Properties of Gaussians

Definition

$$\underline{x} \sim N(\underline{\mu}, \underline{\underline{\Sigma}}) \iff p(\underline{x}) = N(\underline{x}; \underline{\mu}, \underline{\underline{\Sigma}})$$

$$= \frac{1}{\sqrt{\det(2\pi\underline{\underline{\Sigma}})}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \underline{\underline{\Sigma}}^{-1}(\underline{x}-\underline{\mu})}$$

Moments

$$\mathbb{E}_{p(x)}(\underline{x}) = \int \underline{x}\, p(\underline{x})\, dx = \underline{\mu} \quad = \text{mean}$$

$$\mathbb{E}_{p(x)}(\underline{x}\,\underline{x}^T) - \underline{\mu}\,\underline{\mu}^T = \int (\underline{x}-\underline{\mu})(\underline{x}-\underline{\mu})^T p(\underline{x})\, d\underline{x} = \underline{\underline{\Sigma}} \quad = \text{covarian}$$

Marginals are Gaussian

if $\underline{x} \sim N(\underline{\mu}, \underline{\underline{\Sigma}})$ where $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$ $\underline{\mu} = \begin{bmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{bmatrix}$

$$\underline{\underline{\Sigma}} = \begin{bmatrix} \underline{\underline{\Sigma}}_{11} & \underline{\underline{\Sigma}}_{12} \\ \underline{\underline{\Sigma}}_{21} & \underline{\underline{\Sigma}}_{22} \end{bmatrix}$$

then

$$\left.\begin{aligned} p(\underline{x}_1) &= \int p(\underline{x})\, dx_2 = N(\underline{x}_1; \underline{\mu}_1, \underline{\underline{\Sigma}}_{11}) \\ p(\underline{x}_2) &= \int p(\underline{x})\, dx_1 = N(\underline{x}_2; \underline{\mu}_2, \underline{\underline{\Sigma}}_{22}) \end{aligned}\right\}$$

i.e. Gaussians w/ mean & covariance given by corresponding elements of $\underline{\mu}$ & $\underline{\underline{\Sigma}}$

Conditionals are Gaussian

if $\underline{x} \sim N(\underline{\mu}, \underline{\underline{\Sigma}})$ where $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$ $\quad \underline{\mu} = \begin{bmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{bmatrix}$

$$\underline{\underline{\Sigma}} = \begin{bmatrix} \underline{\underline{\Sigma}}_{11} & \underline{\underline{\Sigma}}_{12} \\ \underline{\underline{\Sigma}}_{21} & \underline{\underline{\Sigma}}_{22} \end{bmatrix}$$

$$p(\underline{x}_1 | \underline{x}_2) = N(\underline{x}_1 ; \underline{\mu}_{1|2}, \underline{\underline{\Sigma}}_{1|2})$$

where $\underline{\mu}_{1|2} = \underline{\mu}_1 + \underline{\underline{\Sigma}}_{12} \underline{\underline{\Sigma}}_{22}^{-1} (\underline{x}_2 - \underline{\mu}_2)$

$\underline{\underline{\Sigma}}_{1|2} = \underline{\underline{\Sigma}}_{11} - \underline{\underline{\Sigma}}_{12} \underline{\underline{\Sigma}}_{22}^{-1} \underline{\underline{\Sigma}}_{21}$

Should know this is Gaussian

do not need to learn this identity [will be given if needed]

alternative representation (useful for computing expectations of linear transforms of Gaussian variables quickly)

① $\quad \underline{x} = \underline{\mu} + \underline{\underline{\Sigma}}^{1/2} \underline{\epsilon} \qquad \underline{\epsilon} \sim N(\underline{0}, \underline{\underline{I}})$

$\Updownarrow$

② $\quad p(\underline{x}) = N(\underline{x} ; \underline{\mu}, \underline{\underline{\Sigma}})$

sketch proof

as 1. moments of ① = moments of ②

2. $\underline{x}$ produced from ① must be multivariate Gaussian (see next section)

linear combination of Gaussians

$$\underline{x}_1 \sim N(\underline{\mu}_1, \underline{\underline{\Sigma}}_{11}) \qquad \underline{x}_2 \sim N(\underline{\mu}_2, \underline{\underline{\Sigma}}_{22})$$

$$\underline{z} = \underline{\underline{A}}\, \underline{x}_1 + \underline{\underline{B}}\, \underline{x}_2 + \underline{c}$$

$$\Rightarrow \quad \underline{z} \sim N(\underline{\underline{A}}\, \underline{\mu}_1 + \underline{\underline{B}}\, \underline{\mu}_2 + \underline{c}, \underline{\underline{A}}\, \underline{\underline{\Sigma}}_{11} \underline{\underline{A}}^T + \underline{\underline{B}}\, \underline{\underline{\Sigma}}_{22} \underline{\underline{B}}^T)$$

Product of Gaussian Densities = (unnormalised) Gaussian Density

$$N(\underline{x}; \mu_1, \Sigma_{11})\, N(\underline{x}; \mu_2, \Sigma_{22}) = c\, N(\underline{x}; \underline{\mu}, \underline{\underline{\Sigma}})$$

Gaussian (should know this)

$$\underline{\underline{\Sigma}}^{-1} = \underline{\underline{\Sigma}}_{11}^{-1} + \underline{\underline{\Sigma}}_{22}^{-1}$$

$$\underline{\mu} = \left(\underline{\underline{\Sigma}}_{11}^{-1} + \underline{\underline{\Sigma}}_{22}^{-1} \right)^{-1} \left(\underline{\underline{\Sigma}}_{11}^{-1} \underline{\mu}_1 + \underline{\underline{\Sigma}}_{22}^{-1} \underline{\mu}_2 \right)$$

don't need to know these but should be aware that they can be derived by completing the square