**Задача 97.** Нека случайните величини $X_1, X_2 \sim Exp(\lambda)$ са независими. Да се намери разпределението на случайната величина $Y = X_1/(X_1 + X_2)$.

(97) $X_1, X_2 \overset{iid}{\sim} Exp(\lambda)$

$Y = X_1/(X_1+X_2) \qquad (X_1, X_2) \in [0,\infty)^2$

$X_1 \perp\!\!\!\perp X_2 \iff f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} = \lambda^2 e^{-\lambda(x_1+x_2)}$

Нека $Z := X_1 \qquad Y = \frac{Z}{Z+X_2} \qquad X_2 = \frac{Z}{Y} - Z$

$$f_{Y,Z}(y,z) = f_{X_1,X_2}\left(z, \frac{z}{y} - z\right) |J(y,z)| \, 1_{\{(z, \frac{z}{y}-z) \in (0,\infty)^2\}}$$

$$J(y,z) = \begin{vmatrix} \frac{\partial}{\partial y} x_1 & \frac{\partial}{\partial z} x_1 \\ \frac{\partial}{\partial y} x_2 & \frac{\partial}{\partial z} x_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{z}{y^2} & \frac{1}{y} - 1 \end{vmatrix} = \frac{z}{y^2}$$

$$\Rightarrow f_{Y,Z}(y,z) = \lambda^2 e^{-\lambda \frac{z}{y}} \cdot \frac{z}{y^2} \cdot 1_{\{y \in (0,1)\}} \cdot 1_{\{z \in (0,\infty)\}}$$

$$f_Y(y) = \int_{z=-\infty}^{z=\infty} f_{Y,Z}(y,z)\,dz = \int_{z=0}^{z=\infty} \frac{\lambda^2 z}{y^2} e^{-\lambda \frac{z}{y}} dz \cdot 1_{\{y \in (0,1)\}} =$$

$$\underset{\frac{d}{dz}u = \frac{\lambda}{y}}{\overset{u = \lambda \frac{z}{y}}{=}} \int_{u=0}^{u=\infty} u e^{-u} du = -\int_0^\infty u \, de^{-u} \cdot 1_{\{y \in (0,1)\}} = -\underbrace{\left[u e^{-u}\right]_0^\infty}_{0} \cdot 1_{\{y\in(0,1)\}} + \int_0^\infty e^{-u} du =$$

$$= -\left[e^{-u}\right]_0^\infty 1_{\{y \in (0,1)\}} = 1 \cdot 1_{\{y \in (0,1)\}}, \text{ i.e. } Y \sim Unif(0,1)$$