Taller

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Sección 1.5.7

2. Considere que

$$\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

$$\mathbf{a} = \mathbf{a}(\mathbf{r}) = a^i(x,y,z)\,\hat{\imath}_i \qquad \mathbf{b} = \mathbf{b}(\mathbf{r}) = b^i(x,y,z)\,\hat{\imath}_i$$

$$\phi = \phi(\mathbf{r}) = \phi(x,y,z) \qquad \psi = \psi(\mathbf{r}) = \psi(x,y,z)$$

Utilizando la notación de índices demuestre las siguientes identidades vectoriales:

2a $\nabla(\phi\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi$

Regla producto para derivada

$$(\nabla(\phi\psi))^i = \partial^i(\phi\psi) = (\partial^i\phi)\psi + (\partial^i\psi)\phi$$
$$= (\nabla\phi)\psi + (\nabla\psi)\phi$$

2d $\nabla\cdot(\nabla\times a)$ ¿Qué puede decir de $\nabla\times(\nabla\cdot a)$?

$$(\nabla\cdot(\nabla\times a)) = \partial^i(\nabla\times a)_i = \partial^i\,\varepsilon_{ijk}\,\partial^j a^k$$
$$= \varepsilon_{ijk}\,\partial^i\partial^j a^k$$
$$= \partial^1\partial^2 a^3 + \partial^3\partial^1 a^2 + \partial^2\partial^3 a^1$$
$$- \partial^3\partial^2 a^1 - \partial^1\partial^3 a^2 - \partial^2\partial^1 a^3$$

Teorema de derivadas cruzadas o Clairaut

$\partial^i\partial^j a^k = \partial^j\partial^i a^k$

$$= \partial^1\partial^2 a^3 + \partial^3\partial^1 a^2 + \partial^2\partial^3 a^1 - \partial^2\partial^3 a^1 - \partial^3\partial^1 a^2 - \partial^1\partial^2 a^3$$
$$= 0$$

Se cancelan los términos iguales

→ $\nabla\cdot(\nabla\times a) = 0$ como es de esperarse pues uno de los factores se repite en el triple producto mixto.

→ ¿$\nabla\times(\nabla\cdot a)$?

$= \varepsilon_{ijk}\,\partial^j(\nabla\cdot a)_k$ → Pero $\nabla\cdot a$ no tiene componente k-ésima pues no es un vector, si no, un escalar

→ $\nabla\times(\nabla\cdot a)$ no tiene sentido pues este no es el orden correcto de aplicar los operadores. Otra forma sería $(\nabla\times\nabla)\cdot a = \nabla\cdot(\nabla\times a) = 0$ por propiedades de producto mixto.

2f $\nabla\times(\nabla\times a) = \nabla(\nabla\cdot a) - \nabla^2 a$

$\hookrightarrow$ $(\nabla\times(\nabla\times a))^i = \varepsilon^{ijk}\,\partial_j(\nabla\times a)_k$

$= \varepsilon^{ijk}\,\partial_j\,\varepsilon_{kmn}\,\partial^m a^n$

$= \varepsilon^{ijk}\,\varepsilon_{mnk}\,\partial_j\partial^m a^n$

$= (\delta^i_m\delta^j_n - \delta^i_n\delta^j_m)\,\partial_j\partial^m a^n$

$= \delta^i_m\delta^j_n\,\partial_j\partial^m a^n - \partial^i_n\delta^j_m\,\partial_j\partial^m a^n$

Teorema de Clairâut $\hookrightarrow$

$= \partial_j\partial^i a^j - \partial_j\partial^j a^i$

$= \partial^i(\partial_j a^j) - (\partial_j\partial^j)\,a^i$

$= \partial^i(\nabla\cdot a) - (\nabla\cdot\nabla)\,a^i$

$= \nabla(\nabla\cdot a) - \nabla^2 a$

$\rightarrow$ $\nabla\times(\nabla\times a) = \nabla(\nabla\cdot a) - \nabla^2 a$

Sección 1.6.6

2. Demuestre:

(a) $\cos 3\alpha = \cos^3\alpha - 3\cos\alpha\,\mathrm{sen}^2\alpha$
(b) $\mathrm{sen}\,3\alpha = 3\cos^2\alpha\,\mathrm{sen}\,\alpha - \mathrm{sen}^3\alpha$

$\cos 3\alpha + i\,\mathrm{sen}\,3\alpha = e^{i3\alpha} = (e^{i\alpha})^3 = (\cos\alpha + i\,\mathrm{sen}\,\alpha)^3$

$= \cos^3\alpha + 3\cos^2\alpha(i\,\mathrm{sen}\,\alpha) + 3\cos\alpha(i\,\mathrm{sen}\,\alpha)^2 + (i\,\mathrm{sen}\,\alpha)^3$

$= \cos^3\alpha - 3\cos\alpha\,\mathrm{sen}^2\alpha + 3\cos^2\alpha\,\mathrm{sen}\,\alpha\, i - \mathrm{sen}^3\alpha\, i$

Como tanto del lado derecho como izquierdo tenemos un número complejo, ambos serán iguales cuando sus partes reales e imaginarias sean iguales:

$$\Rightarrow \begin{cases} \cos 3\alpha = \cos^3\alpha - 3\cos\alpha\,\mathrm{sen}^2\alpha \\ \mathrm{sen}\,3\alpha = 3\cos^2\alpha\,\mathrm{sen}\,\alpha - \mathrm{sen}^3\alpha \end{cases}$$

Demostrando ambos incisos

5. Encuentre las raíces de

a) $(2i)^{\frac{1}{2}}$ b) $(1-\sqrt{3}i)^{1/2}$ c) $(-1)^{1/3}$ d) $8^{\frac{1}{6}}$ e) $(-8-8\sqrt{3}i)^{1/4}$

a) $(2i)^{\frac{1}{2}} = \left(2e^{i\left(\frac{\pi}{2}+2\pi n\right)}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} e^{i\left(\frac{\pi}{4}+\pi n\right)}$

Sus dos raíces serán cuando $n=0$, $n=1$

- $z_0 = 2^{\frac{1}{2}} e^{i\left(\frac{\pi}{4}\right)} = 2^{\frac{1}{2}}\left[\cos\frac{\pi}{4} + i\,\mathrm{Sen}\frac{\pi}{4}\right]$

  $= 2^{\frac{1}{2}}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right]$

- $z_1 = 2^{\frac{1}{2}} e^{i\left(\frac{\pi}{4}+\pi\right)} = 2^{\frac{1}{2}}\left[\cos\left(\frac{5\pi}{4}\right) + i\,\mathrm{Sen}\left(\frac{5\pi}{4}\right)\right]$

  $= 2^{\frac{1}{2}}\left[-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right]$

→ z_0, z_1 son iguales pero opuestas en signo algebraico

b) $(1-\sqrt{3}i)^{\frac{1}{2}} = \left(\sqrt{1+3}\; e^{i\tan^{-1}\left(-\frac{\sqrt{3}}{1}\right)}\right)^{1/2}$

$= \left(2e^{i\left(-\frac{\pi}{3}+2\pi n\right)}\right)^{1/2} = \sqrt{2}\; e^{i\left(-\frac{\pi}{6}+\pi n\right)}$

$n=0$ $\quad z_0 = \sqrt{2}\, e^{i\left(-\frac{\pi}{6}\right)} = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\,\mathrm{Sen}\left(-\frac{\pi}{6}\right)\right)$

$= \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$n=1$ $\quad z_1 = \sqrt{2}\, e^{i\left(-\frac{\pi}{6}+\pi\right)} = \sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right) + i\,\mathrm{Sen}\left(\frac{5\pi}{6}\right)\right)$

$= \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

c) $(-1)^{\frac{1}{3}} = \left(e^{i(\pi+2\pi n)}\right)^{\frac{1}{3}} = \left(e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}n\right)}\right)$

$n=0$ $\quad \bullet\, z_0 = e^{i\left(\frac{\pi}{3}\right)} = \cos\frac{\pi}{3} + i\,\mathrm{Sen}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$n=1$ $\quad \bullet\, z_1 = e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)} = \cos\pi + i\,\mathrm{Sen}\,\pi = -1 + i\cdot 0 = -1$

$n=2$ $\quad \bullet\, z_2 = e^{i\left(\frac{\pi}{3}+\frac{4\pi}{3}\right)} = \cos\left(\frac{5\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

d) $8^{\frac{1}{6}} = \left(8e^{i2\pi n}\right)^{\frac{1}{6}} = 8^{\frac{1}{6}} e^{i\frac{\pi}{3}n}$

$n=0$ $\quad \bullet\, z_0 = 8^{\frac{1}{6}}e^0 = 8^{\frac{1}{6}}$

- $z_1 = 8^{\frac{1}{6}} e^{i\frac{\pi}{3}} = 8^{\frac{1}{6}}\left(\cos\frac{\pi}{3} + i\,\mathrm{Sen}\frac{\pi}{3}\right) = 8^{\frac{1}{6}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
- $z_2 = 8^{\frac{1}{6}} e^{i\left(\frac{2\pi}{3}\right)} = 8^{\frac{1}{6}}\left(\cos\left(\frac{2\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{2\pi}{3}\right)\right) = 8^{\frac{1}{6}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
- $z_3 = 8^{\frac{1}{6}} e^{i\pi} = 8^{\frac{1}{6}}(\cos\pi + i\,\mathrm{Sen}\,\pi) = 8^{\frac{1}{6}}(-1) = -8^{\frac{1}{6}}$
- $z_4 = 8^{\frac{1}{6}} e^{i\left(\frac{4\pi}{3}\right)} = 8^{\frac{1}{6}}\left(\cos\left(\frac{4\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{4\pi}{3}\right)\right) = 8^{\frac{1}{6}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
- $z_5 = 8^{\frac{1}{6}} e^{i\left(\frac{5\pi}{3}\right)} = 8^{\frac{1}{6}}\left(\cos\left(\frac{5\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{5\pi}{3}\right)\right) = 8^{\frac{1}{6}}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

e) $(-8-8\sqrt{3}i)^{\frac{1}{4}} = \left[\sqrt{64+64\cdot 3}\; e^{i\tan^{-1}\left(\frac{-8\sqrt{3}}{-8}\right)}\right]^{1/4}$

$= \left[16\; e^{i\left(\frac{\pi}{3}+2\pi n\right)}\right]^{1/4}$

$= 2\; e^{i\left(\frac{\pi}{12}+\frac{\pi}{2}n\right)}$

$n=0 \quad z_0 = 2\,e^{i\left(\frac{\pi}{12}\right)} \approx 1.932 + 0.518i$

$z_1 = 2\,e^{i\left(\frac{7\pi}{12}\right)} \approx -0.518 + 1.932i$

$z_2 = 2\,e^{i\left(\frac{13\pi}{12}\right)} \approx -1.932 - 0.518i$

$z_3 = 2\,e^{i\left(\frac{19\pi}{12}\right)} \approx 0.518 - 1.932i$

→ Como comprobamos, cada raíz es diferente

6. Demuestre que:

a) $\mathrm{Log}(-ie) = 1 - \frac{\pi}{2}i$

↳ $\mathrm{Log}\left(e\; e^{i\left(\frac{3\pi}{2}+2\pi n\right)}\right) = \ln|e| + i\left(-\frac{\pi}{2}+2\pi n\right)$

$= 1 - \frac{\pi}{2}i$ para $n=0$

b) $\mathrm{Log}(1-i) = \frac{1}{2}\ln(2) - \frac{\pi}{4}i$

↳ $\mathrm{Log}\left(\sqrt{2}\; e^{i\left(-\frac{\pi}{4}+2\pi n\right)}\right) \underset{\substack{\uparrow \\ \text{para } n=0}}{=} \ln\sqrt{2} - i\frac{\pi}{4} = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

c) $\mathrm{Log}(e) = 1 + 2\pi n i$

↳ $\mathrm{Log}\left(e\; e^{i2\pi n}\right) = \ln e + i\,2\pi n = 1 + 2\pi n i$

d) $\mathrm{Log}(i) = \left(2n+\frac{1}{2}\right)\pi i$

↳ $\mathrm{Log}\left(1\cdot e^{i\left(\frac{\pi}{2}+2\pi n\right)}\right) = \overset{0}{\ln 1} + i\left(\frac{\pi}{2}+2\pi n\right)$

$= \pi i\left(2n+\frac{1}{2}\right)$