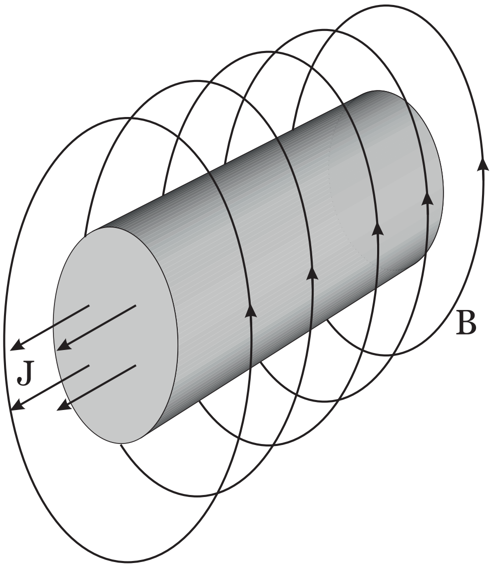
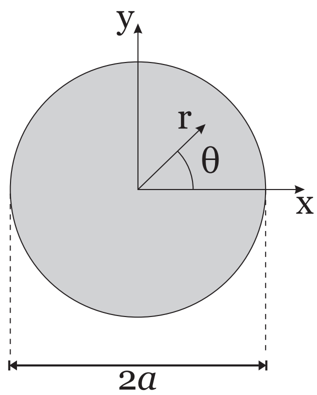
Fusion: MCF - Homework #5

1 In this problem, we will analyse the $m = 0$ stability of the straight z-pinch shown below. The plasma is assumed to obey the usual z-pinch equilibrium

$$\partial_r \left(p + \frac{B_{\theta 0}^2}{2\mu_0} \right) + \frac{B_{\theta 0}^2}{\mu_0 r} = 0,$$

and there is no equilibrium flow. Modes of the usual $\mathbf{v}_1 = \mathbf{V}(r) \exp{(ikz + i\omega t)}$ form will be examined for their stability. For simplicity, we will make the assumption that the plasma is incompressible, i.e. $\nabla \cdot \mathbf{v} = 0$, and assume a simple "rigid surface" boundary condition $V_r(a) = 0$, where a is the column radius.





(a) The perturbed magnetic field can be determined from $i\omega \mathbf{B}_1 = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$. Using this relation, show that

$$\mathbf{B}_1 = \hat{\theta} \frac{1}{i\omega} \left(\frac{B_\theta}{r} - \partial_r B_\theta \right) V_r$$

[4]

(b) The momentum equation can be written in the form

$$i\omega \rho_0 \mu_0 \mathbf{V} = \nabla \cdot (\mathbf{B}_0 \mathbf{B}_1 + \mathbf{B}_1 \mathbf{B}_0) - \nabla (\mu_0 p_1 + \mathbf{B}_0 \cdot \mathbf{B}_1),$$

where $\mathbf{B}_0 \mathbf{B}_1$ is the dyadic product of $\mathbf{B}_0$ and $\mathbf{B}_1$. By evaluating the terms on the right, show that the momentum equation becomes

$$i\omega \rho_0 \mu_0 V_r = -2 \frac{B_\theta B_{\theta 1}}{r} - \partial_r (B_\theta B_{\theta 1} + \mu_0 p_1)$$

and

$$i\omega \rho_0 \mu_0 V_z = -ik (B_\theta B_{\theta 1} + \mu_0 p_1)$$

continued

[4]

(c) Combine the results of parts (a) and (b) with the incompressibility constraint to get a single, second order differential equation for $\mathbf{V}(r)$. Put your result in the form

$$\partial_r \left(\rho_0 \frac{1}{r} \partial_r (r V_r) \right) + k^2 f \left(r, \omega^2 \right) V_r = 0$$

[4]

(d) By multiplying the result you just obtained by rV_r and integrating from $r = 0$ to $r = a$ (where V_r is assumed to vanish), show that the plasma will be unstable if

$$\partial_r B_\theta - \frac{B_\theta}{r} > 0$$

for $0 < r < a$. [4]

If you try to interpret the condition we just found in part (d) in terms of single particle drifts you run into an apparent paradox because this would require the net particle drift (grad-B and curvature) to be in the $-z$ direction, which would be stabilising based on the picture developed in class. The paradox can be resolved by calculating the energy involved in the perturbation. One can show that this is proportional to

$$\frac{2B_\theta^2}{r} + \mu_0 \partial_r p.$$

Now it is interesting to note that *both* of these terms arise from the finite curvature of the magnetic field, so arguments based *only* on "good" or "bad" curvature are incomplete and one must consider stabilising as well as destabilising effects (as explained in our discussion of the energy principle). Nevertheless, a region of bad curvature is *necessary* for instability.

(e) If

$$\frac{2B_\theta^2}{r} + \mu_0 \partial_r p < 0$$

for $0 < r < a$, then the energy argument shows that the plasma will be unstable. Show that this condition is equivalent to that found in part (d). [4]

End of paper