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Convenient Plot for Exponential Functions with Unknown Asymptotes

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FIG. 2. Microphotograph of a typical many-body suspension viewed in the vertical $z-y$ plane. Experimental values $\Omega=60$ cps, $V_{ac}=300$ volts rms, $V_{dc}=0$. The approximate e/m of the particles is 0.01 coulomb/kg. Picture magnification ×10.

produces cubic symmetry by the application of three-phase voltages rather than the axial symmetry obtained with single-phase excitation. In this new system the external electrode structure is arranged in the form of a hollow cube with each of the three sets of opposite faces connected to the terminals of a Y connected three-phase supply. A dc voltage could be connected in series with one or two of the leads of the three-phase supply. The trajectories taken by charged particles confined by the new system are different since in the cubic case, the average restoring field is isotropic about the center of the box. Forced vibrations which are straight lines in the axial case now become elliptical paths. The cubical electrode arrangement further offers certain advantages in simplicity of construction, optical illumination and observation, irradiation of contained dust particles by electron and ion beams, mass injection, and the application of auxilliary fields.

This new system (without dc) can be analyzed directly by solving LaPlace's equation for the static problem of two opposite plates at V_0 volts and the other four plates at $-V_0/2$ volts. Choosing the origin at the center of the cube and using approximations appropriate to a region near this origin one obtains

$$V=5.15\frac{V_0}{a^2}[z^2-\tfrac{1}{2}(x^2+y^2)], \tag{1}$$

where V_0 is the potential of the top and bottom plates and a is the length of the cube. A three-phase potential distribution that satisfies LaPlace's equation is

$$V=A[z^2\cos\Omega t+y^2\cos(\Omega t+2\pi/3)+x^2\cos(\Omega t+4\pi/3)]. \tag{2}$$

Comparing (1) and (2) at $t=0$ we find that $A=5.15V_{ac}/a^2$. To this ac electric potential one can add a dc potential of the form of (1). The equations of motion of a charged particle of mass m and charge e can then be written down and it is found that the motion in each of the three directions is governed by Mathieu's equation,

$$\frac{d^2u}{d\xi^2}+(a-2q\cos 2\xi)u=0. \tag{3}$$

Here u stands for x, y, or z and $2\xi=\Omega t$. In this general case we can specify the dc potentials of two sets of plates, letting the third set (say the set normal to the z axis) be at zero potential. The values of a and q in (3) are found to be

$$\begin{aligned} q_x=q_y=q_z&=4(e/m)\frac{5.15V_{ac}}{a^2\Omega^2}\\ a_z&=-\frac{8}{3\Omega^2}\cdot\frac{5.15}{a^2}\cdot(e/m)\cdot(V_x+V_y)\\ a_y&=\frac{16}{3\Omega^2}\cdot\frac{5.1}{a^2}(e/m)(V_x-\tfrac{1}{2}V_y)\\ a_x&=\frac{16}{3\Omega^2}\cdot\frac{5.15}{a^2}\cdot(e/m)(V_y-\tfrac{1}{2}V_x). \end{aligned} \tag{4}$$

Here V_x and V_y are the dc potentials of the plates normal to the x and y axis. LaPlace's equation moreover requires that

$$a_x+a_y+a_z=0. \tag{5}$$

The resulting motion in each direction is described in some detail in reference 1. The motion is stable and bound for values of a and q which are between certain boundary curves. If $V_x=V_y$ Eq. (5) will give a relation between a_z and a_y or a_x so that the x and y stability extremes can be plotted on the a_z, q_z diagram. The analysis of reference 1 shows that the motion (when $q\lesssim 0.4$) can be separated into a slow large amplitude motion on which is superimposed a rapid jiggle motion at the ac drive frequency. In three-phase containment the forced jiggle motion at the driving frequency results in an elliptical path.

The above theory has been checked out experimentally and found to be quite accurately followed. In particular the extent of the predicted region of stability has been checked. A plot of the extremes of the stability diagram is shown in Fig. 1, together with the experimental points. It is believed that a mechanical dissymmetry of the cube has made the agreement of theory and experiment somewhat inexact. The natural frequency of the large-scale motion has also been measured by exciting such a motion in the z direction by placing a small ac voltage between the two faces normal to the z axis. These checks were all done with single charged particles of aluminum of about 10 microns diameter charged to an e/m of approximately 0.01 coulombs/kg. The ac electric field was obtained from 60- cy or 400-cy sources. The cubical box was $1\frac{1}{4}$ in. long mounted on the same vacuum and injection systems described in reference 1. The e/m of the particles was determined by measuring the natural frequency of motion of the particle at a low value of q and with $a=0$. It is shown in reference 1 that this will enable one to measure q and hence, with known fields, e/m. As in the previous work it was also possible with this apparatus to contain many particles at the same time and if conditions are just right they will also form a "crystalline array." A picture of such a "crystal" is shown in Fig. 2.

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[1] Wuerker, Shelton, and Langmuir, J. Appl. Phys. 30, 342 (1959).

Convenient Plot for Exponential Functions with Unknown Asymptotes

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A GREAT many transient phenomena in the laboratory and in nature are first-order processes with the simple exponential time dependence

$$y=Ae^{-\alpha t}+B, \tag{1}$$

where A, α, and B are all unknown.

In a case of this sort the customary analysis involves subtraction of the quantity B, as nearly as it can be estimated, followed by a logarithmic plot to determine α. A fact which does not seem to be generally recognized or made use of is that Eq. (1) represents a straight line relation between y and $e^{-\alpha t}$, so that y is linearly related to any other variable that is also linear in $e^{-\alpha t}$.

Since α is presumed to be unknown to begin with, the only *a priori* source of such a variable must be another case or another aspect of the same transient phenomena. One occasionally finds this principle used in plotting dy/dt, or even $\Delta y/\Delta t$, against y to obtain a straight line.

We would like to point out that the retarded function $y(t+\Delta t)$ is also just such a variable, so that a plot of $y(t+\Delta t)$ against $y(t)$ with constant Δt throughout, gives a straight line which intersects the line $y(t)=y(t+\Delta t)$ at the point (B,B) and which has a slope of $e^{-\alpha\Delta t}$.

This method is particularly convenient when the value of y can be determined at regular intervals. The method has a number of advantages, one of which is that it permits the use of any ordinary graph paper instead of semilog paper. This also means that the errors in the measured y values are faithfully reflected in the plot. Moreover the method requires no preliminary estimate of any parameter, no subtraction, and only one logarithm to obtain α.

The plot is most easily interpreted if Δt is smaller, but not much smaller, than $1/\alpha$.

Because of the simplicity of this method it is almost certain to have been devised and used before. The purpose of this communication is to bring it to general attention.

Method of Measuring the Distribution of the Easy Axes of Uniaxial Ferromagnetics

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IN connection with the interpretation of the experimental behavior of aligned assemblies of uniaxial ferromagnetic particles it is important to know the spatial distribution of their axes.[1] It is the purpose of this letter to present a new simple method[2] of measuring this distribution for an assembly of uniaxial single-domain particles[3] (not necessarily identical) for which interparticle interaction can be neglected. It is further assumed that the distribution has rotational symmetry. This assumption follows from the routine generally used to align particles.[4]

The method consists of measuring I_p, the component of the maximum remanence in the direction of the magnetizing field, H, as a function of the angle $\bar{\theta}$ between H and the axis of symmetry and calculating the distribution function $f(x)$ from the relation, derived below,

$$f(x)=\sum_{n=0}^{\infty} a_{2n}P_{2n}(x)\int_0^1 \frac{I_p(Z)}{I_s}P_{2n}(Z)dZ \tag{1a}$$

$$\frac{1}{a_{2n}}=\int_0^1 \frac{P_{2n}(y)ydy}{4n+1} \tag{1b}$$

$$Z=\cos\bar{\theta};\quad x=\cos\alpha,$$

where P is the Legendre polynomial, I_s the saturation magnetization and α the polar angle from the symmetry axis. The distribution function $f(\cos\alpha)$ is defined so that $f(\cos\alpha)\sin\alpha d\alpha$ gives the probability of finding the axis of a particle between α and $\alpha+d\alpha$.[5]

The distribution can be evaluated also by measuring I_t, the component of the maximum remanence perpendicular to H in the plane of H and the axis of symmetry, as a function of $\bar{\theta}$. The distribution function is then given by

$$f(x)=1+\sum_{n=1}^{\infty} b_{2n}P_{2n}(x)\int_0^1 \frac{I_t(Z)}{I_s}P_{2n}{}^1(Z)dZ \tag{2a}$$

$$\frac{1}{b_{2n}}=\frac{\int_0^1 P_{2n}{}^1(y)(1-y^2)^{\frac{1}{2}}dy}{4n+1}=\frac{2n(2n+1)}{a_{2n}}. \tag{2b}$$

TABLE I.

n	0	1	2	3
a	2	40	-432	1664
b		$\frac{20}{3}$	$\frac{108}{5}$	$\frac{832}{21}$

Table I gives a and b for $n<4$. It can be seen from this table that relatively high measuring accuracies are needed to evaluate the high terms of $f(x)$.

To prove relation (1) one writes,

$$I_p=\frac{I_s}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} f(\cos\alpha)\cos\theta\sin\theta d\theta d\varphi,$$

expands $f(\cos\alpha)$ as a series in Legendre polynomials, and uses the addition theorem for the Legendre polynomials.[6] Here θ and φ are the polar and meridional coordinates in a spherical coordinate system whose polar axis coincides with the direction of the field. The coordinates of the symmetry axis are $(\bar{\theta},0)$. Relation (2) can be proved similarly.

The fact that $f(x)$ can be evaluated either by the measurement of I_p of I_t shows that for the above model they are related. This relation can be shown to be

$$I_t(\bar{\theta})=dI_p(\bar{\theta})/d\bar{\theta}. \tag{3}$$

It should be noted that the last relation is in fact quite a general one, holding for practically any assembly of noninteracting single-domain particles.

[1] I. S. Jacobs and F. E. Luborsky, Proceedings of the Conference on Magnetism and Magnetic Materials, Boston (October, 1956), p. 145; J. Appl. Phys. **28**, 467 (1957). E. P. Wohlfarth, to be presented before the Conference on Magnetism and Magnetic Materials, Philadelphia (November, 1958).
[2] A more elaborate, less general method was given by E. H. Frei and S. Shtrikman, Proceedings of the Conference on Magnetism and Magnetic Material, Boston (October, 1956), p. 504.
[3] A single-domain particle is defined here as one having uniform magnetization in zero external field.
[4] See, for example, J. H. L. Watson and M. W. Freeman, J. Appl. Phys. **29**, 306 (1958).
[5] This definition refers only to particles having the same volume. When this is not the case, the probability should be referred to a given volume of particles.
[6] H. Morgenau and G. M. Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1956), second edition, p. 112, Eqs. (3)–(60) and (3)–(61).

Microwave Induced Carrier Multiplication in Germanium*

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IN the initial observations of cyclotron resonance[1] in germanium at 4.2°K with 9 kMcps microwave radiation, ionization was observed at rf fields between 1 and 10 v/cm. More recent experiments on impact ionization in germanium have been made using dc fields. Measurements of Koenig and Gunther-Mohr[2] on n-type germanium indicate that above about 4°K where the dominant conductivity mechanism is associated with the conduction band, the field dependence of conductivity is divided roughly into three ranges; an ohmic range below about $E=1$ v/cm is followed by a range where the current density $j(E)$ is nonlinear and, finally, nondestructive breakdown at a critical field E_c. The breakdown is characterized by j being independent of E over several (up to five) decades. According to measurements of Sclar and Burstein[3] the breakdown field E_c is about 7 v/cm for less than 10^{15} impurities/cm^3 and increases linearly to 119 v/cm for 2×10^{16} impurities/cm^3. Each impurity used was an element of either the third or the fifth group of the periodic table, with an activation energy of