Задача 1. (2 т.)

1. (0.5 т.) Теглим 2 карти от стандартно тесте с 52 карти. Нека X е броят на изтеглените аса, а Y на изтеглените купи. Намерете съвместното разпределение и корелацията на X и Y.

2. (1 т.) Нека случайната величина X е цената на даден актив, а случайната величина Y - средният лихвен процент по депозитите (и двете за ден и в безмерни единици). Нека съвместната им плътност е $f_{X,Y}(x,y) = c(x^2 + e^y x)$ за $0 < x, y < 1$ и 0 извън тази област, като c е някаква константа. Намерете константата c и ковариацията на X и Y. Колко е очакването на цената на актива, ако лихвеният процент е 0.5?

3. (0.5 т.) Нека X е случайна величина с плътност $f_X(x) = \frac{1}{4}xe^{-x/2}$ за $x \geq 0$ и 0 иначе. Намерете плътността на случайната величина $Y = -2X + 2$.

① $\cancel{\begin{cases} X \sim Bin(2, \frac{1}{13}) \\ Y \sim Bin(2, \frac{1}{4}) \end{cases}}$ Не са биномни, защото 2-те тегления са зависими

1. $C = \binom{52}{2} = 1326$

$Y \backslash X$	0	1	2
0	$\binom{36}{2}/C$	$\binom{3}{1}\binom{36}{1}/C$	$\binom{3}{2}/C$
1	$\binom{12}{1}\binom{36}{1}/C$	$\frac{\left[\binom{12}{1}\binom{3}{1}+\binom{36}{1}\right]}{C}$	$\binom{3}{1}/C$
2	$\binom{12}{2}/C$	$\binom{12}{1}/C$	0

X	0	1	2
P	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$$\mathbb{E}[X] = \sum_{i=0}^{2} x_i p_i = \frac{34}{221} = \frac{2}{13}$$

Y	0	1	2
P	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{2}{34}$

$$\mathbb{E}[Y] = \sum_{i=0}^{2} y_i p_i = \frac{1}{2}$$

XY	0	1	2	4
	$\frac{413}{442}$	$\frac{12}{221}$	$\frac{5}{442}$	0

$Y \backslash X$	0	1	2	
0	$\frac{105}{221}$	$\frac{18}{221}$	$\frac{1}{442}$	$\frac{19}{34}$
1	$\frac{72}{221}$	$\frac{12}{221}$	$\frac{1}{442}$	$\frac{13}{34}$
2	$\frac{11}{221}$	$\frac{2}{221}$	0	$\frac{2}{34}$
	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$	

$$\mathbb{E}[XY] = 1 \cdot \frac{12}{221} + \frac{2 \cdot 5}{442} = \frac{17}{221} = \frac{1}{13}$$

$$\mathbb{E}[X^2] = 1 \cdot \frac{32}{221} + 2^2 \cdot \frac{1}{221} = \frac{36}{221}$$

$$\mathbb{E}[Y^2] = 1 \cdot \frac{13}{34} + 2^2 \cdot \frac{2}{34} = \frac{21}{34}$$

$$\mathbb{D}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{36}{221} - \left(\frac{2}{13}\right)^2 = \frac{400}{2873}$$

$$\mathbb{D}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{21}{34} - \left(\frac{1}{2}\right)^2 = \frac{25}{68}$$

$$Cor(X,Y) = \frac{\mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y}{\sqrt{\mathbb{D}[X]\mathbb{D}[Y]}} = \frac{\frac{1}{13} - \frac{1}{13}}{\sqrt{\mathbb{D}[X]\mathbb{D}[Y]}} = 0$$

2. $f_{X,Y}(x,y) = c(x^2 + e^y \cdot x)$, $0 < x, y < 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$

$$\int_0^1\int_0^1 c(x^2+e^y \cdot x)\,dy\,dx = c\int_0^1 x \int_0^1 x + e^y\,dy\,dx = c\int_0^1 x\left[\int_0^1 x\,dy + \int_0^1 e^y\,dy\right]dx =$$

$$= c \cdot \int_0^1 x\left[x[y]_0^1 + [e^y]_0^1\right]dx = c \cdot \int_0^1 x[x+e-1]\,dx = c\left[\int_0^1 x^2\,dx + (e-1)\int_0^1 x\,dx\right] =$$

$$= c\left[\frac{x^3}{3}\right]_0^1 + e(e-1)\left[\frac{x^2}{2}\right]_0^1 = c\left[\frac{1}{3} + \frac{e-1}{2}\right] = c \cdot \frac{2+3e-3}{6} = c \cdot \frac{3e-1}{6} = 1 \Rightarrow c = \frac{6}{3e-1}$$

$$f_X(x) = \int_{y=0}^{y=1} f_{X,Y}(x,y)\,dy = \int_0^1 c(x^2+e^y x)\,dy = c\left[x^2\int_0^1 dy + x\int_0^1 e^y dy\right] =$$

$$= c\left[x^2 + x(e-1)\right]$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx = c\int_0^1 x^3+x^2(e-1)\,dx = c\left[\frac{x^4}{4}\Big|_0^1 + (e-1)\left[\frac{x^3}{3}\right]_0^1\right] =$$

$$= c\left[\frac{1}{4} + \frac{e-1}{3}\right] = \frac{c(4e-1)}{12}$$

$$f_Y(y) = \int_0^1 c(x^2+e^y x)\,dx = c\left[\frac{x^3}{3}\Big|_0^1 + e^y\left[\frac{x^2}{2}\right]_0^1\right] = c\left[\frac{1}{3}+e^y\cdot\frac{1}{2}\right] = \frac{c}{6}(2+3e^y)$$

$$\mathbb{E}[Y] = \frac{c}{6}\int_0^1 2y+3ye^y\,dy = \frac{c}{6}\left[2\left[\frac{y^2}{2}\right]_0^1 + 3\int_0^1 ye^y dy\right] = \frac{c}{6}\left[1 + 3\int_0^1 ye^y dy\right]$$

$$\int_0^1 ye^y dy = \int_0^1 y\,de^y = \left[ye^y\right]_0^1 - \int_0^1 e^y dy = e - \left[e^y\right]_0^1 = +1$$

$$\Rightarrow \mathbb{E}[Y] = \frac{c}{6}[1+3] = \frac{2c}{3}$$

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\cdot y\, f_{X,Y}(x,y)\,dy\,dx = c\int_0^1 x\int_0^1 y(x^2+e^y x)\,dy\,dx = c\int_0^1 x\left[x^2\left[\frac{y^2}{2}\right]_0^1 + x\underbrace{\int_0^1 y\,e^y dy}_{1}\right]dx$$

$$= c\int_0^1 x\left[\frac{x^2}{2}+x\right]dx = c\left[\frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1\right] = c\left[\frac{1}{8}+\frac{1}{3}\right] = \frac{11}{24}\cdot c$$

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{11}{24}\cdot c - c^2\cdot\frac{4e-1}{18} \approx -0.0014$$

(3. $\cancel{f_X(x) = \frac{1}{4}x e^{-\frac{x}{2}}\mathbb{1}\{x\geq 0\}}$)

$$\mathbb{E}[X|Y=\tfrac{1}{2}] = \int_0^1 x\cdot f_{X|Y}(x|y=\tfrac{1}{2})\,dx = \int_0^1 x\cdot\frac{f_{X,Y}(x,\frac{1}{2})}{f_Y(\frac{1}{2})} = \int_0^1 x\cdot\frac{c(x^2+x\sqrt{e})}{\frac{c}{6}(2+3\sqrt{e})}\,dx =$$

$$= 6\int_0^1 x\cdot\frac{x^2+x\cdot\sqrt{e}}{2+3\sqrt{e}}\,dx = \frac{6}{2+3\sqrt{e}}\int_0^1 x^3+x^2e^{\frac{1}{2}}\,dx = \frac{6}{2+3\sqrt{e}}\cdot\frac{1}{4} + \frac{6\sqrt{e}}{2+3\sqrt{e}}\int_0^1 x^2\cancel{e^{\frac{1}{2}}}\,dx =$$

$$= \frac{3}{4+6\sqrt{e}} + \frac{2\cancel{6}\sqrt{e}}{2+3\sqrt{e}}\cdot\frac{1}{3} = \frac{3+4\sqrt{e}}{4+6\sqrt{e}} \approx 0.6907$$

3. $f_X(x) = \frac{1}{4} x \cdot e^{-\frac{x}{2}} \cdot \mathbb{1}\{x \geq 0\}$

$y = -2x+2 \qquad x = \frac{2-y}{2}$

$f_Y(y) = f_X\left(\frac{2-y}{2}\right) \left| \underset{\substack{\shortparallel \\ \frac{1}{2}}}{\left(\frac{2-y}{2}\right)'} \right| = \frac{1}{2} \cdot \frac{2-y}{4 \cdot 2} e^{-\frac{2-y}{2} \cdot \frac{1}{2}} = \frac{2-y}{16} \cdot e^{\frac{y-2}{4}}$