Assignment 7.

N1.

a) $f_{\xi,\eta}(x,y) = \begin{cases} \frac{1}{\pi a b}, & \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \\ 0, & \text{otherwise} \end{cases}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \Rightarrow y^2 \le b^2 - \frac{b^2}{a^2}x^2 \Rightarrow$

$\Rightarrow y \in \left[-\sqrt{b^2 - \frac{b^2}{a^2}x^2}; \sqrt{b^2 - \frac{b^2}{a^2}x}\right]$, let $\beta(x) = \sqrt{b^2 - \frac{b^2}{a^2}x^2}$

$f_\xi(x) = \begin{cases} 0, & x \notin [-a; a] \\ \int_{-\infty}^{+\infty} f_{\xi,\eta}(x,y)\,dy, & \text{otherwise} \end{cases} = I_{x\in[-a;a]} \cdot \int_{-\beta(x)}^{\beta(x)} \frac{dy}{\pi a b} = I_{x\in[-a;a]} \cdot \frac{2\sqrt{b^2 - \frac{b^2}{a^2}x^2}}{\pi a b} = \frac{2}{\pi a}\sqrt{1 - \frac{x^2}{a^2}} \cdot$

$\cdot I_{x\in[-a;a]}$

Similarly, $f_\eta(y) = I_{y\in[-b;b]} \cdot \frac{2}{\pi b}\sqrt{1 - \frac{y^2}{b^2}}$

b) $P(0<\xi<a,\ 0<\eta<b) = \int\limits_0^a dx \int\limits_0^b \frac{1}{\pi a b} dy = \frac{1}{\pi}$

$P(0<\xi<a) = \int\limits_0^a \frac{2}{\pi a^2}\sqrt{a^2-x^2}\,dx = \frac{2}{\pi a^2}\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right)\Big|_{x=0}^{a} = \frac{2}{\pi a^2}\cdot\frac{a^2}{2} = \frac{1}{\pi}$

Similarly, $P(0<\eta<b) = \frac{1}{\pi} \Rightarrow P(0<\xi<a)\cdot P(0<\eta<b) = \frac{1}{\pi^2} \neq \frac{1}{\pi} \Rightarrow \xi$ and η are dependent.

c) $E\xi = \int\limits_{-\infty}^{+\infty} x\cdot f_\xi(x)\,dx = \int\limits_{-a}^{a} \frac{2x}{\pi a^2}\sqrt{a^2-x^2}\,dx = 0$, since the integrand is odd and the domain is symmetric w.r.t. 0

Similarly, $E\eta = 0$

$E(\xi\eta) = \int\limits_{-\infty}^{+\infty} dx \int\limits_{-\infty}^{+\infty} xy\cdot f_{\xi,\eta}(x,y)\,dy =$ (law of the unconscious statistician)

$= \int\limits_{-\infty}^{+\infty} x\,dx \int\limits_{-\infty}^{+\infty} \frac{1}{\pi a b}\cdot I_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}\cdot y\,dy = \int\limits_{-a}^{a} x\,dx \int\limits_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{1}{\pi a b}\cdot y\,dy =$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \Rightarrow y^2 \le b^2 - \frac{b^2}{a^2}x^2 \Rightarrow |y| \le \frac{b}{a}\sqrt{a^2-x^2}$

$= \frac{1}{\pi a b}\int\limits_{-a}^{a} x\,dx \underbrace{\int\limits_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy}_{0,\text{ since the integrand is odd and the domain is symmetric w.r.t. } 0} = 0$

Therefore, $\mathrm{Cov}(\xi,\eta) = \mathrm{Cov}(\eta,\xi) = E(\xi\eta) - E(\xi)\cdot E(\eta) = 0$

$E\xi^2 = \int\limits_{-\infty}^{+\infty} x^2\cdot f_\xi(x)\,dx = \frac{2}{\pi a^2}\int\limits_{-\infty}^{+\infty} x^2\cdot\sqrt{a^2-x^2}\cdot I_{x\in[-a;a]}\,dx = \frac{2}{\pi a^2}\int\limits_{-a}^{a} x^2\cdot\sqrt{a^2-x^2}\,dx =$

$= \frac{2}{\pi a^2}\left(\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}\right)\Big|_{x=-a}^{a} = \frac{2}{\pi a^2}\cdot\frac{a^4}{8}\cdot\pi = \frac{a^2}{4}$

Similarly, $E\eta^2 = \frac{b^2}{4}$

$K = \begin{bmatrix} \frac{a^2}{4} & 0 \\ 0 & \frac{b^2}{4} \end{bmatrix}$

d) $f_{\xi|\eta=y}(x) = \frac{f_{\xi,\eta}(x,y)}{f_\eta(y)} = \frac{\frac{1}{\pi a b}\cdot I_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}}{\frac{2}{\pi b^2}\sqrt{b^2-x^2}\cdot I_{y\in[-b;b]}} = \frac{b}{2a\sqrt{b^2-x^2}}\cdot I_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}$

Assuming that we only define $f_{\xi|\eta=y}$ for $y\in[-b;b]$, we can rewrite it:

$f_{\xi|\eta=y}(x) = \frac{b}{2a\sqrt{b^2-x^2}}\cdot I_{x^2\le\frac{a^2}{b^2}(b^2-y^2)}$

$E(\xi|\eta=y) = \int\limits_{-\infty}^{+\infty} x\cdot f_{\xi|\eta=y}(x)\,dx = \int\limits_{-\frac{a}{b}\sqrt{b^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} \frac{b}{2a}\cdot\frac{x\,dx}{\sqrt{b^2-x^2}} = 0$, since the integrand is odd and the domain is symmetric w.r.t. 0

Therefore $E(\xi|\eta) = 0$, similarly, $E(\eta|\xi) = 0$

<u>N2.</u> $S_\Delta = \frac{1}{2}\cdot 4\cdot 10 = 20$, let G be the triangle $((-5;0), (5;0), (0;4))$

$f_{\xi,\eta}(x,y) = \frac{1}{20}\cdot I_{(x;y)\in G}$

$f_\xi(x) = \int\limits_{-\infty}^{+\infty} f_{\xi,\eta}(x,y)\,dy = I_{x\in[-5;0]}\cdot\int\limits_0^{\frac{4}{5}x+4}\frac{1}{20}\cdot dy + I_{x\in[0;5]}\cdot\int\limits_0^{-\frac{4}{5}x+4}\frac{1}{20}\cdot dy = I_{x\in[-5;5]}\cdot\frac{1}{20}\cdot\left(4-\frac{4}{5}|x|\right)$

$f_\eta(y) = \int\limits_{-\infty}^{+\infty} f_{\xi,\eta}(x,y)\,dx = I_{y\in[0;4]}\cdot\int\limits_{\frac{5}{4}y-5}^{-\frac{5}{4}y+5}\frac{1}{20}\,dx = I_{y\in[0;4]}\cdot\frac{1}{10}\cdot\left(5-\frac{5}{4}y\right)$

$$f_{\xi|\eta=y}(x) = \frac{f_{\xi,\eta}(x,y)}{f_\eta(y)} = \frac{\frac{1}{20}\cdot I_{(x;y)\in G}}{\frac{1}{10}\cdot(5-\frac{5}{4}y)\cdot I_{y\in[0;4]}} = \frac{2}{20-5y}\cdot I_{(x;y)\in G}$$

$$f_{\eta|\xi=x}(y) = \frac{f_{\xi,\eta}(x,y)}{f_\xi(x)} = \frac{\frac{1}{20}\cdot I_{(x;y)\in G}}{\frac{1}{20}\cdot(4-\frac{4}{5}|x|)\cdot I_{x\in[-5;5]}} = \frac{5}{20-4|x|}\cdot I_{(x;y)\in G}$$

a) $$\left.\begin{aligned} E(\eta|\xi=2) &= \int_{-\infty}^{+\infty} y\cdot f_{\eta|\xi=2}(y)dy = \int_0^{-2\cdot\frac{4}{5}+4}\frac{5}{12}y\,dy = \frac{5}{12}\cdot\left(\frac{y^2}{2}\right)\Big|_{y=0}^{\frac{12}{5}} = \frac{12}{10} = \frac{6}{5} \\ E(\eta^2|\xi=2) &= \int_{-\infty}^{+\infty} y^2 f_{\eta|\xi=2}(y)dy = \int_0^{-2\cdot\frac{4}{5}+4}\frac{5}{12}y^2dy = \frac{5}{12}\cdot\left(\frac{y^3}{3}\right)\Big|_{y=0}^{\frac{12}{5}} = \frac{48}{25}\end{aligned}\right\} Var(\eta|\xi=2) = \frac{48}{25}-\left(\frac{6}{5}\right)^2 = \frac{12}{25}$$

b) $$\left.\begin{aligned} E(\xi|\eta=2) &= \int_{-\infty}^{+\infty} x\cdot f_{\xi|\eta=2}(x)dx = \int_{-2,5}^{2,5}\frac{2x}{10}dx = \frac{1}{5}\left(\frac{x^2}{2}\right)\Big|_{x=-2,5}^{2,5} = 0 \\ E(\xi^2|\eta=2) &= \int_{-\infty}^{+\infty} x^2\cdot f_{\xi|\eta=2}(x)dx = \int_{-2,5}^{2,5}\frac{2x^2}{10} = \frac{1}{5}\left(\frac{x^3}{3}\right)\Big|_{x=-2,5}^{2,5} = \frac{25}{12}\end{aligned}\right\} \Rightarrow Var(\xi|\eta=2) = \frac{25}{12}-0 = \frac{25}{12}$$

<u>N3.</u> $F_{\xi,\eta}(x,y) = (1-e^{-\lambda x}-e^{-\mu y}+e^{-\lambda x-\mu y})\cdot I_{x>0}\cdot I_{y>0}$

a) $F_\xi(x) = P(\xi<x) = \lim_{y\to+\infty} P(\xi<x, \eta<y) = \lim_{y\to+\infty} F_{\xi,\eta}(x,y) = (1-e^{-\lambda x})\cdot I_{x>0}$

Similarly, $F_\eta(y) = \lim_{x\to+\infty} F_{\xi,\eta}(x,y) = (1-e^{-\mu y})\cdot I_{y>0}$

b) $F_\xi(x)\cdot F_\eta(y) = I_{x>0}\cdot I_{y>0}\cdot(1-e^{-\lambda x})(1-e^{-\mu y}) = (1-e^{-\lambda x}-e^{-\mu y}+e^{-\lambda x-\mu y})\cdot I_{x>0}\cdot I_{y>0} = F_{\xi,\eta}(x,y)$

Therefore, ξ and η are independent.

c) $f_{\xi,\eta}(x,y) = \frac{\partial^2 F_{\xi,\eta}(x,y)}{\partial x\partial y} = \frac{\partial}{\partial x}\left((\mu e^{-\mu y}-\mu e^{-\lambda x-\mu y})\cdot I_{x>0}\cdot I_{y>0}\right) = \mu\lambda e^{-\lambda x-\mu y}\cdot I_{x>0}\cdot I_{y>0}$

$f_\xi(x) = \frac{dF_\xi(x)}{dx} = -\lambda e^{-\lambda x}\cdot I_{x>0}$, $\quad f_\eta(y) = \frac{dF_\eta(y)}{dy} = -\mu e^{-\mu y}\cdot I_{y>0}$

$$E\xi = \int_{-\infty}^{+\infty} x\cdot f_\xi(x)dx = \int_0^{+\infty}\underbrace{x}_{f(x)}\cdot\underbrace{(-\lambda e^{-\lambda x})}_{g'(x)}dx = xe^{-\lambda x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-\lambda x}dx = \frac{1}{\lambda}\cdot e^{-\lambda x}\Big|_0^{+\infty} = \frac{1}{\lambda}$$

$$E\xi^2 = \int_{-\infty}^{+\infty} x^2\cdot f_\xi(x)dx = \int_0^{+\infty}\underbrace{x^2}_{f(x)}\cdot\underbrace{(-\lambda e^{-\lambda x})}_{g'(x)}dx = x^2\cdot e^{-\lambda x}\Big|_0^{+\infty} - \int_0^{+\infty} 2x\cdot e^{-\lambda x}dx = \frac{2}{\lambda}\int_0^{+\infty} x\cdot(-\lambda e^{-\lambda x})dx =$$

$$= \frac{2}{\lambda}xe^{-\lambda x}\Big|_0^{+\infty} - \frac{2}{\lambda}\int_0^{+\infty} e^{-\lambda x}dx = \frac{2}{\lambda^2}e^{-\lambda x}\Big|_0^{+\infty} = \frac{2}{\lambda^2}$$

$Var\,\xi = E\xi^2-(E\xi)^2 = \frac{2}{\lambda^2}-\frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Similarly, $E\eta = \frac{1}{\mu}$, $E\eta^2 = \frac{2}{\mu^2}$, $Var\,\eta = \frac{1}{\mu^2}$

Since ξ and η are independent, $Cov(\xi,\eta)=0 \Rightarrow K = \begin{bmatrix}\frac{1}{\lambda^2} & 0 \\ 0 & \frac{1}{\mu^2}\end{bmatrix}$

d) $f_{\xi|\eta=y}(x) = \frac{f_{\xi,\eta}(x,y)}{f_\eta(y)} \underset{indep.}{\overset{\uparrow}{=}} \frac{f_\xi(x)\cdot f_\eta(y)}{f_\eta(y)} = f_\xi(x) \Rightarrow E(\xi|\eta=y) = E\xi = \frac{1}{\lambda}\ \forall y \Rightarrow E(\xi|\eta) = \frac{1}{\lambda}$

Similarly, $E(\eta|\xi) = E\eta = \frac{1}{\mu}$

N4. $f_{\xi,\eta}(x,y) = \frac{a}{1+x^2+y^2+x^2y^2}$ for some $a \geq 0$

$\int\limits_{-\infty}^{+\infty} dx \int\limits_{-\infty}^{+\infty} f_{\xi,\eta}(x,y)dy = 1$, since $f_{\xi,\eta}(x,y)$ is a valid joint PDF

$\int\limits_{-\infty}^{+\infty} dx \int\limits_{-\infty}^{+\infty} f_{\xi,\eta}(x,y)dy = a\int\limits_{-\infty}^{+\infty} dx \int\limits_{-\infty}^{+\infty} \frac{dy}{(1+x^2)(1+y^2)} = a\int\limits_{-\infty}^{+\infty}\frac{dx}{1+x^2}\cdot\int\limits_{-\infty}^{+\infty}\frac{dy}{1+y^2} = a\cdot \operatorname{arctg} x\Big|_{x=-\infty}^{+\infty}\cdot \operatorname{arctg} y\Big|_{y=-\infty}^{+\infty} = a\pi^2$

Therefore, $a = \frac{1}{\pi^2}$

$f_\xi(x) = \int\limits_{-\infty}^{+\infty} f_{\xi,\eta}(x,y)dy = \frac{1}{\pi^2(1+x^2)}\cdot\int\limits_{-\infty}^{+\infty}\frac{dy}{1+y^2} = \frac{\pi}{\pi^2(1+x^2)} = \frac{1}{\pi(1+x^2)}$, similarly, $f_\eta(y) = \frac{1}{\pi(1+y^2)}$

$f_{\xi,\eta}(x,y) = \frac{1}{\pi^2(1+x^2)(1+y^2)} = f_\xi(x)\cdot f_\eta(y) \Rightarrow \xi$ and η are independent $\Rightarrow \rho_{\xi,\eta} = 0$

N5a) $\xi \sim N(0;1)$, $\eta = \xi^2$

$P(\xi < x) = \Phi(x)$, $\quad P(\eta < x) = P(\xi^2 < x) = I_{x>0}\cdot P(-\sqrt{x} < \xi < \sqrt{x}) = I_{x>0}\cdot(\Phi(\sqrt{x}) - \Phi(-\sqrt{x}))$

$P(\xi < x, \eta < y) = P(\xi < x, \xi^2 < y) = I_{y>0}\cdot P(-\sqrt{y} < \xi < \min\{x, \sqrt{y}\})$

Let's take $y = 0{,}25$, $x = 1$ $\quad$ ($(x;y)$ such that $y>0$ and $y<x^2$)

$$\left.\begin{aligned} &P(\xi<x)\cdot P(\eta<y) = \Phi(1)\cdot(\Phi(\,,5)-\Phi(-0{,}5)) \\ &P(\xi<x,\ \eta<y) = \Phi(0{,}5)-\Phi(-0{,}5) \end{aligned}\right\} \neq,$$ since $\Phi(1) = 0{,}8413 \neq 1$, where Φ is from the standard normal table

Therefore, ξ and η are dependent

б) $E\xi = 0$, since $\xi \sim N(0;1)$

$E(\xi\eta) = E(\xi^3) = \int\limits_{-\infty}^{+\infty} t^3\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = 0$, since the integrand is odd and the domain is symmetric w.r.t. 0

$Cov(\xi,\eta) = E(\xi\eta) - E(\xi)E(\eta) = 0 - 0 = 0 \Rightarrow \rho_{\xi,\eta} = 0 \Rightarrow \xi$ and η are uncorrelated.

N6. $K_{[\xi\,\eta]^T} = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} Var\,\xi & Cov(\xi,\eta) \\ Cov(\eta,\xi) & Var\,\eta \end{bmatrix}$

$Var(-\xi-2\eta) = Var(-\xi) + Var(-2\eta) + 2Cov(-\xi,-2\eta) = (-1)^2 Var\,\xi + (-2)^2 Var\,\eta + 2\cdot 2\,Cov(\xi,\eta) =$
$= 2 + 4\cdot 6 + 4\cdot(-3) = 14$

$Var(3\xi - \eta + 2) = Var(3\xi) + Var\,\eta + Var\,2 - 2Cov(3\xi,\eta) + 2Cov(3\xi,2) - 2Cov(\eta,2) =$
$= 3^2 Var\,\xi + Var\,\eta + 0 - 2\cdot 3\,Cov(\xi,\eta) + 0 - 0 = 3^2\cdot 2 + 6 - 2\cdot 3(-3) = 42$

N7. Let $a, b, c, d \in \mathbb{R}$, $ac \neq 0$.

$\rho(a\xi+b, c\eta+d) = \frac{Cov(a\xi+b,\ c\eta+d)}{\sqrt{Var(a\xi+b)\cdot Var(c\eta+d)}} = \frac{E((a\xi+b)(c\eta+d)) - E(a\xi+b)\cdot E(c\eta+d)}{\sqrt{a^2 Var\,\xi\cdot c^2\cdot Var\,\eta}} =$

$= \frac{E(ac\xi\eta) + E(bc\eta) + E(ad\xi) + bd - E(a\xi)\cdot E(c\eta) - bE(c\eta) - dE(a\xi) - bd}{|a||c|\sqrt{Var\,\xi\cdot Var\,\eta}} = \frac{ac\,Cov(\xi,\eta)}{|a||c|\sqrt{Var\,\xi\cdot Var\,\eta}} = \frac{ac}{|ac|}\rho(\xi,\eta)$.

The sign of the right hand side is the sign of the product ac.

№8. Let $\xi \sim Exp(\lambda)$

a) $\rho(2\xi+3,\ 3\xi-1) = \frac{2\cdot 3}{|2\cdot 3|}\rho(\xi,\xi) = \frac{Var\,\xi}{|Var\,\xi|} = \frac{\frac{1}{\lambda^2}}{|\frac{1}{\lambda^2}|} = 1$, as per №7

б) $\rho(\xi^2,\ \xi^2-\xi) = \frac{Cov(\xi^2,\ \xi^2-\xi)}{\sqrt{Var\,\xi^2\cdot Var(\xi^2-\xi)}}$

$Cov(\xi^2,\xi^2-\xi) = E(\xi^2(\xi^2-\xi)) - E(\xi^2)E(\xi^2-\xi) = E(\xi^4)-E(\xi^3)-(E(\xi^2))^2 - E(\xi^2)E\xi$

$E\xi = \frac{1}{\lambda}$, since $\xi \sim Exp(\lambda)$

$E\xi^2 = Var\,\xi + (E\xi)^2 = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}$

$E\xi^3 = \int\limits_{-\infty}^{+\infty} x^3\cdot\lambda e^{-\lambda x}dx = -\int\limits_0^{+\infty}\underbrace{x^3}_{f(x)}\cdot\underbrace{(-\lambda e^{-\lambda x})}_{g'(x)}dx = -\left(\cancel{x^3 e^{-\lambda x}\Big|_{x=0}^{+\infty}}^{0} - 3\int\limits_0^{+\infty} x^2\cdot e^{-\lambda x}dx\right) = \frac{3}{\lambda}\int\limits_0^{+\infty}x^2\lambda e^{-\lambda x}dx = \frac{3}{\lambda}E\xi^2$

Similarly, $E\xi^4 = \frac{4}{\lambda}E\xi^3 = \frac{24}{\lambda^4}$

$Cov(\xi^2,\ \xi^2-\xi) = \frac{24}{\lambda^4} - \frac{6}{\lambda^3} - \frac{4}{\lambda^4} - \frac{2}{\lambda^3} = \frac{20-4\lambda}{\lambda^4}$

$Var\,\xi^2 = E\xi^4 - (E\xi^2)^2 = \frac{20}{\lambda^4}$, $\quad Var(\xi^2-\xi) = Var\,\xi^2 + Var\,\xi - 2Cov(\xi^2,\xi) =$

$= \frac{20}{\lambda^4} + \frac{1}{\lambda^2} - 2\left(\frac{6}{\lambda^3} - \frac{2}{\lambda^3}\right) = \frac{\lambda^2-8\lambda+20}{\lambda^4}$

$\rho(\xi^2,\ \xi^2-\xi) = \frac{20-4\lambda}{\sqrt{20\cdot(\lambda^2-8\lambda+20)}}$

№9. Let $\xi \sim N(0;1)$

a) $\rho(2\xi,\ \xi^3) = \frac{2}{|2|}\rho(\xi,\xi^3) = \frac{Cov(\xi,\xi^3)}{\sqrt{Var\,\xi\cdot Var\,\xi^3}}$

$Cov(\xi,\xi^3) = E(\xi^4) - E(\xi)\cdot E(\xi^3)$

$E\xi = 0$, since $\xi \sim N(0;1)$

$E\xi^3 = \int\limits_{-\infty}^{+\infty} t^3\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = 0$, since the integrand is odd and the domain is symmetric w.r.t. 0

$E\xi^4 = \int\limits_{-\infty}^{+\infty} t^4\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = -\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\underbrace{t^3}_{f}\cdot\underbrace{(-te^{-\frac{t^2}{2}})}_{g'}dt = -\frac{1}{\sqrt{2\pi}}\left(t^3\cdot e^{-\frac{t^2}{2}}\Big|_{t=-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty}3t^2\cdot e^{-\frac{t^2}{2}}dt\right)$

$\lim\limits_{t\to+\infty} t^3\cdot e^{-\frac{t^2}{2}} = \lim\limits_{t\to+\infty}\frac{t^3}{e^{\frac{t^2}{2}}} = \lim\limits_{t\to+\infty}\frac{3t^2}{te^{\frac{t^2}{2}}} = \lim\limits_{t\to+\infty}\frac{3}{te^{\frac{t^2}{2}}} = 0 = \lim\limits_{t\to-\infty}t^3\cdot e^{-\frac{t^2}{2}}$

$E\xi^4 = 3\int\limits_{-\infty}^{+\infty}t^2\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = 3E\xi^2 = 3(Var\,\xi - (E\xi)^2) = 3(1-0) = 3 \Rightarrow Cov(\xi,\xi^3) = 3$

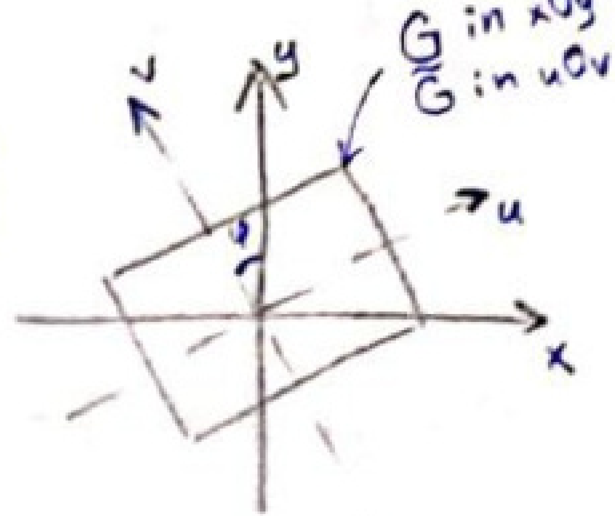
$Var\,\xi^3 = E(\xi^6) - (E(\xi^3))^2 = E\xi^6$

$E\xi^6 = \int\limits_{-\infty}^{+\infty}t^6\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = \quad -\text{"}- \quad = -\frac{1}{\sqrt{2\pi}}\left(\cancel{t^5 e^{-\frac{t^2}{2}}\Big|_{t=-\infty}^{+\infty}}^{0} - \int\limits_{-\infty}^{+\infty}5t^4\cdot e^{-\frac{t^2}{2}}dt\right) = E\xi^4 = 15 = Var\,\xi^3$

$\rho(\xi,\xi^3) = \frac{3}{\sqrt{1\cdot 15}} = \sqrt{\frac{3}{5}}$

б) $\rho(3\xi^2-2,\ 2\xi^2+3) = \frac{3\cdot 2}{|3\cdot 2|}\rho(\xi^2,\xi^2) = \frac{Var\,\xi^2}{|Var\,\xi^2|} = 1$

N10. Let $\begin{bmatrix}\xi\\ \eta\end{bmatrix}$: $\xi \sim N(0;1)$, $\eta \sim N(0;1)$



$P(\begin{bmatrix}\xi\\ \eta\end{bmatrix} \in G) = P(\begin{bmatrix}u\\ v\end{bmatrix} \in \tilde{G})$, where $\begin{matrix}\xi = x\\ \eta = y\end{matrix}$, $\begin{matrix}U = u\\ V = v\end{matrix}$, $\begin{matrix}u = x\cos\varphi - y\sin\varphi\\ v = x\sin\varphi + y\cos\varphi\end{matrix}$

Since $R_\varphi^{-1} = R_\varphi^T$, $\begin{matrix}x = u\cos\varphi + v\sin\varphi\\ y = -u\sin\varphi + v\cos\varphi\end{matrix}$ (R_φ is a rotation matrix)

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left\|\begin{matrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{matrix}\right\| = \left\|\begin{matrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{matrix}\right\| = |\cos^2\varphi + \sin^2\varphi| = 1$$

Since ξ and η are independent, $f_{\xi,\eta}(x,y) = f_\xi(x)\cdot f_\eta(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$

$$f_{u,v}(u,v) = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| f_{\xi,\eta}(x(u,v), y(u,v)) = \frac{1}{2\pi}e^{-\frac{(u\cos\varphi + v\sin\varphi)^2}{2} - \frac{(-u\sin\varphi + v\cos\varphi)^2}{2}} = \frac{1}{2\pi}e^{-\frac{u^2}{2} - \frac{v^2}{2}}$$

$$P(\begin{bmatrix}u\\ v\end{bmatrix} \in \tilde{G}) = \int_{-a}^{a} du \int_{-b}^{b} f_{u,v}(u,v)dv = \frac{1}{2\pi}\int_{-a}^{a} e^{-\frac{u^2}{2}}du \cdot \int_{-b}^{b} e^{-\frac{v^2}{2}}dv = (\Phi(a) - \Phi(-a))(\Phi(b) - \Phi(-b)) =$$
$$= (2\Phi(a) - 1)(2\Phi(b) - 1),$$ where Φ is from the standard normal table

N11. Let $\begin{bmatrix}\xi\\ \eta\end{bmatrix}$: $\xi \sim N(0;1)$, $\eta \sim N(0;1)$, since ξ, η are independent, $f_{\xi,\eta}(x,y) = f_\xi(x)\cdot f_\eta(y) = \frac{1}{2\pi}e^{-\frac{x^2}{2} - \frac{y^2}{2}}$

Let $G_a = \{(x;y) \mid |x| \le 1, |y| \le 1\}$

$$P(\begin{bmatrix}\xi\\ \eta\end{bmatrix} \in G_a) = \int_{-1}^{1} dx \int_{-1}^{1} f_{\xi,\eta}(x,y)dy = \frac{1}{2\pi}\int_{-1}^{1} e^{-\frac{x^2}{2}}dx \cdot \int_{-1}^{1} e^{-\frac{y^2}{2}}dy = (\Phi(1) - \Phi(-1))^2 = (2\Phi(1) - 1)^2 = 0{,}466$$

Let $G_b = \{(x;y) \mid |x| + |y| \le 1\}$

G_b is a square $\sqrt{2} \times \sqrt{2}$ rotated $45°$ ccw and centered around the origin.

From N10 we have $P(\begin{bmatrix}\xi\\ \eta\end{bmatrix} \in G_b) = (2\Phi(\frac{\sqrt{2}}{2}) - 1)^2 = 0{,}312$

$\underline{N12.}$ $\xi_1, \xi_2 \sim U[0;1] \Rightarrow F_{\xi_1}(x) = F_{\xi_2}(x) = \begin{cases} 1, & x \geq 1 \\ x, & x \in (0;1) \\ 0, & x \leq 0 \end{cases}$, $f_{\xi_1}(x) = f_{\xi_2}(x) = \begin{cases} 1, & x \in (0;1) \\ 0, & \text{otherwise} \end{cases}$

$$F_{\xi_1\xi_2}(z) = P(\xi_1\xi_2 \leq z) = \begin{cases} 1, & z \geq 1 \\ \int_0^1 P(\xi_1\xi_2 \leq z \mid \xi_1 = t) \cdot P(\xi_1 = t)\,dt, & z \in (0;1) \\ 0, & z \leq 0 \end{cases}$$

Subsequent lines consider only $z \in (0;1)$

$$P(\xi_1\xi_2 \leq z \mid \xi_1 = t) \cdot P(\xi_1 = t) = \frac{P(\xi_1\xi_2 \leq z,\ \xi_1 = t)}{P(\xi_1 = t)} \cdot P(\xi_1 = t) = P(\xi_1\xi_2 \leq z,\ \xi_1 = t) = P(t\xi_2 \leq z,\ \xi_1 = t) \overset{\text{indep.}}{=}$$
$$= P(\xi_2 \leq \tfrac{z}{t}) \cdot P(\xi_1 = t)$$

$$\int_0^1 P(\xi_1\xi_2 \leq z \mid \xi_1 = t) \cdot P(\xi_1 = t)\,dt = \int_0^1 F_{\xi_2}(\tfrac{z}{t}) \cdot f_{\xi_1}(t)\,dt = \int_0^1 F_{\xi_2}(\tfrac{z}{t})\,dt, \text{ since } t \in (0;1)$$

Let's consider $\frac{z}{t}$ for different $t \in (0;1)$:

1) $\frac{z}{t} \geq 1 \Rightarrow z \geq t \Rightarrow t \leq z$
2) $\frac{z}{t} \in (0;1) \Rightarrow z \in (0;t) \Rightarrow t \geq z \geq 0$
3) $\frac{z}{t} \leq 0 \Rightarrow z \leq 0 \quad \forall\, t \in (0;1)$

$$\int_0^1 F_{\xi_2}(\tfrac{z}{t})\,dt = \int_0^z \underbrace{F_{\xi_2}(\tfrac{z}{t})}_{1,\text{ since } \frac{z}{t} \geq 1}dt + \int_z^1 \underbrace{F_{\xi_2}(\tfrac{z}{t})}_{\frac{z}{t},\text{ since } \frac{z}{t} \in (0;1)}dt = z + z\ln t\Big|_{t=z} = z - z\ln z$$

Thus, $F_{\xi_1\xi_2}(z) = \begin{cases} 1, & z \geq 1 \\ z - z\ln z, & z \in (0;1) \\ 0, & z \leq 0 \end{cases} \Rightarrow f_{\xi_1\xi_2}(z) = F'_{\xi_1\xi_2}(z) = \begin{cases} 0, & z \notin (0;1) \\ -\ln z, & z \in (0;1) \end{cases} = -I_{z \in (0;1)} \ln z$

$$F_{\frac{\xi_2}{\xi_1}}(z) = P(\tfrac{\xi_2}{\xi_1} \leq z) = \begin{cases} \int_0^1 P(\frac{\xi_2}{\xi_1} \leq z \mid \xi_1 = t) \cdot P(\xi_1 = t)\,dt, & z \in [0;+\infty) \\ 0, & z < 0 \end{cases}$$

Subsequent lines only consider $z \geq 0$

$$\int_0^1 P(\tfrac{\xi_2}{\xi_1} \leq z \mid \xi_1 = t) \cdot P(\xi_1 = t)\,dt = \int_0^1 F_{\xi_2}(zt) f_{\xi_1}(t)\,dt = \int_0^1 F_{\xi_2}(zt)\,dt, \quad \text{since } t \in (0;1)$$

Let's consider zt for different $t \in (0;1)$:

1) $zt \geq 1 \Rightarrow t \geq \frac{1}{z}$, assuming $z \neq 0$
2) $zt \in (0;1) \Rightarrow t \in (0;\frac{1}{z})$, —||—
3) $zt \leq 0 \Rightarrow z \leq 0 \quad \forall\, t \in (0;1)$

$$\int_0^1 F_{\xi_2}(zt)\,dt = I_{\frac{1}{z}>1} \cdot \int_0^1 \underbrace{F_{\xi_2}(zt)}_{zt,\text{ since } zt \in (0;1)}dt + I_{\frac{1}{z} \in (0;1)} \cdot \Big(\int_0^{\frac{1}{z}} \underbrace{F_{\xi_2}(zt)}_{zt,\text{ since } zt \in (0;1)}dt + \int_{\frac{1}{z}}^1 \underbrace{F_{\xi_2}(zt)}_{1,\text{ since } zt \geq 1}dt\Big) =$$

$$= I_{z \in (0;1)} \cdot \frac{z}{2} + I_{z>1} \cdot \Big(\frac{1}{2z} + 1 - \frac{1}{z}\Big) = I_{z \in (0;1)} \cdot \frac{z}{2} + I_{z>1} \cdot \Big(-\frac{1}{2z}\Big)$$

Thus, $F_{\frac{\xi_2}{\xi_1}}(z) = \begin{cases} -\frac{1}{2z}, & z \geq 1 \\ \frac{z}{2}, & z \in (0;1) \\ 0, & z \leq 0 \end{cases} \Rightarrow f_{\frac{\xi_2}{\xi_1}}(z) = \begin{cases} \frac{1}{2z^2}, & z > 1 \\ \frac{1}{2}, & z \in (0;1) \\ 0, & z \leq 0 \end{cases}$

N13. $\xi_1, \xi_2 \sim Exp(\lambda)$, let $\varrho_1 = \frac{\xi_2}{\xi_1}$, $\varrho_2 = \frac{\xi_2}{\xi_1+\xi_2}$

$$F_{\xi_1}(x) = F_{\xi_2}(x) = \begin{cases} 1-e^{-\lambda x}, & x \geq 0 \\ 0, & x<0 \end{cases}, \quad f_{\xi_1}(x) = f_{\xi_2}(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x<0 \end{cases}$$

$$F_{\varrho_1}(z) = P\left(\frac{\xi_2}{\xi_1} \leq z\right) = \begin{cases} \int\limits_0^{+\infty} P\left(\frac{\xi_2}{\xi_1} \leq z \mid \xi_1 = t\right) \cdot P(\xi_1 = t)\,dt, & z \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

for $z \geq 0$ $\int\limits_0^{+\infty} P\left(\frac{\xi_2}{\xi_1} \leq z \mid \xi_1 = t\right) \cdot P(\xi_1 = t)\,dt = \int\limits_0^{+\infty} F_{\xi_2}(zt) \cdot f_{\xi_1}(t)\,dt = \int\limits_0^{+\infty} (1-e^{-\lambda z t}) \cdot \lambda e^{-\lambda t}\,dt =$

$$= \underbrace{\int\limits_0^{+\infty} \lambda e^{-\lambda t}\,dt}_{1} - \lambda \int\limits_0^{+\infty} e^{-(z+1)\lambda t}\,dt = 1 + \frac{1}{z+1}\int\limits_0^{+\infty} -\lambda(z+1)e^{-(z+1)\lambda t}\,dt = 1 - \frac{1}{z+1} = \frac{z}{z+1}$$

Thus, $F_{\varrho_1}(z) = \begin{cases} \frac{z}{z+1}, & z \geq 0 \\ 0, & z<0 \end{cases} \Rightarrow f_{\varrho_1}(z) = \begin{cases} \frac{1}{(z+1)^2}, & z \geq 0 \\ 0, & z<0 \end{cases}$

$$F_{\varrho_2}(z) = P\left(\frac{\xi_2}{\xi_1+\xi_2} \leq z\right) = \begin{cases} 1, & z>1 \\ \int\limits_0^{+\infty} P\left(\frac{\xi_2}{\xi_1+\xi_2} \leq z \mid \xi_2 = t\right) \cdot P(\xi_2 = t)\,dt, & z \in (0;1) \\ 0, & z \leq 0 \end{cases},$$

Since $\xi_1, \xi_2 \geq 0$, thus, $\frac{\xi_2}{\xi_1+\xi_2} \leq 1$

for $z \in (0;1)$ $\int\limits_0^{+\infty} P\left(\frac{\xi_2}{\xi_1+\xi_2} \leq z \mid \xi_2 = t\right) \cdot P(\xi_2 = t)\,dt = \int\limits_0^{+\infty} \left(1 - F_{\xi_1}\left(\frac{1-z}{z}t\right)\right) f_{\xi_2}(t)\,dt = \int\limits_0^{+\infty} e^{-\frac{1-z}{z}\lambda t} \cdot \lambda e^{-\lambda t}\,dt =$

$$= \lambda \int\limits_0^{+\infty} e^{-\left(\frac{1-z}{z}+1\right)\lambda t}\,dt = \lambda \int\limits_0^{+\infty} e^{-\frac{\lambda t}{z}}\,dt = -z \int\limits_0^{+\infty} \frac{\lambda}{z} e^{-\frac{\lambda t}{z}}\,dt = -z \cdot e^{-\frac{\lambda t}{z}}\Big|_{t=0}^{+\infty} = z$$

Thus, $F_{\varrho_2}(z) = \begin{cases} 1, & z>1 \\ z, & z \in (0;1] \\ 0, & z \leq 0 \end{cases} \Rightarrow f_{\varrho_2}(z) = \begin{cases} 1, & z \in (0;1] \\ 0, & z \notin (0;1] \end{cases} = I_{z \in (0;1]}$

N14. $\xi_1, \xi_2 \sim N(0;1)$, let $\varrho = \frac{\xi_2}{\xi_1}$, $\eta = \frac{|\xi_2|}{\xi_1}$, $\gamma = \frac{\xi_2}{|\xi_1|}$

$$\begin{matrix} \xi_1 = x \\ \xi_2 = y \end{matrix} \Rightarrow \begin{matrix} u_1 = \frac{y}{x} \\ v_1 = x \end{matrix} \Rightarrow \begin{matrix} x = v_1 \\ y = u_1 v_1 \end{matrix} \qquad \left\|\frac{\partial(x,y)}{\partial(u_1,v_1)}\right\| = \left\|\begin{matrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial v_1} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial v_1} \end{matrix}\right\| = \left\|\begin{matrix} 0 & 1 \\ v_1 & u_1 \end{matrix}\right\| = |v_1|$$

$$f_{\varrho,\xi_1}(u_1,v_1) = f_{\xi_1,\xi_2}(v_1, u_1 v_1) \cdot \left\|\frac{\partial(x,y)}{\partial(u_1,v_1)}\right\| = \frac{|v_1|}{2\pi} e^{-\frac{1}{2}v_1^2 - \frac{1}{2}u_1^2 v_1^2}$$

$$f_{\varrho}(u) = \int\limits_{-\infty}^{+\infty} f_{\varrho,\xi_1}(u_1,v_1)\,dv_1 = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |v_1| e^{-\frac{1}{2}v_1^2(1+u_1^2)}\,dv_1 = \frac{1}{2\pi} \cdot \frac{1}{1+u_1^2} \cdot \left(\int\limits_{-\infty}^{0} -(1+u_1^2)v_1 e^{-\frac{1}{2}(1+u_1^2)v_1^2}\,dv_1 + \int\limits_0^{+\infty} (1+u_1^2) v_1 e^{-\frac{1}{2}(1+u_1^2)v_1^2}\,dv_1\right) =$$

$$= \frac{1}{2\pi} \cdot \frac{1}{1+u_1^2} \cdot \left(e^{-\frac{1}{2}(1+u_1^2)v_1^2}\Big|_{v_1=-\infty}^{0} - e^{-\frac{1}{2}(1+u_1^2)v_1^2}\Big|_{v_1=0}^{+\infty}\right) = \frac{1}{\pi} \cdot \frac{1}{1+u_1^2}$$

$$\begin{matrix} u_2 = \frac{|y|}{x} \\ v_2 = y \end{matrix} \Rightarrow \begin{matrix} x = \frac{|v_2|}{u_2} \\ y = v_2 \end{matrix} \qquad \left\|\frac{\partial(x,y)}{\partial(u_2,v_2)}\right\| = \left\|\begin{matrix} -\frac{|v_2|}{u_2^2} & \frac{v_2}{u_2|v_2|} \\ 0 & 1 \end{matrix}\right\| = \frac{|v_2|}{u_2^2}$$

$$f_{\eta,\xi_2}(u_2, v_2) = f_{\xi_1\xi_2}\left(\frac{|v_2|}{u_2}, v_2\right)\cdot\frac{|v_2|}{u_2^2}$$

$$f_\eta(u_2) = \int_{-\infty}^{+\infty} f_{\eta,\xi_2}(u_2,v_2)\,dv_2 = \frac{1}{2\pi}\cdot\frac{1}{u_2^2}\cdot\int_{-\infty}^{+\infty}|v_2|e^{-\frac{1}{2}\frac{v_2^2}{u_2^2}-\frac{1}{2}v_2^2}dv_2 = \frac{1}{2\pi}\cdot\frac{1}{1+u_2^2}\cdot\int_{-\infty}^{+\infty}\frac{1+u_2^2}{u_2^2}|v_2|e^{-\frac{1}{2}\left(\frac{1+u_2^2}{u_2^2}\right)v_2^2}dv_2^2$$

Further evaluation of this integral is identical to f_ζ, thus, $f_\eta(u_2) = \frac{1}{\pi}\cdot\frac{1}{1+u_2^2}$

$$u_3 = \frac{y}{|x|} \qquad \Rightarrow \qquad x = v_3 \qquad \left|\frac{\partial(x,y)}{\partial(u_3,v_3)}\right| = \left\|\begin{matrix}\frac{\partial x}{\partial u_3} & \frac{\partial x}{\partial v_3}\\ \frac{\partial y}{\partial u_3} & \frac{\partial y}{\partial v_3}\end{matrix}\right\| = \left\|\begin{matrix}0 & 1\\ |v_3| & \frac{u_3 v_3}{|v_3|}\end{matrix}\right\| = |v_3|$$

$$v_3 = x \qquad\qquad y = u_3|v_3|$$

$$f_{\gamma,\xi_1}(u_3, v_3) = f_{\xi_1,\xi_2}(v_3, u_3|v_3|)\cdot|v_3| = \frac{1}{2\pi}|v_3|\cdot e^{-\frac{1}{2}v_3^2-\frac{1}{2}u_3^2v_3^2} = f_{\zeta,\xi_1}(u_3,v_3) \Rightarrow f_\gamma(u_3) = \frac{1}{\pi}\cdot\frac{1}{1+u_3^2}$$

<u>N15.</u> $f_{\xi,\eta}(x,y) = \frac{1}{25\pi}\cdot I_{x^2+y^2\le 25}$, let $\zeta = \frac{\xi}{\eta}$

$$\xi = x \quad\Rightarrow\quad u = \frac{x}{y} \quad\Rightarrow\quad x = uv \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right\| = \left\|\begin{matrix}v & u\\ 0 & 1\end{matrix}\right\| = |v|$$

$$\eta = y \qquad v = y \qquad y = v$$

$$f_{\zeta,\eta}(u,v) = f_{\xi,\eta}(uv, v)\cdot|v| = \frac{|v|}{25\pi}\cdot I_{v^2(1+u^2)\le 25}$$

$$f_\zeta(u) = \int_{-\infty}^{+\infty} f_{\zeta,\eta}(u,v)dv = \frac{1}{25\pi}\int_{-\infty}^{+\infty}|v|\cdot I_{v^2(1+u^2)\le 25}dv = \frac{1}{25\pi}\cdot\int_{-\infty}^{+\infty}|v|\cdot I_{v^2\le\frac{25}{1+u^2}}dv = \frac{1}{25\pi}\int_{-\frac{5}{\sqrt{1+u^2}}}^{\frac{5}{\sqrt{1+u^2}}}|v|dv =$$

$$= \frac{2}{25\pi}\int_0^{\frac{5}{\sqrt{1+u^2}}}v\,dv = \frac{2}{25\pi}\cdot\frac{25}{2(1+u^2)} = \frac{1}{\pi}\cdot\frac{1}{1+u^2}$$

<u>N16.</u> $f_{\xi,\eta}(x,y) = \frac{C\cdot I_{x>0}\cdot I_{y>0}}{(1+x+y)^3}$, let $\zeta = \xi+\eta$, $C: \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}f_{\xi,\eta}(x,y)dy = 1$

$$\xi = x \quad\Rightarrow\quad u = x+y \quad\Rightarrow\quad x = v \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right\| = \left\|\begin{matrix}0 & 1\\ 1 & -1\end{matrix}\right\| = 1$$

$$\eta = y \qquad v = x \qquad y = u-v$$

$$\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}f_{\xi,\eta}(x,y)dy = C\cdot\int_0^{+\infty}dx\int_0^{+\infty}\frac{dy}{(1+x+y)^3} = \frac{C}{2} \Rightarrow C = 2$$

$$f_{\zeta,\xi}(u,v) = f_{\xi,\eta}(v, u-v)\cdot 1 = \frac{C\cdot I_{v>0}\cdot I_{u>v}}{(1+u)^3}$$

$$f_\zeta(u) = \int_{-\infty}^{+\infty}f_{\zeta,\xi}(u,v)dv = C\cdot\int_0^{+\infty}\frac{I_{u>v}}{(1+u)^3}dv = \frac{C}{(1+u)^3}\cdot I_{u>0}\cdot\int_0^u dv = \frac{Cu}{(1+u)^3}I_{u>0} = \frac{2u}{(1+u)^3}I_{u>0}$$

<u>N17.</u> $\xi, \eta \sim U[0;1]$, let $\zeta = \xi+\eta$

$f_{\xi,\eta}(x,y) = f_\xi(x)\cdot f_\eta(y) = I_{x\in(0;1)}\cdot I_{y\in(0;1)}$, since ξ and η are independent

$$\xi = x \quad\Rightarrow\quad u = x+y \quad\Rightarrow\quad x = v \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right\| = \left\|\begin{matrix}0 & 1\\ 1 & -1\end{matrix}\right\| = 1.$$

$$\eta = y \qquad v = x \qquad y = u-v$$

$$f_{\zeta,\xi}(u,v) = f_{\xi,\eta}(v, u-v)\cdot 1 = I_{v\in(0;1)}\cdot I_{v\in(u-1;u)}$$

$$f_\zeta(u) = \int_{-\infty}^{+\infty}f_{\zeta,\xi}(u,v)dv = \int_0^1 I_{v\in(u-1;u)}dv = \begin{cases}0, & u\le 0,\ u\ge 2\\ u, & u\in(0;1]\\ 2-u, & u\in(1;2)\end{cases}$$

N18. $\eta \sim U[a;b]$ - altitude
$\xi \sim Exp(\lambda)$ - base radius, let $V = \pi\xi^2\eta$ be the cylinder's volume

Since η and ξ are independent, $f_{\eta,\xi}(x,y) = f_\eta(x)\cdot f_\xi(y) = \frac{I_{x\in(a;b)}}{b-a}\cdot\lambda e^{-\lambda y}\cdot I_{y>0}$

$$EV = E(\pi\xi^2\eta) = \pi E(\xi^2\eta) = \pi\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}y^2 x f_{\eta,\xi}(x,y)dy =$$ (law of the unconscious statistician)

$$= \pi\frac{1}{b-a}\int_a^b x dx\cdot\int_0^{+\infty}y^2\cdot\lambda e^{-\lambda y}dy = \frac{\pi}{b-a}\cdot\left(\frac{x^2}{2}\Big|_{x=a}^{b}\right)\cdot(E\xi^2) = \frac{\pi(b+a)(b-a)}{(b-a)\cdot 2}\cdot\frac{2}{\lambda^2} = \frac{\pi(b+a)}{\lambda^2}$$

$$EV^2 = E(\pi^2\xi^4\eta^2) = \pi^2E(\xi^4\eta^2) = \pi^2\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}y^4x^2f_{\eta,\xi}(x,y)dy = \frac{\pi^2}{b-a}\int_a^b x^2dx\cdot\int_0^{+\infty}y^4\cdot\lambda e^{-\lambda y}dy =$$

$$= \frac{\pi^2}{b-a}\left(\frac{x^3}{3}\Big|_{x=a}^{b}\right)\cdot\left(-\int_0^{+\infty}y^4\cdot(-\lambda e^{-\lambda y})dy\right) = \pi^2\frac{b^2+ba+a^2}{3}\cdot\left(-y^4e^{-\lambda y}\Big|_{y=0}^{+\infty} + 4\int_0^{+\infty}y^3e^{-\lambda y}dy\right) =$$

$$= \pi^2\frac{b^2+ba+a^2}{3}\cdot\left(-\frac{4}{\lambda}\int_0^{+\infty}y^3(-\lambda e^{-\lambda y})dy\right) = \pi^2\frac{b^2+ba+a^2}{3}\cdot\frac{4}{\lambda}\left(-y^3e^{-\lambda y}\Big|_{y=0}^{+\infty} + 3\int_0^{+\infty}y^2e^{-\lambda y}dy\right) =$$

$$= \pi^2(b^2+ba+a^2)\cdot\frac{4}{\lambda^2}\cdot\int_0^{+\infty}y^2\cdot\lambda e^{-\lambda y}dy = \pi^2(b^2+ba+a^2)\cdot\frac{4}{\lambda^2}\cdot E\xi^2 = \frac{8\pi^2(b^2+ba+a^2)}{\lambda^4}$$

$$Var\,V = EV^2-(EV)^2 = \frac{\pi^2}{\lambda^4}(7b^2+6ba+7a^2)$$

N19. Let the PDFs of ξ and η be $f_\xi(x)$ and $f_\eta(y)$ respectively.

Since ξ and η are independent, $f_{\xi,\eta}(x,y) = f_\xi(x)\cdot f_\eta(y)$

$$E(\xi\eta) = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}xyf_{\xi,\eta}(x,y)dy = \int_{-\infty}^{+\infty}xf_\xi(x)dx\cdot\int_{-\infty}^{+\infty}yf_\eta(y)dy = E\xi\cdot E\eta$$

$$E(\xi^2\eta^2) = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}x^2y^2f_{\xi,\eta}(x,y)dy = \int_{-\infty}^{+\infty}x^2f_\xi(x)dx\cdot\int_{-\infty}^{+\infty}y^2\cdot f_\eta(y)dy = E\xi^2\cdot E\eta^2$$

Thus, $Var(\xi\eta) = E(\xi^2\eta^2)-(E(\xi\eta))^2 = E\xi^2\cdot E\eta^2 - (E\xi\cdot E\eta)^2$ (left hand side)

$$Var\,\xi\cdot Var\,\eta + Var\,\xi\cdot(E\eta)^2 + Var\,\eta\cdot(E\xi)^2 = (E\xi^2-(E\xi)^2)(E\eta^2-(E\eta)^2) + (E\xi^2-(E\xi)^2)(E\eta)^2 +$$
$$+(E\eta^2-(E\eta)^2)(E\xi^2) =$$

$$= E\xi^2\cdot E\eta^2 \underline{-(E\xi)^2E\eta^2} - (E\eta)^2E\xi^2 + (E\xi\cdot E\eta)^2 + E\xi^2\cdot(E\eta)^2 - (E\xi)^2(E\eta)^2 \underline{+ E\eta^2\cdot(E\xi^2)} - (E\eta)^2(E\xi)^2 =$$

$$= E\xi^2\cdot E\eta^2 - (E\xi\cdot E\eta)^2$$ (right hand side)

Therefore, $Var(\xi\eta) = Var\,\xi\cdot Var\,\eta + Var\,\xi\cdot(E\eta)^2 + Var\,\eta\cdot(E\xi)^2$ ∎