ProbStat

Assignment 1.

N1. 343 small cubes $\Rightarrow \sqrt[3]{343} = 7$ cubes along the edge

8 cubes with 3 faces painted

$12 \cdot (7-2) = 60$ cubes with 2 faces painted

$6 \cdot (7-2)(7-2) = 150$ cubes with 1 face painted

$(7-2)(7-2)(7-2) = 125$ cubes with no painted faces

1) probability that a random cube has exactly one painted face: $\frac{150}{343} = 0,437$

2) —||— two painted faces: $\frac{60}{343} = 0,175$

3) —||— at least two painted faces: $\frac{60}{343} + \frac{8}{343} = 0,198$

N2. All permutations of m white and n black balls: $\frac{(m+n)!}{m! \cdot n!} = A$

Permutations with a black ball at position k: $\frac{(m+n-1)!}{m!(n-1)!} = B$

Probability that the k-th ball is black: $\frac{B}{A} = \frac{(m+n-1)!\, m!\, n!}{(m+n)!\, m!\, (n-1)!} = \frac{n}{m+n}$

N3. All permutations of volumes: $6! = A$

Incorrect placements: $A-1$

Probability that at least one volume is misplaced: $\frac{A-1}{A} = \frac{719}{720} = 0,999$

N4. Probability that a black ball disappeared: $\frac{14}{26+14}$

Probability that a random pick is black (provided that a black ball disappeared): $\frac{13}{26+13}$

—||— white: $\frac{14}{25+14}$

Total probability to pick a black ball after disappearance: $\frac{14}{26+14} \cdot \frac{13}{26+13} + \frac{26}{26+14} \cdot \frac{14}{25+14}$
$= 0,35$

N5. All ways to take 4 shoes: $\binom{2 \cdot 10}{4}$

All ways to pick 2 pairs: $\binom{10}{2}$

All ways to pick a pair and 2 other shoes: $\binom{10}{1} \cdot \binom{18}{2} \underbrace{- \binom{10}{2}}_{\text{to not count 2 pairs twice}}$

Probability of forming a pair of 4 random shoes:

$$\frac{\binom{10}{1} \cdot \binom{18}{2} - \binom{10}{2}}{\binom{2 \cdot 10}{4}} = \frac{\frac{10 \cdot 18!}{16! \cdot 2!} - \frac{10!}{8!\, 2!}}{\frac{20!}{16!\, 4!}} = \frac{5 \cdot (17 \cdot 18 - 9)}{5 \cdot 19 \cdot 3 \cdot 17} = 0,306$$

N6. An outcome is a 4-tuple of numbers encoding floors $[2; 9]$

All outcomes: 8^4

Favorable outcomes of everyone getting off at different floors: $\frac{8!}{4!}$

Probability that two people will get off at the same floor: $1 - \frac{8!}{8^4 \cdot 4!} = 0,59$

№7. Given the score 5:3, the probability of the second player winning is $0{,}5^3$
Therefore, the probability of the first player winning is $1-0{,}5^3$
The fair prize distribution is then $1-0{,}5^3 : 0{,}5^3 = 7:1$

№8. An outcome is a 70-element combination of numbers $[1;100]$
All outcomes: $\binom{100}{70}$
Favorable outcomes are where we take 98 and 69 other lesser numbers.
Favorable outcomes: $\binom{97}{69}$
Probability that the largest number is 98: $\frac{\binom{97}{69}}{\binom{100}{70}} = \frac{97!\cdot 70!\cdot 30!}{69!\cdot 28!\cdot 100!} = 0{,}062$

№9. Since the yellow balls are not required, we may omit them to simplify calculations (the ratio favorable : all remains the same)
All permutations: $\frac{(m+n)!}{m!\cdot n!}$
Favorable permutations are the ones where the white ball is the first: $\frac{(m+n-1)!}{(m-1)!\,n!}$
Therefore, the probability that we encounter a white ball before a black one is $\frac{m}{m+n}$

№10. The letters in PORTION are P:1, R:1, O:2, T:1, I:1, N:1
An outcome is a combination of 6 letters (let's assume all letters are different)
All outcomes: $\binom{7}{6} = 7$
Outcomes that allow OPTION are when we take 2"O", 1"P", 1"T", 1"I", 1"N".
Those outcomes are $\binom{2}{2}\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1} = 1$
Outcomes that allow PORT are when we have "P", "O", "R", "T", and two other letters (we can take the second "O" or we can leave it)
Those outcomes are $\binom{1}{1}\binom{2}{2}\binom{1}{1}\binom{1}{1}\cdot\binom{2}{1} + \binom{1}{1}\binom{2}{1}\binom{1}{1}\binom{1}{1}\cdot\binom{2}{2} = 4$
Outcomes that allow RIOT are the same
Answers: a) $\frac{1}{7} = 0{,}143$, b) $\frac{4}{7} = 0{,}571$, c) $\frac{4}{7} = 0{,}571$

№11. An outcome is a set of 10 people that belong to the first group.
All outcomes: $\binom{20}{10}$
Favorable outcomes: $\binom{18}{8} + \binom{18}{8}$ (when they are both in the first or in the second group)
Probability that the two strongest students are in the same group: $\frac{2\binom{18}{8}}{\binom{20}{10}} = \frac{18!\cdot 10!\cdot \cancel{10!}}{\cancel{10!}\cdot 8!\cdot 20!} = 0{,}474$

№12. Let's choose the first number along the y-axis, the second along x-axis.

4	f	f	f	x
3	f	f	x	u
2	f	x	u	u
1	x	u	u	u
	1	2	3	4

"f" means a favorable outcome, "u" means an unfavorable outcome.
"x" means an impossible outcome (numbers are different)
Amount of "f"s: $\frac{100^2-100}{2}$ Amount of "u"s: $\frac{100^2-100}{2}$
Probability of the first number being greater than the second: $\frac{1}{2}$ $\left(\frac{"f"}{"f"+"u"}\right)$

N13. An outcome is a pair of numbers that represents possible dice rolls.
All outcomes: (1,2), (1,5), (2,1), (2,4), (3,3), (3,6), (4,2), (4,5), (5,1), (5,4), (6,3), (6,6) #12

Since only one outcome is favorable, the probability is $\frac{1}{12} = 0{,}083$

N14. An outcome is a 3-tuple of children's genders, where at least two are boys.
All outcomes: (b,b,b), (b,b,g), (b,g,b), (g,b,b)
With a single favorable outcome, the probability of having 3 boys is $\frac{1}{4} = 0{,}25$

N15. Assuming that all balls are different, an outcome is a set of 3 balls.
All outcomes: $\binom{6+4+2}{3}$
Favorable outcomes are all balls are white or black (not enough orange balls)
Such outcomes: $\binom{6}{3} + \binom{4}{3}$
Probability that all three balls are of the same color: $\frac{\binom{6}{3}+\binom{4}{3}}{\binom{6+4+2}{3}} = \frac{\frac{6!}{\cancel{3!}\cdot 3!} + \frac{4!}{\cancel{3!}\cdot 1}}{\frac{12!}{\cancel{3!}\cdot 9!}} = 0{,}109$

N16. $P(A) = \frac{4}{36}$, $P(B) = \frac{18}{36}$, A and B are independent
$P(A+B) = P(A) + P(B) - P(AB) = P(A) + P(B) - P(A)\cdot P(B) = \frac{20}{36} = 0{,}555$

N17. Letters in MASTERPIECE: M:1, A:1, S:1, T:1, E:3, R:1, P:1, I:1, C:1
Assuming that all letters are different, an outcome is a set of 6 letters.
All outcomes: $\binom{11}{6}$
Let R = {REST can be formed from the chosen letters}, S = {STRIP can be formed...}, P = {PEST -"-}

$P(R) = \frac{1\cdot 1\cdot 1\cdot (1+\binom{3}{2}\cdot\binom{5}{1}+\binom{3}{1}\cdot\binom{5}{2})}{\binom{11}{6}}$ (we take R,S,T and then from 1 to 3 "E"'s)

$P(S) = \frac{6}{\binom{11}{6}}$, $P(P) = P(R)$, $P(RS) = \frac{1\cdot 1\cdot 1\cdot 1\cdot 1\cdot \binom{3}{1}}{\binom{11}{6}} = P(SP)$, $P(RP) = \frac{1\cdot 1\cdot 1\cdot 1\,(\binom{3}{2}+\binom{3}{1}\cdot\binom{4}{1})}{\binom{11}{6}}$

$P(RSP) = P(SP) = \frac{3!\cdot 6!\cdot 5!}{2!\cdot 11!} = 0{,}006$ (b)

$P(R+S+P) = P(R) + P(S) + P(P) - P(RS) - P(RP) - P(SP) + P(RSP)$
$= 0{,}099 + 0{,}013 + 0{,}099 - 0{,}006 - 0{,}032 - 0{,}006 + 0{,}006 = 0{,}173$ (a)

N18. $P(A) = P(B) = 0{,}5$

$P(AB) - P(\bar{A}\bar{B}) = P(AB) - P(\overline{A+B}) = P(AB) - (1 - P(A+B)) = P(AB) - (1 - (P(A)+P(B)-P(AB))) =$
$= P(AB) - 1 + P(A) + P(B) - P(AB) = 0$

N19.1 $P(AB) \geq P(A) + P(B) - 1$
$1 \geq P(A) + P(B) - P(AB)$
$1 \geq P(A+B)$, which is true by definition of the probability function

19.2. $P(A_1 A_2 \dots A_n) \geq P(A_1) + P(A_2) + \dots + P(A_n) - (n-1)$ (1)
Proof by induction.
Base. $n=2$. see 19.1
Hypothesis. Let (1) be true for all $n \leq k$ for some k
Step. Let's prove (1) for $n = k+1$
$P((A_1 A_2 \dots A_k) A_{k+1}) \geq P(A_1 A_2 \dots A_k) + P(A_{k+1}) - 1 \geq P(A_1) + \dots + P(A_k) - (k-1) + P(A_{k+1}) - 1$ QED

N20. $P(A)=0{,}7$, $P(B)=0{,}8$, $P(C)=0{,}9$

According to the inequality from 19.2,

$$P(ABC) \geq P(A)+P(B)+P(C)-(3-1) \Rightarrow P(ABC) \geq 0{,}4 \quad \text{(lower bound)}$$

By definition of event product, $AB \subseteq A$, $AB \subseteq B \Rightarrow P(AB) \leq P(A)$, $P(AB) \leq P(B)$

Therefore,

$$P(AB) \leq 0{,}7$$
$$P(AC) \leq 0{,}7 \quad , \quad P(ABC) \leq 0{,}7$$
$$P(BC) \leq 0{,}8$$

Using the inclusion-exclusion principle,

$$P(A+B+C) = P(A)+P(B)+P(C)-P(AB)-P(AC)-P(BC)+P(ABC)$$

$$P(ABC) = \underbrace{P(A+B+C)}_{\leq 1} - P(A)-P(B)-P(C)+\underbrace{P(AB)}_{\leq 0{,}7}+\underbrace{P(AC)}_{\leq 0{,}7}+\underbrace{P(BC)}_{\leq 0{,}8} \leq 0{,}8, \text{ but } P(ABC) \leq 0{,}7$$

Therefore, $P(ABC) \leq 0{,}7$ (upper bound)

N21. $P(A)=0$

$A \triangle B = (A \setminus B) \cup (B \setminus A)$ (symmetric difference)

By definition of event difference, $P(A \setminus B) \leq P(A)$

$(A \setminus B)$ and $(B \setminus A)$ are disjoint, therefore, $P((A \setminus B)+(B \setminus A)) = \underbrace{P(A \setminus B)}_{\leq P(A)=0} + P(B \setminus A)$

By definition of the probability function, $P(A \setminus B) \geq 0 \Rightarrow P(A \setminus B) = 0$

$P(B) = P(B \setminus A) + P(B \cap A)$, $P(B \cap A) \leq P(A) = 0 \Rightarrow P(B) = P(B \setminus A)$

Thus, $P(A \triangle B) = P(B)$

The opposite is not true. Let $\Omega = \{q_1, q_2, q_3\}$, $A = \{q_1, q_2\}$, $B = \{q_2, q_3\}$

$P(A) = \frac{2}{3} \neq 0$, $P(B) = \frac{2}{3}$

$P(A \triangle B) = P(\{q_1, q_3\}) = \frac{2}{3} = P(B)$

N22. $P(A) = \frac{2}{3}$, $P(B) = \frac{3}{4}$

According to the inequality from 19.1,

$$P(AB) \geq P(A)+P(B)-1 = \frac{5}{12}$$

Then $P(A|B) = \frac{P(AB)}{P(B)} \geq \frac{5 \cdot 4}{3 \cdot 12} = \frac{5}{9}$ QED

N23. Letters in SUPERPOSITION:

S	U	P	E	R	O	I	T	N
2	1	2	1	1	2	2	1	1

The probability of getting NOISE is a product of probabilities to pick the right letters and to put them in correct order.

An outcome of a pick is a set of 5 letters.

All outcomes of a pick: $\binom{13}{5}$ $\Rightarrow$ probability of picking the right letters: $\frac{8}{\binom{13}{5}} = 0{,}006$

Favorable outcomes: $1 \cdot \binom{2}{1} \cdot \binom{2}{1} \cdot \binom{2}{1} \cdot 1 = 8$

The probability of putting letters in correct order is $\frac{1}{5!} = 0{,}008$

Thus, the total probability to get NOISE is $0{,}006 \cdot 0{,}008 = 0{,}00005$

N24. Yes, disjoint events may be independent.

Consider $\Omega = \{q_1\}$, $A = \{q_1\}$, $B = \varnothing$

$A \cap B = \varnothing$, $P(A) = 1$, $P(B) = 0$, $P(AB) = P(A) \cdot P(B) = 0$

N25. Let A, B be independent events. Then $P(AB) = P(A)\cdot P(B)$

$$P(A)\cdot P(\overline{B}) = P(A)\cdot(1-P(B)) = \underbrace{P(A)}_{P(A\setminus B)+P(AB)} - \underbrace{P(A)P(B)}_{P(AB)} = P(A\setminus B) = P(A\overline{B})$$

$$P(\overline{A})P(\overline{B}) = (1-P(A))(1-P(B)) = \underbrace{1 - P(A) - P(B) + \underbrace{P(A)\cdot P(B)}_{P(AB)}}_{-P(A+B)} = 1-P(A+B) = P(\overline{A+B}) = P(\overline{A}\overline{B})$$

N26. Let $H_i = \{$i-th sniper has hit the target$\}$, $D = \{$the target has been hit$\}$

$P(H_1) = 0,7$ $\quad P(H_2) = 0,6$, $\quad P(H_3) = 0,5$

$P(\overline{D}) = P(\overline{H}_1\overline{H}_2\overline{H}_3) = P(\overline{H}_1)\cdot P(\overline{H}_2)\cdot P(\overline{H}_3) = 0,3\cdot 0,4\cdot 0,5 = 0,06 \Rightarrow P(D) = 1-0,06 = 0,94$

$\overline{H}_1, \overline{H}_2, \overline{H}_3$ are independent, as shown in N25

N27. As given in the lecture, the 4-faced die with faces $\{R, G, Y, RGY\}$ and events $F_R = \{$the face has some red on it$\}$, F_G and F_Y

N28. Let $H_i = \{$i-th marksman has hit the target$\}$

$P(H_1) = 0,5$, $\quad P(H_2) = 0,4$, $\quad P(H_3) = 0,3$

$D_a = \{$ the target has not been hit$\}$

$D_b = \{$ the target has been hit exactly two times$\}$

$D_c = \{$the target has been hit at least once$\}$

$P(D_a) = P(\overline{H}_1\overline{H}_2\overline{H}_3) = P(\overline{H}_1)\cdot P(\overline{H}_2)\cdot P(\overline{H}_3) = 0,5\cdot 0,6\cdot 0,7 = 0,21$

$$P(D_b) = P(\underbrace{H_1H_2\overline{H}_3 + H_1\overline{H}_2H_3 + \overline{H}_1H_2H_3}_{\text{pairwise disjoint}}) = P(H_1H_2\overline{H}_3) + P(H_1\overline{H}_2H_3) + P(\overline{H}_1H_2H_3) = 0,5\cdot 0,4\cdot 0,7 + 0,5\cdot 0,6\cdot 0,3 + 0,5\cdot 0,4\cdot 0,3 = 0,29$$

$P(D_c) = P(\overline{D}_a) = 1-0,21 = 0,79$

N29. $W_f = \{$first player wins$\} = \bigcup\limits_{i=0}^{\infty} W_{f_i}$

$W_{f_i} = \{$first player wins on the (i+1)-th turn$\}$, $P(W_{f_i}) = 0,5^{2i+1}$

W_{f_i} are disjoint by definition

$$P(W_f) = \sum_{i=0}^{\infty} 0,5^{2i+1} = \frac{1\cdot 0,5}{(1-0,25)} = \frac{2}{3} \Rightarrow P(\overline{W}_f) = 1-\frac{2}{3} = \frac{1}{3} = 0,333$$

N30. $W_f = \{$first player wins$\} = \bigcup\limits_{i=0}^{\infty} W_{f_i}$

$W_{f_i} = \{$first player wins on the (i+1)-th turn$\}$, $P(W_{f_i}) = \left(\frac{48}{52}\right)^{2i}\cdot\frac{4}{52}$

W_{f_i} are disjoint by definition

W_s and W_{s_i} are defined similarly for the second player

$P(W_{s_i}) = \left(\frac{48}{52}\right)^{2i}\cdot\frac{48}{52}\cdot\frac{4}{52}$

$$P(W_f) = \sum_{i=0}^{\infty}\left(\frac{48}{52}\right)^{2i}\cdot\frac{4}{52}, \quad P(W_s) = \sum_{i=0}^{\infty}\left(\frac{48}{52}\right)^{2i}\cdot\frac{48}{52}\cdot\frac{4}{52}$$

Thus $P(W_f) : P(W_s) = 52:48$, so the game is not fair

After changing the rules we have $P(W_{f_i}) = \left(\frac{51}{52}\right)^{2i}\cdot\frac{1}{52}$, $P(W_{s_i}) = \left(\frac{51}{52}\right)^{2i}\cdot\frac{51}{52}\cdot\frac{1}{52}$

Then $P(W_f) : P(W_s) = 52:51$, which makes the game more fair, but still not fair.

N31. To have to flip the coin exactly 6 times, one must have a sequence *-*-h-t-t-t, where "*" is any side, "t" is tails and "h" is heads

All outcomes: 2^6

Favorable outcomes: 2^2

$\Rightarrow$ probability to flip the coin 6 times: $\frac{2^2}{2^6} = \frac{1}{16} = 0,0625$

To have to flip the coin exactly 7 times, the outcome must be *-*-*-h-t-t-t, with at least 2 heads

All outcomes: 2^7

Favorable outcomes: 2^3-1 (cannot have t-t-t)

$\Rightarrow$ probability to flip the coin 7 times: $\frac{7}{2^7} = 0,0547$

N32. To have to flip a coin exactly 6 times, one must have any of the following sequences:

- t-t-h-h-t-h 1 outcome
- *-h-h-h-t-h 2 outcomes
- *-t-t-h-t-h 2 outcomes

5 favorable outcomes out of 2^6 possible outcomes
probability: $\frac{5}{64} = 0{,}078125$

To have to flip a coin exactly 7 times, one must avoid any of the following sequences:

- *-*-h-t-h-t-h 4 outcomes
- *-h-t-h-h-t-h 2 outcomes
- h-t-h-h-h-t-h 1 outcome

7 unfavorable prefixes out of 2^4 possible
2^4-7 favorable outcomes out of 2^7 possible outcomes
probability: $\frac{9}{128} = 0{,}0703125$

N33. An outcome is a 10-tuple of numbers in [1;8].

All outcomes: 8^{10}
Outcomes favoring 1: 6^{10} (not choosing 4 or 5)
Outcomes favoring 2: $7^{10}-6^{10}$ (not choosing 5 minus not choosing 4 or 5)
Outcomes favoring 3 are the outcomes not favoring 1, 2 and the symmetric case of 2,
thus: $8^{10}-6^{10}-2\cdot(7^{10}-6^{10})$
Outcomes favoring 4: $8^{10}-7^{10}-1$ (all except not choosing 4 minus the only case of only stopping on 4s)

Outcomes favoring 5 are all outcomes except possible tuples of equal numbers,
thus: $8^{10}-8$

Probabilities:

1) $\frac{6^{10}}{8^{10}} = 0{,}056$, 2) $\frac{7^{10}-6^{10}}{8^{10}} = 0{,}207$, 3) $\frac{8^{10}-6^{10}-2\cdot(7^{10}-6^{10})}{8^{10}} = 0{,}53$, 4) $\frac{8^{10}-7^{10}-1}{8^{10}} = 0{,}737$

5) $\frac{8^{10}-8}{8} = 0{,}999$