# The Foundations of Mathematics

The foundations of mathematics involves the axiomatic method. This means that in mathematics, one writes down axioms and proves theorems from the axioms. The justification for the axioms (why they are interesting, or true in some sense, or worth studying) is part of the motivation, or physics, or philosophy, not part of the mathematics. The mathematics itself consists of logical deductions from the axioms. Here is example of the axiomatic method.

Example: Group Theory. The group idea, as applied to permutations and algebraic equations, dates from around 1800 (Ruffini 1799, Abel 1824, Galois 1832). The axiomatic treatment is usually attributed to Cayley (1854). We shall list all the group axioms because they are simple and will provide a useful example for us as we go on. A group is a model for the axioms :

Here, we’re saying that G is a set and · is a function from into such that and hold in (with “” meaning “for all ”). Axiom is the associative law. Axiom says that there is an identity element , and that for every , there is an inverse , such that . From the axioms, one proves theorems. For example, the group axioms imply the cancellation rule. We say:

This turnstile symbol “” is read “proves”. This formal presentation is definitely not a direct quote from Cayley, who stated his axioms in English. Rather, it is influenced by the mathematical logic and set theory of the 1900s. Prior to that, axioms were stated in a natural language (e.g., Greek, English, etc.), and proofs were just given in “ordinary reasoning”; exactly what a proof is was not formally analyzed. This is still the case now in most of mathematics. Logical symbols are frequently used as abbreviations of English words, but most math books assume that you can recognize a correct proof when you see it, without formal analysis. This formal analysis makes a clear distinction between syntax and semantics. GP is viewed as a set of two sentences in predicate logic; this is a formal language with precise rules of formation (just like computer languages such as C or java or TEX or html). A formal proof is then a finite sequence of sentences in this formal language obeying some precisely defined rules of inference – for example, the Modus Ponens rule says that from and you can infer . So, the sentences of predicate logic and the formal proofs are syntactic objects. The syntax and semantics are related by the Completeness Theorem which says that iff is true in all groups. After the Completeness Theorem, model theory and proof theory diverge. Proof theory studies more deeply the structure of formal proofs, whereas model theory emphasizes primarily the semantics – that is, the mathematical structure of the models. For example, let be an infinite group. Then has a subgroup which is countably infinite. Also, given any cardinal number , there is a group of size . Proving these statements is an easy algebra exercises if you know some set theory. These statements are part of model theory, not group theory, because they are special cases of the L¨owenheim–Skolem-Tarski Theorem, which applies to models of arbitrary theories. You can also get to satisfy all the first-order properties true in . For example if is non-abelian, then will be also. Likewise for other properties, such as “abelian” or “3-divisible” . The proof, along with the definition of “first-order”, is part of model theory, but the proof uses facts about cardinal numbers from set theory.

[[1]](#footnote-2)

1. Kenneth Kunen (2007) The Foundations of Mathematics,   
    [www.math.wisc.edu/~miller/old/m771-10/kunen770.pdf](http://www.math.wisc.edu/~miller/old/m771-10/kunen770.pdf) [↑](#footnote-ref-2)