Задача 124. *(K2, ВиС 2023)* Нека $X_1, X_2 \sim \Gamma(3,2)$, т.е $f_X(x) = 4x^2e^{-2x}\mathbb{1}_{\{x>0\}}$.

1. Намерете плътността $Y = X_1/(X_1 + X_2)$.

2. Намерете $Cor(X_1, X_2)$ и $Cor(X_1 + X_2, Y)$.

3. Независими ли са $X_1 + X_2$ и Y?

(124) $X_1, X_2 \overset{iid}{\sim} \Gamma(3,2)$

$$f_X(x) = 4x^2 \cdot e^{-2x} \cdot \mathbb{1}_{\{x>0\}}$$

$$f_{X_1,X_2}(x_1,x_2) \overset{iid}{=} f_{X_1}(x_1) f_{X_2}(x_2) = 16 (x_1 x_2)^2 \cdot e^{-2(x_1+x_2)} \cdot \mathbb{1}_{\{x_1,x_2>0\}}$$

1. $y = \dfrac{x_1}{x_1+x_2}$ $\quad y = \dfrac{z}{z+x_2}$ $\quad x_1 = z > 0$

$z = x_1 > 0$ $\quad x_2 = \dfrac{z}{y} - z > 0 \overset{:z}{\iff} \dfrac{1}{y} > 1 \Rightarrow 0 < y < 1 \Rightarrow \left| \begin{array}{l} z \in (0,\infty) \\ y \in (0,1) \end{array} \right.$

$$f_{Y,Z}(y,z) = f_{X_1,X_2}\left(z, \frac{z}{y} - z\right) \cdot |J(y,z)|$$

$$J(y,z) = \begin{vmatrix} \frac{\partial}{\partial y} x_1 & \frac{\partial}{\partial z} x_1 \\ \frac{\partial}{\partial y} x_2 & \frac{\partial}{\partial z} x_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{z}{y^2} & \frac{1}{y} - 1 \end{vmatrix} = \frac{z}{y^2}$$

$$f_{Y,Z}(y,z) = 16 \left(z \left[\frac{z}{y} - z \right] \right)^2 \cdot e^{-2\left(z + \frac{z}{y} - z\right)} \cdot \frac{z}{y^2} = 16 \left(\frac{z^2}{y} - z^2 \right)^2 \cdot e^{-2\frac{z}{y}} \cdot \frac{z}{y^2}$$

$$f_Y(y) = \int_{z=0}^{z=\infty} f_{Y,Z}(y,z)\, dz = 16 \int_0^\infty \left(\frac{z^4}{y^2} - \frac{2z^4}{y} + z^4 \right) \cdot e^{-2\frac{z}{y}} \cdot \frac{z}{y^2}\, dz =$$

$$= 16 \left[\int_0^\infty \frac{z^5}{y^4} \cdot e^{-2\frac{z}{y}} dz - \int_0^\infty \frac{2z^5}{y^3} e^{-2\cdot\frac{z}{y}} dz + \int_0^\infty \frac{z^5}{y^2} \cdot e^{-2\cdot\frac{z}{y}} dz \right] =$$

$$= 16 \left[\left(\frac{1}{y^4} - \frac{2}{y^3} + \frac{1}{y^2} \right) \underbrace{\int_0^\infty z^5 \cdot e^{-2\frac{z}{y}} dz}_{= I} \right] = 16 \left(\frac{1}{y^4} - \frac{2}{y^3} + \frac{1}{y^2} \right) \cdot \frac{15 y^6}{8} =$$

$$= 30 \left(y^2 - 2y^3 + y^4 \right) \cdot \mathbb{1}_{\{y \in (0,1)\}}$$

$$= 30\, y^2 (1-y)^2 \cdot \mathbb{1}_{\{y \in (0,1)\}}$$

$$I=\int_0^\infty z^5\cdot e^{-2\cdot\frac{z}{y}}\,dz=\int_0^\infty z^5\,d\left(-\frac{1}{2}\cdot y e^{-\frac{2z}{y}}\right)=\underbrace{-\frac{y}{2}\cdot z^5\cdot e^{-\frac{2z}{y}}\Big|_{z=0}^{z=\infty}}_{0}+\frac{y}{2}\int_0^\infty e^{-\frac{2z}{y}}\,dz^5=$$

$$=\frac{5y}{2}\int_0^\infty z^4\cdot e^{-\frac{2z}{y}}\,dz=\frac{5y}{2}\int_0^\infty z^4\cdot d\left(-\frac{y}{2}\cdot e^{-\frac{2z}{y}}\right)=$$

$$=\frac{5y}{2}\left[\underbrace{-\frac{y\cdot z^4}{2}e^{-\frac{2z}{y}}\Big|_0^\infty}_{0}+\frac{4y}{2}\int_0^\infty z^3\cdot e^{-\frac{2z}{y}}\,dz\right]=$$

$$=5y^2\left[-\frac{y\cdot z^3}{2}e^{-\frac{2z}{y}}\Big|_0^\infty+\frac{3y}{2}\int_0^\infty z^2\cdot e^{-\frac{2z}{y}}\,dz\right]=$$

$$=5y^3\cdot\frac{3}{2}\left[\underbrace{-\frac{y\cdot z^2}{2}\cdot e^{-\frac{2z}{y}}\Big|_0^\infty}_{0}+\frac{2y}{2}\int_0^\infty z\cdot e^{-\frac{2z}{y}}\,dz\right]=$$

$$=\frac{15y^4}{2}\left[\underbrace{-\frac{y\cdot z}{2}\cdot e^{-\frac{2z}{y}}\Big|_0^\infty}_{0}+\frac{y}{2}\int_0^\infty e^{-\frac{2z}{y}}\,dz\right]=$$

$$=\frac{15y^5}{4}\left[-\frac{y}{2}\cdot e^{-\frac{2z}{y}}\Big|_0^\infty\right]=\frac{15y^6}{8}$$

$Cor(X_1+X_2, Y)$

$Z = X_1+X_2 \quad , Z \in (0,\infty)$

$f_{Y,Z} = 16\left(\frac{Z^2}{Y} - Z^2\right)^2 \cdot e^{-2\cdot\frac{Z}{Y}} \cdot \frac{Z}{Y^2}$

$f_{X_1+X_2} = \int\limits_{0}^{Z} f_{X_1}(x) f_{X_2}(Z-x)\,dx = \frac{8}{15} \cdot e^{-2Z} \cdot Z^5$

$\Rightarrow f_{Z,Y}(Z,Y) = f_Z(Z) \cdot f_Y(Y) \Rightarrow Z$ и Y са независими

$\Rightarrow Cor(X_1+X_2, Y) = 0$

$\mathbb{E}[Z] = \mathbb{E}[X_1+X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{3}{2} + \frac{3}{2} = 3 \quad \mathbb{E}[Z^2] =$

$\mathbb{E}[Y] = \int\limits_0^1 y \cdot 30(y^2(1-y))^2 dy = \frac{1}{2}$

$\mathbb{E}[Y^2] = \int\limits_0^1 y^2 \cdot 30(y^2(1-y))^2 dy = \frac{2}{7}$

$\Rightarrow D[Y] = \frac{2}{7} - \frac{1}{4} = \frac{1}{28}$