Statistics and Stochastic processes

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Outline

- Data
- Probability
- Statistics
- Inference
- Modeling
- Stochastic models
- Stochastic processes

Data

- Variables are random in some way
  - It represents an incompletely, measured variable
  - Sample drawn using random mechanisms
- Data into knowledge:
  - Probability
    - The study of random variables
  - Statistics
    - The discipline of using data samples to support claims about populations.
    - Based on probability
  - Computation
    - A tool well suited to quantitative analyses

Reproducible Research

- Replication
  - Validate findings
  - Some studies cannot be replicated (money/condition)

- Data -> Analytic data -> Reproducible research

- Existing database can be merged into new “mega databases”

- For every field there is a computational field of it

Types of Data Analysis Questions

COMPLEXITY

- Descriptive: First kind of approach, describe a set of data
- Exploratory: Find relationships you didn't know about. No generalizing
- Inferential: Small sample of data to say something about a bigger population
- Predictive: Use data from one object to predict another. No causality
- Causal: To find what happens to one variable when you change another
- Mechanistic: Understand the variables that lead to exact changes for an individual observation

Sources of data

- Census
  - Interested in people
  - Descriptive
- Convenience
  - Depends in how data are sampled
  - Descriptive, Inference and Prediction
  - Highly biased
  - Anecdotal
    - Small number of observations
    - Inaccurate
    - “I heard that vaccines cause autism”

Sources of data

- Observational
  - Measure a group without replacement
  - Inference
- Randomized trial
  - Find a variable that changes other variables
  - Many subgroups without replacement
  - Each group has different conditions
  - Causal analysis
- Prediction study
  - Two data sets: training and test
  - Predictive

Sources of data - Study over time

- Longitudinal
  - It follows along time
  - Inferential and predictive
- Retrospective
  - First and last observation
  - Inferential
  - E.g. Outcome and exposure
- Cross-sectional
  - Taking samples from different types
  - Inferential
  - E.g. Wildtype vs condition

Probability

- All the important results are called Events (E)
- In a success or failure trial:
  - $P(E)$ is the probability of success
  - $P(\neg E)$ is the probability of failure
- Two approaches:
  - Frequentist – Depends on observations amount
  - Bayesian – Depends on degree of knowledge

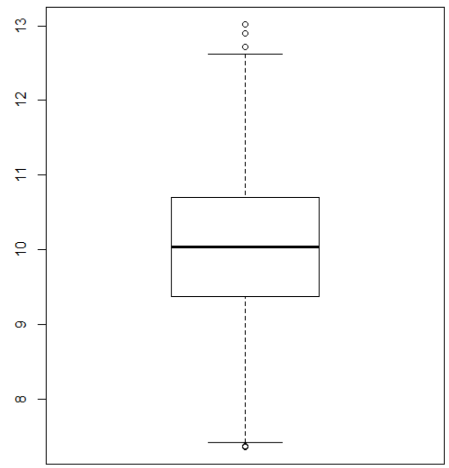
Descriptive statistics

- A small set of parameters can summarize a large amount of data

- Three summary statistics
  - Median
  - Mean
  - Variance

Median

- The value at the center of a sorted dataset
- Value such that the set of values less than itself has a probability of 0.5



Sample mean

- Good description of a set of values

mean≠average

- Average: statistics to describe typical values
- Arithmetic mean is one type of average

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

- At least 1 DOF to compute

Sample variance

- It describes the spread of data
- It is the squared deviation from the mean
  - Biased estimator

$$s_X^2 = \frac{1}{N}\sum_{i=i}^{N}(X_i - \mu)^2$$

  - Unbiased estimator

$$s_X^2 = \frac{1}{N-1}\sum_{i=i}^{N}(X_i - \mu)^2$$

- At least 2 DOF to compute

Probability density function (pdf)

- Also known as probability distribution
- It describes how often a value appears [Frequency]

$$P(a < X \leq b) = \int_a^b f(x)dx$$

- Histogram
  - Frequency of each value

- Probability mass function (pmf)
  - It describes a discrete random variable

$$P(X = a)$$

Probability density function (pdf)

- Example: loaded die

Cumulative distribution function

- The CDF is the function that maps values to their percentile rank in a distribution

$$P(X \leq x)$$

- The CDF is a function of X, where X is any value that might appear in the distribution

$$\lim_{X \to -\infty} cdf(X) = 0$$

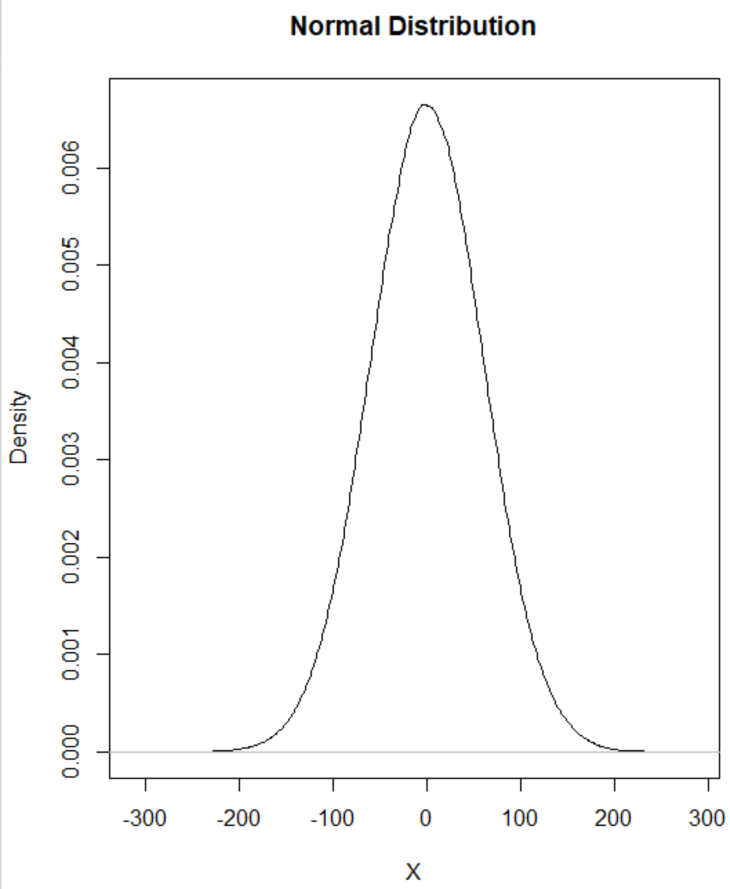
$$\lim_{X \to \infty} cdf(X) = 1$$

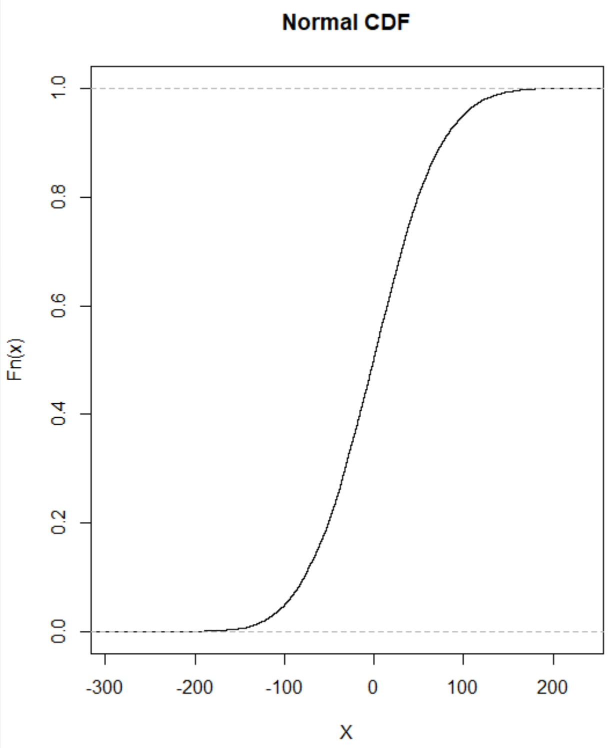
- Cumulative mass function (cmf)
  - It describes a discrete random variable

Cumulative distribution function

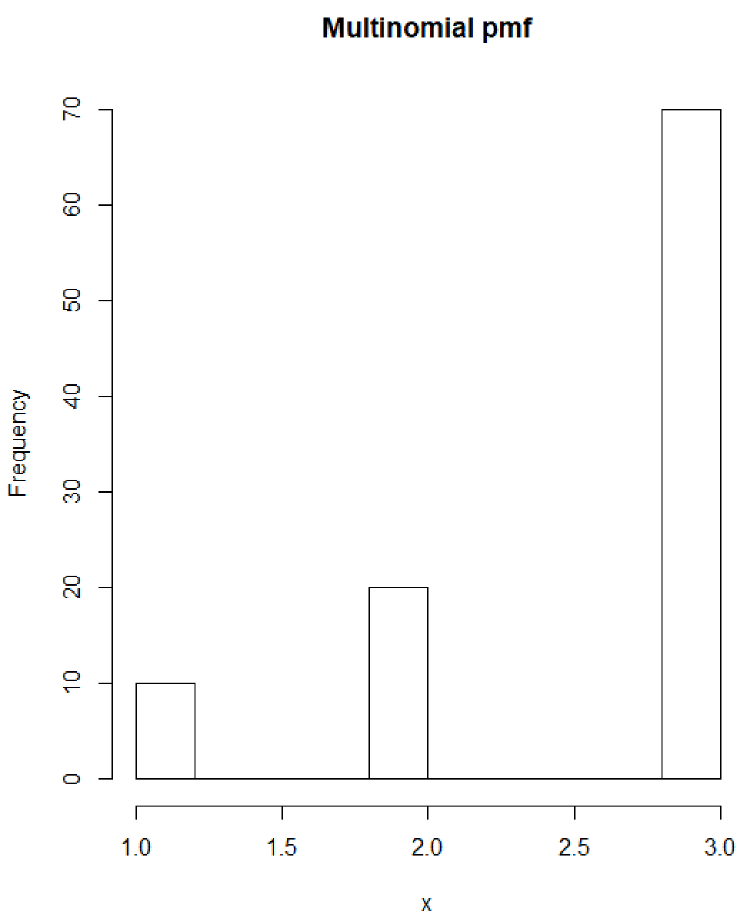
- Example: loaded die

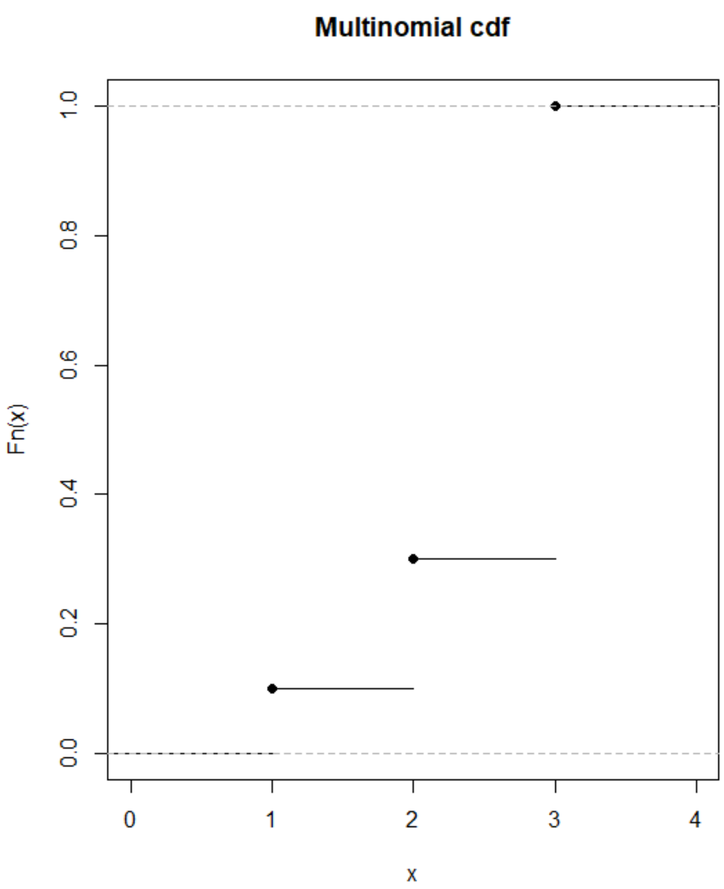
Example - Normal distribution





Example - Multinomial distribution





Law of large numbers

- The law of large numbers describes the result of performing the same experiment a large number of times

- Strong law of large numbers states that the sample average converges almost surely to the expected value

$$Average(X_{1:n}) \to \mu \qquad \text{when } n \to \infty$$

Central Limit Theorem

- This explains the prevalence of normal distribution in the real world

- The characteristics we measure are the sum of a huge number of small effects
  - Therefore, the distribution tends to be normal

Example

- Bernoulli Trial -> Binomial distribution -> Normal distribution

Hypothesis testing

- The fundamental question we want to address is whether the effects are real or due to randomness

- Two steps:
  - Effect is significant, didn't happen by chance
  - Interpret the result as an answer to the original question

Example

- Testing a difference in Means
  - Null hypothesis – the distribution for the two groups are the same. Difference are due to chance

$$\begin{cases} H_o & \mu_X = \mu_{null} \\ H_A & \mu_X \neq \mu_{null} \end{cases}$$

  - Example – Height in different cities

Statistical significance

- Null hypothesis: Assumption that the apparent effect was actually due to chance (H_o)

- Alternative Hypothesis : The experiment that we are measuring

Statistical significance

- P-value: Probability of the apparent effect under the null hypothesis
  - If the p-value is low enough, the null hypothesis unlikely true

- Interpretation: Based on the p-value, we conclude if the effect is real or not
  - i.e. The effect is false until there is a contradiction. If there is a contradiction, then the effect is true

Statistical significance

- What is significant and what am I measuring?
- Example: p-value 0.05

Choosing a threshold

- Hypothesis testing error
  - False positive – accept hypothesis when it is false
  - False negatives – reject hypothesis when it is true

		Real values	
	Total population	Condition positive	Condition negative
Prediction	Predicted positive	True positive (Power)	False positive Type I error
	Predicted negative	False negative Type II error	True negative

Choosing a threshold

		Real values	
	Total population	Condition positive	Condition negative
Prediction	Predicted positive	True positive (Power)	False positive Type I error
	Predicted negative	False negative Type II error	True negative

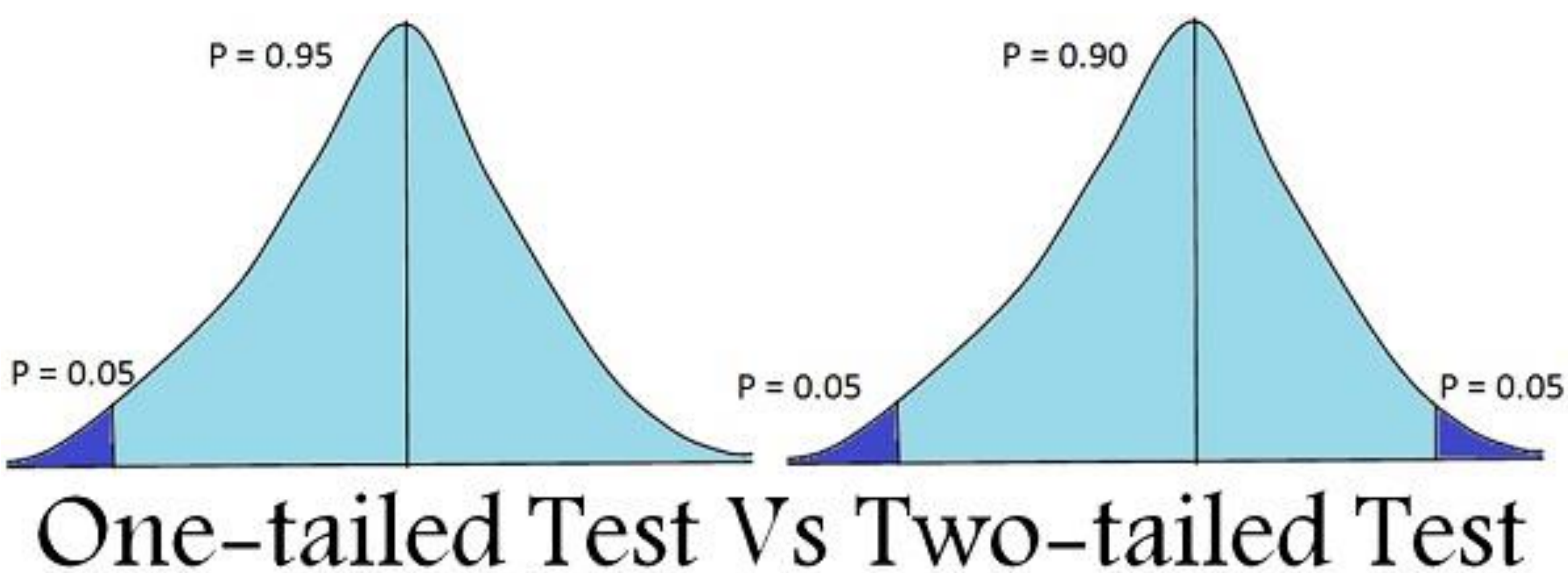
Choosing a threshold

- Statistical Power – It is the probability that the test will be positive if the null hypothesis is false
- False Discovery Rate (FDR) – Rate of false positives and number of true values predicted
- Precision - Rate of true positives and number of true values predicted
- Sensitivity – Rate of true positive and real true values

Choosing a threshold

- Choose an α threshold for p-values and to accept as significant when p-value < α
- Common choice: $\alpha \leq 5\%$
- The probability of a false positive is α
- If lower alpha then it is lower the chance of false positive
  - However, it may reject a valid hypothesis
- Trade-off between false positives and false negatives

Choosing a threshold



Choosing a threshold

- Hypothesis testing relational operator: $<$, $>$, $\neq$

Interpreting the result

- Classical
  - If p-value$<\alpha$, then it is statistically significant

- Practical
  - The lower the p-value, the higher the confidence the effect is real

Statistic test/Contrast test

- They are used to verify or reject a hypothesis from data
- They must have:
  - Data
  - Null hypothesis
  - Alternative hypothesis
  - Contrast statistic – p-value

- Type of contrasts:
  - Parametric
  - Non-parametric

T-test (Univariate)

- Parametric test
- It contrasts the mean of a population
- The population follows a Normal distribution
  - But the variance is unknown
- Hypothesis

$$\begin{cases} H_o: \mu_1 = \mu_0 \\ H_A: \mu_1 \neq \mu_0 \end{cases}$$

Mann-Whitney U Test

- Non-Parametric test
  - $N < 25$
- It contrasts the centrality of a population (median)
- Symmetric distribution
- Hypothesis

$$\begin{cases} H_o: Median(X) = Median_0 \\ H_A: Median(X) \neq Median_0 \end{cases}$$

T-test (2 Samples)

- Parametric test
  - N<25
- It contrasts the mean of two populations
  - Independent variables
- Both populations follow a Normal distribution
  - But the variance is unknown in both
- Hypothesis

$$\begin{cases} H_o: \mu_1 = \mu_2 \\ H_A: \mu_1 \neq \mu_2 \end{cases}$$

Wilcoxon Test

- Non-Parametric test
  - Small sample
  - Paired data
- It contrasts the centrality of a population (median)
- Symmetric distribution
- Hypothesis

$$\begin{cases} H_o: Median(X) = Median_0 \\ H_A: Median(X) \neq Median_0 \end{cases}$$

Z-test

- Parametric test
  - N >= 25
- It contrasts the mean of two populations
  - Independent variables
- Both populations follow a Normal distribution
- Hypothesis

$$\begin{cases} H_o: \mu_1 = \mu_2 \\ H_A: \mu_1 \neq \mu_2 \end{cases}$$

Correlation test

- Contrast to test for independence between two variables
- If data follows a normal distribution
- Hypothesis

$$\begin{cases} H_o: \rho = 0 \\ H_A: \rho \neq 0 \end{cases}$$

- If data does not follows a normal distribution a Kendall's Tau correlation coefficient is used

χ^2-test/ Categoric data test

- Contrast to test for homogeneity and/or independence
- Two-way tables
- For each factor the events are summed and are compared to the expected value
- Hypothesis

$$\begin{cases} H_o: Homogeneous \\ H_A: Non-homogeneous \end{cases}$$

Example

- In the dataset "Popular Kids," students in grades 4-6 were asked whether good grades, athletic ability, or popularity was most important to them.

Original Table

Goals	Grade 4	Grade 5	Grade 6	Total
Grades	49	50	69	168
Popular	24	36	38	98
Sports	19	22	28	69
Total	92	108	135	335

Expected Values

Goals	Grade 4	Grade 5	Grade 6
Grades	46.1	54.2	67.7
Popular	26.9	31.6	39.5
Sports	18.9	22.2	27.8

- DOF: 4 and $\chi^2 = 1.51 \therefore p - value = 0.8244$

Example

- Dataset from "Popular kids", now associated by type of school

	School Area			
Goals	Rural	Suburban	Urban	Total
Grades	57	87	24	168
Popular	50	42	6	98
Sports	42	22	5	69
Total	149	151	35	335

- DOF: 4, $\chi^2 = 18.564 \therefore p - value = 0.001$

Statistic test/Contrast test

- Which test do I need?
  - Number of samples

Statistic test/Contrast test

- Which test do I need?
  - Paired data

Statistic test/Contrast test

- Which test do I need?
  - Depth of the contrast – one or two parameters to compare

Statistic test/Contrast test

- Which test do I need?
  - Correlation bewtween two variables

Statistic test/Contrast test

- Which test do I need?
  - Correlation bewtween two variables

Statistic test/Contrast test

- Which test do I need?
  - Do I have count data or continuous data

Bootstraping

- As the population is unknown, the true error in a sample statistic against its population value is unknown.

- In bootstrap we resample the sample, assuming the sample is the total population

- A great advantage of bootstrap is its simplicity

- We use it to avoid bias

Bootstraping

- Example

Modeling

- Model
  - A system's representation
  - It incorporates the knowledge of the system
- Constraints:
  - Are the system variables quantifiable?
- Requirements:
  - Representation
  - Learning
  - Inference

Stochastic models

- Stochastic models are used to model the relationships between random variables
- To model relationships they use independence and probability distributions
- Stochastic modeling is needed when the studied system can be only measured partially

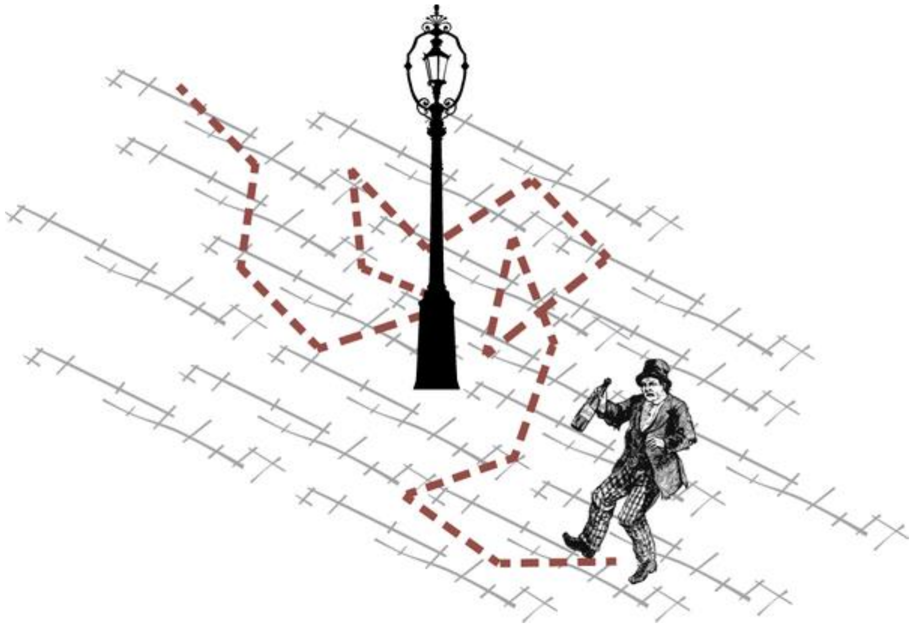
Stochastic processes

- A stochastic or random process refers to a collection of random variables that are associated or are indexed by another variable
  - i.e. A variable depend on a position or time

- Most of the sciences use stochastic processes
  - Physics
  - Biology
  - Engineering

- Stochastic Process Realization vs Random Variable
  - Example: Random walks or Brownian motion

Random walk



HW

- R code: (50%)
  - With the Dataset.csv, filtered by "Drug use disorders" and "Deaths per 100 000 population (standardized rates)“ apply a statistical test to see if the deaths in 2014 are significantly different than in 2003.
    - Use all the data for this test
    - Use a bootstrapping strategy with 100 resamples of 75% of the data per resample
    - Justify your answer and also justify the use of the statistical test

HW

- R code: (50%)
  - Investigate the Mann-Whitney U Test and code it
    - Everything must be in a R-Markdown
  - Test versus t-test and also test versus wilcox.test with the parameter paired=F
  - Example data: From the faraway library the pima data.
  - https://en.wikipedia.org/wiki/Mann%E2%80%93Whitney_U_test#Calculations

Pima data