Frequentist Inference Tools (Cont'd)

⑤ Pivotal statistics

- Recall we're trying to estimate $\theta = E_F(X)$

→ would be desirable if we had $\hat{\theta} = t(\underline{X})$ such that it does not depend on F.

(No need to estimate other parameters of F in order to estimate the parameter we want).

→ A statistic w/ this property is called a pivotal statistic.

Example:

- Say we have two samples:

$\underline{X}_1 = (X_{11}, X_{12}, X_{13}, \dots, X_{1n_1})$; $\underline{X}_2 = (X_{21}, X_{22}, \dots X_{2n_2})$

n_1 is ~~size~~ cardinality of $\underline{X}_1$

- Assume (to make things easy)

$$\boxed{X_{1i} \overset{iid}{\sim} N(\mu_1, \sigma^2) \quad i = 1, \dots, n_1}$$

$$\boxed{X_{2i} \overset{iid}{\sim} N(\mu_2, \sigma^2) \quad i = 1, \dots, n_2}$$

- We wish to test null hypothesis $H_0: \mu_1 = \mu_2$

- The difference of means test statistic

$\hat{\theta} = \bar{X}_2 - \bar{X}_1$ has distribution:

$$\boxed{\hat{\theta} \sim N\left(0, \sigma^2 \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)}$$

we don't know this. (of course, we can use ① plug-in principle)

Digression:

Let X be a $N(\mu, \sigma^2)$ r.v.

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\boxed{V(aX) = a^2 \cdot V(X)}$$

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \mu = n \cdot \mu$$

$$E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n} \cdot E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{\cancel{n}} \cancel{n}\, \mu = \mu$$

$\rightarrow \bar{X}$ is unbiased estimator of μ

$$V\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n^2} \cdot V\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(X_i) = \frac{\cancel{n} \cdot \sigma^2}{n^{\cancel{2}}} = \frac{\sigma^2}{n}$$

$\rightarrow$ We could estimate σ^2 with $\hat{\sigma}^2$

$$\hat{\sigma}^2 = \frac{\left[\sum_{i=1}^{n_1} (X_{1i} - \bar{X}_1)^2 + \sum_{i=1}^{n_2} (X_{2i} - \bar{X}_2)^2\right]}{(n_1 + n_2 - 2)}$$

However, there is a better way.

Student, instead of using $\hat{\theta}$, proposed testing H_0 using $t = \frac{\bar{X}_2 - \bar{X}_1}{\hat{sd}}$; $\boxed{\hat{sd} = \hat{\sigma}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{1/2}}$

$\boxed{t \sim t_{n_1+n_2-2}}$ (No other parameter to estimate).

- Under H_0, t statistic is pivotal. (σ^2 doesn't matter).

Then, $\bar{X}_2 - \bar{X}_1 \pm \boxed{t_{\nu,\alpha}} \cdot \hat{sd}$ is an exact confidence interval on $\mu_2 - \mu_1$

+ critical value

$\nu = n_1+n_2-2$

Level of significance.

no need to estimate σ^2 with $\hat{\sigma}^2$.

- Most times pivotality is not available, but we still got the other frequentist tools :).

Frequentist Optimality

Neyman-Pearson lemma for optimal hypothesis testing.

(Simple Version) NP Lemma:

Assumptions

- Trying to decide between 2 possible densities for observed data $\underline{x}$, a null hyp. density $f_0(\underline{x})$ and alternative density $f_1(\underline{x})$

A testing rule $t(\underline{x})$ says which choice, 0 or 1, we will make having $\underline{x}$.

For any decision we make, there will be two associated errors α & β.

Type I Type II.

$$\alpha = \underbrace{Pr_{f_0}}_{\text{The null density is true}} \{ t(\underline{x}) = 1 \}$$ — choosing 1 when f_0 generated $\underline{x}$.

Rejecting f_0 Incorrectly

$$\beta = Pr_{f_1} \{ t(\underline{x}) = 0 \}$$ — Choosing 0 when f_1 generated $\underline{x}$.

Incorrectly failing to reject f_0

Let $L(\underline{x})$ be the likelihood ratio

$$L(\underline{x}) = f_1(\underline{x}) / f_0(\underline{x})$$

and define testing rule $t_c(\underline{x})$ by

$$t_c(\underline{x}) = \begin{cases} 1 & \text{if } L(\underline{x}) \geq c \\ 0 & \text{if } L(\underline{x}) < c \end{cases}$$

a fixed cutoff value.

$\Updownarrow$ Alternatively

$$t_c'(\underline{x}) = \begin{cases} 1 & \text{if } \log L(\underline{x}) \geq c' \\ 0 & \text{if } \log L(\underline{x}) < c' \end{cases}$$

N-P Lemma: Only rules of this form can be optimal.

That is : $\alpha_c < \alpha$ and $\beta_c < \beta$ for some c

where α, β are generated by any other kind of rule.

Example: we'd like to get a c such that our Type I error rate is equal to α_0 and Type II error β is minimized for $\boxed{\begin{array}{l} f_0 \sim N(0,1) \\ f_1 \sim N(\frac{1}{2}, 1) \end{array}}$

If we have sample $\underline{x} = (x_1, x_2, \dots, x_n)$

$$f_0(x_i) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x_i^2}{2}} \quad [\text{Normal}(0,1) \text{ pdf}]$$

$$\Rightarrow f_0(\underline{x}) = \left[\frac{1}{\sqrt{2\pi}}\right]^n \cdot \prod_{i=1}^n e^{\frac{-x_i^2}{2}} = \left[\frac{1}{\sqrt{2\pi}}\right]^n \cdot e^{-\frac{1}{2}\cdot\sum_{i=1}^n x_i^2}$$

If A and B are indep $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$$\Rightarrow f_1(\underline{x}) = \left[\frac{1}{\sqrt{2\pi}}\right]^n \cdot \prod_{i=1}^n e^{\frac{-(x_i - \frac{1}{2})^2}{2}} = \left[\frac{1}{\sqrt{2\pi}}\right]^n \cdot e^{-\frac{1}{2}\sum_{i=1}^n (x_i - \frac{1}{2})^2}$$

$$L(\underline{x}) = \frac{f_1(\underline{x})}{f_0(\underline{x})} = \frac{e^{-\frac{1}{2}\cdot\sum_{i=1}^n (x_i - \frac{1}{2})^2}}{e^{-\frac{1}{2}\cdot\sum_{i=1}^n x_i^2}} = e^{-\frac{1}{2}\left[\sum_{i=1}^n (x_i - \frac{1}{2})^2 - \sum_{i=1}^n x_i^2\right]}$$

$$= e^{-\frac{1}{2}\left[-\sum_{i=1}^n x_i + \sum_{i=1}^n \frac{1}{4}\right]} = e^{-\frac{1}{2}\left[-n\cdot\bar{x} + \frac{n}{4}\right]}$$

$$= e^{\frac{1}{2}\left[n\bar{x} - \frac{n}{4}\right]}$$

Then, $\boxed{L(\underline{x}) > C} \Rightarrow e^{\frac{1}{2}\left[n\bar{x} - \frac{n}{4}\right]} > C$

$$\Rightarrow \frac{1}{2} n\bar{x} - \frac{n}{8} > \log(C) = c'$$

$$\Rightarrow \boxed{\bar{X} > c_2}$$

We want compare $\bar{x}$ vs. a constant.

- Most Powerful hyp. test for 2 Normals is to compare $\bar{X}$ vs. a constant.

Now, how to relate c, α and β for this test?

$$\alpha = \Pr(\underbrace{\bar{X} > c}_{\text{we reject}} \mid \underbrace{\mu = 0}_{\text{Null density is true.}})$$

Then $z = \dfrac{\overbrace{\bar{X} - 0}^{\text{Your typical z-statistic}}}{1/\sqrt{n}} = \bar{X}\sqrt{n} \sim N(0,1)$

Thus,

$$\begin{aligned}
\alpha &= \Pr(\bar{X} > c \mid \mu = 0) \\
&= \Pr(\bar{X}\sqrt{n} > c \cdot \sqrt{n} \mid \mu = 0) \\
&= 1 - \boxed{\Pr(\underbrace{\bar{X}\sqrt{n} \leq c\sqrt{n}}_{z \leq c\sqrt{n}} \mid \mu = 0)}
\end{aligned}$$

Value of a $N(0,1)$ cdf at $c\sqrt{n}$

$$\alpha = 1 - \underbrace{\Phi}_{\text{standard Normal cdf.}}(c\sqrt{n})$$

$$\Rightarrow \Phi(c\sqrt{n}) = 1-\alpha$$

$$c\sqrt{n} = \Phi^{-1}(1-\alpha) \qquad \boxed{\overset{!!}{0}\ !}$$

$$c = \frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}}$$

(Btw, in $\underset{\text{for 2 normals}}{\underline{\text{general}}}$ $\boxed{c = \mu_0 + \frac{1}{\sqrt{n}} \cdot \Phi^{-1}(1-\alpha)}$

$$\beta = P(\underbrace{\bar{X} < c}_{\text{we do not reject } H_0} \mid \underbrace{\mu = \tfrac{1}{2}}_{H_1 \text{ is true.}})$$

Then $\dfrac{\bar{X} - \frac{1}{2}}{1/\sqrt{n}} \sim N(0,1)$

Following $\underline{\text{same reasoning.}}$

$$\beta = Pr\left[(\bar{X} - \tfrac{1}{2}) \cdot \sqrt{n} \le (c - \tfrac{1}{2}) \cdot \sqrt{n} \mid \mu = \tfrac{1}{2}\right]$$

$$\Rightarrow \boxed{\beta = \Phi\left[(c - 1/2) \cdot \sqrt{n}\right]}$$

$$\Phi^{-1}(\beta) = \left(c - \tfrac{1}{2}\right)\sqrt{n}$$

Smallest type 2 error for a hyp. test

with type I error equal to α.