Exercise for MA-INF 2201 Computer Vision WS22/23
Submission deadline 5.01.2023
Christmas Special

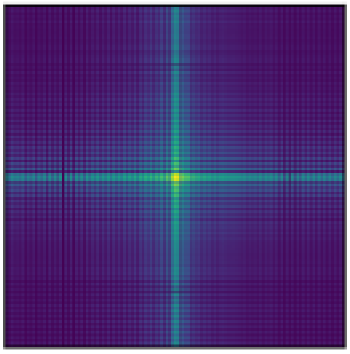
1. **Convolution and Fourier Transform:**

   (a) Compute the Fourier transform of the following function

   $$r(t) = \begin{cases} k, \; |t-C| \leq a/2 \\ 0, \; |t-C| > a/2 \end{cases}$$

   (2 Points)

   (b) The image below displays the Fourier transform of a function. The frequency magnitudes are color such that lighter corresponds to a higher value.

   How did the original image look like? Explain why.
   (1.5 Points)

   (c) With which function does convolution keep the frequencies of a signal $s(t)$ (or image $I(x,y)$) unchanged? Why?
   (1 Point)

   (d) For the following 5×5 filter, determine if it is separable. If yes, provide a separation. If not, argue why not.

   $$A = \begin{pmatrix} 3 & 1 & -9 & -2 & 0 \\ 5 & 2 & 2 & 3 & -1 \\ 9 & 4 & -9 & -8 & 1 \\ 2 & 10 & -20 & -20 & 0 \\ 4 & 8 & 4 & -6 & 0 \end{pmatrix}$$

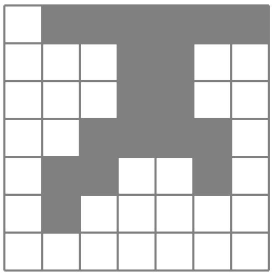
   (1 Point)

   (e) For the following 5×5 filter, determine if it is separable. If yes, provide a separation. If not, argue why not.

   $$A = \begin{pmatrix} -21 & 6 & 3 & 12 & 9 \\ 7 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 35 & -10 & -5 & -20 & -15 \\ -14 & 4 & 2 & 8 & 6 \end{pmatrix}$$

(1 Point)

(f) Compute the 2D distance transform of the below image by hand.



Provide the result after the initialization, after the forward pass, and after the backward pass.
(2.5 Points)

2. **EM-Algorithm and Factor Analysis**: When working with images, a normal distribution with a full covariance matrix is usually prohibitive since images have a very high dimensionality. A small image of 100×100 pixel image requires a $10,000 \times 10,000$ covariance matrix. Using a diagonal covariance matrix only can be too strong a limitation. *Factor analysis* provides a compromise by adding additional degrees of freedom to the model without using the full covariance matrix. Assuming D-dimensional observations, a matrix $\mathbf{\Phi} \in \mathbb{R}^{D \times K} (K \ll D)$ is used to extend the diagonal covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{D \times D}$. The final model then looks as follows:

$$Pr(x) = \mathcal{N}_x(\mu, \mathbf{\Phi}\mathbf{\Phi}^T + \mathbf{\Sigma}). \tag{1}$$

We define

$$Pr(x|h) = \mathcal{N}_x(\mu + \mathbf{\Phi}h, \mathbf{\Sigma}), \tag{2}$$
$$Pr(h) = \mathcal{N}_h(0, \mathbf{I}). \tag{3}$$

Then, Equation (1) can be rewritten as a marginalization by introducing a K-dimensional hidden variable h,

$$\begin{aligned} Pr(x) &= \int Pr(x|h)Pr(h)dh \\ &= \int \mathcal{N}_x(\mu + \mathbf{\Phi}h, \mathbf{\Sigma})\mathcal{N}_h(0, \mathbf{I})dh. \end{aligned} \tag{4}$$

Note that Equation (1) and (4) are equivalent formulations of the same problem. Equation (4) allows us to optimize the model parameters using the EM-Algorithm.

(a) Given observations $x_1, \dots, x_i, \dots, x_I$, derive the E-Step of the EM-Algorithm for factor analysis, *i.e.* compute

$$\hat{q}_i(h_i) = Pr(h_i|x_i, \theta),$$

where $\theta = (\mu, \mathbf{\Phi}, \mathbf{\Sigma})$ denotes the set of model parameters. *Hint:* Terms that are independent of h_i are irrelevant later in the M-Step, so you can just represent them in a constant.
(2 Points)

(b) Show that the update rules are

$$\tilde{\mu} = \frac{1}{I}\sum_{i=1}^{I}\big(x_i - \tilde{\mathbf{\Phi}}\mathbb{E}(h_i)\big),$$
$$\tilde{\mathbf{\Phi}} = \Big(\sum_{i=1}^{I}(x_i - \tilde{\mu})\mathbb{E}(h_i)^T\Big)\Big(\sum_{i=1}^{I}\mathbb{E}(h_i h_i^T)\Big)^{-1},$$
$$\tilde{\mathbf{\Sigma}} = \frac{1}{I}\sum_{i=1}^{I}\operatorname{diag}\Big[(x_i - \tilde{\mu})(x_i - \tilde{\mu})^T - \tilde{\mathbf{\Phi}}\mathbb{E}(h_i)(x_i - \tilde{\mu})^T\Big].$$

To make it easier, you may use that:

$$\begin{aligned}&\arg\max_{\tilde{\theta}}\Big\{\sum_{i=1}^{I}\int \hat{q}_i(h_i)\log Pr(x_i, h_i|\tilde{\theta})dh_i\Big\}\\ =&\arg\max_{\tilde{\theta}}\Big\{\sum_{i=1}^{I}\mathbb{E}\big[-\log|\tilde{\mathbf{\Sigma}}| - (x_i - \tilde{\mu} - \tilde{\mathbf{\Phi}}h_i)^T\tilde{\mathbf{\Sigma}}^{-1}(x_i - \tilde{\mu} - \tilde{\mathbf{\Phi}}h_i)\big]\Big\}\end{aligned}$$

and

$$\mathbb{E}(h_i) = (\mathbf{\Phi}^T\mathbf{\Sigma}^{-1}\mathbf{\Phi} + \mathbf{I})^{-1}\mathbf{\Phi}^T\mathbf{\Sigma}^{-1}(x_i - \mu),$$
$$\mathbb{E}(h_i h_i^T) = (\mathbf{\Phi}^T\mathbf{\Sigma}^{-1}\mathbf{\Phi} + \mathbf{I})^{-1} + \mathbb{E}(h_i)\mathbb{E}(h_i)^T,$$

where $\mathbb{E}$ is the expectation taken with respect to $Pr(h_i|x_i, \theta)$.
(6 Points)

(c) What happens to the update rules if μ is initialized with the empirical mean, *i.e.* $\mu^{(0)} = \frac{1}{I}\sum_{i=1}^{I} x_i$?
(2 Points)

(d) In order to start with a good initialization, one might want to initialize the model $\mathcal{N}_x(\mu, \mathbf{\Phi}\mathbf{\Phi}^T + \mathbf{\Sigma})$ such that it is a normal distribution with diagonal covariance, *i.e.*

$$\mu^{(0)} = \frac{1}{I}\sum_{i=1}^{I} x_i, \quad \mathbf{\Phi}^{(0)} = \mathbf{0}, \quad \mathbf{\Sigma}^{(0)} = \frac{1}{I}\sum_{i=1}^{I}\operatorname{diag}\big[(x_i - \mu)(x_i - \mu)^T\big].$$

Is this beneficial? Why/why not?
(1 Point)