**Probability & Statistics. Assignment 6**

1. Prove Chebyshev's inequality ($\epsilon \geq 0, t \geq 0$):

   (a) $P(|\xi| \geq \epsilon) \leq \frac{E|\xi|}{\epsilon}$;

   (b) $P(|\xi - E\xi| \geq \epsilon) \leq \frac{Var\xi}{\epsilon^2}$;

   (c) $P(|\xi| \geq \epsilon) \leq \frac{E|\xi|^t}{\epsilon^t}$.

2. Let $\xi$ be a random variable with a finite variance. Prove the inequality

   $P(|\xi - E\xi| \leq 3\sqrt{Var\xi}) \geq \frac{8}{9}$.

3. Compare the estimate of probability $P(|\xi - E\xi| \leq 3\sqrt{Var\xi})$ from the previous task with the exact probability given that

   (a) $\xi \sim N(\mu, \sigma^2)$ (normal distribution);

   (b) $\xi \sim Exp(\lambda)$ (exponential distribution);

   (c) $\xi \sim U(a, b)$ (uniform distribution).

   **Answer**: (a) $2\Phi(3) - 1$ or $2\Phi_0(3)$; (b) $1 - e^{-4}$; (c) 1.

4. The average water consumption on a factory is equal to 50000 gallons per day. Estimate the probability that the real water consumption on the factory is no more than triple average consumption.

   **Answer**: $P(\xi \leq 3E\xi) \geq \frac{2}{3}$.

5. Let $\xi > 0$ and $0 < p < \frac{1}{2}$ and $\xi \sim \begin{pmatrix} -\epsilon & 0 & \epsilon \\ p & 1-2p & p \end{pmatrix}$.

   Show that Chebyshev's inequality turns into an equality for this distribution:
   $P(|\xi - E\xi| \geq \epsilon) = \frac{Var\xi}{\epsilon^2}$.

6. Prove that if random variable $\xi$ is non-negative, integer, and its expected value $E\xi$ is finite, then

   $E\xi = \sum_{k=1}^{\infty} P(\xi \geq k)$.

7. Calculate the expected value of $\xi \sim G(p)$ using the result from the previous task.

   **Answer**: $E\xi = \frac{1}{p}$, *hint*: $P(\xi \geq k) = (1-p)^{k-1}$.

8. $N$ fair dice are rolled. Random variable $\xi$ is the smallest digit obtained.

   (a) Calculate expected value of $\xi$ if $N = 6$;

   (b) What happens to this expected value as $N \to \infty$?

   **Answer**: (a) $\sum_{k=1}^{6} (\frac{k}{6})^6 \approx 1.4397$; (b) $\sum_{k=1}^{6} (\frac{k}{6})^N \to 1$.

9. Ten people enter the lift at the ground floor of a nine-storey building. Random variables $\xi$ and $\eta$ are the numbers of floors where the lift made its first and last stops respectively. Find their expected values.

   **Answer**: $E\xi \approx 1.3295$, $E\eta \approx 6.6705$.

10. Joint distribution of random variables $\xi$ and $\eta$ is provided in a table below:

| $\xi \setminus \eta$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| -1 | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{7}{24}$ |
| 1 | $\frac{1}{3}$ | $\frac{1}{6}$ | 0 |

(a) Calculate conditional variances $Var(\eta|\xi = 1)$ and $Var(\xi|\eta = 0)$ ;

(b) find the distribution of $\xi + \eta$;

(c) find the distribution of $\xi\eta$;

(d) find the distribution of vector $(\xi\eta, \xi + \eta)^T$.

**Answer**:

(a) $Var(\eta|\xi = 1) = \frac{2}{9}$, $Var(\xi|\eta = 0) = \frac{8}{9}$;

(b) $\xi + \eta \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{3}{24} & \frac{2}{24} & \frac{15}{24} & \frac{4}{24} & 0 \end{pmatrix}$;

(c) $\xi\eta \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{15}{24} & \frac{6}{24} & \frac{3}{24} \end{pmatrix}$;

(d)

| $\xi + \eta \setminus \xi\eta$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| -2 | 0 | 0 | $\frac{3}{24}$ |
| -1 | 0 | $\frac{2}{24}$ | 0 |
| 0 | $\frac{15}{24}$ | 0 | 0 |
| 1 | 0 | $\frac{4}{24}$ | 0 |

11. A marksman hits the target with probability $0.8$. The marksman is shooting until he misses the target at least once and hits the target at least once. Find the expected value of the number of shoots.

    **Answer**: $E\xi = 5.25$.

12. Random variable $\zeta$ is uniformly distributed on set $\{-1; 0; 1\}$. Let us consider random variables $\xi = 1 - \zeta^{1000}$ and $\eta = 1 - \zeta^{1001}$.

    (a) Determine if $\xi$ and $\eta$ are independent.

    (b) Determine if $\xi$ and $\eta$ are correlated.

    **Answer**: (a) dependent; (b) $\rho = 0$.

13. Two independent random variables $\xi, \eta$ have geometric distribution with parameter $p$. Proof that $P(\xi = k|\xi + \eta = n) = \frac{1}{n-1}$.

14. Two independent random variables $\xi_1, \xi_2$ have geometric distribution with parameters $p_1$ and $p_2$. Find the distribution law of random variable $\xi = min(\xi_1, \xi_2)$.

    **Answer**: $\xi \sim \begin{pmatrix} 1 & 2 & \dots & k & \dots \\ 1 - q_1 q_2 & q_1 q_2 (1 - q_1 q_2) & \dots & (q_1 q_2)^{k-1} (1 - q_1 q_2) & \dots \end{pmatrix}$.

15. Expected value $\mu$ and covariance matrix $\mathcal{K}$ of random vector $\xi = (\xi_1, \xi_2, \xi_3)^T$ are provided below:

    $$\mu = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 35 \end{pmatrix}.$$

    Calculate expected value and variance of

    (a) $\eta = \xi_1 - \xi_3$;

    (b) $\eta = 2\xi_1 - \xi_2 + 3\xi_3$;

    (c) $\eta = -2\xi_1 + 3\xi_2 - \xi_3$.

    **Answer**: (a) $E\eta = -1, Var\eta = 42$; (b) $E\eta = 6, Var\eta = 314$; (c) $E\eta = -10, Var\eta = 66$.

16. Covariance matrix

    $$\mathcal{K} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & \lambda \\ 1 & \lambda & 2 \end{pmatrix}$$

    of random vector $\xi = (\xi_1, \xi_2, \xi_3)^T$ depends on parameter $\lambda$. Find the value of $\lambda$ such that variance of random variable $\zeta = \lambda\xi_1 + 2\xi_2 - \xi_3$ reaches its minimum.

    **Answer**: $\lambda_{1,2} = 5 \pm \sqrt{11}$.

17. It is known that $\xi_1, \xi_2, \ldots, \xi_{100} \sim Exp(\lambda)$ are independent identically distributed *(i.i.d)* random variables. Find expected value and covariance matrix of random vector $\eta = (\eta_1, \eta_2)^T$ given that $\eta_1 = \sum_{k=1}^{100} \xi_k$, $\eta_2 = \sum_{k=1}^{30} \xi_k$.

    **Answer**: $E\eta = \begin{pmatrix} \frac{100}{\lambda} \\ \frac{30}{\lambda} \end{pmatrix}$, $Var\eta = \begin{pmatrix} \frac{100}{\lambda^2} \\ \frac{30}{\lambda^2} \end{pmatrix}$.

18. $N$ letters have been written and envelopes have been inscribed for these letters. An absent-minded secretary places the letters into envelopes at random and sends them with the evening post. Find the $Var\xi$, where $\xi$ is the number of letters that reached their destination.

    **Answer**: $Var\xi = 1$.

19. Ten numbers are chosen at random from the the first 100 natural numbers (without replacement). The sum of those numbers is random variable $\xi$. Find expected value and variance of $\xi$.

    **Answer**: $E\xi = 505$, $Var\xi = 7575$.