Fusion: MCF - Homework #2

1 The point of this problem is to examine the motion of charged particles in the geomagnetic field to the analogy of banana orbits in neoclassical theory. For sufficiently small distances away from the Earth, its magnetic field is well approximated as a dipole, given by

$$\mathbf{B} = \frac{\mu_0 M_{\mathrm{E}}}{4\pi r^3}\left(\hat{\mathbf{r}}\, 2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right)$$

where $M_{\mathrm{E}} = 8\times 10^{22}\mathrm{Am}^2$ is the magnetic dipole moment of the Earth, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors of the radius and polar angle, respectively, in the standard right-handed spherical polar coordinate system (r, θ, ϕ), and r is measured from the center of the Earth. For simplicity, you may assume that the axis of the magnetic dipole is aligned with that of Earth's rotation. Our sample particle is a $\mathcal{E} = 1\mathrm{MeV}$ proton that is injected into the magnetic field at a distance $r_0 = 3R_{\mathrm{E}}$ from the center of the Earth, highly magnetized, with $v_\perp^{\mathrm{eq}}/v_{||}^{\mathrm{eq}} = 3$, where $v_\perp^{\mathrm{eq}}$ and $v_{||}^{\mathrm{eq}}$ are the proton's velocities at the equator perpendicular and parallel to the magnetic field, respectively. The radius of Earth is $R_{\mathrm{E}} = 6.37\times 10^6\mathrm{m}$.

(a) Show that the proton is trapped. [2]

(b) Derive an equation for the *latitudes* at which it is reflected, including only known quantities. Do **not** try to solve the equation.

[Hint: This will require you to calculate the equation of a field line; the differential displacement in spherical coordinates is

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\phi\hat{\boldsymbol{\phi}}.]$$

[5]

(c) (i) Calculate the drifts arising from the inhomogeneity of the magnetic field at the injection point of the proton. Make sure to clearly indicate the direction. How long would it take to travel around the equator at the particle's altitude at this speed? [3]

(ii) Does this drift lead to a current? Why/Why not? [2]

(iii) Is there an analogous current in a tokamak? What if the density and temperature profiles are totally uniform? [2]

(d) (i) The period of the bounce motion τ_{b} cannot be integrated analytically in general. Strong magnetization, however, causes particles to stay close to the equator. Estimate τ_{b}, using a local parabolic approximation of the magnetic field near this region. [4]

continued

(e) (i) How does τ_b compare to the time it takes to go around the Earth at the drift speed? Judging only by this, do you think

$$J = \oint p_{||} \mathrm{d}s$$

is a good adiabatic invariant? Here, $p_{||}$ is the particle's canonical momentum, analogous to the toroidal canonical momentum in a tokamak. [2]