Flaws in frequentist inference & Bayesian vs. Frequentist Comparison

- **First Example:**

- Say that an ongoing experiment is being run. Each month an independent Normal variate is observed $\underbrace{X_t}_{\text{month index}} \sim N(\mu, 1)$

- You run a hypothesis test on each month to test $\boxed{H_0: \mu = 0 \quad vs. \quad H_a: \mu > 0}$

Thus, every month you compute test statistic

$$\underline{z_i} = \frac{\sum_{t=1}^{i} X_t / i - 0}{1/\sqrt{i}} = \frac{\sum_{t=1}^{i} X_t}{\sqrt{i}}$$

which is a z-score based on data up to month i.

- Now, say that at month 80, the scheduled end of the experiment $z_{80} = 1.66 > \underbrace{1.645}_{\text{One-sided 95\% for a } N(0,1) \text{ distr.}}$

Then, we would reject H_0. OK, nothing funky up to this point.

However, say they checked the data after month 20 looking for significance. If significance is observed, then we would stop the experiment. If not, we would continue.

- If $z_{20} = 0.79$ for example, we would continue up to time 30 and we would have concluded significance, since $z_{30} = 1.66$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The issue here is that algorithm $f(\underline{x})$ has type I error:

Union

$$\alpha = P(\text{Rejecting } H_0 \text{ Incorrectly})$$
$$= P(z_{20} > 1.645 \cup z_{30} > 1.645 \mid \mu = 0)$$
$$= P(z_{20} > 1.645 \mid \mu = 0) + P(z_{30} > 1.645 \mid \mu = 0)$$
$$- P(z_{20} > 1.645 \cap z_{30} > 1.645 \mid \mu = 0)$$
$$= 0.05 + 0.05 - \text{Intersection}$$
$$\approx \underline{0.074}.$$

Details in book

Thus, under the algorithm $f(\underline{x})$, frequentists would have concluded the result is not at 5% significance level!!

Conversely, on Bayesian Inference we always use the same algorithm. (Bayes' Rule) in which the likelihood function of $\underline{x} = (x_1, x_2, \dots, x_{30})$ is always $f_\mu(x) = \prod_{i=1}^{30} e^{-\frac{1}{2}(x_i - \mu)^2} \cdot C$ regardless of stopping the experiment early or not.

The stopping rule has no impact on posterior $g(\mu \mid \underline{x})$; as it only depends on $\underline{x}$ through the likelihood!