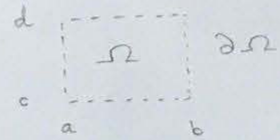
Aleks Lyubenov

Homework 3

Exercise 1: Let $\Omega \subset R^2$ open, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$,

$$\begin{cases} -\triangle u(x) = f & \text{for } x \in \Omega \\ u(x)\big|_{\Gamma_1} = g_1(x) \\ \frac{\partial u}{\partial n}(x)\big|_{\Gamma_2} = g_2(x) \end{cases}$$

Domain : $\partial\Omega = \Gamma_D \cup \Gamma_N$



$\Gamma_D = [a,b] \times \{d\}$ (Top)

$\Gamma_N = \{a\} \times [c,d] \cup [a,b] \cup \{b\} \times [c,d]$ (left $\cup$ bottom $\cup$ Right)

(D) : Find u st $-\Delta u(x,y) = f(x,y)$ on Ω

$(\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y))$

Boundary conditions : $u|_{\Gamma_D} = u_1(x,y)$ $\quad \frac{\partial u}{\partial \vec{n}}\Big|_{\Gamma_N} = g_2(x,y)$

($\vec{n}$: outer normal vector)

$V_0 := \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$

$V_A := \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = u_1(x,y) \}$

Let $v \in V_0$

$-\Delta u(x,y) = f(x,y) \Rightarrow -\iint_\Omega v \cdot \Delta u \, dA = \iint_\Omega v \cdot f \, dA \quad \forall v \in V_0$

By Green's formula $\Rightarrow \iint_\Omega \nabla u \nabla v \, dA - \int_{\partial\Omega} (\nabla u \, \vec{n}) v \, ds = \iint_\Omega v \cdot f \, dA$

Since $v|_{\Gamma_D} = 0 \Rightarrow -\int_{\partial\Omega} \underbrace{(\nabla u \cdot \vec{n})}_{= \frac{\partial}{\partial \vec{n}} u} v \, ds = -\int_{\Gamma_N} g_2 \cdot v \, ds$

$\therefore \iint_\Omega \nabla u \nabla v \, dA - \int_{\Gamma_N} g_2 v \, ds = \iint_\Omega f \cdot v \, dA \quad \forall v \in V_0$

(V) Find $u \in V_A$ st $a(u,v) = b(v)$ where

$a(u,v) = \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$ and $b(v) = \langle f, v \rangle_{L^2(\Omega)} + \langle g_2, v \rangle_{L^2(\Gamma_N)}$

$\forall v \in V_0$

(D) $\Rightarrow$ (V)

$$\iint_\Omega \nabla u \nabla v \, dA = \iint_\Omega f v \, dA + \int_{\Gamma_N} g_2 v \, ds \quad \forall v \in V_0$$

Note: $$-\iint_\Omega \Delta u \cdot v \, dA = \iint_\Omega \nabla u \cdot \nabla v \, dA - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds$$

$$= \iint_\Omega \nabla u \cdot \nabla v \, dA - \left[\int_{\Gamma_N} \frac{\partial u}{\partial n} v \, ds + \underbrace{\int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds}_{=0} \right]$$

$$\iint_\Omega \nabla u \cdot \nabla v \, dA - \iint_\Omega f v \, dA - \left[\int_{\Gamma_N} g_2 v \, ds + \underbrace{\int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds}_{=0} \right] = 0$$

Let $v \in H_0^1(\Omega)$, then:

$$\iint_\Omega \nabla u \nabla v \, dA - \iint_\Omega f v \, dA - \left[\int_{\Gamma_N} g_2 v \, ds + \int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds \right] = 0$$

$$\Rightarrow \iint_\Omega \nabla u \nabla v \, dA - \iint_\Omega f v \, dA = 0 \qquad (*)$$

$$\iint_\Omega \nabla u \cdot \nabla v \, dA - \iint_\Omega f v \, dA - \left[\int_{\Gamma_N} g_2 v \, ds + \int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds \right] = 0$$

by $(*)$: $$\int_{\Gamma_N} g_2 v \, ds + \int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds = 0$$

$$\therefore \int_{\partial\Omega} \left(g_2 v + \frac{\partial u}{\partial n} \right) v \, ds = 0 \Rightarrow \int_{\partial\Omega} g_2 v \, ds = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds$$

$$\therefore \left. \frac{\partial u}{\partial n} \right|_{\Gamma_N} = g_2$$

$(V) \Rightarrow (D)$

Let $u = \tilde{u} + g$ where $\tilde{u} \in V_0 = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$ and (3)

$u, g \in V_A = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = u_1(x,y) \}$

Consider the variational problem:

(V):

Find $u \in V_A$ st $a(u,v) = b(v) \ \forall \ v \in V_0$

Let $\tilde{u} \in V_0$ and $g \in V_A$ with $Ag = 0$

Substituting $\tilde{u}+g$ for u, we obtain:

We then have: $a(\tilde{u}+g, v) = b(v) \ \forall \ v \in V_0$

$$\Rightarrow a(\tilde{u}, v) + a(g, v) = b(v)$$

$$\Rightarrow \langle \nabla \tilde{u}, \nabla v \rangle_{L^2(\Omega)} = \langle f, v \rangle_{L^2(\Omega)} + \langle g_2, v \rangle_{L^2(\Gamma_N)} - \langle \nabla g, \nabla v \rangle_{L^2(\Omega)}$$

$(\tilde{V})$: Find $\tilde{u} \in V_0$ st $a(\tilde{u}, v) = b(v) - a(g,v) \ \forall \ v \in V_0$

WTS: $|a(\tilde{u}, v)| \leq \alpha \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$

$|a(\tilde{u}, v)| = \langle \nabla \tilde{u}, \nabla v \rangle_{L^2(\Omega)} \leq \|\nabla \tilde{u}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$ (Cauchy Schwarts)

<u>Note:</u> $\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}$

$$\therefore \ |a(\tilde{u}, v)| \leq \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$a(\tilde{u}, v)$ is bounded.

WTS: $|a(v,v)| \geq \gamma \|v\|^2_{H^1(\Omega)}$

$|a(v,v)| = |\langle \nabla v, \nabla v \rangle_{L^2(\Omega)}| = \|\nabla v\|^2_{L^2(\Omega)} = \frac{1}{2}\left[\|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \right]$

Poincaré inequality: $\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$

$$\therefore \ \|u\|^2_{L^2(\Omega)} \leq C^2 \|\nabla u\|^2_{L^2(\Omega)}$$

$$\therefore \ \frac{1}{C^2} \|u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)}$$

$$|a(v,v)| = \|\nabla v\|^2_{L^2(\Omega)} \geq \frac{1}{2}\left[\frac{1}{C^2}\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \right] \quad \text{(Poincaré)}$$

$$\geq \frac{1}{2}\left[\frac{1}{C^2}\|v\|^2_{L^2(\Omega)} + \frac{1}{C^2}\|\nabla v\|^2_{L^2(\Omega)} \right]$$

$$= \frac{1}{2C^2} \|v\|^2_{H^1(\Omega)}$$

$\therefore$ $a(\tilde{u}, v)$ is coercive.

$$|\ell(v)| = |b(v) - a(g,v)| = |\langle f,v\rangle_{L^2(\Omega)} + \langle g_2,v\rangle_{L^2(\Gamma_N)} - \langle \nabla g, \nabla v\rangle_{L^2(\Omega)}$$

$$\leq |\langle f,v\rangle_{L^2(\Omega)}| + |\langle g_2,v\rangle_{L^2(\Gamma_N)}| + |\langle \nabla g, \nabla v\rangle_{L^2(\Omega)}|$$

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$$\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g_2\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} + \|\nabla g\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

$\|v\|_{L^2(\Gamma_N)} \leq C\|v\|_{H^1(\Omega)}$ for some $C \in \mathbb{R}$ (Trace Theorem)

$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$

$\therefore \|v\|^2_{H^1(\Omega)} \geq \|v\|^2_{L^2(\Omega)}$ and $\|v\|^2_{H^1(\Omega)} \geq \|\nabla v\|^2_{L^2(\Omega)}$

$$\leq \left(\|f\|_{L^2(\Omega)} + \frac{1}{C}\|g_2\|_{L^2(\Gamma_N)} + \|\nabla g\|_{L^2(\Omega)}\right) \|v\|_{H^1(\Omega)}$$

$\therefore$ $\ell(v)$ is bounded.

Therefore, by Lax Milgram, $\exists!$ $\tilde{u}$ satisfying $(\tilde{V})$:

Find $\tilde{u} \in V_0$ st $a(\tilde{u},v) = b(v) - a(g,v)$ $\forall v \in V_0$ and some $g \in V_A$

$a(\tilde{u},v) = \langle \nabla\tilde{u}, \nabla v\rangle_{L^2(\Omega)}$

$\ell(v) \equiv b(v) - a(g,v) = \langle f,v\rangle_{L^2(\Omega)} + \langle g_2,v\rangle_{L^2(\Gamma_N)} - \langle \nabla g, \nabla v\rangle_{L^2(\Omega)}$

Since $\tilde{u} \in V_0$ and $g \in V_A$ are both unique, $\exists\, u \in V_A$ st $u = \tilde{u} - g$

Such a u satisfies (V):

Find $u \in V_A$ st $a(u,v) = b(v)$ $\forall v \in V_0$

$a(u,v) = \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}$

$b(v) = \langle f,v\rangle_{L^2(\Omega)} + \langle g_2,v\rangle_{L^2(\Omega)}$

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Exercise 2: Let $\Omega \subset R^2$ open, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$,

$$\begin{cases} u - \Delta u(x) = f & \text{for } x \in \Omega \\ u(x)\big|_{\Gamma_1} = g_1(x) & \\ \frac{\partial u}{\partial n}(x)\big|_{\Gamma_2} = g_2(x) & \end{cases}$$

(D): $u - \Delta u = f$ in Ω with $u|_{\Gamma_D} = u_1(x,y)$ and $u|_{\Gamma_N} = g_2(x,y)$

$\partial\Omega = \Gamma_D \cup \Gamma_N$

$V_0 = \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}$ and $V_A = \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = u_1(x,y)\}$

choose $v \in V_0$

$u - \Delta u = f \Rightarrow \iint_\Omega u \cdot v \, dA - \iint_\Omega \Delta u \, v \, dA = \iint_\Omega f v \, dA \quad \forall v \in V_0$

By Green's First Identity, we obtain:

$$\Rightarrow \iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA - \int_{\partial\Omega} (\nabla u \cdot \hat{n}) v \, ds = \iint_\Omega f v \, ds$$

$$\Rightarrow \iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA = \iint_\Omega f v \, dA + \int_{\Gamma_N} g_2 v \, ds \quad \forall v \in V_0$$

(V) Find $u \in V_A$ st $a(u,v) = b(v) \quad \forall v \in V_0$ where

$a(u,v) = \iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA$ and

$b(v) = \iint_\Omega f v \, dA + \int_{\Gamma_N} g_2 v \, ds$

(D) $\Rightarrow$ (V)

$$\iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA = \iint_\Omega f v \, dA + \int_{\Gamma_N} g_2 v \, ds \quad \forall v \in V_0$$

$$\iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA - \iint_\Omega f v \, dA - \int_{\Gamma_N} g_2 v \, ds = 0$$

$$\Rightarrow \iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA - \iint_\Omega f \cdot v \, dA - \left[\int_{\Gamma_N} g_2 v \, ds + \int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds \right] = 0$$

let $v \in H_0^1(\Omega)$, then:

$$\iint_\Omega uv \, dA + \iint_\Omega \nabla u \nabla v \, dA - \iint_\Omega f v = 0 \quad (*)$$

$$\Rightarrow \text{By } (*): \int_{\Gamma_N} g_2 v \, ds + \int_{\Gamma_D} \frac{\partial u}{\partial n} v \, ds = 0 \Rightarrow \int_{\partial\Omega} \left(g_2 + \frac{\partial u}{\partial n}\right) v \, ds = 0$$

$$\Rightarrow \int_{\partial\Omega} g_2 v \, ds = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds = 0 + \int_{\Gamma_N} \frac{\partial u}{\partial n} v \, ds$$

$$\Rightarrow \frac{\partial u}{\partial n}\Big|_{\Gamma_N} = g_2 \qquad \therefore (V) \Rightarrow (D)$$

Consider the variational problem: (V):

Find $u \in V_A$ st $a(u,v) = b(v) \ \forall \ v \in V_0$ where

$a(u,v) = \langle u,v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$

$b(v) = \langle f,v \rangle_{L^2(\Omega)} + \langle g_2, v \rangle_{L^2(\Gamma_N)}$

Recall the definitions of our spaces:

$$V_0 = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \}$$
$$V_A = \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = u_1(x,y) \}$$

let $\tilde{u} \in V_0$, let $g \in V_A$ with $\Delta g = 0$

Substituting $\tilde{u} + g$ for u, we obtain:

$a(\tilde{u}+g, v) = a(\tilde{u},v) + a(g,v) = b(v) \quad \forall \ v \in V_0$

Define a new variational problem $(\tilde{V})$:

Find $\tilde{u} \in V_0$ st $a(\tilde{u},v) = \ell(v) \ \forall \ v \in V_0$ where

$a(\tilde{u},v) = \langle \tilde{u}, v \rangle_{L^2(\Omega)} + \langle \nabla \tilde{u}, \nabla v \rangle_{L^2(\Omega)}$

$$\ell(v) = b(v) - a(g,v) = \langle f,v \rangle_{L^2(\Omega)} + \langle g_2, v \rangle_{L^2(\Gamma_N)} - \langle g, v \rangle_{L^2(\Omega)} - \langle \nabla g, \nabla v \rangle_{L^2(\Omega)}$$

WTS: $|a(\tilde{u},v)| \leq \alpha \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$ for some $\alpha \in \mathbb{R}$

$$|a(\tilde{u},v)| = |\langle \tilde{u},v \rangle_{L^2(\Omega)} + \langle \nabla \tilde{u}, \nabla v \rangle_{L^2(\Omega)}| = |\langle \tilde{u}, v \rangle_{H^1(\Omega)}|$$

$$\leq \|\tilde{u}\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \text{(Cauchy Schwartz)}$$

$\therefore$ $a(\tilde{u},v)$ is bounded

WTS : $|a(v,v)| \geqslant \gamma \|v\|^2_{H^1(\Omega)}$

$|a(v,v)| = |\langle v,v\rangle_{L^2(\Omega)} + \langle \nabla v, \nabla v\rangle_{L^2(\Omega)}| = |\langle v,v\rangle_{H^1(\Omega)}|$

$= \|v\|^2_{H^1(\Omega)}$

$\therefore$ $a(u,v)$ is coercive.

WTS : $|l(v)| \geqslant \beta \|v\|_{H^1(\Omega)}$ for some $\beta \in \mathbb{R}$

$|l(v)| = |b(v) - a(g,v)|$

$= |\langle f,v\rangle_{L^2(\Omega)} + \langle g_2,v\rangle_{L^2(\Gamma_N)} - \langle g,v\rangle_{L^2(\Omega)} - \langle \nabla g, \nabla v\rangle_{L^2(\Omega)}|$

$\leq |\langle f,v\rangle_{L^2(\Omega)}| + |\langle g_2,v\rangle_{L^2(\Gamma_N)}| + |\langle g,v\rangle_{L^2(\Omega)}| + |\langle \nabla g, \nabla v\rangle_{L^2(\Omega)}|$

$\leqslant \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g_2\|_{L^2(\Gamma_N)}\|v\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$

$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}$

$\Rightarrow \|v\|_{H^1(\Omega)} \geqslant \|v\|_{L^2(\Omega)}$ and $\|v\|_{H^1(\Omega)} \geqslant \|v\|_{L^2(\Omega)}$

By the trace theorem: $\|v\|_{L^2(\Gamma_N)} \leq C\|v\|_{H^1(\Omega)}$, $C \in \mathbb{R}$

$\leq \left(\|f\|_{L^2(\Omega)} + \frac{1}{C}\|g_2\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)}\right)\|v\|_{H^1(\Omega)}$

$\therefore$ $l(v)$ is bounded.

Therefore : by Lax Milgram, $\exists! \; \tilde{u} \in V_0$ st $a(\tilde{u},v) = l(v) \; \forall \; v \in V_0$

where $a(\tilde{u},v) = \langle \tilde{u},v\rangle_{L^2(\Omega)} + \langle \nabla\tilde{u}, \nabla v\rangle_{L^2(\Omega)}$ and

$l(v) = \langle f,v\rangle_{L^2(\Omega)} + \langle g_2,v\rangle_{L^2(\Gamma_N)} - \langle g,v\rangle_{L^2(\Omega)} - \langle \nabla g, \nabla v\rangle_{L^2(\Omega)}$

with $g \in V_A$

Since $\tilde{u} \in V_0$ and $g \in V_A$ are unique, $\exists! \; u \in V_0$ satisfying

$a(u,v) = b(v) \; \forall \; v \in V_0$ where

$a(u,v) = \langle u,v\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}$ & $b(v) = \langle f,v\rangle_{L^2(\Omega)} + \langle g_2,v\rangle_{L^2(\Gamma_N)}$