Plasma Physics for Fusion

Assignment 1

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Part (a)

An ionised gas is considered to be a plasma if it satisfies two criteria. First condition is an ionised gas has to be "quasi-neutral", number of negatively and positively charged particles in the system should be approximately equal each other. To satisfy this condition the size of the system must be significantly bigger than size of the Dexbye sphere: $\boxed{\lambda_D << L}$

Second condition is an ionised gas has to demonstrate "collective behaviour", like for example the cyclotron motion. For this, the free time between particles' collision τ_c should be significantly bigger than oscillation period τ_p: $\boxed{\tau_c >> \tau_p}$

In addition, number of particles in the Debye sphere has to be sufficient $\boxed{N_D >> 1.}$

If a small positively charged sphere is placed into a quasi-neutral plasma, it will attract opposite charged particles (electrons) and repeal positively charged particles (ions). As a result of this, electrons will drift towards the sphere and ions will move in opposite direction. Eventually a negatively charged cloud of electrons will appear around the charged sphere and will neutralize or "screen out" the positevely charged sphere. This phenomena is called "Debye screening".

The "Debye screening" will block the positevely charged sphere and reduce the electrostatic field to zero. So outside the Debye Sphere will be no electrostatic field, no potential, no eletrostatic force, and electrons will stop moving towards the positevely charged sphere.

(i) The main characteristic of the Debye screening is its radius which is called as Debye length.

In another words, the Debye length is the length scale over which mobile carries (electrons) screen out the electric field. It describes a screening distance beyond which charges are unaware of another charges inside the Debye sphere.

(ii) The Debye length depends on some plasma parametres, particulary, temperature T and dengity n.

The higher temperature T, the more difficult to screen out the external field Φ, and the size of the Debye Sphere needs to be bigger. So, the higher temperature, the longer Debye length:

$$\boxed{\lambda_D \propto T^{\alpha} \; (\alpha > 0)}$$

Additionally, the higher density n, the more plasma particles are available to screen out the external field Φ. Therefore, the Debye length is invergely proportional to the plasma density:

$$\boxed{\lambda_D \propto \frac{1}{n^{\alpha}} \; (\alpha > 0)}$$

(iii) If the plasma particles have a Maxwellian velocity distribution $f(u) = A \cdot \exp(-\frac{1}{2} m u^2 / u_{th}^2)$ and Boltzmann relation $n = n_0 \exp(-\Phi / k_B T)$ applies to the situation:

$$n = A n_0 \exp\left(-\frac{1}{2} m u^2 / u_{th}^2 - \Phi / k_B T\right)$$

we can conclude following:

Just outside the Debye sphere some quick particles have opportunity to escape from the electric field. It means they have enough energy to overcome electrostatic force, imposed by the electric field with potential Φ.

So we can roughly estimate the order of magnitude of the electric field outside the Debye Sphere.

$$-\frac{1}{2} m u^2 / u_{th}^2 - \Phi / k_B T = 0 \Rightarrow$$

$$\Phi = \frac{1}{2} \frac{m v^2}{u_{th}^2} kT \approx \frac{3}{2} k_B T$$

$$\boxed{\Phi = \frac{3}{2} k_B T}$$

Part (b) A Maxwellian distribution for plasma particles is $f(u) = A \cdot \exp(-\frac{1}{2} mu^2/k_B T)$

In the presence of a potential energy $ez\Phi$ we can rewrite the equation: $f(u) = A\exp\left[-\frac{\frac{1}{2}mu^2}{KT} + \frac{ez\Phi}{KT}\right]$

because the total energy for the particles is: $\mathcal{E} = \frac{1}{2}mu^2 + ez\Phi$, where $\frac{1}{2}mu^2$ is a kinetic energy, and $ez\Phi$ is a potential energy.

So density n will be an integral of $f(u)$

$$n(\Phi) = \int_{-\infty}^{\infty} f(u)\,du =$$

$$= A \cdot \int_{-\infty}^{+\infty} \exp\left[-\left(\tfrac{1}{2}mu^2 + ez\Phi\right)/k_bT\right] du =$$

$$= A\exp[-ez\Phi/kT] \cdot \frac{1}{\sqrt{\pi}} \; ; \left\{ \int_{-\infty}^{\infty} \exp(-\xi^2)\,d\xi = \frac{1}{\sqrt{\pi}} \right\}$$

If we set the border condition

$n(\Phi \to \infty) \Rightarrow n_0$, n_0 is the density far away.

$$\frac{1}{\sqrt{\pi}} A\exp[-ez\Phi/kT] = n_0$$

$$\Rightarrow A = \frac{n_0}{\sqrt{\pi}} \Rightarrow$$

$$\boxed{n(\Phi) = n_0\exp\left[-\frac{ze\Phi}{kT}\right] \text{ – a Boltzmann relation.}}$$

The Poisson equation is

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} = \frac{Ze}{\varepsilon_0}[n_i - n_e]$$

$$\nabla^2 \Phi = \frac{eZ}{\varepsilon_0}\left[n_0 - n_0 \exp\left[-\frac{Ze\Phi}{KT}\right]\right]$$

$$\left\{ \begin{array}{l} \text{If } e\Phi/KT << 1, \ \exp\left[-\frac{Ze\Phi}{KT}\right] = \\ = 1 - \frac{Ze\Phi}{KT} - \frac{1}{2}\left(\frac{e\Phi}{KT}\right)^2 + \dots \end{array} \right\}$$

$$\nabla^2 \Phi = \frac{eZ n_0}{\varepsilon_0}\left[1 - \left(1 - \frac{Ze\Phi}{KT}\right)\right] = \frac{e^2 Z^2 n_0}{\varepsilon_0 KT}\Phi$$

In spherical coordinates

$$\nabla^2 \Phi = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right)\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial^2}{\partial^2 \varphi}\Phi, \quad \left\{ \begin{array}{l} \text{If we have symetry} \\ \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \varphi} = 0. \end{array} \right\}$$

$$\nabla^2 \Phi = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right)\right)$$

$$\left\{ \text{Also} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) \right\}$$

Also we call λ_D as $\boxed{\lambda_D = \sqrt{\frac{\varepsilon_0 KT}{e^2 n_0}}}$

λ_D is the Debye length

So finally we got Poisson equation in spherical coordinates

$$\frac{\partial}{\partial r^2}(r\,\Phi) = \frac{Z^3}{\lambda_D^2}(r\,\Phi)$$

A general solution for this equation will be:

$$\Phi(r) = A \cdot Z^2 \frac{1}{r} \exp\left(-\frac{r}{\lambda_D}\right)$$

To find A we need to set the initial conditions:

$$\Phi(r=R) = A Z^2 \frac{1}{R} \exp\left(-R/\lambda_D\right) =$$

$$= eZ/(4\pi\varepsilon_0 R)$$

where $eZ/(4\pi\varepsilon_0 R)$ – is potential near the charged sphere.

$$A = \frac{e}{4\pi\varepsilon_0 Z} \exp\left(\frac{R}{\lambda_D}\right)$$

$$\Phi(r) = \frac{e}{4\pi\varepsilon_0 Z} \exp\left(R/\lambda_D\right) \cdot \frac{Z^2}{r^2} \exp\left(-\frac{r}{\lambda_D}\right)$$

$$\boxed{\Phi(r) = \frac{eZ}{4\pi\varepsilon_0} \cdot \frac{1}{r} \cdot \exp\left[-\frac{(r-R)}{\lambda_D}\right]}$$