Задача 66. Нека $(X_1, \dots, X_n)$ е случайна пермутация на числата от множеството $\{1, 2, \dots, n\}$ и $S = X_1 + \dots + X_n$.

1. Намерете $\mathbb{E}S$ и DS.
2. Докажете, че за две случайни величини X и Y е изпълнено $D(X+Y) = DX + DY + 2Cov(X,Y)$.
3. Изразете $\mathbb{E}S$ чрез $\mathbb{E}X_i$. Намерете $\mathbb{E}X_i$, $\mathbb{E}X_i^2$ и DX_i за всяко i.
4. Изразете DS чрез DX_i и $Cov(X_i, X_j)$. Намерете $Cov(X_i, X_j)$ за всеки i, j.

(66) $S = X_1 + \ldots + X_n$

$\mathbb{E}[S] = \mathbb{E}[X_1 + \ldots + X_n]$

$\mathbb{E}[X_i] = \sum_{i=1}^{n} i \cdot \frac{1}{n} = \frac{1}{n}(1+2+\ldots+n) = \frac{n(n+1)}{n.2} = \frac{n+1}{2}$

$S = 1+2+3+\ldots+n$ - фиксирано

1. $\mathbb{E}[S] = n.E[X_i] = \frac{n(n+1)}{2}$

$\mathbb{E}[X_i^2] = \frac{(n+1)(2n+1).n}{6.n} = \frac{(n+1)(2n+1)}{6}$

$\mathbb{E}[S^2] = \frac{n(n+1)(2n+1)}{6}$

$\mathbb{E}[(S-\mathbb{E}S)^2] = 0$, защото S е константа

$\mathbb{D}[S] = \mathbb{E}[S^2] - (\mathbb{E}[S])^2 = \cancel{\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} = \frac{n^2-1}{12}}$

2. $\mathbb{D}(X+Y) = \mathbb{D}X + \mathbb{D}Y + 2\,Cov(X,Y)$

$\mathbb{D}(X+Y) \overset{X \perp Y}{=} \mathbb{D}X + \mathbb{D}Y$

$\mathbb{D}(X+Y) \overset{X \not\perp Y}{=} \mathbb{D}X + \mathbb{D}Y + 2.Cov(X,Y)$

$\mathbb{D}(X+Y) = \mathbb{E}[(X+Y)^2] - (E[X+Y])^2 = \mathbb{E}[X^2+2XY+Y^2] - (\mathbb{E}[X]+\mathbb{E}[Y])^2 =$

$= \mathbb{E}[X^2] - (\mathbb{E}X)^2 + \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 - 2\,[\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]]$

$= \mathbb{D}X + \mathbb{D}Y - 2.Cov(X,Y)$

3. $\mathbb{E}[S] = n.\mathbb{E}[X_i]$

$\mathbb{E}[X_i] = \frac{n+1}{2}$

$\mathbb{E}[X_i^2] = \frac{(n+1)(2n+1)}{6}$

$\mathbb{D}X_i = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = \frac{(n+1)(2n+1)}{6} - \frac{n^2+2n+1}{4} = \frac{n^2-1}{12}$

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4. $\mathbb{D}[S] = 0 = \mathbb{D}[X_1 + X_2 + \dots + X_n] = \mathbb{D}X_1 + \mathbb{D}X_2 + \dots + \mathbb{D}X_n + 2\sum_{i>j}^{n} Cov(X_i, X_j)$

$Cov(X_i, X_j) = \mathbb{E}[X_iX_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]$

$$\mathbb{D}[S] = \mathbb{D}(X_1 + \dots + X_n) = n.\mathbb{D}(X_1) + 2.\binom{n}{2} Cov(X_1, X_2)$$

$$= n \cdot \frac{n^2+1}{12} + n(n+1)\cancel{\frac{(n-1)}{12}}\, Cov(X_1, X_2) = 0$$

$$\Rightarrow Cov(X_1, X_2) = \frac{-n+1}{12}$$

$$Cor(X_1, X_2) = \frac{-\frac{n+1}{12}}{\sqrt{\mathbb{D}X_1\mathbb{D}X_2}} = -\frac{\frac{n+1}{12}}{\mathbb{D}X_1} = -\frac{\frac{n+1}{12}}{\frac{n^2-1}{12}} = -\frac{1}{n+1} \xrightarrow{n\to\infty} 0$$