Задача 98. Нека случайните величини $X_1, X_2 \sim U(0,1)$ са независими. Да се намери разпределението на случайната величина $Y = X_1 + X_2$.

(98) $X_1, X_2 \overset{iid}{\sim} Unif(0,1)$

$Y = X_1 + X_2$

Нека $Z = X_1$ $\quad Y = Z + X_2 \quad X_2 = Y - Z$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = 1_{\{x_1 \in (0,1)\}} \cdot 1_{\{x_2 \in (0,1)\}}$$

$$f_{Y,Z}(y,z) = f_{X_1,X_2}(z, y-z) \cdot |J(y,z)| = 1_{\{z \in (0,1)\}} \cdot 1_{\{(y-z) \in (0,1)\}}$$

$$J(y,z) = \begin{vmatrix} \frac{\partial}{\partial y} x_1 & \frac{\partial}{\partial z} x_1 \\ \frac{\partial}{\partial y} x_2 & \frac{\partial}{\partial z} x_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\,dz = \int_{-\infty}^{\infty} 1_{\{\underset{x_1}{z} \in (0,1)\}} \cdot 1_{\{(y-z) \in (0,1)\}}\,dz =$$

$$\overset{1)}{=} \int_{z=0}^{z=y} dz = y, \quad y \in [0,1)$$

$$\overset{2)}{=} \int_{y-1}^{1} dz = 2 - y, \quad y \in [1,2)$$

$f_Y(y) = 0$ при $y \notin (0,2)$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & , y \le 0 \\ \int_0^y u\,du = \frac{y^2}{2} & , y \in (0,1) \\ \int_0^1 u\,du + \int_{y-1}^{1} (2-u)\,du = \frac{-y^2+4y-2}{2} & , y \in (1,2] \\ 1 & , y > 2 \end{cases}$$

(98) $X_1, X_2 \overset{iid}{\sim} Unif(0,1)$

$Y = X_1 + X_2$

$Z = X_1 \quad Y = Z + X_2 \quad X_2 = Y - Z$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = 1 \cdot \{x_1 \in (0,1)\} \cdot 1\{x_2 \in (0,1)\}$$

$$f_Y(y) = f_{X_1+X_2}(y) = \int_{\mathbb{R}} f_{X_1,X_2}(z, y-z)\,dz = \int_{\mathbb{R}} f_{X_1}(z) \cdot f_{X_2}(y-z)\,dz \overset{(*)}{=} \int_{\mathbb{R}} 1\{z\in(0,1)\} \cdot 1\{(y-z)\in(0,1)\}\,dz =$$

$$\overset{1)}{=} \int_0^{z=y} 1 \cdot \{(y-z) \in (0,1)\}\,dz = y, \; y \in (0,1]$$

$$\overset{2)}{=} \int_{y-1}^{1} 1 \cdot \{(y-z) \in (0,1)\}\,dz = 2-y, \; y \in [1,2)$$

$f_{X_1+X_2}(y) = 0$, при $y \notin (0,2)$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & , y \le 0 \\ \int_0^y u\,du = \frac{y^2}{2} & , y \in (0,1) \\ \int_0^1 u\,du + \int_1^y 2-y\,dy = -\frac{y^2+4y-2}{2} & , y \in (1,2] \\ 1 & , y > 2 \end{cases}$$