EXAM

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①

Statistics

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1.1

Poisson distribution comes from Binominal distribution

$$P_B(r; p, n) = p^r (1-p)^{n-r} \frac{n!}{r!(n-r)!}$$

But instead of having 0 or 1 in Binominal distribution, Poisson dėstribution has outcome λ, λ are natural numbers. $\lambda = pn$

$$P(r, \frac{\lambda}{n}, n) = \left(\frac{\lambda}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n-r} \frac{n!}{r!(n-r)!}$$

After some assumptions and simplifications Poisson distribution will look

$$P(r, \frac{\lambda}{n}, n) = \frac{e^{-\lambda}\lambda^r}{r!}$$

We need to prove that

1) $0 < P(r, \frac{\lambda}{n}, n) < 1$ or

$$0 < \frac{e^{-\lambda}\lambda^{n}}{r!} < 1$$

2) $$\sum_{r}^{\infty}\left(\frac{e^{-\lambda}\lambda^{n}}{r!}\right) = 1$$

I start with 2)

$$\sum_{r}^{\infty}\left(\frac{e^{-\lambda}\lambda^{n}}{r!}\right) = e^{-\lambda}\underbrace{\sum_{r}^{\infty}\frac{\lambda^{n}}{r!}}_{e^{+\lambda}} =$$

$$= e^{-\lambda}\cdot e^{\lambda} = 1$$

1) $\frac{e^{-\lambda}\lambda^{n}}{r!}$ - is a positive figure and it is > 0.

It is a term of ~~So~~ a sum which equals 1. But each term of the sum should be $\leq$ the sum.

So. $$0 < \frac{e^{-\lambda}\lambda^{n}}{r!} < 1$$

1.2.

Poisson distribution $P(y;\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!}$

Where λ frequency of visits.

y number of visits per day.

$y=0$, no visit

$P(0,1) = \frac{e^{-1}\cdot 1^{0}}{0!} = e^{-1} = \underline{0.3679}$

$y=1$ One visit

$P(1,1) = \frac{e^{-1}\,1^{1}}{1!} = e^{-1} = \underline{0.3679}$

$y=2$ 2 visits =

$P(2,1) = \frac{e^{-1}\,1^{2}}{2!} = \frac{1}{2}e^{-1} = \underline{0.1840}$

Propability of having at least 1 visit in 1 day.

$P(y>1;1) = 1 - P(0,1) = 1 - e^{-1} = \underline{0.6321}$

1.3

$$\text{PDF} - f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$\mu = \int_0^1 x f(x)\,dx = \int_0^1 x(a+bx^2)\,dx =$$

$$= \int_0^1 (ax + bx^3)\,dx = \left(\frac{1}{2}ax^2 + \frac{1}{4}bx^4\right)\Big|_0^1 =$$

$$= \frac{1}{2}a + \frac{1}{4}b = \frac{3}{5}$$

$$\int_0^1 f(x)\,dx = \int_0^1 (a+bx^2)\,dx = a + \frac{1}{3}b = 1$$

$$\begin{cases} \frac{1}{2}a + \frac{1}{4}b = \frac{3}{5} \\ a + \frac{1}{3}b = 1 \end{cases} \Leftrightarrow \begin{cases} a + \frac{1}{2}b = \frac{6}{5} \\ -a - \frac{1}{3}b = -1 \end{cases}$$

$$\left(\frac{1}{2} - \frac{1}{3}\right)b = \frac{6}{5} - 1$$

$$\begin{cases} \frac{1}{6}b = \frac{1}{5} \\ a = 1 - \frac{b}{3} \end{cases} \Rightarrow \begin{cases} b = \frac{6}{5} \\ a = \frac{3}{5} \end{cases}$$

Check results.

$$f(x) = \frac{3}{5} + \frac{6}{5} x^2$$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{3}{5} + \frac{6}{5} x^2\right) dx =$$

$$= \frac{3}{5} + \frac{1}{3}\,\frac{6}{5} = \frac{3}{5} + \frac{2}{5} = \underline{1}$$

$$\int_0^1 f(x)\,x\,dx = \int_0^1 \left(\frac{3}{5} x + \frac{6}{5} x^3\right) dx =$$

$$= \frac{1}{2}\,\frac{3}{5} + \frac{1}{4}\,\frac{6}{5} = \frac{3}{2 \cdot 5} + \frac{3}{2 \cdot 5} = \underline{\frac{3}{5}}$$

$$\boxed{\text{PDF} - \quad f(x) = \begin{cases} \frac{3}{5} + \frac{6}{5} x^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}}$$

2.1

To find the MLE for the probability p of Bernoulli distribution $f(x_i, p) = p^{x_i}(1-p)^{1-x_i}$ we will use the ML formalism.

$$\ln L = -\sum_{i=1}^{N} f(x_i, p) =$$

$$= -\sum_{i=1}^{N} \ln\left(p^{x_i}(1-p)^{1-x_i}\right) =$$

$$= -\sum_{i=1}^{N} \left(\ln p^{x_i} + (1-x_i)\ln(1-p)\right) =$$

$$= -\sum_{i=1}^{N} \ln p^{x_i} - \sum_{i=1}^{N} (1-x_i)\ln(1-p) =$$

$$= -\sum_{i=1}^{N} x_i \ln p - \ln(1-p)\sum_{i=1}^{N}(1-x_i) =$$

$$= -\ln p \sum_{i=1}^{N} x_i - \ln(1-p)\sum_{i=1}^{N}(1-x_i)$$

To find the function $\ln \mathcal{L}(x_i, p)$ minimum we will differentiate it regarding p and equate it to zero. Then we will solve the equation regarding p.

$$\frac{\partial(\ln \mathcal{L}(x_i,p))}{\partial p} = -\frac{1}{(p)}\sum_{i=1}^{N} x_i + \frac{1}{(1-p)}\sum_{i=1}^{N}(1-x_i)$$

$$\frac{\partial \ln L}{\partial p} = 0$$

$$-\frac{1}{p}\sum_{i=1}^{N} x_i + \frac{1}{1-p}\sum_{i=1}^{N}(1-x_i) = 0$$

$$\frac{1-p}{p} = \frac{\sum_{i=1}^{N}(1-x_i)}{\sum_{i=1}^{N} x_i} \qquad \left\{\sum_{i=1}^{N} 1 = N\right\}$$

$$\frac{1}{p} - 1 = \frac{N}{\sum_{i=1} x_i} - 1$$

$$\boxed{\hat{p} = \frac{1}{N}\sum_{i=1}^{N} x_i}$$

$\hat{p}$ is the estimator of the pdf.

2.2

To find the MLE for the mean μ of the long-normal distribution

$$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-(\ln(x)-\mu)^2/2\sigma^2}$$

we will use the MLE formalism:

$$\ln L = -\sum_{i=1}^{N} \ln(f(x_i))$$

$$= -\sum_{i=1}^{N} \left(\ln\left(\frac{1}{\sigma x_i \sqrt{2\pi}} e^{-(\ln(x_i)-\mu)^2/2\sigma^2}\right)\right) =$$

$$= \sum_i \left(\ln(\sigma x_i \sqrt{2\pi}) + \frac{(\ln(x_i)-\mu)^2}{2\sigma^2} \right)$$

To find minimum of the function $\ln L(\mu, x)$ we need to find its derivative regarding μ and equate it to zero.

$$\frac{\partial \ln L}{\partial \mu} = \sum_{i=0}^{N} \left(0 + \frac{(\ln(x_i)-\mu)^{2-1} \cdot 2 \cdot (-1)}{2\sigma^2} \right)$$

We equate $\dfrac{\partial \ln L}{\partial \mu}$ to zero and solve the equation regarding μ.

$$\sum_{i=0}^{N} \left(\left(\ln(x_i) - \mu \right) \right) = 0$$

$$\sum_{i=1}^{N} \ln(x_i) - \sum_{i=1}^{N} \mu = 0, \quad \left\{ \sum_{i=1}^{N} \mu = N\mu \right\}$$

$$\boxed{\mu = \frac{1}{N} \sum_{i=1}^{N} \left(\ln(x_i) \right)}$$

At this μ the function $\ln \mathcal{L}(\mu, x)$ will have the smallest value.

2.3

a) As I understand the origins of the data, there was a set of N numbers of independand experiments, for escample, of the radiactive nucleous decay.

Each nucleus decayed at some moment t_i. For example, 1st neicleus decayed at $t_1 = 0.15$ sec, 2nd at $t_2 = 3.51$ sec, nnd at $t_n = 2.13$ sec, ...)

We rearanged these t_i and got the data provided in the file.

At the first, we will find MLE of $P(t_i, \tau) = \frac{1}{\tau} e^{-t_i/\tau}$ analitically.

$P(t_i, \tau)$ - is a probability of an event at time t_i but τ - is a parametor.

We will use the MLE formalism

$$\ln L = \sum_{i=1}^{N} \left(\ln \left(\frac{1}{\tau} e^{-t_i/\tau} \right) \right) =$$

$$= \sum_{i=1}^{N} \left(\ln \frac{1}{\tau} - \frac{t_i}{\tau} \right) =$$

$$= \sum_{i=1}^{N} \left(-\ln \tau - \frac{t_i}{\tau} \right)$$

Differentiating the function $\ln L(t_i, \tau)$ with respect of τ and equate it to zero, we will find the estimator $\hat{\tau}$

$$\frac{\partial}{\partial \tau} \left(\ln L(t_i, \tau) \right) = \frac{\partial}{\partial \tau} \left(\sum_{i=1}^{N} \left(-\ln \tau - \frac{t_i}{\tau} \right) \right) =$$

$$= \sum_{i=1}^{N} \left(-\frac{1}{\tau} + \frac{t_i}{\tau^2} \right) = -\frac{N}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^{N} t_i$$

$$\frac{\partial}{\partial \tau} \left(\ln L(t_i, \tau) \right) = 0$$

$$-\frac{N}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^{N} t_i = 0 \Rightarrow \boxed{\hat{\tau} = \frac{1}{N} \sum_{i=1}^{N} t_i}$$

The error on MLE can be found as $\hat{\sigma}_{\hat{\tau}} = \left(\left\langle \frac{\partial^2}{\partial \tau^2} \ln L(t_i, \tau) \right\rangle \right)^{-1}$

$$\frac{\partial^2}{\partial \tau^2} \ln L(t_i, \tau) = \frac{\partial}{\partial \tau} \left(-\frac{N}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^{N} t_i \right) =$$

$$= \frac{N}{\tau^2} - \frac{2 \cdot \sum_{i=1}^{N} t_i}{\tau^3}$$

So, $\hat{\sigma}_{\hat{\tau}} = \left(\left\langle \frac{N}{\tau^2} - \frac{2 \sum_{i=1}^{N} t_i}{\tau^3} \right) \right)^{-1}$

Using the data provided in file 'exercise23.cvs' we find

$\sum_{i=1}^{N} t_i = 170{,}71$, $N = 87$

$\hat{\tau} = \frac{1}{87} \sum_{i=1}^{87} t_i = \frac{1}{87} 170.71 = \underline{1.9622}$

$$\hat{\sigma}_{\hat{\tau}} = \left(\frac{87}{(1.9622)^2} - \frac{2 \cdot 170.71}{(1.9622)^3} \right)^{-1} = 0.044$$

$$\boxed{\hat{\tau} = 1.9622 \pm 0.044}$$

We see that numerical results are the same as analitical.

From this point I can assume that the code is working properly.

I have difficulties to find $\frac{\partial^2}{\partial \tau^2} \ln L(t_i, \tau)$ in symbolic expression.

For estimating $\frac{\partial^2}{\partial \tau^2} \ln L$ I use a numerical estimation

$$\frac{\partial^2}{\partial \tau^2} f(\tau) = \frac{f(\tau + 2\Delta\tau) - 2f(\tau + \Delta\tau) - f(\tau)}{(\Delta\tau)^2}$$

I got $\frac{\partial^2}{\partial \tau^2} \ln L \approx 19.5$

So, $\hat{\sigma}_{\hat{\tau}} \approx \frac{1}{19.5} \approx 0.052$

So, $\boxed{\hat{\tau} = 1.962 \pm 0.052}$

However if a pdf is more comprehensive, we may have difficulties to find analitical solution.

In this case we may use a code in MATLAB to find a numerical solution.

I wrote a MATLAB code,
I loaded data from file
'exercise 23. cvs'
I defined a function

$fs = @(\tau)\ sum(\log 1/\tau - t_i/\tau)$

Then I found minimum of this function by using

$[\tau_{min}, fs_{min}] = fminsearch(fs, 4)$

$\tau = 1.9622$ and $\ln \mathcal{L} = 145.65$

2.3 b)

I evaluated the acceptance time $T = 10$ sec.

It means after time $t_i > T$ propability $P(t_i, \tau) = + \frac{1}{\tau} e^{-t/\tau}$ will be close to zero.

$$P = \frac{1}{1.96} \times e^{-\frac{10}{1.96}} \approx 0.003$$

In the real experiment it would mean that no events will happen afte the time $T = 10$ sec.
(or even some rare events happen, but we will neglect them because they are to rare).

2.3. c)

For finding value of $\hat{\tau}$ with T we will use the similar procedure as we did before.

But instead of pdf $p = \frac{1}{\tau} e^{-t/\tau}$

we will use pdf $p = \frac{1}{\tau} e^{-t/\tau} \frac{1}{1 - e^{-T/\tau}}$

$$\ln L = \sum_{i=1}^{N} \ln\left(\frac{1}{\tau} e^{-t_i/\tau} \frac{1}{1 - e^{-T/\tau}}\right) =$$

$$= \sum_{i=1}^{N} \left(- \ln \tau - t_i/\tau - \ln\left(1 - e^{-T/\tau}\right)\right)$$

In contraste to previus estimation a new term $\left(- \ln\left(1 - e^{-T/\tau}\right)\right)$ appears.

As before we need to find min. of function $\sum \ln \mathcal{L}(t_i, \tau)$ regarding τ

As before we will use <u>fminsearch</u>

2.3 d)

I found $\hat{\tau}$ for a few T

$T = 10$ sec $\quad \hat{\tau} = 2.03$ sec $\quad \sigma = 0.073$ sec

$T = 4$ sec $\quad \hat{\tau} = 35$ sec $\quad \sigma = 1000$ sec

$T = 4.5$ sec $\quad \hat{\tau} = 5.8$ sec $\quad \sigma = 9$ sec

$T = 5$ sec $\quad \hat{\tau} = 3.76 \quad \sigma = 1.4$ sec

$T = 6$ sec $\quad \hat{\tau} = 2.674 \quad \sigma = 0.30$ sec

$T = 7$ sec $\quad \hat{\tau} = 2.325 \quad \sigma = 0.15$ sec

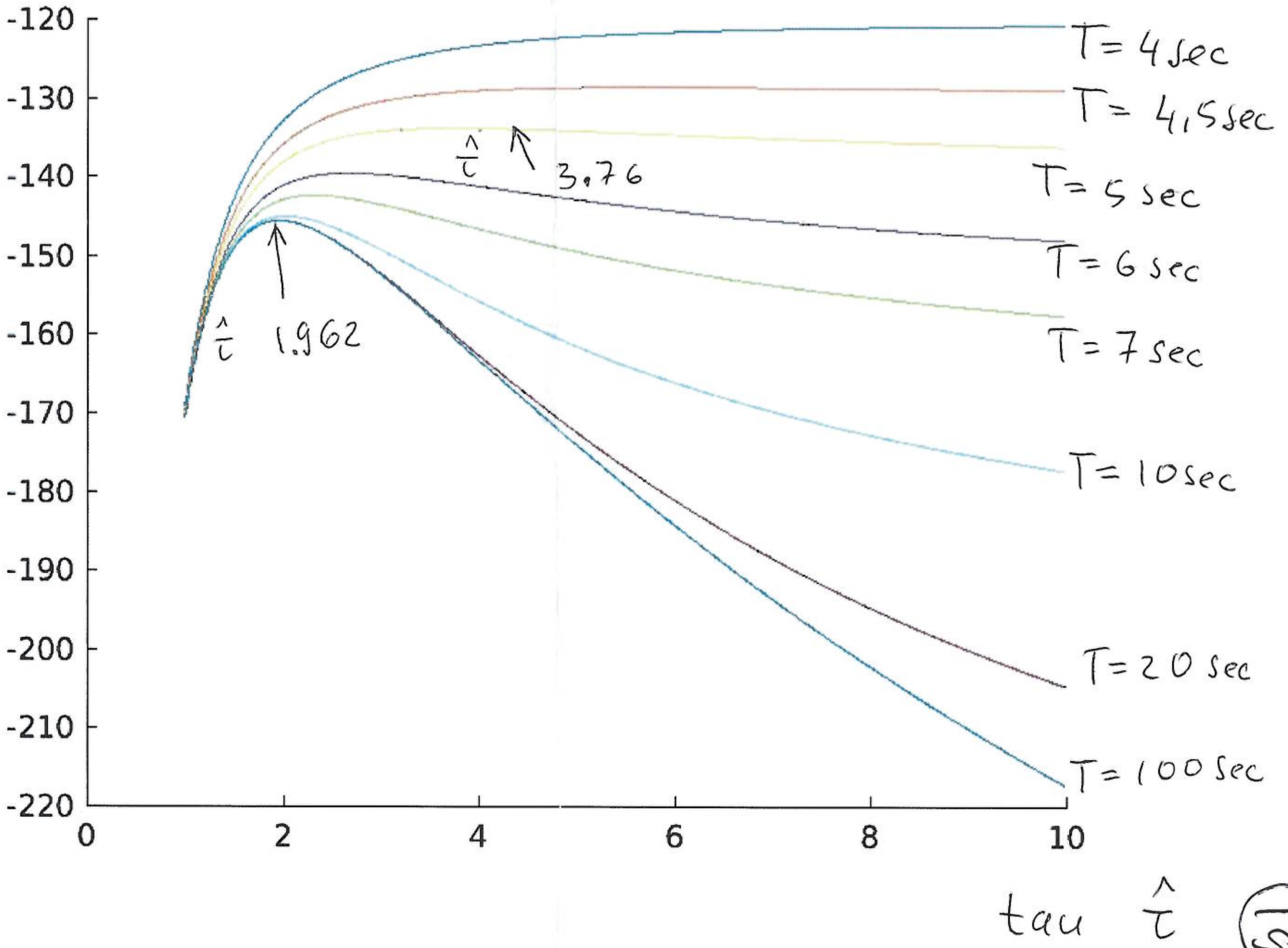
$T = 10$ sec $\quad \hat{\tau} = 2.03 \quad \sigma = 0.073$ sec

$T = 20$ sec $\quad \hat{\tau} = 1.9632 \quad \sigma = 0.055$ sec

$T = 100$ sec $\quad \hat{\tau} = 1.9622 \quad \sigma = 0.053$ sec

Plots of LMEs for different T is show on fig 3.

LME with different time acceptance



19

3.1

$$\sigma_y^2 = \left(\frac{\partial\, y(x)}{\partial x}\right)^2 \sigma_x^2 =$$

$$= \left(\frac{\partial}{\partial x}\frac{1}{\sqrt{4-x^2}}\right)^2 \sigma_x^2 = \left(\frac{\partial}{\partial x}\left(4-x^2\right)^{-\frac{1}{2}}\right)^2 \sigma_x =$$

$$= \left(2x\left(-\frac{1}{2}\right)\left(4-x^2\right)^{-\frac{3}{2}}\right)^2 \sigma_x^2 =$$

$$= \left(4-x^2\right)^{-3} x^2 \sigma_x^2$$

$$\boxed{\sigma_y^2 = x^2\left(4-x^2\right)^{-3}\sigma_x^2}$$ Error for $y(x)$

To find the value of x at which $\sigma_y^2(x)$ has a minimum, we will take its derivative and put it to zero.

$$\frac{\partial}{\partial x}\sigma_y^2(x) = \frac{\partial}{\partial x}\left(x^2(4-x^2)^{-3}\sigma_x^2\right)$$

$$= \Big(x^2(4-x^2)^{-4}(-3)(-2x) +$$
$$+ 2x(4-x^2)^{-3}\Big)\sigma_x^2 =$$

$$= \left(6x^3(4-x^2)^{-4} + 2x(4-x^2)^{-3}\right)\sigma_x^2 =$$

$$= \underline{2x}(4-x^2)^{-3}\left(3x^2(4-x^2)^{-1}+1\right)\sigma_x^2 = 0$$

$$\begin{cases} 3x^2(4-x^2)^{-1}+1 = 0 \quad \text{or} \quad 2x = 0 \end{cases}$$

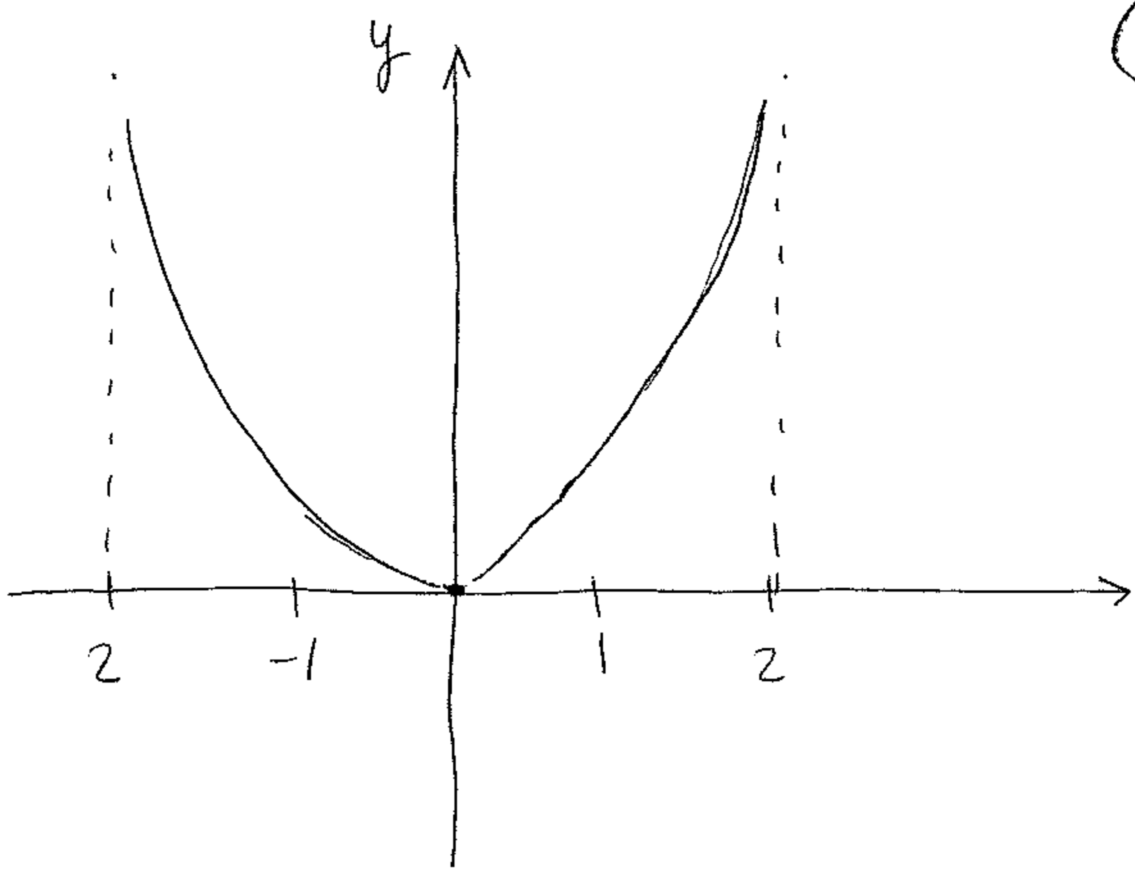
$$3x^2 + (4-x^2) = 0$$

$$3x^2 + 4 - x^2 = 0$$

$2x^2 + 2 = 0 \Rightarrow$ no solution for x

This function $\sigma_y^2(x)$ doesn't have a maximum or minimum between $-2 < x < 0$; $0 < x < 2$

The function has a minimum at $x = 0$ only.



So, at $x=0$ $\boxed{\sigma_y^2 = \frac{x^2}{(4-x^2)^3}\sigma_x^2 = 0}$

3.2 a)

From formula

$$R_i I_{Ri} = \frac{1}{\frac{1}{R_i} + \frac{1}{R}} I_{RRi} \quad , i = 1,2$$

we can find R_i

$$\left(\frac{1}{R_i} + \frac{1}{R}\right) R_i I_i = I_{RRi}$$

$$\frac{R + R_i}{\cancel{R_i} R} \cancel{R_i} I_{R_i} = I_{RRi}$$

$$R + R_i = R \frac{I_{RRi}}{I_{Ri}}$$

$$\boxed{R_i = R\left(\frac{I_{RRi}}{I_{Ri}} - 1\right) = \frac{R I_{RRi}}{I_{Ri}} - R}$$

$i = 1,2$

Covariance matrix

$$V = \begin{pmatrix} \sigma_R^2 & 0 & 0 & 0 & 0 \\ 0 & \left(\begin{matrix}\sigma_{I_{R1}} \cdot \sigma_{I_{R1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{R1}} \sigma_{I_{RR1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{R1}} \cdot \sigma_{I_{R2}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{R1}} \cdot \sigma_{I_{RR2}} \\ + \sigma_s^2\end{matrix}\right) \\ 0 & \left(\begin{matrix}\sigma_{I_{RR1}} \cdot \sigma_{I_{R1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{RR1}} \cdot \sigma_{I_{RR1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{RR1}} \cdot \sigma_{I_{R2}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{RR1}} \cdot \sigma_{I_{RR2}} \\ + \sigma_s^2\end{matrix}\right) \\ 0 & \left(\begin{matrix}\sigma_{I_{R2}} \cdot \sigma_{I_{R1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{R2}} \cdot \sigma_{I_{RR1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{R2}} \cdot \sigma_{I_{R2}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{R2}} \sigma_{I_{RR2}} \\ + \sigma_s^2\end{matrix}\right) \\ 0 & \left(\begin{matrix}\sigma_{RR2} \cdot \sigma_{R1} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{RR2}} \cdot \sigma_{I_{RR1}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{RR2}} \cdot \sigma_{I_{R2}} \\ + \sigma_s^2\end{matrix}\right) & \left(\begin{matrix}\sigma_{I_{RR2}} \cdot \sigma_{I_{RR2}} \\ + \sigma_j^2\end{matrix}\right) \end{pmatrix}$$

$$\sigma_{I_R}^2 = (470\ \Omega \times 5\%)^2 = 552\ \Omega$$

$$\sigma_{I_{R1}} = (5\% \times 12.2) = 0.61\ (mA)^2$$

$$\sigma_{I_{RR1}} = (5\% \times 25.8) = 1.29\ (mA)^2$$

$$\sigma_{I_{R2}} = (5\% \times 72.4) = 3.62\ (mA)^2$$

$$\sigma_{I_{RR2}} = (5\% * 76.3) = 3.82\ (mA)^2$$

$$\sigma_S = (0.5)^2 = 0.25\ (mA)^2$$

$$V = \begin{pmatrix} 552 & 0 & 0 & 0 & 0 \\ 0 & 0.61 & 1.03 & 2.45 & 2.55 \\ 0 & 1.03 & 1.81 & 1.91 & 5.15 \\ 0 & 2.45 & 1.91 & 13.2 & 13.92 \\ 0 & 2.55 & 5.15 & 13.92 & 14.70 \end{pmatrix}$$

3.2 c)

$$R_1 = R\left(\frac{I_{RR1}}{I_{R1}} - 1\right)$$

$$R_2 = R\left(\frac{I_{RR2}}{I_{R2}} - 1\right)$$

Error matrix for R_1 and R_2

$$G = \begin{pmatrix} \frac{\partial R_1}{\partial R} & \frac{\partial R_1}{\partial I_{R1}} & \frac{\partial R_1}{\partial I_{RR1}} & \frac{\partial R_1}{\partial I_{R2}} & \frac{\partial R_1}{\partial I_{RR2}} \\ \frac{\partial R_2}{\partial R} & \frac{\partial R_2}{\partial I_{R1}} & \frac{\partial R_2}{\partial I_{RR1}} & \frac{\partial R_2}{\partial I_{R2}} & \frac{\partial R_2}{\partial I_{RR2}} \end{pmatrix}$$

$$\frac{\partial R_1}{\partial R} = \frac{I_{RR1}}{I_{R1}} = \frac{25.8}{12.2} = 1.11$$

$$\frac{\partial R_1}{\partial I_{R1}} = -\frac{R\, I_{RR1}}{(I_{R1})^2} = \frac{470 \times 25.8}{12.2^2} = -81.5$$

$$\frac{\partial R}{\partial I_{RR1}} = \frac{R}{I_{RR1}} = \frac{470}{25.8} = 18.21$$

$$\frac{\partial R_2}{\partial R} = \left(\frac{I_{RR2}}{I_{R2}} - 1\right) = \left(\frac{76.3}{72.4} - 1\right) = 0.053$$

$$\frac{\partial R_2}{\partial I_{R2}} = -\frac{R\, I_{RR2}}{(I_{R2})^2} = -\frac{470 \cdot 76.3}{(72.4)^2} = -6.8$$

$$\frac{\partial R_2}{\partial I_{RR2}} = \frac{R}{I_{R2}} = \frac{470}{72.4} = 6.5$$

$$G = \begin{pmatrix} 1.11 & -8.15 & 18.21 & 0 & 0 \\ 0.053 & 0 & 0 & -6.8 & 6.5 \end{pmatrix}$$

$$G * V * G^{-1} = \begin{pmatrix} 2,367 & 29.10 \\ 29.10 & 12.31 \end{pmatrix}$$

$$R_1 = 470 \times \left(\frac{25.8}{12.2} - 1 \right) = 523.9 \pm 48.7 \,\Omega$$

$$R_2 = 470 \times \left(\frac{76.3}{72.4} - 1 \right) = 25.31 \pm 3.51 \,\Omega$$

$$\boxed{\begin{aligned} R_1 &= 524 \pm 49 \,\Omega \\ R_2 &= 25.3 \pm 3.5 \,\Omega \end{aligned}}$$

3.2 d)

Error of R_2 will be a sum of errors of its variables.

$$\sigma_{R_2}^2 = \left(\frac{\partial R_2}{\partial R}\right)^2 \sigma_R^2 + \left(\frac{\partial R}{\partial I_{R2}}\right)^2 \sigma_{I_{R2}}^2 + \dots$$

However if we look at

$$\frac{\partial R_2}{\partial I_{RR1}} = \frac{I_{RR_2}}{I_{R_2}} - 1 .$$

This value will be very close to zero, because $I_{RR2}/I_{R2} \approx 1$ and $1 - 1 \approx 0$.

It will increase uncertanty in the experiment.

We need to increase I_{RR_2} or dicrease I_{R_2} and make I_{RR2}/I_{R2} different from one.

4. a)

I loaded data from the file 'exercise4.cvs' and ploted it. I was asked to fit this data by function

$f(x) = a(1-e^{-bx})$.

I ploted this function $f(x)$ on the same plot and compared them. and started to vary parameters a and b.

At <u>$a = 4$</u> and <u>$b = 0.07$</u> visually $f(x) = a(1-e^{-bx})$ fitted the data at the best.

Two plots – data and $f(x)$ shown on the fig. below.

4. a)

At $a = 4$ and $b = 0.07$

$\chi^2(a, b) = 14{,}36$

and degree of freedom

of $19 - 2 = 17$

$$\sqrt{17 \times 2 - 1} = \sqrt{33} = 5{,}74$$

$$\sqrt{2 \times 14{,}36} = 5{,}36$$

χ^2 is $0{,}38\,\sigma$ away from

the expected centroid

$1 - 0{,}38\,\sigma = 0.62\,\sigma$

The fit should not be excluded.

4. b)

To find exact values of parameters a and b we need to use numerical method, for example, χ^2, "chi square", We define χ^2 as a function of two variables a and b. and find its minimum

$$\chi^2(a, b) = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left(y_i - f(x_i, a, b) \right)^2$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{N} \left(y_i - a\left(1 - e^{-bx_i}\right) \right)^2$$

Where (x_i, y_i) - the data provided from the file 'exercise 4'.cvs' and $\sigma = 0.12$ is proved.

The 3 D plot of $\chi^2(a,b)$ shown bellow.

4. b)

Using MATLAB I calculated parameters a and b

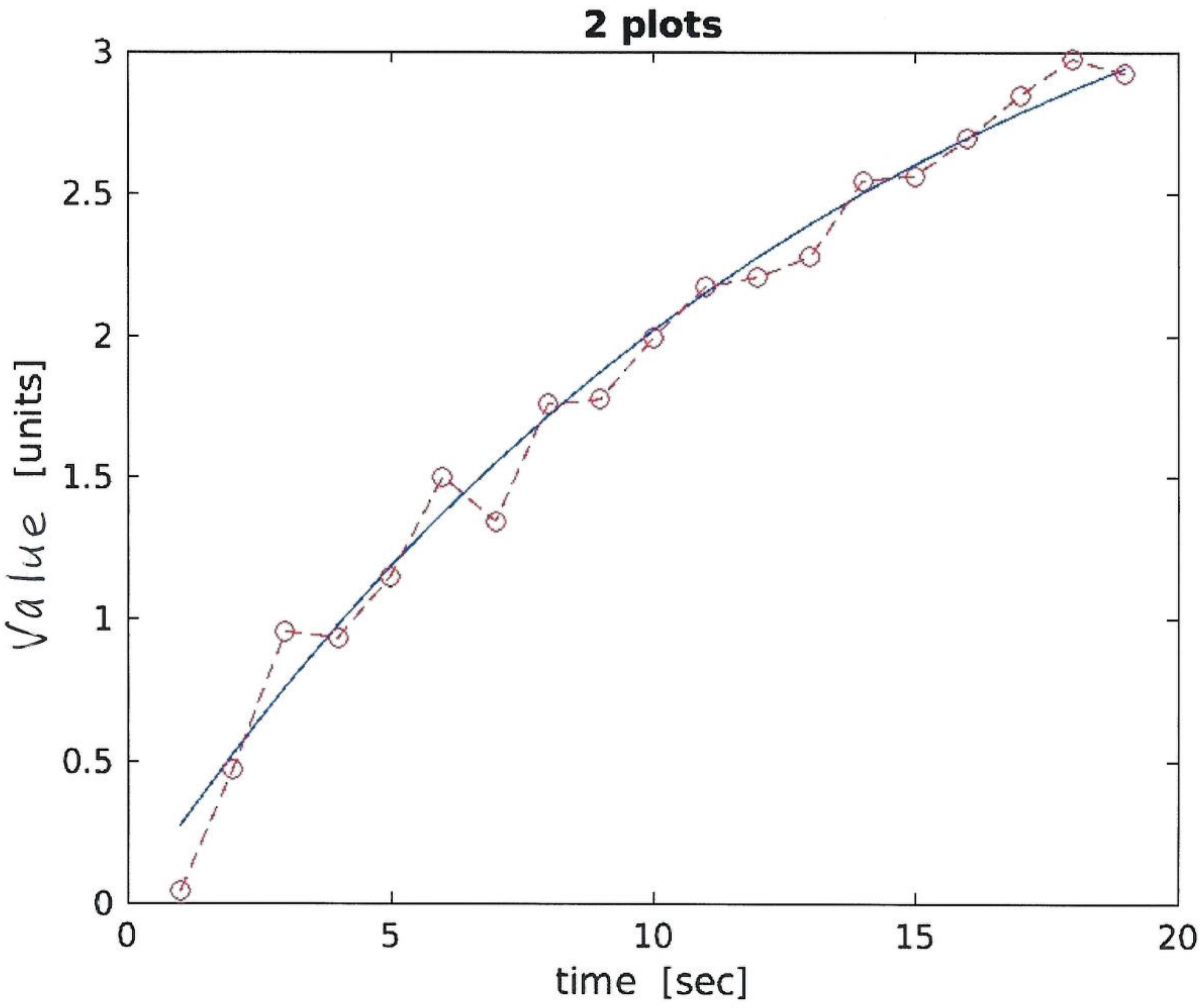
$$\boxed{\begin{aligned} a &= 4.2549 \\ b &= 0.0630 \end{aligned}}$$

So the function

$$f(x_i) = 4.2549 \times \left(1 - e^{-0.063 x_i}\right)$$

fits the data at the best possible way.

The data and the fit function shown bellow.

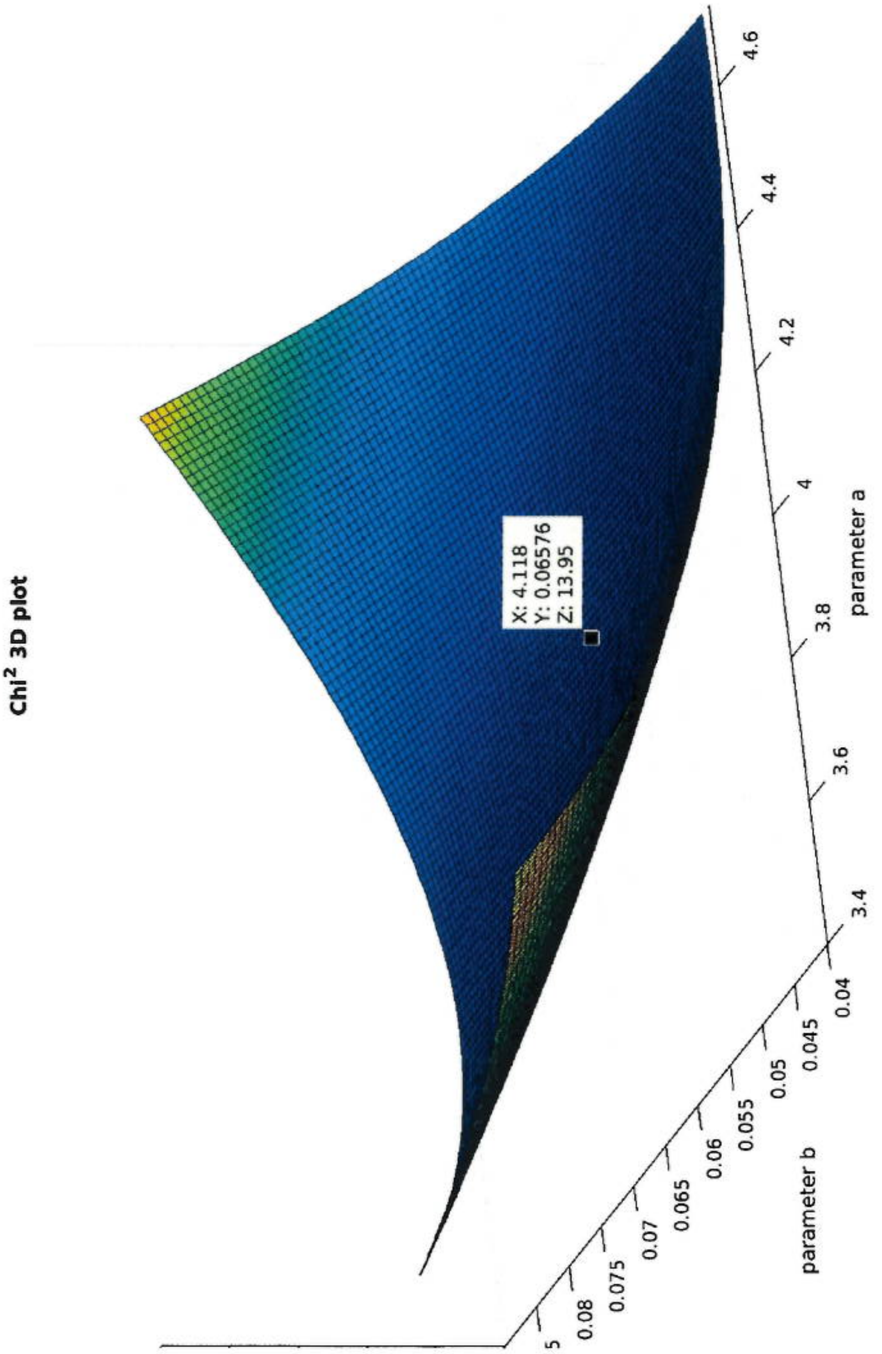


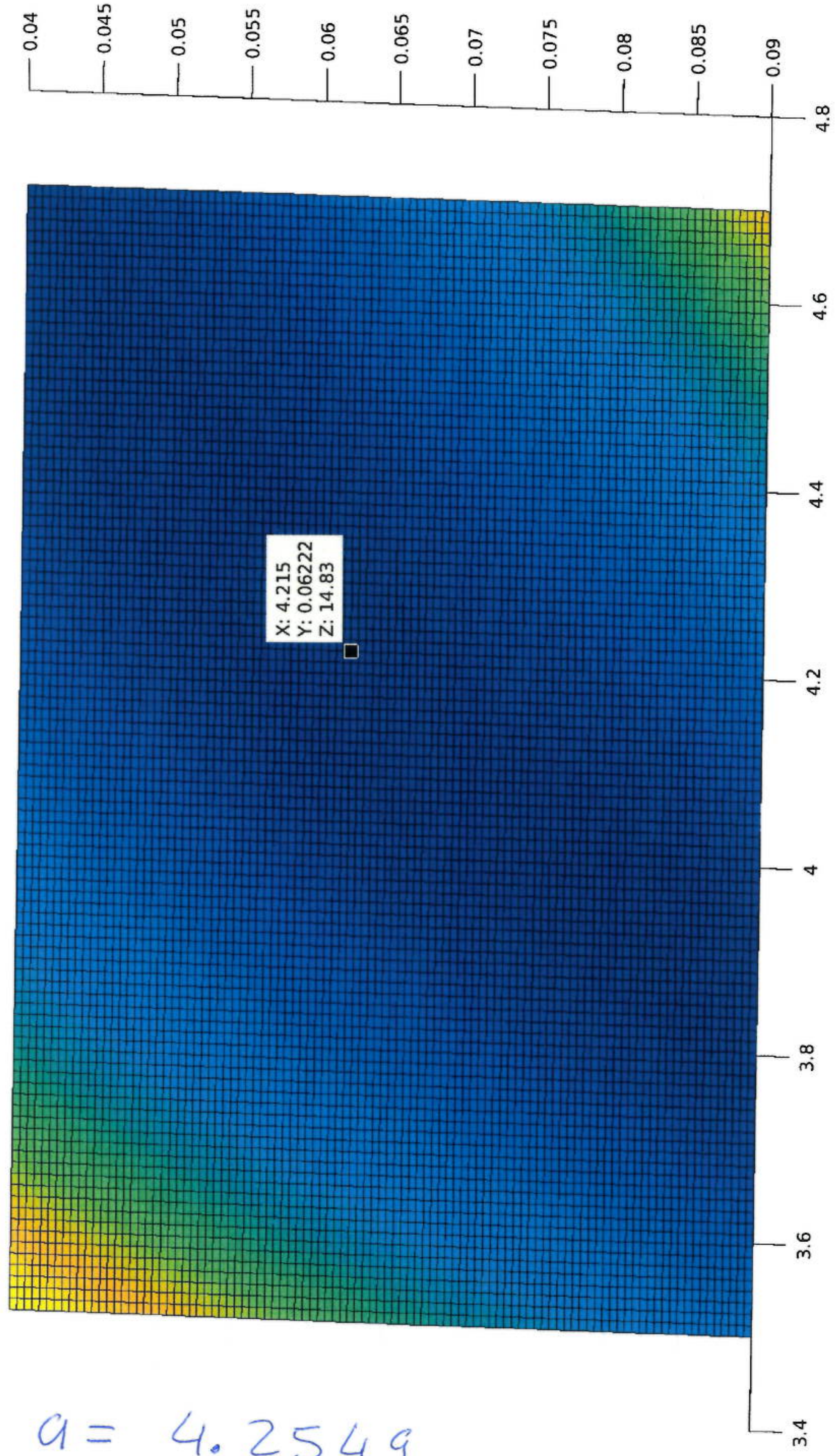
$a = 4.2549$

$b = 0.063$

$a = 4.2549$

$b = 0.0630$





$a = 4.2549$

$b = 0.0630$

4. c) Covarience matrix of a, b

$$V(a,b) = \frac{1}{2}\begin{pmatrix} \frac{\partial^2\chi^2}{\partial a^2} & \frac{\partial^2\chi^2}{\partial a\partial b} \\ \frac{\partial^2\chi^2}{\partial a\partial b} & \frac{\partial^2\chi^2}{\partial a\partial b} \end{pmatrix}$$

$$V(a,b) = \frac{1}{2}\begin{pmatrix} 37.11 & 248.8 \\ 248.8 & 195.85 \end{pmatrix}$$

$$V^{-1} = V^{-1}(a,b) = 10^{-3} \times \begin{pmatrix} 58.9 & -0.748 \\ -0.748 & 0.112 \end{pmatrix}$$

4. d)

Parameters a and b are correlated.

a and be are dependant from each other by formula

$$a = \frac{\sum_i^N y_i}{\sum_i^N (1 - e^{-bx})}$$

This formula comes from solving equations.

$$\begin{cases} \frac{\partial \chi^2}{\partial a} = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left(y_i - a(1-e^{-bx})\right)(1 - e^{-bx}) \\ \frac{\partial \chi^2}{\partial b} = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left(y_i - a(1-e^{-bx})\right)(a b e^{-bx}) \end{cases}$$

For example a and b in area (4.25; 0.063)

b	0.050	0.055	0.060	0.065	0.070	0.075
a	4.98	4.66	4.38	4.157	3.96	3.79

4. e)

If we define a new error $\sigma = 0.012$ instead of $\sigma = 0.12$ the parameters a and b will not change.

It is because σ doesn't depent from the parameters.

For example if find the parameters a and b analitically by solving a system of equations.

$$\begin{cases} \frac{\partial \chi^2}{\partial a} = \frac{\partial}{\partial a}\left(\frac{1}{\sigma^2}\sum_{i=1}^{N}(y_i - f_i)^2\right) = 0 \\ \frac{\partial \chi^2}{\partial b} = \frac{\partial}{\partial b}\left(\frac{1}{\sigma^2}\sum_{i=1}^{N}(y_i - f_i)^2\right) = 0 \end{cases}$$

error σ will be simply connselled and the system won't depend from the error. So a and b don't depent from σ and won't change.