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Assignment 1

2. i) $$f_B = \frac{n_B}{\pi^{3/2} V_{TB}^3} \exp\left[-\frac{|\bar{V} - V_B \hat{e}_x|^2}{V_{TB}^2}\right]$$

$$|\vec{V} - V_B \hat{e}_x|^2 = \left| \left(\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} - \begin{pmatrix} v_B \\ 0 \\ 0 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} - \begin{pmatrix} v_B \\ 0 \\ 0 \end{pmatrix} \right) \right|$$

$$= \left| \begin{pmatrix} v_x - v_B \\ v_y \\ v_z \end{pmatrix} \begin{pmatrix} v_x - v_B \\ v_y \\ v_z \end{pmatrix} \right| =$$

$$= (v_x - v_B)^2 + v_y^2 + v_z^2$$

$$f_B = \frac{n_B}{\pi^{3/2} V_{TB}^3} \exp\left[-\left(\frac{(v_x - v_B)^2 + v_y^2 + v_z^2}{v_{TB}^2}\right)\right]$$

$$F_{B_0} = \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{\infty} dv_z \, f_B(v_x, v_y, v_z)$$

$$F_{B_0} = \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} \frac{n_B}{\pi^{3/2} V_{TB}^3} \exp\left[-\left(\frac{(v_x - v_B)^2 + v_y^2 + v_z^2}{v_{TB}^2}\right)\right.$$

$$F_{Bo} = \frac{n_B}{\pi^{3/2} v_{TB}^3} \exp\left[-\frac{(v_x - v_B)^2}{v_{TB}^2}\right] \times$$

$$\times \int_{-\infty}^{+\infty} dv_y \exp\left[-\frac{v_y^2}{v_{TB}^2}\right] \times \int_{-\infty}^{\infty} dv_z \exp\left(-\frac{v_z^2}{v_{TB}^2}\right) =$$

$$\left\{ \begin{array}{l} \text{using standard integral } \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \\ \text{and } a = \frac{1}{V_{TB}^2}, \quad X = v_y, v_z \end{array} \right\}$$

$$F_{Bo} = \frac{n_B}{\pi^{3/2} v_{TB}^3} \exp\left[-\frac{(v_x - v_B)^2}{v_B^2}\right] (\sqrt{\pi} \cdot V_{TB})(\sqrt{\pi}\, V_{TB})$$

So,
$$\boxed{F_{Bo} = \frac{n_B}{\sqrt{\pi}\, V_{TB}} \exp\left(-(v_x - v_B)^2 / v_{TB}^2\right)}$$

$$F_{M0} = \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z \, f_M =$$

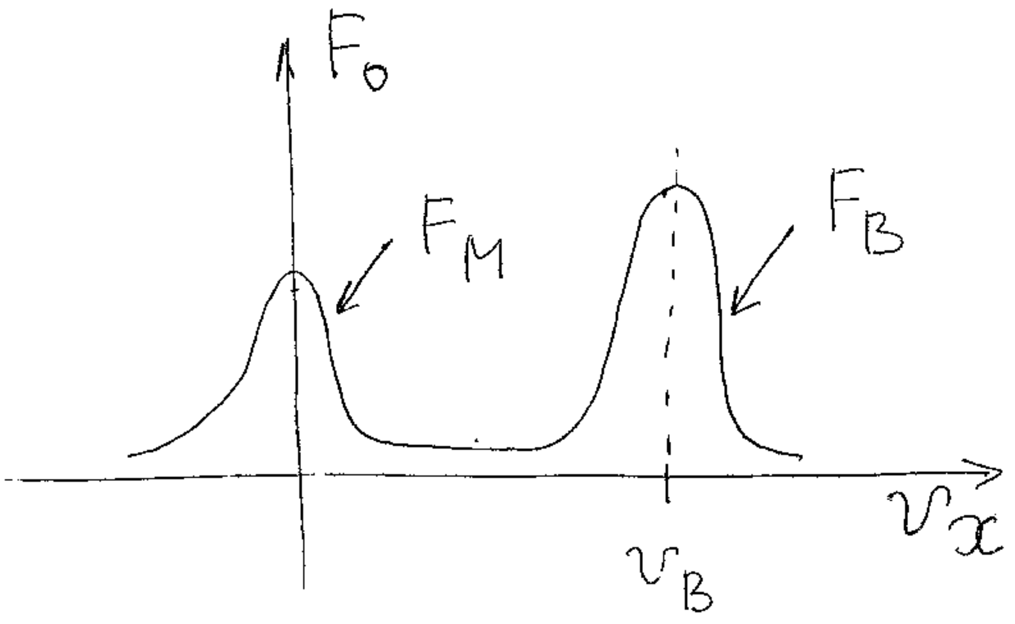
$$= \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z \left(\frac{n_M}{\pi^{3/2} v_{TM}^3} \right) \exp\left[- \frac{v_x^2 + v_y^2 + v_z^2}{v_{TM}^2} \right] =$$

Integrating in the same way as we did before we will get

$$\boxed{F_{M0} = \frac{n_M}{\sqrt{\pi}\, V_{TM}} \exp\left(- \frac{v_x^2}{v_{TM}^2} \right)}$$

$$F_0 = F_{B0} + F_{M0}$$

ii)



2. (iii) Calculating damping rate.

$$\omega_I = \frac{\pi \omega_{pe}^2}{2k^2 n_e} \left. \frac{dF_0}{dv_x} \right|_{v_x = \omega/k}$$

If $\omega/k \gg v_{TM}$

$$F_0 \approx F_{B0} = \frac{n_B}{\sqrt{\pi}\, v_{TB}} \exp\left(-\frac{(v_x - v_B)^2}{v_{TB}^2}\right)$$

$$\frac{dF_0}{dv_x} = \frac{d}{dv_x}\left(\frac{n_B}{\sqrt{\pi}\, v_{TB}} \exp\left(-\frac{(v_x - v_B)^2}{v_{TB}^2}\right)\right)$$

$$= \frac{n_B}{\sqrt{\pi}\, v_{TB}}\left(-\frac{2(v_x - v_B)}{v_{TB}^2} \exp\left(-\frac{(v_x - v_B)^2}{v_{TB}^2}\right)\right) =$$

$$= \frac{-2 n_B (v_x - v_B)}{\sqrt{\pi}\, v_{TB}^3} \exp\left(-\frac{(v_x - v_B)^2}{v_{TB}^2}\right)$$

$$\omega_I = \frac{\pi \omega_{pe}^2}{2 k^2 n_e} \left(\frac{-2 n_B (v_x - v_B)}{\sqrt{\pi}\, v_{TB}^2} \right)$$

$$\times \exp\left(- \frac{(v_x - v_B)^2}{v_{TB}^2} \right)$$

$$\boxed{\left. \omega_I \right|_{v_x = \omega/k} = \frac{-\sqrt{\pi}\, \omega_{pe}^2 n_B (\omega/k - v_B)}{k^3 n_e}}$$

3. The starting point is Vlasov equation

$$\frac{\partial f}{\partial t} + \underline{V} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{Ze}{m}\left(\underline{E} + \underline{V} \times \underline{B}\right) \cdot \frac{\partial f}{\partial \underline{V}} = -\nu_c (f - f_M)$$

Krook collision operator $f = f_M + \delta f$ where f_M is maxweilian distribution and $\delta f << f_M$.

Insert f into Vlasov equation, we get

$$\underbrace{\frac{\partial (f_M + \delta f)}{\partial t}}_{①} + \underbrace{\frac{\underline{V} \cdot \partial (f_M + \delta f)}{\partial \underline{x}}}_{②} +$$

$$+ \underbrace{\frac{Ze}{m}\left(\underline{E} + \underline{V} \times \underline{B}\right) \cdot \frac{\partial (f_M + \delta f)}{\partial \underline{V}}}_{③} = \underbrace{-\nu_c (f_M + \delta f - f_M)}_{④}$$

In ① term $\frac{\partial f}{\partial t} = 0$ because of the plasma equilibrium

④ $-\nu_c(\cancel{f_M} + \delta f - \cancel{f_M}) = -\nu_c \, \delta f$

In term 2 $\delta f = 0$ and $(f_M + \delta f) = f_M$

Also $\frac{\partial f}{\partial \underline{x}} = \frac{\partial n}{\partial \underline{x}} \frac{\partial f}{\partial n} = \frac{\partial n}{\partial x} \frac{f}{n}$

$\left(\frac{\partial f}{\partial n} = \frac{f}{n} \right)$

So, term 2

$$v \cdot \frac{\left(\partial (f_M + \delta f)\right)}{\partial \underline{x}} = \frac{f_M}{n} \underline{v} \cdot \frac{\partial n}{\partial \underline{x}}$$

In term 3 δf can't be ignored

$$\frac{\partial f}{\partial \underline{v}} = \frac{\underline{v}}{v} \frac{\partial f_M}{\partial v}$$

Also $(\underline{E} + \underline{v} \times \underline{B}) = (\underline{v} \times \underline{B})$

because $\underline{E} = 0$.

$$(\underline{v} \times \underline{B}) \frac{\partial f_M}{\partial v} = 0$$

So term 3 will be

$$\frac{Ze}{m} (\underline{v} \times \underline{B}) \cdot \frac{\partial \delta f}{\partial \underline{v}}$$

Finally taking changes in terms ①, ②, ③ and ④ Vlasov eqn will be.

$$\boxed{\frac{f_M}{n} \underline{v} \cdot \frac{\partial n}{\partial \underline{x}} + \frac{ze}{m}(\underline{v} \times \underline{B}) \cdot \frac{\delta f}{\partial \underline{v}} = -\nu_c \delta f}$$

(ii) From definition of flux

$$\Gamma = n \langle v \rangle$$

but $n = \int d^3\underline{v} f$

$$\Gamma = \int d^3\underline{v}\, \underline{v} f$$

But $f = f_M + \delta f$

$$\int f_M \underline{v}\, d^3\underline{v} = 0$$

because f_M is symmetric function.

$$\boxed{\Gamma = \int \underline{v}\, \delta f\, d^3\underline{v}}$$

(iii)

For calculating particle flux we will use Vlasov eqn as a starting point

$$\frac{f_M}{n} \underline{V} \cdot \frac{\partial n}{\partial \underline{x}} + \frac{Ze}{m} (\underline{V} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{V}} = -\nu_c \delta f$$

Gradient density $\nabla n = \begin{pmatrix} \frac{\partial n}{\partial x} \\ 0 \\ \frac{\partial n}{\partial z} \end{pmatrix}$

and velocity

$$\underline{v} = \begin{pmatrix} v_x \\ 0 \\ v_z \end{pmatrix}$$

So, $\begin{pmatrix} v_x \\ 0 \\ v_z \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial n}{\partial x} \\ 0 \\ \frac{\partial n}{\partial x} \end{pmatrix} = v_x \frac{\partial n}{\partial x} + v_z \frac{\partial n}{\partial z}$

$$\Rightarrow \frac{f_M}{n} v_x \frac{\partial n}{\partial x} + \frac{f_M}{n} v_z \frac{\partial n}{\partial z} +$$

$$+ \frac{Ze}{m} (\underline{V} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{V}} = -\nu_c \delta f$$

$$\left(\underline{v} \times \underline{B} = v_y B \hat{i} - v_x B \hat{j} \right)$$

$$(*) \quad \frac{\cancel{\partial} f_M}{n} v_x \frac{\partial n}{\partial x} + \frac{f_M}{n} v_z \frac{\partial n}{\partial z} + \omega \left(v_y \frac{\partial \delta f}{\partial v_x} - v_x \frac{\partial \delta f}{\partial v_y} \right) = -\nu_c \delta f. \quad (*)$$

We multiply eqn (*) by $\underline{v}$ separatelly for x, y, z and integrate over $\underline{v}$

$$\int d^3\underline{v}\, \frac{\cancel{\partial} f_M}{n} \frac{\partial n}{\partial x} v_x^2 + \overbrace{\int d^3 v \frac{f_M}{n} v_z v_x \frac{\partial n}{\partial z}}^{=0} +$$

$$+ \omega \int d^3\underline{v} \left(v_y v_x \frac{\partial \delta f}{\partial v_x} - v_x^2 \frac{\partial \delta f}{\partial v_y} \right) =$$

$$= -\int d^3\underline{v}\, \nu_c v_x \delta f$$

Taking into account $\Gamma_x = \int \delta f\, v_x d^3\underline{v}$

and $$\int d^3v \frac{f_M}{n}\frac{\partial n}{\partial x} v_x^2 = \frac{T}{m}\frac{\partial n}{\partial x}$$

(we need to use integrating Gaussian integrals)

we get

$$\frac{T}{m}\frac{\partial n}{\partial x} - \omega \Gamma_y = -\nu_c \Gamma_x \qquad (1)$$

We multiply $(*)$ by v_y and integrate over $\underline{v}$ space.

$$\int \overbrace{d^3\underline{v}\, \frac{f_M}{n}\frac{\partial n}{\partial x} v_x v_y}^{=0} + \int \overbrace{d^3\underline{v}\, \frac{f_M}{n} v_z v_y \frac{\partial n}{\partial z}}^{=0} +$$

$$+\omega \int d^3\underline{v} \left(v_y^2 \frac{\partial \delta f}{\partial v_x} - \frac{v_x v_y \partial \delta f}{\partial v_y} \right) =$$

$$= -\nu_c \int d^3\underline{v}\, v_y \delta f$$

We get $\omega \tau_c \Gamma_x = -\Gamma_y \qquad (2)$

Now we multiply $\underline{(*) \text{ by } v_z}$
and make the same procedures.

$$\underbrace{\int d^3\underline{v}\ \frac{f_M}{n} v_x \frac{\partial n}{\partial x} v_z}_{\overset{\shortparallel}{0}} + \underbrace{\int d^3\underline{v}\ \frac{f_M}{n} v_z^2 \frac{\partial n}{\partial z}}_{\overset{\shortparallel}{\frac{T}{m}\frac{\partial n}{\partial z}}}$$

$$+\omega \int d^3\underline{v} \left(\overbrace{v_y v_z \frac{\partial \delta f}{\partial v_x}}^{=0} - \overbrace{v_x v_z \frac{\partial \delta f}{\partial v_y}}^{0} \right)$$

$$= -\nu_c \underbrace{\int d^3\underline{v}\ v_z \delta f}_{\Gamma_z}$$

$$\Rightarrow \quad \frac{T}{m} \frac{\partial n}{\partial z} = -\nu_c \Gamma_z \qquad (3)$$

There fore taking into account our consideration about ①, ②, ③, ④ tems, Vlasov equation will be

$$\boxed{\frac{f_M}{n}\underline{v}\cdot\frac{\partial n}{\partial \underline{v}} + \frac{Ze}{m}(\underline{v}\times\underline{B})\cdot\frac{\partial \delta f}{\partial \underline{v}} = -\nu_c\,\delta f}$$

(ii) From definition of flux

$\Gamma = n \cdot \langle v \rangle$,

but $\quad n = \int f\, d^3\underline{v}$

So, $\quad \Gamma = \int \underline{v}\, f\, d^3\underline{v}$

We defined $\quad f = f_M + \delta f$

$\Gamma = \int \underline{v}(f_M + \delta f)\, d^3 v =$

$= \underbrace{\int \underline{v} f_M\, d^3 v}_{\text{this term}=0} + \int \underline{v}\, \delta f\, d^3 v$

because f_M is symmetric

So $\quad \boxed{\Gamma = \int \underline{v}\, \delta f\, d^3\underline{v}}$

Finally we got eqn (1), (2), (3)

$$\begin{cases} \frac{T}{m}\frac{\partial n}{\partial x} - \omega \Gamma_y = -\nu_c \Gamma_x \\ \omega T \Gamma_x = -\Gamma_y \\ \frac{T}{m}\frac{\partial n}{\partial z} = -\nu_c \Gamma_z \end{cases}$$

$$\nu_c = \frac{1}{\tau_c}$$

Solving the s-m of eqn (1), (2), (3)
we get

$$\begin{cases} \Gamma_x = \frac{\tau_c T}{m(1+(\omega\tau_c)^2)}\frac{\partial n}{\partial x} \\ \Gamma_y = \omega\tau_c \frac{\tau_c T}{m(1+(\omega\tau_c)^2)}\frac{\partial n}{\partial y} \\ \Gamma_z = \frac{\tau_c T}{m}\frac{\partial n}{\partial z} \end{cases}$$