Homework #2

Aleks Lyubenov
MATH/CSCI 360 - Mathematical Modeling
Lake Forest College

April 20, 2020

Contents

1 Section 1: Set Up

1.1 Building Matrices

Defining the Basis Functions:

Let $\mathcal{P}^q(I)$ be the vector space defined by the set of polynomials $p(x) = \sum_{i=0}^{q} c_i x^i,\ x \in I$

We need to define two vector spaces of piece-wise polynomials over the interval $I = (a, b)$

- Discontinuous piece-wise polynomials on I: $W_h^{(q)} = \{v : v \in \mathcal{P}^q(I)\}$
- Continuous piece-wise polynomials on I: $V_h^{(q)} = \{v \in W_h^{(q)} : v \in \mathcal{C}(I)\}$

We would like to define a basis for $W_h^{(q)}$ in terms of the following functions:

$$\lambda_{i,0}(x) = \frac{x_i - x}{x_i - x_{i-1}} = \frac{x_i - x}{h_i}$$
$$\lambda_{i,1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} = \frac{x - x_{i-1}}{h_i}$$

We can now define the basis for $W_h^{(q)}$ by:

$$\phi_{(i,j)}(x) = \begin{cases} 0 & x \neq [x_{i-1}, x_i] \\ \lambda_{i,j} & x \in [x_{i-1}, x_i] \end{cases}$$

with $i = \{1, 2, ..., m+1\}$ and $j = 0, 1$.

For $V_h^{(q)}$, the basis functions we define need to be continuous.

$$\phi_i(x) = \begin{cases} 0 & x \neq [x_{i-1}, x_{i+1}] \\ \lambda_{i,1} & x \in [x_{i-1}, x_i] \\ \lambda_{i+1,0} & x \in [x_i, x_{i+1}] \end{cases}$$

Note: The matrices A and R are sparse, as $A_{ij} = R_{ij} = 0 \; \forall i, j$ such that $|i - j| > 1$

1.1.1 Constructing the matrix A:

$$
\begin{aligned}
A_{ii} = \langle \phi_i', \phi_i' \rangle &= \int_a^b (\phi_i')^2(x)dx = \int_{x_{i-1}}^{x_i} (\lambda_{i,1}')^2(x)dx + \int_{x_i}^{x_{i+1}} (\lambda_{i+1,0}')^2(x)dx \\
&= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{1}{h_{i+1}}\right)^2 dx = \frac{1}{h_i} + \frac{1}{h_{i+1}}
\end{aligned}
$$

$$
A_{ii+1} = \langle \phi_i', \phi_{i+1}' \rangle = \int_a^b \phi_i'(x)\phi_{i+1}'(x)dx = \int_{x_i}^{x_{i+1}} \frac{-1}{h_{i+1}} \frac{1}{h_{i+1}} dx = \frac{-1}{h_{i+1}}
$$

$$
A_{ii-1} = \langle \phi_i', \phi_{i-1}' \rangle = \int_a^b \phi_i'(x)\phi_{i-1}'(x)dx = ... = \frac{-1}{h_i}
$$

1.1.2 Constructing the matrix R:

$$
\begin{aligned}
R_{ii} = \langle \phi_i, \phi_i \rangle &= \int_a^b \phi_i^2(x)dx = \int_{x_{i-1}}^{x_i} \lambda_{i,1}^2(x)dx + \int_{x_i}^{x_{i+1}} \lambda_{i+1,0}^2(x)dx \\
&= \int_{x_{i-1}}^{x_i} \frac{(x - x_{i-1})^2}{h_i^2} dx + \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - x)^2}{h_{i+1}^2} dx \\
&= \frac{1}{h_i^2} \left[\frac{(x - x_{i-1})^3}{3} \right]_{x_{i-1}}^{x_i} + \frac{1}{h_{i+1}^2} \left[\frac{-(x_{i+1} - x)^3}{3} \right]_{x_i}^{x_{i+1}} = \frac{h_i}{3} + \frac{h_{i+1}}{3}
\end{aligned}
$$

$$
\begin{aligned}
R_{ii+1} = \langle \phi_i, \phi_{i+1} \rangle &= \int_a^b \phi_i(x)\phi_{i+1}(x)dx \\
&= \int_{x_i}^{x_{i+1}} \lambda_{i+1,0}(x)\lambda_{i+1,1}(x)dx \\
&= \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - x)}{h_{i+1}} \frac{(x - x_i)}{h_{i+1}} dx \\
&= \frac{1}{h_{i+1}^2} \int_{x_i}^{x_{i+1}} (x_{i+1}x - x_{i+1}x_i - x^2 + xx_i)dx \\
&= \frac{1}{h_{i+1}^2} \left[\frac{x_{i+1}x^2}{2} - x_{i+1}x_i x - \frac{x^3}{3} + \frac{x^2 x_i}{2} \right]_{x_i}^{x_{i+1}} \\
&= \frac{1}{6h_{i+1}^2} (x_{i+1}^3 - 3x_{i+1}^2 x_i + 3x_{i+1}x_i^2 - x_i^3) \\
&= \frac{1}{6h_{i+1}^2} (x_{i+1} - x_i)^3 = \frac{h_{i+1}}{6}
\end{aligned}
$$

$$
R_{ii-1} = \langle \phi_i, \phi_{i-1} \rangle = \int_a^b \phi_i(x)\phi_{i-1}(x)dx = ... = \frac{h_i}{6}
$$

1.2 Tests

1.2.1 Testing R:

$$(Ru)_i = \sum_{j=0}^{n} \langle \phi_i, \phi_j \rangle_{L^2} \cdot u_j = \langle \sum_{j=0}^{n} u_j \phi_j, \phi_i \rangle_{L^2}$$

$$\sum_{i=0}^{n} (Ru)_i = \sum_{i=0}^{n} \langle \sum_{j=0}^{n} u_j \phi_j, \phi_i \rangle_{L^2} = \langle \sum_{j=0}^{n} u_j \phi_j, \sum_{i=0}^{n} \phi_i \rangle_{L^2} = \langle u, 1 \rangle_{L^2} = \int_a^b u(x)dx$$

1.2.2 Result of Test:

The chosen function: $u(x) = (3 - 5\pi + \pi^2)x + (x^2 - 4x)sin(x) - 1$

$$\begin{aligned}
\sum_{i=0}^{n} (Ru)_i &= -23.845073804269454 \\
\int_a^b u(x)dx &= \int_a^b (3 - 5\pi + \pi^2)x + (x^2 - 4x)sin(x) - 1dx \\
&= \pi^2 - 5\pi + \frac{\pi^4 - 5\pi^3 + 3\pi^2}{2} - 4 \approx -23.84509\ldots
\end{aligned}$$

1.2.3 Testing A:

$$(Au)_i = \sum_{j=0}^{n} \langle \phi_i', \phi_j' \rangle_{L^2} \cdot u_j = \langle \sum_{j=0}^{n} u_j \phi_j', \phi_i' \rangle_{L^2}$$

$$\sum_{i=0}^{n} (Au)_i = \sum_{i=0}^{n} \langle \sum_{j=0}^{n} u_j \phi_j', \phi_i' \rangle_{L^2} = \langle \sum_{j=0}^{n} u_j \phi_j', \sum_{i=0}^{n} \phi_i' \rangle_{L^2} = \langle \sum_{j=0}^{n} u_j \phi_j', 0 \rangle_{L^2} = 0$$

1.2.4 Result of Test:

The chosen function: $u(x) = (3 - 5\pi + \pi^2)x + (x^2 - 4x)sin(x) - 1$

$$\sum_{i=0}^{n} (Au)_i = -1.2079226507921703e^{-12} \approx 0$$

2 Section 2: Steady State Heat Equation $-u_{xx}(x) = f(x)$

2.1 Variational Forms:

2.1.1 Dual Dirichlet

Differential Equation (D): $-u'' = f$ on (a, b) with $u(a) = c_1,\ u(b) = c_2$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a, b) : v(a) = c_1,\ v(b) = c_2\}$ such that $a(u, v) = b(v)\ \forall v \in H_0^1(a, b)$ where:

$$a(u, v) = \int_a^b u'v'\,dx \text{ and } b(v) = \int_a^b fv\,dx$$

Let V_h be the space of continuous piece-wise linear functions, that satisfy the boundary conditions specified in the problem above.
$V_h = \{v \in V_h^{(1)} : v(a) = c_1,\ v(b) = c_2\}$

We can represent any function in this space over a mesh with n internal nodes as:
$v(x) = \sum_{i=1}^{n} v_i\phi_i(x) \quad \forall v \in V_h$ where $v_i = v(x_i)$ and ϕ_i is a nodal basis $\forall i = \{1, 2, ..., n\}$
The solution u in this space can be represented as:
$u(x) = \sum_{j=1}^{n} u_j\phi_j(x)$, with $u_j = u(x_j)$

Substituting these linear combinations into our linear and bilinear functionals above, we obtain:
$$\int_a^b u'v'dx = \int_a^b fvdx \Rightarrow \sum_{j=1}^{n} u_j \int_a^b \phi_j'(x)\phi_i'(x)dx = \int_a^b f(x)\phi_i(x)dx, \quad i = \{1, 2, ..., n\}$$
This corresponds to the matrix equation $Ax = b$ with $A_{ij} = \langle \phi_j', \phi_i' \rangle,\ x_j = u_j$ and $b_i = \langle f, \phi_i \rangle$
To implement the boundary conditions:

- set $A_{0,0} = 1,\ A_{0,1}\ ...\ A_{0,n} = 0$ and $b_0 = c_1$
- set $A_{n,n} = 1,\ A_{n,0}\ ...\ A_{n,n-1} = 0$ and $b_n = c_2$

2.1.2 Dirichlet Neumann

Differential Equation (D): $-u'' = f$ on (a, b) with $u(a) = c_1,\ u'(b) = c_2$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a,b) : v(a) = c_1\}$ such that
$a(u, v) = b(v)\ \forall v \in V_0 = \{v \in H^1(a,b) : v(a) = 0\}$ where:

$$a(u,v) = \int_a^b u'v'\,dx \quad \text{and} \quad b(v) = \int_a^b fv\,dx + c_2 v(b)$$

Let V_h be the space of continuous piece-wise linear functions, that satisfy the boundary conditions specified in the problem above.
$V_h = \{v \in V_h^{(1)} : v(a) = c_1\}$

We can represent any function in this space over a mesh with n internal nodes as:
$v(x) = \sum_{i=1}^{n} v_i \phi_i(x) \quad \forall v \in V_h$ where $v_i = v(x_i)$ and ϕ_i is a nodal basis $\forall i = \{1, 2, ..., n\}$
The solution u in this space can be represented as:
$u(x) = \sum_{j=1}^{n} u_j \phi_j(x)$, with $u_j = u(x_j)$

Substituting these linear combinations into our linear and bilinear functionals above, we obtain:
$\int_a^b u'v'dx = \int_a^b fvdx + c_2 v(b) \Rightarrow \sum_{j=1}^{n} u_j \int_a^b \phi_j'(x)\phi_i'(x)dx = \int_a^b f(x)\phi_i(x)dx + c_2 v(b),$
$i = \{1, 2, ..., n\}$

This corresponds to the matrix equation $Ax = b$ with $A_{ij} = \langle \phi_j', \phi_i' \rangle,\ x_j = u_j$ and
$b_i = \langle f, \phi_i \rangle, b_n = \langle f, \phi_n \rangle + c_2$
To implement the boundary conditions:

- set $A_{0,0} = 1,\ A_{0,1}\ ...\ A_{0,n} = 0$ and $b_0 = c_1$
- add c_2 to b_n

2.1.3 Neumann Dirichlet

Differential Equation (D): $-u'' = f$ on (a, b) with $u'(a) = c_1,\ u(b) = c_2$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a,b) : v(b) = c_2\}$ such that $a(u,v) = b(v)\ \forall v \in V_0 = \{v \in H^1(a,b) : v(b) = 0\}$ where:

$$a(u,v) = \int_a^b u'v'\,dx \quad \text{and} \quad b(v) = \int_a^b fv\,dx - c_1 v(a)$$

Let V_h be the space of continuous piece-wise linear functions, that satisfy the boundary conditions specified in the problem above.
$V_h = \{v \in V_h^{(1)} : v(a) = c_1\}$

We can represent any function in this space over a mesh with n internal nodes as:
$v(x) = \sum_{i=1}^{n} v_i \phi_i(x) \quad \forall v \in V_h$ where $v_i = v(x_i)$ and ϕ_i is a nodal basis $\forall i = \{1, 2, ..., n\}$
The solution u in this space can be represented as:
$u(x) = \sum_{j=1}^{n} u_j \phi_j(x)$, with $u_j = u(x_j)$

Substituting these linear combinations into our linear and bilinear functionals above, we obtain:
$\int_a^b u'v'dx = \int_a^b fvdx + c_2 v(b) \Rightarrow \sum_{j=1}^{n} u_j \int_a^b \phi_j'(x)\phi_i'(x)dx = \int_a^b f(x)\phi_i(x)dx - c_1 v(a),$
$i = \{1, 2, ..., n\}$

This corresponds to the matrix equation $Ax = b$ with $A_{ij} = \langle \phi_j', \phi_i' \rangle,\ x_j = u_j$ and $b_i = \langle f, \phi_i \rangle, b_0 = \langle f, \phi_n \rangle - c_1$
To implement the boundary conditions:

- set $A_{n,n} = 1,\ A_{n,0}\ ...\ A_{n,n-1} = 0$ and $b_n = c_2$
- subtract c_1 from b_0

2.2 Test Case:

$$\begin{aligned}\text{The exact solution } &: u(x) = (3 - 5\pi + \pi^2)x + (x^2 - 4x)sin(x) - 1 \\ \text{The RHS function } &: f(x) = (8 - 4x)cos(x) - (2 + 4x - x^2)sin(x) \\ \text{Domain } &: x \in (0, \pi)\end{aligned}$$

The solution is computed for $n = \{100, 200, ..., 1500\}$ for each of the three types of boundary conditions:

- Dual Dirichlet:
  $u(0) = -1$
  $u(\pi) = -1 + 3\pi - 5\pi^2 + \pi^3$

- Dirichlet Neumann:
  $u(0) = -1$
  $u'(\pi) = 3 - \pi$

- Neumann Dirichlet:
  $u'(0) = 3 - 5\pi + \pi^2$
  $u(\pi) = -1 + 3\pi - 5^2 + \pi^3$

2.2.1 Accuracy Metrics:

L^2 Norm:

$$\|u\|_{L^2} = \sqrt{\sum_{i=0}^{n} Ru^2} \quad \text{where } u^2 \text{ is the element-wise square of the vector } u.$$

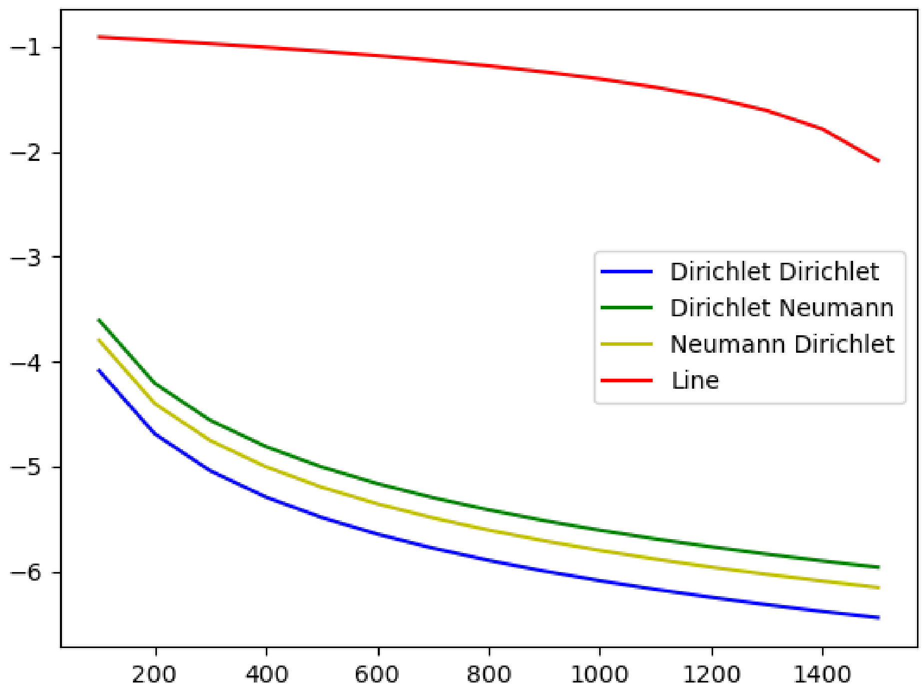
Relative Error:

$$\frac{\|u_{comp} - u_{exact}\|_{L^2}}{\|u_{exact}\|_{L^2}} = \frac{\sqrt{\sum_{i=0}^{n} R(u_{comp} - u_{exact})^2}}{\sqrt{\sum_{i=0}^{n} R(u_{exact})^2}}$$

2.2.2 Test Case Results:

Below is a plot showing:

- $log_{10}(error)$ for the **relative error** of each type of **boundary condition**
- $log_{10}(line)$ where $line$ is a linear equation with a slope of -1
- n, the number of **nodes** on the x axis



Below is the **log slope** and the **regular slope** of each curve (calculated at $n = 1500$):

$$log\ slope = \frac{log_{10}(error_n) - log_{10}(error_{n-1})}{log_{10}(n) - log_{10}(n-1)}$$

$$regular\ slope = \frac{error_n - error_{n-1}}{n - (n-1)}$$

```
Log Slope: Dirichlet Dirichlet =  -2.001140455288914
Log Slope: Dirichlet Neumann =  -1.9969029813534258
Log Slope: Neumann Dirichlet =  -2.0237282575936693

Log Slope: Line =  -10.046649249713294
Reg Slope: Line =  -1.0

Reg Slope: Dirichlet Dirichlet =  -0.0005996061847146006
Reg Slope: Dirichlet Neumann =  -0.0005983365009337493
Reg Slope: Neumann Dirichlet =  -0.0006063742183752296
```

3 Section 3: Time Dependent Heat Equation: $u_t(x,t) - u_{xx}(x,t) = f(x,t)$

3.1 Variational Forms

First, we approximate the time derivative:

$u_t(x,t) = \frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t}$

Our equation now becomes:

$\frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t} - u_{xx}(x,t_j) = f(x,t_j) \Rightarrow u(x,t_j) - \Delta t u_{xx}(x,t_j) = u(x,t_{j-1}) + \Delta t f(x,t_j)$

3.1.1 Dual Dirichlet

Differential Equation (D): $-u'' = f$ on (a,b) with $u(a) = c_1,\ u(b) = c_2$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a,b) : v(a) = g_1(t),\ v(b) = g_2(t)\}$ such that $a(u,v) = b(v)\ \forall v \in H_0^1(a,b)$ where:

$$\frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t} - u_{xx}(x,t_j) = f(x,t_j) \Rightarrow u(x,t_j) - \Delta t u_{xx}(x,t_j) = u(x,t_{j-1}) + \Delta t f(x,t_j)$$
$$\Rightarrow \int_a^b u^j v\, dx - \Delta t \int_a^b u_x^j v_x\, dx = \int_a^b u^{j-1} v\, dx + \Delta t \int_a^b f^j v\, dx + [u_x^j v]_a^b$$

$$a(u,v) = \int_a^b u^j v\, dx - \Delta t \int_a^b u_x^j v_x\, dx \ \text{ and } \ b(v) = \int_a^b u^{j-1} v\, dx + \Delta t \int_a^b f^j v\, dx$$

3.1.2 Dirichlet Neumann

Differential Equation (D): $-u'' = f$ on (a,b) with $u(a) = g_1(t),\ u'(b) = g_2(t)$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a,b) : v(a) = g_1(t)\}$ such that $a(u,v) = b(v)\ \forall v \in V_0 = \{v \in H^1(a,b) : v(a) = 0\}$ where:

$$\frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t} - u_{xx}(x,t_j) = f(x,t_j) \Rightarrow u(x,t_j) - \Delta t u_{xx}(x,t_j) = u(x,t_{j-1}) + \Delta t f(x,t_j)$$
$$\Rightarrow \int_a^b u^j v\, dx - \Delta t \int_a^b u_x^j v_x\, dx = \int_a^b u^{j-1} v\, dx + \Delta t \int_a^b f^j v\, dx + [u_x^j v]_a^b$$

$$a(u,v) = \int_a^b u^j v\, dx - \Delta t \int_a^b u_x^j v_x\, dx \ \text{ and } \ b(v) = \int_a^b u^{j-1} v\, dx + \Delta t \int_a^b f^j v\, dx + \Delta t g_2(t) v(b)$$

3.1.3 Neumann Dirichlet

Differential Equation (D): $-u'' = f$ on (a,b) with $u'(a) = g_1(t),\ u(b) = g_2(t)$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a,b) : v(b) = g_2(t)\}$ such that $a(u,v) = b(v)\ \forall v \in V_0 = \{v \in H^1(a,b) : v(b) = 0\}$ where:

$\frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t} - u_{xx}(x,t_j) = f(x,t_j) \Rightarrow u(x,t_j) - \Delta t u_{xx}(x,t_j) = u(x,t_{j-1}) + \Delta t f(x,t_j)$
$\Rightarrow \int_a^b u^j v\,dx - \Delta t \int_a^b u_x^j v_x\,dx = \int_a^b u^{j-1} v\,dx + \Delta t \int_a^b f^j v\,dx + [u_x^j v]_a^b$

$$a(u,v) = \int_a^b u^j v\,dx - \Delta t \int_a^b u_x^j v_x\,dx \ \text{ and } \ b(v) = \int_a^b u^{j-1} v\,dx + \Delta t \int_a^b f^j v\,dx - \Delta t g_1(t) v(a)$$

3.1.4 Neumann Neumann

Differential Equation (D): $-u'' = f$ on (a,b) with $u'(a) = g_1(t),\ u'(b) = g_2(t)$

Variational Form (V): Find $u \in V_A = \{v \in H^1(a,b)\}$ such that $a(u,v) = b(v)\ \forall v \in V_0 = \{v \in H^1(a,b)\}$ where:

$\frac{u(x,t_j)-u(x,t_{j-1})}{\Delta t} - u_{xx}(x,t_j) = f(x,t_j) \Rightarrow u(x,t_j) - \Delta t u_{xx}(x,t_j) = u(x,t_{j-1}) + \Delta t f(x,t_j)$
$\Rightarrow \int_a^b u^j v\,dx - \Delta t \int_a^b u_x^j v_x\,dx = \int_a^b u^{j-1} v\,dx + \Delta t \int_a^b f^j v\,dx + [u_x^j v]_a^b$

$$a(u,v) = \int_a^b u^j v\,dx - \Delta t \int_a^b u_x^j v_x\,dx$$

and

$$b(v) = \int_a^b u^{j-1} v\,dx + \Delta t \int_a^b f^j v\,dx + \Delta t g_2(t) v(b) - \Delta t g_1(t) v(a)$$

3.2 Modifying the Linear System

We need only modify the Steady State Linear System:

$A = (R + \Delta tD)$ and $b = Ru^{j-1} + \Delta tRf^j$ where R and D are the stiffness matrix and mass matrices used in the steady state problem.
Our system therefore becomes: $Au^j = b \Leftrightarrow (R + \Delta tD)u^j = Ru^{j-1} + \Delta tRf^j$
To implement the boundary conditions:

- Dual Dirichlet:
  - set $A_{0,0} = 1,\ A_{0,1} ... A_{0,n} = 0$ and $b_0 = g_1(t)$
  - set $A_{n,n} = 1,\ A_{n,0} ... A_{n,n-1} = 0$ and $b_n = g_2(t)$
- Dirichlet Neumann
  - set $A_{0,0} = 1,\ A_{0,1} ... A_{0,n} = 0$ and $b_0 = c_1$
  - add $\Delta tg_2(t)$ to b_n
- Neumann Dirichlet
  - set $A_{n,n} = 1,\ A_{n,0} ... A_{n,n-1} = 0$ and $b_n = g_2(t)$
  - subtract $\Delta tg_1(t)$ from b_0
- Neumann Neumann
  - subtract $\Delta tg_1(t)$ from b_0
  - add $\Delta tg_2(t)$ to b_n

3.3 Test Case:

$$\begin{aligned}
\text{The exact solution } &: u(x,t) = t^2cos(x) + sin(x^2) \\
\text{The RHS function } &: f(x,t) = 2tcos(x) + t^2cos(x) + 4x^2sin(x^2) - 2cos(x^2) \\
\text{Domain } &: x \in (0, \pi) \\
\text{Timestep } &: 1e-07 \\
\text{Final Time } &: 1e-05
\end{aligned}$$

The solution is computed for $n = \{100, 200, ..., 600\}$ for each of the four types of boundary conditions:

- Dual Dirichlet:
  $u(0,t) = t^2$
  $u(\pi,t) = sin(\pi^2) - (t^2)$
- Dirichlet Neumann:
  $u(0,t) = t^2$
  $u_x(\pi,t) = 2\pi cos(\pi^2)$
- Neumann Dirichlet:
  $u_x(0,t) = 0$
  $u(\pi) = sin(\pi^2) - (t^2)$
- Dual Neumann:
  $u_x(0,t) = 0$
  $u_x(\pi,t) = 2\pi cos(\pi^2)$

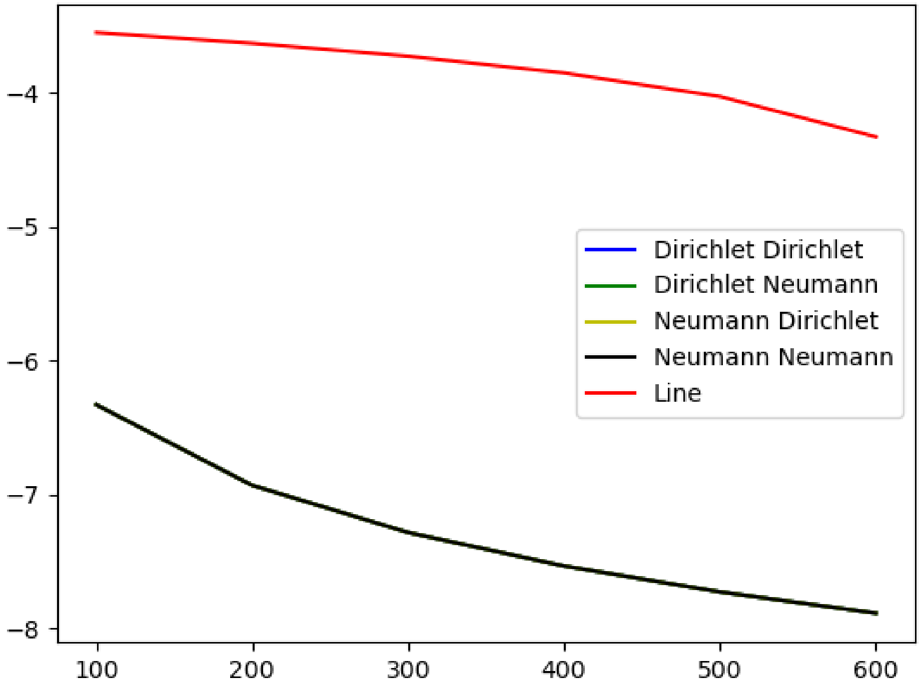
3.3.1 Accuracy Metrics

The accuracy metrics used for the Time Dependent Test Case are the same as those used for the Steady State Test Case.

3.3.2 Test Case Results:

Below is a plot showing:

- $log_{10}(error)$ for the **relative error** of each type of **boundary condition**
- $log_{10}(line)$ where $line$ is a linear equation with a slope of -1
- n, the number of **nodes** on the x axis



Below is the **log slope** and the **regular slope** of each curve (calculated at $n = 600$):

$$log\ slope = \frac{log_{10}(error_n) - log_{10}(error_{n-1})}{log_{10}(n) - log_{10}(n-1)}$$

$$regular\ slope = \frac{error_n - error_{n-1}}{n - (n-1)}$$

```
Log Slope: Dirichlet Dirichlet =  -1.9998823323188266
Log Slope: Dirichlet Neumann =  -1.9998985569198566
Log Slope: Neumann Dirichlet =  -1.9998823323075534
Log Slope: Neumann Neumann =  -1.9998985569085836

Log Slope: Line =  -3.8017840169239387
Reg Slope: Line =  -1.0

Reg Slope: Dirichlet Dirichlet =  -0.0015835317502163448
Reg Slope: Dirichlet Neumann =  -0.0015835445970576068
Reg Slope: Neumann Dirichlet =  -0.0015835317502074188
Reg Slope: Neumann Neumann =  -0.0015835445970486806
```