Задача 56. A и B играят последователно партии, като A печели една партия с вероятност $2/3$, а B - с $1/3$. Равни партии не са възможни. Играта продължава докато някой спечели две последователни партии. Нека X е случайната величина „брой на изиграните партии". Да се определи разпределението и математическото очакване на X.

56) A и B играят до 2 последователни победи

$\mathbb{P}(A)=\frac{2}{3}$ и $\mathbb{P}(B)=\frac{1}{3}$

$\mathbb{P}(A\cap B)=0$

I сл. A печели

2 игри: $\left(\frac{2}{3}\right)^2$ 5 игри: $\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\left(\frac{2}{3}\right)^2=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$

3 игри: $\frac{1}{3}\left(\frac{2}{3}\right)^2$ 6 игри: $\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\left(\frac{2}{3}\right)^2=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$

4 игри: $\frac{2}{3}\cdot\frac{1}{3}\left(\frac{2}{3}\right)^2=\frac{1}{3}\cdot\left(\frac{2}{3}\right)^3$

Нека X = A печели при k хода:

$$\mathbb{P}(X=2k+2)=\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{k+2}=\frac{4}{9}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k$$

$$\mathbb{P}(X=2k+1)=\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{k+1}=\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k-\frac{2}{3}$$

Нека Y = B печели при k хода

$$\mathbb{P}(Y=2k+2)=\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k}\left(\frac{1}{3}\right)^{k+2}=\frac{1}{9}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k$$

$$\mathbb{P}(Y=2k+1)=\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k}\left(\frac{1}{3}\right)^{k+1}=\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k-\frac{1}{3}$$

Нека Z = брой изиграни партии => $Z=X\cup Y$

Z	2	3	$2k+1$	$2k+2$	----
$\mathbb{P}$	$\frac{5}{9}$	-----	$\sum_{k=1}^{\infty}\left(\frac{2}{9}\right)^k$	$\frac{5}{9}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k$	

$\mathbb{P}(Z=2)=\frac{4}{9}+\frac{1}{9}=\frac{5}{9}$

$\mathbb{P}(Z=3)=\frac{2}{3}\sum_{k=0}^{1}\left(\frac{2}{9}\right)^k-\frac{2}{3}+\frac{1}{3}\sum_{k=0}^{1}\left(\frac{2}{9}\right)^k-\frac{1}{3}$

$$\mathbb{E}[Z]=\frac{5}{9}\sum_{k=0}^{\infty}(2k+2)\left(\frac{2}{9}\right)^k+\underbrace{\sum_{k=1}^{\infty}(2k+1)\left(\frac{2}{9}\right)^k}_{\sum_{k=0}^{\infty}(2k+1)\left(\frac{2}{9}\right)^k-1}=$$

$$=\frac{10}{9}\sum_{k=0}^{\infty}k\left(\frac{2}{9}\right)^k+\frac{10}{9}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k+2\sum_{k=0}^{\infty}k\cdot\left(\frac{2}{9}\right)^k+\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^k-1=$$

$$= \frac{28}{9}\sum_{k=0}^{\infty} k\cdot\left(\frac{2}{9}\right)^{k} + \frac{19}{9}\sum_{k=0}^{\infty}\left(\frac{2}{9}\right)^{k} - 1$$

$$(*)\ \sum_{k=0}^{\infty} k\cdot x^{k} = x\cdot\frac{d}{dx}\sum_{k=0}^{\infty} x^{k} = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^{2}}$$

$$\overset{(*)}{\Longrightarrow}\ \frac{28}{9}\cdot\frac{\frac{2}{9}}{\left(\frac{7}{9}\right)^{2}} + \frac{19}{9}\cdot\frac{1}{1-\frac{2}{9}} - 1 = \frac{\overset{8}{\cancel{56}}}{\cancel{9^{2}}}\cdot\frac{\cancel{9^{2}}}{\cancel{7}^{2}} + \frac{19}{\cancel{9}}\cdot\frac{\cancel{9}}{7} - 1 =$$

$$= \frac{8}{7} + \frac{12}{7} = \frac{20}{7}$$