November 2, 2020

Maximum likelihood

$X \sim \text{Poisson}(\theta)$

$$P(X=k) = e^{-\theta} \cdot \frac{\theta^k}{k!}$$

Likelihood function

$$\leftarrow f_{\vec{X}}(\vec{x})$$

$$L(\vec{x}, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdot \ldots \cdot f(x_n, \theta) > 0$$

probability densities (for continuous distributions)

$P(X_1 = x_1, \theta)$ (for discrete distributions)

$$L(\vec{x}, \theta) \longrightarrow \max_{\theta}$$

$\theta \to$ maximum likelihood estimator

$$\ln L(\vec{x}, \theta) \longrightarrow \max_{\theta}$$

$\longrightarrow$ logarithm of likelihood

$X_1, X_2, \ldots, X_n$ — simple sample out of Poisson (θ) distribution

$$L(\vec{x}, \theta) = \prod_{k=1}^{n} \left(e^{-\theta} \cdot \frac{\theta^{x_k}}{x_k!} \right) = e^{-n\theta} \cdot \theta^{\sum_{k=1}^{n} x_k} \cdot \frac{1}{\prod_{k=1}^{n} x_k!}$$

$P(X_k = x_k)$

$$\ln L = -n\theta + \sum_{k=1}^{n} x_k \cdot \ln\theta - \sum_{k=1}^{n} \ln(x_k!)$$

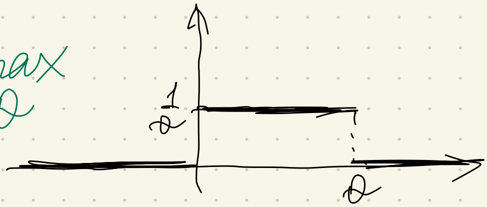
$$\frac{\partial \ln L}{\partial \theta} = -n + \frac{\sum_{k=1}^{n} x_k}{\theta} = 0 \longrightarrow \theta = \frac{\sum_{k=1}^{n} x_k}{n} = \bar{x}$$

$X_1, X_2, \ldots, X_n$ — a simple sample out of uniform distribution on $[0;\theta]$

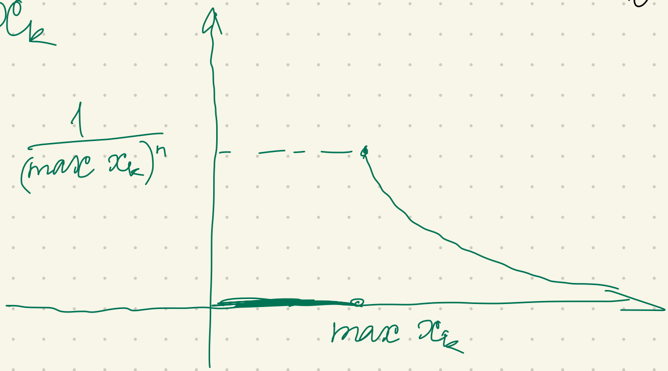
$$L(\vec{x}, \theta) = \frac{1}{\theta} I(0 \le x_1 \le \theta) \cdot \frac{1}{\theta} I(0 \le x_2 \le \theta) \cdot \ldots \cdot$$

$$\cdot \frac{1}{\theta} I(0 \le x_n \le \theta) =$$

$$= \frac{1}{\theta^n} \cdot I\left(\max_{1 \le k \le n} x_k \le \theta\right) \longrightarrow \max_{\theta}$$



$$\hat{\theta} = \max_{1 \le k \le n} x_k$$



$X_1, X_2, \ldots, X_n$ — i.i.d. r.v.,

$X_k \sim U[\theta; \theta+1]$

$$L(\vec{x}, \theta) = \prod_{k=1}^{n} I(\theta \le x_k \le \theta + 1) \Leftrightarrow$$

$$\theta \le x_1 \le \theta+1$$

$$\theta \le x_2 \le \theta+1$$

$$\ldots \ldots$$

$$\theta \le x_n \le \theta+1$$

$$\begin{cases} \theta \le \min\limits_{1 \le k \le n} x_k \\ \max\limits_{1 \le k \le n} x_k \le \theta + 1 \end{cases}$$

$$\Leftrightarrow I(\max x_k - 1 \le \theta \le \min x_k)$$

$$\max x_k - 1 > \min x_k$$

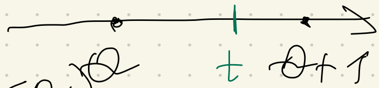
$$\max x_k - \min x_k > 1$$

$$\theta^* = \frac{\min X_k + \max X_k - 1}{2}$$

$$E\theta^* = \frac{\left(\theta + \frac{1}{n+1}\right) + \left(\theta + \frac{n}{n+1}\right) - 1}{2} = \theta$$

$$F_{\max X_k}(t) = P(\max X_k < t) = P(X_1 < t, X_2 < t, \dots, X_n < t) = P(X_1 < t)\cdot P(X_2 < t)\dots P(X_n < t) =$$

$$= (t-\theta)^n, \quad \theta \le t \le \theta + 1$$



$$f_{\max X_k}(t) = n(t-\theta)^{n-1}\, I(\theta \le t \le \theta+1)$$

$$E \max X_k = \int_{\theta}^{\theta+1} n(t-\theta)^{n-1}\cdot t\,dt = \int_{\theta}^{\theta+1} t\, d(t-\theta)^n =$$

$$= t(t-\theta)^n \Big|_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1} (t-\theta)^n dt = (\theta+1) - \frac{(t-\theta)^{n+1}}{n+1}\Big|_{\theta}^{\theta+1} =$$

$$= \theta + 1 - \frac{1}{n+1} = \theta + \frac{n}{n+1}$$

$$F_{\min X_k}(t) = P(\min X_k < t) = 1 - P(\min X_k \ge t) =$$

$$= 1 - P(X_1 \ge t, X_2 \ge t, \dots, X_n \ge t) = 1 - (1+\theta-t)^n,$$

$$\theta \le t \le \theta + 1$$

$$f_{\min X_k}(t) = n(1+\theta-t)^{n-1}\cdot I(\theta \le t \le \theta+1)$$

$$E \min X_k = \int_{\theta}^{\theta+1} t\cdot n(1+\theta-t)^{n-1} dt = -\int_{\theta}^{\theta+1} t\, d(1+\theta-t)^n =$$

$$= -t(1+\theta-t)^n \Big|_{t=\theta}^{\theta+1} + \int_{\theta}^{\theta+1} (1+\theta-t)^n dt = \theta +$$

$$+ \frac{(1+\theta-t)^{n+1}}{n+1}\Big|_{\theta+1}^{\theta} = \theta + \frac{1}{n+1}$$

$X_1, X_2, \ldots, X_n$ — i.i.d. r.v.

$\sim N(\mu; \sigma^2)$

/both parameters are unknown/

$$L(\vec{x}, \theta) = \prod_{k=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_k-\mu)^2}{2\sigma^2}} \right) =$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} \cdot e^{-\sum_{k=1}^{n} \frac{(x_k-\mu)^2}{2\sigma^2}}$$

$$\ln L = -\frac{n}{2}\ln(2\pi) - n\ln\sigma - \sum_{k=1}^{n} \frac{(x_k-\mu)^2}{2\sigma^2}$$

$$\begin{cases} \frac{\partial \ln L}{\partial \mu} = \sum_{k=1}^{n} \frac{x_k-\mu}{\sigma^2} = 0 \\ \frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{k=1}^{n} \frac{(x_k-\mu)^2}{\sigma^3} = 0 \end{cases}$$

sample mean

$$\sum_{k=1}^{n} x_k - n\mu = 0 \Rightarrow \mu = \frac{1}{n}\sum_{k=1}^{n} x_k = \bar{x}$$

$$-n\sigma^2 + \sum_{k=1}^{n}(x_k - \bar{x})^2 = 0 \Rightarrow \sigma^2 = \underbrace{\frac{1}{n}\sum_{k=1}^{n}(x_k-\bar{x})^2}_{S^2}$$

sample variance

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma} = 2\sum_{k=1}^{n} \frac{\mu - x_k}{\sigma^3} = \frac{2}{\sigma^3}\sum_{k=1}^{n}(\bar{x} - x_k) = \frac{2}{\sigma^3}\left(n\bar{x} - \sum_{k=1}^{n} x_k\right) = 0$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{n}{\sigma^2} - 3\sum_{k=1}^{n} \frac{(x_k-\mu)^2}{\sigma^4} = \frac{n}{S^2} - 3\sum_{k=1}^{n}\frac{(x_k-\bar{x})^2}{S^4} =$$

$$= \frac{n}{S^2} - \frac{3}{S^4}\cdot nS^2 = -\frac{2n}{S^2}$$