1. Propagation model:

$$\begin{bmatrix} x \\ y \end{bmatrix}_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_{t-1} + \begin{bmatrix} \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}_t + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t$$

1) $X = \begin{bmatrix} x \\ y \end{bmatrix}$ - state vector

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ - dynamic matrix

$B = \begin{bmatrix} \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}$ - input gain matrix

$u = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ - control input.

$\eta = \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0{,}1 & 0 \\ 0 & 0{,}1 \end{bmatrix}\right)$ - noise vector

Hence: $X_t = AX_{t-1} + Bu_t + \eta$

2) Assume: $X_{t-1} \sim N\left(\begin{bmatrix} \mu_{x_{t-1}} \\ \mu_{y_{t-1}} \end{bmatrix}, \begin{bmatrix} \Sigma_{x_{t-1}} & 0 \\ 0 & \Sigma_{y_{t-1}} \end{bmatrix}\right)$, x, y are independent

$$\Sigma_{X_{t-1}} = E\{(X_{t-1} - \mu_{X_{t-1}})(X_{t-1} - \mu_{X_{t-1}})^T\}$$

3) Noise model: $\eta \sim N(\mu_\eta, \Sigma_\eta)$

$$\Sigma_\eta = E\{(\eta - \mu_\eta)(\eta - \mu_\eta)^T\}$$

4) Cross-covariance X_{t-1} and η:

$$\Sigma_{X_{t-1}\eta} = E\{(X_{t-1} - \mu_{X_{t-1}})(\eta - \mu_\eta)^T\}$$

5) Properties of X_t:

$$\mu_{X_t} = E\{X_t\} = E\{AX_{t-1} + Bu_t + \eta\} = E\{AX_{t-1}\} + E\{Bu_t\} + E\{\eta\} \Rightarrow$$

$$\mu_{X_t} = A\mu_{X_{t-1}} + Bu_t + \mu_\eta$$

$$\Sigma_{X_t} = E\{(X_t - \mu_{X_t})(X_t - \mu_{X_t})^T\} = E\{(AX_{t-1} + \cancel{Bu_t} + \eta - A\mu_{X_{t-1}} - \cancel{Bu_t} - \mu_\eta) \cdot$$

$$\cdot (AX_{t-1} + \cancel{Bu_t} + \eta - A\mu_{X_{t-1}} - \cancel{Bu_t} - \mu_\eta)^T\} \Rightarrow$$

$$\Sigma_{X_t} = E\{[\underbrace{A(X_{t-1} - \mu_{X_{t-1}})}_{a} + \underbrace{(\eta - \mu_\eta)}_{b}][A(X_{t-1} - \mu_{X_{t-1}}) + (\eta - \mu_\eta)]^T\} =$$

$$= E\{[a+b][a+b]^T\} = E\{[\overset{①}{aa^T} + \overset{③}{2ab^T} + \overset{②}{bb^T}]\}$$

① $aa^T = A\underbrace{(X_{t-1} - \mu_{X_{t-1}})(X_{t-1} - \mu_{X_{t-1}})^T}A^T$

$E\{aa^T\} = A\,\Sigma_{X_{t-1}}A^T$

② $bb^T = \underbrace{(\eta - \mu_\eta)(\eta - \mu_\eta)^T}$

$E\{bb^T\} = E\{\ \} = \Sigma_\eta$

③ $2ab^T = \underbrace{2 \cdot (X_{t-1} - \mu_{X_{t-1}})(\eta - \mu_\eta)}$

$E\{2ab^T\} = 2 \cdot E\{\ \} = 2\,\Sigma_{X_{t-1}\eta}$

Thus:

$\cancel{X}$ $\Sigma_{X_t} = A\,\Sigma_{X_{t-1}}A^T + \Sigma_\eta + \cancelto{0}{2\,\Sigma_{X_{t-1}\eta}}$ as state X_{t-1} and η at t time step are ~~independent~~ uncorrelated.

$$\Sigma_{X_t} = A\,\Sigma_{X_{t-1}}A^T + \Sigma_\eta$$