Assignment 4

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1) (a) $i\omega \bar{B}_1 = \nabla \times (\bar{V}_1 \times \bar{B}_0)$

$$(\bar{V}_1 \times \bar{B}_0) = \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ V_r & V_\theta & V_z \\ 0 & B_\theta & 0 \end{pmatrix} =$$

$$= \hat{r}(-B_\theta V_z) + \hat{z}(V_r B_\theta) + \hat{\theta} \cdot 0$$

$$\nabla \times \begin{pmatrix} -B_\theta V_z \\ 0 \\ B_\theta V_r \end{pmatrix} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) & \frac{\partial}{\partial z} \\ (-B_\theta V_z) & 0 & (V_r B_\theta) \end{vmatrix}$$

We are interested only in θ component because of symmetry, as r and z components $= 0$

So, at $\hat{\theta}$ $\left(-\frac{\partial}{\partial r}(V_r B_\theta) - \frac{\partial}{\partial z}(B_\theta V_z)\right) =$

$$= -\underbrace{\frac{\partial B_\theta}{\partial z}}_{=0} V_z - B_\theta \frac{\partial V_z}{\partial z} - \frac{\partial V_r}{\partial r} B_\theta - \frac{\partial B_0}{\partial r} V_r$$

$$\nabla \cdot \bar{v} = 0$$

$$\nabla \cdot \bar{v} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r)\right) + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{= 0} + \frac{\partial v_z}{\partial z} = 0$$

$$\nabla \cdot \bar{v} = \frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \cancel{\frac{1}{r}}\frac{\partial v_z}{\partial z} = 0$$

$$\frac{\partial v_r}{\partial r} = -\frac{v_r}{r} - \frac{\partial v_z}{\partial z}$$

$$i \omega B_1 = -B_\theta \frac{\partial v_z}{\partial z} - \frac{\partial v_r}{\partial r} B_\theta - \frac{\partial B_\theta}{\partial r} v_r =$$

$$= -B_\theta \cancel{\frac{\partial v_z}{\partial z}} - \left(-\frac{v_r}{r} - \cancel{\frac{\partial v_z}{\partial z}}\right) B_\theta - \frac{\partial B_\theta}{\partial r} v_r =$$

$$= \frac{v_r}{r} B_\theta - \frac{\partial B_\theta}{\partial r} v_r = \left(\frac{B_\theta}{r} - \frac{\partial B_\theta}{\partial r}\right) v_r$$

$$\boxed{B_1 = \hat{\theta}\,\frac{1}{i\omega}\left(\frac{B_\theta}{r} - \frac{\partial B_\theta}{\partial r}\right) v_r}$$

$$B_0 B_1 + B_1 B_0 = 2 B_0 B_1 =$$

Because of symmetry

$$= 2 \begin{pmatrix} 0 \\ B_{0\theta} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ B_{1\theta} \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{0\theta} B_{1\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the formula provided below we can see at r component there is only $S_{\theta\theta}$ in the matrix.

Also r component of ∇ will be $\frac{\partial}{\partial r}$

$$\nabla \left(\mu_0 p_1 + \vec{B}_0 \cdot \vec{B}_1 \right) =$$

$$\nabla \left(\mu_0 p_1 + B_{0\theta} B_{1\theta} \right) \Big|_r = \frac{\partial}{\partial r} \left(B_0 B_{01} + \mu_0 p_1 \right)$$

$$\bar{\nabla}\cdot\bar{S} = \frac{\partial S_{rr}}{\partial r}\hat{e}_r + \frac{\partial S_{r\theta}}{\partial r}\hat{e}_\theta + \frac{\partial S_{rz}}{\partial z}\hat{e}_z$$

$$+\frac{1}{r}\left[\frac{\partial S_{\theta r}}{\partial \theta} + (S_{rr} - S_{\theta\theta})\hat{e}_r\right] +$$

$$+\frac{1}{r}\left[\frac{\partial S_{\theta\theta}}{\partial \theta} + (S_{r\theta} + S_{\theta r})\right]\hat{e}_\theta +$$

$$+\frac{1}{r}\left[\frac{\partial S_{\theta z}}{\partial \theta} + S_r\right]\hat{e}_z$$

$$+\frac{\partial S_{zr}}{\partial z}\hat{e}_r + \frac{\partial S_{z\theta}}{\partial z}\hat{e}_\theta + \frac{\partial S_{zz}}{\partial z}\hat{e}_z$$

$$\Rightarrow \nabla\cdot(2\,B_0 \cdot B_1) = -2\frac{B_\theta B_{\theta 1}}{r}$$

$$\boxed{i\omega\rho_0\mu_0 V_r = -2\frac{B_\theta B_{\theta 1}}{r} - \partial_r(B_\theta B_1 + \mu_0 p_1)}$$

c) $\nabla \cdot \bar{V} = 0$

$$\nabla \cdot V = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{r \cdot V_r}{\cancel{\partial}}\right)\right) + \underbrace{\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}}_{\overset{\shortparallel}{0}} + \frac{\partial V_z}{\partial z} = 0$$

$$\frac{\partial}{\partial r}\left(\rho_0 \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \cdot V_r\right)\right)\right) + \frac{\partial}{\partial r}\left(\rho_0 \frac{\partial V_z}{\partial z}\right) = 0$$

$$\frac{\partial V_z}{\partial z} = i k V_z$$

$$\left\{ \cancel{i}\omega \rho_0 \mu_0 V_z = -\cancel{i}k\left(B_0 B_1 + \mu_0 p_1\right)' \Rightarrow \right.$$

$$\left. \Rightarrow V_z = -\frac{k}{\omega}\frac{1}{\rho_0 \mu_0}\left(B_0 B_1 + \mu_0 p_1\right) \right\}$$

$$\frac{\partial V_z}{\partial z} = -ik\left(\frac{k}{\omega}\frac{1}{\rho_0 \mu_0}\left(B_0 B_1 + \mu_0 p_1\right)\right) =$$

$$= -\frac{k^2}{\omega \rho_0 \mu_0}\left(B_0 B_1 + \mu_0 p_1\right)$$

$$\left\{ i\omega \rho_0 \mu_0 V_r = -\frac{2 B_0 B_1}{r} - \frac{\partial}{\partial r}\left(B_0 B_1 + \mu_0 p_1\right) \right.$$

$$\frac{\partial}{\partial r}\left(B_0 B_1 + \mu_0 p_1\right) = -\frac{2 B_0 B_1}{r} - i\omega \rho_0 \mu_0 V_r \}$$

$$\frac{\partial}{\partial r}\frac{\partial V_z}{\partial z} = \frac{-k^2}{\omega \rho_0 \mu_0}\frac{\partial}{\partial r}\left(B_0 B_1 + \mu_0 p_1\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial r}\frac{\partial V_z}{\partial z}\right) = \frac{-k^2}{\omega \rho_0 \mu_0}\left(-\frac{2 B_0 B_1}{r} - i\omega \rho_0 \mu_0 V_r\right)$$

$$\left\{ B_1 = \frac{1}{i\omega}\left(\frac{B_0}{r} - \frac{\partial}{\partial r} B_\theta\right) V_r \right\}$$

$$\left(\frac{\partial}{\partial r}\frac{\partial V_z}{\partial z}\right) = \frac{-k^2}{\omega \rho_0 \mu_0}\left(-\frac{2 B_0}{r}\left(\frac{1}{i\omega}\left(\frac{B_\theta}{r} - \frac{\partial}{\partial r} B_\theta\right)\right) V_r - i\omega \rho_0 \mu_0 V_r\right) =$$

$$= \frac{-k^2}{\omega \rho^2 \mu_0}\left(-\frac{2 B_0}{r}\left(\frac{1}{i\omega}\left(\frac{B_\theta}{r} - \frac{\partial B_\theta}{\partial r}\right)\right) - i\omega \rho_0 \mu_0\right) V_r$$

$$\frac{\partial^2 V_z}{\partial r \partial z} = -\frac{k^2}{\omega \rho^2 \mu_0}\left(-\frac{2B_\theta}{r}\left(\frac{1}{i\omega}\left(\frac{B_\theta}{r} - \frac{\partial B}{\partial r}\right.\right.\right.$$

$$\left. - i\omega \rho_0 \mu_0\right) V_r =$$

$$= k^2 f(r, \omega, B_\theta)$$

$$\boxed{\begin{aligned} &\rho_0 \frac{\partial}{\partial r}\left(\nabla \cdot \bar{V}\right) = \\ &= \frac{\partial}{\partial r}\left(\rho_0 \frac{1}{r}\frac{\partial}{\partial r}(r V_r)\right) + k^2 f(r, \omega^2, B_\theta) = 0 \end{aligned}}$$