Further Plasma Physics

Introduction to plasma kinetic theory

Dr C.P. Ridgers

Course Structure

- 6 Lectures – Plasma kinetic theory
- 12 Lectures – High energy-density physics

  OR
- 12 Lectures – Low temperature plasmas

Learning outcomes

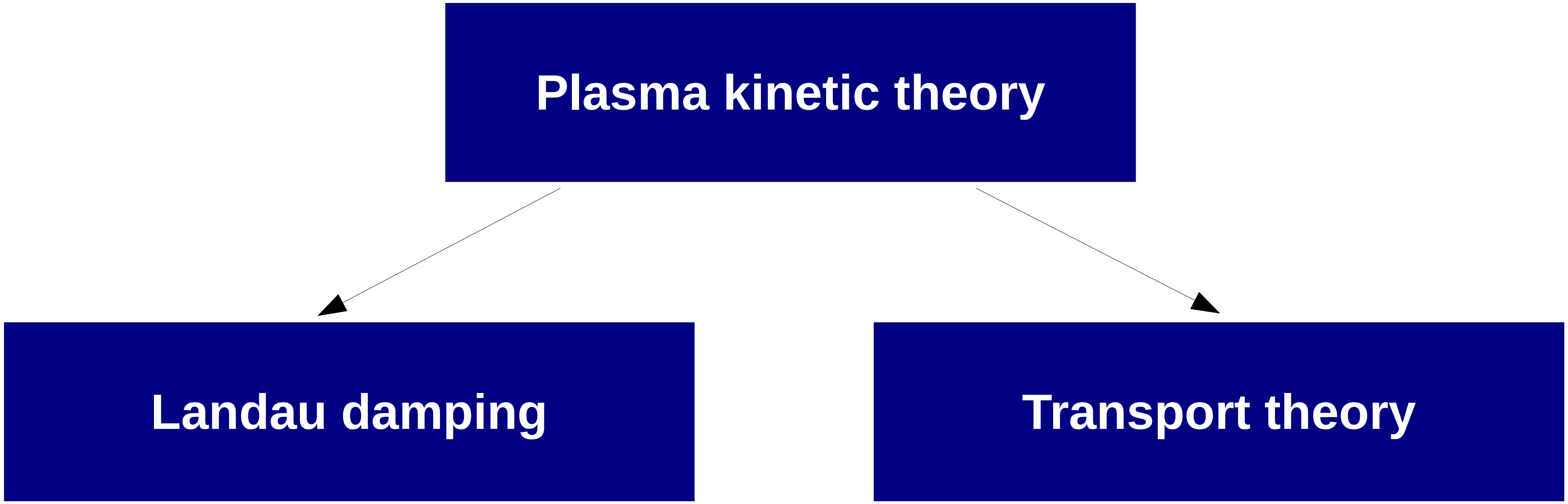
1. Derive the Vlasov equation and understand the need for a collision operator in the context of Debye shielding.
2. Linearise the Vlasov equation to obtain the plasma dielectric function and understand how the form of the dielectric function gives rise to Landau damping.
3. Write down the form of the Krook and Fokker-Planck collision operators.
4. Derive the diffusion coefficients for a magnetised plasma and use this derivation to illustrate the need to close the fluid equations.
5. Explain the origin of the Braginskii transport relations

Snappier learning outcome

What can we do with plasma kinetic theory?....

….Kinetic theory → *an explanation of the macroscopic properties of matter in terms of the motion of its constiuent microscopic particles*

Course outline



Recommended reading

Plasma kinetic theory

Landau damping

Transport theory

'Complete' derivation of Vlasov equation.
Far too much detail for this course – only
for the real enthusiast!

Shkarofsky, Bachyinskii & Johnston
Particle Kinetics of Plasmas

Landau & Lifschitz
Physical Kinetics

Boyd & Sanderson
Physics of Plasmas

Liboff
Kinetic theory: classical, quantum and relativistic description

N-body description of a plasma

Complete description of a plasma:

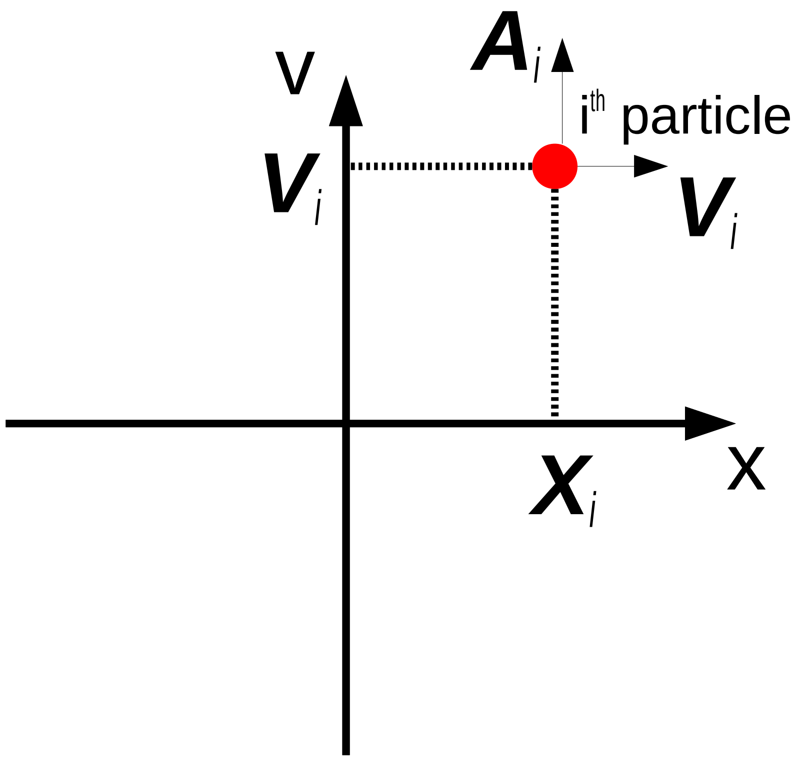
- Positions and velocities of *every* particle ($\boldsymbol{X}_i, \boldsymbol{V}_i$)
- Obey:

$$\frac{d\boldsymbol{X}_i}{dt} = \boldsymbol{V}_i \qquad \frac{d\boldsymbol{V}_i}{dt} = \frac{q}{m}\left[\boldsymbol{E}(\boldsymbol{X}_i) + \boldsymbol{V}_i \times \boldsymbol{B}(\boldsymbol{X}_i)\right]$$

- $\boldsymbol{X}_i, \boldsymbol{V}_i$ functions of time only
- **E** and **B** fields from *all other particles*. Self consistently determined by Maxwell's equations

Phase space

- **x** and **v** are coordinates and are independent



N(**x**,**v**,t)

- Particles represented by points moving around in phase space
- Moves along ***x***-axis with 'speed' $\boldsymbol{V}_i$ and along ***v***-axis with 'speed' $\boldsymbol{A}_i$ (where $\boldsymbol{A}_i$ is the acceleration of the i^{th} particle)

Klimontovich equation

- Density of particles in phase space

$$N(\boldsymbol{x},\boldsymbol{v},t)=\sum_i \delta(\boldsymbol{x}-\boldsymbol{X}_i)\delta(\boldsymbol{v}-\boldsymbol{V}_i) \quad \delta(\boldsymbol{x}-\boldsymbol{X}_i)\begin{cases}\rightarrow\infty & \boldsymbol{x}=\boldsymbol{X}_i\\ =0 & \text{otherwise}\end{cases}$$

- Satisfies the Klimontovich equation

$$\frac{\partial N}{\partial t}+\boldsymbol{v}\cdot\nabla_x N+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_v N=0$$

- This is exact but has problems:

1. Need to know exact state of all particles at t=0 (i.e. all the initial positions and velocities)

2. Need to track motion of huge number of particles

→ **no easier to solve than N-body problem**

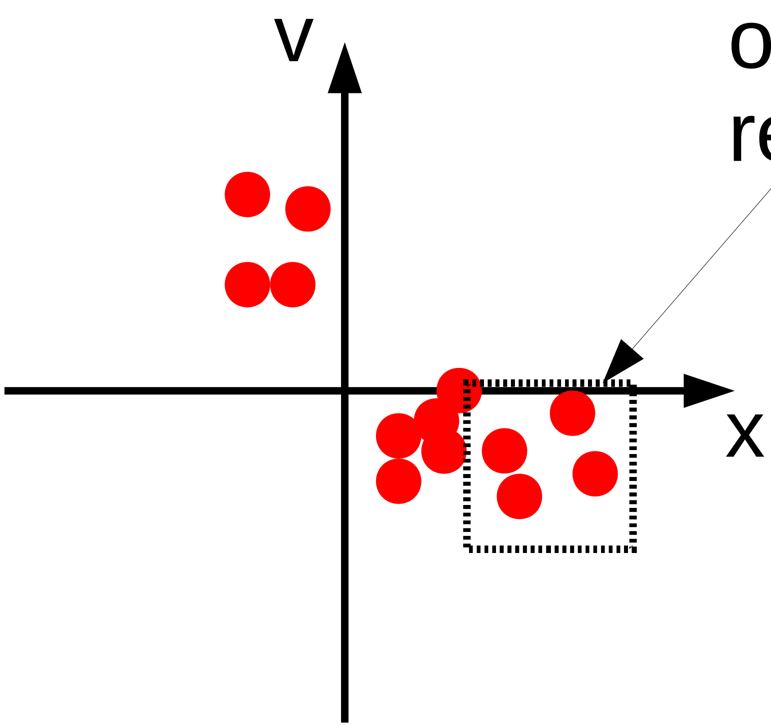
The distribution function

- $N(\mathbf{x},\mathbf{v},t)$ tells us if a particle is found at phase space coordinates $(\mathbf{x},\mathbf{v})$
- Define the distribution function $f(\mathbf{x},\mathbf{v},t)d^3\mathbf{x}d^3\mathbf{v}$ – tells us how many particles are found in phase-space volume $d^3\mathbf{x}d^3\mathbf{v}$, i.e. how many particles have position between $\mathbf{x} \rightarrow \mathbf{x}+d^3\mathbf{x}$ and velocity between $\mathbf{v} \rightarrow \mathbf{v}+d^3\mathbf{v}$.
- $f(\mathbf{x},\mathbf{v},t)$ is related to $N(\mathbf{x},\mathbf{v},t) \rightarrow f(\mathbf{x},\mathbf{v},t) = \{N(\mathbf{x},\mathbf{v},t)\}$
- {...} refers to the ensemble average over all states of N consistent f..... or more helpfully f is what we get when we diffuse point particles within $d^3\mathbf{x}d^3\mathbf{v}$ over that volume

The distribution function

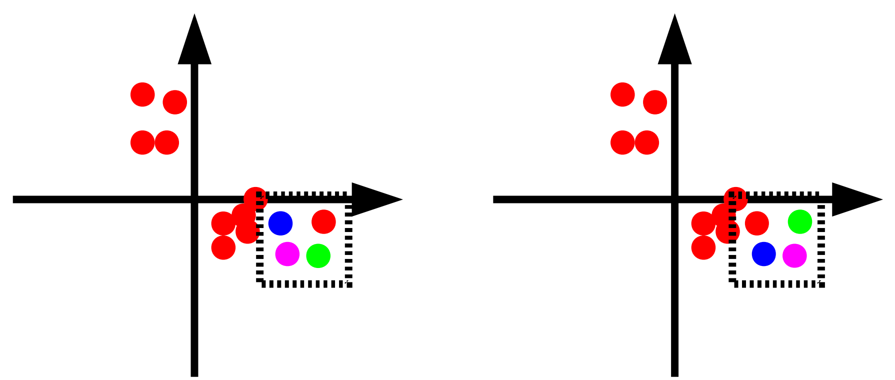
Pictorially.....

Average over this region



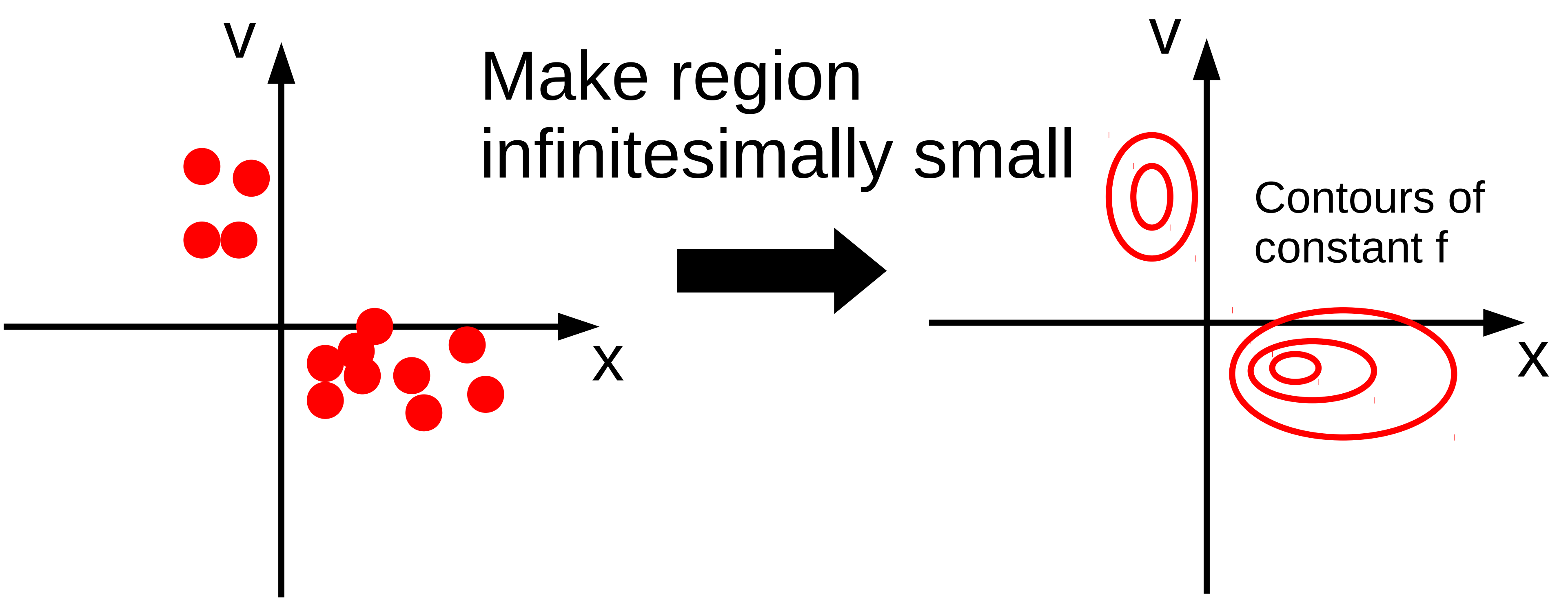
N(**x**,**v**,t)

Ensemble average is over all N that give the same f. Some examples:



The distribution function

Pictorially.....



N(**x**,**v**,t)

f(**x**,**v**,t)

The Vlasov equation

- Substitute into the Klimontovich equation

$$N(\boldsymbol{x},\boldsymbol{v},t)=f(\boldsymbol{x},\boldsymbol{v},t)+\delta N(\boldsymbol{x},\boldsymbol{v},t)$$

$$\boldsymbol{E}=\boldsymbol{E}_s+\delta\boldsymbol{E} \qquad \boldsymbol{B}=\boldsymbol{B}_s+\delta\boldsymbol{B}$$

- δN is the fluctuating part of the distribution. δ**E** & δ**B** are the fluctuating fields associated with δN

- Making this substitution and performing the ensemble average yields

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_x f+\frac{q}{m}(\boldsymbol{E}_s+\boldsymbol{v}\times\boldsymbol{B}_s)\cdot\nabla_v f=-\frac{q}{m}\{(\delta\boldsymbol{E}+\boldsymbol{v}\times\delta\boldsymbol{B})\cdot\nabla_v\delta N\}$$

The Vlasov equation

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_{x}f+\frac{q}{m}\left(\boldsymbol{E_s}+\boldsymbol{v}\times\boldsymbol{B_s}\right)\cdot\nabla_{v}f=\left(\frac{\partial f}{\partial t}\right)_{coll}$$

- Left-hand side insensitive to discreet particle nature of plasma
- Fields $\mathbf{E}_s$ & $\mathbf{B}_s$ are the 'smoothed' fields due to so-called *collective effects*
- Right-hand side depends on the discreet particle nature of the plasma - the '*collision operator*'
- To get *Vlasov equation* ignore collisions (drop subscript 's' – from now on **E** & **B** will refer to fields from collective processes)

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_{x}f+\frac{q}{m}\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_{v}f=0$$

Advanced Plasmas

Landau Damping

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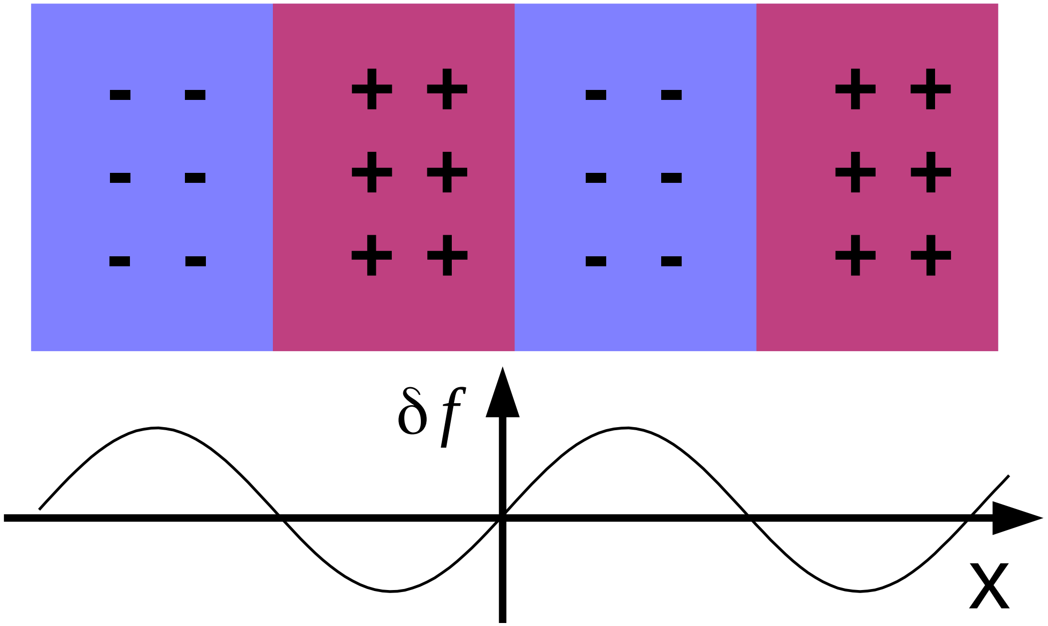
Linearising the Vlasov equation

- Consider an electrostatic wave in a plasma

  $\delta \boldsymbol{E} \propto e^{i(kx-\omega t)}$ is the (small) electric field due to collective motion



- Induces a small oscillating component in distribution function $f(\boldsymbol{x},\boldsymbol{v},t)=f_0(\boldsymbol{v})+\delta f(\boldsymbol{x},\boldsymbol{v},t)$ $\quad f_0 \gg \delta f$
- Keep terms in Vlasov equation up to first order in small quantities

$$\frac{\partial \delta f}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\delta f-\frac{e}{m_e}\delta \boldsymbol{E}\cdot\nabla_{\boldsymbol{v}} f_0=0$$

Linearising the Vlasov equation

- Put this together with Gauss's law

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E_1} = -\frac{e}{\epsilon_0} \int \delta f \, d^3 \boldsymbol{v}$$

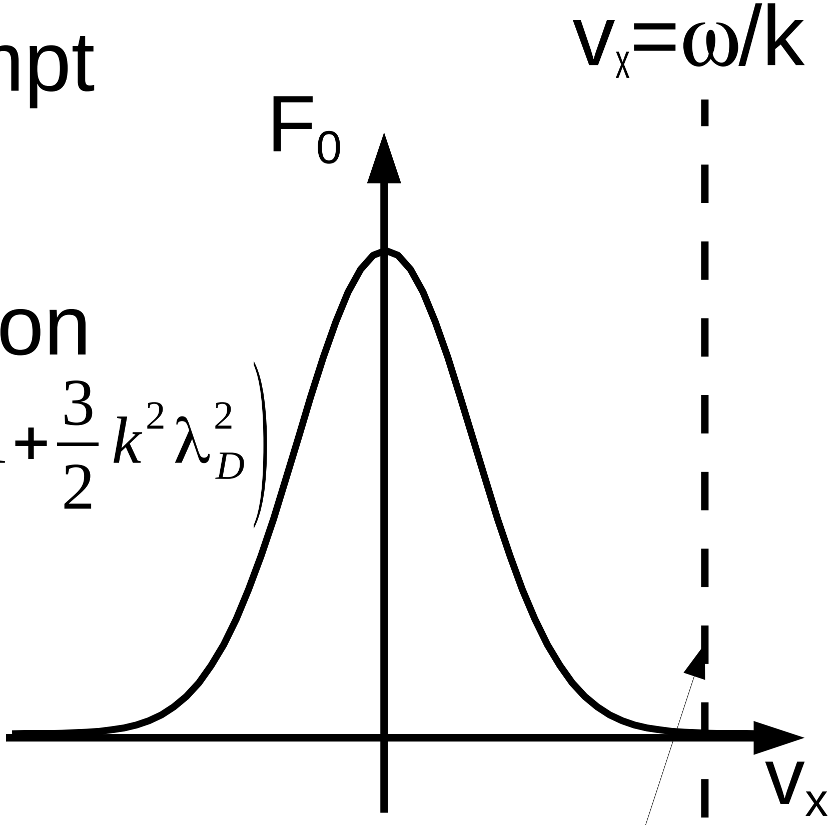
- Arrive at...

$$1 - \frac{\omega_{pe}^2}{n_e k^2} \int \frac{1}{v_x - \omega/k} \frac{dF_0}{dv_x} dv_x = 0 \quad (1) \qquad F_0(v_x) = \int f_0 \, dv_y \, dv_z$$

- Here we have assumed that the k-vector of the electrostatic wave is in the x-direction

Langmuir waves

- Integral has a singularity at $v_x=\omega/k$
- Ignore this for now and attempt the integral in equation (1)
- Recover hot plasma dispersion relation $\omega^2=\omega_{pe}^2+\frac{3}{2}k^2v_T^2 \Rightarrow \omega=\omega_{pe}\left(1+\frac{3}{2}k^2\lambda_D^2\right)$
- Assumed F_0 is Maxwellian, i.e. $F_0(v_x)=\frac{n_{e0}}{\sqrt{\pi}\,v_T}e^{-v_x^2/v_T^2}\quad v_T=\sqrt{\frac{2k_bT_e}{m_e}}$
- Also assumed $\omega/k \gg v$ for for all v's where the distribution is non-negligible, I.e. ω/k is here



Landau damping

- Can't ignore singularity in (1)
- To do the integral properly requires complex analysis.
- I will just quote the required result

Principal part of integral, i.e. the integral without worrying about the pole!

$$\lim_{\alpha\to 0}\left[\int_{-\infty}^{\infty}\frac{f(x)}{x-i\alpha}dx\right]=P\left[\int_{-\infty}^{\infty}\frac{f(x)}{x}dx\right]+i\pi f(0)$$

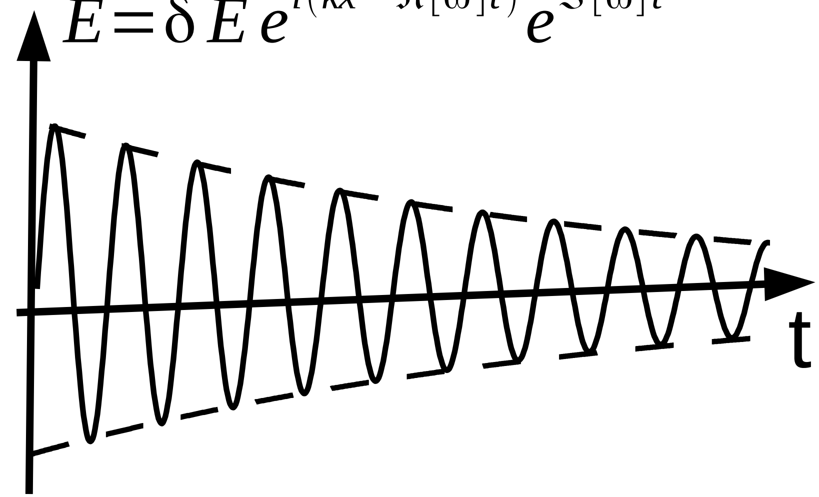
- This gives the following dispersion relation

$$\omega=\omega_{pe}\left(1+\frac{3}{2}k^2\lambda_D^2\right)+i\frac{\pi\omega_{pe}^2}{2\mathrm{k}^2 n_{e0}}\left(\frac{dF_0}{dv_x}\right)_{v_x=\omega/k}$$

Landau damping

- Imaginary part of dispersion relation means that the electrostatic wave is *damped*

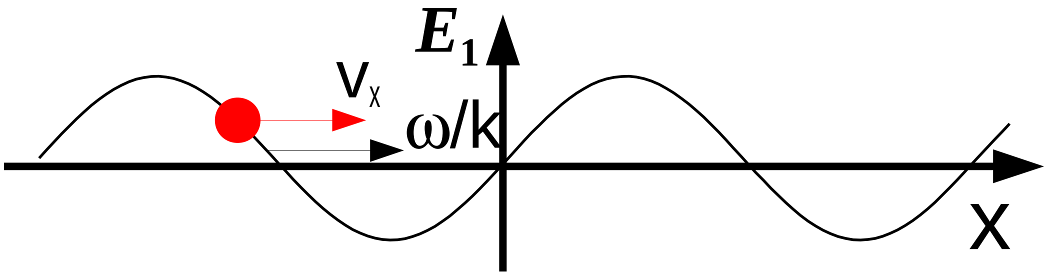
$$E=\delta E\, e^{i(kx-\Re[\omega]t)}\, e^{\Im[\omega]t}$$



- Damping rate is given by $-\Im[\omega]=-\frac{\pi\omega_{pe}^{2}}{2k^{2}n_{e0}}\left(\frac{dF_0}{dv_x}\right)_{v_x=\omega/k}$
- Depends on sign of dF_0/dv_x at $v_x=\omega/k$
- F_0 is Maxwellian → dF_0/dv_x is negative and wave damps

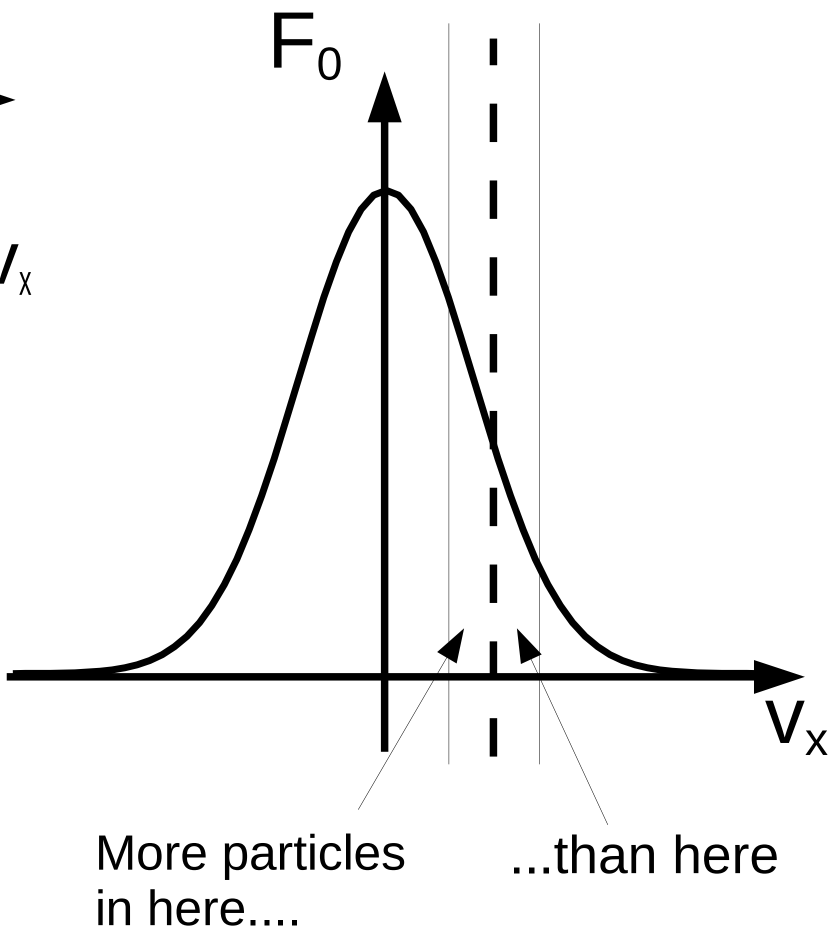
Physical picture

- Particle with speed close to the phase speed of the electrostatic wave can gain energy from or lose energy to the wave.



- Damping rate depends on dF_0/dv_x

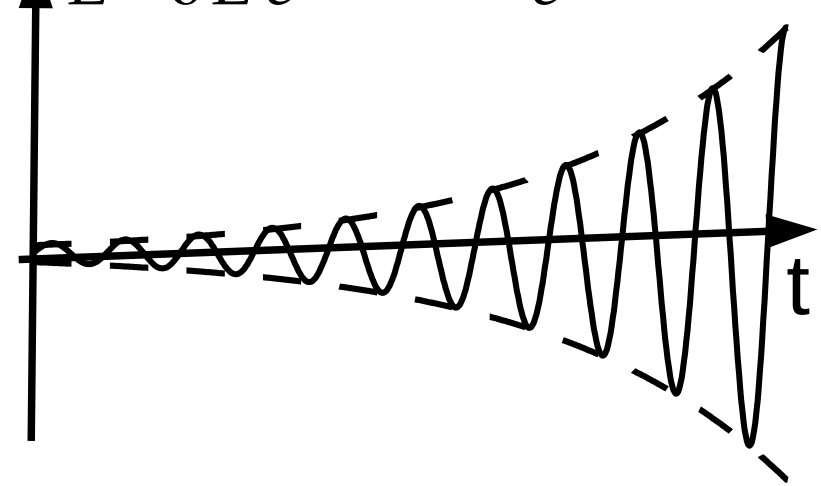
- If $dF_0/dv_x < 0$ the damping rate is negative and the wave is damped. This is because there are more particles which can take energy from the wave than can give energy to the wave

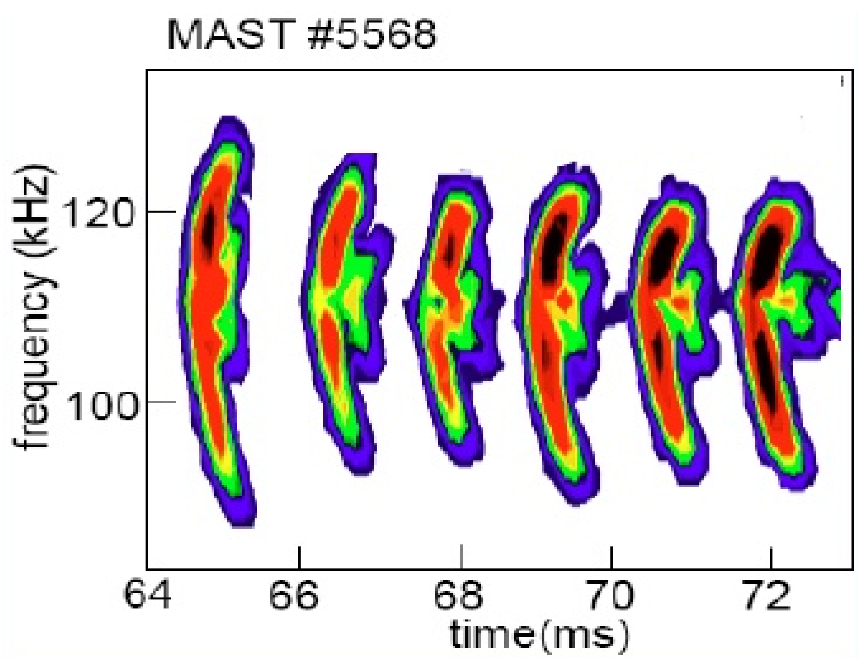


Example: bump on tail instability

- What if $dF_0/dv_x > 0$?
- Wave grows → instability!
- Important in tokamaks

$$E = \delta E\, e^{i(kx - \Re[\omega]t)}\, e^{\Im[\omega]t}$$





- Examples.... Neutral beam heating causes energetic populations of ions, Alpha heated plasmas

Advanced Plasmas

Collisions

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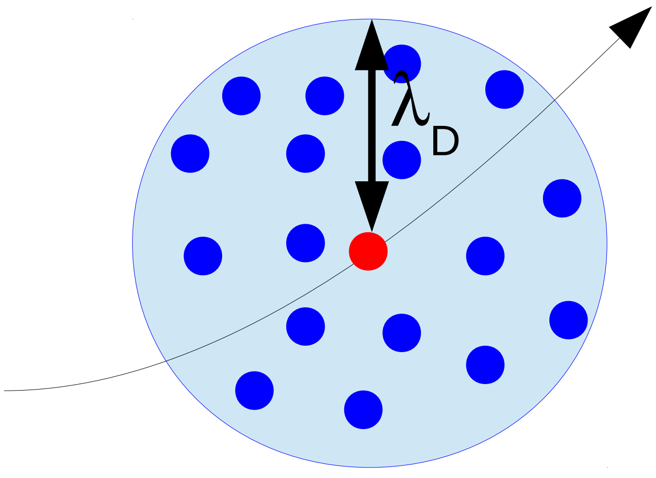
Collisions

- Fluctuations in the particle density and fields swept into a 'collision operator'

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_x f+\frac{q}{m}\left(\boldsymbol{E}_s+\boldsymbol{v}\times\boldsymbol{B}_s\right)\cdot\nabla_v f=\left(\frac{\partial f}{\partial t}\right)_{coll}$$

- Fluctuations important over distances shorter than the Debye length



- Example – put ion (charge Ze) into a plasma – potential drops exponentially outside Debye Sphere

$$\Phi(r)=\frac{Ze}{4\pi\epsilon_0 r}e^{-r/\lambda_D}\qquad \lambda_D=\sqrt{\frac{\epsilon_0 k_B T_e}{n_e e^2}}=\frac{v_T}{2\omega_p}$$

Collisions

- The collision operator relates to fluctuations

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\frac{q}{m}\left\{\left(\delta \boldsymbol{E} + \boldsymbol{v} \times \delta \boldsymbol{B}\right) \cdot \nabla_{\boldsymbol{v}} \delta N\right\}$$

- How do we calculate the fluctuations $\delta \boldsymbol{E}$ etc?
- These fluctuations correspond to corrections to the smoothed particle distribution function *f* due to discreet particle effects
- To calculate these fluctuations exactly we must go back to the N-body equations – impractical!
- Possible to get a tractable solution by noting that if the number of particles in the Debye sphere (discussed later) is large then the fluctuations evolve rapidly through a transient and reach a statistical steady state
- Ignore the transients and assume fluctuations are always in their statistical steady state - Bogoluibov hypothesis

Krook Collision Operator

- Bogoluibov hypothesis breaks time reversibility and makes collision operator entropy producing
- Collisions relax the distribution function to the maximum entropy state - a Maxwellian
- Krook operator is the simplest collision operator that does this

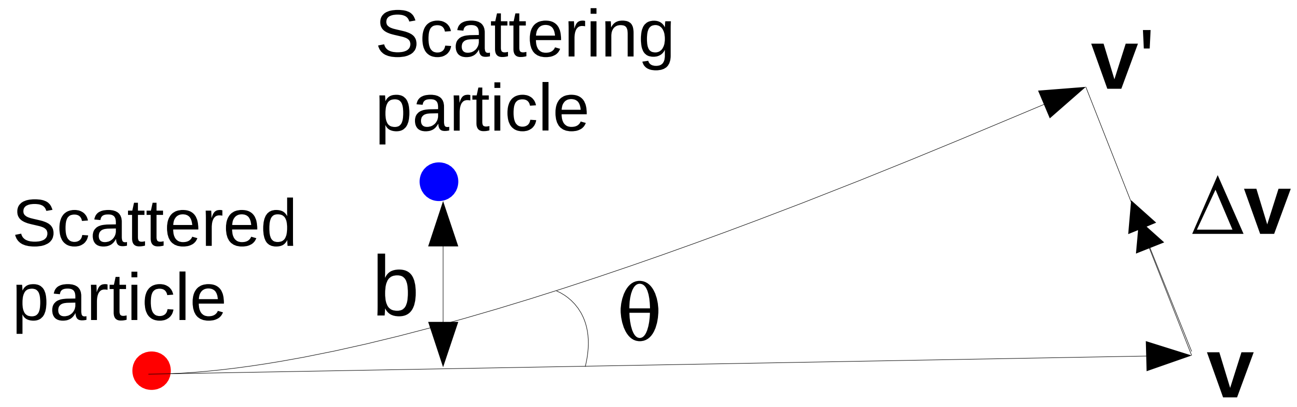
Collision frequency

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\nu_c \left(f - f_M\right) \qquad f_M = \frac{n_e}{\sqrt{\pi}\, v_T^3} e^{-v^2/v_T^2}$$

- Objections – not derived from first principles, doesn't tell us what ν_c actually is!
- First principles collision operator - Fokker-Planck

Doing collisions properly: Coulomb collisions

- First we consider an individual Coulomb collision
- Schematic:



- Impact parameter = b

  Scattering angle = θ

  Electron's initial velocity = **v**

  Final velocity = **v'**

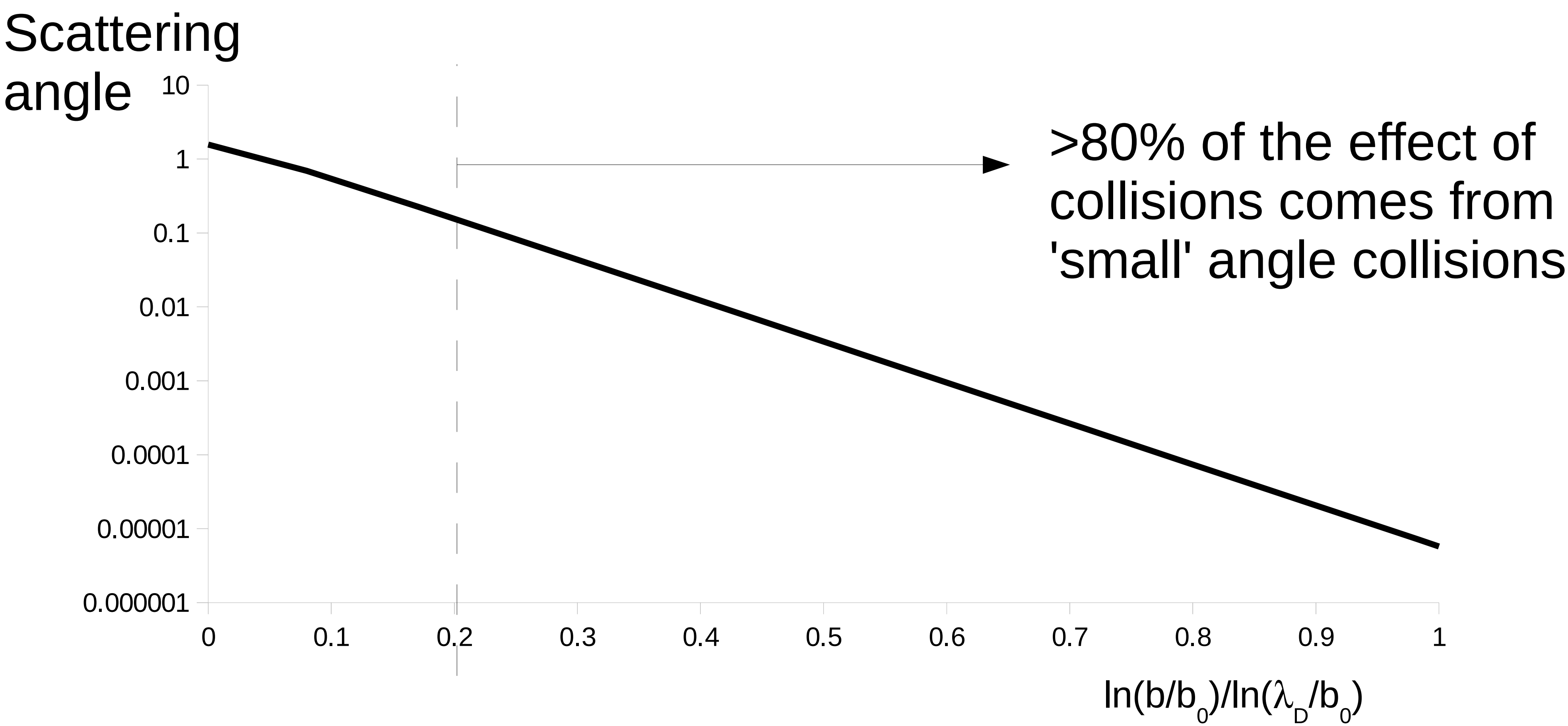
Small vs large angle scattering

- Effect of collisions from impact parameter from b_0 up to b goes as $\ln(b/b_0)$ – b is impact parameter of collision & b_0 is impact parameter for 90^0 scattering
- Maximum impact parameter is the Debye length (limit of range of fluctuations due to collisions)
- We are interested in weakly-coupled plasmas with many particles in the Debye sphere (N)

$N = n_e \lambda_D^3 >> 1$ and $N \propto \lambda_D / b_0 \rightarrow \lambda_D >> b_0$

Small vs large angle collisions

- Plot of relative importance of collisions of a given scattering angle (10keV plasma, electron number density $=10^{20} m^{-3}$, Z=1)



Small vs large angle collisions

- Require $\ln\Lambda >> 1$, where $\Lambda=\lambda_D/b_0$ is the **Coulomb logarithm**, for the scattering angle to drop quickly on the previous plot
- Therefore we require $\ln\Lambda >> 1$ for small angle collisions to dominate.
- This is more stringent than the condition than the condition that we have many particles in the Debye sphere, i.e. $\Lambda >> 1$.

Weakly-coupled plasmas

- Without $\ln\Lambda \gg 1$ for the separation of collisional and collective effects becomes much harder! We are stuck having to solve the N-body problem again!
- Fortunately weakly-coupled plasmas (those with $\Lambda \gg 1$) are ubiquitous

Plasma	n_e/m^{-3}	Te/keV	$n_e\lambda_D{}^3$	$\ln\Lambda$
Solar corona	10^{12}	0.1	10^8	20
Ionosphere	10^{12}	10^{-4}	10^4	10
MCF	10^{20}	10	10^7	15
ICF	10^{28}	10	10^3	8

Bogoliubov's three temporal stages

- Interaction timescale τ_1. In this phase particles propagate without collisional interaction and we need to solve N-body problem to determine motion
- Kinetic timescale τ_2. Longer than interaction timescale. Long enough that the collisional process becomes *Markovian.* i.e. a large number of collisions has taken place and the memory of the initial state is lost.
- Thermodynamic timescale τ_3. The timescale over which the large scale thermodynamic properties of the system vary
- In general $\tau_1 << \tau_2 << \tau_3$

Bogoliubov's three temporal stages

- Now we can understand the importance of Bogoliubov's hypothesis!
- Assume we are not interested in phenomena varying on the interaction timescale but only on the kinetic timescale.
- In this case we can ignore the transient fluctuations occurring on timescale τ_1. Good! To calculate these fluctuations we would need to solve the N-body problem.
- We can treat collisions statistically on the kinetic timescale in the standard way one treats a Markovian statistical process – no solution to the N-body problem is required!

Fokker-Planck collision operator

- Define probability that velocity is changed by Δ**v** in a collision as p(**v**,Δ**v**)
- Distribution function after collisions is given by

$$f(\boldsymbol{r},\boldsymbol{v},t)=\int f(\boldsymbol{r},\boldsymbol{v}-\boldsymbol{\Delta v},t)\,p(\boldsymbol{v}-\boldsymbol{\Delta v},\boldsymbol{\Delta v})\,d^3\Delta\boldsymbol{v}$$

- Small angle collisions so assume Δ**v**<<**v** get to Fokker-Planck collision operator

$$\left(\frac{\delta f}{\delta t}\right)_{coll}=-\sum_i\frac{\partial}{\partial v_i}\left(f\frac{\langle\Delta v_i\rangle}{\Delta t}\right)+\frac{1}{2}\sum_i\sum_j\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\left(f\frac{\langle\Delta v_i\Delta v_j\rangle}{\Delta t}\right)$$

Here $\langle\Delta v_i\rangle=\int d^3(\Delta\boldsymbol{v})\,P\,\Delta v_i \qquad \langle\Delta v_i\Delta v_j\rangle=\int d^3(\Delta\boldsymbol{v})\,P\,\Delta v_i\Delta v_j$

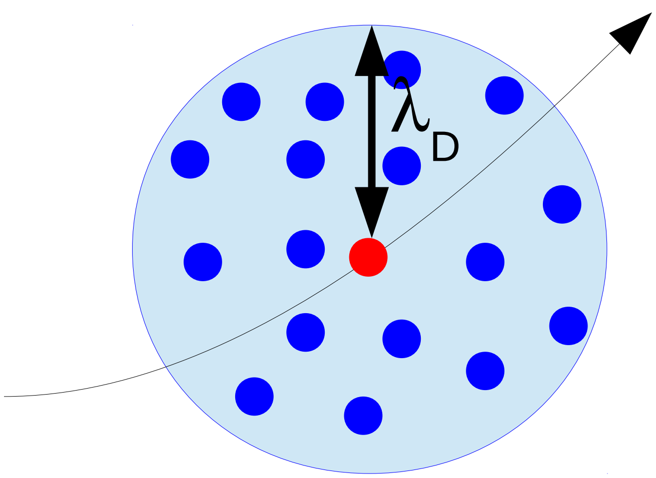
Coulomb collisions

1. Collisions occur due to fluctuating Coulomb interactions within the Debye sphere and there are many particles within this sphere
2. Coulomb forces long ranged
3. 1 & 2 imply that collisions are necessarily many body
4. BUT small angle collisions dominate and we can assume:

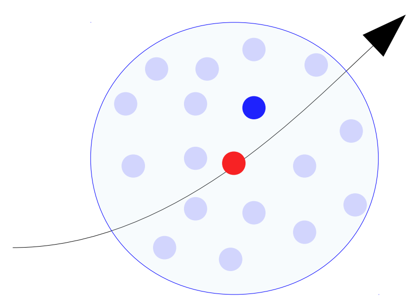
MANY-BODY INTERACTIONS ≡ SUM OF BINARY INTERACTIONS

Coulomb collisions

MANY-BODY INTERACTIONS ≡ SUM OF BINARY INTERACTIONS



Interaction with this particle



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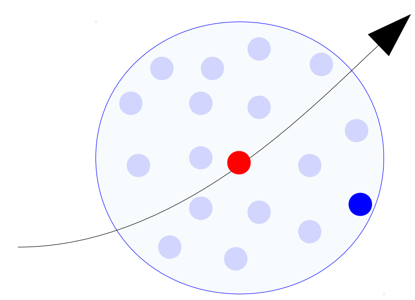
Interaction with this particle



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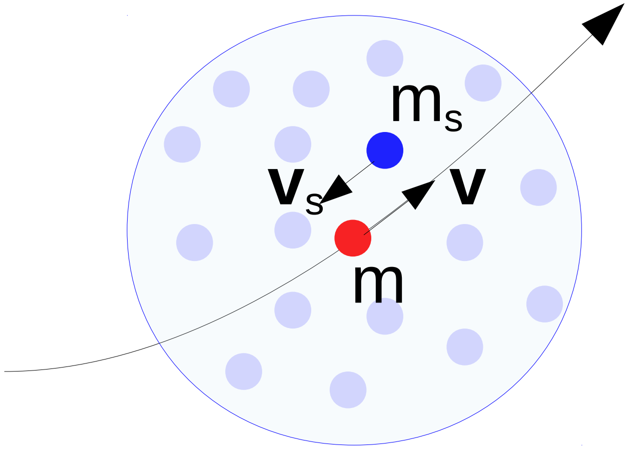
Coulomb collisions

- Relative velocity $\mathbf{g}=\mathbf{v}-\mathbf{v}_s$
- Centre of mass velocity

  $\mathbf{V}=(m\mathbf{v}+m_s\mathbf{v}_s)/(m+m_s)$



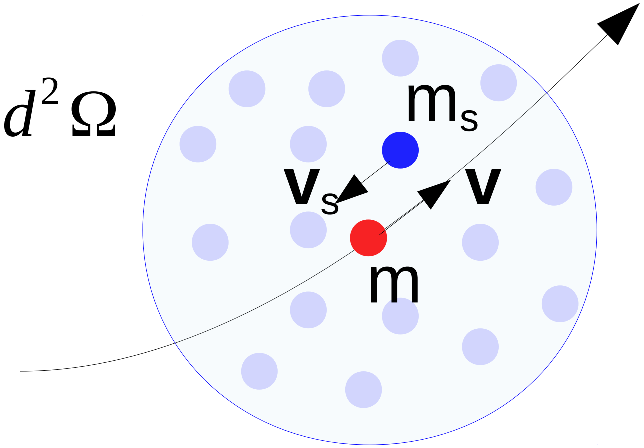
- Probability particle velocity changes by $\Delta\mathbf{v}$ in summed over collisions with all particles in Debye sphere

$$p(\boldsymbol{v},\Delta\boldsymbol{v})=\int f_s(\boldsymbol{v_s})d^3\boldsymbol{v_s}\int|\boldsymbol{g}|\Delta t\,\sigma(|\boldsymbol{g}|,\theta)d^2\Omega$$

Coulomb collisions

- How do we get

$$p(\boldsymbol{v}, \Delta \boldsymbol{v}) = \int f_s(\boldsymbol{v}_s) d^3 \boldsymbol{v}_s \int |\boldsymbol{g}| \Delta t \, \sigma(|\boldsymbol{g}|, \theta) d^2 \Omega$$



- Centre of mass frame - scattering is purely angular, i.e. $|\mathbf{g}|$ doesn't change
- σ is cross-section for scattering by angle θ
- Rutherford cross-section $\sigma(|\boldsymbol{g}|, \theta) = \dfrac{Z Z_s e^2}{8 \pi \epsilon_0 m |\boldsymbol{g}|^2 \sin^2(\theta/2)}$

- Integrate over all possible scattering angles angles
- Integrate over all possible scatterer velocities

Collision frequency & Coulomb logarithm

- Electron ion collision frequency can be derived by substitution of p(**v**,Δ**v**) and Δ**v** (derived on supplementary handout).

$$\nu_{ei}(v)=4\pi\left(\frac{Ze^2}{4\pi\epsilon_0 m_e}\right)^2\frac{n_i\ln\Lambda}{v^3}$$

- Ratio of frequencies

$$\nu_{ei}:\nu_{ee}:\nu_{ii}:\nu_{ie} \qquad 1:\frac{1}{Z}:Z^2\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}:Z\left(\frac{m_e}{m_i}\right)\left(\frac{T_e}{T_i}\right)$$

- Depends on Coulomb logarithm lnΛ. Where $\Lambda=\lambda_D/b_0$ (or λ_D/λ_{db} if the De-Broglie wavelength λ_{db} is larger than b_0)

Not same thing as the 'energy exchange time'!

- Collision time defined as time taken to deflect particle by ~90^0
- Energy exchange time defined as time taken for energy to change by amount comparable to original energy
- Ratio of energy exchange times

$$\nu_{ee}^{E} : \nu_{ii}^{E} : \nu_{ei}^{E} \sim \nu_{ie}^{E} \qquad 1 : \left(\frac{m_e}{m_i}\right)^{1/2} : \left(\frac{m_e}{m_i}\right)$$

- Energy exchange between like species in the plasma is much faster than unlike species due to the large difference in mass of electrons and ions

Example: non-local transport

- Fact that $\nu_{ei} \propto 1/v^3$ very important for energy transport
- Heat flow carried primarily by electrons moving at 3-4 times the thermal speed. Mean free path of these electrons is

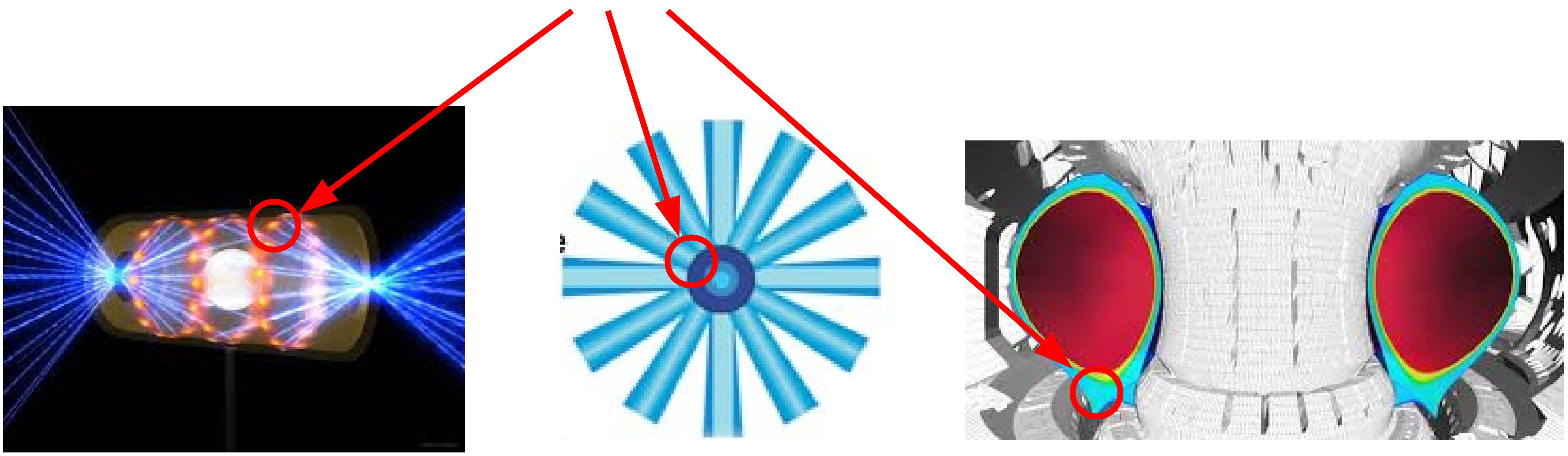
$$\lambda_{ei}(v=3-4\mathrm{v}_T)=\frac{(3-4)^4 v_T^4}{Y\,n_i \ln\Lambda}=(81-256)\,\lambda_{ei}(v=v_T)$$

- Classical transport theory (discussed later) only works if mean free path OF IMPORTANT SPECIES is small (compared to scale lengths).
- For heat transport this can be O(100) times mean free path of thermal particles.

Example: non-local transport

- 'Non-local' thermal transport is one of the major unknowns in modelling of experiments on the NIF and is important in the Scrape-Off Layer in tokamaks.

$$L_T/\lambda_{ei/ee}<100$$



Advanced Plasmas

Transport Theory

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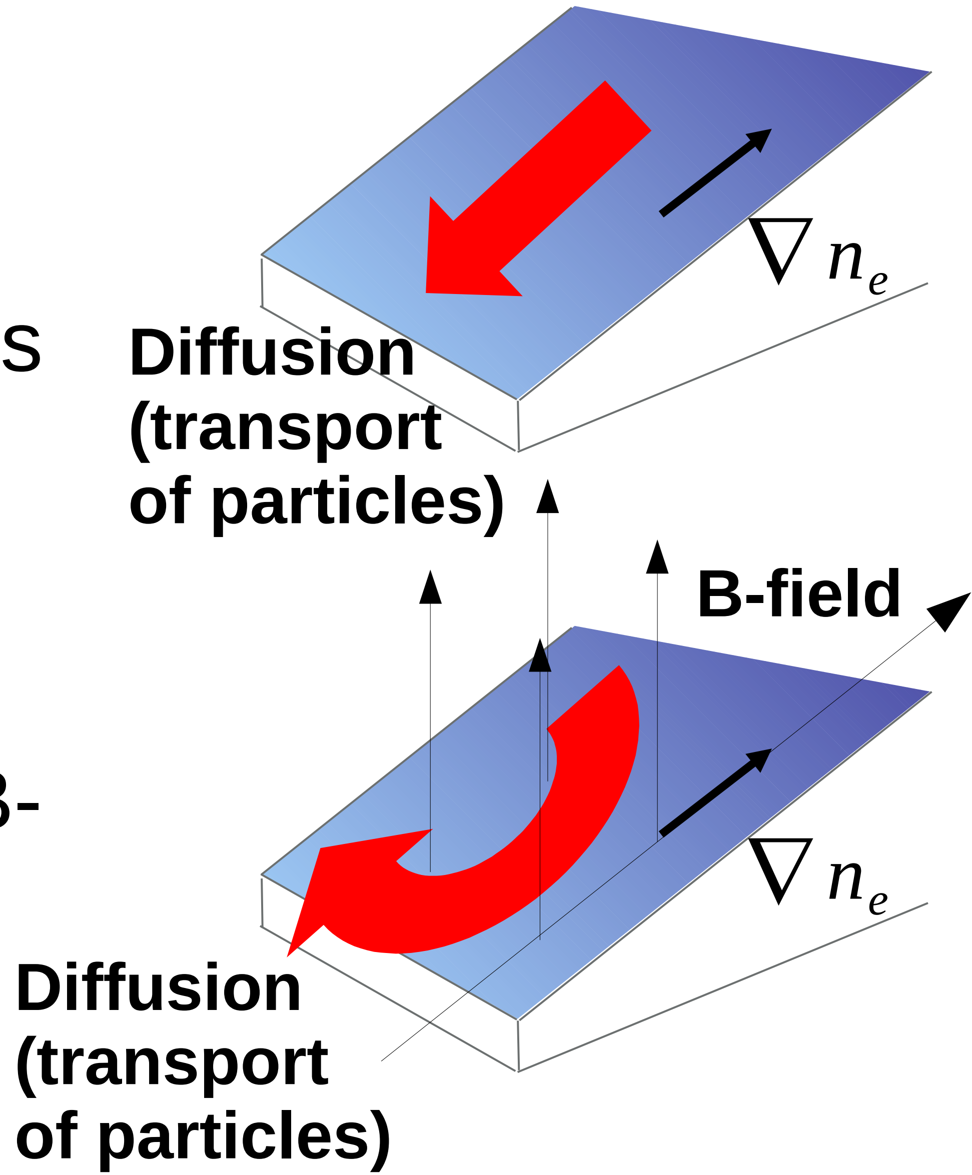
What is transport

- Transport of energy, momentum, particles driven by spatial gradients

- Particles causing the transport are charged → transport is modified by B-fields



Transport equations

- Transport equations derived by taking 'velocity moments' of distribution function.
- Moments relate to fluid quantities

$$n(\boldsymbol{x},t)=\int f(\boldsymbol{x},\boldsymbol{v},t)\,d^3\boldsymbol{v} \qquad T(\boldsymbol{x},t)=\frac{1}{n}\int \frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2 f(\boldsymbol{x},\boldsymbol{v},t)\,d^3\boldsymbol{v}$$

- Transport quantities also related to moments, for example momentum &heat flow

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{n}\int \boldsymbol{v} f(\boldsymbol{x},\boldsymbol{v},t)\,d^3\boldsymbol{v} \quad \boldsymbol{q}(\boldsymbol{x},t)=\frac{1}{n}\int \frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2\boldsymbol{v} f(\boldsymbol{x},\boldsymbol{v},t)\,d^3\boldsymbol{v}$$

Diffusion in a constant B-field

- Start from Vlasov equation + collisions

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}} f=\left(\frac{\partial f}{\partial t}\right)_{coll}$$

- For simplicity will use Krook collision operator. Let $f=f_M+\delta f$ but this time have spatial gradients in density, i.e. $f_M(x,z,v,t)=\frac{n(x,z,t)}{\pi^{3/2}v_T^3}\exp\left(\frac{-v^2}{v_T^2}\right)$
- Where $\nabla_{\boldsymbol{x}} n=\left(\frac{\partial n}{\partial x},0,\frac{\partial n}{\partial z}\right)$
- B-fields in z-direction $\boldsymbol{B}=(0,0,B)$

Diffusion in a constant B-field

- Linearised equation is

$$\frac{f_M}{n}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} n + \frac{Ze}{m}(\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}\delta f = -\nu_c\delta f$$

- Diffusion is transport of particles. Particle flux is $\boldsymbol{\Gamma}=n\boldsymbol{u}=\int \boldsymbol{v} f\, d^3 v$
- Multiplying by **v** and integrating over velocity space yields the following equations for the components of the particle flux

$$\Gamma_x = -\frac{\tau_c T}{m\left[1+(\omega\tau)^2\right]}\frac{\partial n}{\partial x} \qquad \Gamma_z = -\frac{\tau_c T}{m}\frac{\partial n}{\partial z}$$

$$\Gamma_y = \omega\tau\frac{\tau_c T}{m\left[1+(\omega\tau)^2\right]}\frac{\partial n}{\partial x}$$

ω is the gyro-frequency and $\tau_c=1/\nu_c$ is the collision time

Transport coefficients

- Particle flux related to density gradient by diffusion coefficient $\mathbf{\Gamma} = -\underline{\underline{D}} \cdot \nabla_{\boldsymbol{x}} n$
- Diffusion coefficient is an example of a *transport coefficient*
- Diffusion is different parallel and perpendicular to B-field → diffusion coefficient is a tensor

$$\underline{\underline{D}} = \begin{pmatrix} D_{\perp} & -D_{\wedge} & 0 \\ -D_{\wedge} & D_{\perp} & 0 \\ 0 & 0 & D_{\parallel} \end{pmatrix}$$

Magnetisation

- In example considered (where $\partial n / \partial y = 0$)

$$D_\perp = \frac{\tau_c T}{m\left[1 + (\omega \tau_c)^2\right]} \qquad D_\| = \frac{\tau_c T}{m}$$

Note $D_\perp = D_\|$ when **B**=0 (as it should!)

- Consider following limits

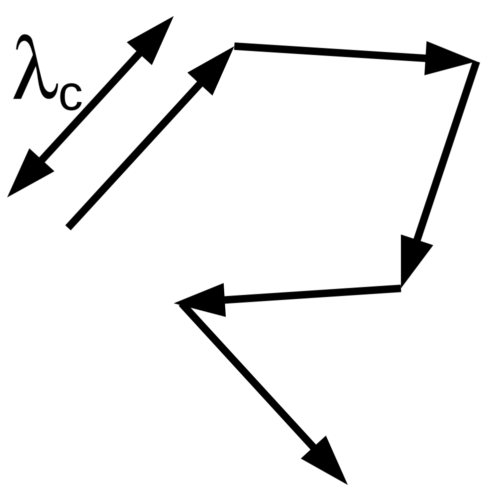
$$\omega \tau_c \ll 1 \quad D_\perp \approx \frac{(\lambda_c / \sqrt{2})^2}{\tau_c} \qquad \omega \tau_c \gg 1 \quad D_\perp \approx \frac{(r_g / \sqrt{2})^2}{\tau_c}$$

λ_c is the collisional mean-free-path and r_g is the gyro-radius

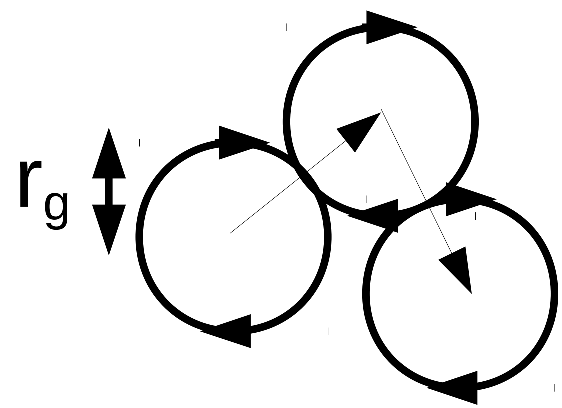
Magnetisation

- Diffusion distance goes as L^2/t where L is the characteristic diffusion length anf t the diffusion time
- We have the following two cases

$\omega\tau_c \ll 1$

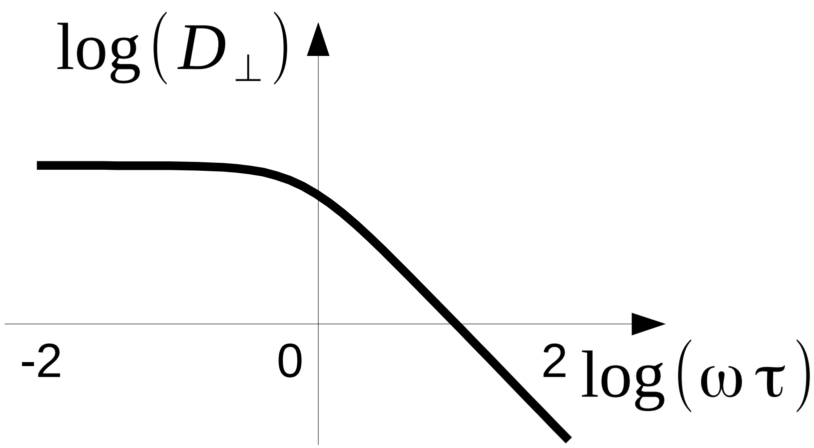


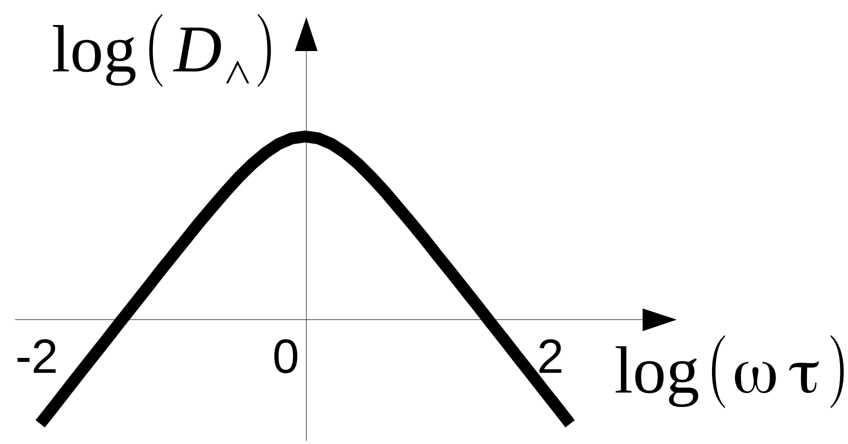
$\omega\tau_c \gg 1$



Magnetisation

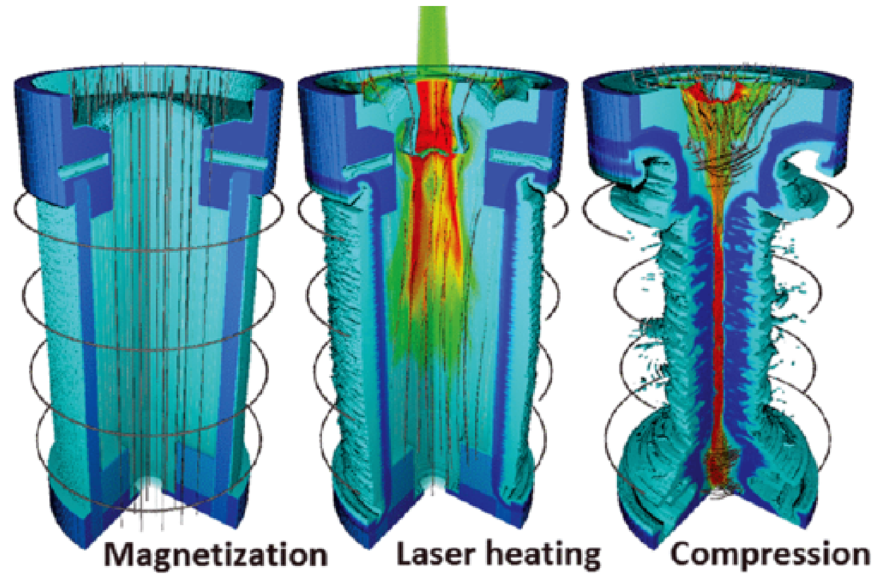
- Which of these regimes we are in depends on the *magnetisation* of the plasma $\omega\tau$
- Also get diffusion perpendicular to the density gradient!
- Controlled by $D_{\wedge} = \omega\tau \dfrac{\tau_c T}{m\left[1+(\omega\tau)^2\right]}$
- Illustrate generic way transport coefficients depend on the magnetisation of the plasma

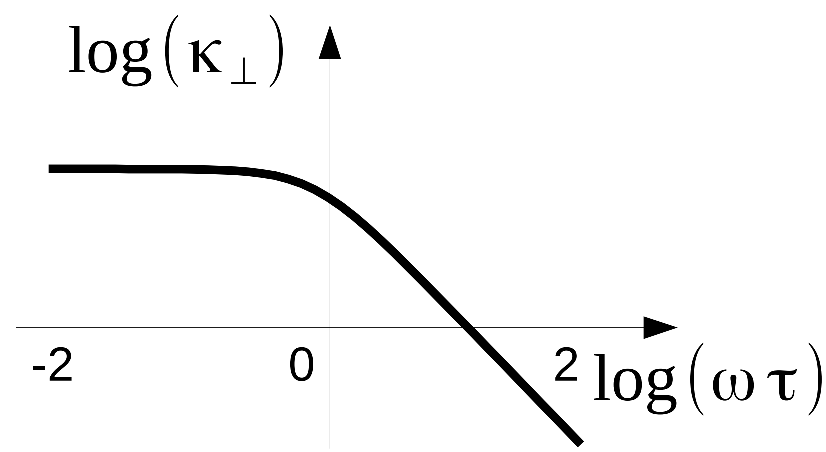




Example: MagLIF

- Magnetised liner inertial fusion → magnetise fuel to inhibit thermal conduction
- How big does the B-field need to be to do this?
- Later we will see that thermal conductivity scales similarly to diffusion coefficient
- Therefore need $\omega\tau > 1$





Braginskii 'classical' transport theory

- Fluid equations are derived by taking moments of the kinetic equation
- But on doing this we don't get a closed system of equations
- For example equation for rate of change of temperature (2nd moment)...

$$T(\boldsymbol{x},t)=\frac{1}{n}\int\frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2 f(\boldsymbol{x},\boldsymbol{v},t)\,d^3\boldsymbol{v}$$

- ...depends on the divergence of the heat flow (3rd moment)

$$\boldsymbol{q}(\boldsymbol{x},t)=\frac{1}{n}\int\frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2\boldsymbol{v}\, f(\boldsymbol{x},\boldsymbol{v},t)\,d^3\boldsymbol{v}$$

Braginskii 'classical' transport theory

- Similarly the equation for the third moment (heat flow) will depend on the fourth and so on ad infinitum
- Need to close the system of equations and this is usually done by deriving an equation for the heat flow in terms of lower moments (temperature, current).
- Also need an equation for the electric field → an Ohm's law
- This is done in a similar way to the diffusion coefficient previously derived.

Braginskii 'classical' transport theory

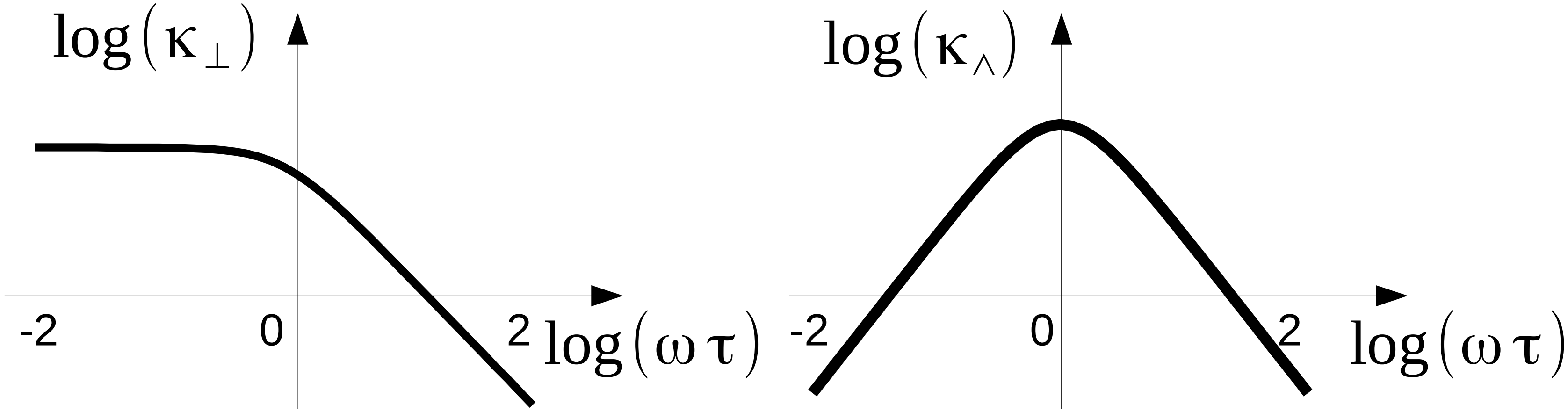
- Braginskii's classical heat flow equation and Ohm's law are:

$$\boldsymbol{q} = -\underline{\underline{\kappa}} \cdot \nabla_{\boldsymbol{x}} T_e - \underline{\underline{\beta}} \cdot \boldsymbol{j} \frac{T_e}{e} \qquad e n_e \boldsymbol{E} = -\nabla_{\boldsymbol{x}} P_e + \frac{\underline{\underline{\alpha}} \cdot \boldsymbol{j}}{n_e e} - n_e \underline{\underline{\beta}} \cdot \nabla_{\boldsymbol{x}} T_e$$

- Transport coefficients:
  - $\underline{\underline{\alpha}}$ → resistivity
  - $\underline{\underline{\beta}}$ → thermoelectric coefficient
  - $\underline{\underline{\kappa}}$ → thermal conductivity

Braginskii 'classical' transport theory

- Components of thermal conductivity scale similarly to diffusion coefficient previously derived



- Resistivity and thermoelectric coefficient scale similarly

Example: B-field generation in laser-plasma interactions

- Rate of change of magnetic field from Faraday's law and Ohm's law.
- Consider pressure gradient term in Braginskii's Ohm's law

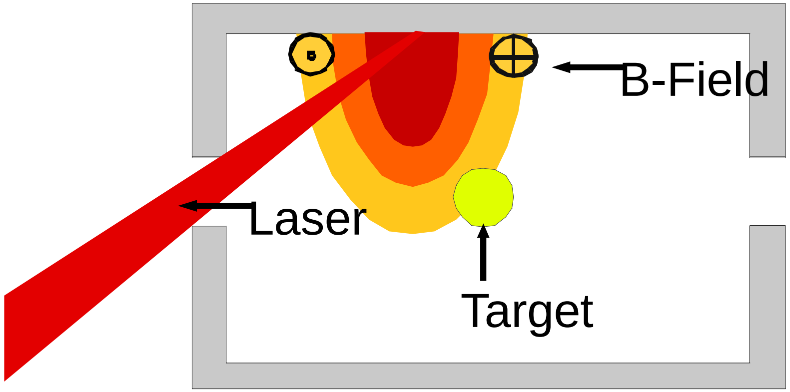
$$\left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_{gen} = -\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_{\boldsymbol{grad\,P}} \qquad \boldsymbol{E}_{\boldsymbol{grad\,P}} = -\frac{\nabla_{\boldsymbol{x}} P_e}{e n_e}$$

- Results in 'Biermann battery' term

$$\left(\frac{\partial \boldsymbol{B}}{\partial t}\right)_{gen} = -\frac{1}{e\,n_e} \nabla_{\boldsymbol{x}} n_e \times \nabla_{\boldsymbol{x}} T_e$$

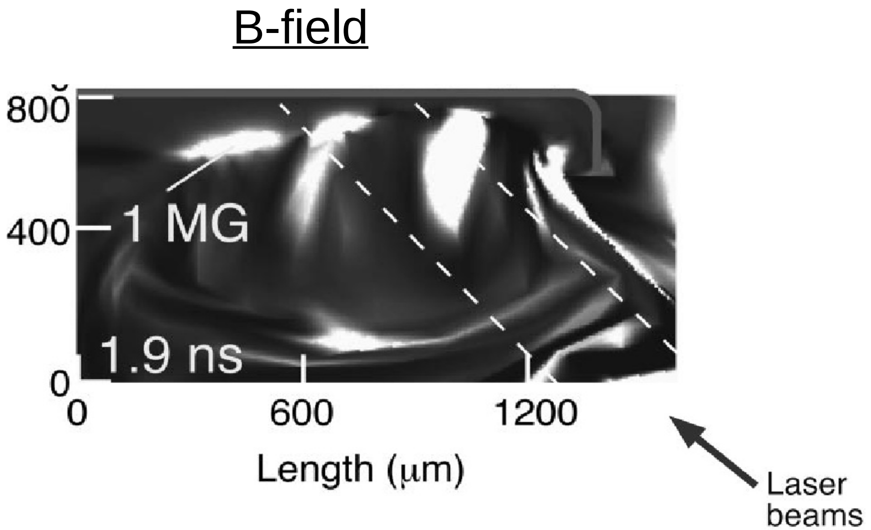
Example: B-field generation in laser-plasma interactions

- B-fields generated by the Biermann battery may effect hohlraum conditions



- Density and temperature gradients not aligned

- Simulations suggest MG fields may be generated at hohlraum wall



Advanced Plasmas

Summary

Dr C.P. Ridgers

Learning outcomes

1. Derive the Vlasov equation and understand the need for a collision operator in the context of Debye shielding.

What you need to know....

(i) Phase-space and the N-particle distribution. Derivation of Klimontovich equation

$$\frac{\partial N}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} N+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}} N=0$$

(ii) Derivation of Vlasov (+ collisions) equation from Klimontovich equation

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f+\frac{q}{m}(\boldsymbol{E}_{\boldsymbol{s}}+\boldsymbol{v}\times\boldsymbol{B}_{\boldsymbol{s}})\cdot\nabla_{\boldsymbol{v}} f=-\frac{q}{m}\{(\delta\boldsymbol{E}+\boldsymbol{v}\times\delta\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}\delta N\}$$

(iii) Identification of collisions with fluctuating part

Learning outcomes

2. Linearise the Vlasov equation to obtain the plasma dielectric function and understand how the form of the dielectric function gives rise to Landau damping.

What you need to know....

(i) The derivation of Landau damping up to...

$$1+\frac{\omega_{pe}^2}{n_e k^2}\int \frac{1}{\omega/k-v_x}\frac{dF_0}{dv_x}dv_x=0$$

(ii) That this integral has an imaginary part and understand that the imaginary part gives the damping rate of the wave

(iii) That whether the wave damps or grows depends on whether $(dF_0/dv_x)_{\omega/k}$ is positive or negative

Learning outcomes

3. Write down the form of the Krook and Fokker-Planck collision operators.

What you need to know...

(i) That collisions relax the distribution to the maximum entropy state and that the Maxwellian is this state. The Krook operator relaxes to Maxwellian by design and is

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\nu_c \left(f - f_M\right)$$

(ii) Small angle scattering dominates over large angle scattering and know that in this case the collision operator is the Fokker-Planck operator

$$\left(\frac{\delta f}{\delta t}\right)_{coll} = -\sum_i \frac{\partial}{\partial v_i}\left(f \frac{\langle \Delta v_i \rangle}{\Delta t}\right) + \frac{1}{2}\sum_i \sum_j \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\left(f \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t}\right)$$

Learning outcomes

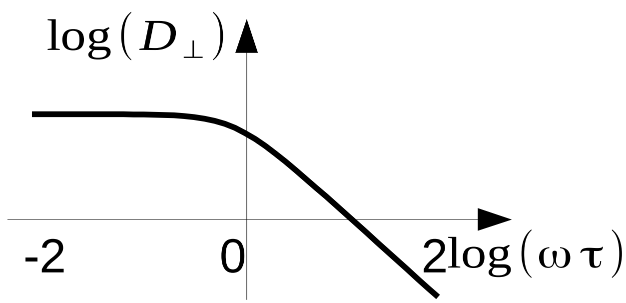
4. Derive the diffusion coefficients for a magnetised plasma and use this derivation to illustrate the need to close the fluid equations.

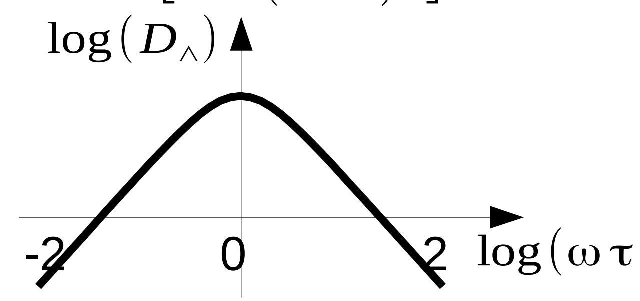
What you need to know...

(i) The derivation the diffusion coefficients in a constant magnetic field, i.e.

$$D_{\perp}=\frac{\tau_c T}{m\left[1+\left(\omega\tau_c\right)^2\right]} \qquad D_{\parallel}=\frac{\tau_c T}{m} \qquad D_{\wedge}=\omega\tau\frac{\tau_c T}{m\left[1+\left(\omega\tau\right)^2\right]}$$

(ii) How these coefficients scale with ωτ





(iii) Taking moments of the Vlasov equation gives the fluid equations, but the equation for the n^th^ moment depends on the (n+1)^th^. Therefore transport equations are required to close the system

Learning outcomes

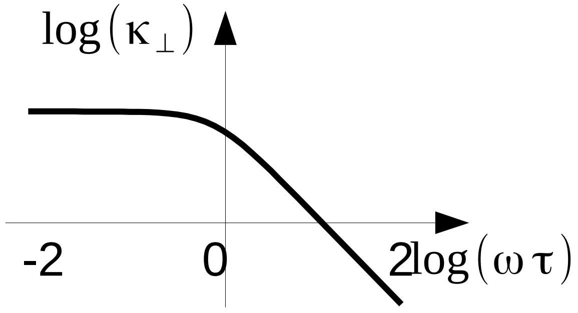
5. Explain the origin of the Braginskii transport relations

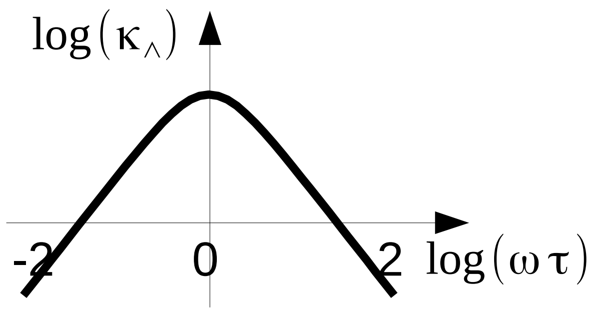
What you need to know...

(i) The Braginskii heat flow and Ohm's law

$$\boldsymbol{q} = -\underline{\underline{\kappa}} \cdot \nabla_{\boldsymbol{x}} T_e - \underline{\underline{\beta}} \cdot \boldsymbol{j} \frac{T_e}{e} \qquad e n_e \boldsymbol{E} = -\nabla_{\boldsymbol{x}} P_e + \frac{\underline{\underline{\alpha}} \cdot \boldsymbol{j}}{n_e e} - n_e \underline{\underline{\beta}} \cdot \nabla_{\boldsymbol{x}} T_e$$

(ii) How $\kappa_\perp$, $\kappa_\wedge$ coefficients scale with $\omega\tau$





Example: neo-classical transport

- B-field in a tokamak is curved and inhomogeneous
- Plasma is highly magnetised BUT drifts and banana orbits control diffusion step length
- Diffusion coefficient given by

$$D_{\perp} \approx \frac{?}{\tau_c}$$

$$D_{\perp} \approx \frac{\left(q r_g / \sqrt{2}\right)^2}{\tau_c}$$