Задача 91. Нека случайната величина $X \sim Exp(\lambda)$. Да се намерят плътностите на случайните величини

- $Y = -X$;
- $Y = 2X - 1$;
- $Y = \sqrt{X}$;
- $Y = X^{\alpha}$ за $\alpha > 0$.

(9.1) $X \sim Exp(\lambda)$

Каква е плътността на Y:

а) $Y = -X$
б) $Y = 2X - 1$
в) $\sqrt{X}$

Знаем $f_X(x) \Rightarrow f_g(x) = ?$

$P(X<x) \to P(Y<x)$ чрез $P(X<x)$

$f_{g(x)}(x) = ?$

$P(g(X)<x) = P(X < g^{-1}(x)) = F_X(g^{-1}(x))$

$\Rightarrow f_{g(x)}(x) = \frac{d}{dx} F_{g(x)}(x) = \frac{d}{dx}\left[F_X(g^{-1}(x))\right] =$

$= F_X'(g^{-1}(x)) \cdot g^{-1}(x)'$

$= f_X(g^{-1}(x)) \cdot g^{-1}(x)'$

а) $Y = -X$

$P(Y<y) = P(-X<y) = P(X>-y) = 1 \cdot 1_{\{-y \le 0\}} + e^{+\lambda y} 1_{\{-y>0\}}$

$\Rightarrow f_Y(y) = \frac{d}{dy} P(Y<y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}\left(1 \cdot 1_{\{y>0\}} + e^{\lambda y} 1_{\{y \le 0\}}\right) = \lambda \cdot e^{\lambda \cdot y} 1_{\{y \le 0\}}$

б) $P(Y<y) = P(X>-y) = \begin{cases} e^{\lambda \cdot y}, & y \le 0 \\ 1, & y>0 \end{cases}$

$\Rightarrow f_Y(y) = \begin{cases} \lambda e^{\lambda y}, & y \le 0 \\ 0, & y>0 \end{cases}$

б) $Y = 2X - 1$

$P(Y<y) = P(2X-1<y) = P(X < \frac{1+y}{2}) = \begin{cases} 1 - e^{-\lambda \cdot \frac{1+y}{2}}, & \frac{1+y}{2} \ge 0 \\ 0, & \text{иначе} \end{cases}$

$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2}\lambda \cdot e^{-\lambda \cdot \frac{1+y}{2}}, & \frac{1+y}{2} \ge 0 \\ 0, & \text{иначе} \end{cases}$

$f_Y(y) = \frac{1}{2}\lambda \cdot e^{-\lambda\left(\frac{1+y}{2}\right)} \cdot 1_{\{y \ge -1\}}$

в) $Y = \sqrt{X}$

I начин:

$P(Y<y) = P(X^{\frac{1}{2}}<y) = P(X<y^2) = \begin{cases} 1 - e^{-\lambda \cdot y^2}, & y>0 \\ 0, & y \le 0 \end{cases}$

$f_Y(y) = \frac{d\, F_X(y^2)}{dy} = \begin{cases} 2y e^{-\lambda \cdot y^2}, & y>0 \\ 0, & y \le 0 \end{cases}$

II начин:

$g(x) = \sqrt{x}$ е растяща и диференцируема и $g^{-1}(y) = y^2$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy}(g^{-1}(y)) = f_X(y^2) \frac{d}{dy} y^2 = 2y \cdot f_X(y^2) = \begin{cases} 2y \cdot e^{-\lambda y^2}, & y>0 \\ 0, & y \le 0 \end{cases}$

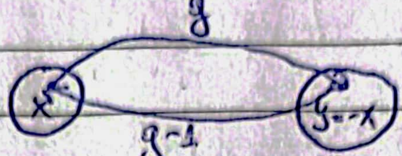
г) $g(x) = x^{\alpha}, \alpha > 0$

g е растяща и диференцируема и $g^{-1}(y) = y^{\frac{1}{\alpha}}$

$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) = f_X(y^{\frac{1}{\alpha}}) \cdot \frac{d}{dy} \cdot y^{\frac{1}{\alpha}} =$

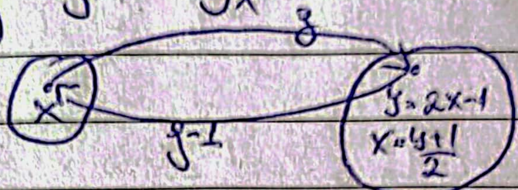
$$= \frac{1}{\alpha} \cdot y^{\frac{1-\alpha}{\alpha}} \cdot f_X(y^{\frac{1}{\alpha}}) = \begin{cases} \frac{1}{\alpha} \cdot y^{\frac{1-\alpha}{\alpha}} \cdot e^{-\lambda \cdot \frac{1}{\alpha}}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

$X \sim Exp(\lambda)$



1. $g(x) = y = -x$

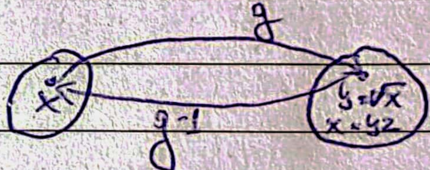
$$f_{g(x)}(y) = f_X(g^{-1}(y)) \cdot |g^{-1}(y)|' = f_X(\overset{-y}{x}) \cdot x' = \lambda e^{-\lambda x} = \lambda e^{\lambda y} \cdot 1_{\{y \le 0\}}$$



2. $g(x) = y = 2x - 1$

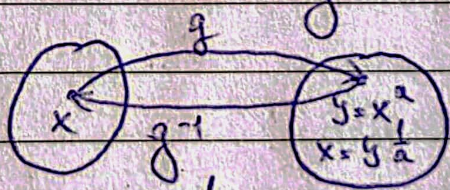
$$f_{g(x)}(y) = f_X(g^{-1}(y)) \, |(g^{-1}(y))'| = f_X\left(\frac{y+1}{2}\right) \left|\left(\frac{y}{2} + \frac{1}{2}\right)'\right| =$$

$$= \frac{1}{2} \cdot \lambda e^{-\lambda \cdot \frac{y+1}{2}} \cdot 1_{\{\frac{y+1}{2} \ge 0\}}$$



3. $g(x) = y = \sqrt{x}$

$$f_{g(x)}(y) = f_X(g^{-1}(y)) \, |(g^{-1}(y))'| = f_X(y^2) \frac{d}{dy} y^2 = 2y \lambda e^{-\lambda y^2} 1_{\{y > 0\}}$$



4. $g(x) = y = x^a \quad , a > 0$

$$f_{g(x)}(y) = f_X(y^{\frac{1}{a}}) \left| \frac{d}{dy} y^{\frac{1}{a}} \right| = \lambda e^{-\lambda y^{\frac{1}{a}}} \frac{1}{a} y^{\frac{1-a}{a}} = \frac{\lambda}{a} y^{\frac{1-a}{a}} e^{-\lambda y^{\frac{1}{a}}} 1_{\{y > 0\}}$$