Задача 4. Нека съвместната плътност на X и Y е $f_{X,Y}(x,y) = cx^3 + 1$ за $x, y \geq 0, x + 4y \leq 1$ и 0 извън тази област, където c е някаква константа. Намерете:

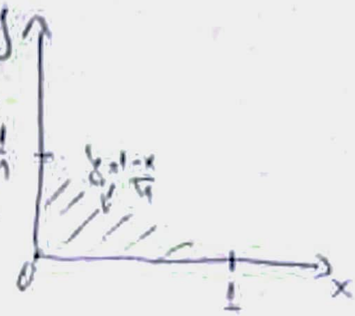
1. (0.5 т.) c, плътността на X и очакването на Y;
2. (0.25 т.) $\mathbb{E}(Y|X = 1/2)$;
3. (0.25 т.) плътността на случайната величина $Z = X + 2Y$.

④ $f_{X,Y}(x,y) = cx^3+1$ $\quad \begin{cases} x,y\ge 0 \\ x+4y\le 1\end{cases} \iff \begin{cases} y\le \frac{1-x}{4} \\ x\ge 0\end{cases} \Rightarrow \begin{matrix} x\in(0,\infty) \\ y\in(0,\frac{1-x}{4})\end{matrix} \quad \begin{matrix}\frac{1-x}{4}>0 \\ x<1\end{matrix}$

1. $\int_0^1\int_0^{\frac{1-x}{4}} f_{X,Y}(x,y)\,dydx = \int_0^1\int_0^{\frac{1-x}{4}} cx^3+1\,dydx = \int_0^1 \left[cx^3\int_0^{\frac{1-x}{4}}dy + \int_0^{\frac{1-x}{4}}dy\right]dx =$

$= \int_0^1 \left[cx^3[y]_0^{\frac{1-x}{4}} + [y]_0^{\frac{1-x}{4}}\right]dx = \int_0^1 \frac{1-x}{4}(cx^3+1)\,dx =$

$= \int_0^1 \frac{cx^3}{4} - \frac{cx^4}{4} + \frac14 - \frac x4\,dx = \frac c4\left[\frac{x^4}{4}\right]_0^1 - \frac c4\left[\frac{x^5}{5}\right]_0^1 + \frac14[x]_0^1 - \frac14\left[\frac{x^2}{2}\right]_0^1$



$= \frac{c}{16} - \frac{c}{20} + \frac14 - \frac18 = \frac{c}{80} + \frac{10}{80} \Rightarrow c = 70$

$f_X(x) = \int_0^{\frac{1-x}{4}} 70x^3+1\,dy = 70x^3[y]_0^{\frac{1-x}{4}} + [y]_0^{\frac{1-x}{4}} = \frac{1-x}{4}(70x^3+1)\cdot 1\{x\in(0,1)\}$

$E[Y] = \int_0^{\frac{1-x}{4}} y\cdot f_Y(y)\,dy = \frac{37}{2}\int_0^{\frac{1-x}{4}} y\,dy = \frac{37}{2}\left[\frac{y^2}{2}\right]_0^{\frac{1-x}{4}} = \frac{37}{4\cdot 16}(1-x)^2$

$f_Y(y) = \int_0^1 70x^3+1\,dx = 70\left[\frac{x^4}{4}\right]_0^1 + [x]_0^1 = \frac{70+4}{4} = \frac{74}{4} = \frac{37}{2}$

2. $E[Y|X=\frac12] = \int_0^{\frac{1-x}{4}} y\, f_{Y|X}(y|x=\tfrac12)\,dy = \int_0^{\frac{1-x}{4}} y\cdot\frac{f_{X,Y}(\frac12,y)}{f_X(\frac12)}\,dy = \int_0^{\frac{1-x}{4}} y\cdot \frac{\frac18 c+1}{\frac12\cdot\frac18(\frac{70}{8}+1)}\,dy$

$= \int_0^{\frac{1-x}{4}} y\cdot\frac{\overset{70}{c}+8}{78}\,dy = \int_0^{\frac{1-x}{4}} y\,dy = \left[\frac{y^2}{2}\right]_0^{\frac{1-x}{4}} = \frac{(1-x)^2}{32} \overset{x=\frac12}{=} \frac{1}{128}$

3. $Z = X+2Y$, $W = X$ $\quad \begin{matrix} X=W \\ Y=\frac{Z-W}{2}\end{matrix} \Rightarrow \begin{cases} w>0 \\ \frac{z-w}{2}\in(0,\frac{1-w}{4})\end{cases} \Rightarrow \begin{cases} \frac{z-w}{2}>0 \\ \frac{z-w}{2}<\frac{1-w}{4} \\ w>0\end{cases} \Rightarrow \begin{cases} w<z \\ 2z-2w<1-w \\ w>0\end{cases} \Rightarrow \begin{cases} w<z \\ w>0 \\ z<\frac{1+w}{2}\end{cases}$

$f_{Z,W}(z,w) = f_{X,Y}(w,\frac{z-w}{2})\,|J(z,w)| = \frac{cw^3+1}{2}$ $\quad \Rightarrow w\in(0,z)$, $z\in(0,\frac{1+w}{2})$

$J(z,w) = \det\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w}\end{vmatrix} = \begin{vmatrix}0 & 1\\ \frac12 & -\frac12\end{vmatrix} = \frac12$

$f_Z(z) = \int_0^z f_{Z,W}(z,w)\,dw = \int_0^z \frac{cw^3+1}{2}\,dw = \frac c2\left[\frac{w^4}{4}\right]_0^z + \frac12[w]_0^z = \frac{cz^4}{8}+\frac z2 = \frac{70z^4+4z}{8}\cdot 1\{z\in(0,\frac{1+w}{2})\}$