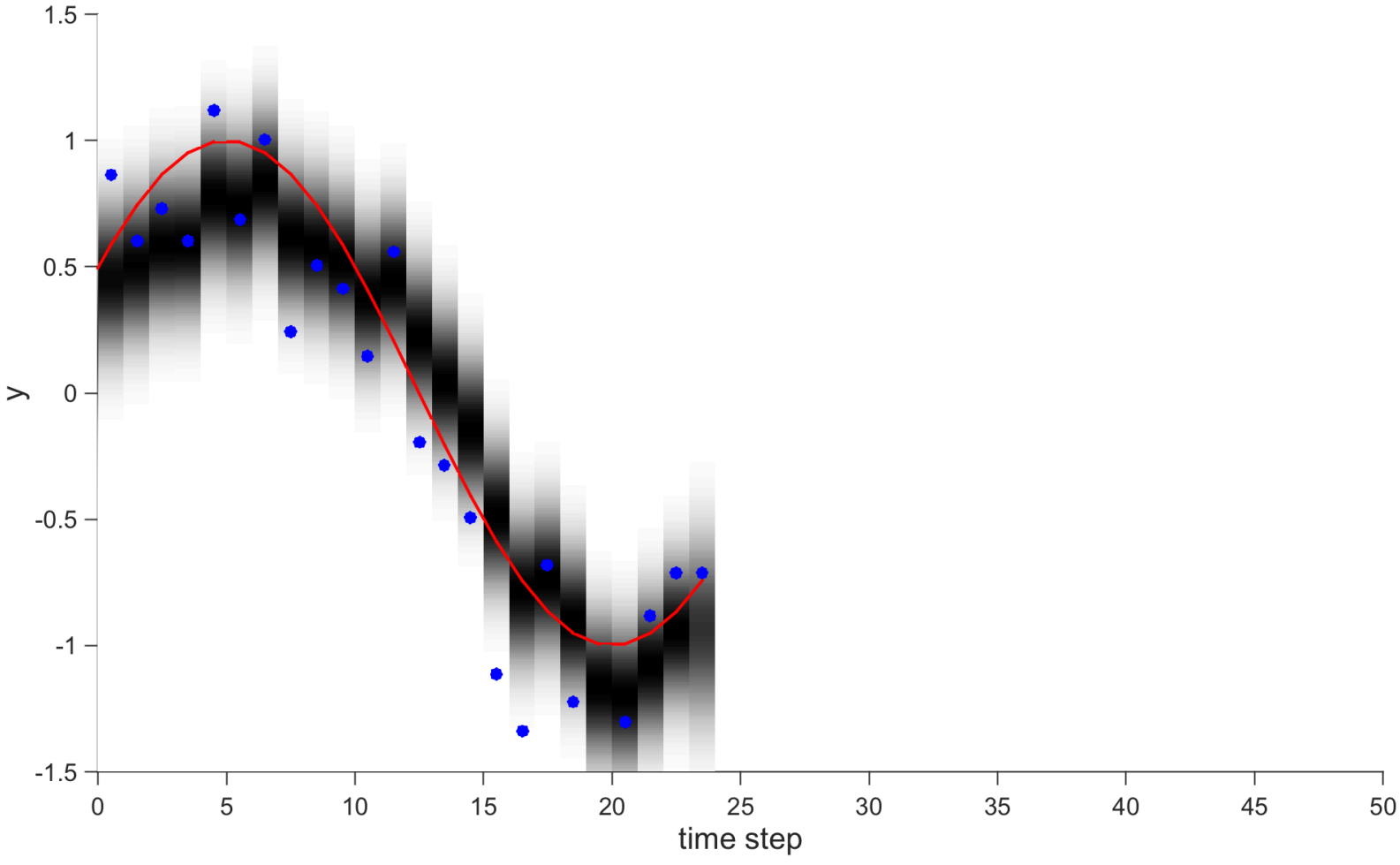

Kalman Filter Demo

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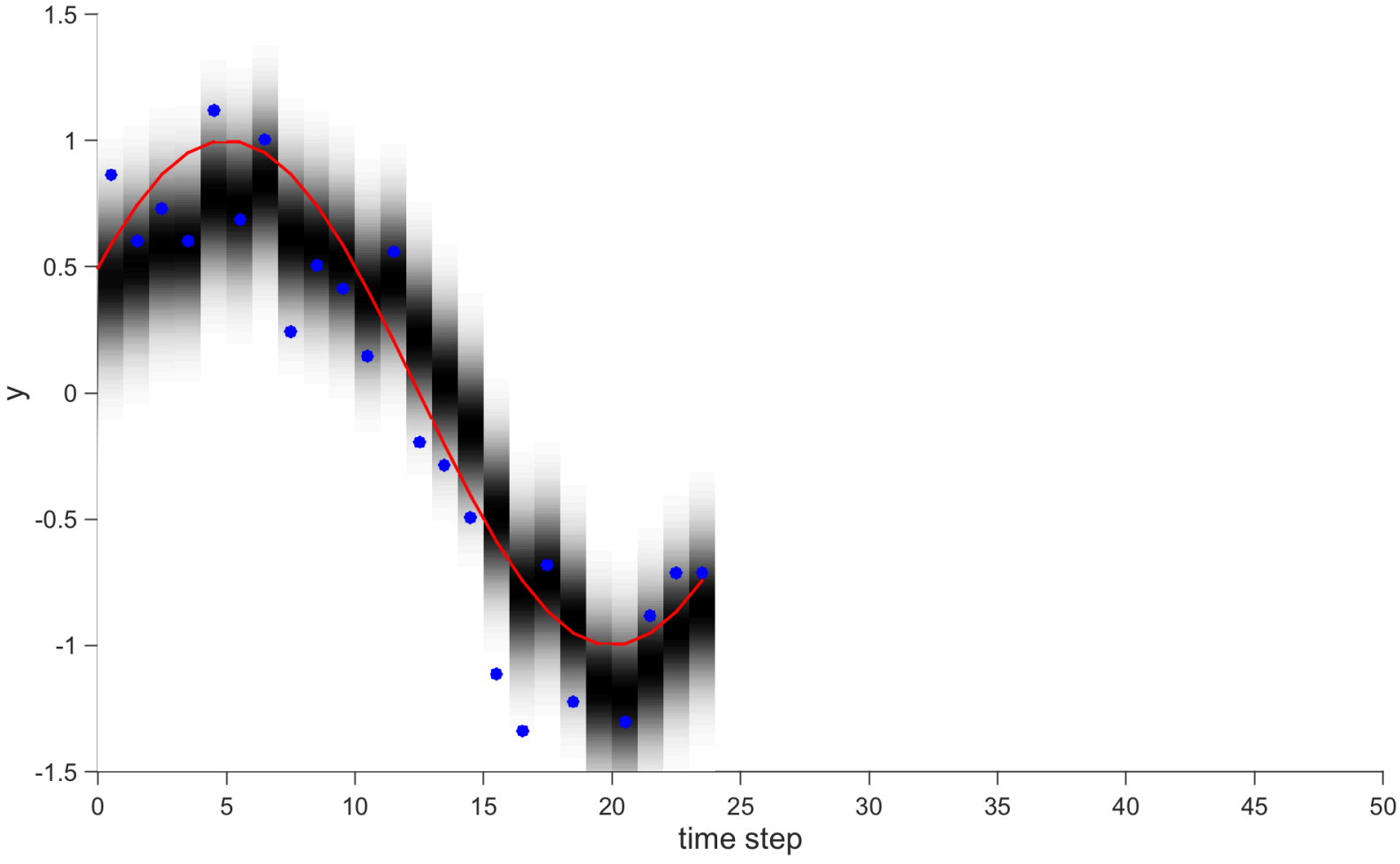

Kalman Filter Demo

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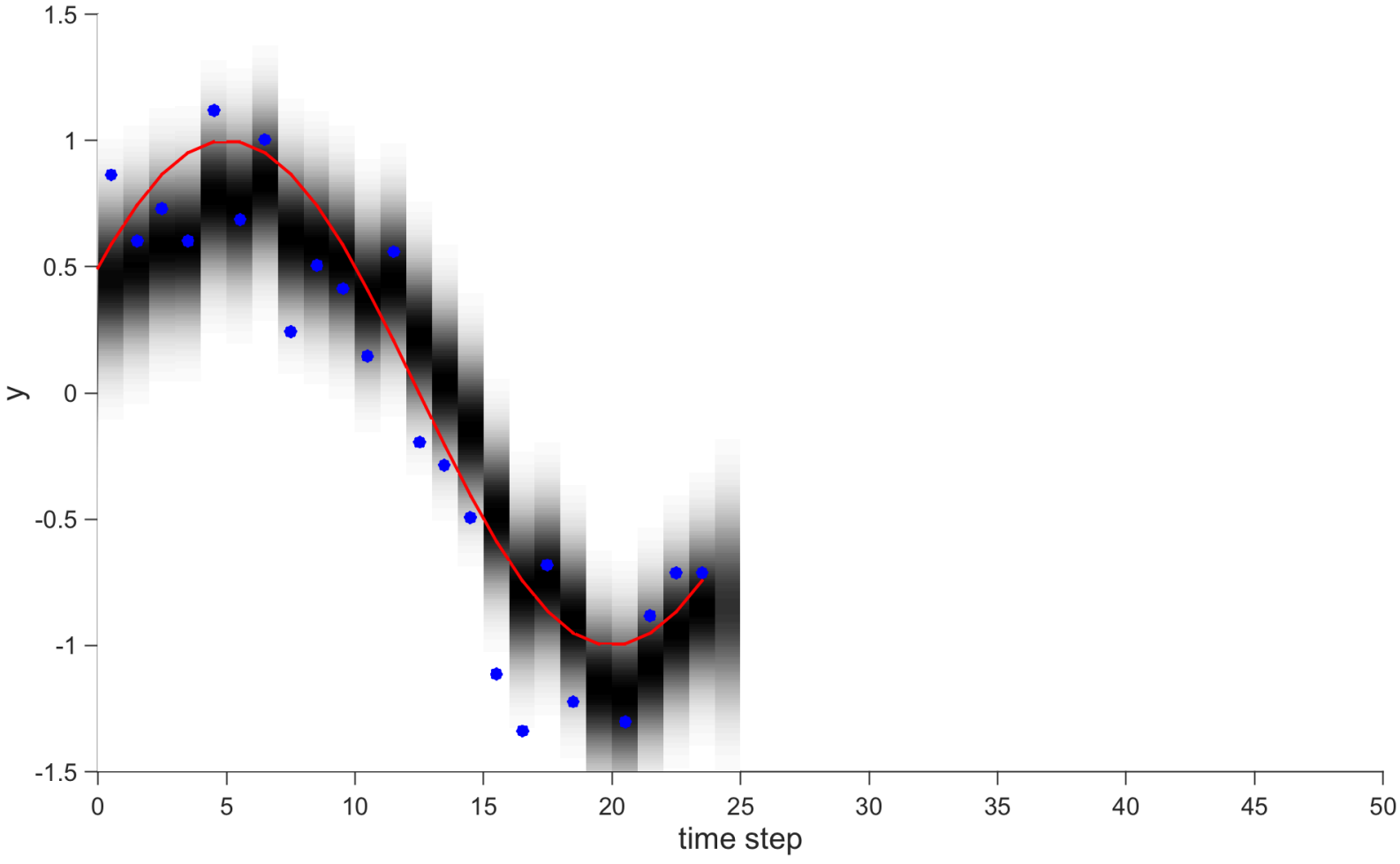

Kalman Filter Demo

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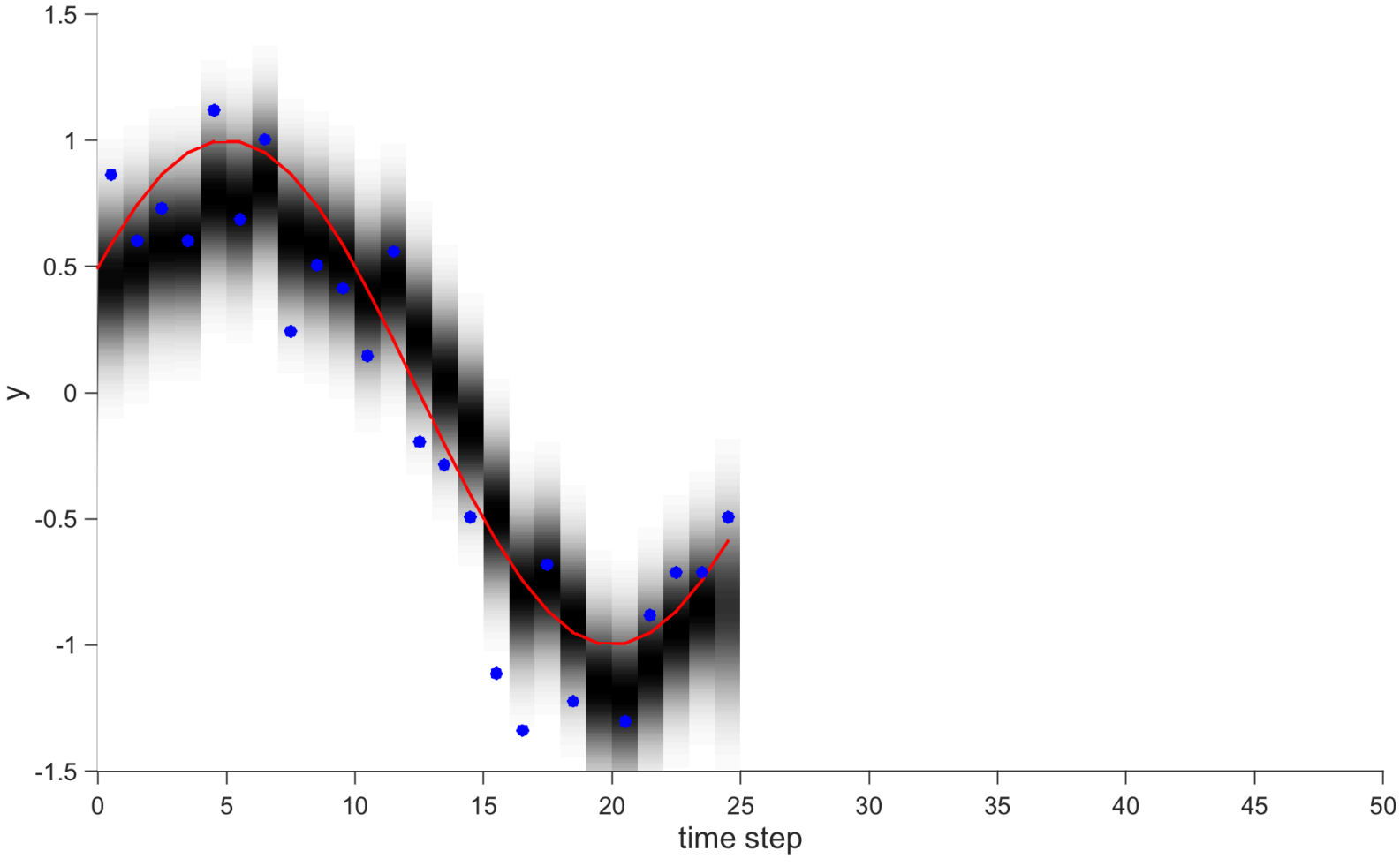

Kalman Filter Demo

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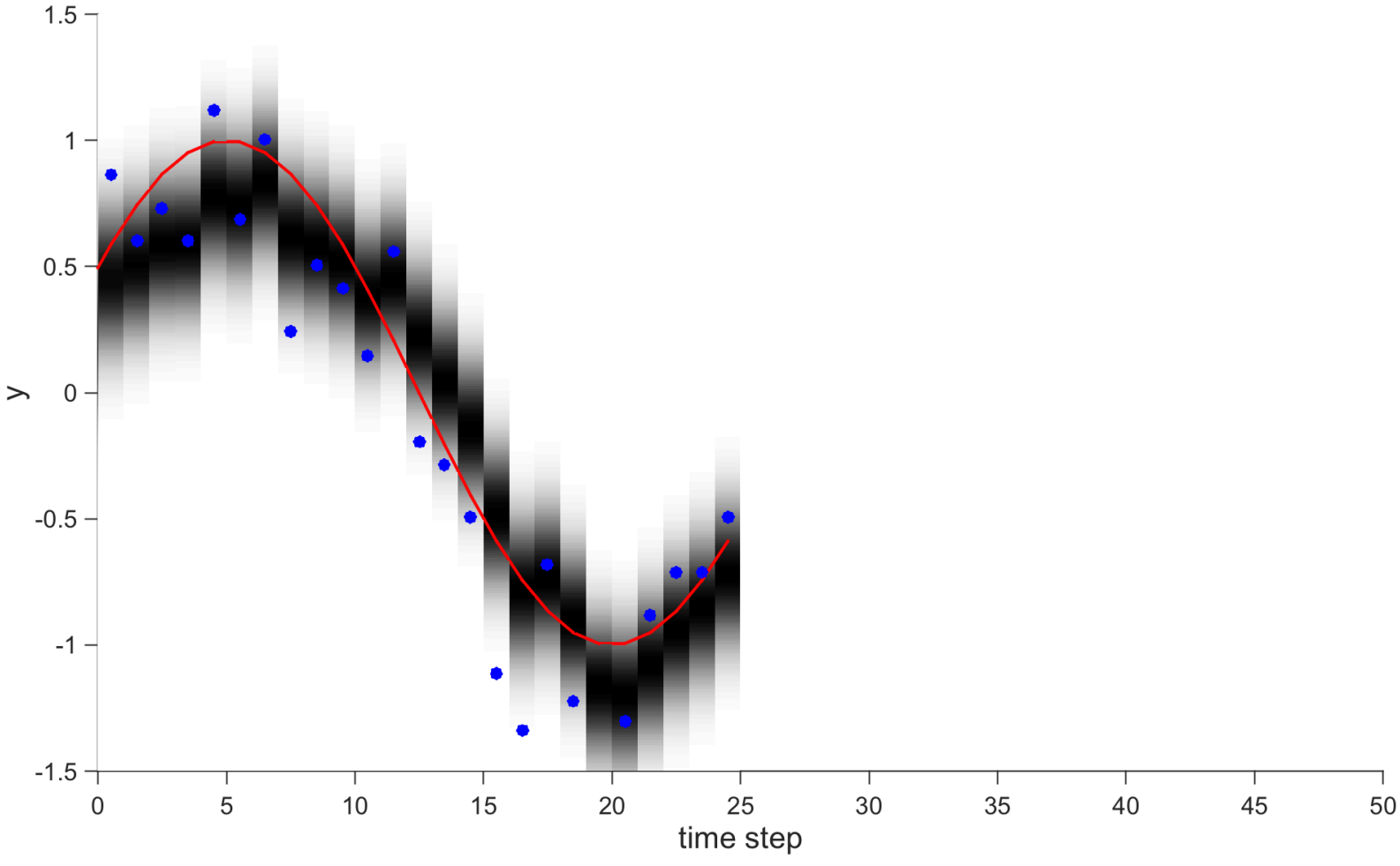
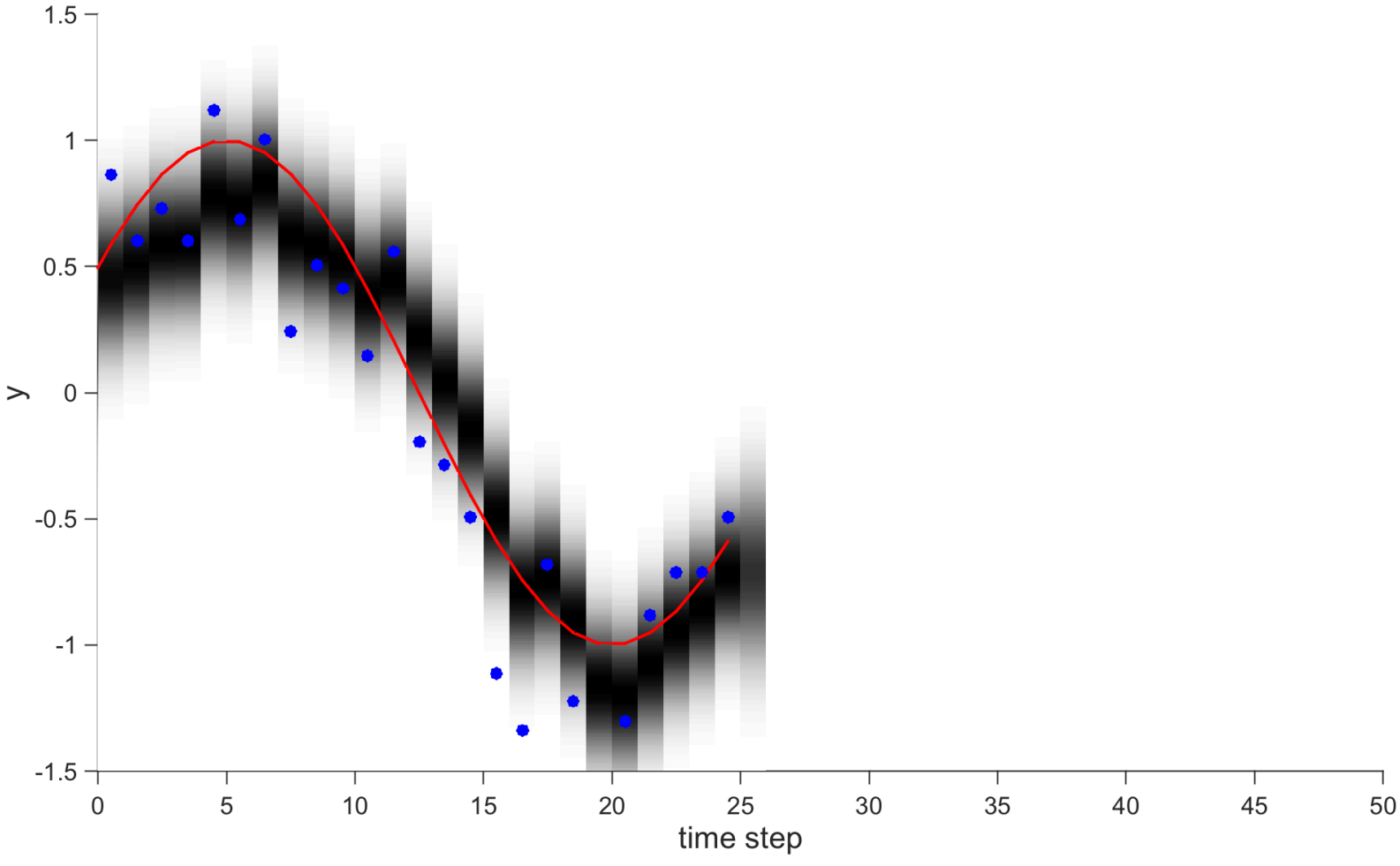


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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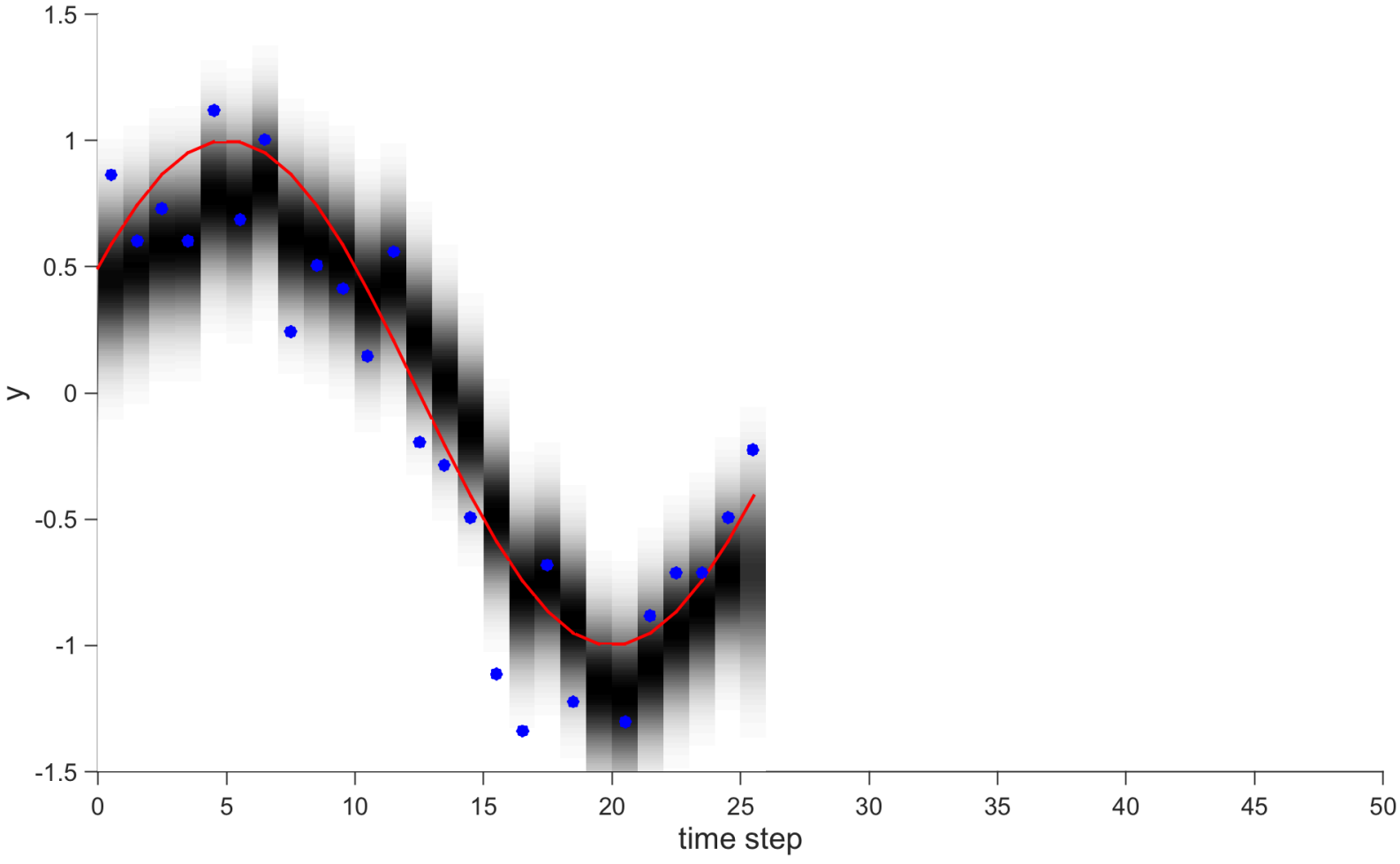

Kalman Filter Demo

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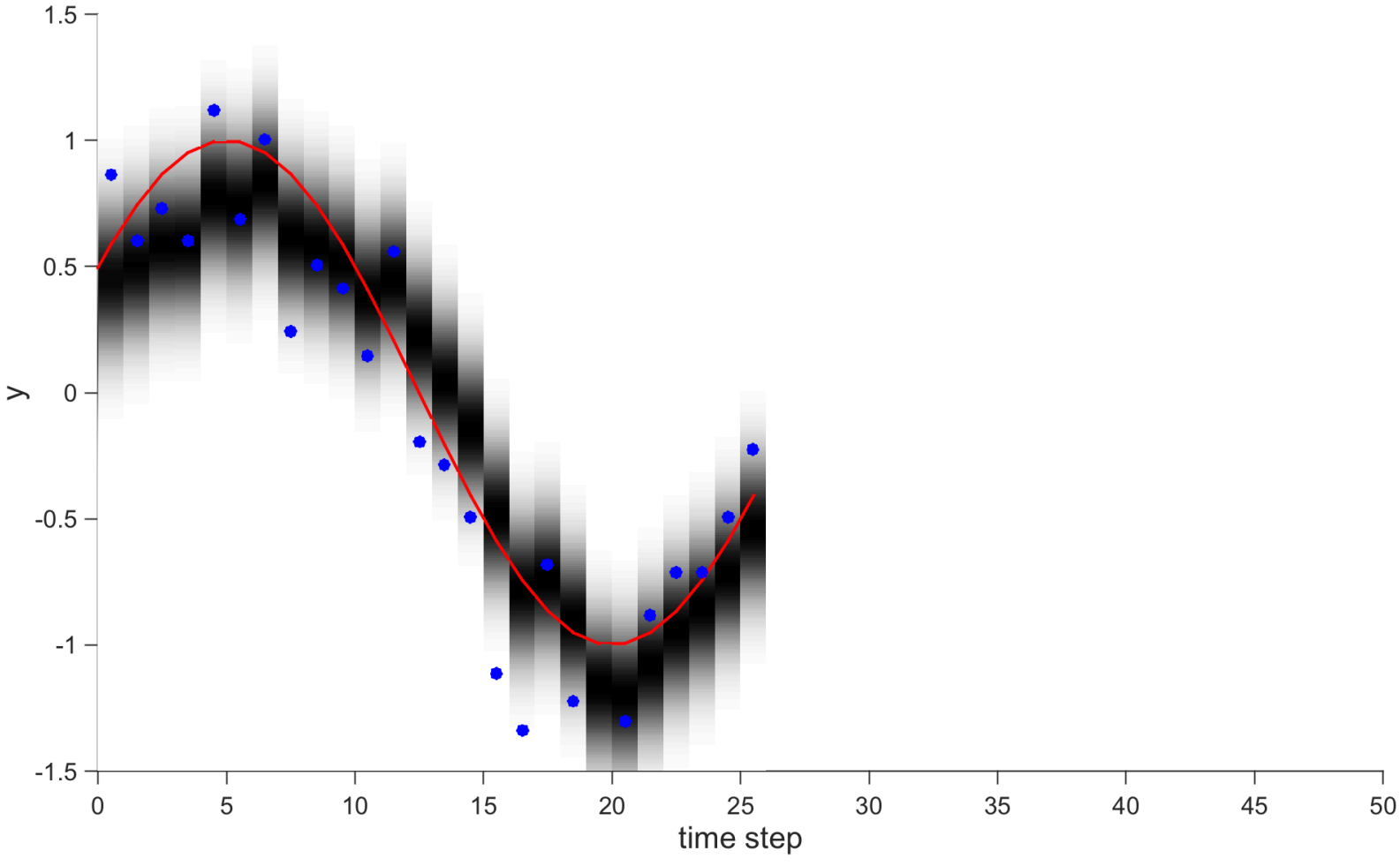

Kalman Filter Demo

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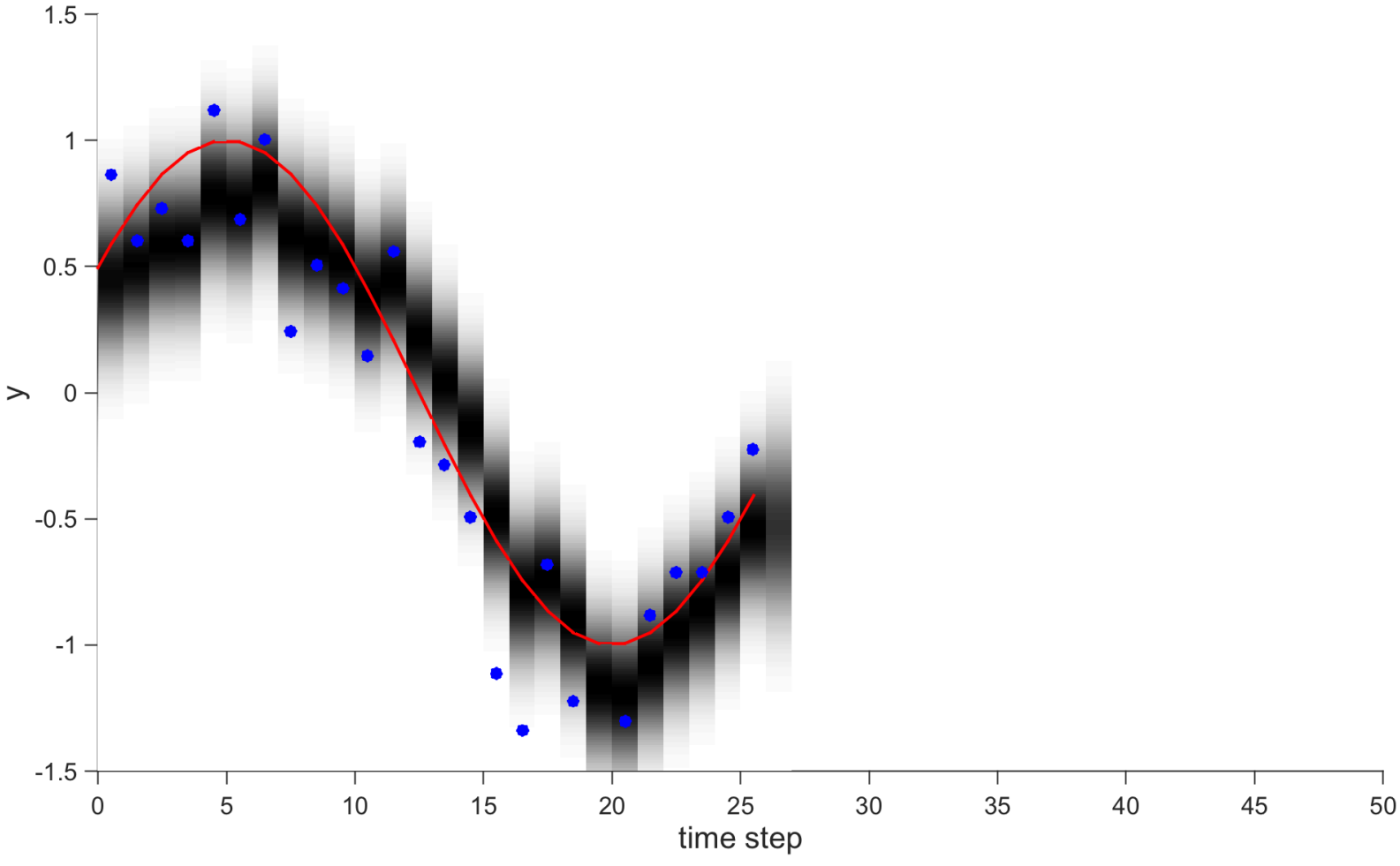

Kalman Filter Demo

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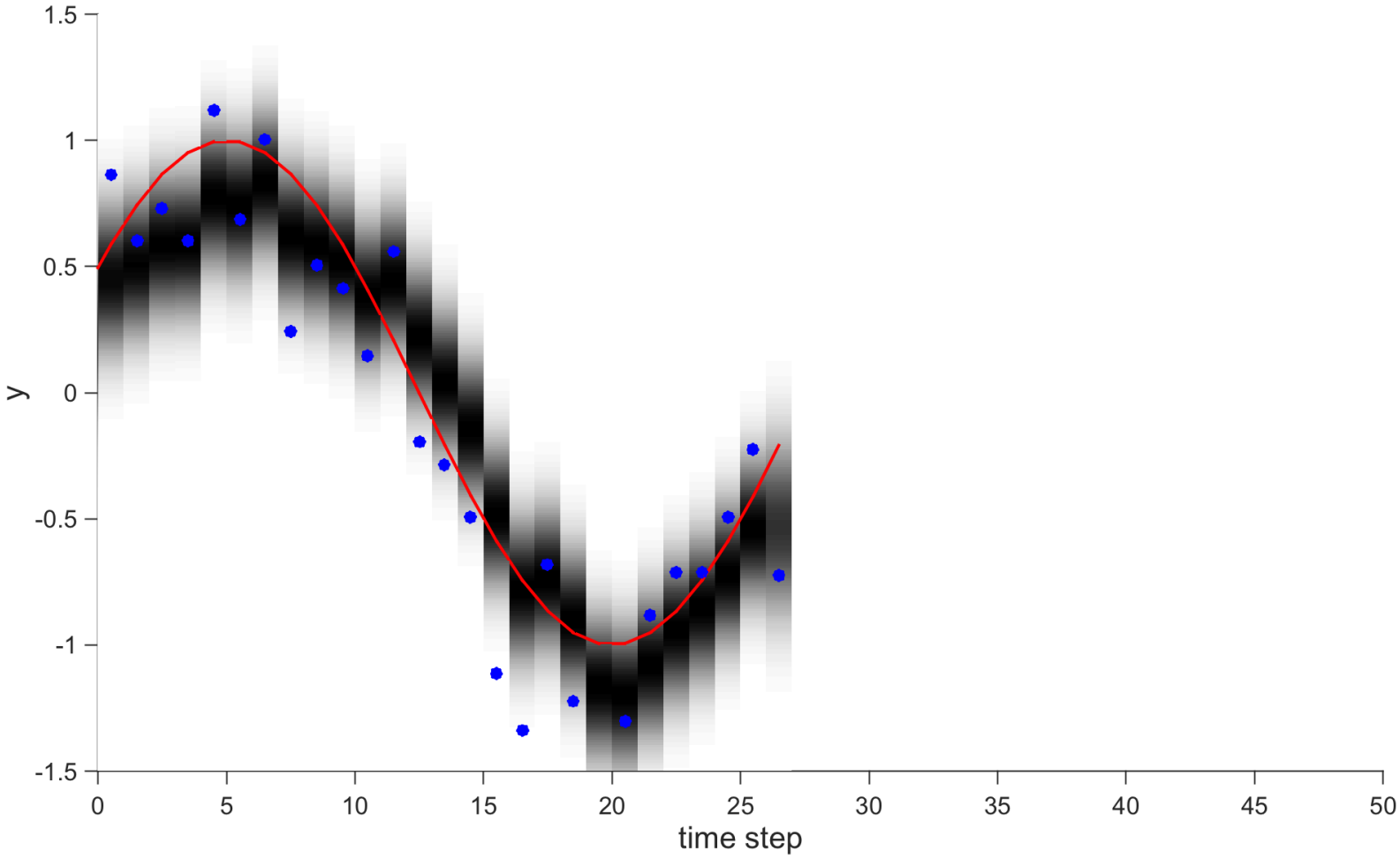

Kalman Filter Demo

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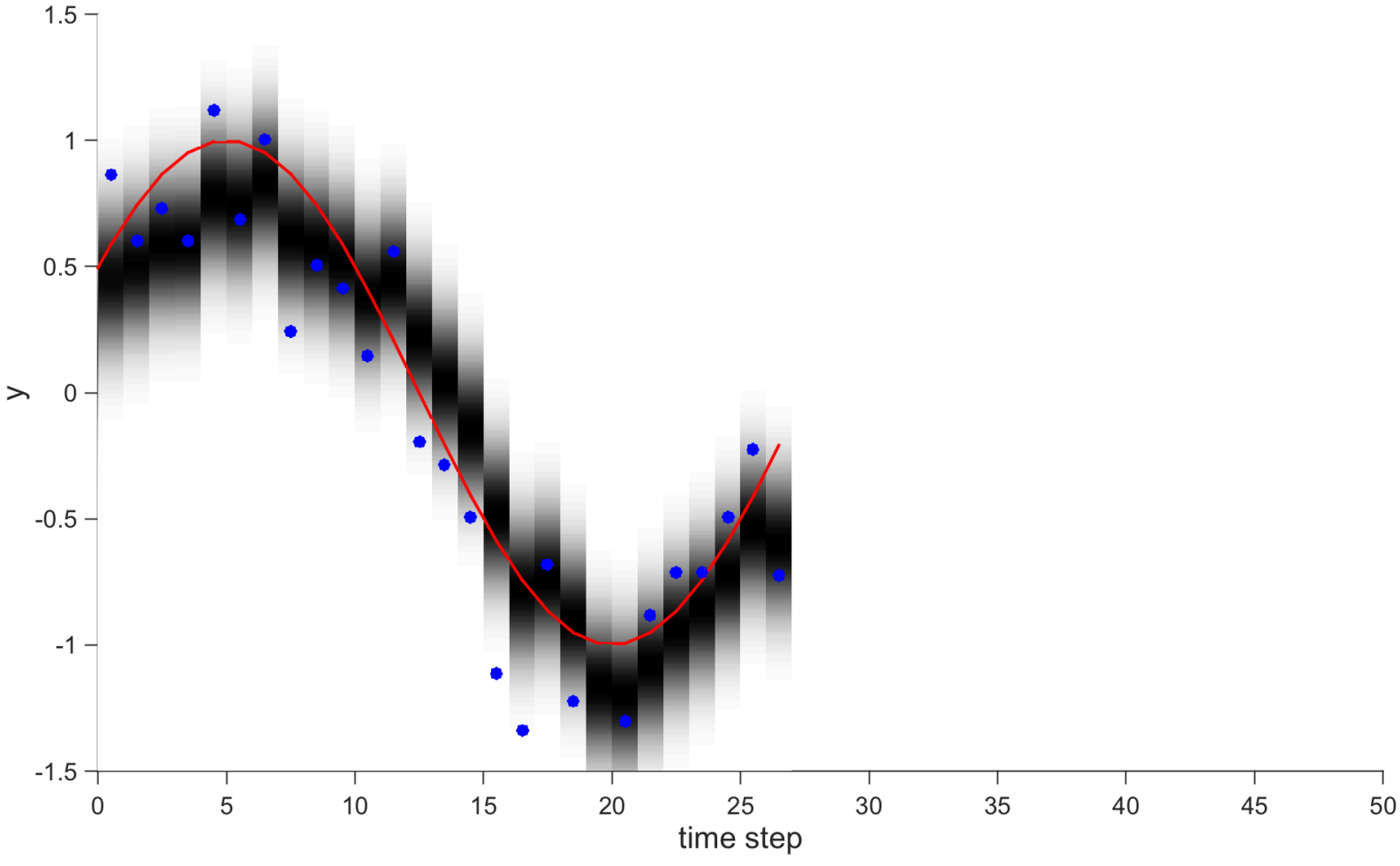

Kalman Filter Demo

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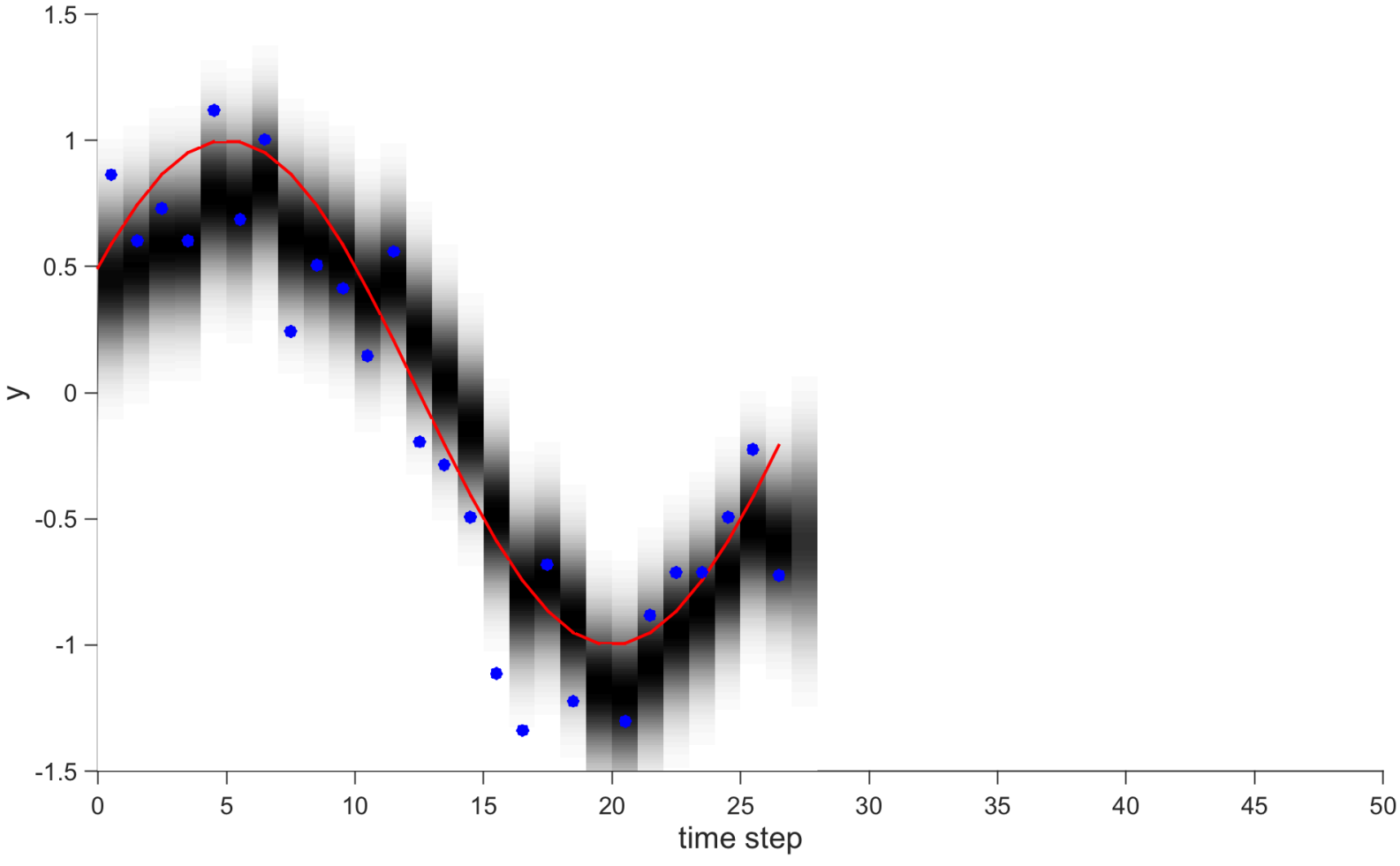
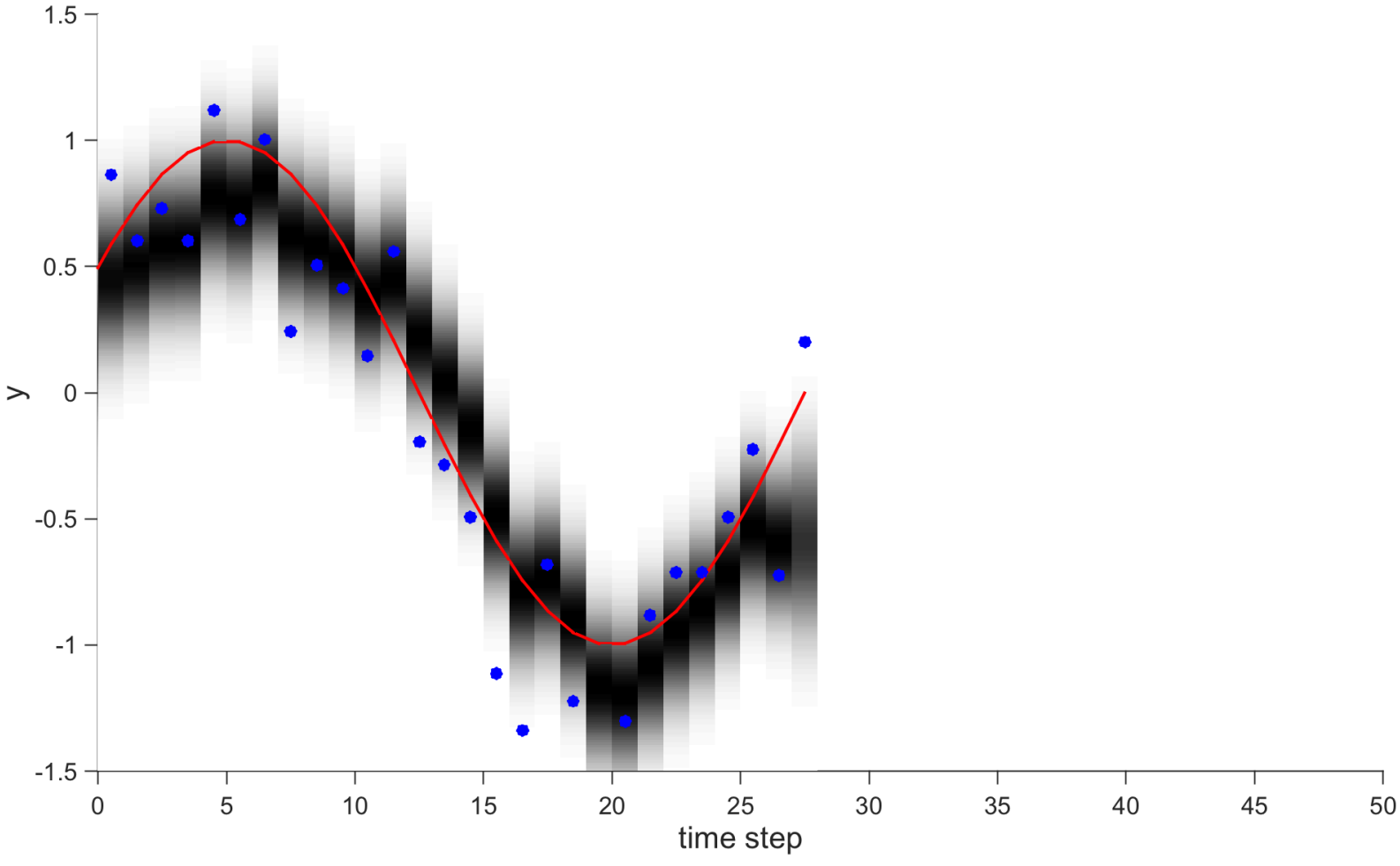


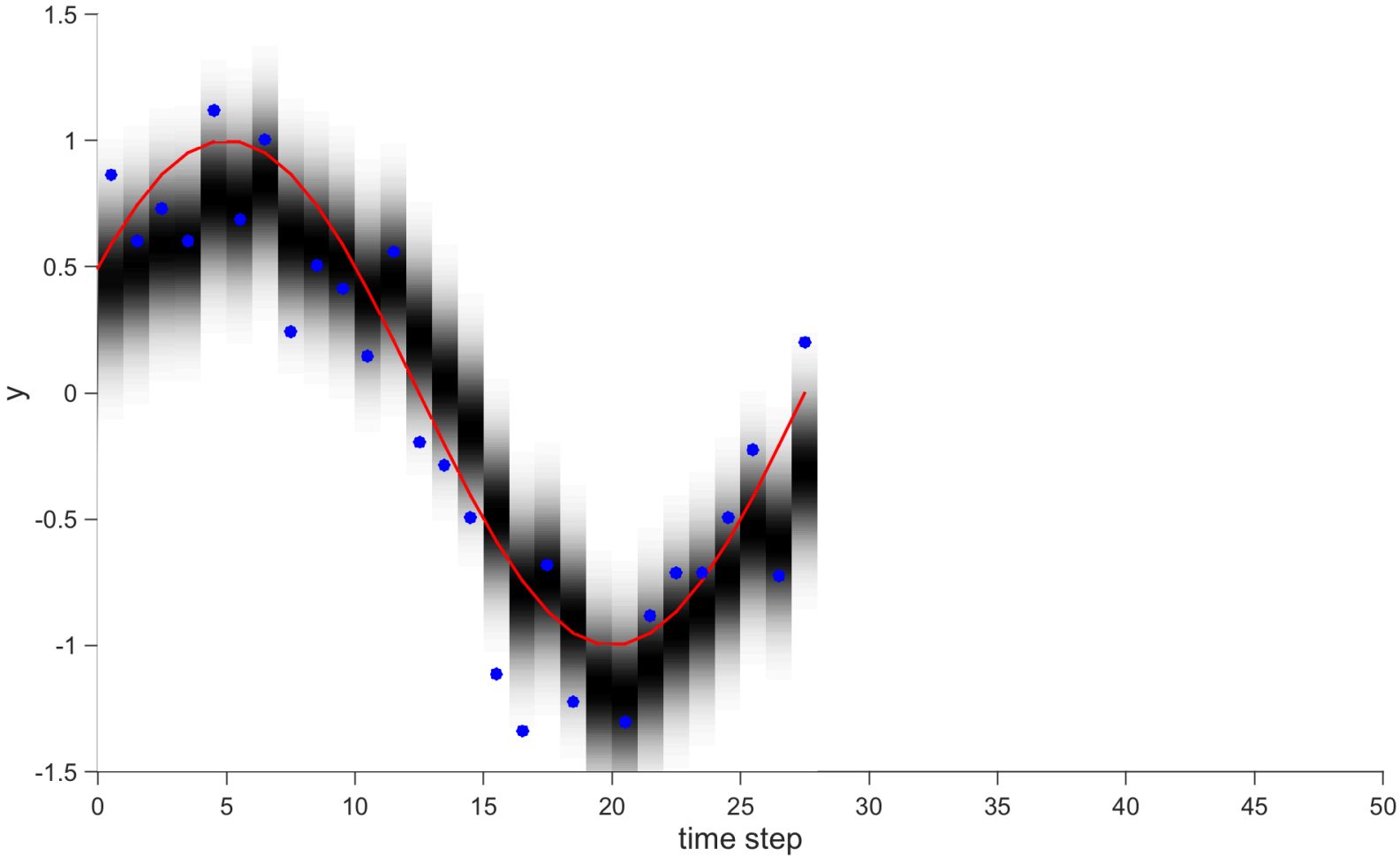
Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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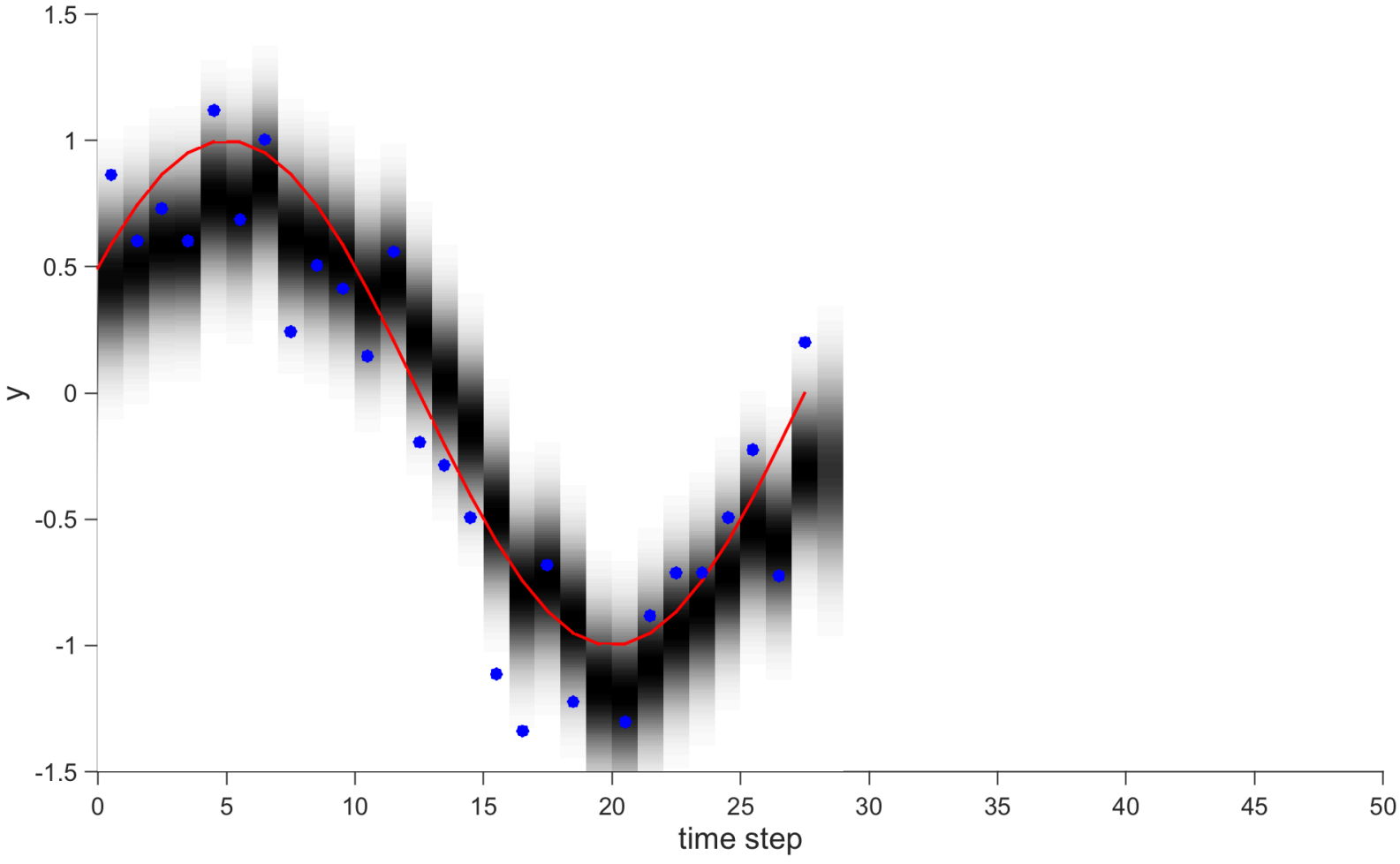
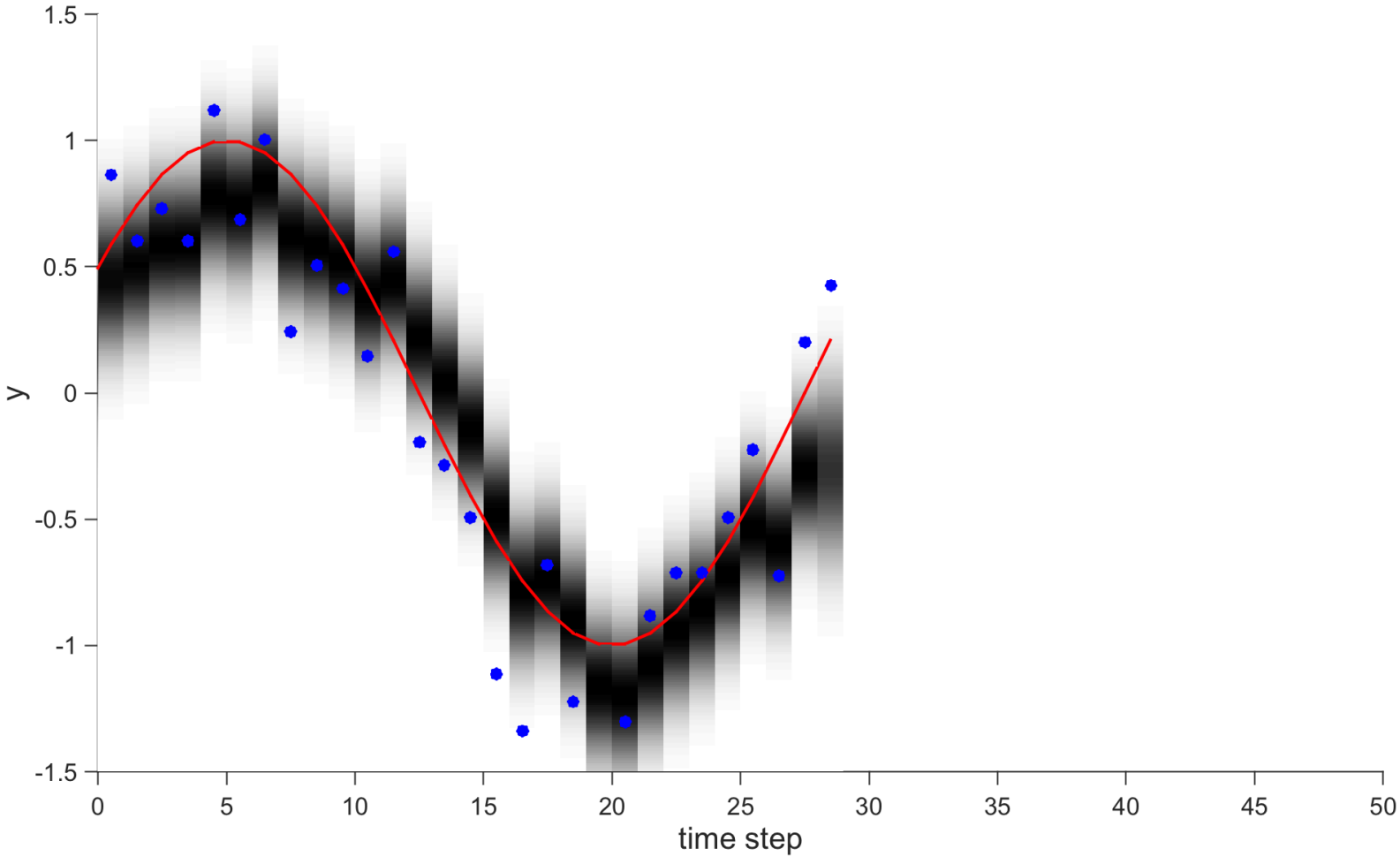


Kalman Filter Demo

observed noisy data y_t, ground truth sinusoid



Kalman Filter Demo

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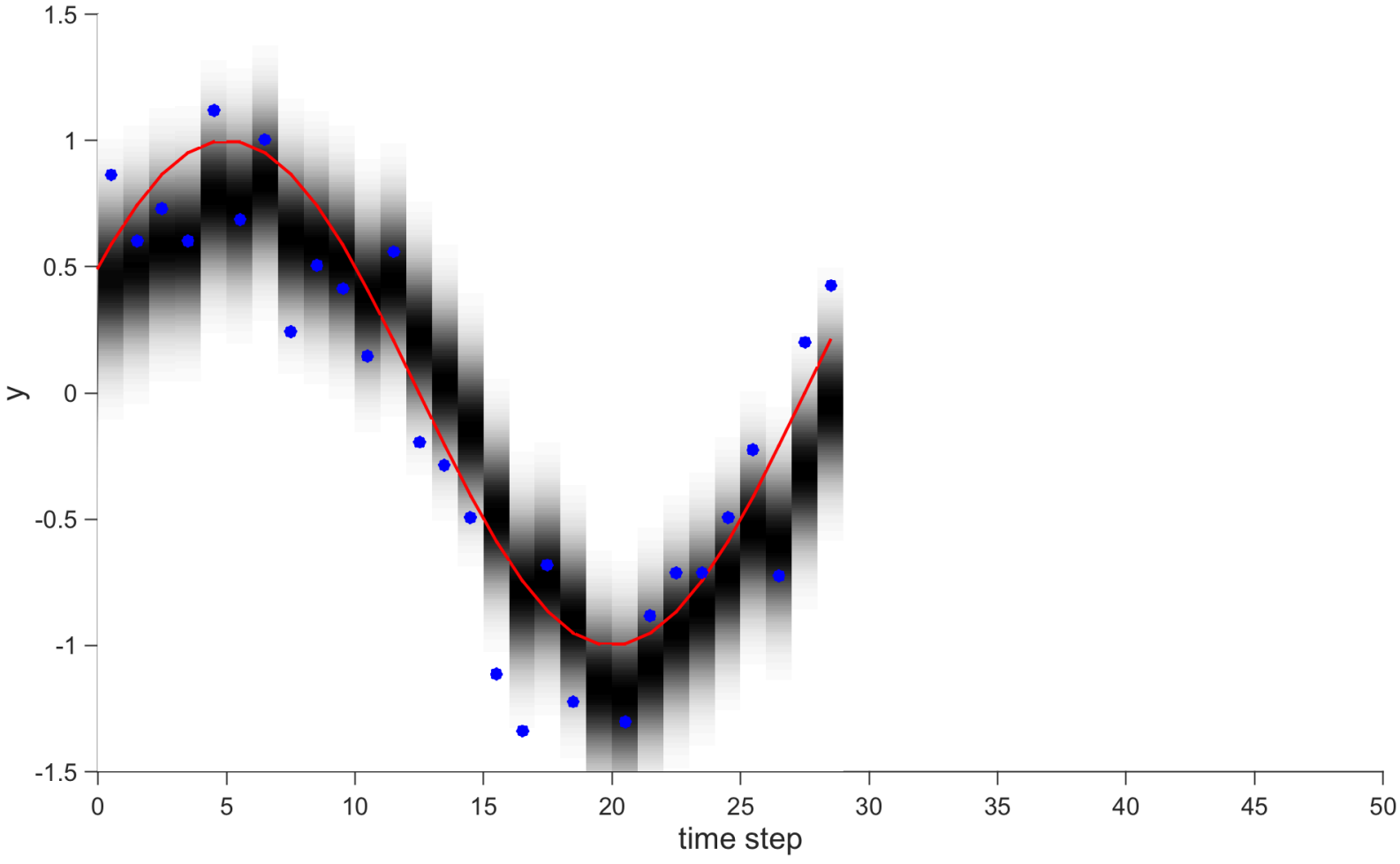

Kalman Filter Demo

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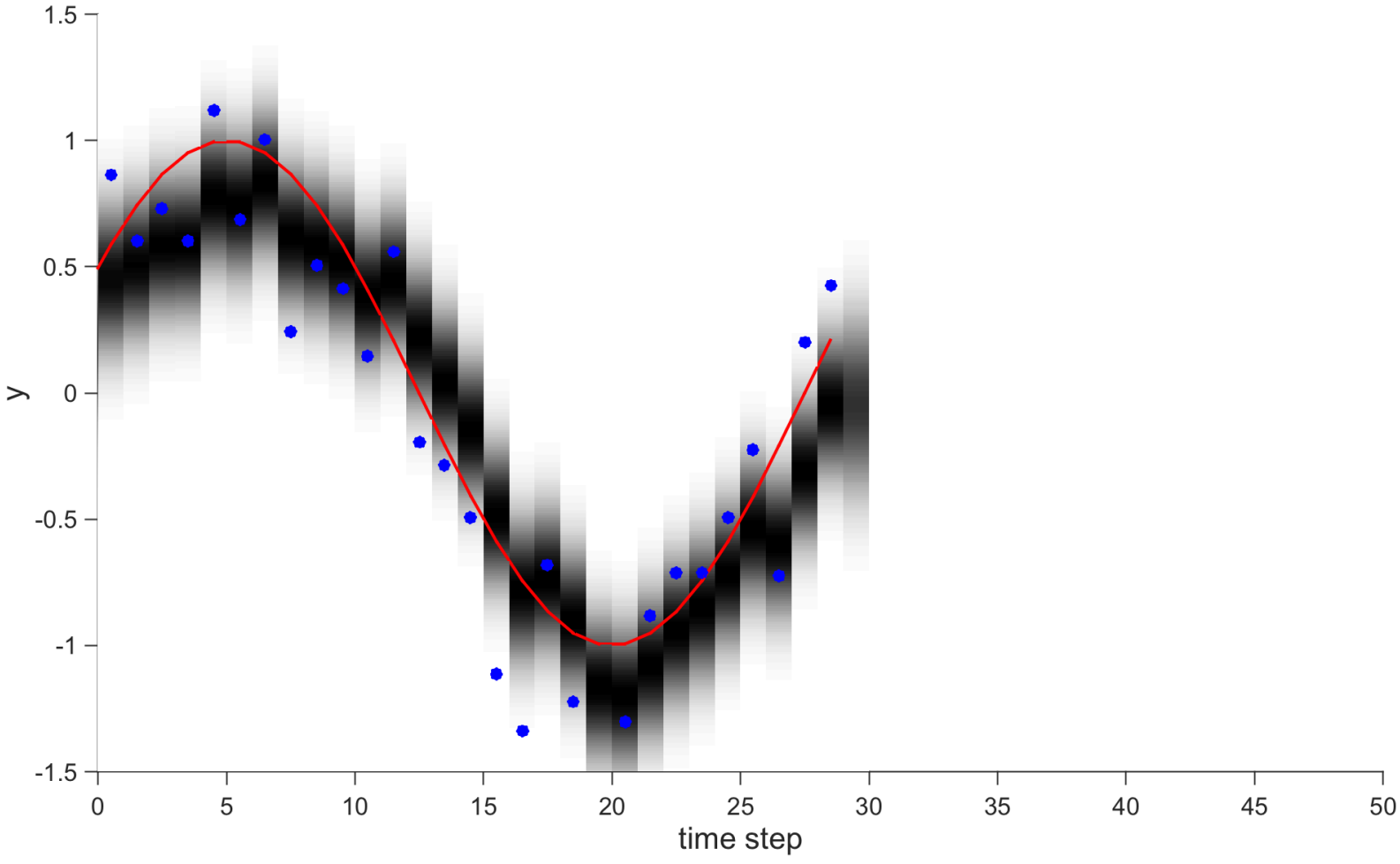

Kalman Filter Demo

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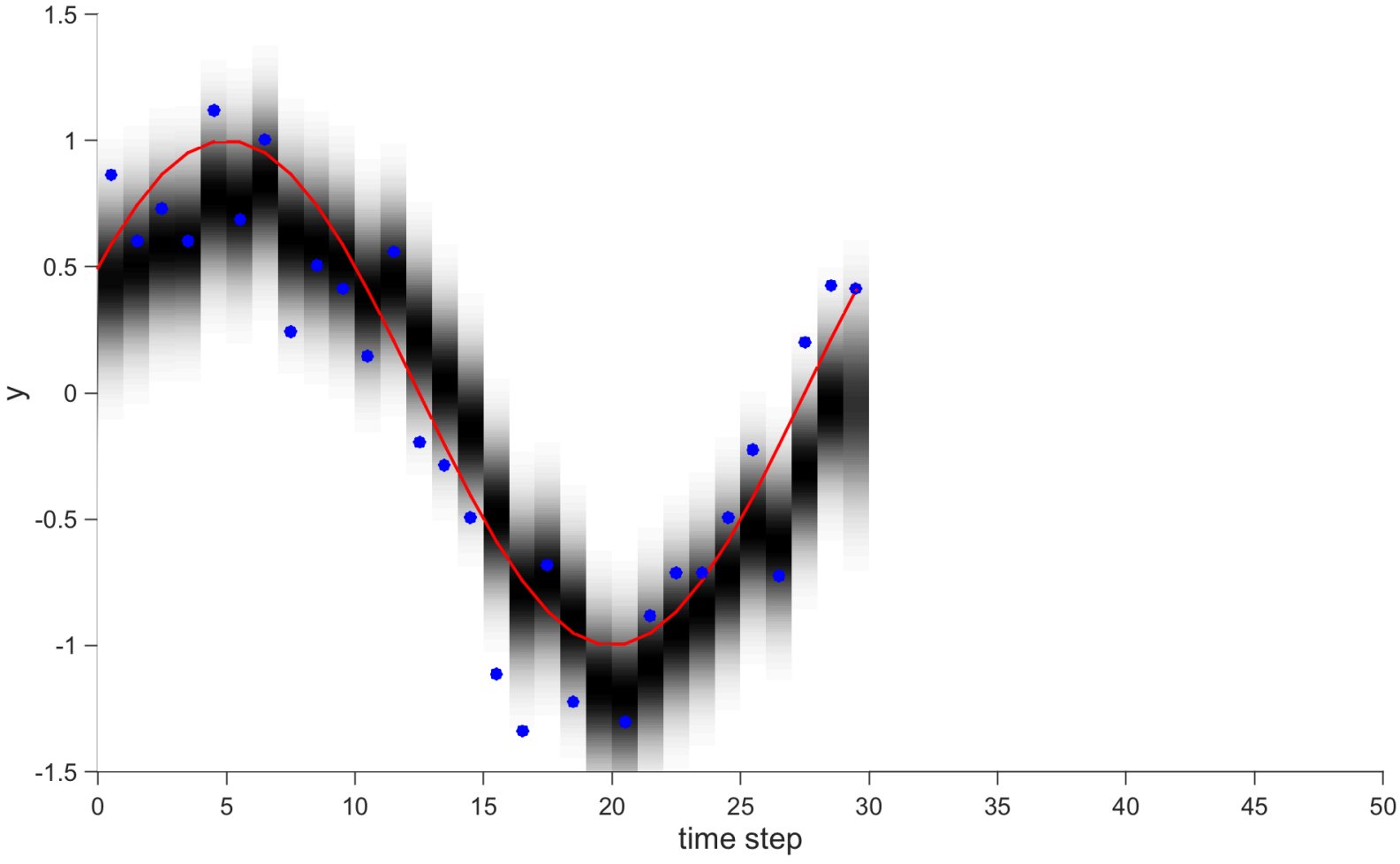

Kalman Filter Demo

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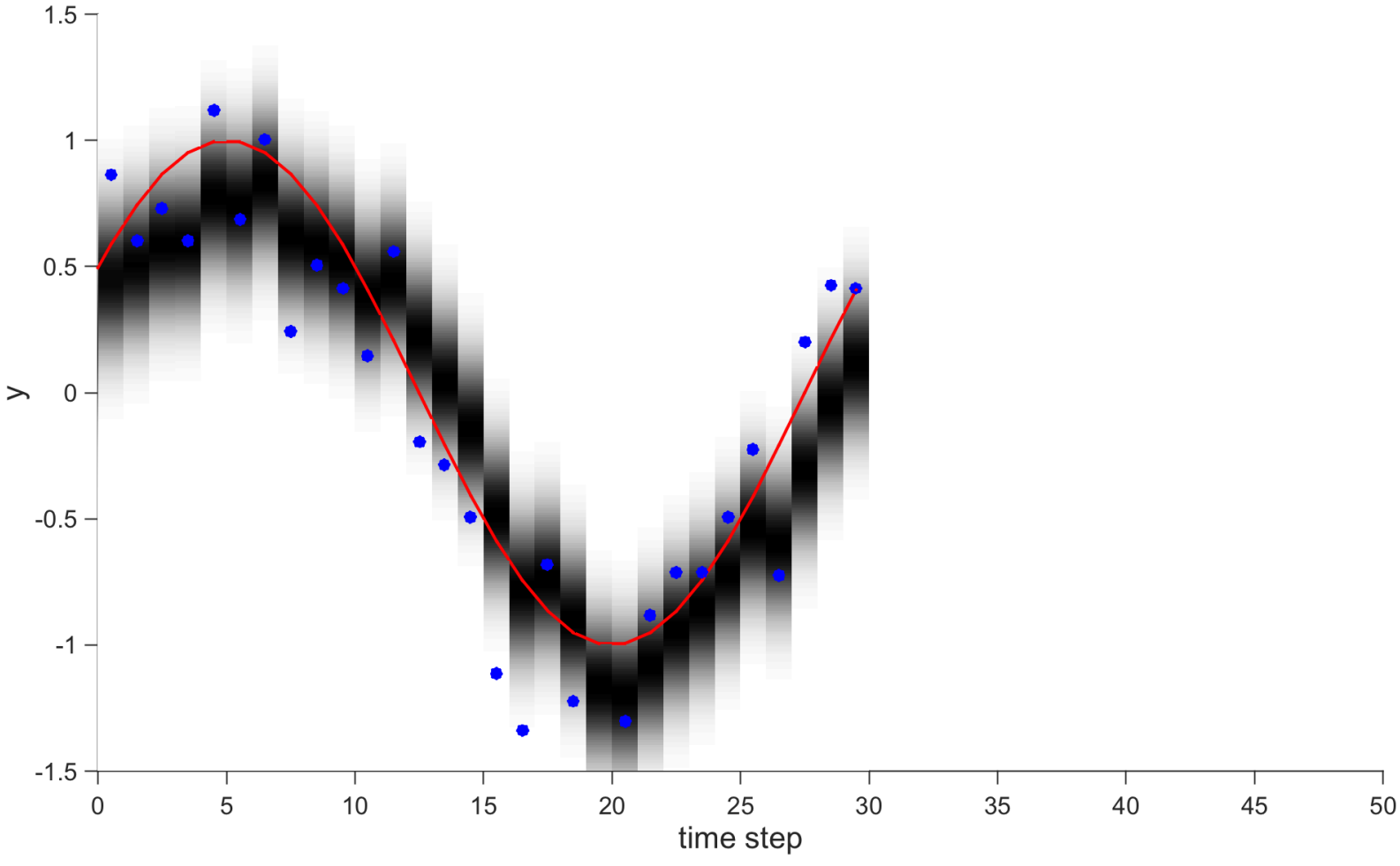

Kalman Filter Demo

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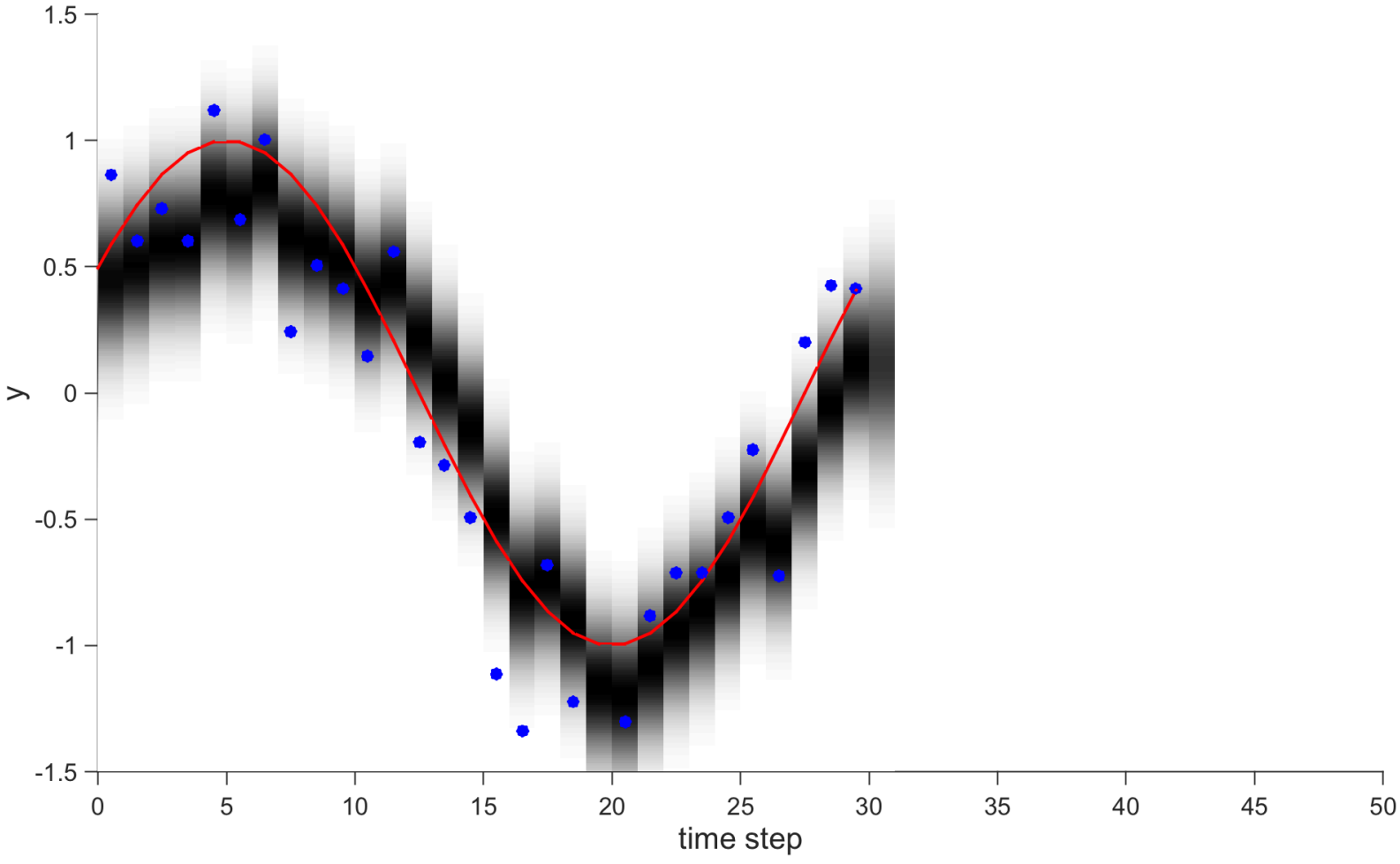

Kalman Filter Demo

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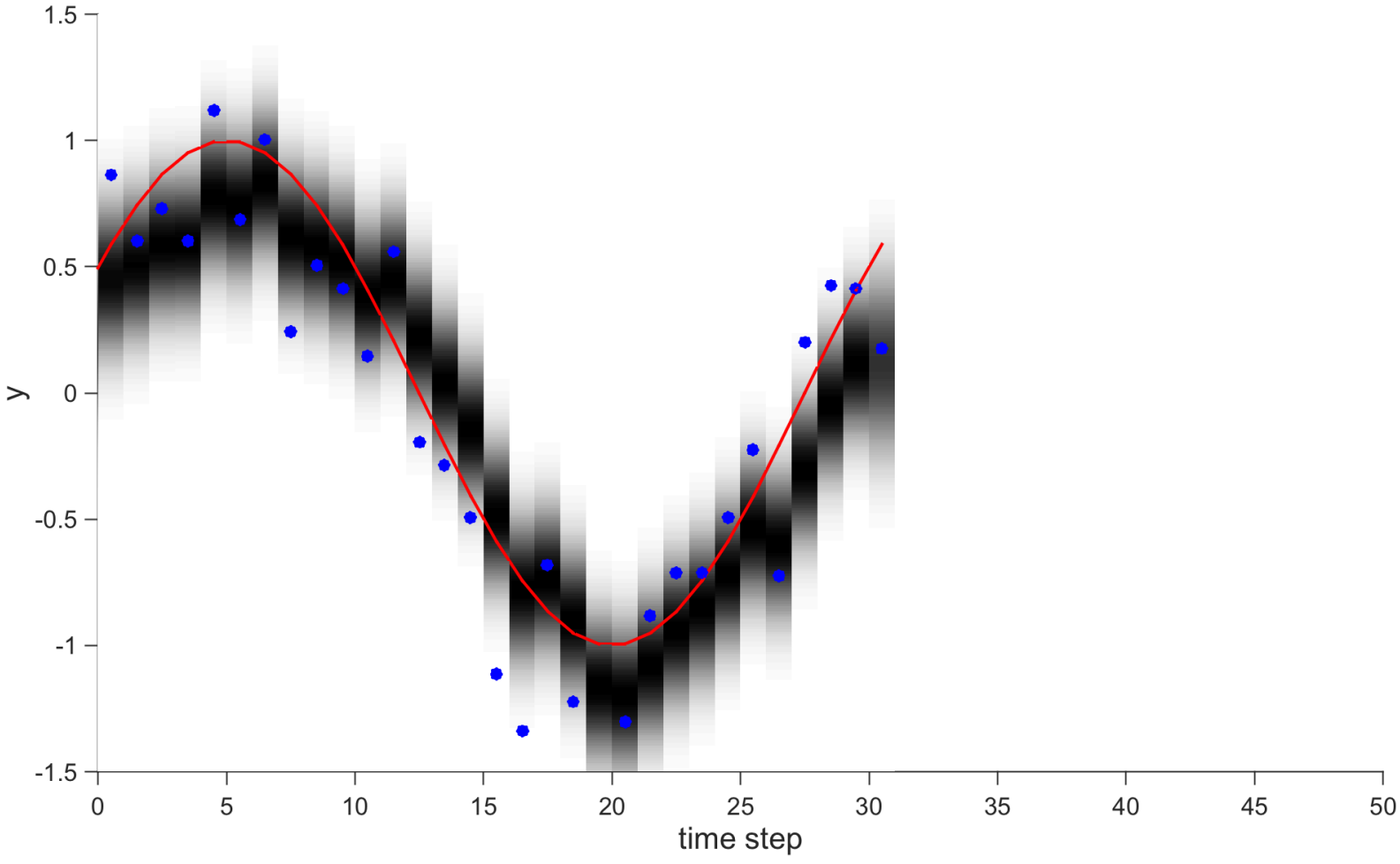

Kalman Filter Demo

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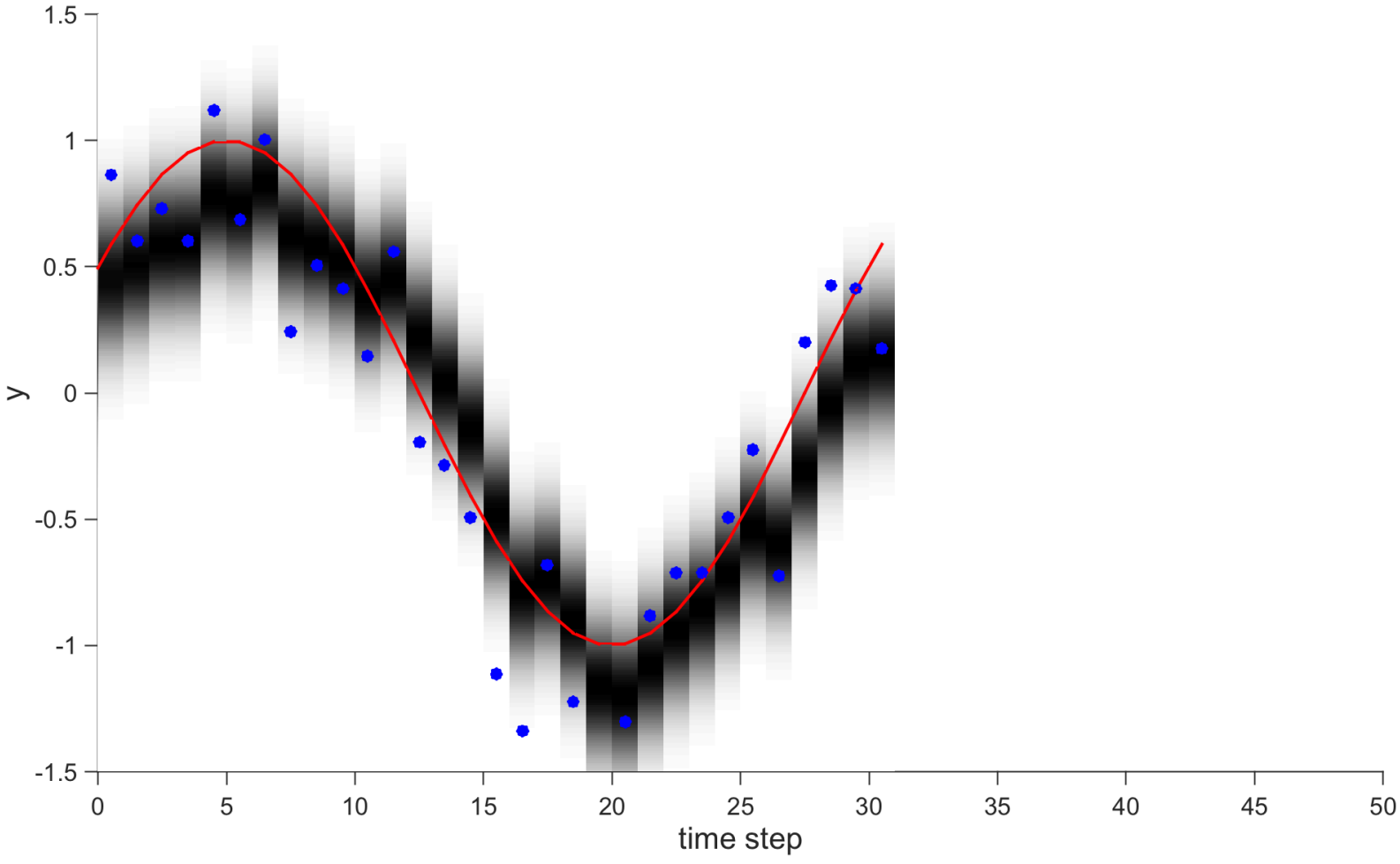

Kalman Filter Demo

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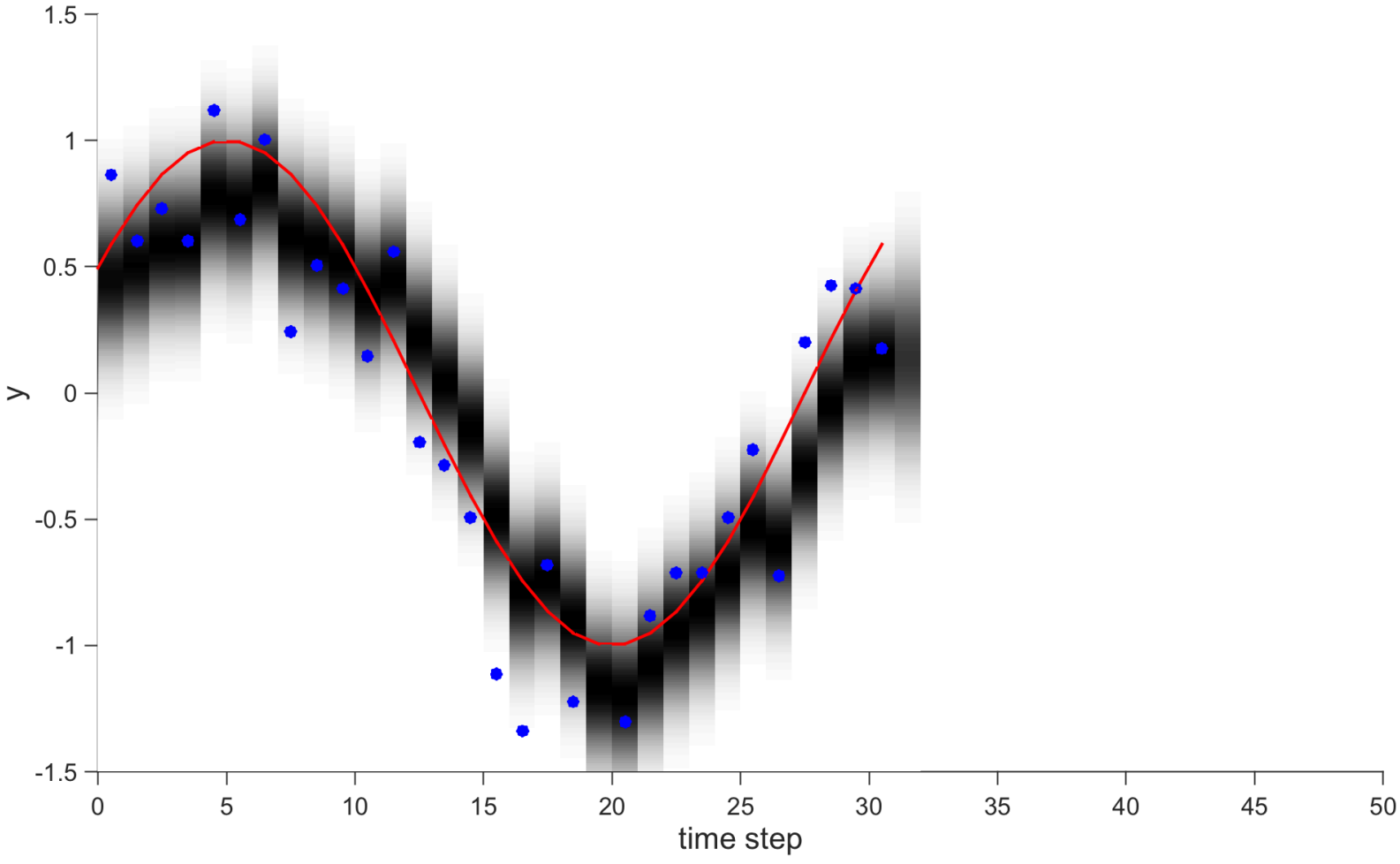

Kalman Filter Demo

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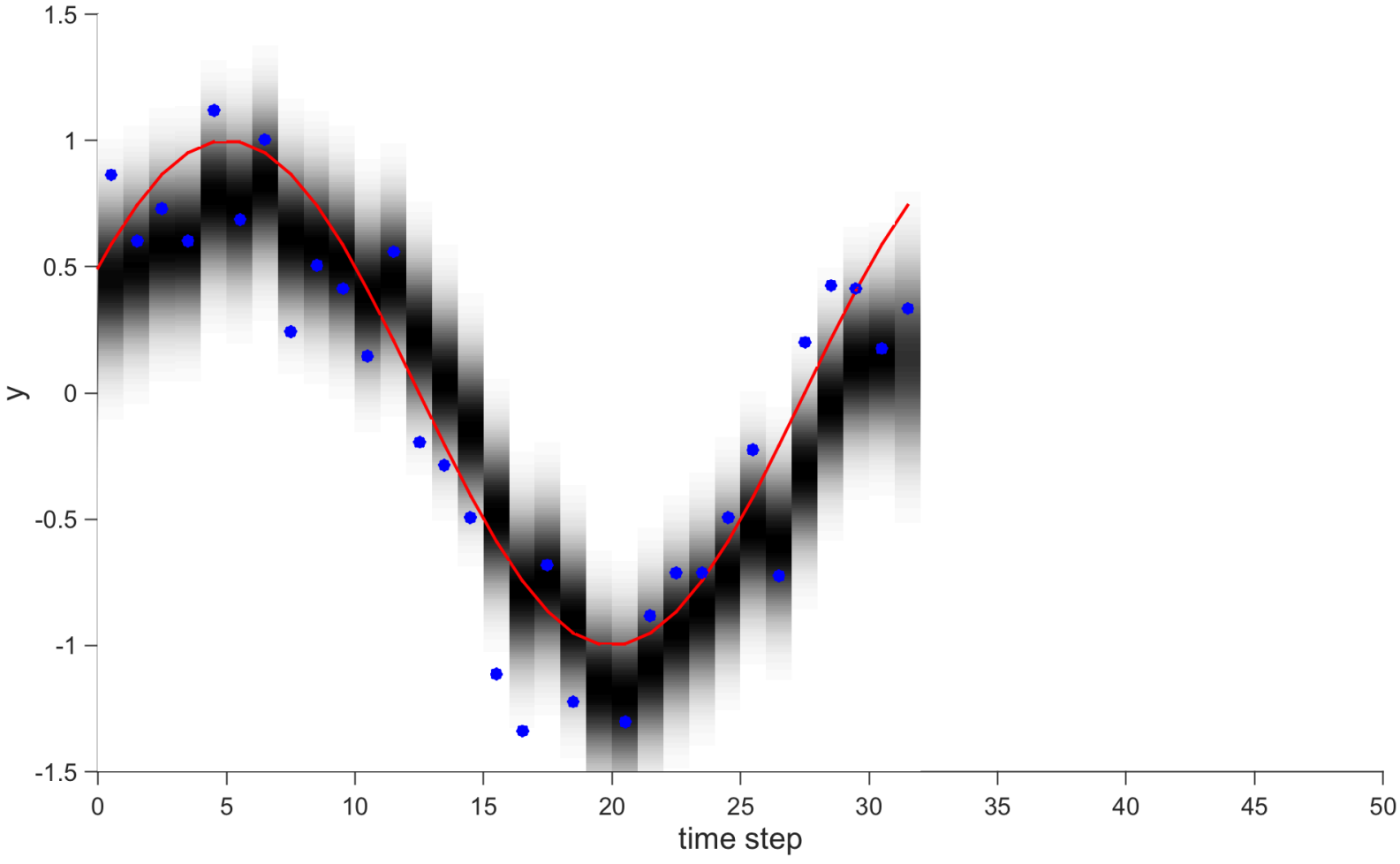

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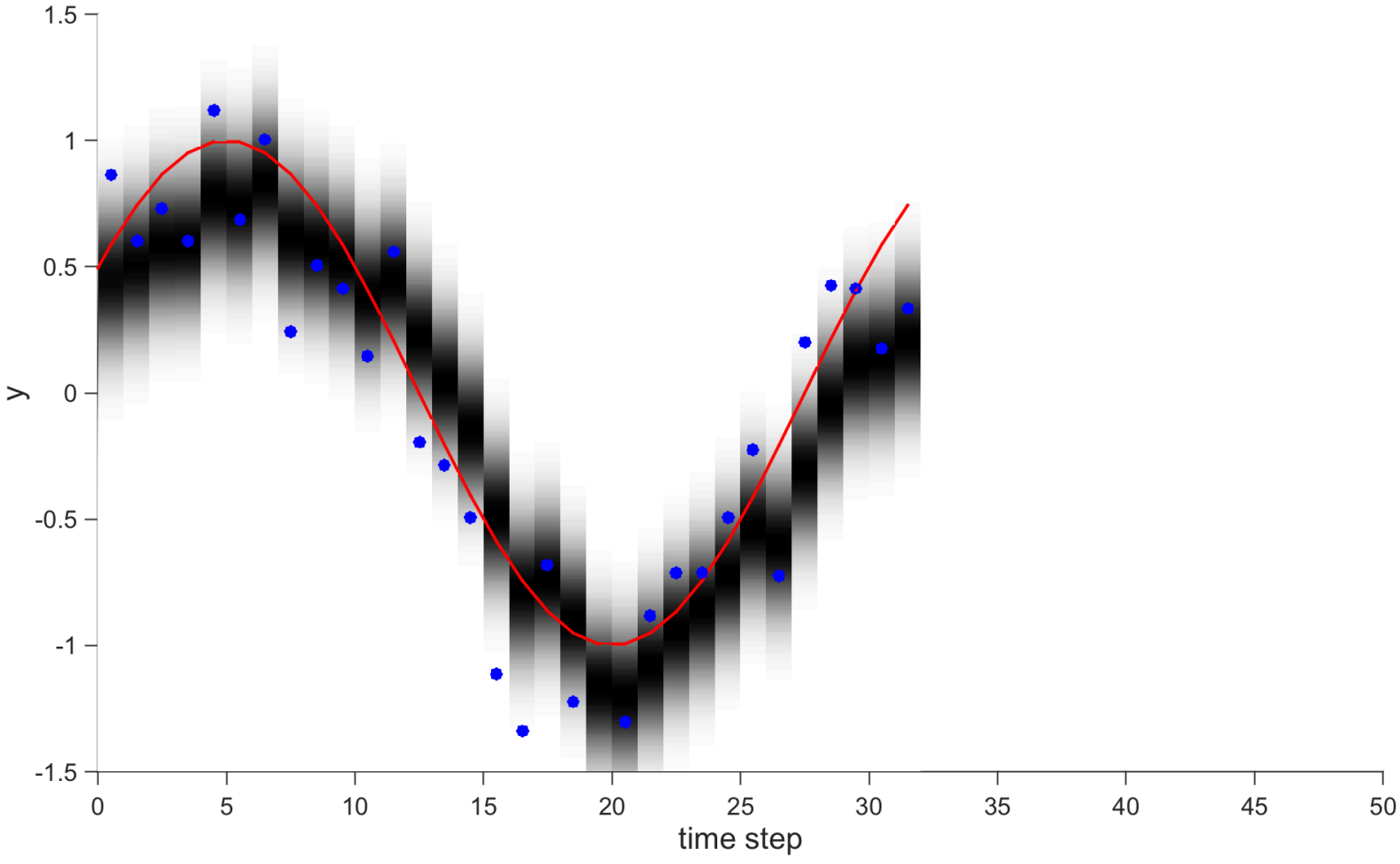

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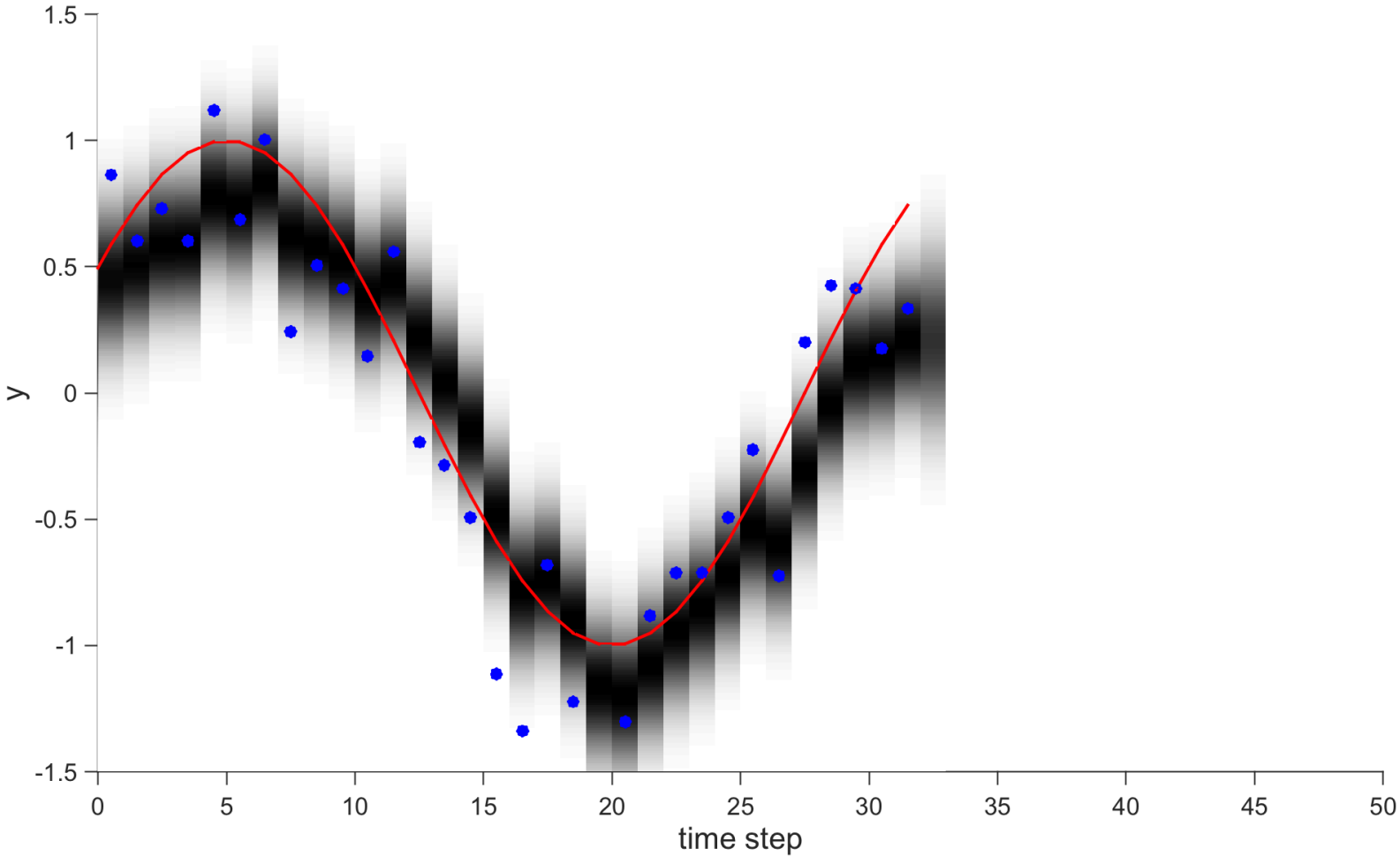

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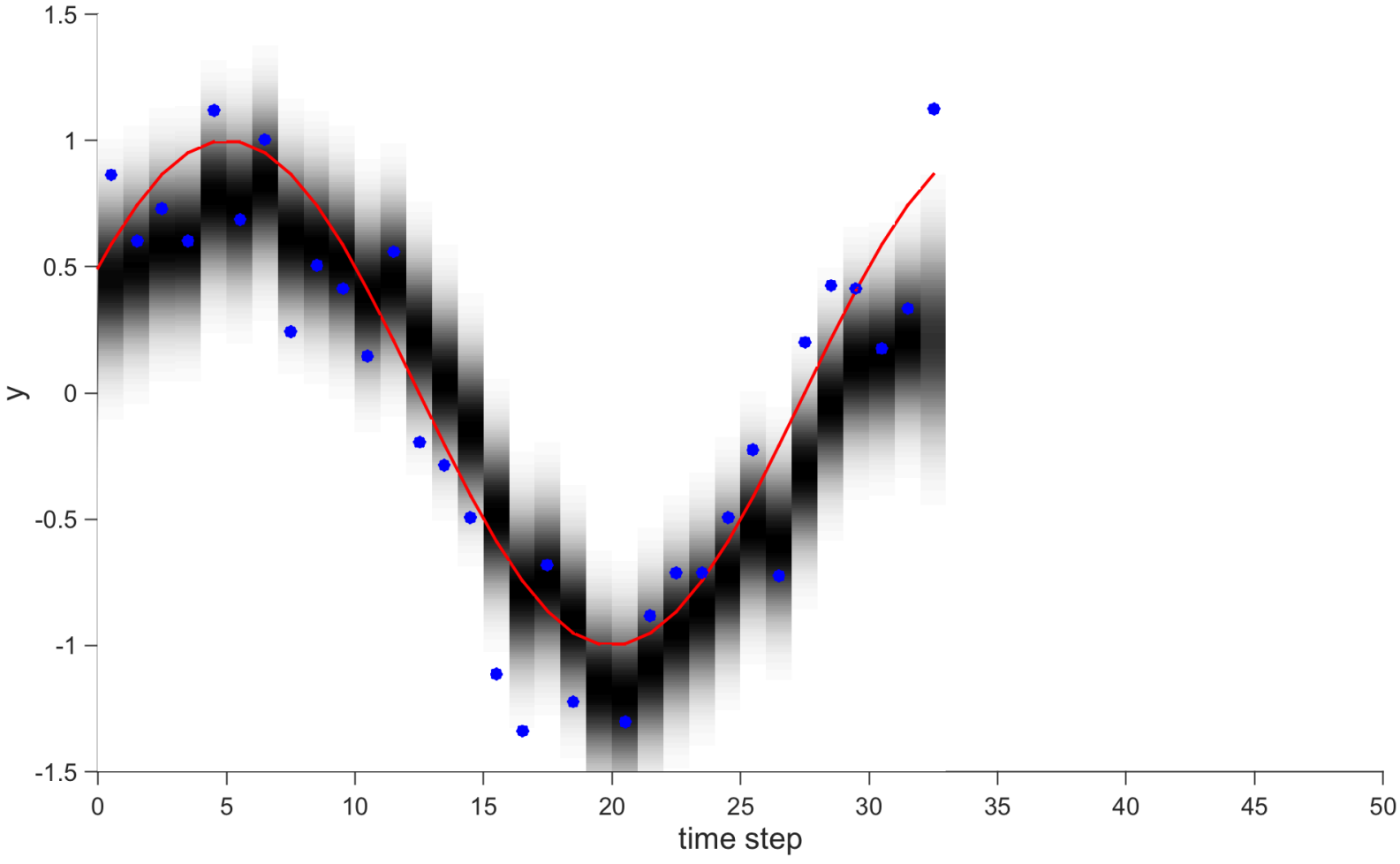

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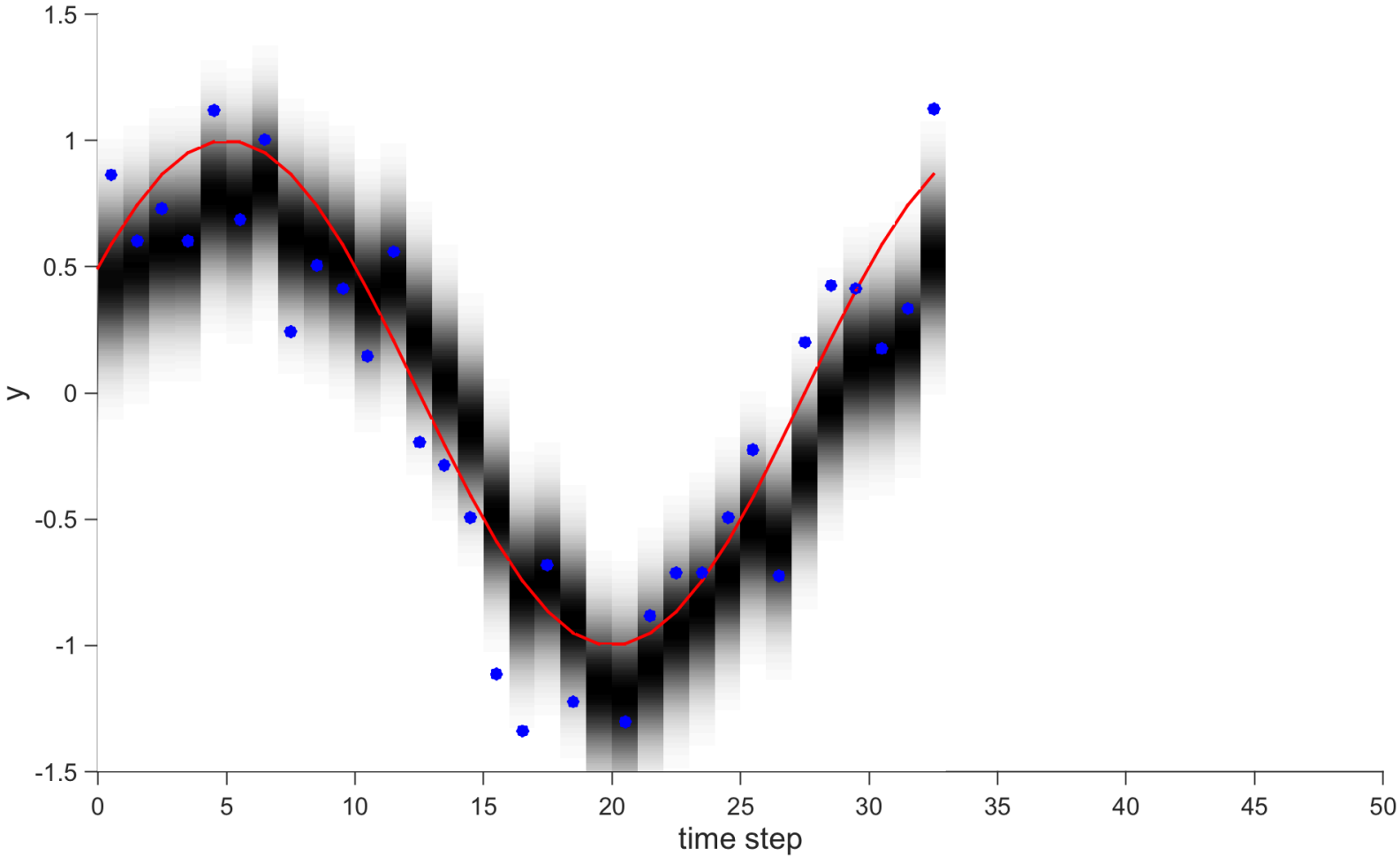

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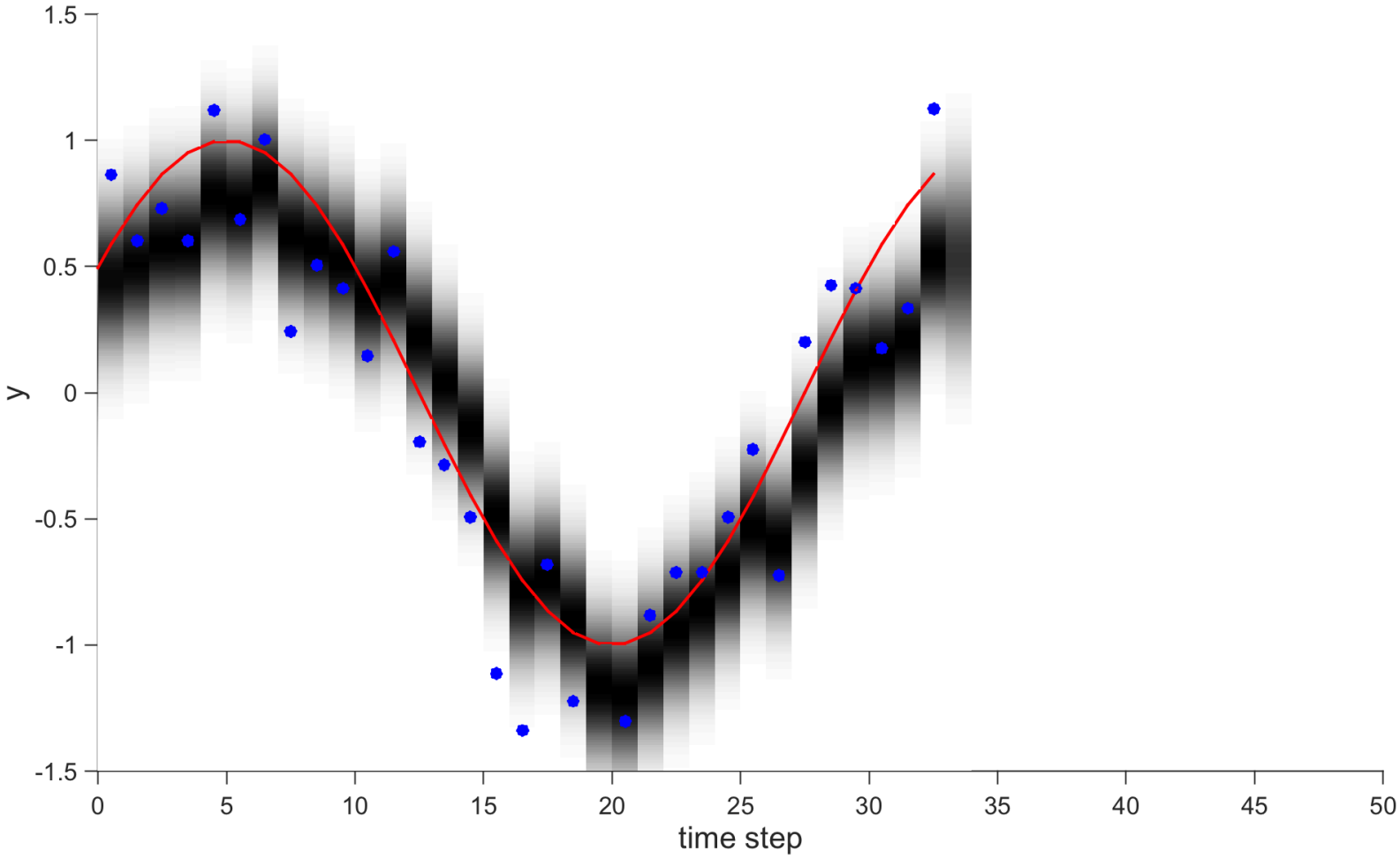

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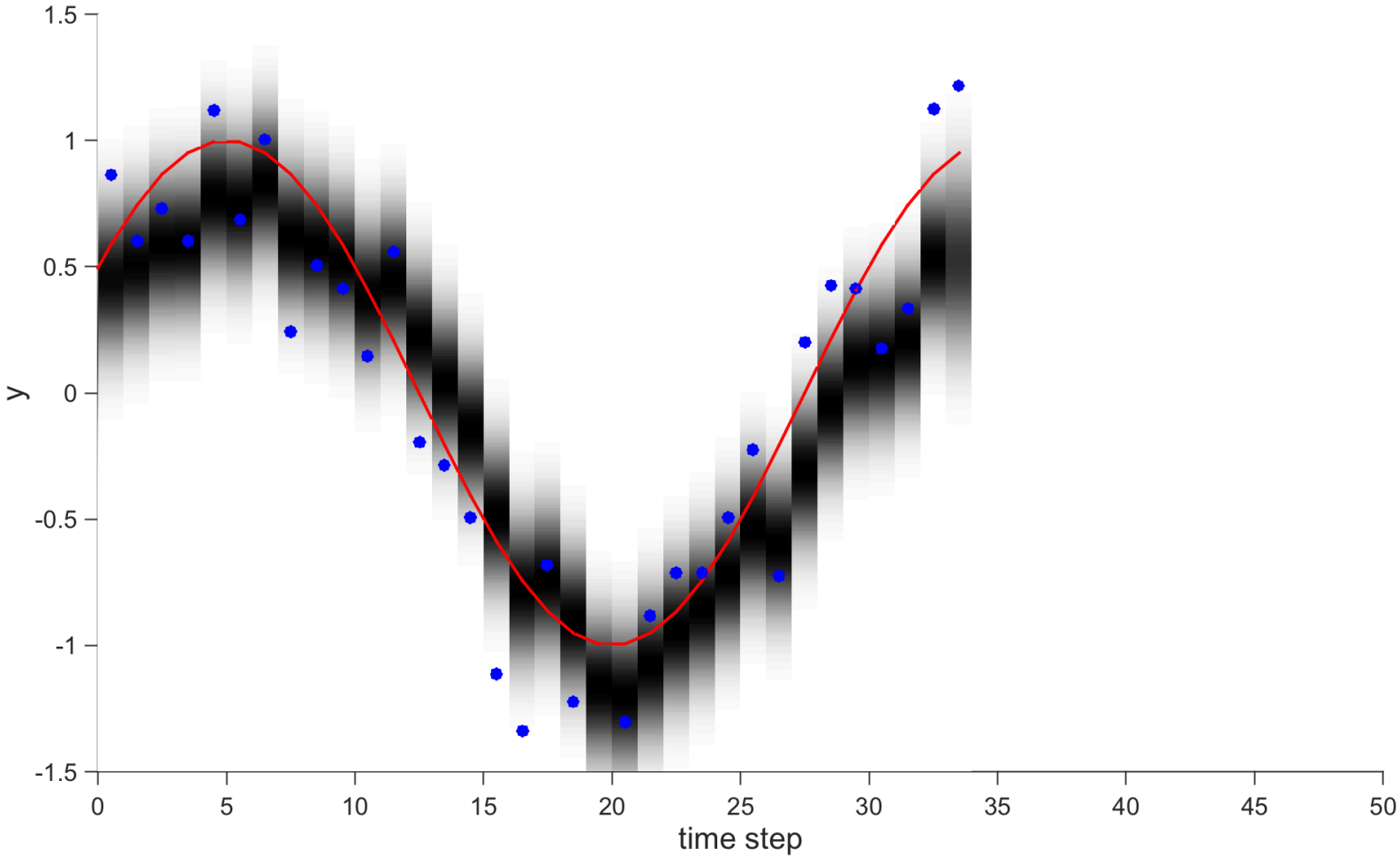

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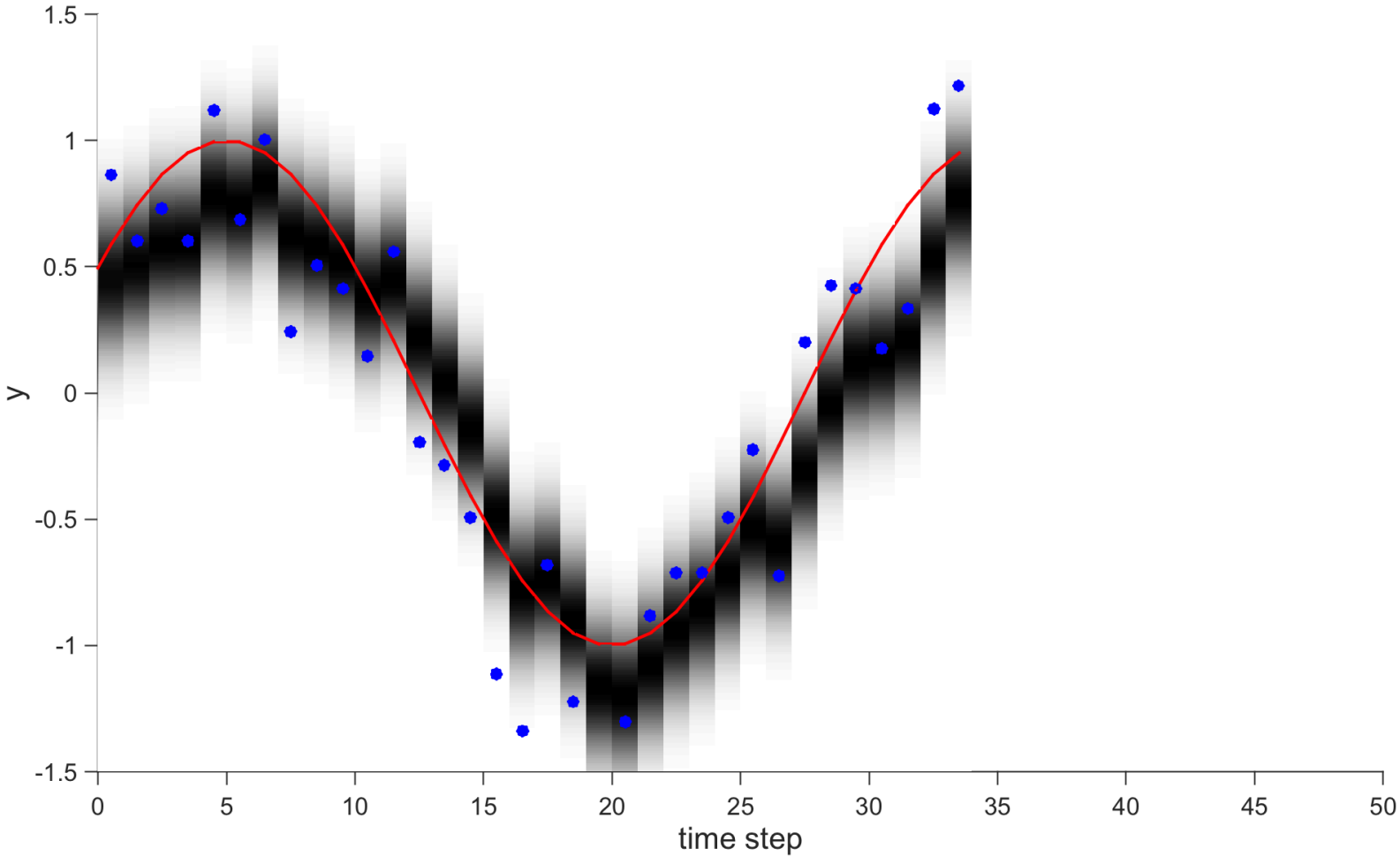

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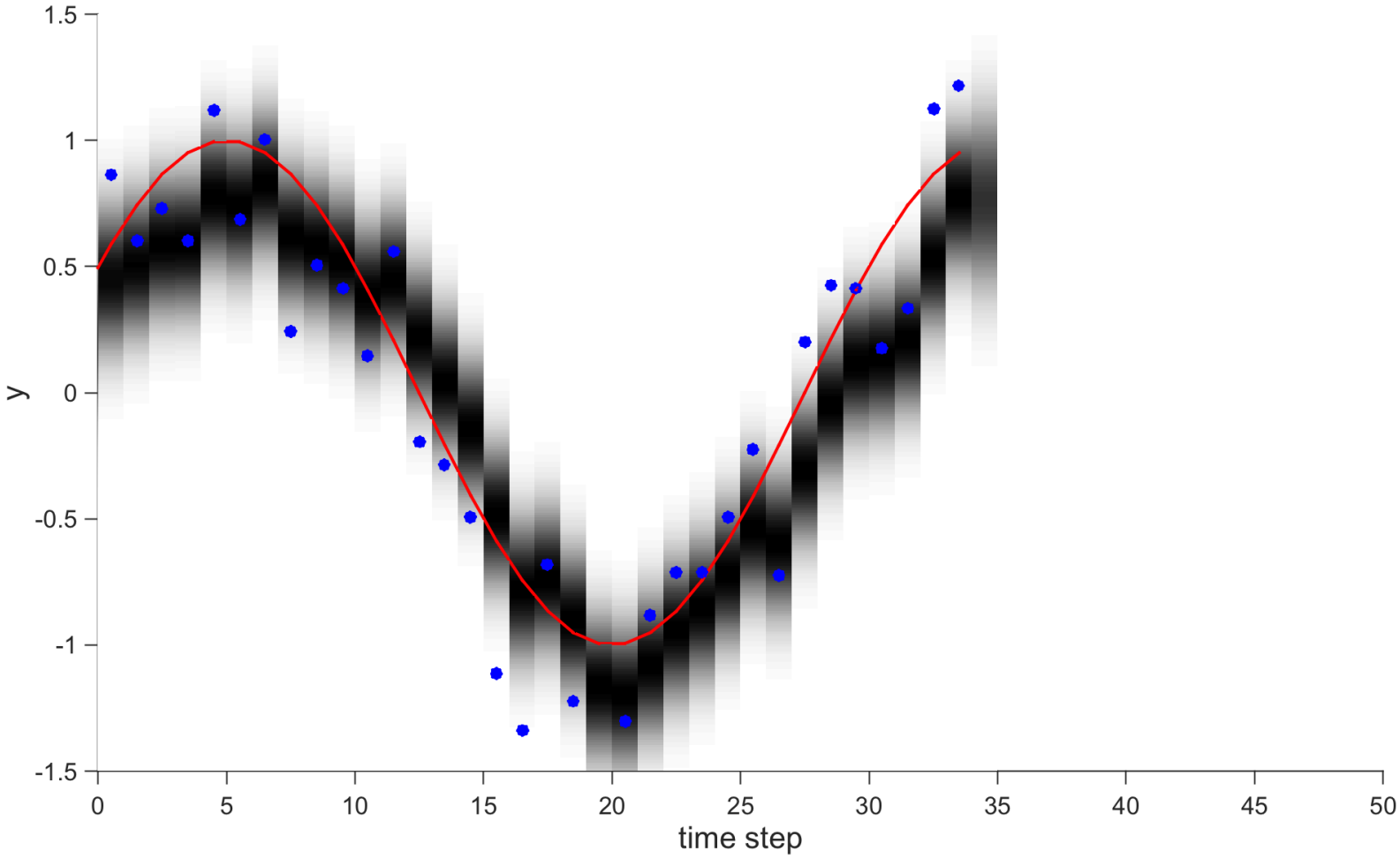

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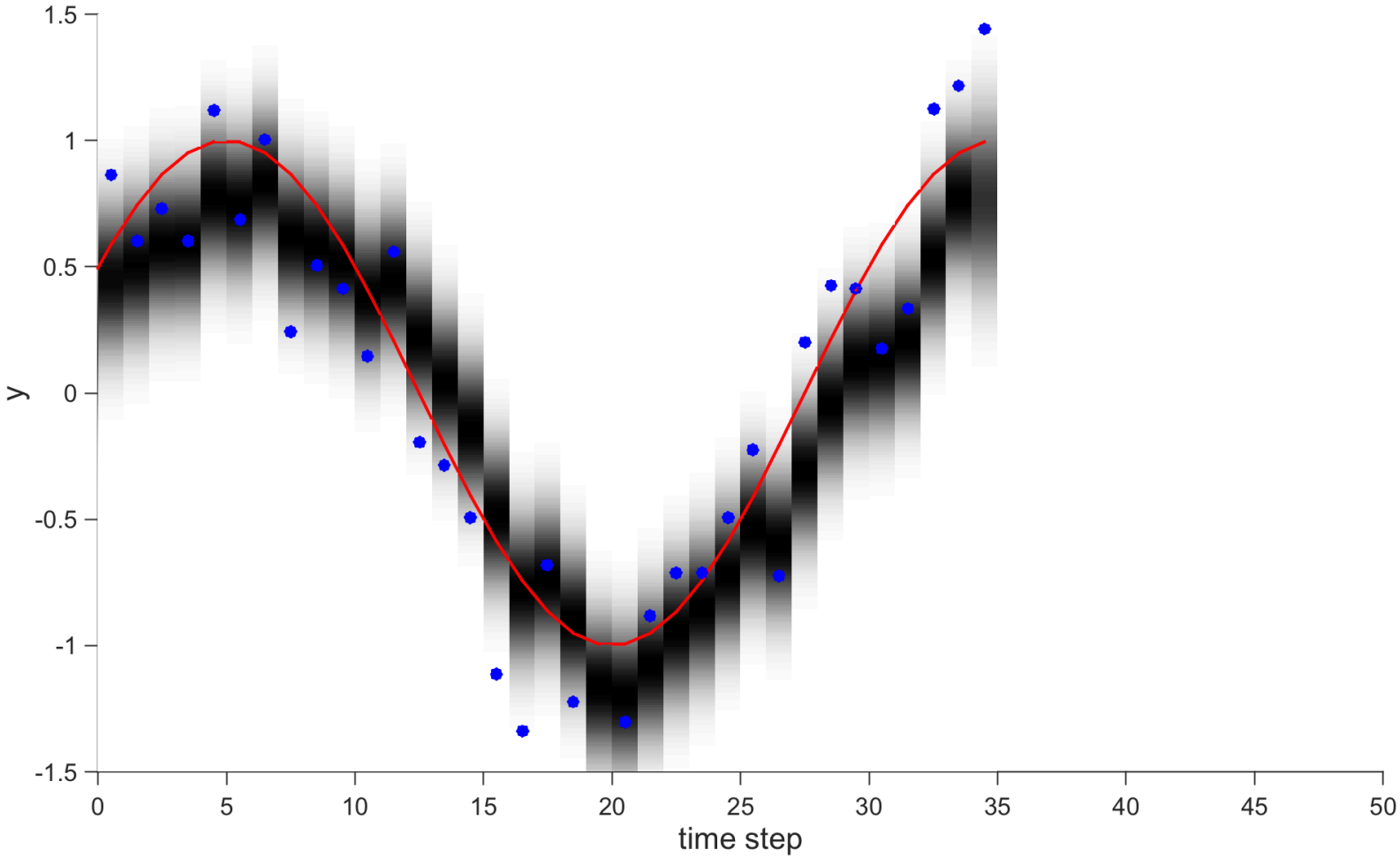

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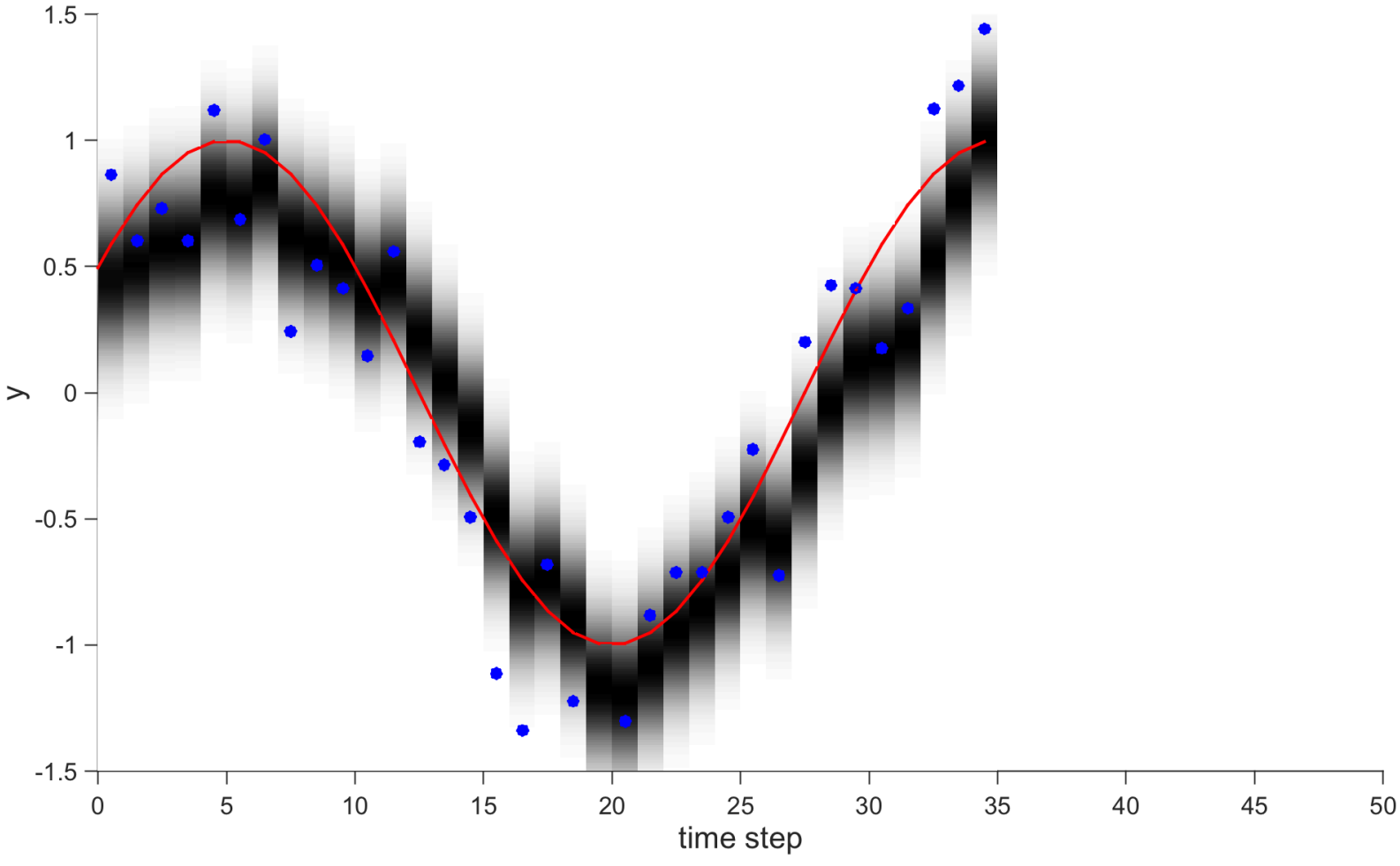

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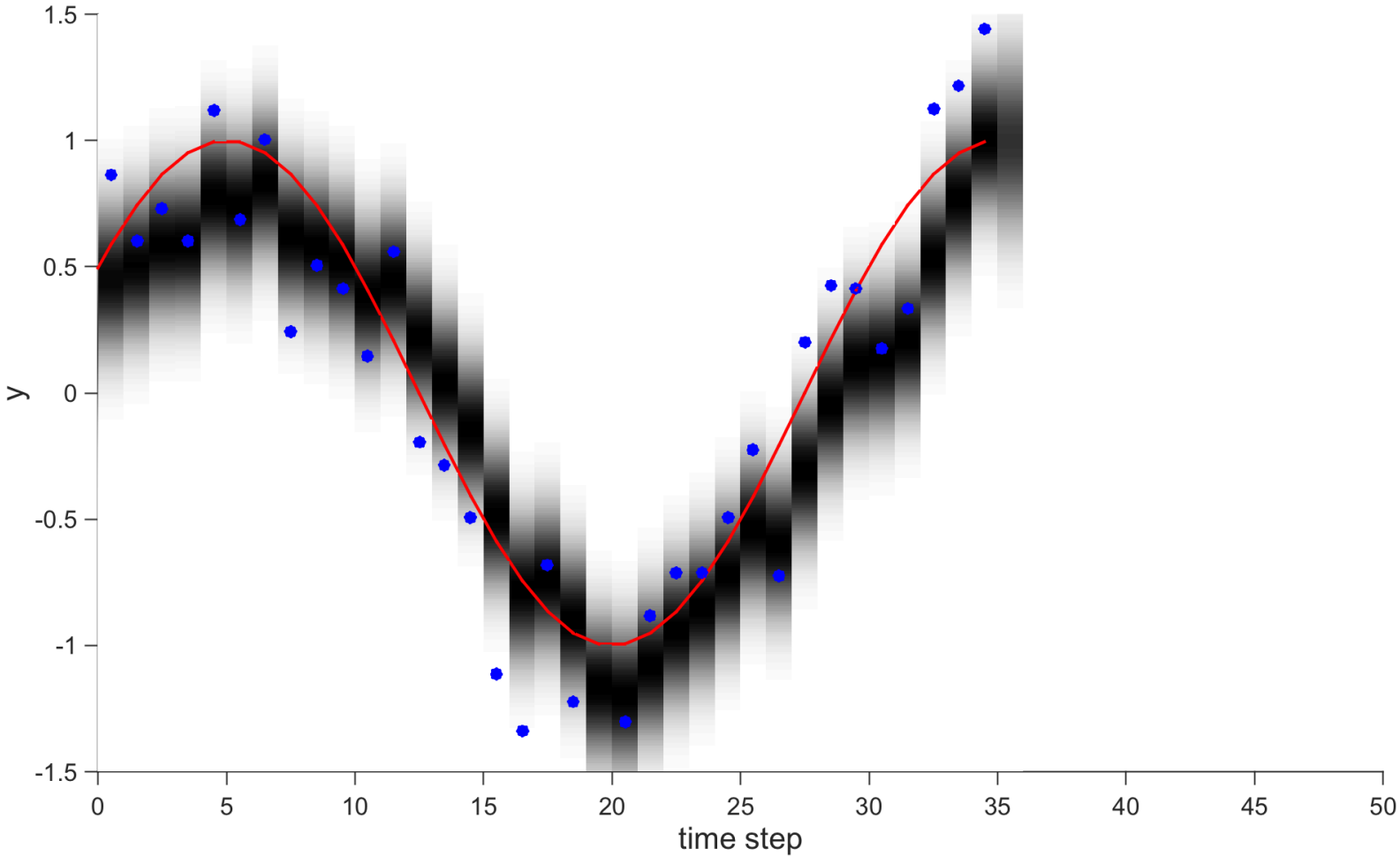

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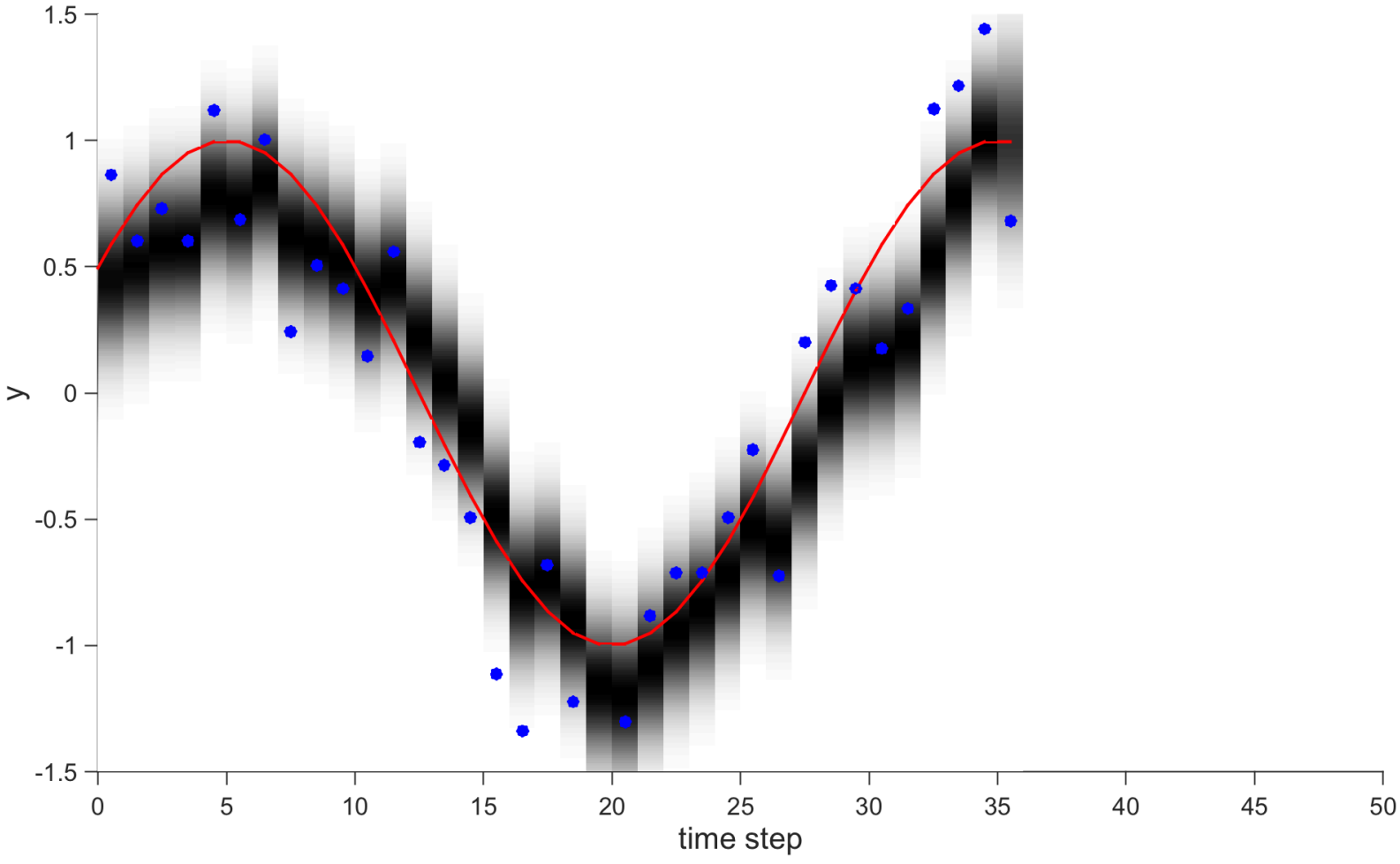

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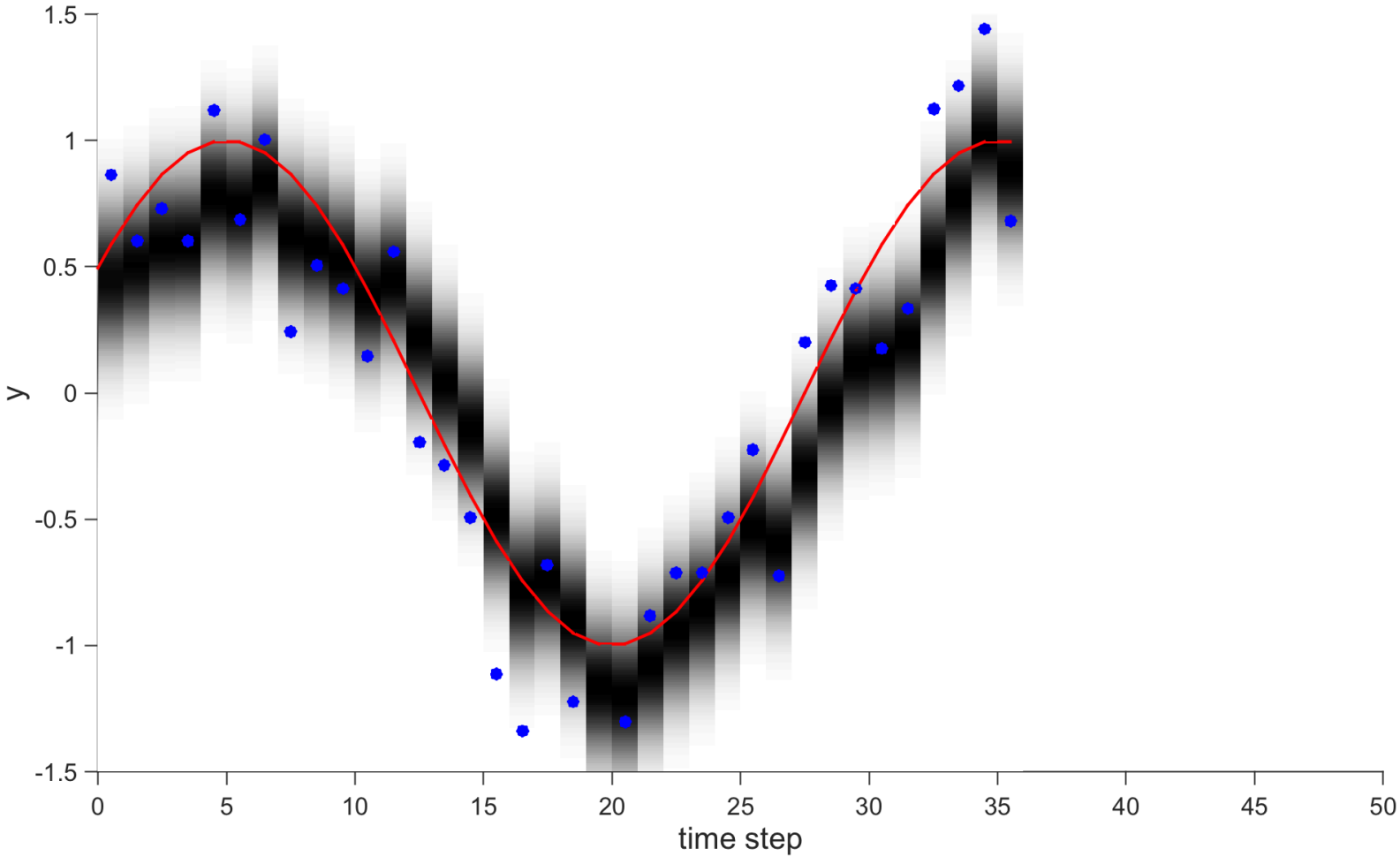

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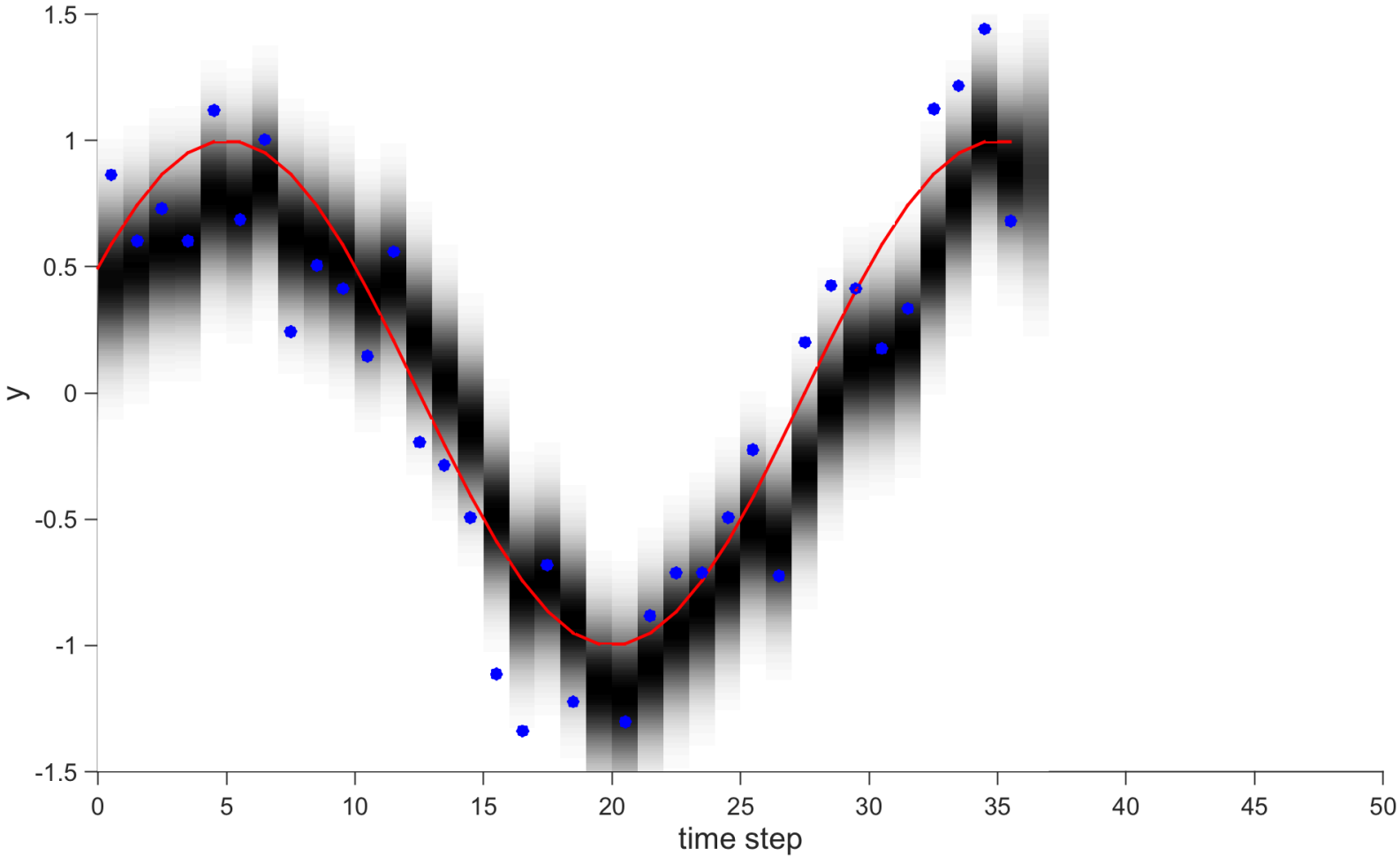

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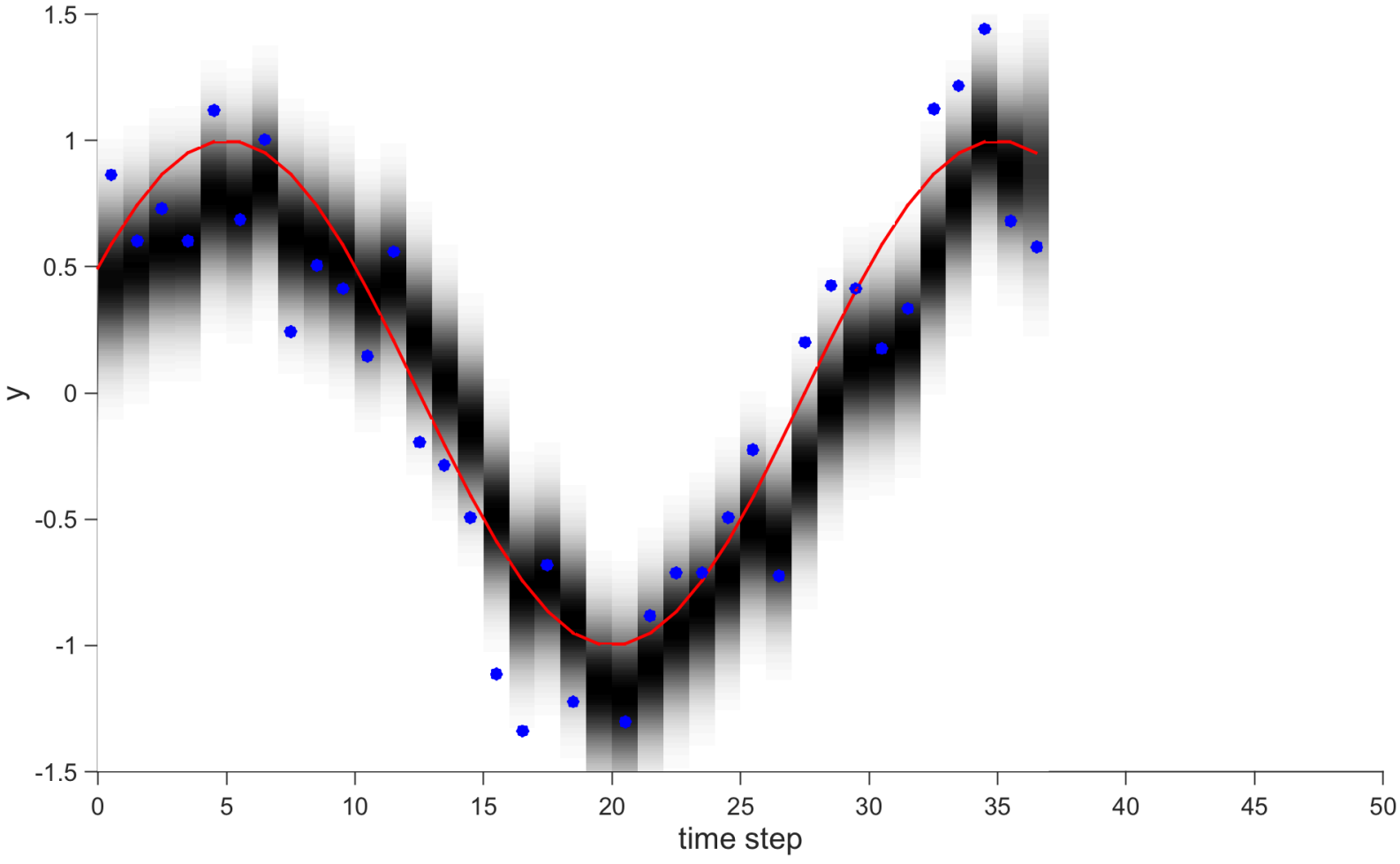

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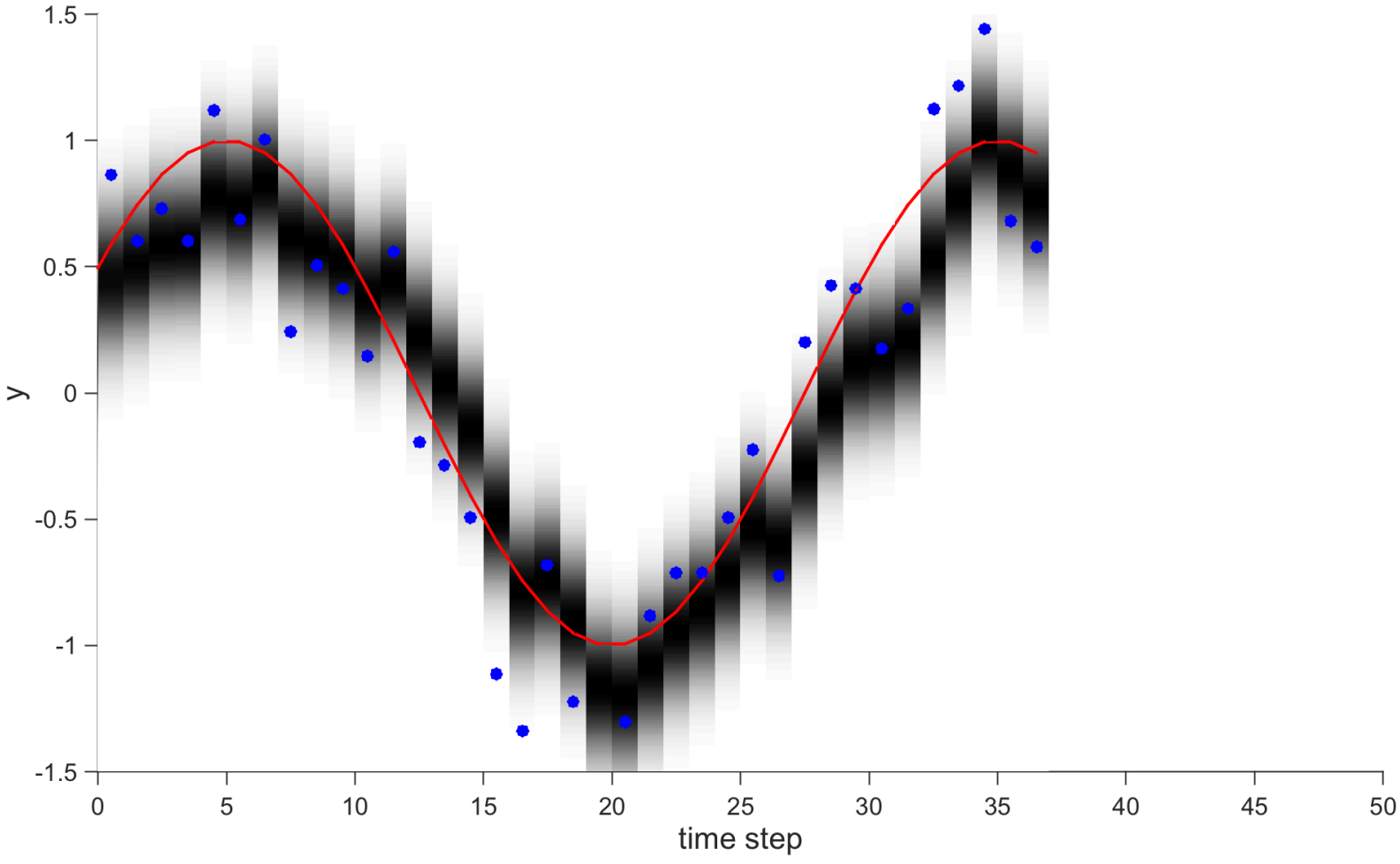

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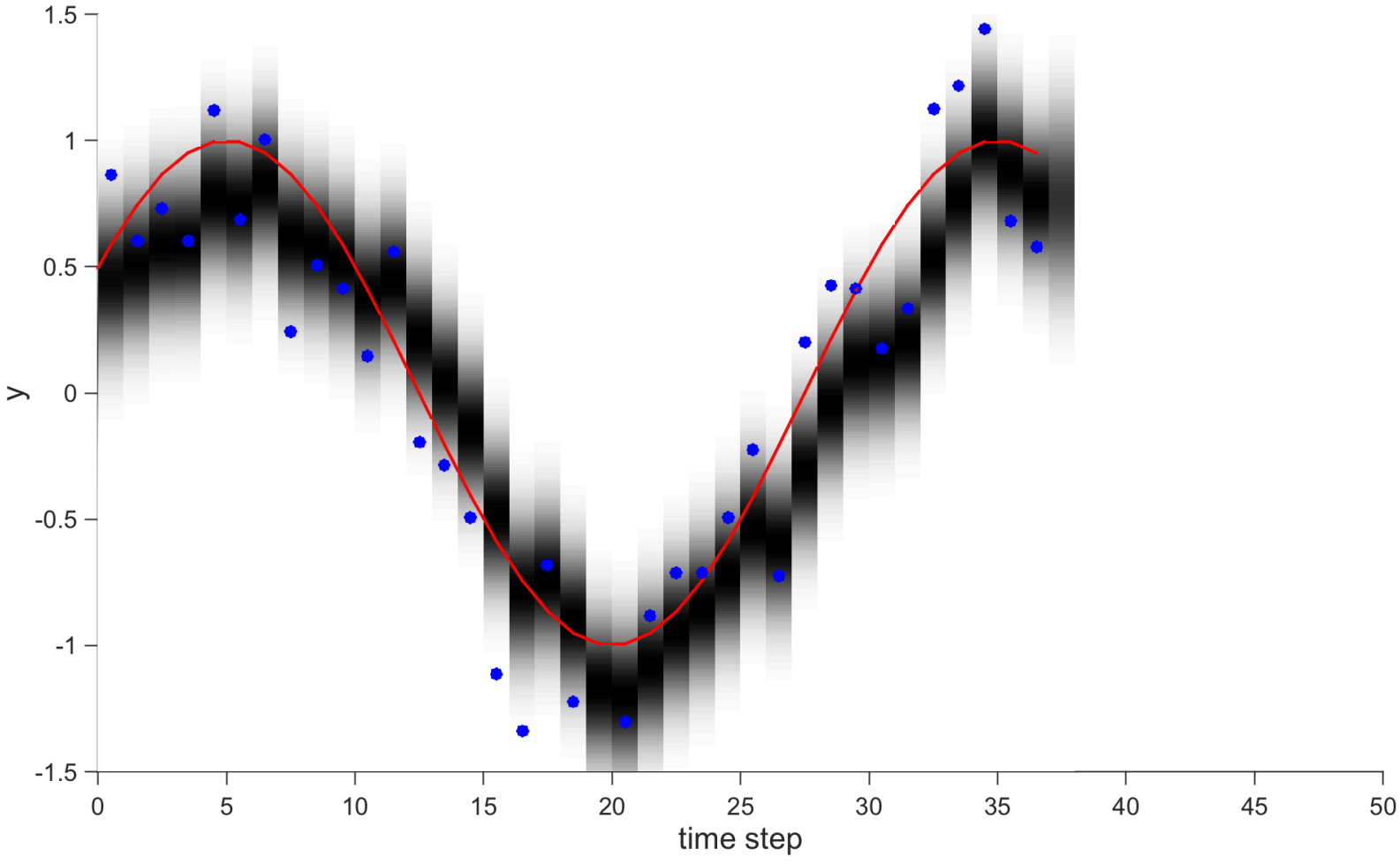

Kalman Filter Demo

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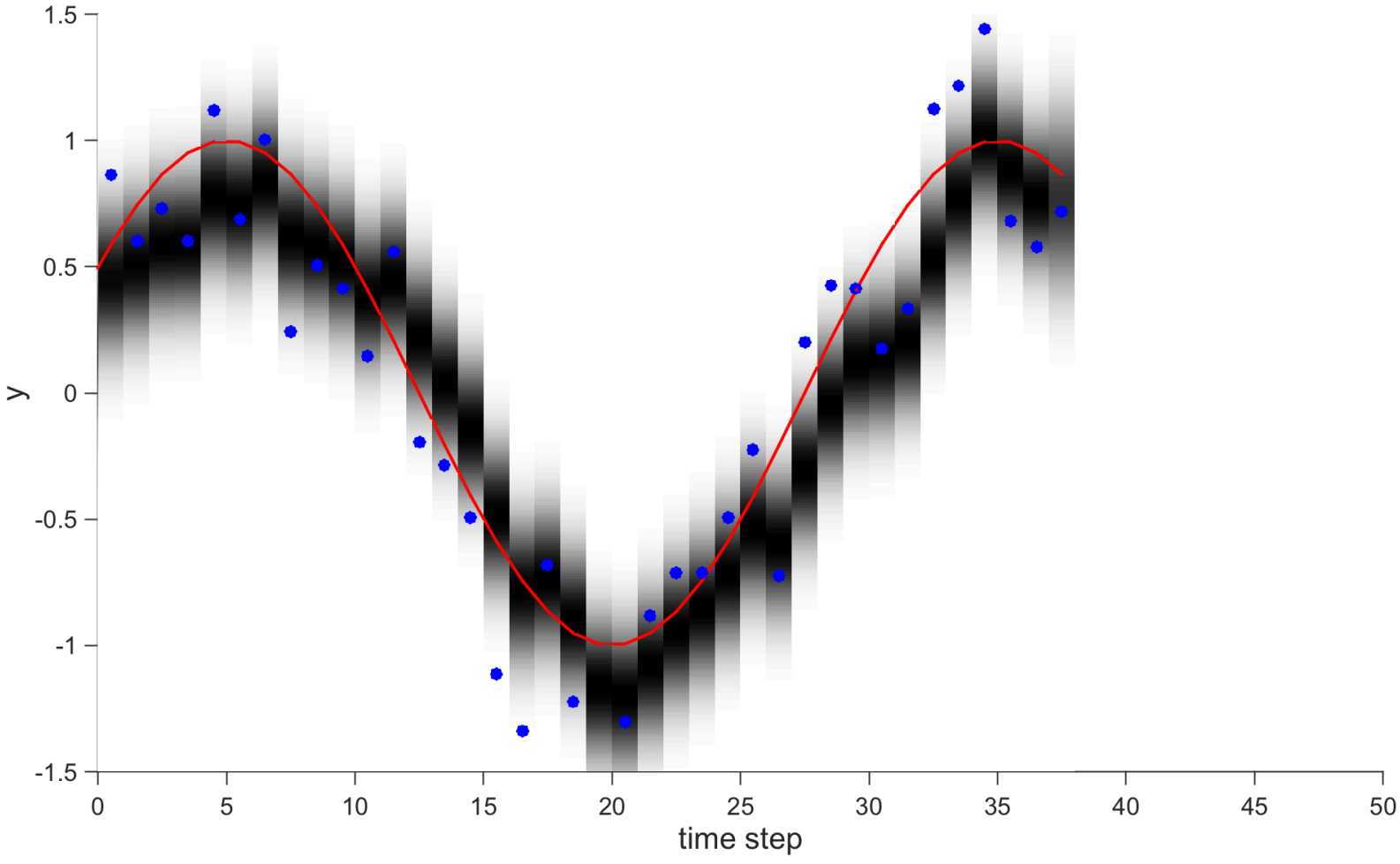

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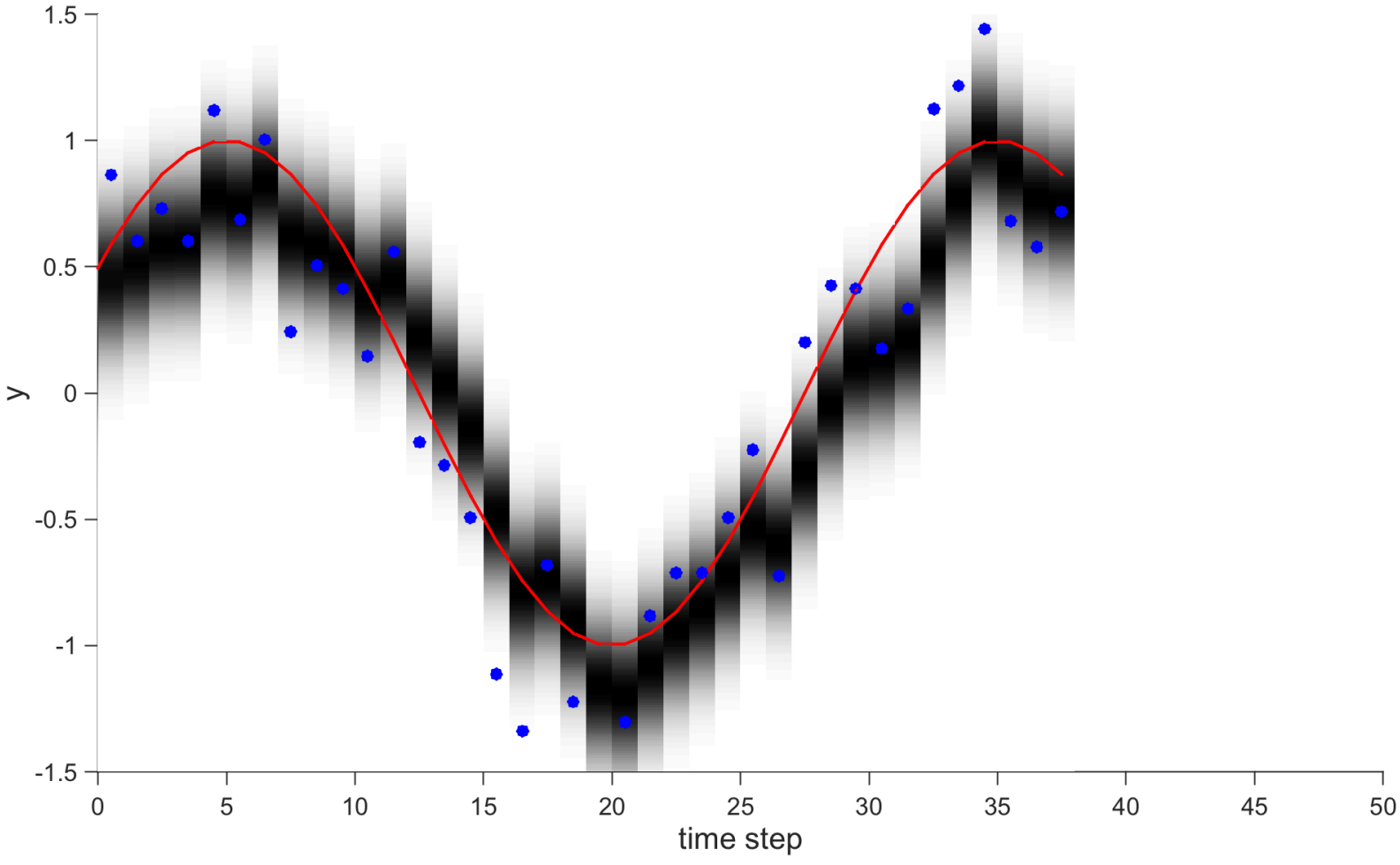

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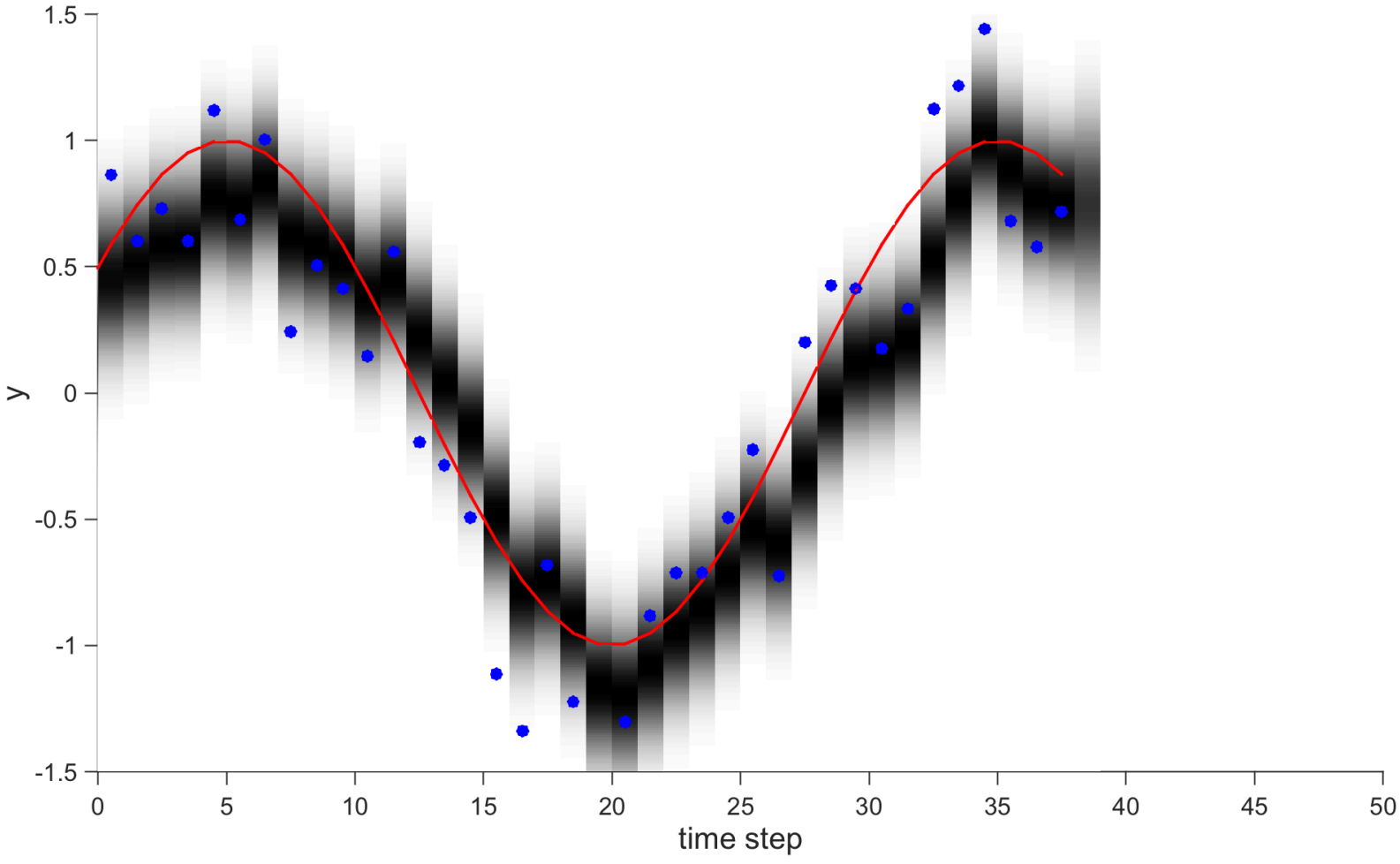

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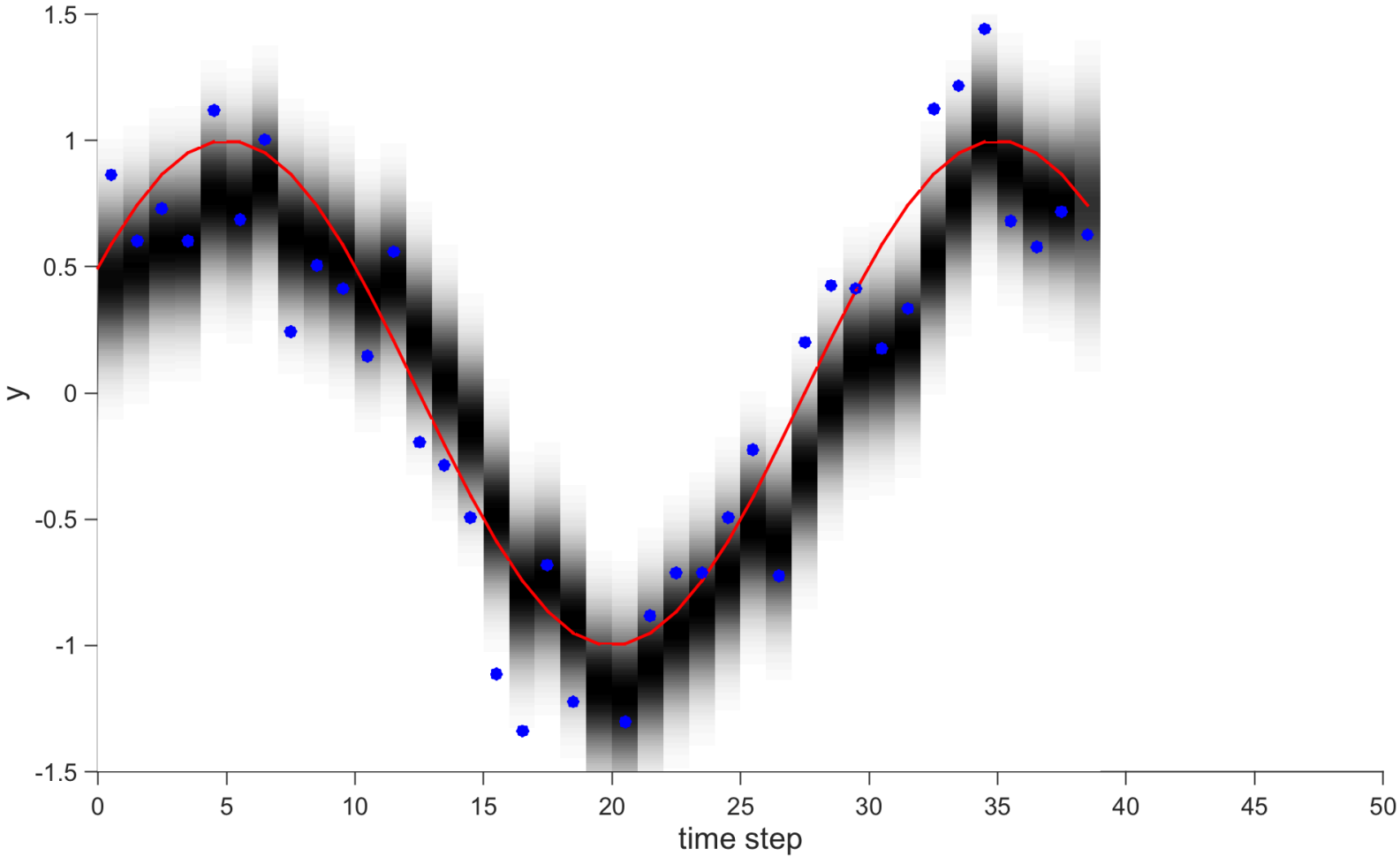

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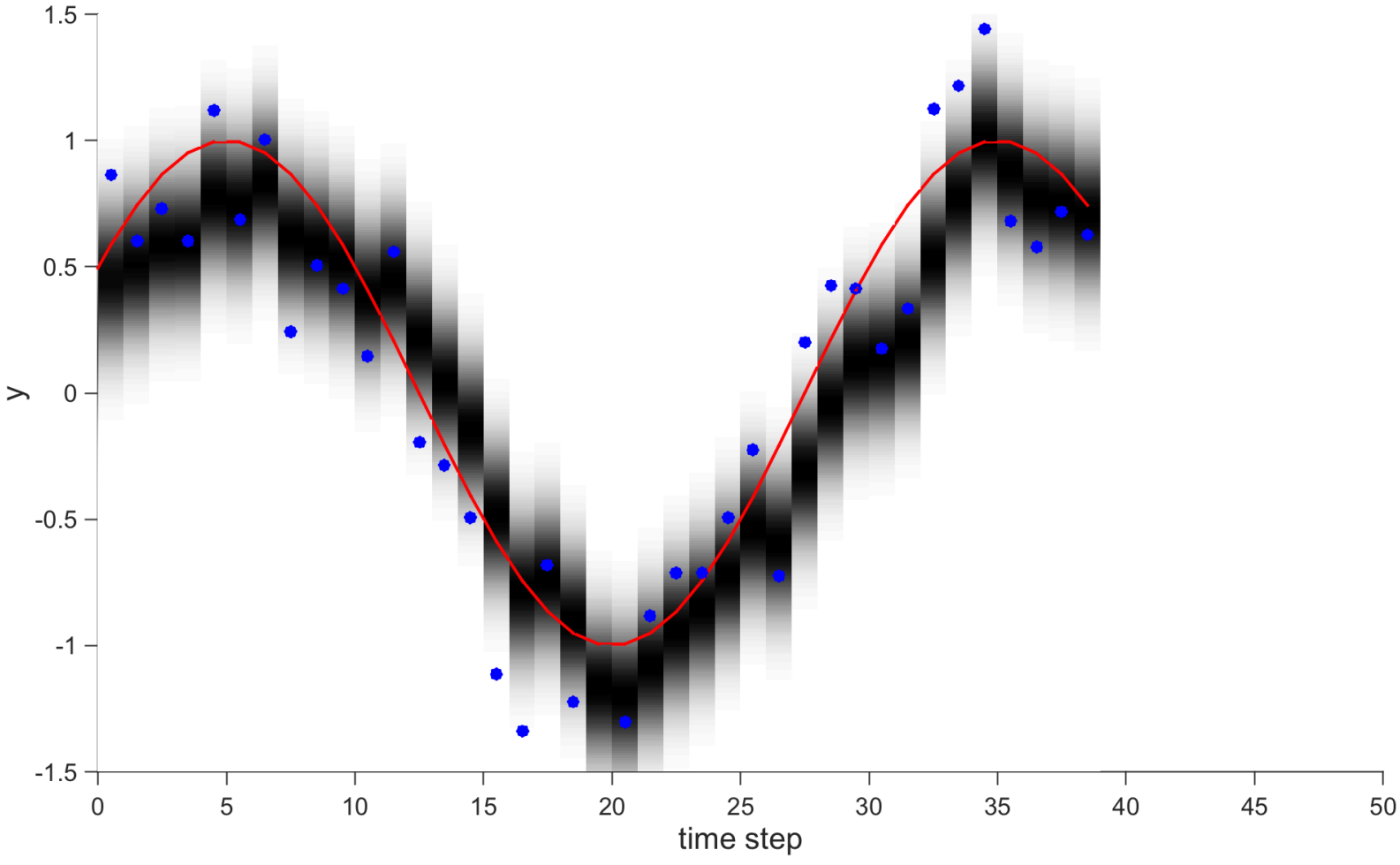

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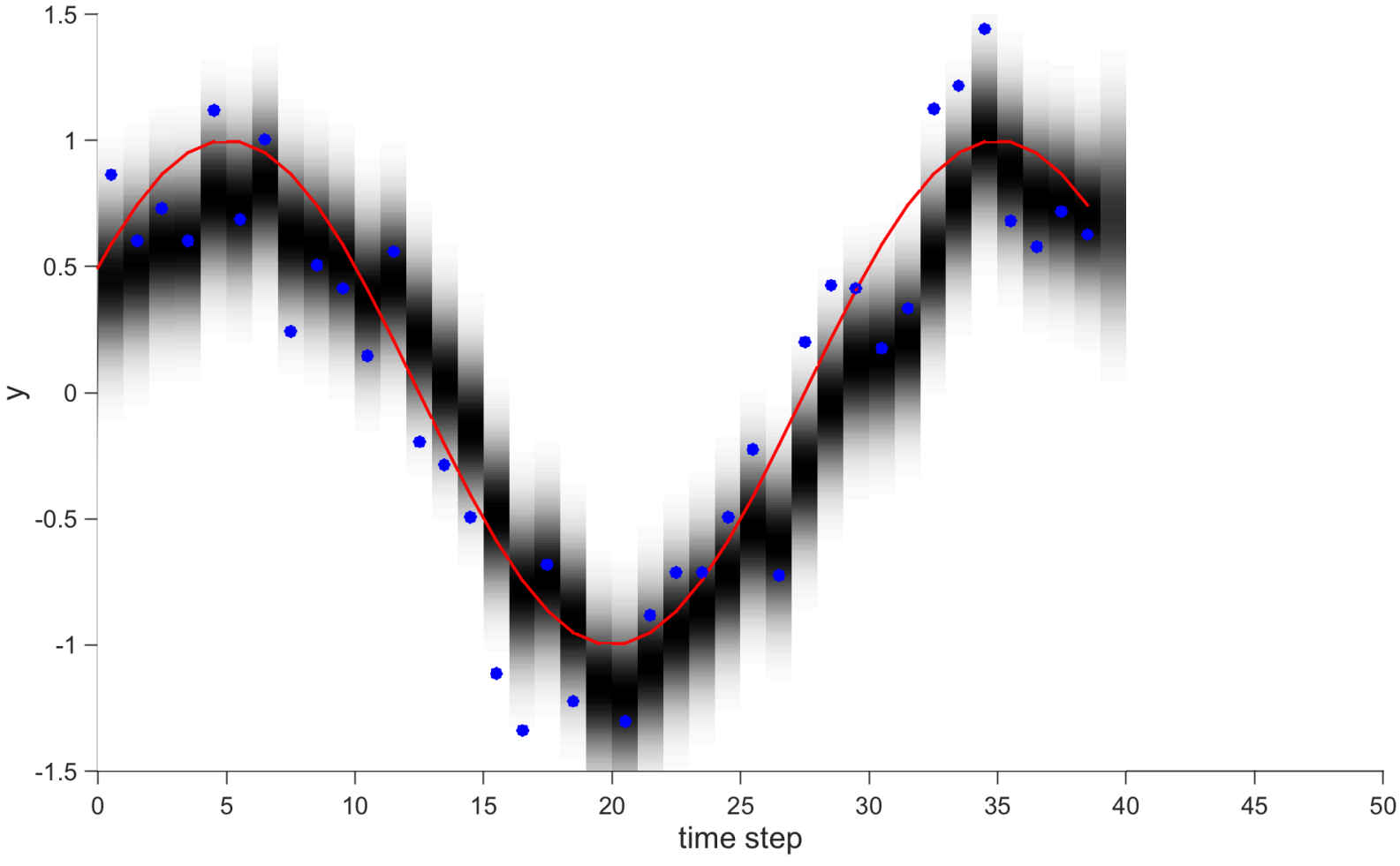

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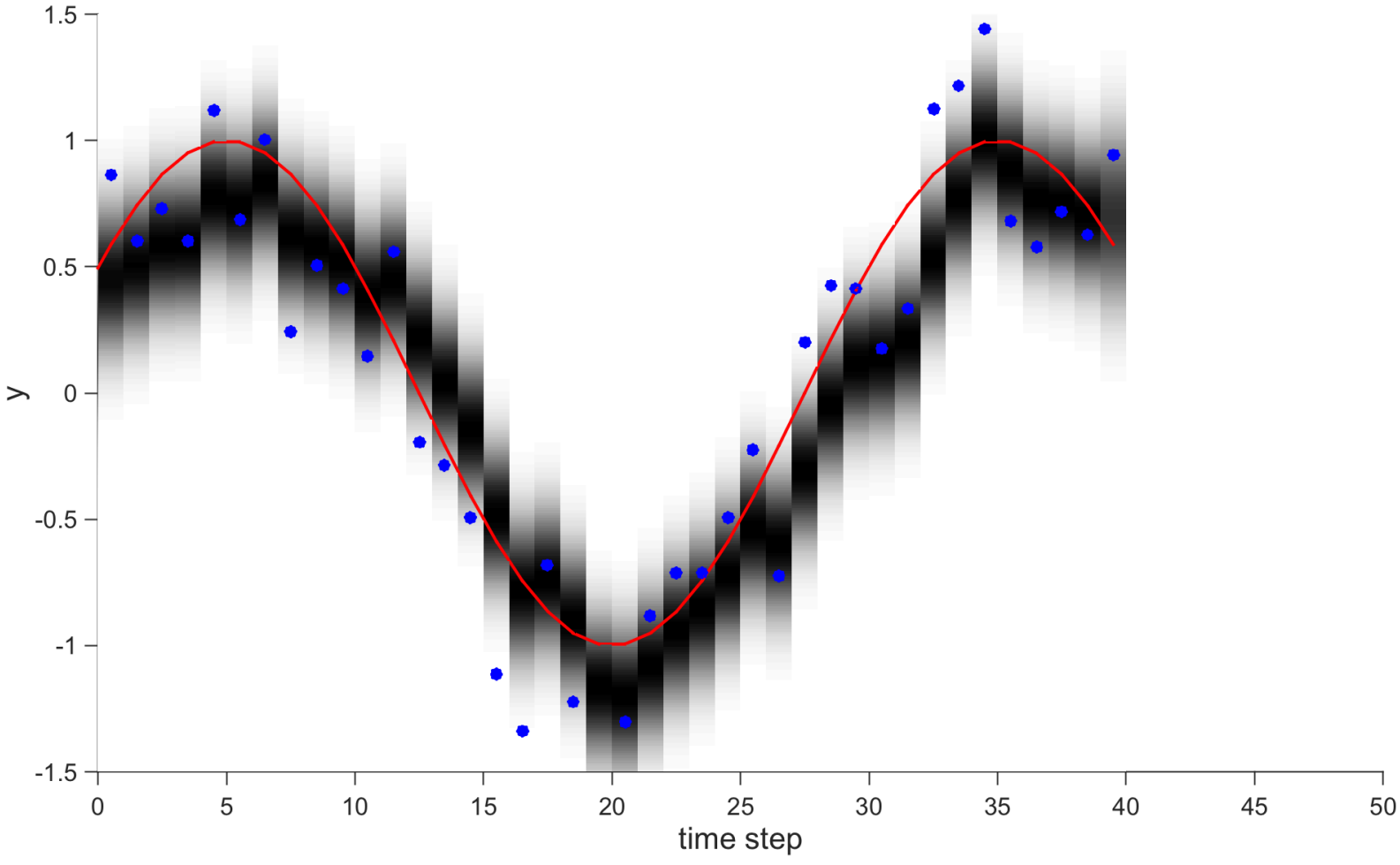

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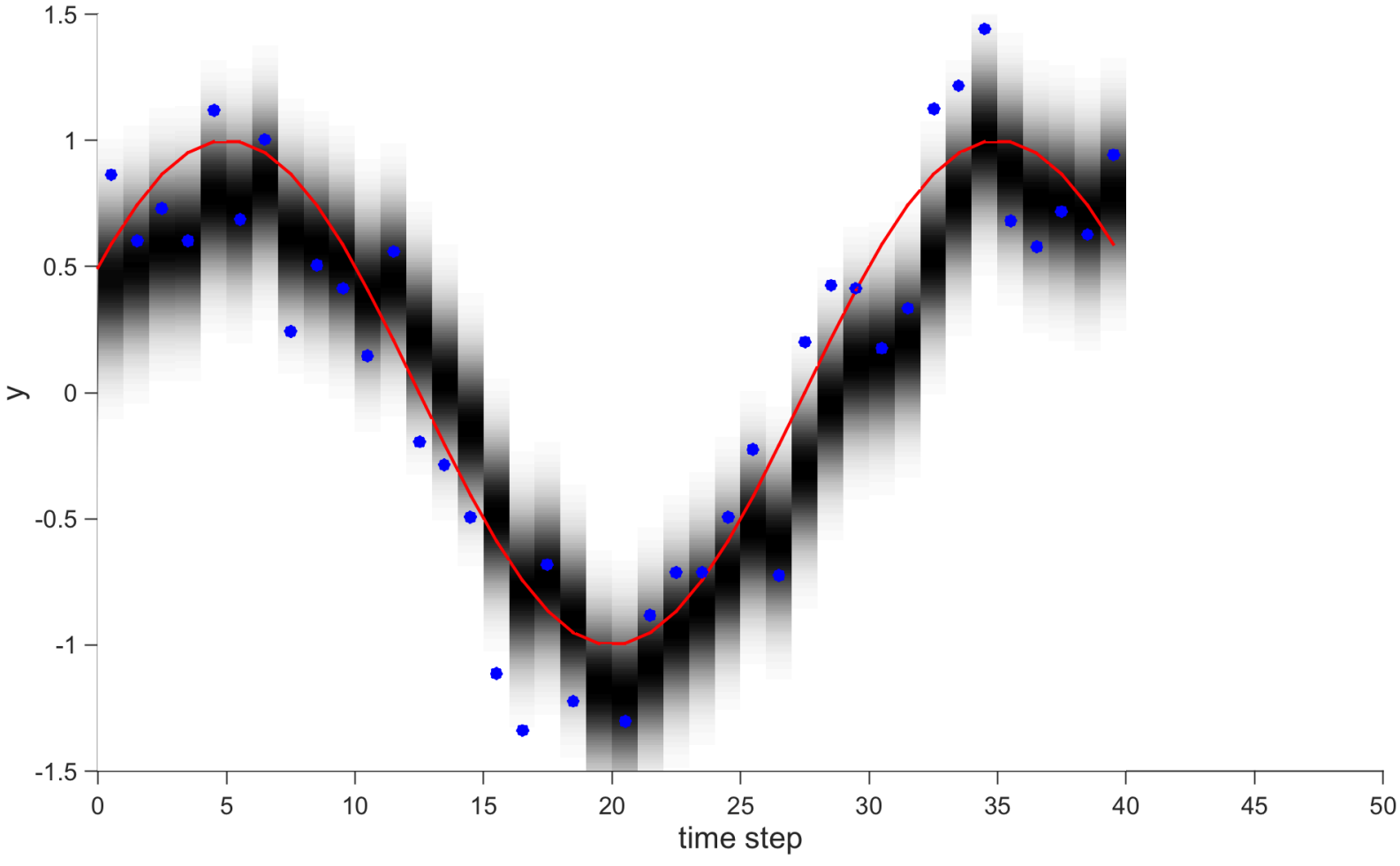

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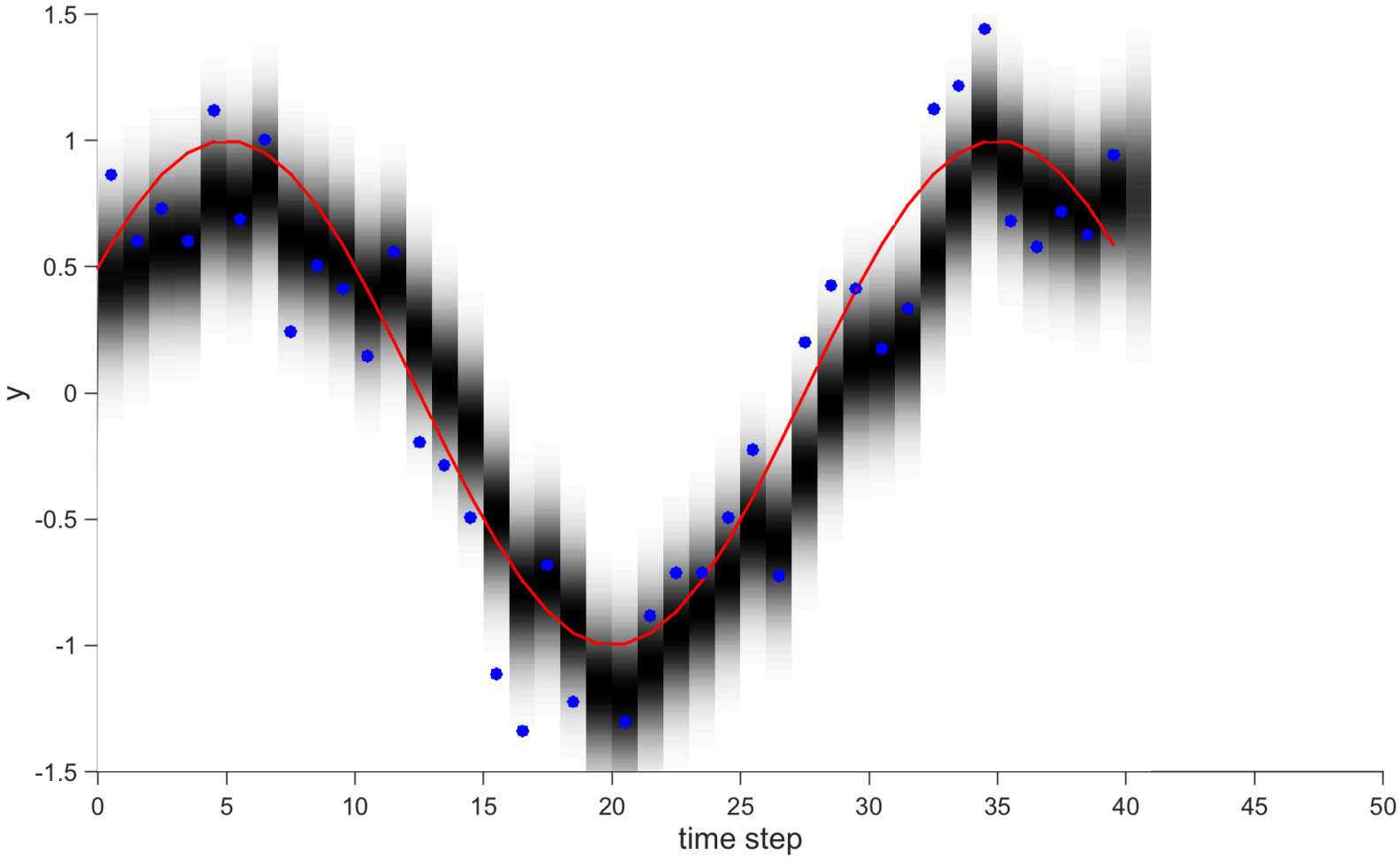

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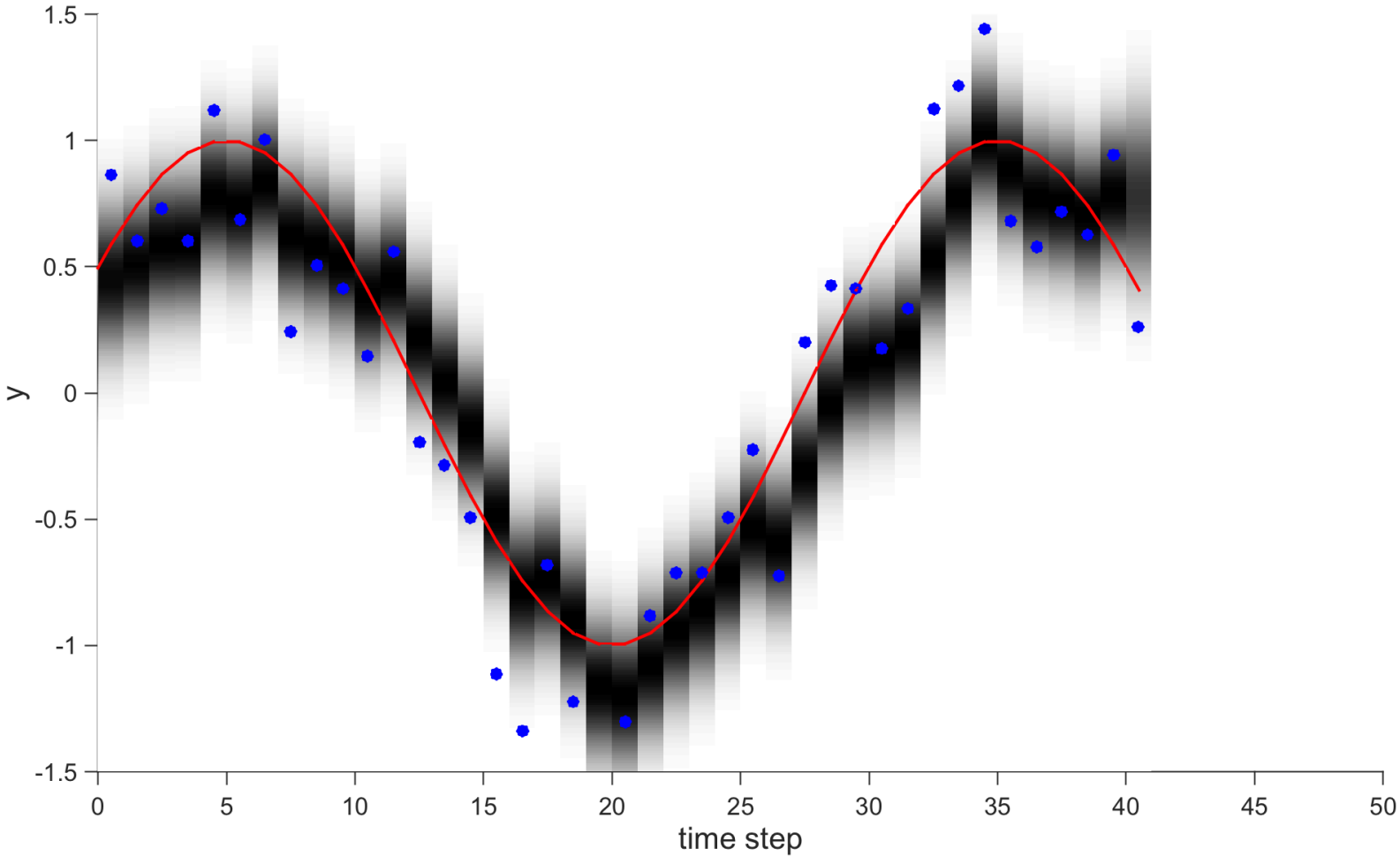

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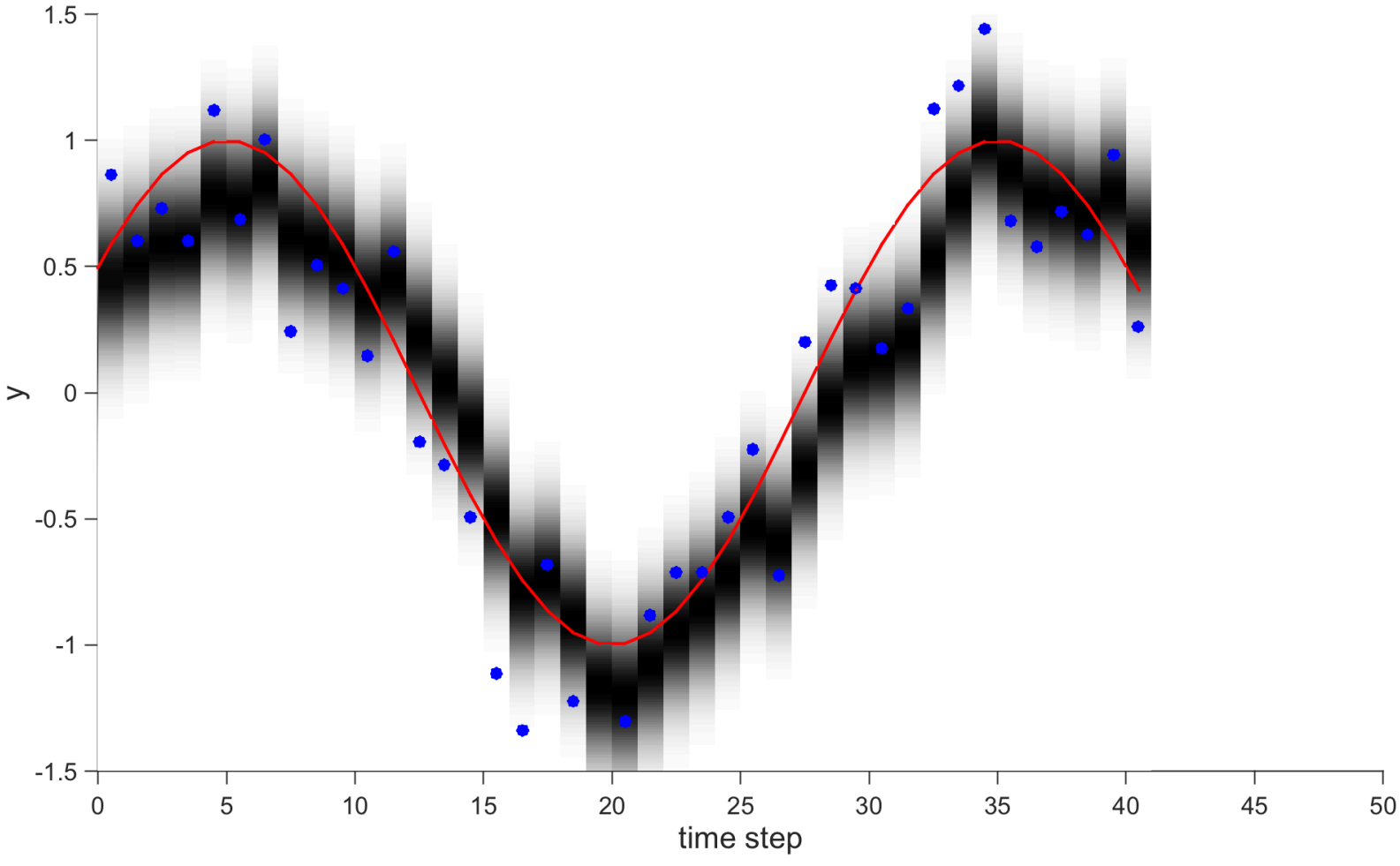

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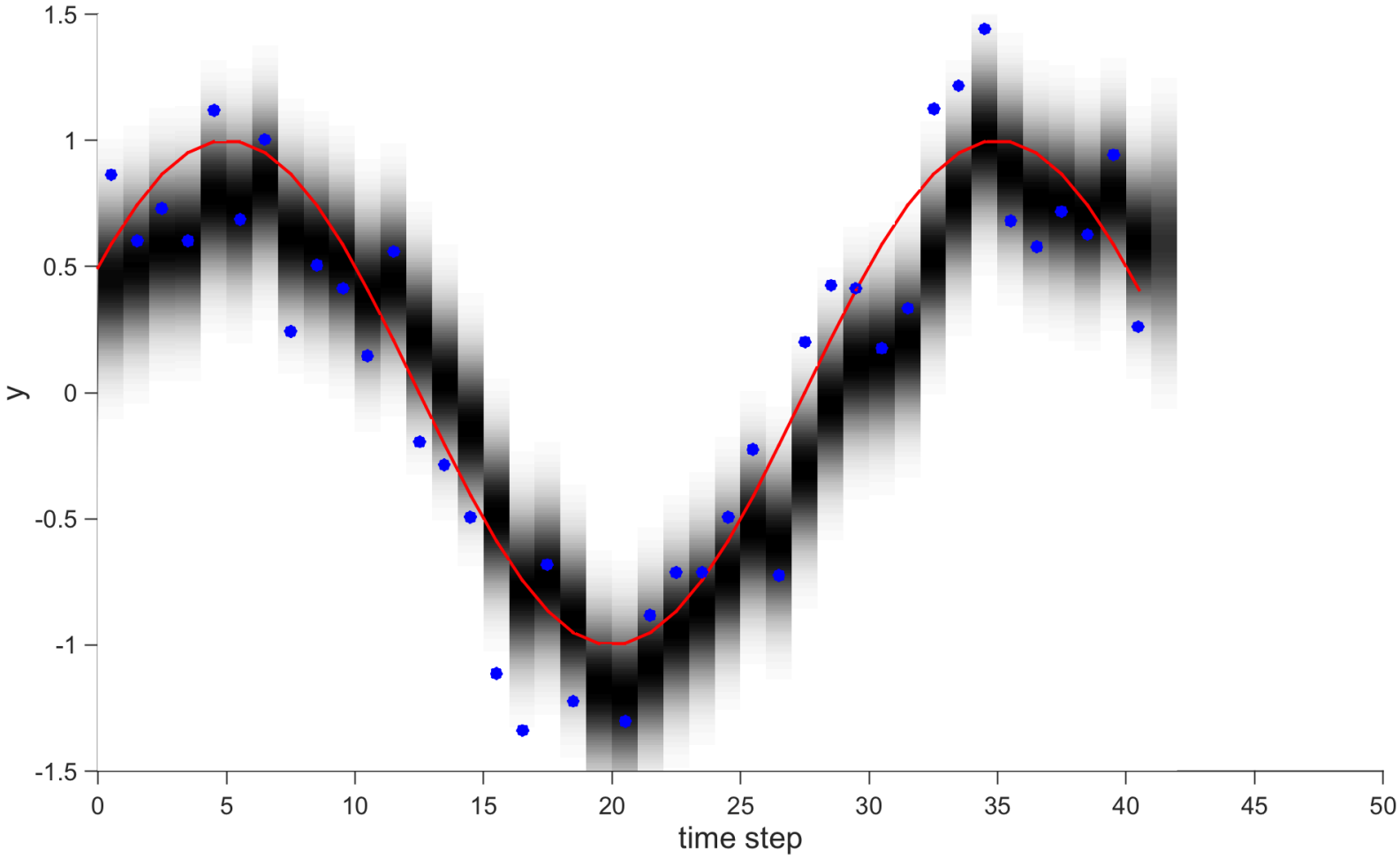

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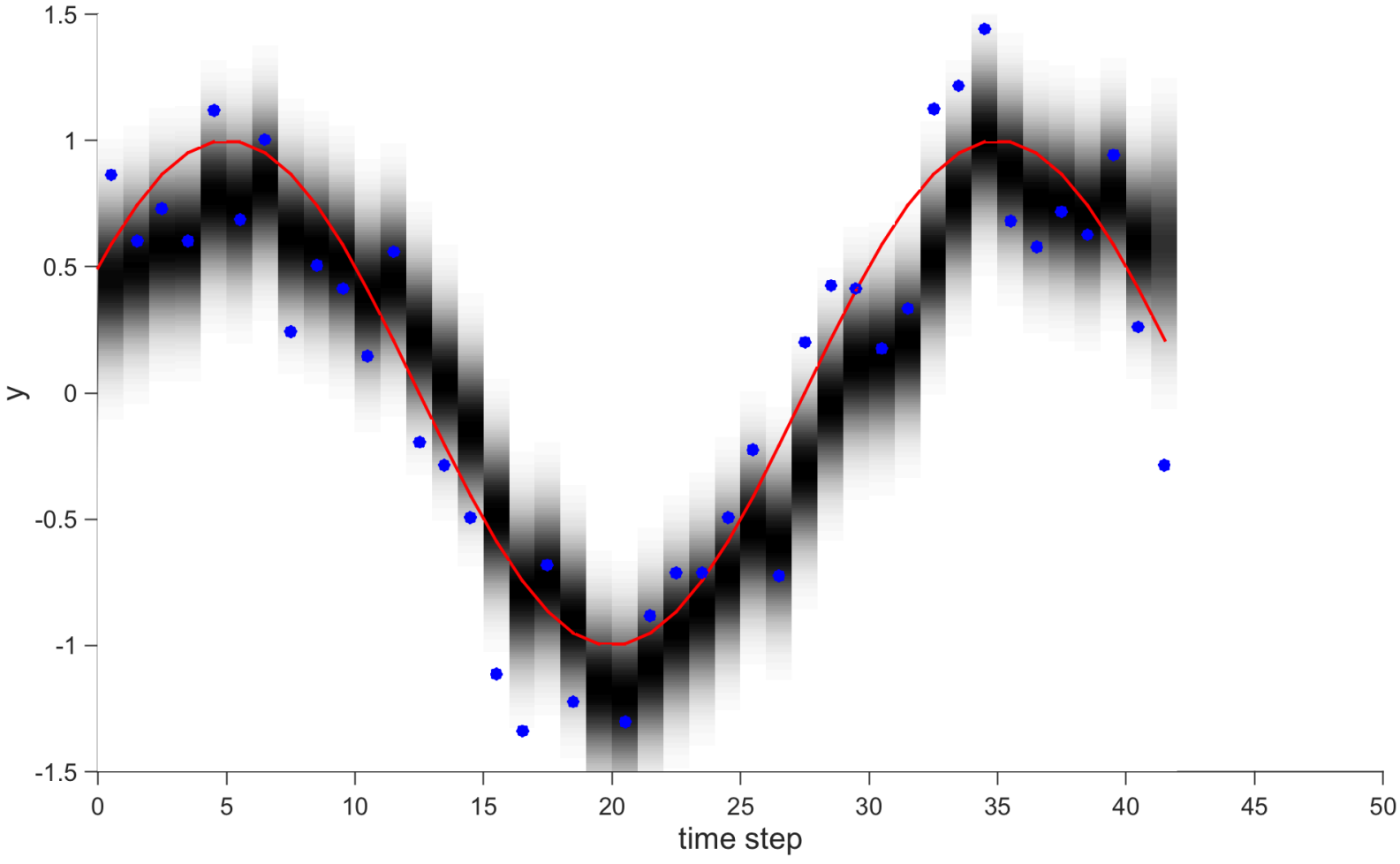

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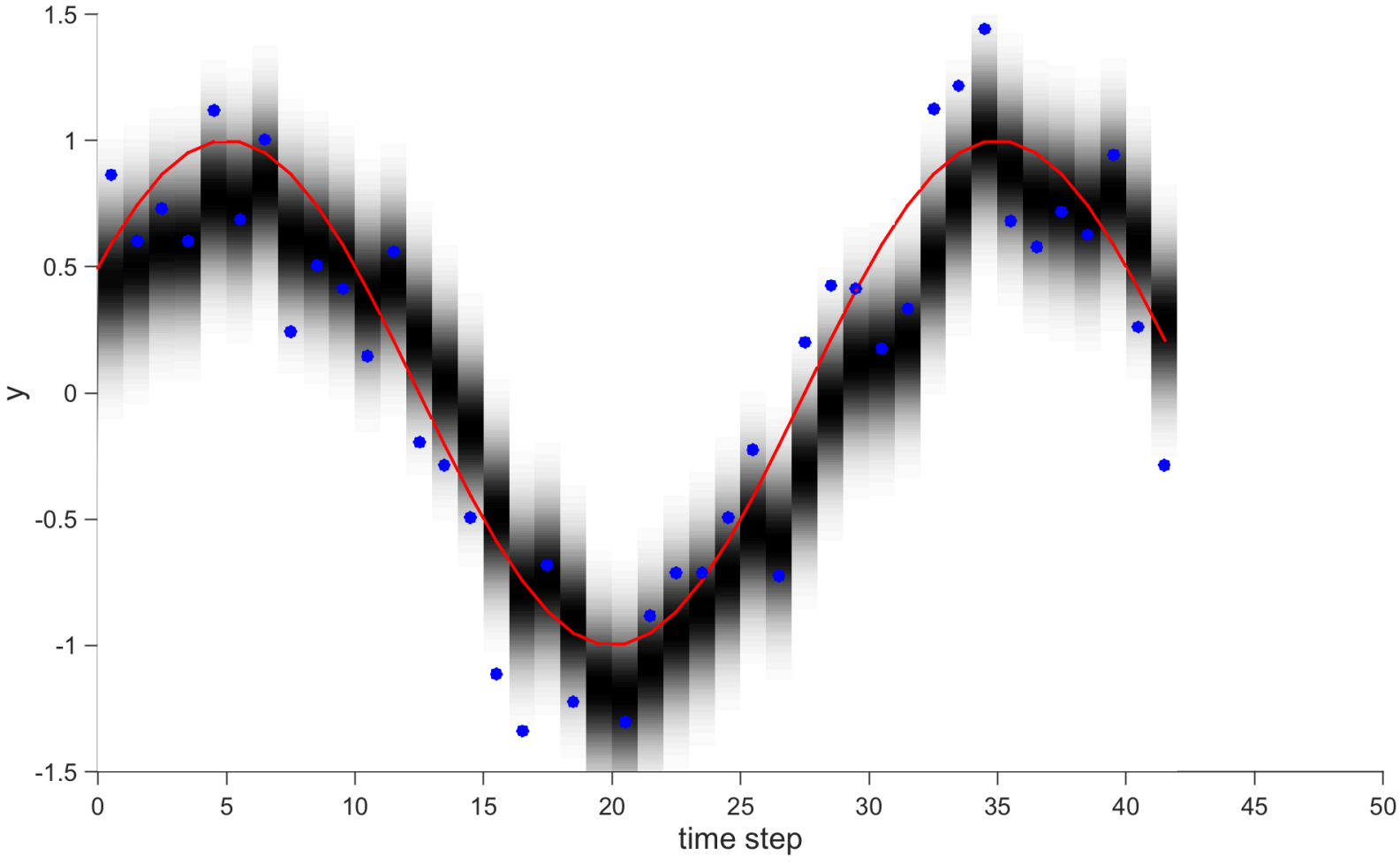

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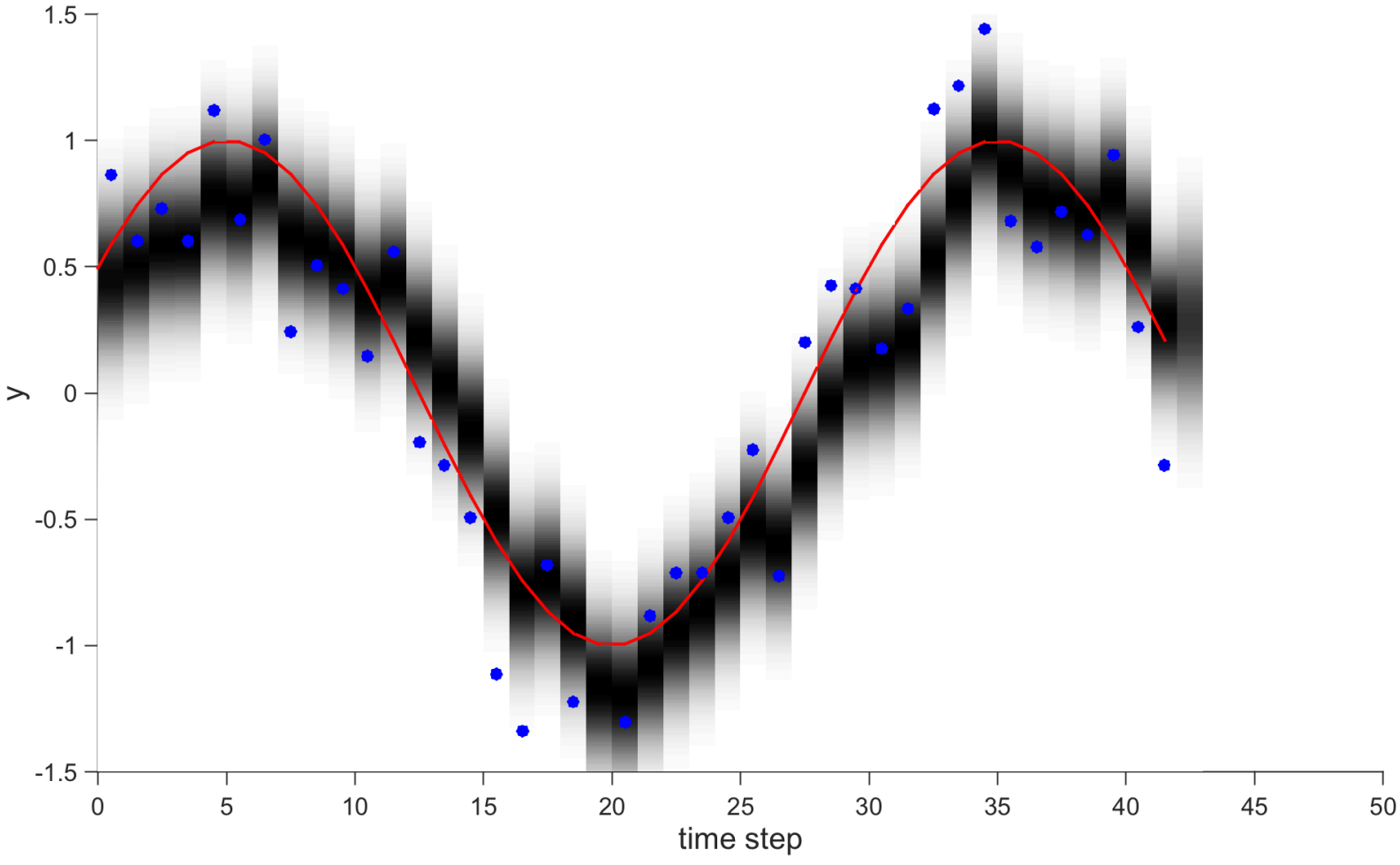

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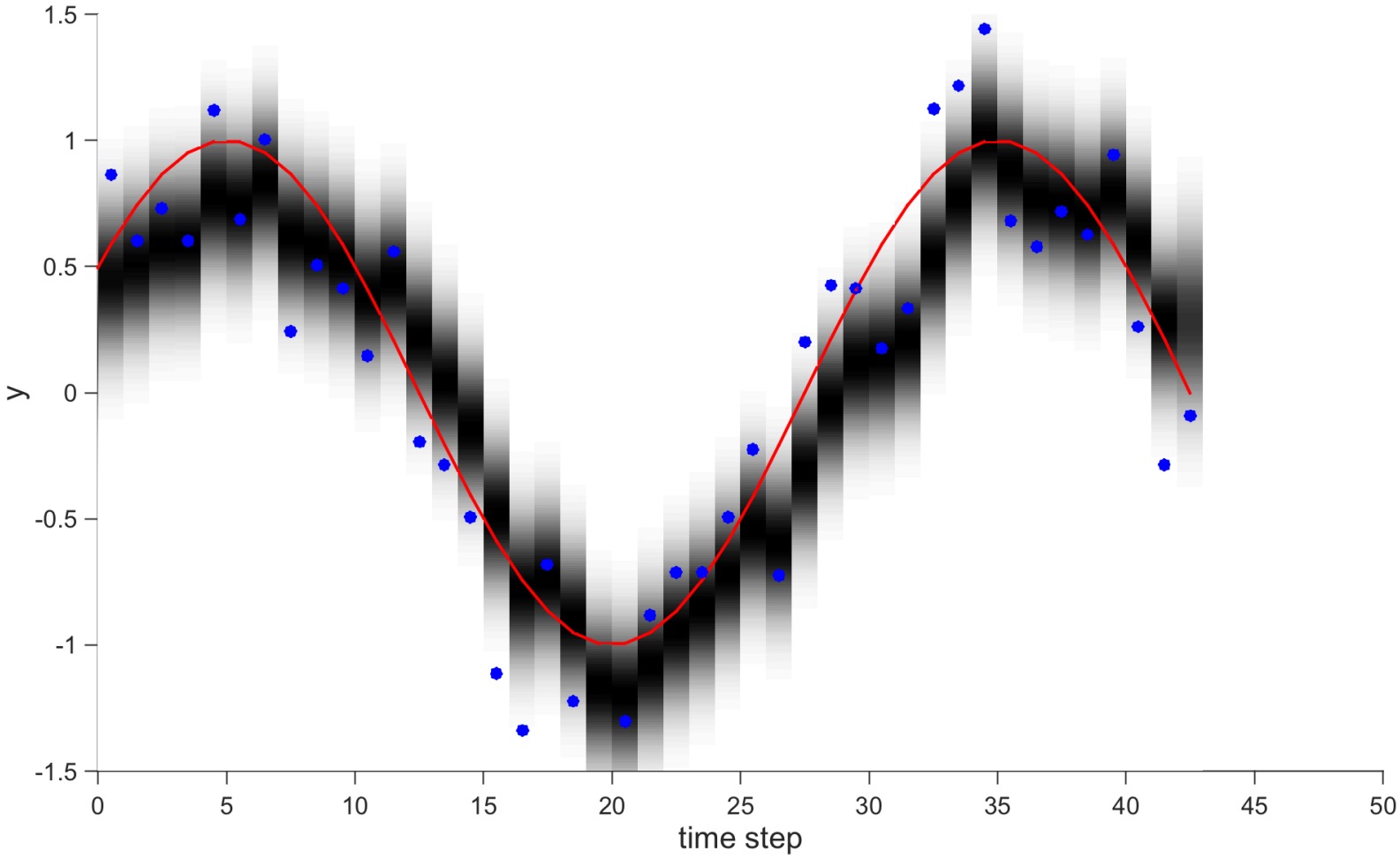

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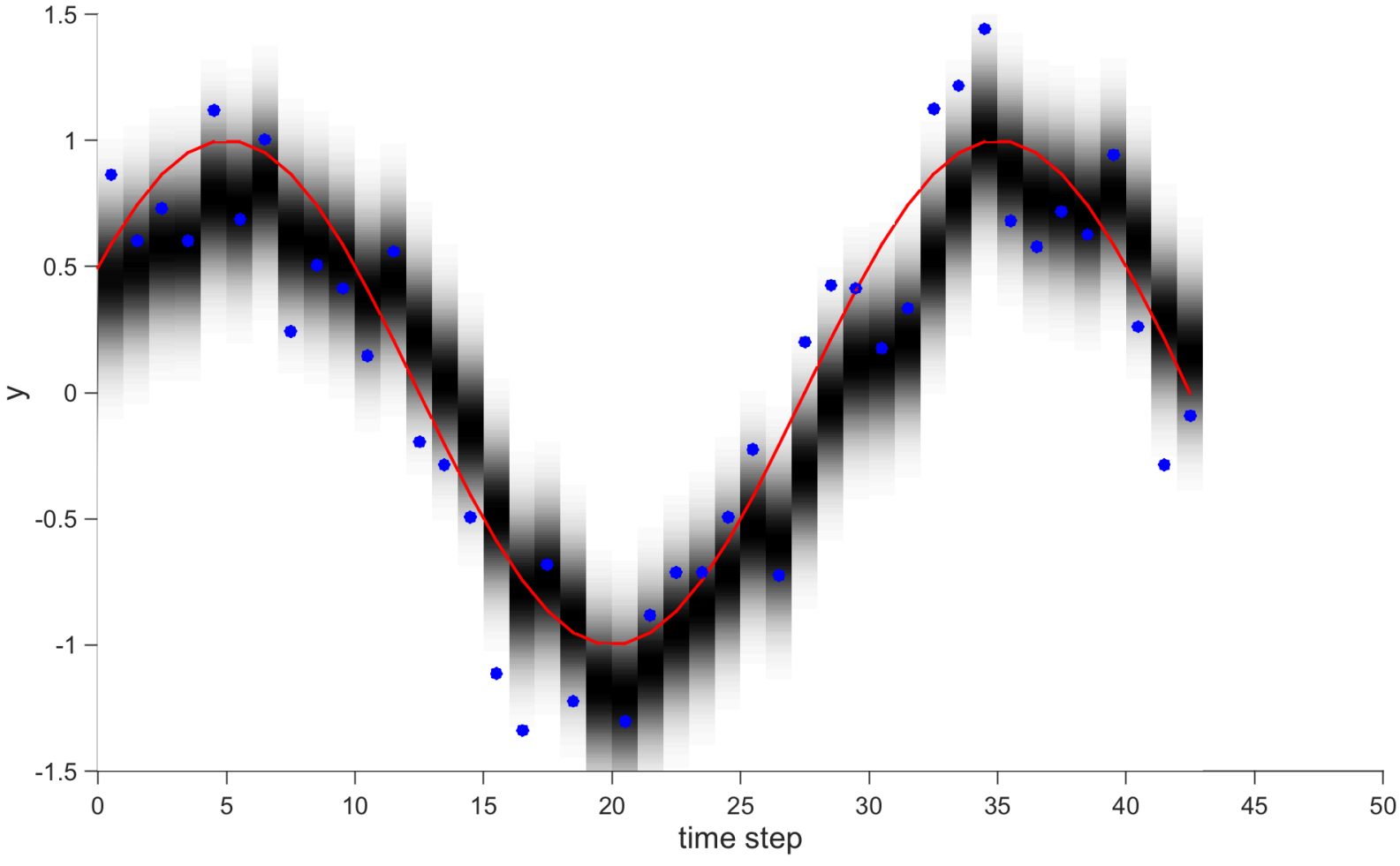

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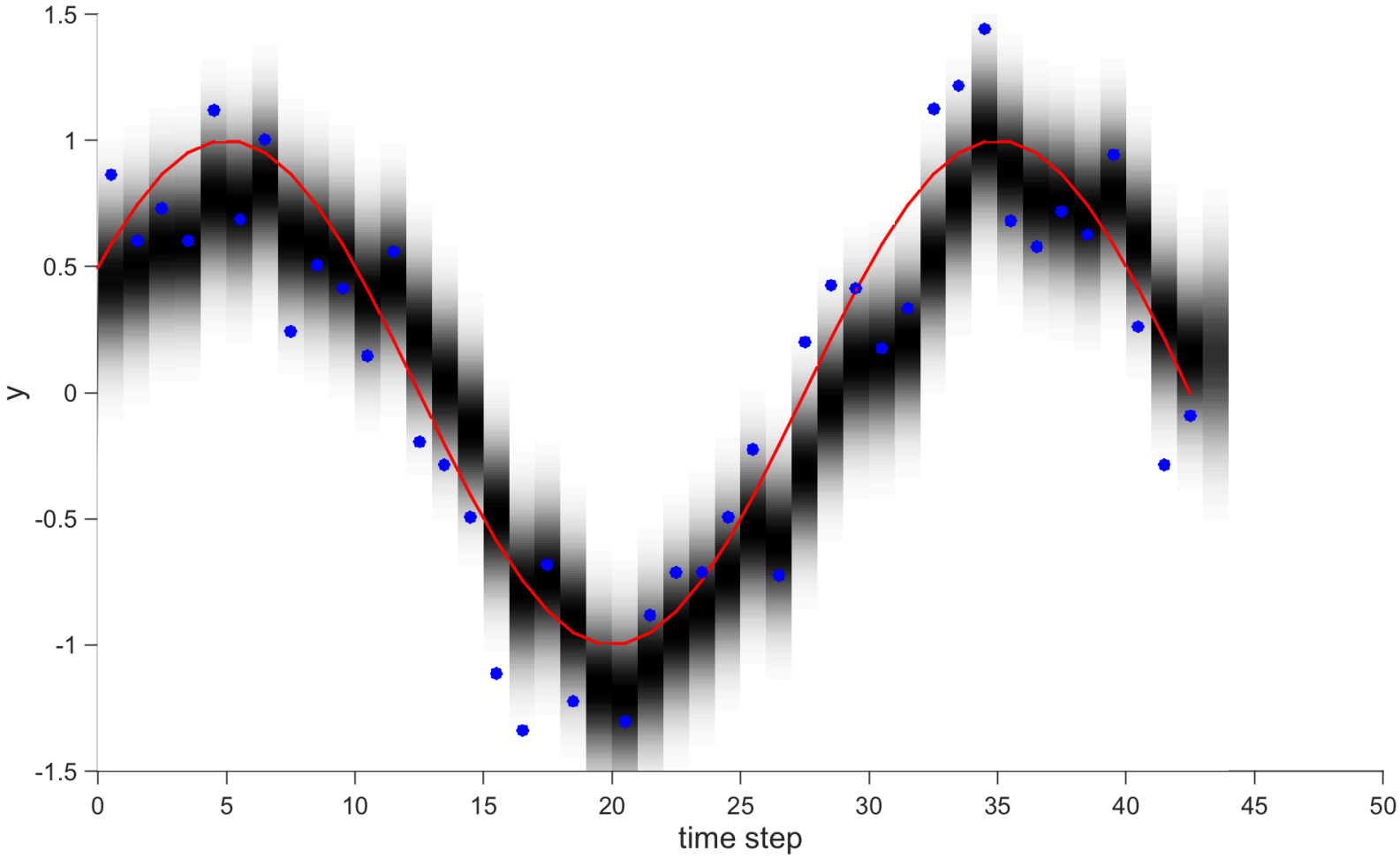

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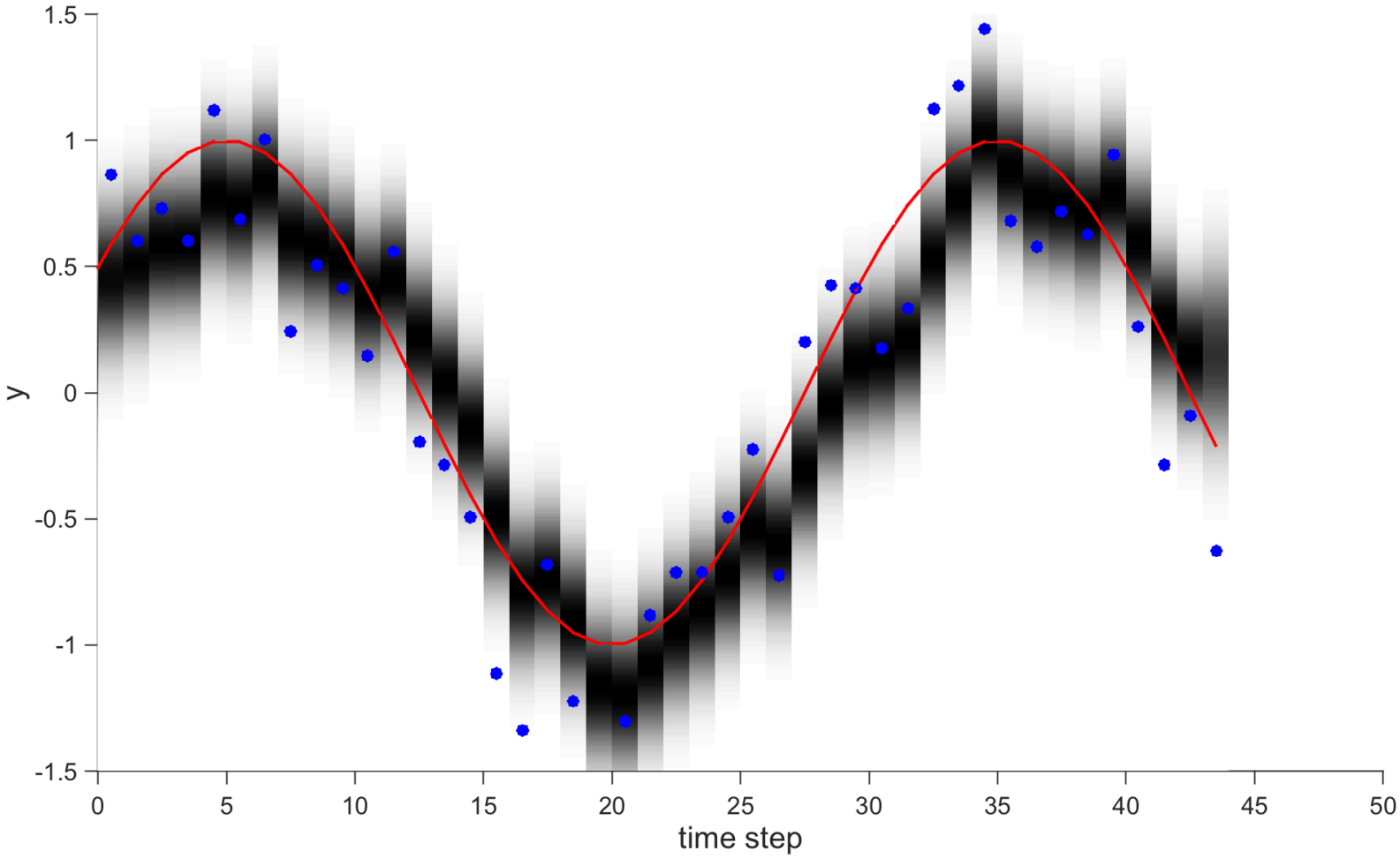

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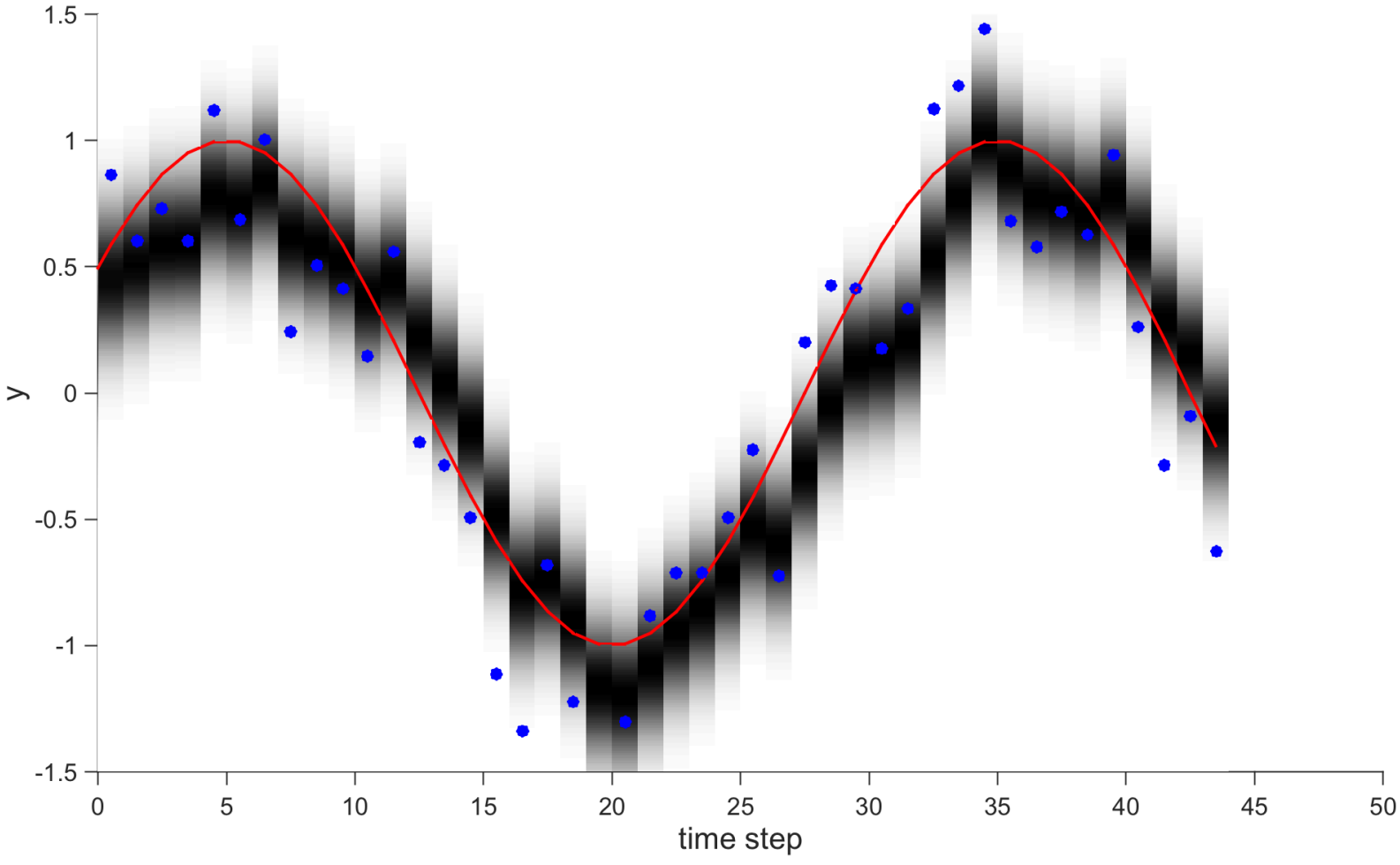

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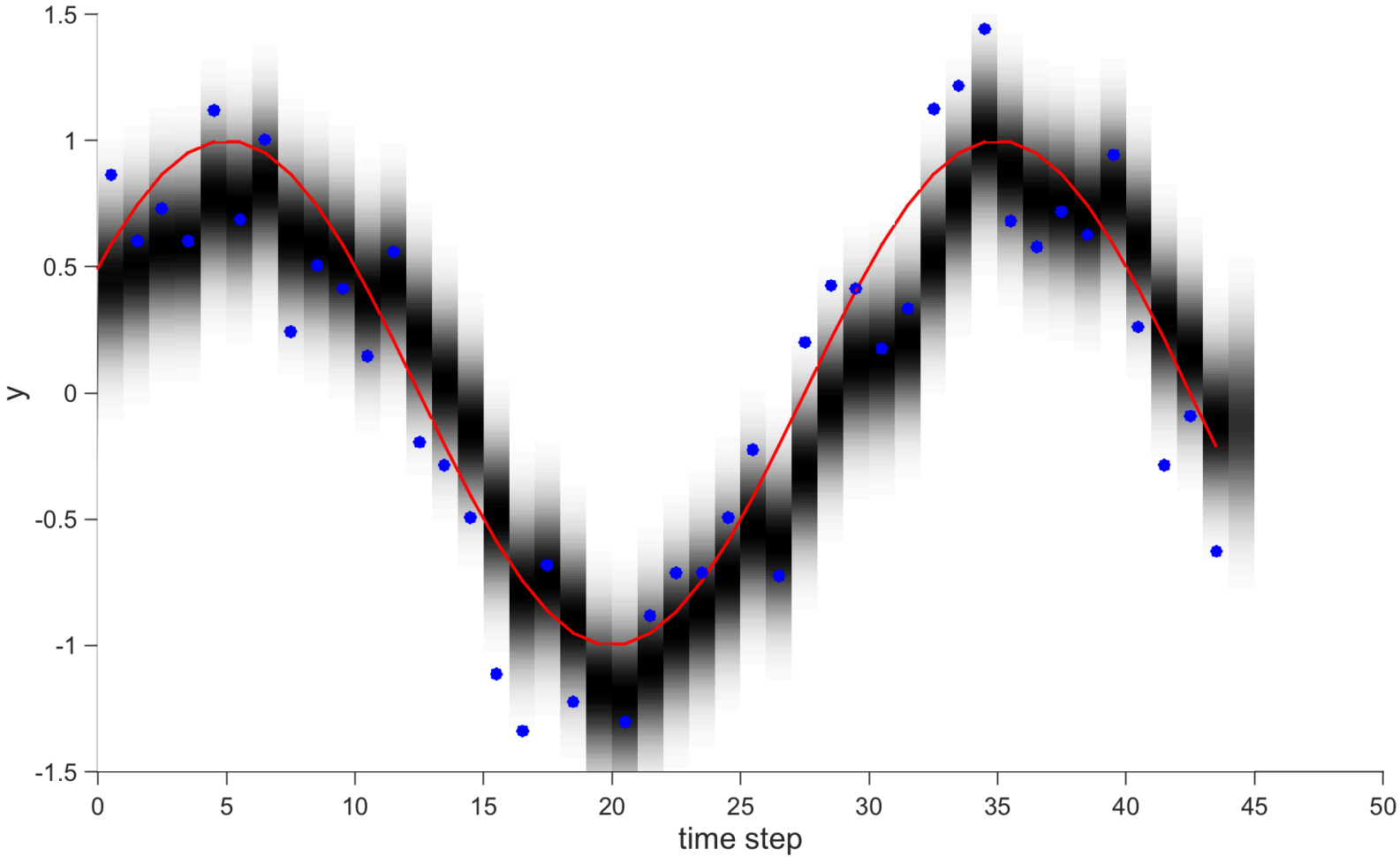

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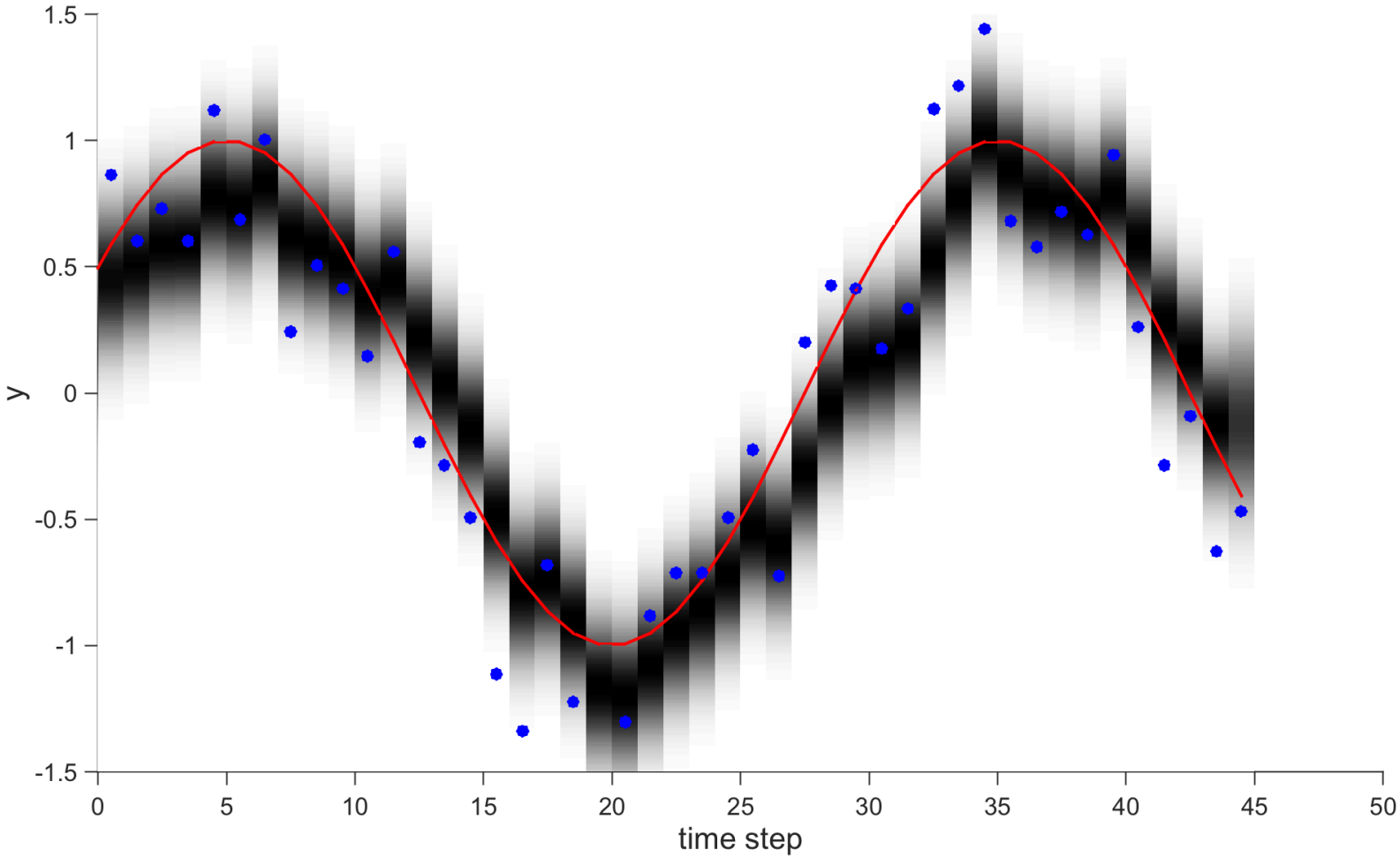

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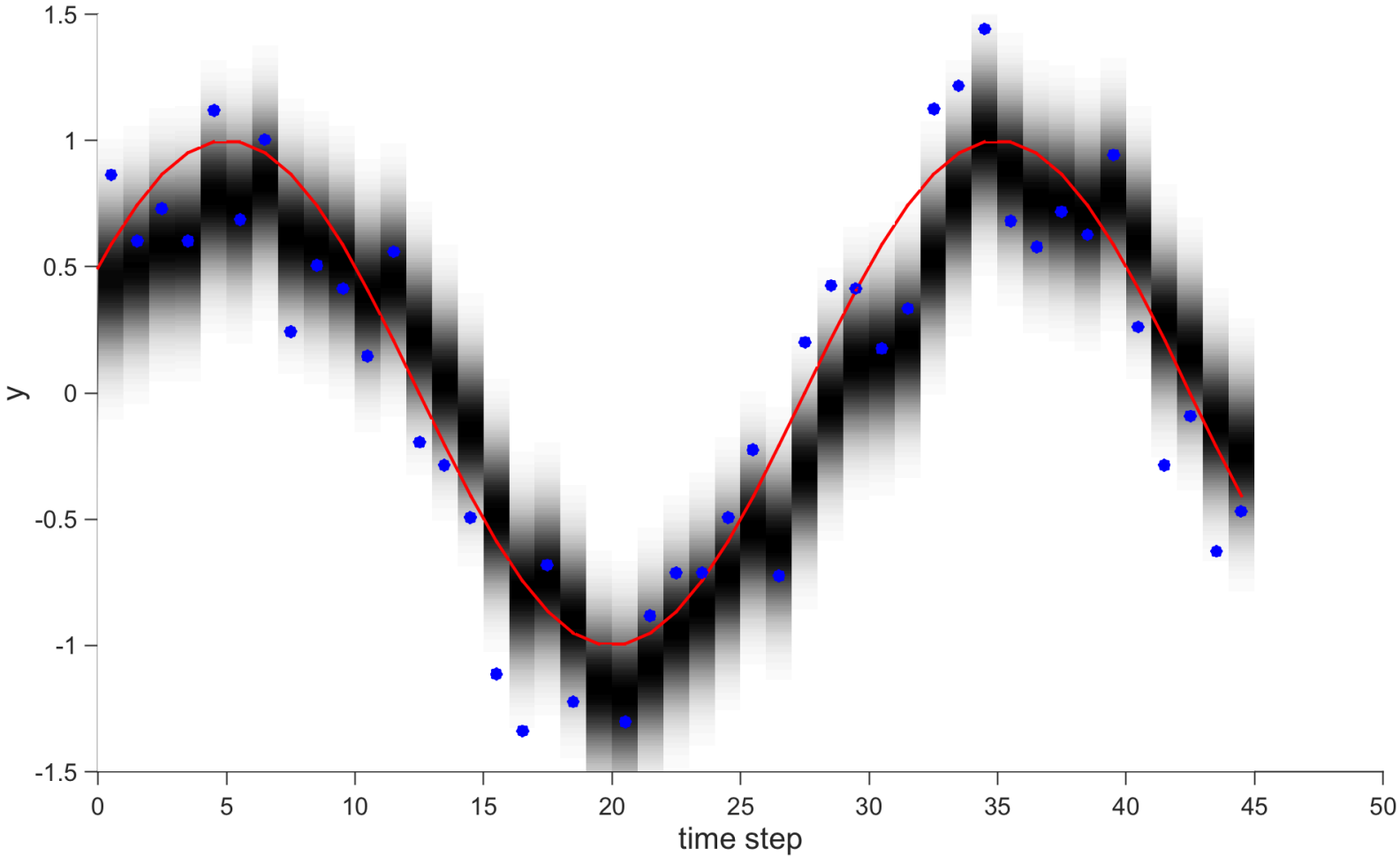

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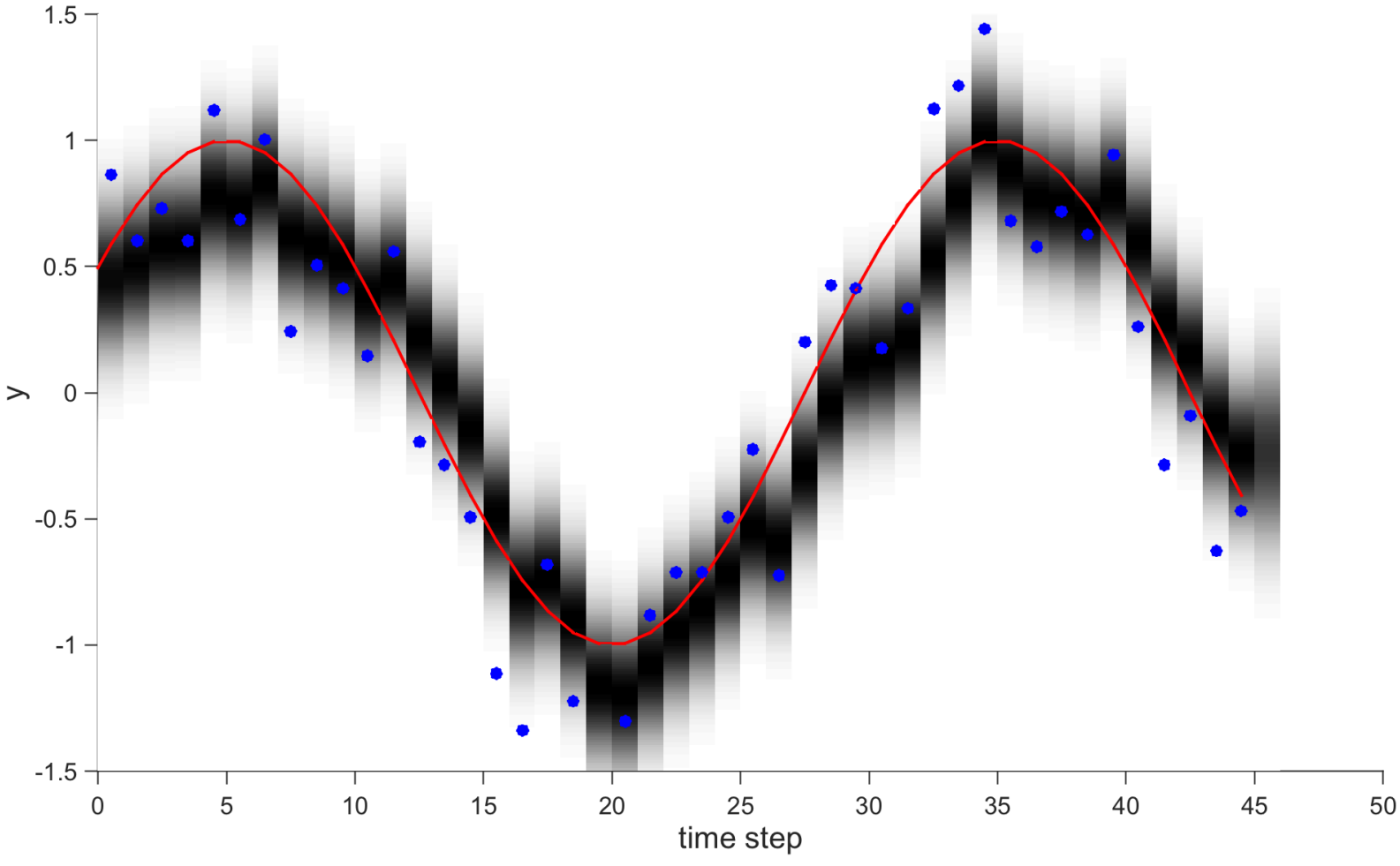

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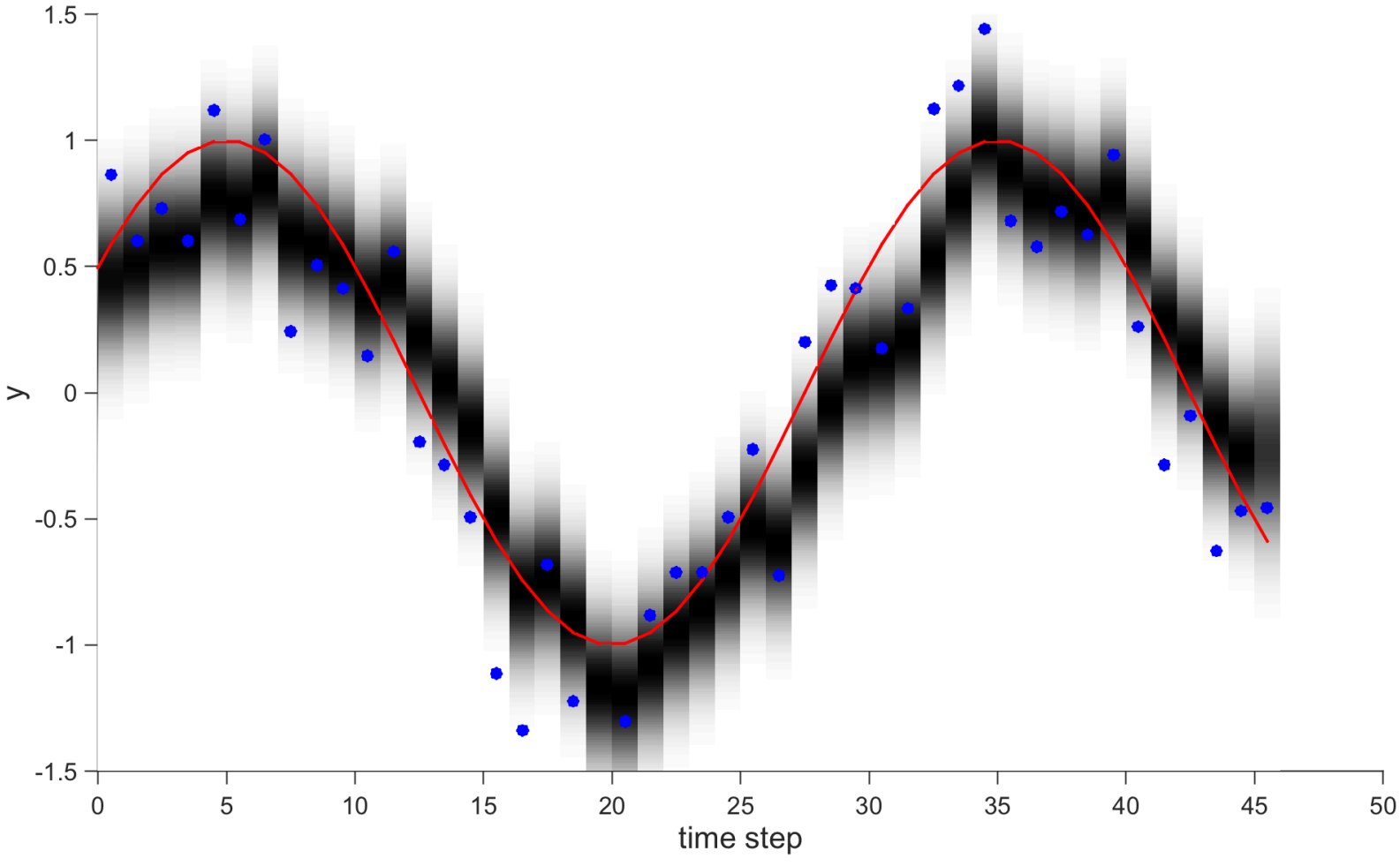

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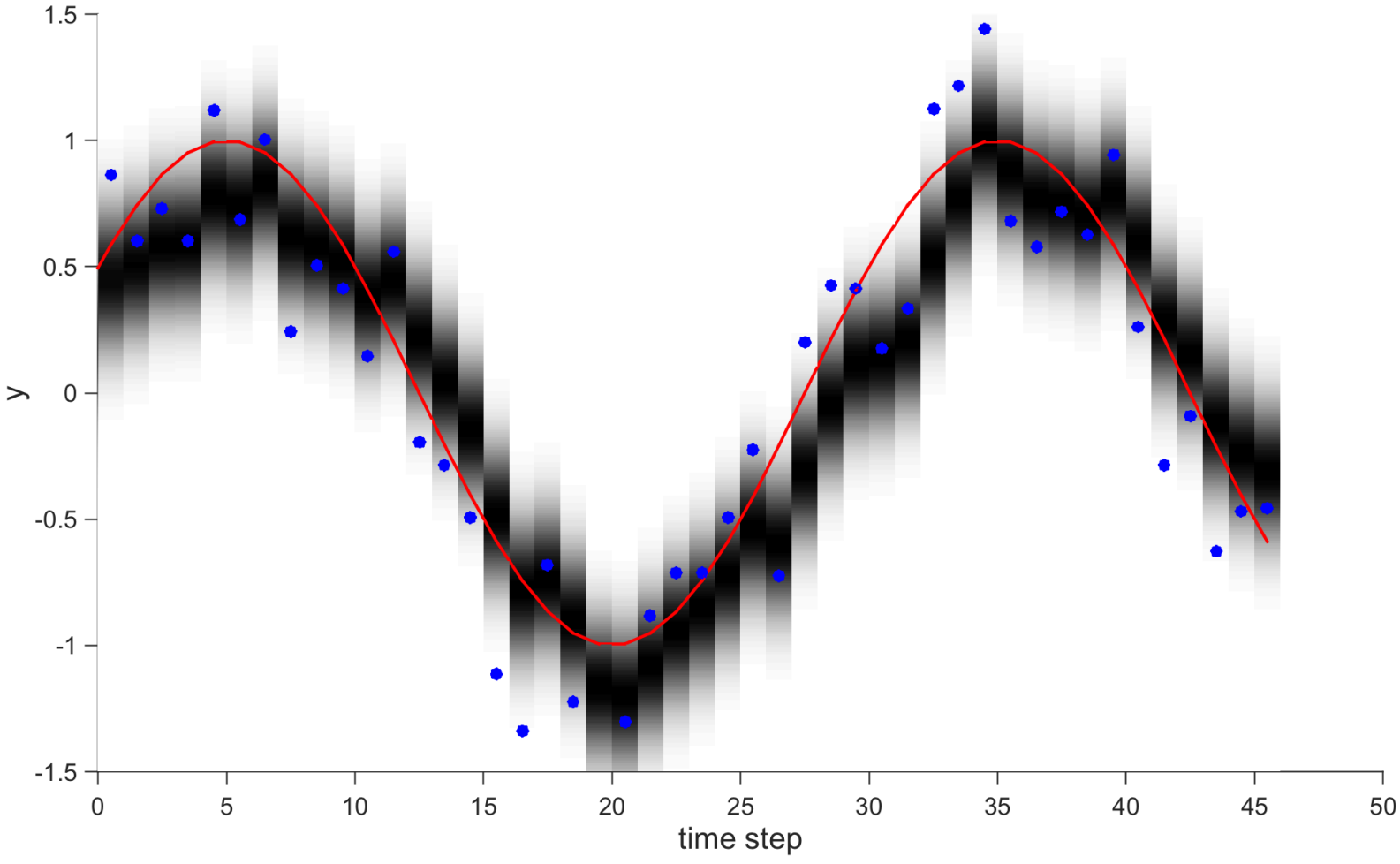

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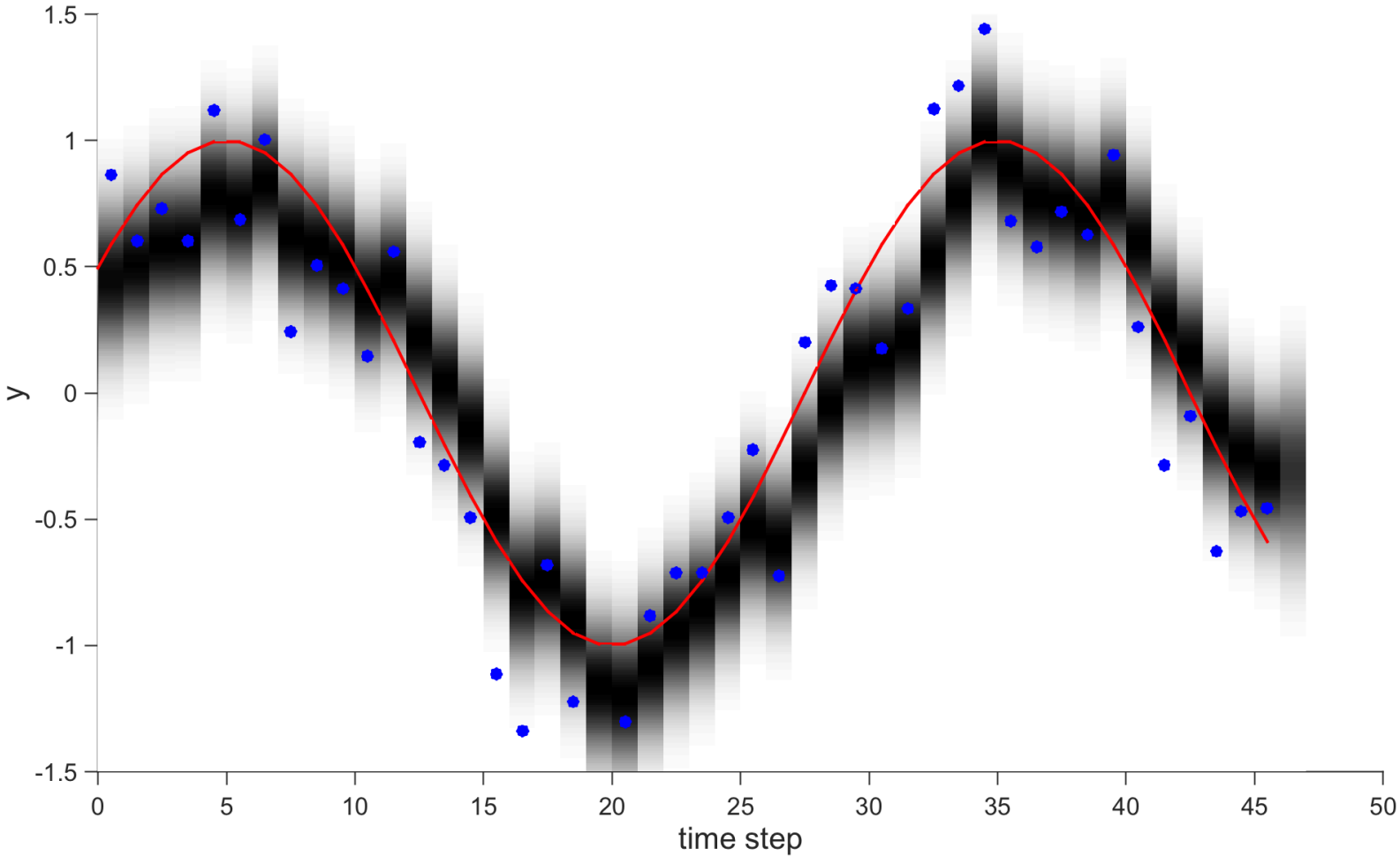

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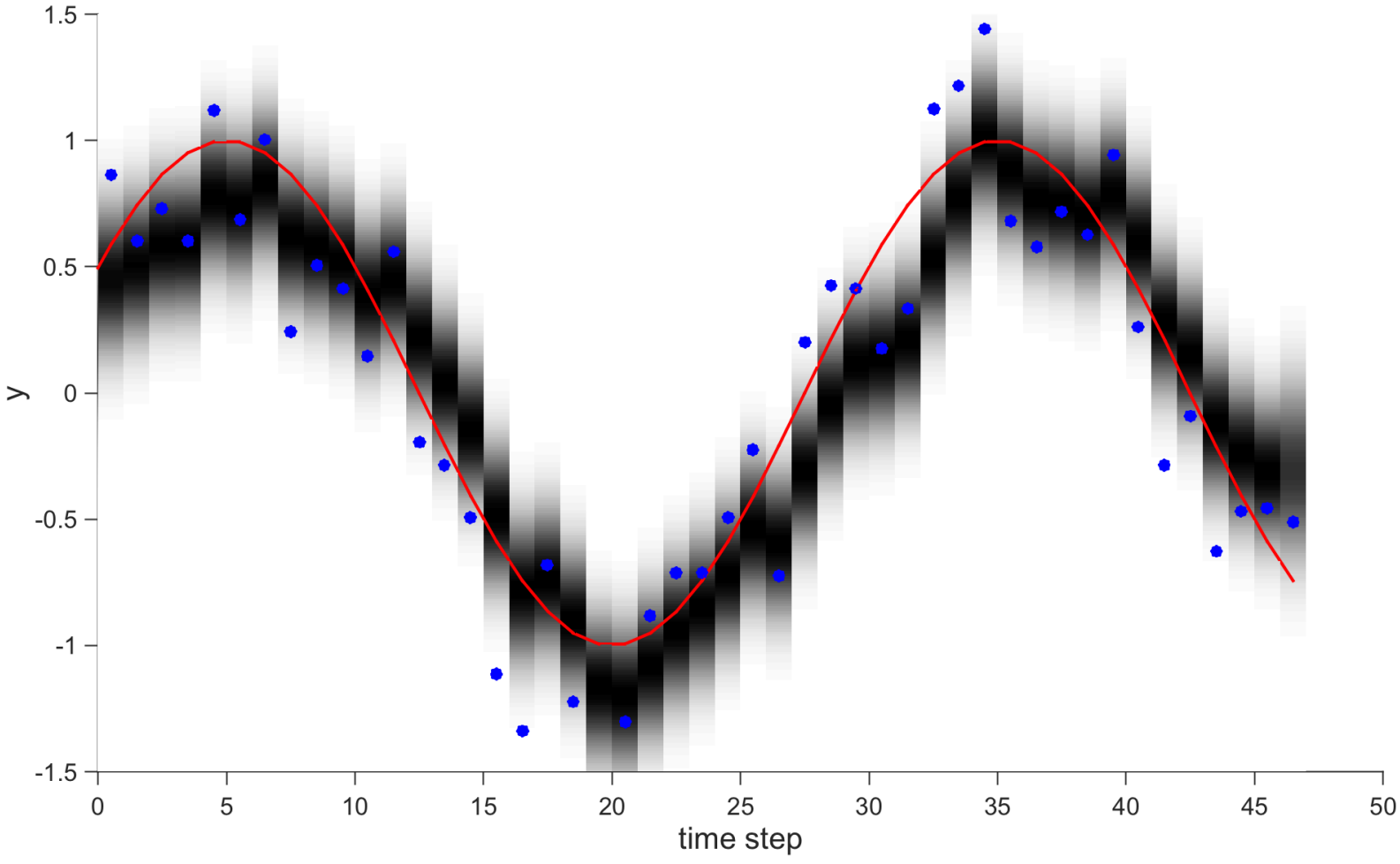

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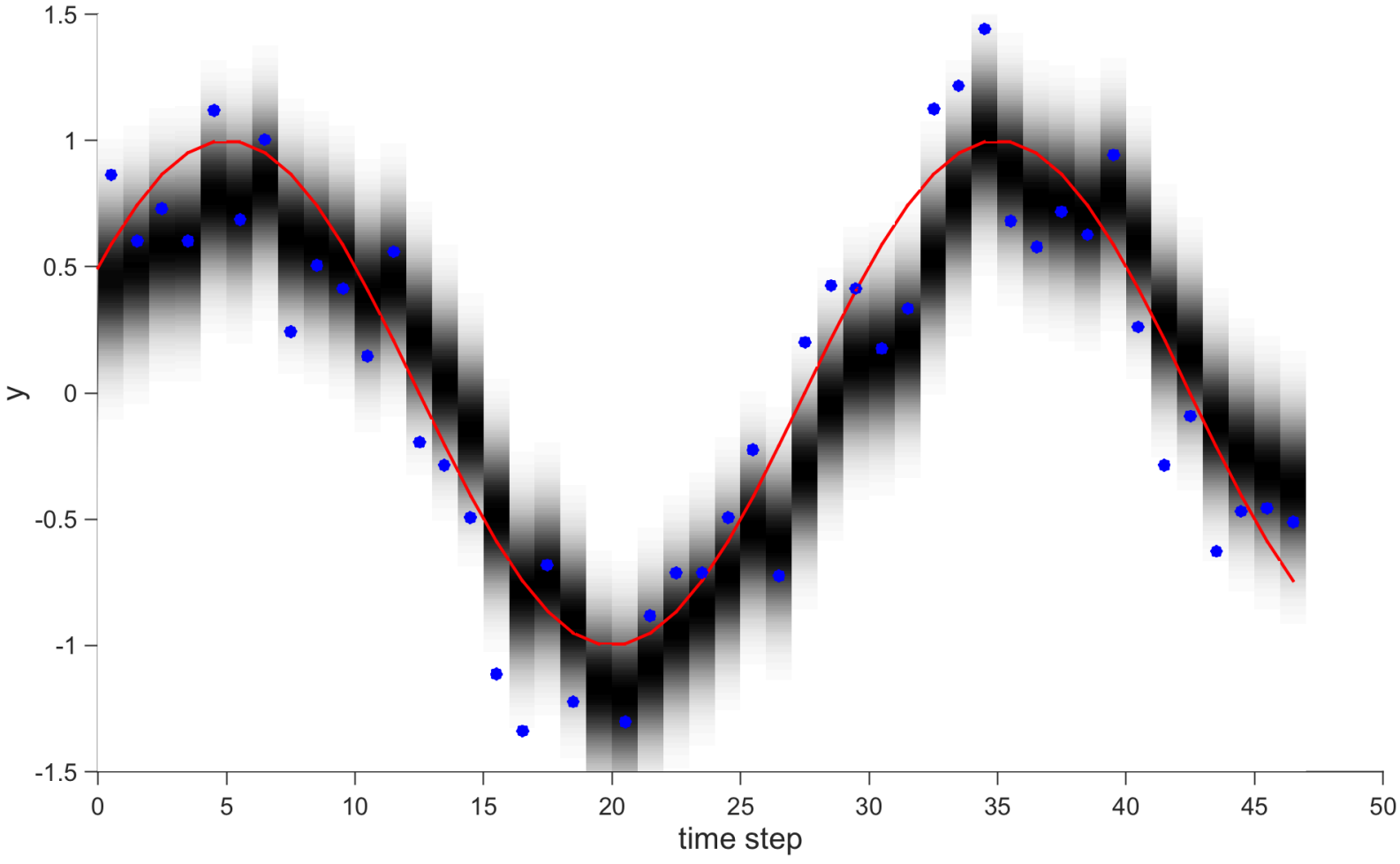

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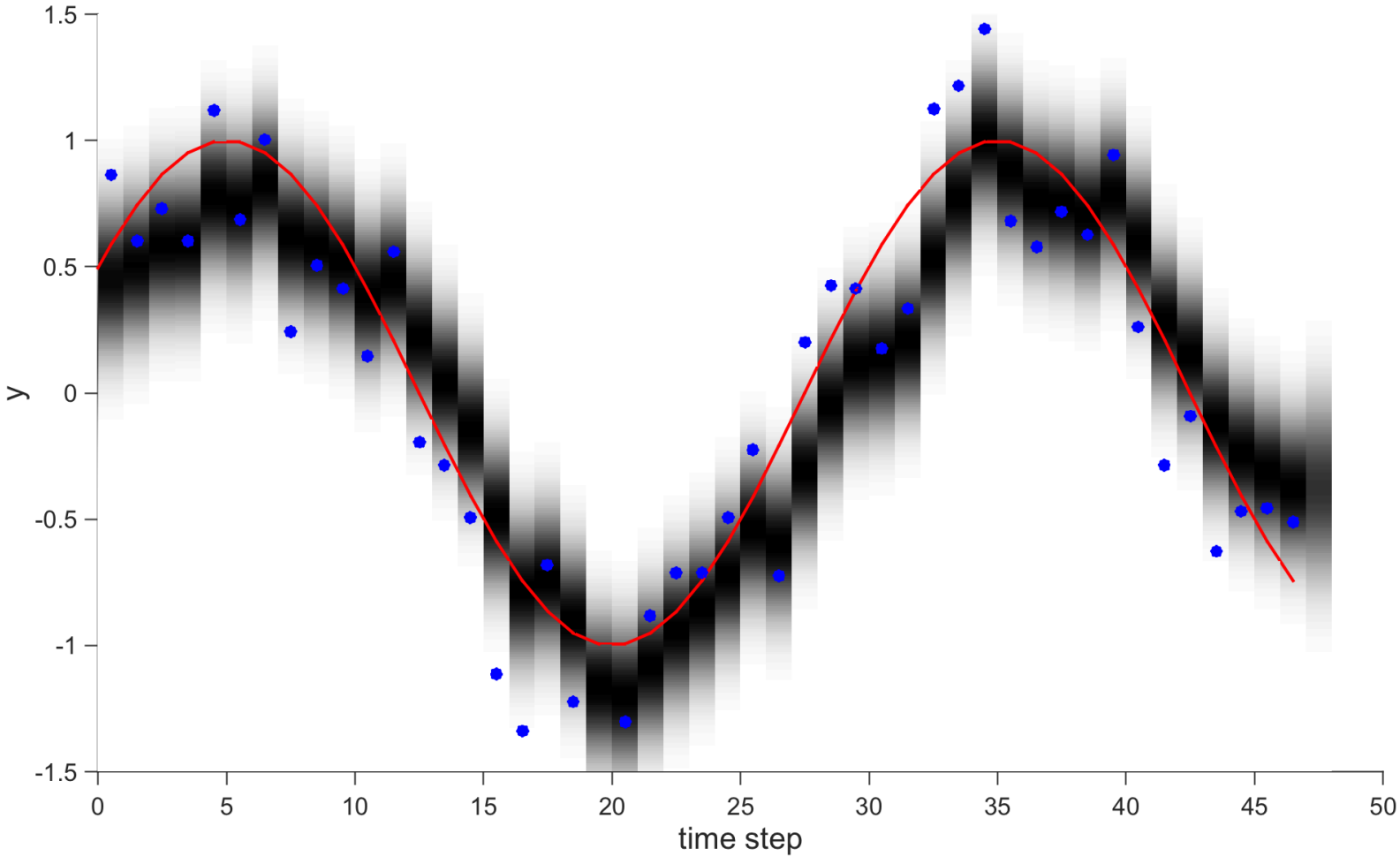

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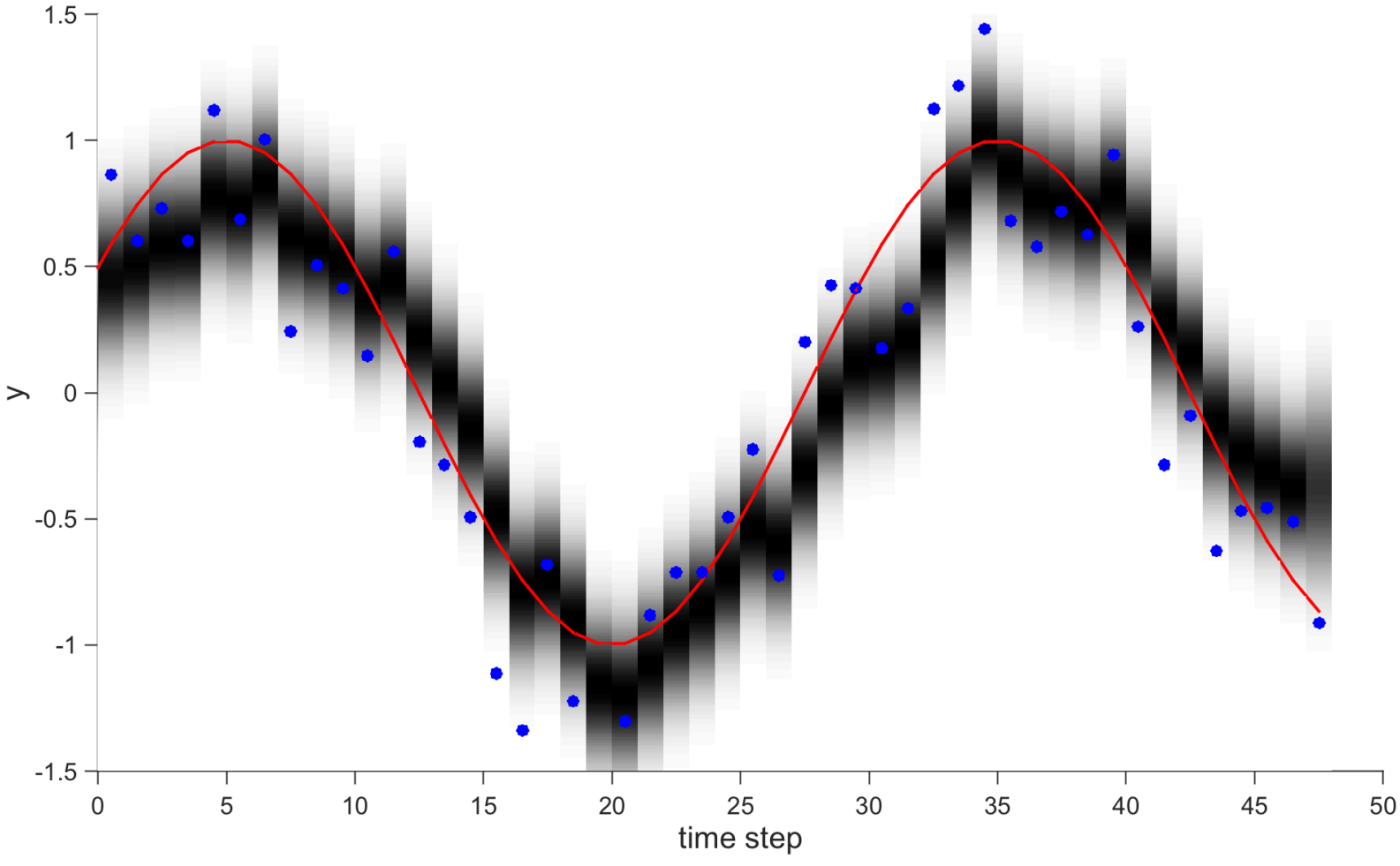

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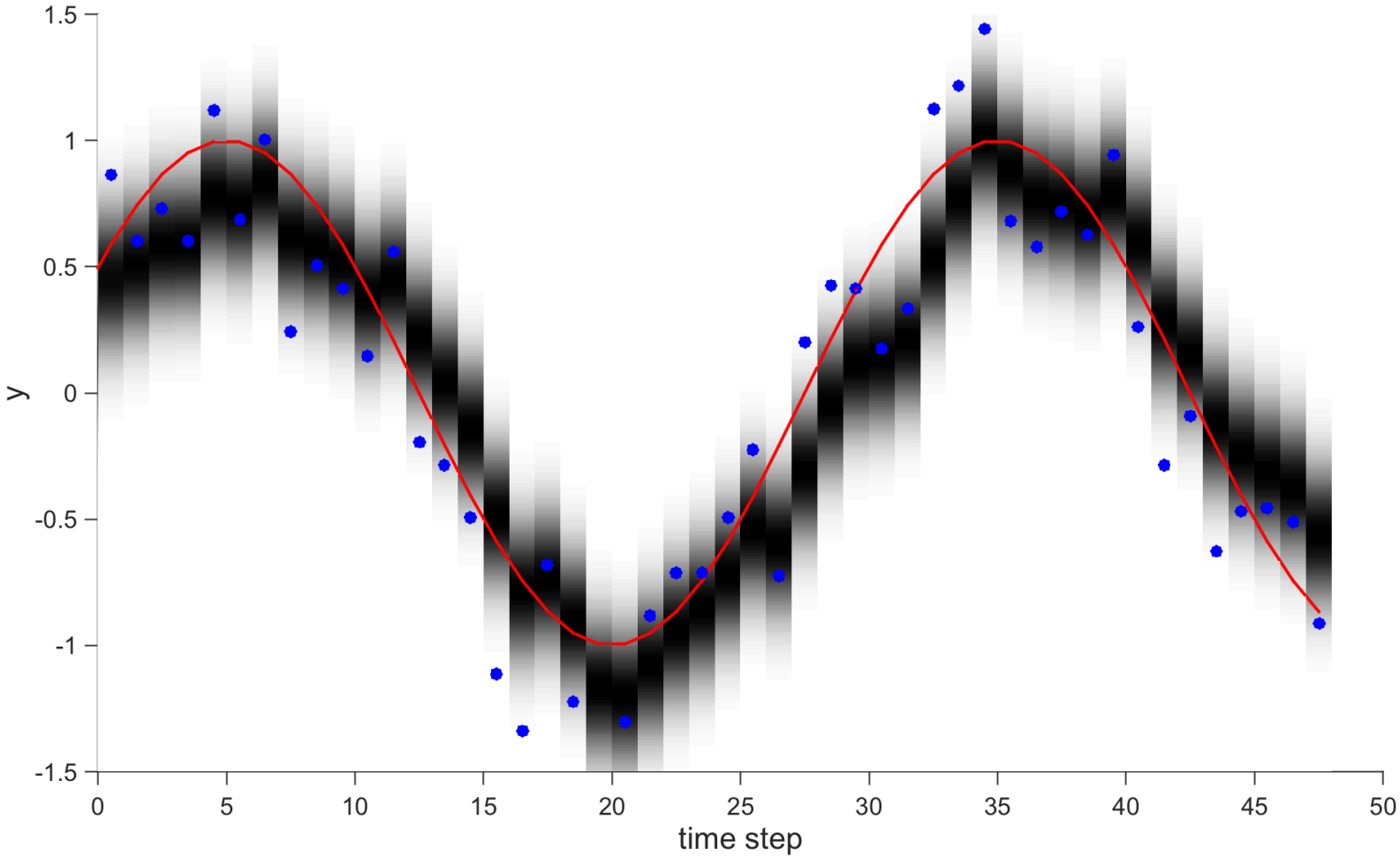

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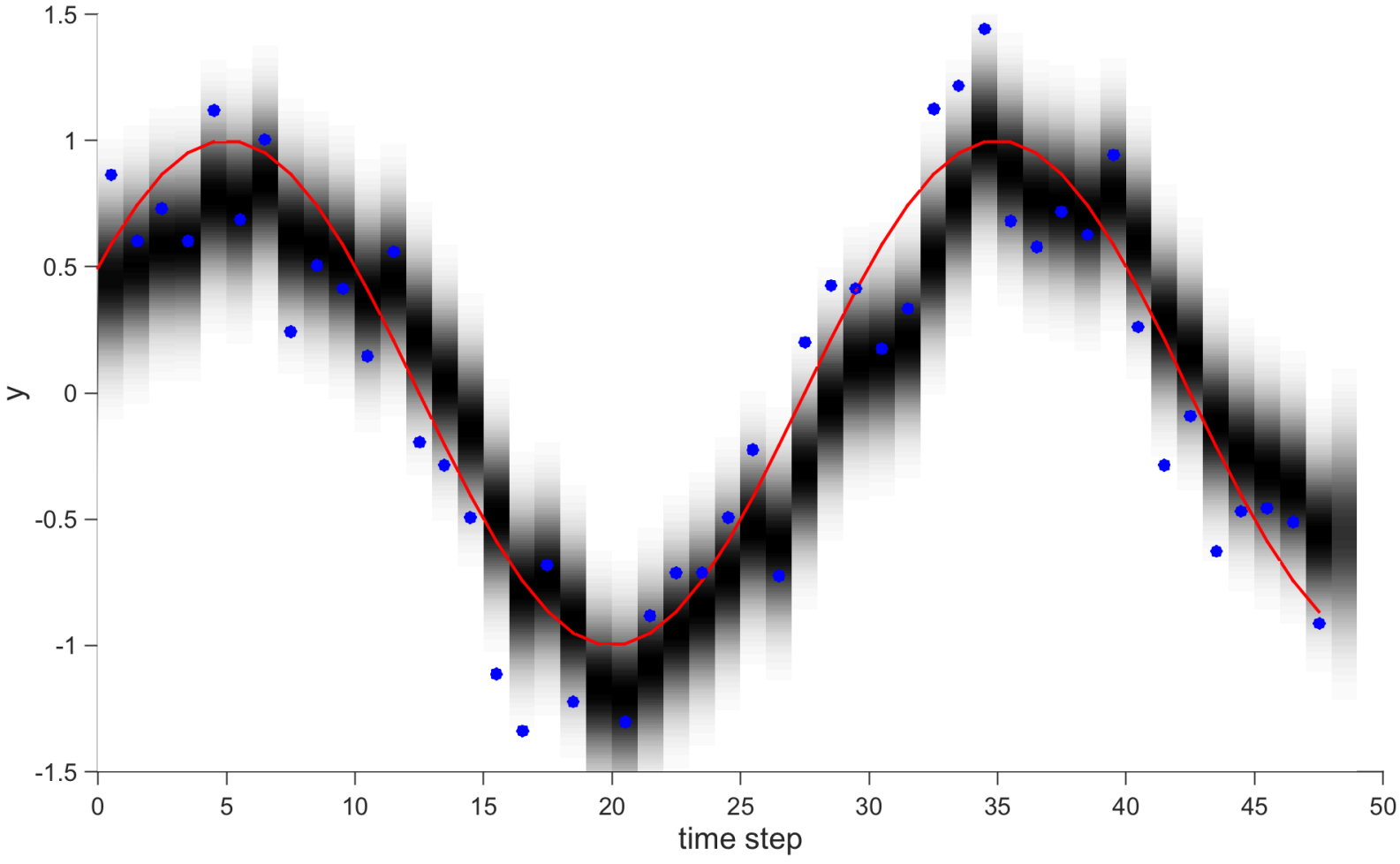

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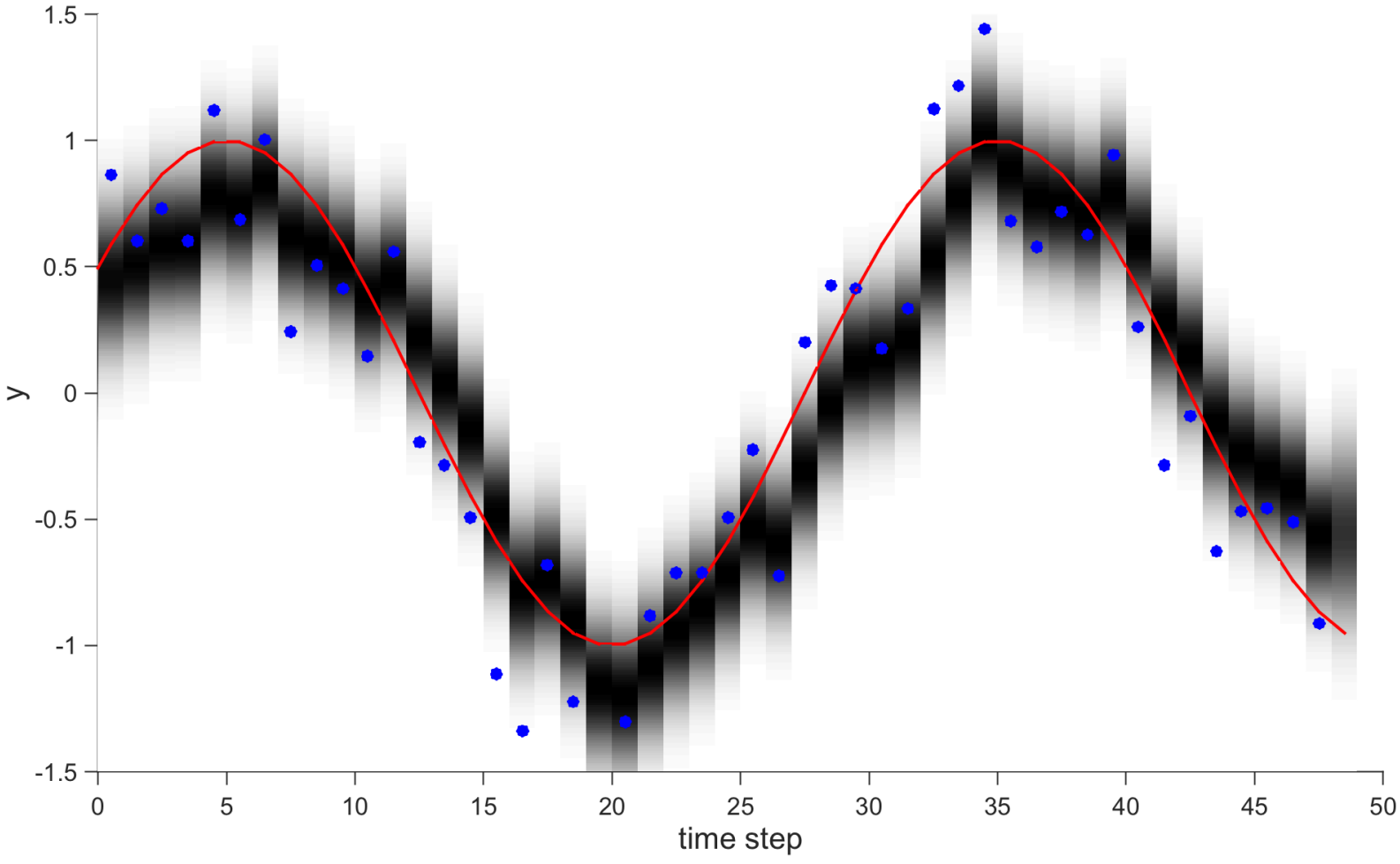

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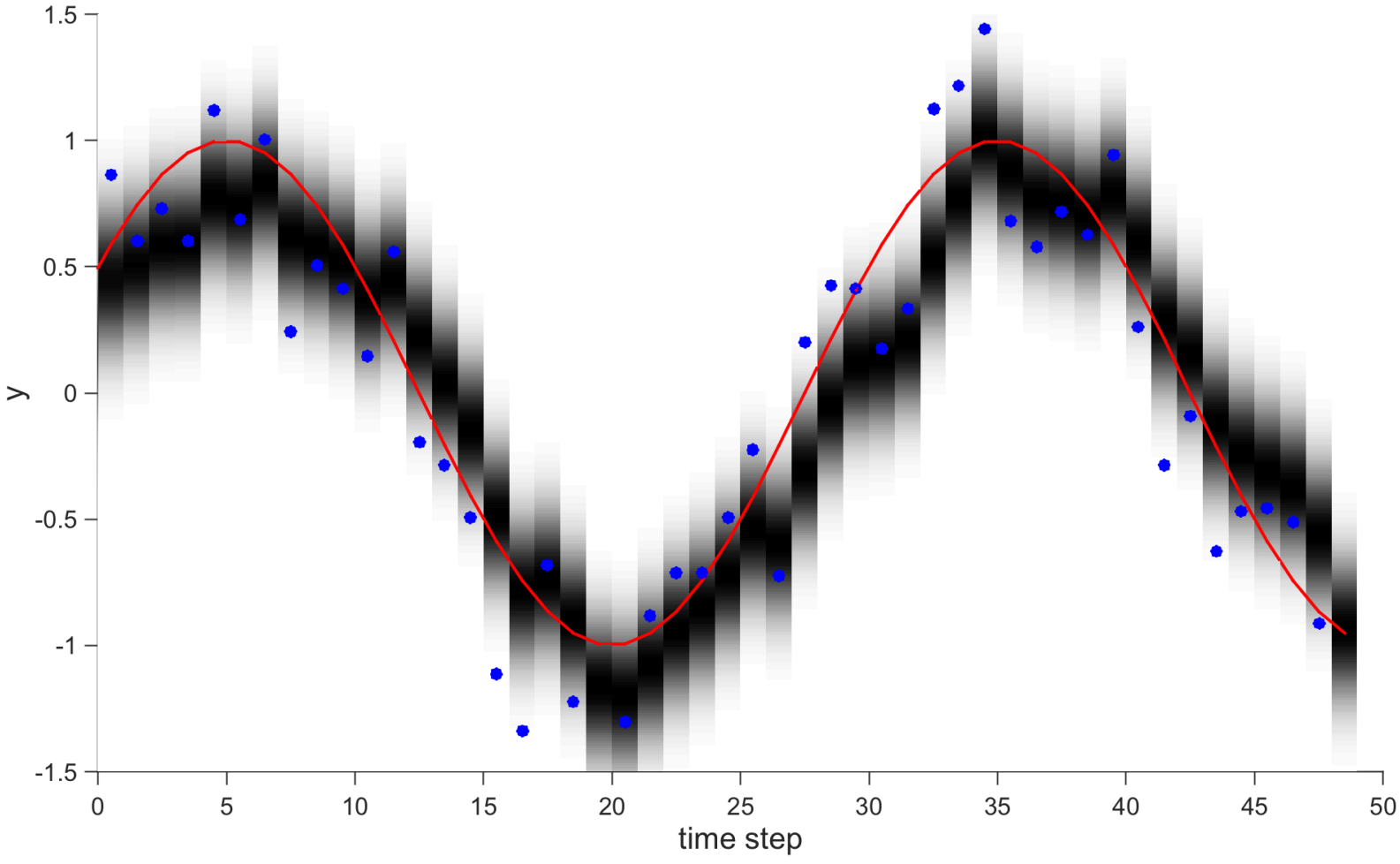

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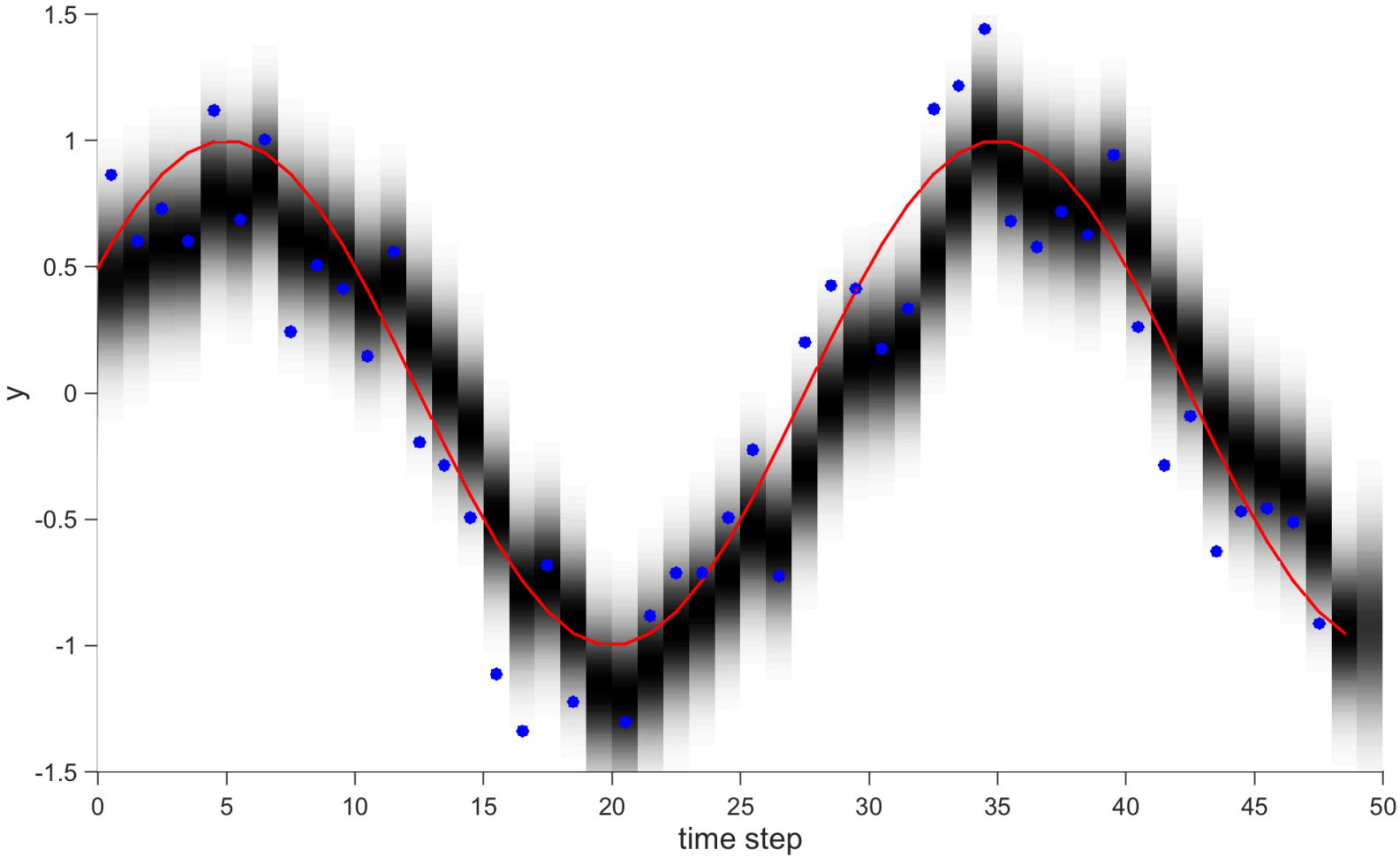

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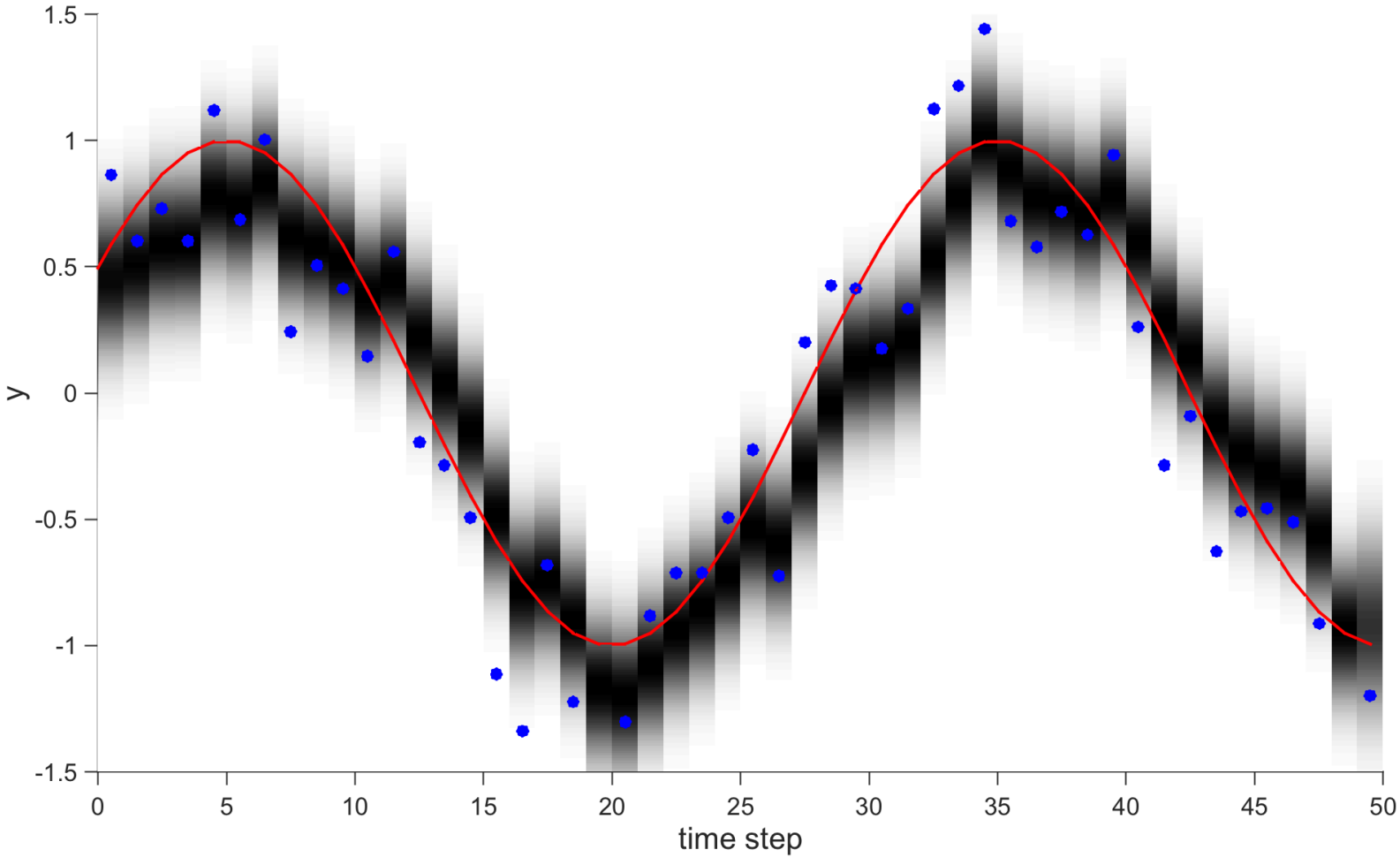

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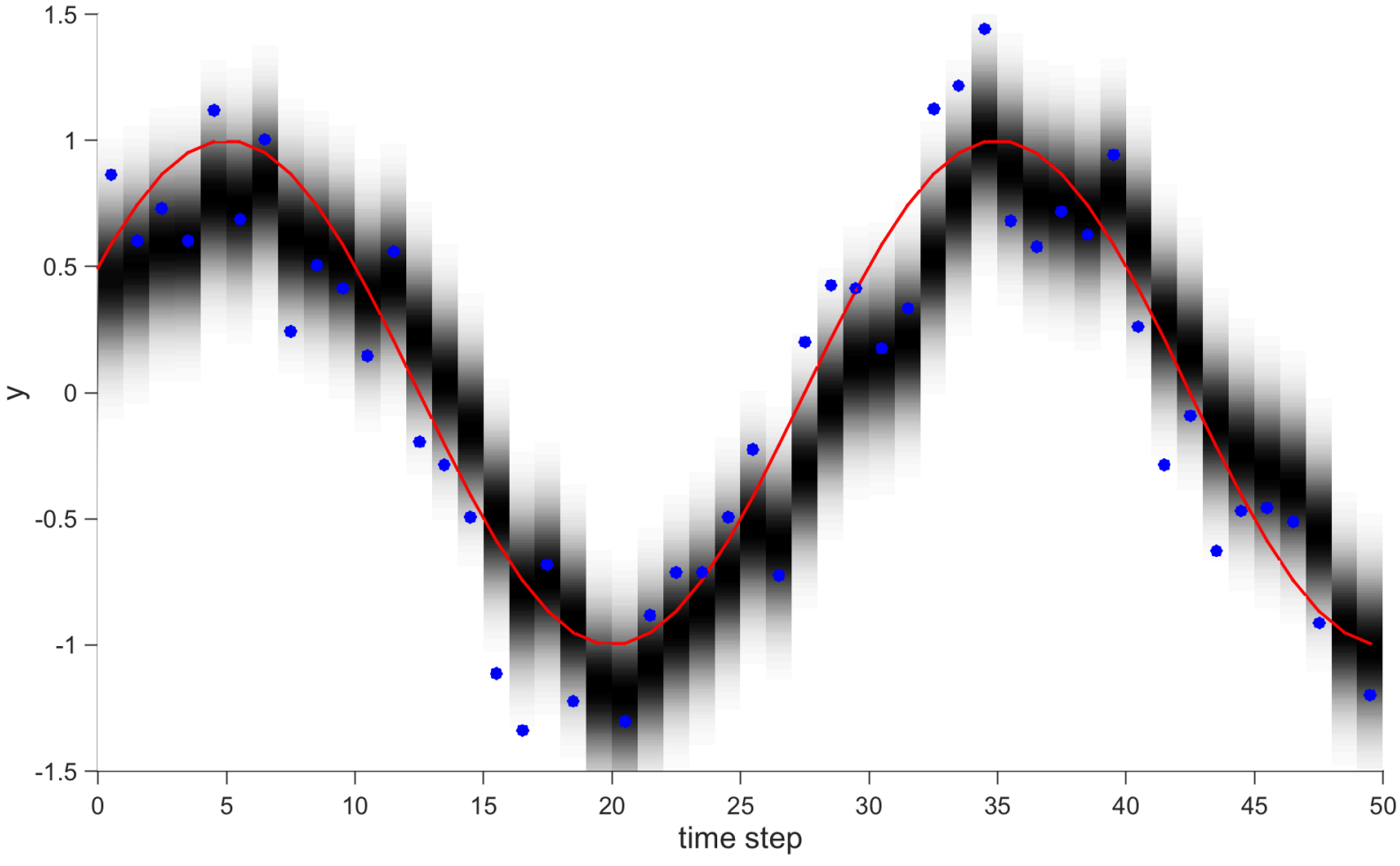
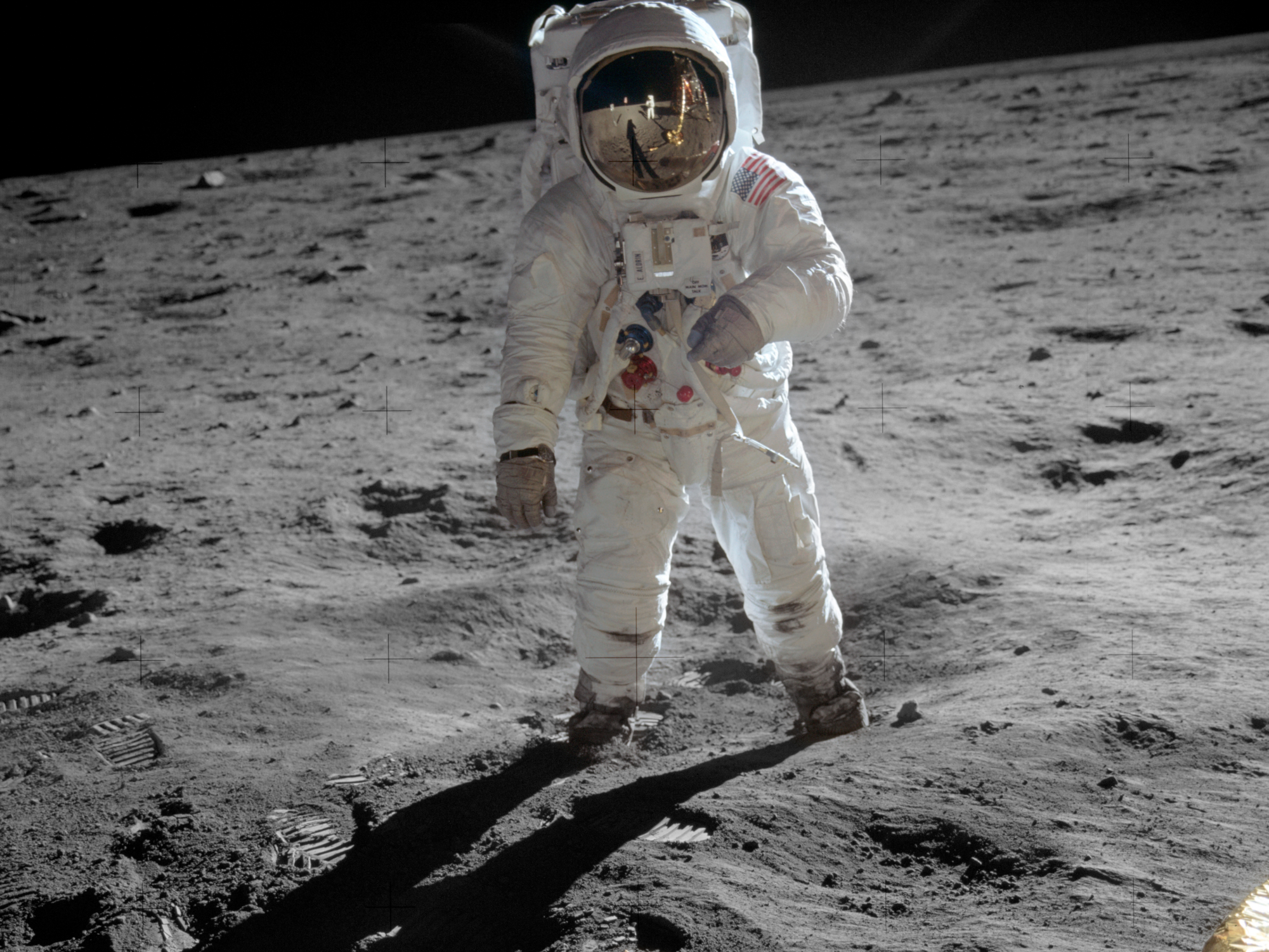




Course Survey: please complete this!

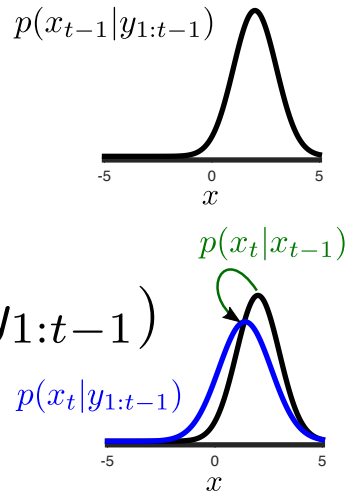
https://tinyurl.com/3F8survey2020

Inference: Forward Algorithm

$$p(x_{t-1} = k | y_{1:t-1})$$

diffuse via dynamics

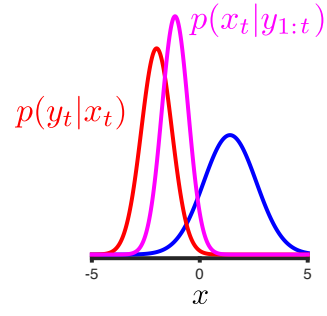
$$p(x_t = k | y_{1:t-1}) = \sum_{l=1}^{K} p(x_t = k | x_{t-1} = l) p(x_{t-1} = l | y_{1:t-1})$$



combine with likelihood

$$p(x_t = k | y_{1:t}) \propto p(x_t = k | y_{1:t-1}) p(y_t | x_t = k)$$

prior — likelihood

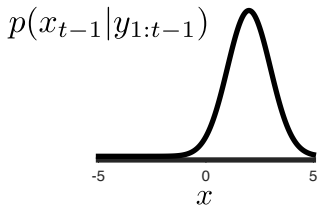


Inference: Forward Algorithm

$$p(x_{t-1} = k | y_{1:t-1}) = \rho_{t-1}^{t-1}(k)$$

most recent data used in prediction

variable being predicted



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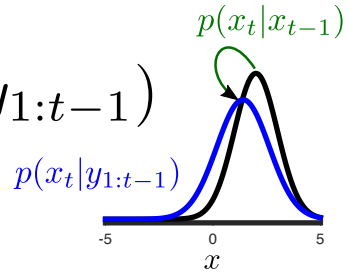

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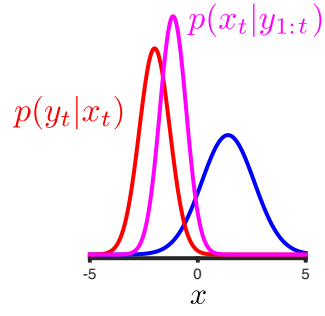

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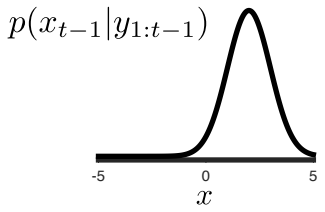

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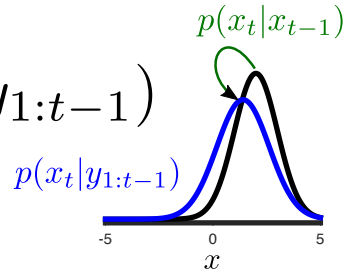

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combine with likelihood

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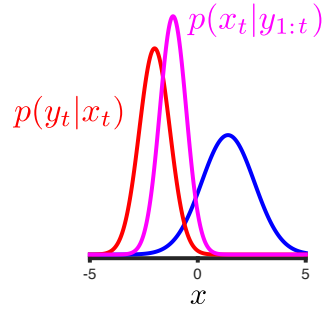

Inference: Forward Algorithm

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most recent data used in prediction

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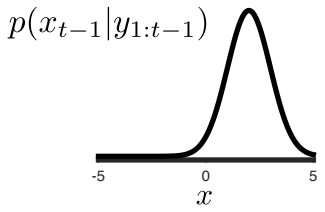

diffuse via dynamics

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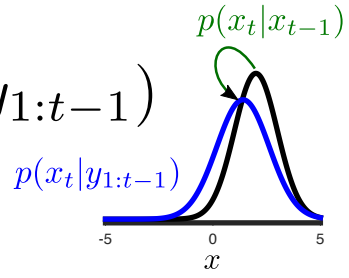

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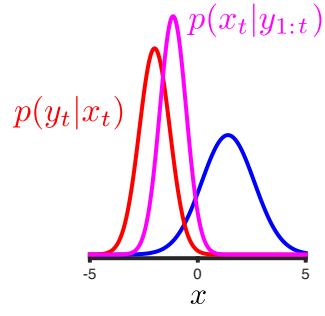

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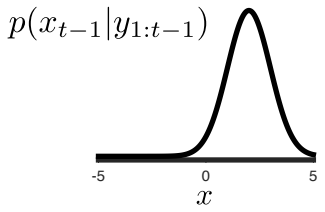

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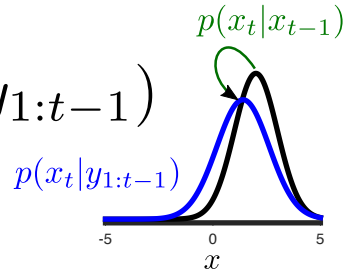

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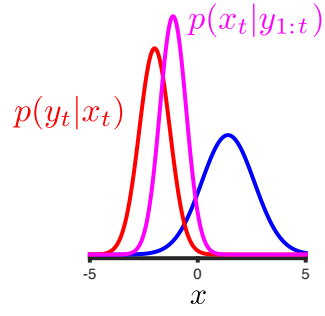

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When implementing, take care with numerical underflow/overflow.

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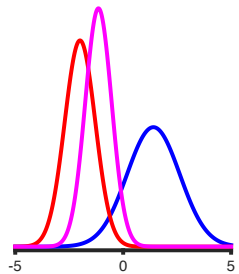

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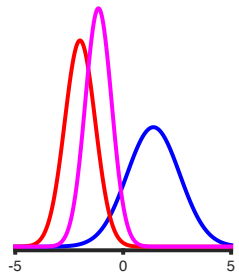

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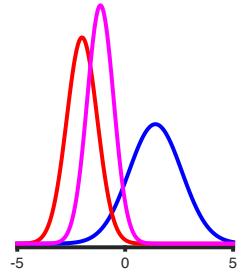

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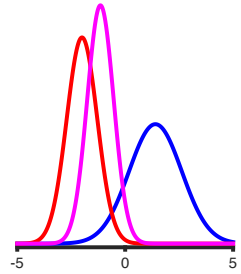

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HMM: Viterbi Decoding

The magic of the Forward Algorithm: Dynamic Programming

What's going on here?

In discrete case, likelihood involves sum over all sequences: $x_{1:T}^{(k)}$

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Trellis diagram represents possible sequences:

$K = 4$

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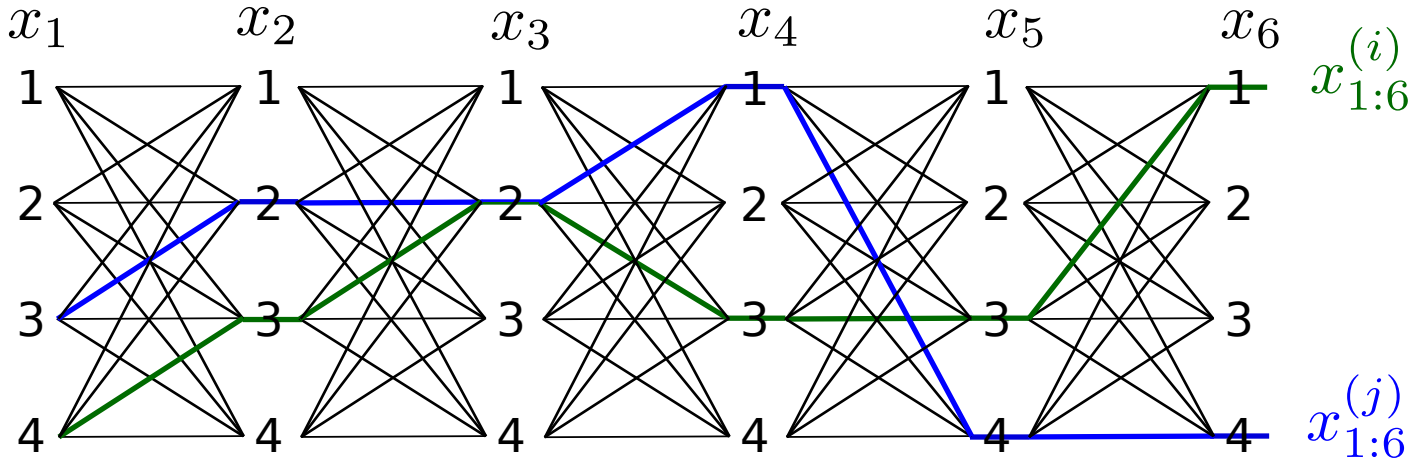

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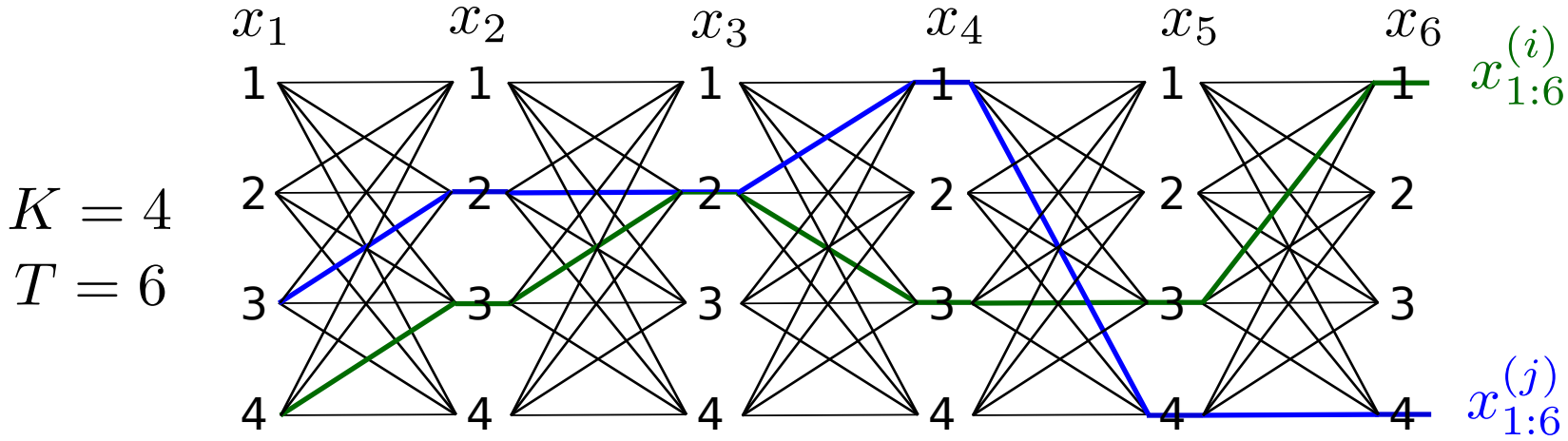

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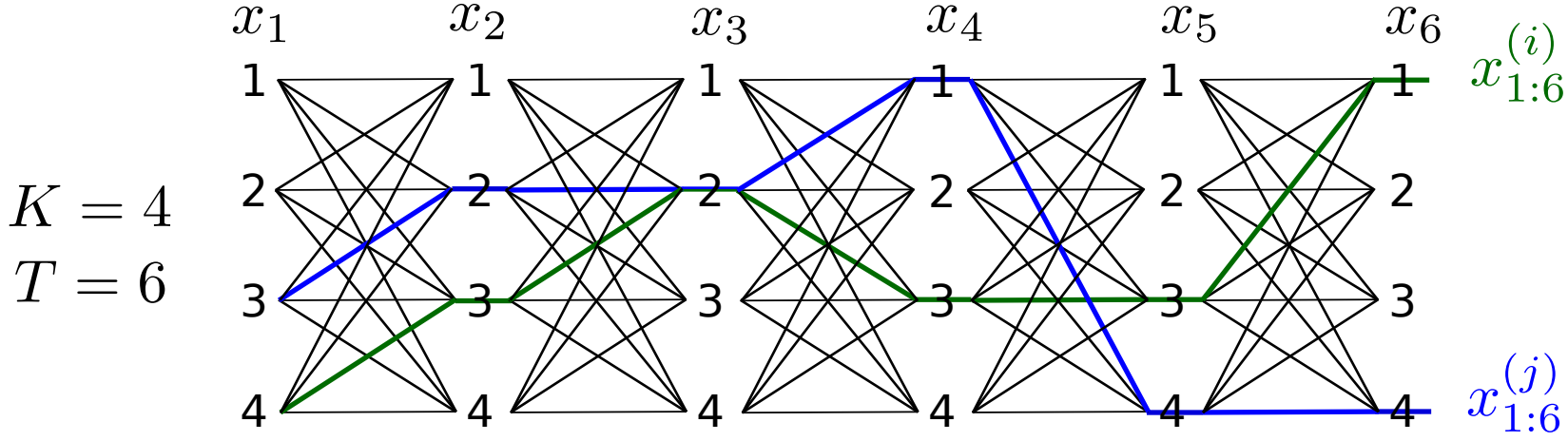

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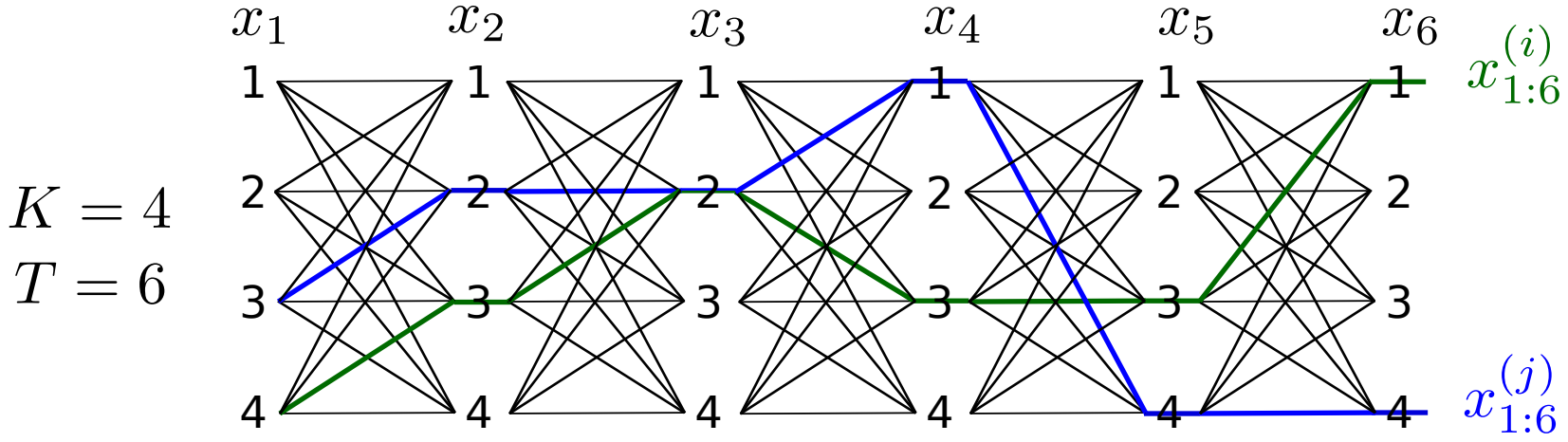

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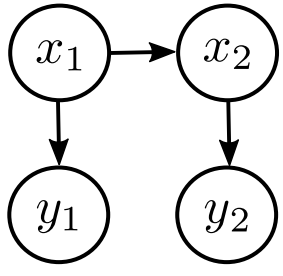

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Markov property means we can forget history of previous states: just remember last one (dynamic programming/belief propagation)

Maximum Likelihood Learning of HMMs: simple once inference is solved

log-likelihood: $$\log p(y_{1:T}|\theta) = \log \int p(y_{1:T}, x_{1:T}|\theta) dx_{1:T}$$

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requires posterior moments: marginals and pairwise marginals

Summary of Sequence Modelling

- **Markov models**: class of probabilistic models for sequence data
  - **N-Gram models** (discrete data) and **Gaussian auto-regressive models** (continuous data)
  - simple to perform maximum likelihood fitting
  - unnatural for many tasks e.g. removing additive noise, separating signals, data containing latent variables
- **Hidden Markov Models**: more flexible class of probabilistic model
  - generalise Markov models and naturally support a wider range of tasks (removal of noise, source separation, representation learning)
  - different varieties: discrete vs. continuous latent variables (**discrete HMMs** and **linear Gaussian state space models**)
  - inference in these models requires **dynamic programming** / message passing e.g. smoothing via the **forwards-backwards** recursions or **Kalman filtering-smoothing** recursions
  - maximum-likelihood fitting requires smoothing as a subroutine

Hidden Markov Models for unsupervised high dimensional video understanding

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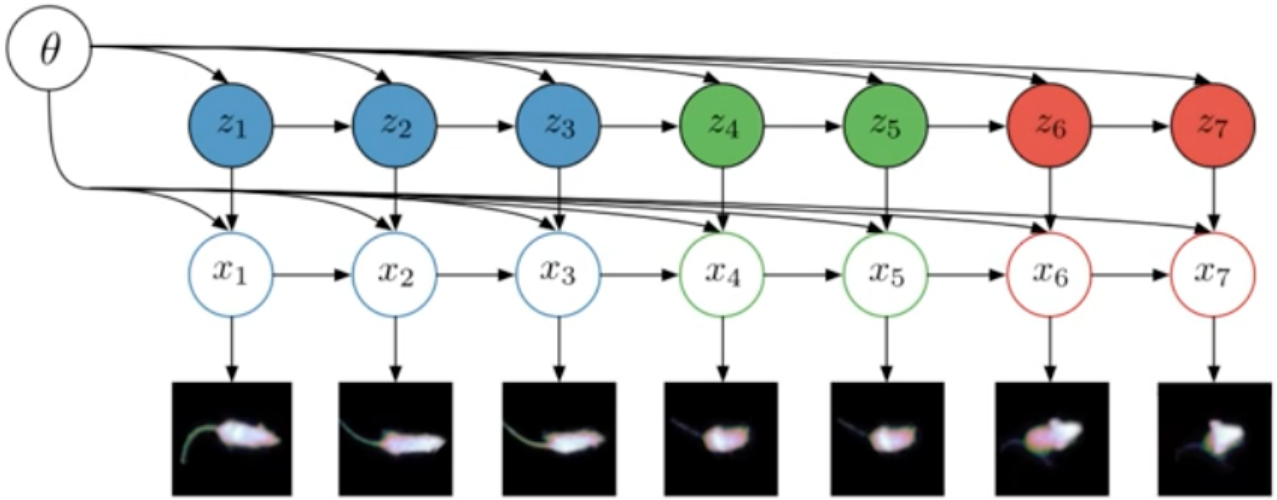

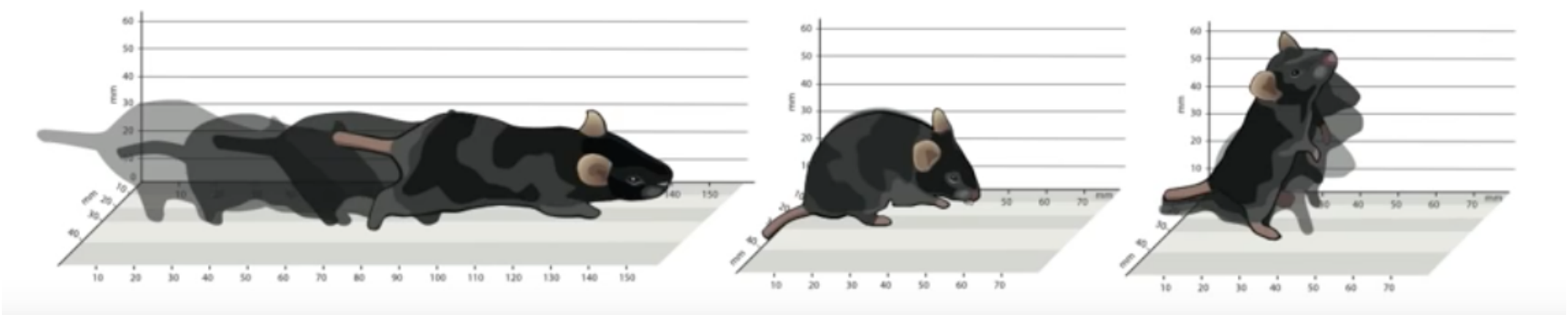

https://www.youtube.com/watch?v=btr1poCYIzw