UNIVERSITY OF YORK

ADVANCED PLASMA PHYSICS

NOTES

High Energy Density Plasmas

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Contents

1 Introduction

The physics of plasmas at high energy density requires an understanding of equilibrium relationships and some of the concepts used in solid state physics. For example, equilibrium ionisation populations and equilbrium radiation distributions are often present. The Fermi energy associated with the near-full occupancy of electron quantum states can be important. Radiation transport often needs to be evaluated as the plasmas can be highly opaque.

High energy density is sometimes defined to refer to energy densities exceeding 10^{11} J m^{-3}, which for a plasma is approximately equivalent to pressures exceeding 1 megabar (Mbar) $= 10^{11}$ Nm^{-2}. Pressure has the same dimensions as energy density per unit volume and even at high energy density, a plasma is often close to being a perfect gas where numerically the gas pressure is equal to 2/3 of the energy density per volume.

1.1 The ion-ion plasma coupling constant

The ion-ion plasma coupling constant reflects the ratio of the potential energy between adjacent ions and the average kinetic energy. It gives a good guide if high energy density plasma conditions are important. We can write for the ion-ion coupling constant

$$\Gamma_{ii} = \frac{Z_{av}^2 e^2}{4\pi\epsilon_0 r_i k T_i}$$

where Z_{av} is the average degree of ionisation, r_i is the average spacing between ions and T_i is the ion temperature. We are using SI units so that the parameters e, ϵ_0 and k are the electron charge, electric dielectric constant and Boltzmann constant respectively. The volume occupied by an ion is on average $\frac{4}{3}\pi r_i^3 = 1/n_i$ where n_i is the ion number density, so we can write that

$$r_i = (\frac{4\pi}{3}n_i)^{-1/3}.$$

We then have

$$\Gamma_{ii} = \frac{1}{4\pi\epsilon_0}(\frac{4\pi n_i}{3})^{1/3}\frac{Z_{av}^2 e^2}{k T_i}.$$

The average degree of ionisation Z_{av} is calculated from the ratio of the number density of electrons n_e to ions (i.e. $Z_{av} = n_e/n_i$). If $\Gamma_{ii} \geq 10$, plasmas are referred to as strongly coupled and if $\Gamma_{ii} \leq 10$, they are said to be weakly coupled.

1.2　Equation of state for a plasma

In a plasma, pressure P is related to the mass density ρ and temperature T by adding up the electron and ion pressure given by Boyles law. We can write

$$P = \frac{\rho(1 + Z_{av})kT}{Am_p} \tag{1}$$

where A is the average atomic mass and m_p is the mass of the proton.

　　Changes of mass density ρ and the velocity $\mathbf{u}$ of a plasma fluid are related by the continuity equation (representing conservation of mass) and the equation of motion (representing a fluid version of Newton's law that force is equal to mass times acceleration). We can write for these two equations respectively

$$\frac{\partial \rho}{\partial t} + \nabla.\rho\mathbf{u} = 0,$$

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u}) = -\nabla P.$$

For variations in mass density, velocity and pressure in one dimension x, the continuity equation and equation of motion can be written such that

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0,$$

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}) = -\frac{\partial P}{\partial x}.$$

Assuming a small time varying deviation of density $\rho = \rho_0 + \rho_1$, velocity $u = u_0 + u_1$ and pressure $P = P_0 + P_1$ from steady state values ρ_0, u_0 and P_0, it is possible to show that to a good approximation the one-dimensional continuity equations combine to give an equation for the propagation of the deviation of density such that

$$\frac{\partial^2 \rho_1}{\partial x^2} + \frac{1}{(\frac{\partial P}{\partial \rho})}\frac{\partial^2 \rho_1}{\partial t^2} = 0.$$

This equation has the form of a wave equation where the speed c_s of the wave is given by

$$c_s^2 = (\frac{\partial P}{\partial \rho}). \tag{2}$$

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The propagation of a disturbance in density is usually known as a sound wave, so this equation shows that the sound speed is given by the square root of the rate of change of pressure with density changes. (The partial derivative here means that other parameters such as entropy and energy density are held constant).

Statistical mechanics tells us that for a gas (or other system) characterised by a temperature T, the average energy per degree of freedom per particle is equal to $(1/2)kT$. A degree of freedom can be represented by translational motion in one direction (giving three degrees of freedom for a monatomic gas or a plasma species such as the electrons), but can also include, for example, vibrational degrees of freedom for polyatomic gases.

Rather than consider degrees of freedom in a plasma, it is often more convenient to define a parameter γ using the relationship between pressure and energy density. We introduce the energy density per unit mass (ϵ) and write for the energy density per unit volume $\rho\epsilon$ that

$$\rho\epsilon = \frac{P}{\gamma - 1} \tag{3}$$

which then defines γ. For an ideal gas with n degrees of freedom, if we write

$$\gamma = 1 + \frac{2}{n}$$

then

$$\epsilon = \frac{3}{2}\frac{(1 + Z_{av})kT}{Am_p}$$

upon substituting into equation (3), using our Boyles law expression for the pressure (equation (1)) and setting $n = 3$. This energy per unit mass expression is equivalent to the one that can be obtained by counting $(1/2)kT$ energy per degree of freedom assuming that there are three degrees of freedom for both the electrons and ions.

Differentiating the equation that defines γ (equation (3)), we get

$$\frac{\partial P}{\partial \rho} = (\gamma - 1)(\epsilon + \rho\frac{\partial \epsilon}{\partial \rho}). \tag{4}$$

The change of energy content $\Delta\epsilon$ per unit mass of a gas is given by the summation of energy added (Δq) minus the work done by the gas due to volume changes

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$(-P\Delta V)$: a statement often know as the first law of thermodynamics. We can write that

$$\Delta\epsilon = \Delta q - P\Delta(1/\rho) \tag{5}$$

as the volume change ΔV is equal to the change of $1/\rho$. As

$$\frac{d(1/\rho)}{d\rho} = -\frac{1}{\rho^2},$$

the partial derivative of the energy content per unit mass with respect to density can now be evaluated from equation (5). We use a partial derivative which means that quantities other than density are held constant (so the heat flow $\Delta q = 0$) and obtain

$$\frac{\partial\epsilon}{\partial\rho} = \frac{P}{\rho^2}.$$

Substituting into equation (4) and using equation (3) gives another expression for the sound speed

$$c_s^2 = \frac{\partial P}{\partial\rho} = \frac{\gamma P}{\rho}. \tag{6}$$

Interestingly, we see that any factor that affects the relationship between energy density and pressure (equation(3)) will affect the speed of sound in the plasma. For example, the degree of ionisation in a plasma affects this relationship, so we have the seemingly perverse result that different models of ionisation can cause changes in the calculated speed of sound in a plasma.

2 The Fermi energy and pressure

Fermi-Dirac statistics govern the occupancy of electronic states in high density plasmas as the Pauli Exclusion Principle does not allow more than one electron in each quantum state. In high density plasmas, the probability of occupancy of a free electron quantum state becomes high, so the Pauli Exclusion Principle does need to be invoked in order to determine energy density and pressure calculations.

2.1 Free electron speed distributions

The probability $P(E)$ of occupancy by a particle of a quantum state of energy E is given by the proportionality

$$P(E) \propto \exp\left(\frac{N(\mu - E)}{kT}\right) \tag{7}$$

where N is the number of particles occupying the state with energy E and μ is the chemical potential (which is equal to the Fermi energy E_F at zero degrees Kelvin). If the state is not occupied by a particle $P(E) \propto 1$ as $N = 0$. As electrons are fermions, a state can only be occupied by one electron, or it can be unoccupied. If occupied, the probability relationship is $P(E) \propto \exp((\mu - E)/kT)$. The proportionality constants to turn the probabilities into absolute probabilities are the same for both occupied and not occupied states, so the average occupancy $n(E)$ of a state of energy E is given by the ratio of the probabilities here for $P_{occupied}/(P_{occupied} + P_{notoccupied})$ giving

$$n(E) = \frac{\exp\left(\frac{\mu - E}{kT}\right)}{\exp\left(\frac{\mu - E}{kT}\right) + 1} = \frac{1}{1 + \exp\left(\frac{E - \mu}{kT}\right)}.$$

This average occupancy of a quantum state can be immediately utilised to obtain an expression for the distribution of speeds of electrons. The number of electrons per unit volume $f_v(v)dv$ with speed between v and $v + dv$ is given by the proportionality

$$f_v(v)dv \propto 4\pi v^2 n(E)dv$$

where $E = (1/2)mv^2$ is the electron energy for electron mass m. The factor $4\pi v^2 dv$ is the velocity space volume corresponding to the speed range v to $v + dv$ given by the volume of a shell of radius v and thickness dv. The expression for the electron distribution of speeds can then be written as

$$f_v(v)dv \propto 4\pi v^2 \frac{dv}{1 + \exp\left(\frac{(1/2)mv^2 - \mu}{kT}\right)}.$$

To convert the proportionality constant here to an absolute value of the distribution of speeds requires normalisation. We choose to require that integrating over all possible speeds will give the total electron number density n_e per unit volume. We then have that

$$\int_0^\infty f_v(v)dv = n_e.$$

The probability distribution function with this normalisation will give the number of electrons per unit volume with speeds between v and $v + dv$. An alternative normalisation with $\int_0^\infty f_v(v)dv = 1$ would give the probability of finding an electron with speed in the range v to $v + dv$ (not the number of electrons). Unfortunately, the integrations to do the normalisation are not straightforward, except when the chemical potential is large and negative so that the occupancy of the free electron quantum states is low. We then have the familiar Maxwellian distribution of speeds with

$$f_v(v)dv = n_e \left(\frac{m}{2\pi kT} \right)^{\frac{3}{2}} 4\pi v^2 \exp \left(-\frac{mv^2}{2kT} \right) dv. \tag{8}$$

2.2 The density of quantum states for free electrons

Electrons bound by the central potential of the nuclear charge of an ion are clearly in discrete quantum state and there are large energy gaps between some of the quantum states. The energies of the quantum states of free electrons (not bound by an atomic or ionic potential) are close together, but still represent different quantum states rather than a true continuum of states.

Solving the time-independent Schrodinger equation shows that a free electron (i.e. an electron not in a potential energy field) has a sinusoidally oscillating wavefunction with a wavelength λ_{DB} given by h/p, where h is Planck's constant and p is the electron momentum. The wavelength λ_{DB} is known as the de Broglie wavelength. To calculate the density of quantum states for free electrons, we need to evaluate how many wavefunctions with a certain de Broglie wavelength can occupy a volume.

Consider a cubic volume with sides L. If electrons are bound within this volume, a steady state requirement is that only a half-integer number of de Broglie wavelengths occur between the walls of the cube, otherwise the oscillating nature of the free electron wavefunction will cause cancellation of wavefunction (in a similar manner to interference effects with light). The cube walls can be imagined to be infinitely high, infinitely steep potential barriers for this 'thought' evaluation.

The requirement that $m\lambda_{DB}/2 = L$ in the three directions between the cube walls, where m is an integer, means that the momentum associated with each m value is $mh/(2L)$ and the momentum spacing between $m+1$ and m is $h/(2L)$. The volume in p-space (momentum space) occupied by one electron quantum state is

thus $(h/(2L))^3$.

We can now evaluate the number $g(p)dp$ of free electron states per unit volume with momentum in the range p to $p + dp$. The volume of p-space to be considered is $(1/8)4\pi p^2 dp$ which is the volume of an octant of p-space with momentum in the range p to $p+dp$ corresponding to positive values of p in three Cartesian co-ordinate directions. Negative momentum components have no physical meaning here, so only the volume of an octant of p-space, where the p_x, p_y and p_z components in the directions x, y and z are all positive should be considered. The number of free electron quantum states is then given by

$$g(p)dp = 2\frac{(1/8)4\pi p^2 dp}{(h/(2L))^3 L^3} \qquad (9)$$

after allowing for the two possible electron spin states of the electron. Equation (9) divides the allowed p-state volume for free electrons by the volume in p-space occupied by one electron state and the volume L^3 of the cube. Simplifying equation (9) gives the density $f(p)$ of free electron states per unit volume per unit of momentum. The number of free electron quantum states between momentum p and $p + dp$ is given by

$$g(p)dp = \frac{8\pi p^2}{h^3} dp. \qquad (10)$$

Using $p = mv$, $dp = mdv$ and $g(p)dp = g(v)dv$, we can immediately write down an expression for the number $g(v)dv$ of free electron quantum states with speeds between v and $v + dv$:

$$g(v)dv = 8\pi v^2 \left(\frac{m}{h}\right)^3 dv. \qquad (11)$$

2.3   Calculation of Fermi energy and pressure

To gain an estimate of the Fermi energy and pressure of the free electrons, we assume a temperature T approaching zero. The distribution of speeds (discussed in section 2.1) becomes straightforward as there is then full occupancy of the free electron quantum states for electron energies less than the Fermi energy and zero occupancy for electron energies greater than the Fermi energy. Using equation

(10), the distribution of electrons with momentum p to $p + dp$ is such that

$$g(p)dp = \frac{8\pi p^2}{h^3}dp$$

for $p^2/(2m) < E_F$. while

$$g(p)dp = 0$$

for $p^2/(2m) > E_F$. To convert this to an electron energy distribution, we use $E = p^2/(2m)$ to write that $p = \sqrt{2mE}$ and $dp = \sqrt{m/(2E)}dE$, so that for electron energies less than the Fermi energy, the number of electrons (at zero temperature) with energy in the range E to $E + dE$ per unit volume is given by

$$g(E)dE = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} E^{1/2}dE.$$

The electron density associated with the Fermi energy E_F can be found by integrating this expression over energy. We only need to integrate up to the Fermi energy as at zero temperature there is zero occupancy of electrons at higher energy. The electron density for a particular Fermi energy is given by

$$n_e = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} \int_0^{E_F} E^{1/2}dE = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} \frac{2}{3}E_F^{3/2}. \tag{12}$$

It is more common for the electron density to be known from estimation or a measurement rather than the Fermi energy, so we manipulate this expression to give an expression for the free electron Fermi energy as a function of the electron density. We obtain

$$E_F = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m}n_e^{2/3}. \tag{13}$$

In practical units, if we measure the electron density in units of cm^{-3}, the Fermi energy in electron volts (eV) is given by

$$E_F(eV) = 0.3646 \left(\frac{n_e}{10^{21}}\right)^{2/3}.$$

The average energy U of Fermi degenerate electrons per unit volume is calculated by integrating the electron energy multiplied by the number density of electrons with that energy. We have

$$U = \int_0^{E_F} Ef_E(E)dE = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} \int_0^{E_F} E^{3/2}dE = 4\pi \left(\frac{2m}{h^2}\right)^{3/2} \frac{2}{5}E_F^{5/2}.$$

Using equation (12), we see that $U = (3/5)n_e E_F$.

Equation (3) relates energy density per unit volume $U = \rho \epsilon$ to pressure P. Even for Fermi degenerate electrons, the assumption that $\gamma = 5/3$ (see equation (3)) as for an ideal gas is a good approximation, so we find that the pressure is related to the energy density by $P = (2/3)U$. This means that our calculation of the average energy U of the Fermi degenerate electrons corresponds to a Fermi pressure where

$$P_F = (2/5)n_e E_F = \frac{2}{5}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}n_e^{5/3}.$$

In practical units, the Fermi pressure in Mbar ($= 10^{11}$ Nm^{-1}) is related to the electron density in units of cm^{-3} by

$$P_F(Mbar) = 0.5\left(\frac{n_e}{10^{23}}\right)^{5/3}.$$

3 The degree of ionisation

We have represented the degree of ionisation of a plasma by $Z_{av} = n_e/n_i$, where the electron density is n_e and the ion density is n_i. For many calculations such as the evaluation of plasma pressure and the determination of the plasma sound speed, a single value of Z_{av} is sufficient. However, to calculate Z_{av} it is necessary to evaluate the populations of a range of ionisation stages. In lower density plasmas, this is a major undertaking requiring a model of the collisional and radiative processes populating the discrete quantum states. For high density plasmas, it is often sufficient to assume that the populations are given by an equilibrium relationship known as the Saha-Boltzmann equation. This can be derived by extending the Boltzmann population ratio used to infer the equilibrium ratio of populations between quantum states within a single ionisation stage or atom.

At high densities, electron-ion collisional processes dominate over radiative processes in the populating and de-populating processes between quantum states. The electron-ion collisions cause a transfer of an electron from one bound state to another, or by collisional ionisation, cause a transition of an electron from a bound state to a free electron quantum state. The inverse of this last process, where free electrons interact by colliding with an ion to cause one electron to move to a bound state, while another electron absorbs the energy of the recombining

electron is known as three-body recombination. The important issue is that each collisional excitation or ionisation process, for a system in equilibrium, proceeds at the same rate as the inverse collisional de-excitation or three-body recombination process. This equality of rate for each process is known as the principle of detailed balance. Provided that all important process and their inverses occur, this means that quantum state populations will be in equilbrium. At low densities, radiative decay rates become significant compared to the electron collisional decay rates, but the radiative process of spontaneous decay is often not balanced by the other detailed radiative processes of photo-absorption and stimulated emission. This can cause the populations for low density plasmas to differ from the equilibrium populations.

3.1 The Saha-Boltzmann equation

Consider Z and $(Z + 1)$ charged ions with population densities n_Z and n_{Z+1} in two discrete energy states. We want to calculate the population ratio n_{Z+1}/n_Z for a plasma in equilibrium. We will assume that the $Z + 1$ ion energy state is the ground state, so that the energy difference between the two states is equal to the ionisation energy E_{ion} of the Z charged ion. We assume the two energy states have degeneracies of g_Z and g_{Z+1}: i.e. the number of quantum states and hence the maximum number of electrons per ion that can exist at the energies associated with the two states are g_Z and g_{Z+1}.

The Boltzmann population ratio between two quantum states (upper u and lower l) in the same ion can be written as

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} \exp\left(-\frac{\Delta E}{kT}\right)$$

where g_u and g_l represent the degeneracies of the upper and lower states respectively and ΔE is the energy difference between the quantum states. This ratio arises from Fermi-Dirac statistics (equation (7)) when the average occupancy N of a quantum state is such as that $N << 1$.

In the ionisation process of converting a Z charged ion to a $(Z + 1)$ charged ion, a free electron with a speed (say v) is created. The Boltzmann ratio becomes

$$\frac{f_v(v)dv}{n_Z} = \frac{g_{Z+1}(1/n_{Z+1})g(v)dv}{g_Z} \exp\left(-\frac{E_{ion} + (1/2)mv^2}{kT}\right) \qquad (14)$$

where $g(v)dv$ is the density of free electron states (see equation (11)) and $f_v(v)dv$ is the number density of electrons with speeds in the range v to $v + dv$. The quantity $g_{Z+1}(1/n_{Z+1})g(v)dv$ represents the degeneracy associated with the upper energy state for electrons with speed v to $v + dv$. It comprises the number of quantum states in the volume of one $Z + 1$ ion $((1/n_{Z+1})f(v)dv)$ multiplied by the degeneracy of the $Z + 1$ energy state (g_{Z+1}). The density of speeds is given by the Maxwellian distribution (see equation (8)). Substituting equations (11) and (8) into equation (14), we obtain the Saha-Boltzmann equation for the population ratio n_{Z+1}/n_Z such that

$$\frac{n_{Z+1}}{n_Z} = \frac{g_{Z+1}}{g_Z}\frac{2}{n_e}\left(\frac{m}{h}\right)^3\left(\frac{2\pi kT}{m}\right)^{3/2}\exp\left(-\frac{E_{ion}}{kT}\right). \tag{15}$$

The exponentiated electron kinetic energy $(1/2)mv^2$ and a v^2dv term cancel on both sides. The Saha-Boltzmann equation is sometimes written in terms of the thermal de Broglie wavelength Λ defined by $\Lambda = h/\sqrt{2\pi mkT}$ such that

$$\frac{n_{Z+1}}{n_Z} = \frac{g_{Z+1}}{g_Z}\frac{2}{n_e\Lambda^3}\exp\left(-\frac{E_{ion}}{kT}\right).$$

The quantity $2/(n_e\Lambda^3)$ can be regarded as representing the average degeneracy of the free electrons created in the ionisation process.

3.2 Continuum lowering

Ions in a high density plasma have wavefunctions and energy levels which are perturbed by neighbouring ions. An estimate of the radius R_0 from an ion nucleus where effects due to neighbouring ions dominate is obtained by associating the volume of a sphere of radius R_0 with the average volume occupied by an ion. We can write

$$\frac{4\pi}{3}R_0^3 = \frac{1}{n_i}$$

where n_i is the ion density. If we make the assumption that electron wavefunctions enclosed within R_0 are similar and the electrons have similar energies to those in an isolated ion, the effect of the neighbouring ions is to eliminate some higher principal quantum number n states occuring at higher radius, while leaving lower quantum states n largely unperturbed. This is, in fact, what is seem experimentally. Higher

n quantum levels dissappear so that the ionisation energy is reduced and the continuum of free electron states subsumes some of the higher energy states that are discrete in an isolated ion.

An estimate of the ionisation potential depression ΔE_{ion} in a dense plasma can be obtained by using the energy and wavelength scaling applicable to hydrogen-like ions. The higher energy levels of ions with large wavefunction radius see a central potential which is close to being hydrogen-like (i.e. a nuclear charge with a few shielding electrons close to the nucleus). Atomic physics calculations show that the radius of the wavefunction for a hydrogen-like ion of charge Z varies as $a_0 n^2/Z$, where a_0 is the Bohr radius. The energies of levels for hydrogen-like ions scale as $E_H Z^2/n^2$, where E_H is the ionisation energy of the hydrogen ground state. The degree of ionisation potential depression ΔE_{ion} is inversely proportional to R_0 and we can fix the scale of ΔE_{ion} absolutely by taking a ratio to the ground state ionisation energy E_{ion} and wavefunction radius a_0/Z. We write

$$\frac{\Delta E_{ion}}{E_{ion}} = \frac{\Delta E_{ion}}{Z_{av}^2 E_H} = \frac{a_0/Z_{av}}{R_0}$$

where Z_{av} is the average degree of ionisation. Substituting for R_0, we obtain an expression for the ionisation potential depression ΔE_{ion} due to neighbouring ions:

$$\frac{\Delta E_{ion}}{E_{ion}} = \frac{a_0}{Z_i} \left(\frac{4\pi n_i}{3} \right)^{1/3}.$$

Expressions where the ionisation energy occurs need to be modified to take account of the ionisation potential depression. For example, in the Saha-Boltzmann equation (equation (15)), we have

$$\frac{n_{Z+1}}{n_Z} = \frac{g_{Z+1}}{g_Z} \frac{2}{n_e} \left(\frac{m}{h} \right)^3 \left(\frac{2\pi kT}{m} \right)^{3/2} \exp\left(-\frac{E_{ion} - \Delta E_{ion}}{kT} \right).$$

4   Radiation and radiative transfer

The radiation intensity within a high density plasma exchanges energy with the bound and free electrons by emission and absorption processes. Electron transitions between bound discrete quantum states associated with the energy states of an ion or atom produce spectral line emission, while transitions from free electrons to other free quantum states or to bound quantum states produce continuum

emission. The emission processes can be represented by an emission coefficient ϵ which is the radiated power per unit volume into unit solid angle in a frequency range ω to $\omega + d\omega$. Radiation can be absorbed by electrons bound within ions or atom or by free electrons with resulting conversion of the photon energy to kinetic energy (for free electrons) or a transition of an electron to a higher quantum states (for bound electrons). The rate of radiation absorption can be represented by an absorption coefficient K with dimensions of inverse length.

Radiation intensity I is often measured in units of power per unit solid angle per unit of frequency per unit of area. The term irradiance or radiation flux is used to refer to the power per unit area integrated over angle (though sometimes confusingly the term intensity can be used here as well). With our definition of intensity, we can write an expression for the change in intensity dI within a small length dz along a line of sight through a medium:

$$dI = \epsilon dz - KI dz. \tag{16}$$

We can define a quantity known as the optical depth τ by considering the optical depth change $d\tau = K dz$ over an incremental length (dz) along the line of sight. We then have

$$\frac{dI}{d\tau} = \frac{\epsilon}{K} - I = S - I \tag{17}$$

where $S = \epsilon/K$ is known as the source function. In a large uniform plasma with $dI/dz = 0$, the radiation intensity $I = \epsilon/K = S$. For a system in thermal equilibrium, the radiation intensity is given by the Planck radiation law and the source function is also equal to the Planck radiation intensity. The Planck radiation intensity is often referred to as black-body radiation as a black object at room temperature emits a spectrum of radiation close to that for an equilibrium system - with a peak of emission in the infra-red (hence the black colour in the visibile).

For a plasma with populations in equilibrium, but not necessarily an equilibrium radiation field, the relation $I = \epsilon/K$ can be used to deduce the value of the absorption coefficient K if, say, the emission coefficient ϵ is known. The intensity I is set to the Planck radiation intensity for radiation within a black-body cavity at the electron temperature of the plasma. Similarly, the emissivity ϵ could be deduced from the absorption coefficient K. The condition where populations

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of bound and free electrons are in equilbrium, but the radiation field is not in equilibrium is known as local thermodynamic equilbrium (often abbreviated to LTE).

4.1 The Planck radiation law

If the radiation and electrons in a medium have many interactions through emission and absorption, the radiation field will become thermalised and can be regarded as having a temperature equal to that of the electrons. In this section, we calculate the form of the thermalised radiation known as the Planck radiation intensity.

The radiation field is a collection of photons which follow the rules of statistical mechanics for a collection of bosons. In a similar way to the earlier calculation of the density of quantum states for free electrons (which are fermions), we need to calculate the density of modes for light in a steady state system. We imagine a cube in space with perfectly conducting (and hence perfectly reflecting) walls and sides of length L. Such a volume with an assumed radiation energy in equilbirium with the walls is known as a black-body cavity (though in inertial fusion work the German word 'hohlraum' is often used).

4.1.1 The density of modes

The wavelengths λ of light that can exist in the cube are quantised by the condition that a half-integer number of wavelengths only can exist in directions perpendicular to the walls. Otherwise, the oscillating electric field for the light causes transient interference effects and a steady state solution for the radiation field does not exist. This condition is best quantified in terms of wavenumber $k = 2\pi/\lambda$, so that k for each direction x, y and z satisfies $k_m = m\pi/L$ and $k_{m+1} - k_m = \pi/L$, where m is an integer. The density of modes per unit volume of k-space is thus $(\pi/L)^3$.

The number $\rho_k dk$ of modes with wavenumber between k and $k + dk$ per unit volume of the cube is given by

$$\rho_k dk = 2\frac{(1/8)4\pi k^2 dk}{(\pi/L)^3 L^3}$$

after allowing for a factor 2 increase in the number of modes due to the two polarisation components of light. The numerator here represents the volume of

k-space for modes in the range k to $k + dk$ which is an octant of a sphere of radius k and thickness dk. The octant $1/8$ factor arises because only positive k_x, k_y and k_z values of k have meaning as distinct modes. The denominator is the k-space taken up by a single mode ($(\pi/L)^3$) multipied by the volume of the cube (L^3). We can simplify the above and obtain an equation for the number of modes with wavenumber between k and $k + dk$ per unit volume:

$$\rho_k dk = \frac{k^2}{\pi^2} dk. \tag{18}$$

We can convert this expression to the density of modes per unit frequency, by noting that angular frequency $\omega = ck$ and hence $d\omega = cdk$, where c is the speed of light. The number of modes $\rho_\omega d\omega$ with frequency between ω and $\omega + d\omega$ per unit volume is given by

$$\rho_\omega d\omega = \frac{\omega^2}{c^3 \pi^2} d\omega. \tag{19}$$

4.1.2 Quantisation of the radiation field

The propagation of light within the cube introduced in section (4.1.1) is governed by the wave equation which determines the values of electric field $\mathbf{E}$ associated with the radiation. We have for the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{20}$$

which has solutions of the form $\mathbf{E}(t) = \mathbf{E}_0(t) \exp(i\mathbf{k}.\mathbf{r})$ where $\mathbf{r}$ is a position vector. Substituting this solution into the wave equation (equation (20)) gives

$$-k^2 \mathbf{E}_0(t) - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_0(t)}{\partial t^2} = 0$$

which can be re-written as

$$\frac{\partial^2 \mathbf{E}_0(t)}{\partial t^2} = -\omega^2 \mathbf{E}_0(t). \tag{21}$$

Solutions of this equation take the form $\mathbf{E}_0(t) = \mathbf{E}_0 \exp(i\omega t)$. More importantly here, we can recognise that equation (21) describes harmonic oscillation of the electric field in time. To find the correct energy of the radiation field, the harmonic oscillation needs to be treated quantum mechanically. Solving the Shrodinger

equation in one-dimension with a parabolic potential energy variation gives quantised harmonic oscillator energies E_{n_p} such that

$$E_{n_p} = (n_p + 1/2)\hbar\omega \qquad (22)$$

where n_p is an integer such that $n_p = 0, 1, 2, 3, ...$. The energies associated with the electric and magnetic fields of the radiation for each ω are quantised in this way with each unit $\hbar\omega$ of quantisation referred to as a photon.

The energy of $(1/2)\hbar\omega$ in equation (22) is known as the vacuum energy or zero-point energy as it exists even when the number of photons n_p is zero. Real effects arise from the vacuum energy. The Casimir effect arises when two unequal black-body cavities are next to each other. There is a measurable force between the cavities as the zero-point energy is different in the two cavities. It is thought that dark energy responsible for the initial acceleration of the expansion of the Universe is also associated with the vacuum energy.

In thermal equilbrium at temperature T, we can calculate the probability of the radiation field for a particular mode to have n_p photons using the probability distribution (equation (7)). Photons are bosons, so the number n_p of photons at each energy $\hbar\omega$ can range from 0, 1, 2, ... upwards and their chemical potential μ is zero. The absolute probability P_{n_p} that n_p photons are present is given by

$$P_{n_p} = \frac{\exp(-n_p\hbar\omega/kT)}{\Sigma_{n'}\exp(-n'\hbar\omega/kT)}$$

upon cancelling $\exp((1/2)\hbar\omega/kT)$ in the numerator and denominator. If we set $U = \exp(-\hbar\omega/kT)$ then

$$P_{n_p} = \frac{U_p^n}{\Sigma_{n'}U^{n'}}$$

The infinite series $\Sigma_{n'}U^{n'} = 1/(1-U)$ so that

$$P_{n_p} = (1-U)U^{n_p}.$$

The mean number n_{av} of photons

$$n_{av} = \Sigma_{n_p}n_pP_{n_p}.$$

This equation for n_{av} can be simplified as follows:

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$$n_{av} = (1-U)\Sigma_{n_p} n_p U^{n_p} = (1-U)U\frac{d(\Sigma_{n_p} U^{n_p})}{dU} = (1-U)U\frac{d(1-U)^{-1}}{dU} = \frac{U}{1-U}.$$

This quantity is known as the mean number of photons per mode. Substituting in our definition of U gives

$$n_{av} = \frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}. \tag{23}$$

The total energy density of radiation can be evaluated by multiplying our expression for the density of modes (equation (19)) by the mean number of photons per modes (equation (23)) and the energy per photon ($\hbar\omega$) . We have for the equilibrium radiation energy $W_p(\omega)d\omega$ per unit volume in the frequency range ω to $\omega + d\omega$:

$$W_p(\omega)d\omega = n_{av}\hbar\omega\rho_\omega d\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{d\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}. \tag{24}$$

This is the Planck radiation energy density inside a black-body cavity.

4.1.3 The Planck radiation flux and intensity

The intensity of radiation $< I_p(\omega) >$ represented by the radiation power per unit area integrated over all angles (4π steradian) associated with the Planck radiation energy density is found by multiplying the radiation energy density $W_p(\omega)$ by the speed of light c. We can write that

$$< I_p(\omega) >= cW_p(\omega) = \frac{\hbar\omega^3}{\pi^2 c^2}\frac{d\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}. \tag{25}$$

We can define the radiation intensity $J_p(\omega)$ per unit steradian by setting $J_p(\omega) =< I_p(\omega) > /(4\pi)$.

There are several other ways of representing the intensity of black-body radiation. A small aperture in a black-body cavity wall will emit a flux $F_p(\omega)$ of radiation which we can define as the radiated power into 2π steradians per unit area. Consider a black body cavity with photons impinging on a plane wall. We need to calculate the fraction of photons inside the cavity that impinge on the plane wall in a time Δt to evaluate $F_p(\omega)d\omega$ from the radiation energy density

$W_p(\omega)d\omega$. Photons initially at depth $z\Delta t$ along a normal from the wall surface will reach the wall if they are travelling within an angle θ to the wall normal such that $\cos\theta = z/(c\Delta t)$, where c is the speed of light. This means that photons travelling within a cone with half-angle θ will reach the wall in our specified time Δt. The solid angle Ω of such a cone is given by $\Omega = 2\pi(1-\cos\theta)$. The fraction of photons reaching the wall in time Δt from a depth of z is thus $(1/2)(1 - z/(c\Delta t))$. The radiation energy per unit area impinging on the black body cavity wall in a time Δt is given by $F_p(\omega)\Delta t$ which is evaluated from the radiation energy density by multiplying by an integration of the fraction of photons reaching the wall with respect to distance z. We have

$$F_p(\omega)\Delta t = W_p(\omega) \int_{z=0}^{c\Delta t} (1/2)(1 - z/(c\Delta t))dz = \Delta t \frac{c}{4}W_p(\omega)$$

The flux $F_p(\omega)d\omega$ per unit area into 2π steradian over the frequency range ω to $\omega + d\omega$ is thus given by

$$F_p(\omega)d\omega = \frac{1}{4}cW_p(\omega)d\omega = \frac{\hbar\omega^3}{4\pi^2 c^2} \frac{d\omega}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \tag{26}$$

The radiation intensity per unit steradian $J_p(\omega)$ is related to this flux of radiation impinging on the walls of a black-body by

$$J_p(\omega) = \frac{1}{\pi}F_p(\omega). \tag{27}$$

We can convert the expression for $F_p(\omega)$ into the radiation intensity $I_{Pedge}(\nu)d\nu$ per unit area per steradian for a frequency range ν to $\nu + d\nu$ with ν measured in Hertz by dividing by 2π (as the flux is into 2π steradian) and noting that $\omega = 2\pi\nu$ and $d\omega = 2\pi d\nu$. The resulting intensity $I_{Pedge}(\nu)$ is such that $I_{Pedge}(\nu)d\nu$ represents the power per unit area per steradian in the spectral range ν to $\nu + d\nu$ impinging on the walls of a black-body cavity. We have

$$I_{Pedge}(\nu)d\nu = \frac{h\nu^3}{c^2} \frac{d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}. \tag{28}$$

For a surface inside a black-body cavity, the intensity incident on the surface is twice this value (with radiation impinging from two directions). The intensity $I_{Pint}(\nu)$ such that $I_{Pint}(\nu)d\nu$ represents the power per unit area per steradian in

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the spectral range ν to $\nu + d\nu$ impinging from both sides on a surface inside a black-body cavity is given by

$$I_{Pint}(\nu)d\nu = \frac{2h\nu^3}{c^2} \frac{d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} = J_p(\nu)d\nu. \tag{29}$$

The parameter $J_p(\nu)$ is the radiation intensity per unit steradian as defined below equation (25), but over a frequency range of ν to $\nu + d\nu$ (rather than angular frequency).

The total energy density W_{tot} per unit volume integrated over all frequencies can be readily evaluated by integrating equation (24). Setting $x = \hbar\omega/kT$ we have

$$W_{tot} = \int_0^\infty W(\omega)d\omega = \frac{(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3 dx}{\exp(x) - 1}.$$

The integral with respect to x here is given in standard tabulations and has a value equal to $\pi^4/15$. The total radiation energy per unit volume for an equilbrium radiation field is then given by

$$W_{tot} = \frac{\pi^2 (kT)^4}{15 c^3 \hbar^3}. \tag{30}$$

The spectrally integrated radiation flux F_{tot} representing power per unit area impinging on the walls of a black-body cavity and hence the flux escaping through an aperture in a black-body cavity wall is again found using the result that $1/4$ of the radiation energy density over the distance $c\Delta t$ from the wall will impinge on the wall in a time Δt. We have

$$F_{tot} = \frac{c}{4} W_{tot} = \frac{\pi^2 (kT)^4}{60 c^2 \hbar^3} = \sigma_{SB} T^4 \tag{31}$$

where σ_{SB} is known as the Stefan-Boltzmann constant (numerically $\sigma_{SB} = 5.67$ x 10^{-8} Wm^{-2} K^{-4}).

4.2   Radiative transfer

We have seen that the equation of radiative transfer for the change of radiation intensity I in a medium is given by equation(17) with

$$\frac{dI}{d\tau} = S - I$$

where S is the source function and $d\tau$ is the optical depth change over a spatial distance dz such that $d\tau = Kdz$ where K is the absorption coefficient. Instead of thinking of real space z, to evaluate radiative transfer equations, we usually need to think of how the source function S varies in optical depth τ space. The formal solution of the radiative transfer equation becomes

$$I = \int_0^\tau S(\tau') \exp(-\tau')d\tau' + I(0)\exp(-\tau) \tag{32}$$

where τ is the optical depth measured through the plasma along a line of sight and $I(0)$ is the intensity incident onto the plasma. For many laboratory plasmas, there is no external source of radiation so that $I(0) = 0$. If the source function is contant with optical depth then $S(\tau') = S_0$ everywhere and

$$I = S_0(1 - \exp(-\tau)).$$

For a large plasma with large τ, we see that $I = S_0$. The equilbrium radiation intensity for a large optical depth plasma is the Planck intensity given by equation (28) for radiation incident from the surface of the plasma. We have that $S_0 = I_{Pint}$ and as the source function is given by the ratio of of the emission coefficient ϵ to the absorption coefficient K, it is possible to relate the emission and absorption coefficients to each other using the Planck radiation intensity. We can write that

$$S = \frac{\epsilon}{K} = I_{Pint}.$$

As the emission and absorption coefficients are atomic physics parameters, this relationship between them will hold outside of an equilbrium plasma. This is sometimes known as Kirchhoff's law.

If a source function varies linearly with optical depth from a surface, we can write for the source function

$$S(\tau) = S_0 + f\tau$$

The equation of radiative transfer then becomes

$$I = S_0 \int_0^\tau \exp(-\tau')d\tau' + f \int_0^\tau \tau' \exp(-\tau')d\tau' = S_0 + f + \exp(-\tau)(f(\tau - 1) - S_0).$$

If τ is large

$$I \approx S_0 + f.$$

This expression tells us that the intensity of radiation with a large linearly increasing source function as optical depth increases from the edge of the medium is equal to the source function at an optical depth $\tau = 1$. This is known as the Eddington-Barbier relation. Allied to this result is the concept of the mean optical depth given by

$$\tau_{av} = \int_0^\infty \tau' \exp(-\tau') d\tau' = 1$$

The 'average' or mean optical depth can be regarded as unity.

4.2.1 Intensities in a planar geometry

Consider a planar geometry where radiation is emitted from a plane within a plasma. For the radiative transfer calculations, the spatial dimensions are in units of optical depth τ typically measured from the edge of the plasma closest to the observer (where $\tau = 0$). The radiation propagating at an angle of θ to the normal to the plane subtends a solid angle given by $2\pi\tau \sin\theta d\theta$, where τ is the optical depth of the planar element along the angle θ. The intensity of radiation emitted from the plane of thickness $d\tau$ at an optical depth of τ in the angular range θ to $\theta + d\theta$ is $S(\tau)\exp(-\tau)2\pi\tau \sin\theta d\theta d\tau$.

We can evaluate an average intensity $< I >$ of radiation by integrating. Integrating over optical depth distance and over angle, we obtain

$$< I >= \int_{\theta=0}^{\theta=\pi} \int_{\tau=0}^{\infty} S(\tau) \exp(-\tau) 2\pi\tau \sin\theta d\theta d\tau.$$

The integration over angle includes radiation propagating in all directions from the plasma. Integrals as above are more readily solved by letting $u = \cos\theta$, so that $\sin\theta d\theta = -du$. We have

$$< I >= 2\pi \int_{u=-1}^{u=1} \int_{\tau=0}^{\infty} S(\tau)\tau \exp(-\tau) d\tau du \tag{33}$$

For a constant source function S_0, $< I >= 4\pi S_0$ as

$$\int_0^\infty \tau \exp(-\tau) d\tau = 1$$

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and

$$\int_{u=-1}^{u=1} du = 2.$$

We see that the angle integrated intensity for a constant source function is equal to the source function value times 4π steradian.

The total flux F escaping per unit area from the plasma is calculated by integrating $S(\tau)\cos\theta\exp(-\tau)2\pi\tau\sin\theta d\theta d\tau$ over optical depth distance and over angle. This is a similar integration to that for equation (33) except with an additional $\cos\theta$ term so as to evaluate the component of the radiation intensity in a plane parallel to the assumed planar geometry. The $\cos\theta$ term adjusts for the radiation being distributed over an area on a plane which increases as $1/\cos\theta$ with increasing angle θ to the plane normal. We obtain

$$F = 2\pi \int_{u=-1}^{u=1} \int_{\tau=0}^{\infty} S(\tau)u\tau\exp(-\tau)d\tau du \tag{34}$$

If we let $I(u) = \int_{\tau=0}^{\infty} S(\tau)\tau\exp(-\tau)d\tau$, then

$$< I >= 2\pi \int_{u=-1}^{u=1} I(u)du = 4\pi J$$

and

$$F = 2\pi \int_{u=-1}^{u=1} I(u)u du = 4\pi H$$

upon defining integrals

$$J = \frac{1}{2} \int_{u=-1}^{u=1} I(u)du \tag{35}$$

and

$$H = \frac{1}{2} \int_{u=-1}^{u=1} I(u)u du \tag{36}$$

4.2.2 Radiation pressure in a planar geometry

For any particle with kinetic energy much greater than its rest mass energy (mc^2), the kinetic energy is given by pc where the particle momentum is p. Considering photons as such particles means that each photon has a momentum equal to the photon energy divided by c, so the rate of energy arrival (i.e. intensity) divided by c gives the momentum change upon absorption which is equivalent to pressure.

Consequently, radiation of intensity I exerts a pressure of $I\cos^2\theta/c$ when absorbed onto a surface at angle θ to the radiation propagation direction. The $\cos^2\theta$ variation arises because (i) the radiation is distributed over an area on the plane which increases as $1/\cos\theta$ with increasing angle θ to the plane normal (and hence the intensity per unit area on the surface drops as $\cos\theta$) and (ii) the component of the radiation momentum normal to the plane varies as $\cos\theta$. The average radiation pressure for radiation emanating from a planar surface is thus

$$P_{rad} = \frac{2\pi}{c}\int_{u=-1}^{u=1} I(u)u^2 du = \frac{4\pi}{c}K$$

where

$$K = \frac{1}{2}\int_{u=-1}^{u=1} I(u)u^2 du. \tag{37}$$

If we assume that the radiation intensity $I(u)$ is a power series in u with terms only up to linear, we can write that

$$I(u) = a + bu$$

Evaluating the J, H and K integrals introduced (equations (35), (36) and (37)) gives

$$J = \frac{1}{2}\int_{-1}^{1}(a+bu)du = a,$$

$$H = \frac{1}{2}\int_{-1}^{1}(au+bu^2)du = \frac{1}{3}b$$

and

$$K = \frac{1}{2}\int_{-1}^{1}(au^2+bu^3)du = \frac{1}{3}a.$$

We see that $K = (1/3)J$ which is known as the Eddington approximation. We have that the average radiation pressure ($P_{rad} = (4\pi/c)K$) is related to the average intensity ($< I >= 4\pi J$) such that

$$\frac{P_{rad}}{< I >} = \frac{1}{3c}.$$

4.2.3 Radiation diffusion in a planar geometry

It is possible to re-write the equation of radiative transfer ($dI/d\tau = S - I$) for a planar geometry using angle integrated parameters. We need to consider the optical depth (say τ_n) in a direction normal to the planes of symmetry. For any radiation propagation angle to the normal of the planes of symmetry, we then have an optical depth $\tau = \tau_n/u$ and the equation of radiative transfer becomes

$$\frac{d(Iu)}{d\tau_n} = S - I.$$

Integrating over all angles, we have

$$\int_{-1}^{1} \frac{d(Iu)}{d\tau_n} du = \int_{-1}^{1} S du - \int_{-1}^{1} I du.$$

Multiplying throughout by $1/2$ and substituting our definitions of J (equation (35)) and H (equation (36)), gives that

$$\frac{dH}{d\tau_n} = S - J \tag{38}$$

assuming that the source function S is isotropic (independent of angle).

In a similar manner if we multiply the equation of radiative transfer by u and integrate over all angles, we obtain for the 'first moment' of the equation of radiative transfer

$$\int_{-1}^{1} \frac{d(Iu)}{d\tau_n} u du = \int_{-1}^{1} S u du - \int_{-1}^{1} I u du,$$

which simplies to

$$\frac{dK}{d\tau_n} = -H \tag{39}$$

upon substituting for K (equation (37)) and H (equation (36)). The integration of Su over all angles for an isotropic source function S is zero.

If we use the Eddington approximation ($K = (1/3)J$) and equation (39), we can obtain

$$\frac{1}{3} \frac{d^2 J}{d\tau_n^2} = -\frac{dH}{d\tau_n} = J - S \tag{40}$$

and

$$H = -\frac{1}{3}\frac{dJ}{d\tau_n} \qquad (41)$$

upon substituting equation (38) for $dH/d\tau_n$. Equation (40) has the form of a diffusion equation for J.

4.2.4   The Rosseland mean opacity

For many applications, we require the spectrally and angularly integrated flux F_{tot} of radiation impinging on a surface. Considering the definition of H (equation (36)), we can write

$$F_{tot} = 4\pi \int_{\omega=0}^{\infty} H(\omega)d\omega$$

where the integration is over all frequencies ω. The optical depth change $d\tau_n$ is related to distances z which are normal to our assumed planes of symmetry by $d\tau_n = Kdz$, where K is the absorption coefficient. Using equation (41), we then have for the spectrally and angularly integrated flux

$$F_{tot} = -\frac{4\pi}{3} \int_{\omega=0}^{\infty} \frac{1}{K}\frac{dJ}{dz}d\omega = -\frac{4\pi}{3} \int_{\omega=0}^{\infty} \frac{1}{K}\frac{dJ}{dT}\frac{dT}{dz}d\omega.$$

The plasma temperature T variation with distance z does not depend on frequency ω, so can be taken outside the integration to give

$$F_{tot} = -\frac{4\pi}{3}\left[\int_{\omega=0}^{\infty}\frac{1}{K}\frac{dJ}{dT}d\omega\right]\frac{dT}{dz}.$$

The spectrally and angularly integrated flux of radiation can thus be expressed as

$$F_{tot} = -\frac{4\pi}{3K_{Ros}}\left[\int_{\omega=0}^{\infty}\frac{dJ}{dT}d\omega\right]\frac{dT}{dz} \qquad (42)$$

where we define a mean absorption coefficient (known as the Rosseland mean absorption coefficient) by

$$\frac{1}{K_{Ros}} = \left[\int_{\omega=0}^{\infty}\frac{1}{K}\frac{dJ}{dT}d\omega\right] / \left[\int_{\omega=0}^{\infty}\frac{dJ}{dT}d\omega\right]. \qquad (43)$$

This evaluation for the Rosseland mean absorption coefficient gives a value of $1/K_{Ros}$ which is a spectral average of $1/K$. For the denominator in the definition

of the Rosseland mean absorption coefficient, we can undertake the integration of the angle integrated radiation flux J with frequency ω before we differentiate with respect to temperature T.

For plasmas with high opacity, the radiation flux will approach a Planckian distribution and then $J = F_p/\pi$ (see equation (27)) and the integration over all frequency gives

$$\int_{\omega=0}^{\infty} F_p d\omega = \sigma_{SB} T^4$$

where σ_{SB} is the Stefan-Boltzmann constant. Differentiating with respect to temperature gives that

$$\frac{d\left(\int_{\omega=0}^{\infty} F_p d\omega\right)}{dT} = 4\sigma_{SB} T^3.$$

We can now write an expression for the spectrally and angularly integrated flux of radiation for a plasma with radiation field close to Planckian. We have

$$F_{tot} = -\frac{16\sigma_{SB}}{3K_{ros}} \frac{dT}{dz} \tag{44}$$

with the Rosseland mean absorption coefficient K_{Ros} given by

$$\frac{1}{K_{Ros}} = \frac{1}{4\sigma_{SB} T^3} \int_{\omega=0}^{\infty} \frac{1}{K} \frac{dF_p}{dT} d\omega. \tag{45}$$

We have the flux of radiation determined by the gradient of temperature with distance (dT/dz).