Задача 117. *(K2, СЕМ 2022)* Нека $X_0 \sim Exp(1)$ и $X_n = 2X_{n-1} + \epsilon_n$ за $n \in \mathbb{N}$, където ϵ_n са независими $N(0,1)$ случайни величини.

1. Намерете $\mathbb{E}X_n$ и DX_n.
2. Нека $S_{n,1} = \sum_{i=1}^{n}(X_n - \rho X_{n-1})^2$ и $S_{n,2} = \sum_{i=1}^{n}|X_n - \rho X_{n-1}|$. Намерете $\mathbb{E}S_{n,1}$ и $\mathbb{E}S_{n,2}$.
3. Можете ли да отговорите на въпросите от 1. и 2., когато $\epsilon_i \sim N(1,2)$?

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(117) $X_0 \sim Exp(1)$ и $X_n = 2X_{n-1} + \varepsilon_n$ за $n \in \mathbb{N}$, където $\varepsilon_n \overset{iid}{\sim} N(0,1)$

1. $\mathbb{E}[X_n] = \mathbb{E}[2X_{n-1} + \varepsilon_n] = 2\mathbb{E}[X_{n-1}] + \underset{0}{\underbrace{\mathbb{E}[\varepsilon_n]}} = 2\mathbb{E}[X_{n-1}] =$

$$= 2\mathbb{E}[2X_{n-2} + \varepsilon_n] = 4\mathbb{E}[X_{n-2}] = 2^2\mathbb{E}[X_{n-2}] = 2^n\mathbb{E}[X_0] = 2^n$$

$\mathbb{D}[X_n] = \mathbb{D}[2X_{n-1} + \varepsilon_n] = 4\mathbb{D}[X_{n-1}] + 1 \quad , \mathbb{D}[X_0] = 1$

Нека $\mathbb{D}[X_n] =: a_n$ $\qquad t^2 - 5t + 4 = 0$

$a_n = 4a_{n-1} + 1$ $\qquad t_{1,2} = 1, 4$

$a_{n-1} = 4a_{n-2} + 1$

$a_n - 5a_{n-1} + 4a_{n-2} = 0$

$\Rightarrow a_n = C_1 . 4^n + C_2$

$a_0 = 1, a_1 = 5$

$\Rightarrow \begin{cases} C_1 + C_2 = 1 \\ 4C_1 + C_2 = 5 \end{cases} \Rightarrow C_1 = \frac{4}{3} \quad C_2 = -\frac{1}{3} \Rightarrow a_n = \frac{4}{3}.4^n - \frac{1}{3} = \frac{1}{3}(4^{n+1} - 1) = \mathbb{D}[X_n]$

2. $S_{n,1} = \sum_{i=1}^{n} (X_i - 2X_{i-1})^2 = \sum_{i=1}^{n} (2X_{i-1} + \varepsilon_n - 2X_{i-1})^2 = n.\varepsilon_n^2$

$\mathbb{E}[S_{n,1}] = \mathbb{E}[n.\varepsilon_n^2] = n$

$S_{n,2} = \sum_{i=1}^{n} |X_i - 2X_{i-1}| = \sum_{i=1}^{n} |\varepsilon_n| = n.|\varepsilon_n|$

$\mathbb{E}[S_{n,2}] = n.\int_{-\infty}^{\infty} |\varepsilon_n| . \frac{1}{\sqrt{2\pi}} . e^{-\frac{\varepsilon_n^2}{2}} = n\int_{-\infty}^{0} -\varepsilon_n \frac{1}{\sqrt{2\pi}} e^{-\frac{\varepsilon_n^2}{2}} d\varepsilon_n + n\int_{0}^{\infty} \varepsilon_n . \frac{1}{\sqrt{2\pi}} . e^{-\frac{\varepsilon_n^2}{2}} d\varepsilon_n \overset{x = \varepsilon_n}{=}$

$$= \frac{n}{\sqrt{2\pi}}\left[-\int_{-\infty}^{0} x . e^{-\frac{x^2}{2}} dx + \int_{0}^{\infty} x . e^{-\frac{x^2}{2}} dx\right] =$$

$$= \frac{n}{\sqrt{2\pi}}\left[-\int_{-\infty}^{0} e^{-\frac{x^2}{2}} d\frac{x^2}{2} + \int_{0}^{\infty} e^{-\frac{x^2}{2}} d\frac{x^2}{2}\right] = \frac{n}{\sqrt{2\pi}}\left[\underset{1}{\left[e^{-\frac{x^2}{2}}\right]_{-\infty}^{0}} + \underset{+1}{\left[-e^{-\frac{x^2}{2}}\right]_{0}^{\infty}}\right] =$$

$$= \frac{2n}{\sqrt{2\pi}} = \frac{\sqrt{2}n}{\sqrt{\pi}}$$

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3. $\mathbb{E}[X_n] = \mathbb{E}[2X_{n-1}+\varepsilon_n] = 2\mathbb{E}[X_{n-1}]+1$

$\mathbb{E}[X_0] = 1$

$\mathbb{E}[X_1] = 3$

Нека $a_n := \mathbb{E}[X_n]$

$a_n = 2a_{n-1}+1$ $\qquad$ $t^2-3t+2=0$

$a_{n-1} = 2a_{n-2}+1$ $\qquad$ $t_{1,2} = 1, 2$

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

$\Rightarrow a_n = c_1.2^n + c_2 \Rightarrow \begin{cases} c_1+c_2 = 1 \\ 2c_1+c_2 = 3 \end{cases} \Leftrightarrow \begin{cases} c_2 = 1-c_1 \\ c_1 = 2 \end{cases} \Leftrightarrow \begin{cases} c_1 = 2 \\ c_2 = -1 \end{cases}$

$\Rightarrow a_n = 2^{n+1} - 1 = \mathbb{E}[X_n]$

$\mathbb{D}[X_n] = 4\mathbb{D}[X_{n-1}]+2$

$\mathbb{D}[X_n] = c_1.4^n + c_2$

$\mathbb{D}[X_0] = 1$

$\mathbb{D}[X_1] = 6$ $\Rightarrow \begin{cases} c_1+c_2 = 1 \\ 4c_1+c_2 = 6 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{5}{3} \\ c_2 = -\frac{2}{3} \end{cases}$

$\Rightarrow \mathbb{D}[X_n] = \frac{1}{3}(5.4^n - 2)$

$\mathbb{E}[S_{n,1}] = n.\mathbb{E}[\varepsilon_n^2] = n[\underbrace{\mathbb{D}[\varepsilon_n]}_{=2} + (\underbrace{\mathbb{E}[\varepsilon_n]}_{=1})^2] = 3n$

$\mathbb{E}[S_{n,2}] = \mathbb{E}[n.|\varepsilon_n|]$ Нека $Z \sim N(0,1)$ $\frac{\varepsilon_n - 1}{\sqrt{2}} = Z \Rightarrow \varepsilon_n = Z\sqrt{2}+1$

$\mathbb{E}[S_{n,2}] = n.\mathbb{E}[|Z\sqrt{2}+1|] = n.\int_{-\infty}^{\infty}|z\sqrt{2}+1|.\frac{1}{\sqrt{2\pi}}.e^{-\frac{z^2}{2}}dz$

$z\sqrt{2}+1>0$
$z > -\frac{\sqrt{2}}{2}$ $\Rightarrow \mathbb{E}[S_{n,2}] = -n\int_{-\infty}^{-\frac{\sqrt{2}}{2}}(z\sqrt{2}+1)\frac{1}{\sqrt{2\pi}}.e^{-\frac{z^2}{2}}dz + n\int_{-\frac{\sqrt{2}}{2}}^{\infty}(z\sqrt{2}+1)\frac{1}{\sqrt{2\pi}}.e^{-\frac{z^2}{2}}dz =$

$= \frac{n}{\cancel{\sqrt{2\pi}}}\left[\int_{-\frac{\sqrt{2}}{2}}^{\infty}(z\sqrt{2}+1)\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz - \int_{-\infty}^{-\frac{\sqrt{2}}{2}}(z\sqrt{2}+1).\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right] = \frac{n}{\sqrt{2\pi}}\left[\sqrt{2}\int_{-\frac{\sqrt{2}}{2}}^{\infty} z.\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz + \underbrace{\int_{-\frac{\sqrt{2}}{2}}^{\infty}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz}_{\Phi(\frac{\sqrt{2}}{2})} - \sqrt{2}\int_{-\infty}^{-\frac{\sqrt{2}}{2}} z f_z dz + \underbrace{\int_{-\infty}^{-\frac{\sqrt{2}}{2}} f_z dz}_{\Phi(-\frac{\sqrt{2}}{2})}\right.$

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$$= n\Big[\Phi\Big(\frac{\sqrt{2}}{2}\Big) - \underbrace{\Phi\Big(-\frac{\sqrt{2}}{2}\Big)}_{1-\Phi(\frac{\sqrt{2}}{2})} + \sqrt{2}\int_{-\frac{\sqrt{2}}{2}}^{\infty} z \cdot \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz - \sqrt{2}\int_{-\infty}^{-\frac{\sqrt{2}}{2}} z \cdot \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz\Big] =$$

$$= n\Big[2\Phi\Big(\frac{\sqrt{2}}{2}\Big) - 1 + \frac{1}{\sqrt{\pi}}\Big[\int_{-\frac{\sqrt{2}}{2}}^{\infty} e^{-\frac{z^2}{2}}\,d\frac{z^2}{2} - \int_{-\infty}^{-\frac{\sqrt{2}}{2}} e^{-\frac{z^2}{2}}\,d\frac{z^2}{2}\Big]\Big] =$$

$$= n\Big[2\Phi(0.71) - 1 + \frac{1}{\sqrt{\pi}}\Big[-e^{-\frac{z^2}{2}}\Big|_{-\frac{\sqrt{2}}{2}}^{\infty} + e^{-\frac{z^2}{2}}\Big|_{-\infty}^{-\frac{\sqrt{2}}{2}}\Big] =$$

$$= n\Big[2\Phi(0.71) - 1 + \frac{1}{\sqrt{\pi}}\Big[+e^{-\frac{1}{4}} + e^{-\frac{1}{4}}\Big] =$$

$$= n\Big[2\Phi(0.71) - 1 + \frac{2}{\sqrt{\pi}} \cdot e^{-\frac{1}{4}} \simeq 1.3993 \cdot n$$

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