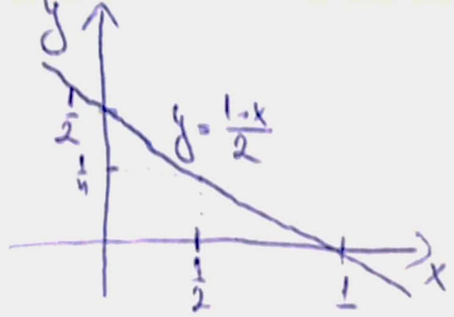
Задача 1. Нека съвместната плътност на X и Y е $f_{X,Y}(x,y) = cx^2 + 1$ за $x, y \geq 0, x + 2y \leq 1$ и 0 извън тази област, където c е някаква константа. Намерете:

1. (0.5 т.) c, плътността на X и очакването на Y;
2. (0.25 т.) $\mathbb{E}(Y|X = 1/2)$;
3. (0.25 т.) плътността на случайната величина $Z = X + 2Y$.

① 1. $f_{X,Y}(x,y) = c\cdot x^2 + 1$, $\left|\begin{array}{l} x,y \ge 0 \\ x+2y \le 1 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x,y \ge 0 \\ y \le \frac{1-x}{2} \end{array}\right.$



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$

$$\int_0^1\int_0^{\frac{1-x}{2}} c\cdot x^2+1\,dy\,dx = \int_0^1 \left[cx^2\int_0^{\frac{1-x}{2}}dy + \int_0^{\frac{1-x}{2}}dy\right]dx = \int_0^1\left[cx^2[y]_0^{\frac{1-x}{2}} + [y]_0^{\frac{1-x}{2}}\right]dx =$$

$$= \int_0^1\left[cx^2\cdot\frac{1-x}{2} + \frac{1-x}{2}\right]dx = \frac{1}{2}\int_0^1 cx^2 - cx^3 + 1 - x\,dx = \frac{1}{2}\left[c\cdot\frac{x^3}{3}\Big|_0^1 - c\left[\frac{x^4}{4}\right]_0^1 + [x]_0^1 - \left[\frac{x^2}{2}\right]_0^1\right] =$$

$$= \frac{1}{2}\left[\frac{c}{3} - \frac{c}{4} + 1 - \frac{1}{2}\right] = \frac{1}{2}\left[\frac{c}{12} + \frac{6}{12}\right] = \frac{c}{24} + \frac{1}{4} = 1 \Rightarrow c = \frac{3}{4}\cdot \overset{6}{\cancel{24}} = 18$$

2. $\mathbb{E}[Y|X=\frac{1}{2}] = \cancel{\int_{-\infty}^{\infty}}\int_{-\infty}^{\infty} y\cdot f_{Y|X}(\frac{1}{2},y)\,dy = \int_0^{\frac{1}{4}} y\,\frac{f_{X,Y}(\frac{1}{2},y)}{f_X(\frac{1}{2})}\,dy$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{\frac{1-x}{2}} c\cdot x^2+1\,dy = cx^2[y]_0^{\frac{1-x}{2}} + [y]_0^{\frac{1-x}{2}} = (cx^2+1)\left(\frac{1-x}{2}\right)$$

$$\Rightarrow \mathbb{E}[Y|X=\tfrac{1}{2}] = \int_0^{\frac{1}{4}} y\cdot\frac{\cancel{(\frac{1}{4}c+1)}}{\cancel{(\frac{1}{4}c+1)}(\frac{1}{4})}\,dy = 4\left[\frac{y^2}{2}\right]_0^{\frac{1}{4}} = \frac{4}{32} = \frac{1}{8}$$

$$\mathbb{E}[Y] = \int_0^1 \int_0^{\frac{1-x}{2}} y\, f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^{\frac{1-x}{2}} y\left[cx^2+1\right] dy\, dx =$$

$$= \int_0^1 \left[c\cdot x^2 \left[\frac{y^2}{2}\right]_0^{\frac{1-x}{2}} + \left[\frac{y^2}{2}\right]_0^{\frac{1-x}{2}} \right] dx = \int_0^1 \frac{cx^2}{8}(1-2x+x^2) + \frac{1}{8}(1-2x+x^2)\, dx =$$

$$= \frac{1}{8} c \int_0^1 x^2 - 2x^3 + x^4\, dx + \frac{1}{8}\int_0^1 1-2x+x^2\, dx = \frac{c}{8}\left[\frac{x^3}{3}\Big|_0^1 - 2\cdot\frac{x^4}{4}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1 \right] + \frac{1}{8}\left[x\Big|_0^1 - 2\cdot\frac{x^2}{2}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 \right] =$$

$$= \frac{c}{8}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] + \frac{1}{8}\left[\cancel{1} - \cancel{1} + \frac{1}{3}\right] = \frac{1}{24} + \frac{c}{8}\cdot\frac{1}{30} = \frac{1}{24} + \frac{\overset{9\ 3}{\cancel{18}}}{\underset{4\ \ 10}{8\cdot 30}} = \frac{1}{24} + \frac{3}{40} = \frac{7}{60}$$

③ $Z = X + 2Y$, Нека $W = X$ $\Rightarrow$ $X = W$, $y = \frac{Z-W}{2}$

$y \le \frac{1-W}{2} \Leftrightarrow \frac{Z-W}{2} \le \frac{1-W}{2} \Rightarrow Z \le 1$, $Z \ge 0$

$y = \frac{Z-W}{2}$, $y \ge 0 \Rightarrow Z - W \ge 0 \Rightarrow W \le Z$, $W \ge 0$

$$f_{Z,W}(z,w) = f_{X,Y}\left(w, \frac{z-w}{2}\right) \cdot \left| J(z,w) \right|$$

$$J(z,w) = \det \begin{vmatrix} \frac{\partial}{\partial w} x & \frac{\partial}{\partial z} x \\ \frac{\partial}{\partial w} y & \frac{\partial}{\partial z} y \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\Rightarrow f_{Z,W}(z,w) = \frac{cw^2+1}{2} \cdot \mathbb{1}\{w \in (0,z)\} \cdot \mathbb{1}\{z \in (0,1)\}$$

$$f_Z(z) = \int_{w=-\infty}^{w=\infty} f_{Z,W}(z,w)\,dw = \frac{1}{2}\int_0^z c \cdot w^2 + 1\,dw = \frac{1}{2} c \left[\frac{w^3}{3}\right]_0^z + \frac{1}{2}\left[w\right]_0^z =$$

$$= 3z^3 + \frac{z}{2} \cdot \mathbb{1}\{z \in (0,1)\}$$