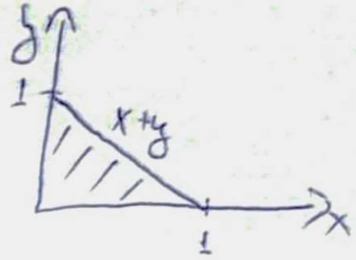
Задача 1. *Нека съвместната плътност на X и Y е $f_{X,Y}(x,y) = cx + 1$ за $x, y \geq 0, x + y \leq 1$ и 0 извън тази област, където c е някаква константа. Намерете:*

1. *(0.75 т.) c и $Cov(X, Y)$;*
2. *(0.25 т.) $\mathbb{E}(X|Y = 1/2)$.*

① $f_{X,Y}(x,y) = cx+1$

$\begin{cases} x,y \geq 0 \\ x+y \leq 1 \end{cases} \Leftrightarrow \begin{cases} x \in (0,1) \\ y \in (0,1-x) \end{cases}$



$c = ?$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1$$

$$\int_0^1\int_0^{1-x} cx+1 \, dy\,dx = \int_0^1 \left[cx[y]_0^{1-x} + [y]_0^{1-x} \right] dx = \int_0^1 (1-x)(cx+1)\,dx =$$

$$= \int_0^1 cx\,dx + \int_0^1 dx - \int_0^1 cx^2\,dx - \int_0^1 x\,dx = c\left[\frac{x^2}{2}\right]_0^1 + [x]_0^1 - c\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1 =$$

$$= \frac{c}{2} + 1 - \frac{c}{3} - \frac{1}{2} = \frac{c}{6} + \frac{1}{2} \Rightarrow \boxed{c = 3}$$

$$E[XY] = \int_0^1 \int_0^{1-x} xy(cx+1)\,dy\,dx = \int_0^1 \int_0^{1-x} cx^2y + xy\,dy\,dx = \int_0^1 \left[cx^2 \left[\frac{y^2}{2}\right]_0^{1-x} + x\left[\frac{y^2}{2}\right]_0^{1-x} \right] dx =$$

$$= \int_0^1 \frac{cx^2+x}{2}(1-x)^2\,dx = \frac{1}{2}\int_0^1 (cx^2+x)(1-2x+x^2)\,dx = \frac{1}{2}\int_0^1 cx^2 - 2cx^3 + cx^4 + x - 2x^2 + x^3\,dx =$$

$$= \frac{1}{2}\left(c\left[\frac{x^3}{3}\right]_0^1 - 2c\left[\frac{x^4}{4}\right]_0^1 + c\left[\frac{x^5}{5}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1 \right) =$$

$c=3$

$$= \frac{1}{2}\left(\cancel{1} - \cancel{\frac{3}{2}} + \frac{3}{5} + \cancel{\frac{1}{2}} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{2}\left(\frac{36-40+15}{60} \right) = \frac{11}{120}$$

$$\mathbb{E}[X] = \int_0^1\int_0^{1-x} x(cx+1)\,dy\,dx = \int_0^1\int_0^{1-x} cx^2+x\,dy\,dx = \int_0^1 \left[cx^2(1-x)+x(1-x)\right]dx =$$

$$= \int_0^1 cx^2 - cx^3 + x - x^2\,dx = c\left[\frac{x^3}{3}\right]_0^1 - c\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 =$$

$$\stackrel{c=3}{=} \cancel{1} - \frac{3}{4} + \frac{1}{2} - \frac{1}{3} = \frac{5}{12}$$

$$\mathbb{E}[Y] = \int_0^1\int_0^{1-x} y(3x+1)\,dy\,dx = \int_0^1\int_0^{1-x} 3xy+y\,dy\,dx = \int_0^1\left[3x\left[\frac{y^2}{2}\right]_0^{1-x} + \left[\frac{y^2}{2}\right]_0^{1-x}\right]dx =$$

$$= \int_0^1 \frac{(3x+1)(1-2x+x^2)}{2}\,dx = \frac{1}{2}\int_0^1 3x - 6x^2 + 3x^3 + 1 - 2x + x^2\,dx =$$

$$= \frac{1}{2}\left(\left[\frac{x^2}{2}\right]_0^1 - 5\left[\frac{x^3}{3}\right]_0^1 + 3\left[\frac{x^4}{4}\right]_0^1 + [x]_0^1\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{5}{3} + \frac{3}{4} + 1\right) =$$

$$= \frac{1}{2}\left(\frac{9}{4} - \frac{5}{3}\right) = \frac{7}{24}$$

$$\Rightarrow Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{11}{120} - \frac{5}{12}\cdot\frac{7}{24} \approx -0.02986$$

2. $$\mathbb{E}[X \mid Y = \tfrac{1}{2}] = \int_0^1 x \cdot f_{X|Y}(x \mid y=\tfrac{1}{2})\,dx = \int_0^1 x \cdot \frac{f_{X,Y}(x,\frac{1}{2})}{f_Y(\frac{1}{2})}\,dx$$

$$f_Y(y) = \int_0^{1-y} 3x+1\,dx = 3\left[\frac{x^2}{2}\right]_0^{1-y} + [x]_0^{1-y} = \frac{3}{2}(1-2y+y^2) + 1 - y = \frac{5}{2} - 4y + \frac{3}{2}y^2 \cdot \mathbb{1}_{\{y\in(0,1)\}}$$

$$f_Y(\tfrac{1}{2}) = \frac{5}{2} - \frac{4}{2} + \frac{3}{8} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$$

$$f_{X,Y}(x,\tfrac{1}{2}) = cx+1$$

$$\Rightarrow \mathbb{E}[X \mid Y=\tfrac{1}{2}] = \int_0^{\frac{1}{2}} x\cdot\frac{3x+1}{\frac{7}{8}}\,dx = \frac{8}{7}\int_0^{\frac{1}{2}} 3x^2 + x\,dx = \frac{8}{7}\left[3\left[\frac{x^3}{3}\right]_0^{\frac{1}{2}} + \left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}\right] =$$

$$= \frac{8}{7}\left[\frac{1}{8} + \frac{1}{8}\right] = \frac{\cancel{8}}{7}\cdot\frac{2}{\cancel{8}} = \frac{2}{7}$$