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Assignment 2

1. M
2. M
3. Marginal

$$f(x) = (\frac{1}{\sqrt{2\pi D}}\exp\left(-\frac{(x-\mu)^2}{2D}\right)) * (\frac{1}{\sqrt{2\pi A}}\exp\left(-\frac{(\mu-M)^2}{2A}\right))$$
$$= \exp\left(-\frac{(x-\mu)^2}{2D} - \frac{(\mu-M)^2}{2A}\right)$$
$$= \exp\left(-\frac{1}{2DA}(x^2A - 2x\mu A + \mu^2 A + \mu^2 A - 2\mu MD + M^2 D)\right)$$
$$= \exp\left(-\frac{1}{2DA}\mu^2(D+A) - 2\mu(xA+MD) + (x^2A + \mu^2 D)\right)$$
$$= \exp\left(-\frac{D+A}{2DA}\left(\mu^2 - \frac{2M(xA+MD)}{D+A} + \frac{x^2A+M^2D}{D+A}\right)\right)$$
$$= \exp\left(-\frac{D+A}{2DA}\left(\mu - \frac{xA+MD}{D+A}\right)^2\right) * \exp\left(-\frac{D+A}{2DA}\left(\mu - \frac{xA+MD}{D+A}\right)^2\right)$$

4. Marginal density is α= (1-p)/p and β=n
   The conditional density is μ~gamma
5. The F distribution in this case relates to the Beta distribution with the equation

$$\frac{x_1}{x_1+x_2}$$

6. By transforming it into a multivariate distribution making x_2 ~ Bin(20, 0.5*θ) so we define

$$f(20, \theta_1, \theta_2) = \frac{20!}{x_1!\ x_2!}\ {\theta_1}^{x_1}{\theta_2}^{x_2}$$

We obtain the fisher information for both θ's their combination an the hessian matrix

$$I_\theta = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log\left(\frac{20!}{x_1!\ x_2!}\ {\theta_1}^{x_1}{\theta_2}^{x_2}\right)\right\}$$
$$= -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log(20!) - \log(x_1!) - \log(x_2!) + \log({\theta_1}^{x_1}) + \log({\theta_2}^{x_2})\right\}$$
$$= -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log({\theta_1}^{x_1}) + \log({\theta_2}^{x_2})\right\}$$

$$I_{\theta_1} = -E_{\theta_1}\left\{\frac{\partial^2}{\partial\theta_1^2}\ x_1\log(\theta_1)\right\} = -E_{\theta_1}\left\{\frac{\partial^2}{\partial\theta_1^2}\frac{x_1}{\theta_1}\right\} = E_{\theta_1}\left\{\frac{x_1}{\theta_1^2}\right\} = \frac{1}{\theta_1}$$

$$I_{\theta_2} = -E_{\theta_2}\left\{\frac{\partial^2}{\partial\theta_2^2}\, x_2\log(\theta_2)\right\} = -E_{\theta_2}\left\{\frac{\partial^2}{\partial\theta_2^2}\frac{x_2}{\theta_2}\right\} = E_{\theta_2}\left\{\frac{x_2}{\theta_2^2}\right\} = \frac{1}{\theta_2}$$

$$\begin{aligned} I_{\theta_{12}} = I_{\theta_{21}} = -E_{\theta_{21}}&\left\{\frac{\partial^2}{\partial\theta_2\theta_1}\log({\theta_1}^{x_1}) + \log({\theta_2}^{x_2})\right\} \\ &= -E_{\theta_{12}}\left\{\frac{\partial^2}{\partial\theta_2\theta_1}\log({\theta_1}^{x_1}) + \log({\theta_2}^{x_2})\right\} \\ &= -E_{\theta_{21}}\left\{\frac{\partial^2}{\partial\theta_2\theta_1}x_1\log(\theta_1) + \log x_2(\theta_2)\right\} \\ &= -E_{\theta_{12}}\left\{\frac{\partial^2}{\partial\theta_2\theta_1}x_1\log(\theta_1) + \log x_2(\theta_2)\right\} = -E_{\theta_{21}}\left\{\frac{\partial}{\partial\theta_2}\left(\frac{x_1}{\theta_1}\right)\right\} \\ &= -E_{\theta_{21}}\left\{\frac{\partial}{\partial\theta_1}\left(\frac{x_2}{\theta_2}\right)\right\} = -E_{\theta}\{0\} = 0 \end{aligned}$$

$$I_\theta = \begin{bmatrix} \frac{1}{\theta_1} & 0 \\ 0 & \frac{1}{\theta_2} \end{bmatrix}$$

Giving

$$var(\theta) \geq \frac{1}{n * I_\theta}$$

7. The relationship between them is that when the flat prior density g(θ) = 1 then the posterior expectation of θ will be the same as the MLE θ
8. T
9. $$\frac{fx_1(x)}{fx_2(x)} = \frac{\frac{n!}{x!(n-x)!}x_1^x(1-x_1)^{n-x}}{\frac{n!}{x!(n-x)!}x_2^x(1-x_2)^{n-x}} = \frac{x_1^x(1-x_1)^{n-x}}{x_2^x(1-x_2)^{n-x}} = \left(\frac{x_1}{x_2}\right)^x * \frac{(1-x_1)^{n-x}}{(1-x_2)^{n-x}} = \left(\frac{x_1}{x_2}\right)^x * \frac{\left(\frac{1-x_1}{1-x_2}\right)^n}{\left(\frac{1-x_1}{1-x_2}\right)^x} = \left(\frac{x_1(1-x_2)}{x_2(1-x_1)}\right)^x * \left(\frac{1-x_1}{1-x_2}\right)^n$$

$$\alpha = \log\left(\frac{x_1(1-x_2)}{x_2(1-x_1)}\right) \qquad \Psi(\alpha) = \left(\left(\frac{x_1}{x_2}\right) * \left(\frac{1}{e^\alpha}\right)\right)^n$$

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10. With a Bayessian approach the way to estimate theta would be to first use the known probability of 0.5 heads and 0.5 tails as a prior in which case the conjugate prior of the binomial is the beta

$$\mathrm{P}(\theta) = \frac{1}{B(h_1, t_1)}\theta^{h_1-1}(1-\theta)^{t_1-1}$$

Where h_1 is the times that the coin landed in heads and t_1 the times that it landed on tails.

For the probability of the event happening because we have a likelihood with binomial distribution and a beta prior we obtain that:

$$\mathrm{P(D)} = {}_{h_2+t_2}C_{h_2} \frac{B(h_2+h_1, t_2+t_1)}{B(h_1, t_1)}$$

After obtaining this we join it to obtain the probability of obtaining heads that would be

$$\mathrm{P(\theta|D)} = \frac{1}{B(h_2+h_1, t_2+t_1)} \theta^{h_2-1}(1-\theta)^{t_2-1}$$