Assignment 4.

N1. Given a circle of radius R and center O with a random point M inside,

Let $\xi = OM$.

$$F_\xi(x) = P(\xi < x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{R^2}, & x \in (0; R) \\ 1, & x \geq R \end{cases} \Rightarrow f_\xi(x) = F'_\xi(x) = \begin{cases} 0, & x \le 0 \\ \frac{2x}{R^2}, & x \in (0; R) \\ 0, & x \geq R \end{cases}$$

$$E\xi = \int_{-\infty}^{+\infty} x f_\xi(x)dx = \int_{-\infty}^{0} x \cdot 0 \cdot dx + \int_{0}^{R} x \cdot \frac{2x}{R^2} \cdot dx + \int_{R}^{+\infty} x \cdot 0 \cdot dx = \frac{2}{R^2}\int_{0}^{R} x^2 dx = \frac{2}{R^2} \cdot \frac{x^3}{3}\Big|_0^R = \frac{2}{3}R$$

$$E\xi^2 = \int_{-\infty}^{+\infty} x^2 f_\xi(x)dx = \int_0^R \frac{2x^3}{R^2}dx = \frac{1}{2}R^2$$

$$Var\ \xi = E\xi^2 - (E\xi)^2 = \frac{1}{2}R^2 - \frac{4}{9}R^2 = \frac{1}{18}R^2$$

<u>N2</u>. Given a sphere of radius R and center O with a random point M inside,

Let $\xi = R - OM$

$$F_\xi(x) = P(\xi < x) = P(OM > R-x) = \begin{cases} 0, & x<0 \\ \frac{R^3-(R-x)^3}{R^3}, & x\in(0;R) \\ 1, & x>R \end{cases} \Rightarrow f_\xi(x) = F'_\xi(x) = \begin{cases} 0, & x<0 \\ \frac{3(R-x)^2}{R^3}, & x\in(0;R) \\ 0, & x>0 \end{cases}$$

$$E\xi = \int_0^R x\cdot\frac{3(R-x)^2}{R^3}dx = \int_0^R \frac{3x}{R}dx - \int_0^R \frac{6x^2}{R^2}dx + \int_0^R \frac{3x^3}{R^3}dx = \frac{3}{2}R - 2R + \frac{3}{4}R = \frac{R}{4}$$

$$E\xi^2 = \int_0^R x^2\frac{3(R-x)^2}{R^3}dx = \int_0^R \frac{3x^2}{R}dx - \int_0^R \frac{6x^3}{R^2}dx + \int_0^R \frac{3x^4}{R^3}dx = R^2 - \frac{6}{4}R^2 + \frac{3}{5}R^2 = \frac{R^2}{10}$$

$$Var\ \xi = E\xi^2 - (E\xi)^2 = \frac{3}{80}R^2$$

N3.

$$F_\xi(x) = \begin{cases} 0, & x \le 1 \\ 1 - \frac{C}{x}, & x > 1 \end{cases} \Rightarrow f_\xi(x) = F'_\xi(x) = \begin{cases} 0, & x \le 1 \\ \frac{C}{x^2}, & x > 1 \end{cases}$$

For $F_\xi(x)$ to be a valid CDF, $f_\xi(x)$ needs to be a valid PDF, so:

$$\begin{cases} f_\xi(x) \geq 0 \quad \forall x & \text{true for } C \geq 0 \\ \int_{-\infty}^{+\infty} f_\xi(x)dx = 1 & \int_1^{+\infty} \frac{C}{x^2}dx = -\frac{C}{x}\Big|_1^{+\infty} = \lim_{x\to+\infty}\left(-\frac{C}{x}\right) + C = C, \text{ equal to 1 for } C = 1 \end{cases}$$

$E\xi = \int_1^{+\infty} x \cdot \frac{C}{x^2}dx = C \cdot \ln x \Big|_1^{+\infty}$, the integral diverges $\Rightarrow E\xi$ d.n.e.

N4.

a) $f(x) = \begin{cases} Ce^{-2x}, & x > 0 \\ 0, & x < 0 \end{cases}$

For $f(x)$ to be a probability density function, two conditions must be met:

$\begin{cases} f(x) \geq 0 \quad \forall x \in \mathbb{R} \\ \int\limits_{-\infty}^{+\infty} f(x)dx = 1 \end{cases}$
true for $C \geq 0$

$\int\limits_0^{+\infty} Ce^{-2x}dx = -\frac{C}{2}\int\limits_0^{+\infty} e^{-2x}d(-2x) = -\frac{C}{2}e^{-2x}\Big|_0^{+\infty} = \frac{C}{2}$, equal to 1 for $C=2$

$f(x)$ can be a PDF for $C=2$, let ξ be a random variable which has it as a PDF

$E\xi = \int\limits_{-\infty}^{+\infty} xf(x)dx = \int\limits_0^{+\infty} x \cdot 2e^{-2x}dx = -\left(xe^{-2x}\Big|_0^{+\infty} - \int\limits_0^{+\infty} e^{-2x}dx\right) = -\frac{1}{2}e^{-2x}\Big|_0^{+\infty} - xe^{-2x}\Big|_0^{+\infty} =$

$= \frac{1}{2} - \lim\limits_{x\to+\infty} xe^{-2x} = \frac{1}{2}$

$\lim\limits_{x\to+\infty} xe^{-2x} = \lim\limits_{x\to+\infty} \frac{x}{e^{2x}} = \lim\limits_{x\to+\infty} \frac{(x)'}{(e^{2x})'} = \lim\limits_{x\to+\infty} \frac{1}{2e^{2x}} = 0$

$E\xi^2 = \int\limits_0^{+\infty} x^2 \cdot 2e^{-2x}dx = -\left(x^2e^{-2x}\Big|_0^{+\infty} - \underbrace{\int\limits_0^{+\infty} 2x \cdot e^{-2x}dx}_{E\xi = \frac{1}{2}}\right) = \frac{1}{2} - \lim\limits_{x\to+\infty} x^2e^{-2x} = \frac{1}{2}$

$\lim\limits_{x\to+\infty} x^2e^{-2x} = \lim\limits_{x\to+\infty} \frac{x^2}{e^{2x}} = \lim\limits_{x\to+\infty} \frac{(x^2)'}{(e^{2x})'} = \lim\limits_{x\to+\infty} \frac{2x}{2e^{2x}} = 0$, as found previously

$Var\,\xi = E\xi^2 - (E\xi)^2 = \frac{1}{4}$

б) $f(x) = Ce^{-|x|}$

For $f(x)$ to be a probability density function, two conditions must be met:

$\begin{cases} f(x) \geq 0 \quad \forall x \in \mathbb{R} \\ \int\limits_{-\infty}^{+\infty} f(x)dx = 1 \end{cases}$
true for $C \geq 0$

$\int\limits_{-\infty}^{0} Ce^{x}dx + \int\limits_0^{+\infty} Ce^{-x}dx = C \cdot \left(e^{x}\Big|_{-\infty}^{0} - e^{-x}\Big|_0^{+\infty}\right) = 2C$, equal to 1 for $C = \frac{1}{2}$

$f(x)$ can be a PDF for $C=\frac{1}{2}$, let ζ be a random variable which has it as a PDF

$E\zeta = \int\limits_{-\infty}^{+\infty} xf(x)dx = \int\limits_{-\infty}^{0} \frac{1}{2} \cdot x \cdot e^{x}dx + \int\limits_0^{+\infty} \frac{1}{2} \cdot x \cdot e^{-x}dx$ (*)

$\int\limits_{-\infty}^{0} \frac{1}{2}x \cdot e^{x}dx = \frac{1}{2}\left(xe^{x}\Big|_{-\infty}^{0} - \int\limits_{-\infty}^{0} e^{x}dx\right) = -\frac{1}{2}\int\limits_{-\infty}^{0} e^{x}dx = -\frac{1}{2}e^{x}\Big|_{-\infty}^{0} = -\frac{1}{2}$

$\lim\limits_{x\to-\infty} xe^{x} = \lim\limits_{x\to-\infty} \frac{x}{e^{-x}} = \lim\limits_{x\to-\infty} \frac{(x)'}{(e^{-x})'} = \lim\limits_{x\to-\infty} \frac{1}{-e^{-x}} = 0$

$\int\limits_0^{+\infty} \frac{1}{2}x \cdot e^{-x}dx = -\frac{1}{2}\left(xe^{-x}\Big|_0^{+\infty} - \int\limits_0^{+\infty} e^{-x}dx\right) = \frac{1}{2}\int\limits_0^{+\infty} e^{-x}dx = -\frac{1}{2}e^{-x}\Big|_0^{+\infty} = \frac{1}{2}$

$\lim\limits_{x\to+\infty} xe^{-x} = \lim\limits_{x\to+\infty} \frac{x}{e^{x}} = \lim\limits_{x\to+\infty} \frac{(x)'}{(e^{x})'} = \lim\limits_{x\to+\infty} \frac{1}{e^{x}} = 0$

$E\zeta = -\frac{1}{2} + \frac{1}{2} = 0$

$E\zeta^2 = \int\limits_{-\infty}^{+\infty} x^2 f(x)dx = \int\limits_{-\infty}^{0} \frac{1}{2}x^2e^{x}dx + \int\limits_0^{+\infty} \frac{1}{2}x^2e^{-x}dx$

$\int\limits_{-\infty}^{0} \frac{1}{2}x^2e^{x}dx = \frac{1}{2}\left(x^2e^{x}\Big|_{-\infty}^{0} - \int\limits_{-\infty}^{0} 2xe^{x}dx\right) = -\int\limits_{-\infty}^{0} xe^{x}dx = 1$, as computed previously

$$\lim_{x\to-\infty} x^2e^{x} = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{(x^2)'}{(e^{-x})'} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{(2x)'}{-(e^{-x})'} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

$$\int_0^{+\infty} \frac{1}{2}x^2e^{-x}dx = -\frac{1}{2}\left(x^2e^{-x}\Big|_0^{+\infty} - \int_0^{+\infty} 2xe^{-x}dx\right) = \int_0^{+\infty} xe^{-x}dx = 1$$

$$\lim_{x\to+\infty} x^2e^{-x} = \lim_{x\to+\infty} \frac{x^2}{e^{x}} = \lim_{x\to+\infty} \frac{2x}{e^{x}} = \lim_{x\to+\infty} \frac{2}{e^{x}} = 0$$

$$E\xi^2 = 1 + 1 = 2$$

$$Var\,\xi = E\xi^2 - (E\xi)^2 = 2 - 0 = 2$$

N5. a) $f(x) = \frac{C}{1+x^2}$

For $f(x)$ to be a probability density function, two conditions must be met:

$$\begin{cases} f(x) \geq 0 \quad \forall x \in \mathbb{R} & \text{true for } C \geq 0 \\ \int_{-\infty}^{+\infty} f(x)dx = C\int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx = C\cdot \operatorname{arctg} x \Big|_{-\infty}^{+\infty} = C\pi = 1, & \text{true for } C = \frac{1}{\pi} \end{cases}$$

b) $$f(x) = \begin{cases} 0, & |x| \leq 1 \\ \frac{1}{2x^2}, & |x| > 1 \end{cases}$$

It is a valid PDF since it satisfies the conditions below:

$$\begin{cases} f(x) \geq 0 \quad \forall x \in \mathbb{R} \\ \int_{-\infty}^{+\infty} f(x)dx = 1 \end{cases} \qquad \int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{-1} \frac{dx}{2x^2} + \int_1^{+\infty} \frac{dx}{2x^2} = \int_1^{+\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{+\infty} = 1$$

However, $E\xi$ d.n.e., where ξ is a random variable with $f(x)$ as a PDF.

$$E\xi = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{-1} \frac{1}{x}dx + \int_1^{+\infty} \frac{1}{x}dx = \ln x\Big|_{-\infty}^{-1} + \ln x\Big|_1^{+\infty} \quad \text{d.n.e.}$$

N6. $\xi \sim U[0;4]$, $f_\xi(x) = \begin{cases} \frac{1}{4}, & x \in [0;4] \\ 0, & x \notin [0;4] \end{cases}$

$E\xi = \frac{0+4}{2} = 2$, $Var\,\xi = \frac{4^2}{12} = \frac{4}{3}$

$$F_\xi(x) = \int_{-\infty}^{x} f_\xi(x)dx = \begin{cases} 0, & x \leq 0 \\ \frac{1}{4}x, & x \in (0;4] \\ 1, & x > 4 \end{cases}$$

$$P(\xi < E\xi) = F_\xi(E\xi) = \frac{1}{2}$$

$$P(\xi > \sqrt{Var\,\xi}) = 1 - P(\xi \leq \sqrt{Var\,\xi}) = 1 - F_\xi(\sqrt{Var\,\xi}) = 1 - \frac{1}{2\sqrt{3}}$$

$$P(-5 \leq \xi \leq 5) = F_\xi(5) - F_\xi(-5) = 1$$

N7. $Y \sim U[a;b]$, $EY = 3$, $Var\,Y = 3$

$$\begin{cases} \frac{a+b}{2} = 3 \Rightarrow a = 6 - b \\ \frac{(b-a)^2}{12} = 3 \Rightarrow b - a = \pm 6 \\ a \leq b \end{cases} \Rightarrow a = 0,\ b = 6$$

N8. $\xi \sim U[a;b]$

$P(\xi \in [1;2)) = P(\xi < 2) - P(\xi < 1) = F_\xi(2) - F_\xi(1) = \frac{1}{6}$, where $\underset{\text{CDF}}{F_\xi(x)} = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & x \in (a;b] \\ 1, & x > b \end{cases}$

$$F_\xi(2) - F_\xi(1) = \frac{1}{b-a} = \frac{1}{6} \Rightarrow b - a = 6, \quad F_\xi(1) = \frac{1-a}{b-a} = \frac{1}{2} \Rightarrow a = -2,\ b = 4 \Rightarrow F_\xi(x) = \begin{cases} 0, & x \leq -2 \\ \frac{x+2}{6}, & x \in (-2;4] \\ 1, & x > 4 \end{cases}$$

$f_\xi(x) = F'_\xi(x) = \begin{cases} 0, & x \le -2 \\ \frac{1}{6}, & x \in (-2;4] \\ 0, & x > 4 \end{cases}$

$E\xi = \frac{4-2}{2} = 1, \quad Var\,\xi = \frac{6^2}{12} = 3$

N9. $Z \sim u[a;b], \quad F_Z(1) = \frac{1}{3}, \quad F_Z(4) = 1$

$P(1 \le Z < 4) = F_Z(4) - F_Z(1) = \frac{2}{3}$

Since $P(Z<4) = F_Z(4) = 1$, $b \le 4$, since $P(Z<1) = F_Z(1) > 0$, $a < 1$

$F_Z(4) = F_Z(b)$, Because $b \le 4$

Then we have the following conditions for a, b:

$\begin{cases} F_Z(b) - F_Z(1) = \frac{2}{3} \Leftrightarrow \frac{b-a}{b-a} - \frac{1-a}{b-a} = \frac{2}{3} \Leftrightarrow \frac{b-1}{b-a} = \frac{2}{3} \Leftrightarrow b = 3-2a \\ a < 1 \\ b \le 4 \Rightarrow 3-2a \le 4 \Rightarrow a \ge -\frac{1}{2} \end{cases}$

$Var\,Z = \frac{(b-a)^2}{12} = \frac{(3-3a)^2}{12}, \quad (Var\,Z)(a) = \frac{3(1-a)^2}{4}$, a parabola with branches up

$\max\limits_{[-\frac{1}{2};1)} (Var\,Z)(a) = (Var\,Z)(-\frac{1}{2}) = \frac{27}{16}$, since $a=1$ is the apex of the parabola

N10. $Z \sim u[a;b], \quad P(0<Z<1) = \frac{2}{3}, \quad P(1<Z<2) = \frac{1}{3}$

$P(0<Z<2) = P(0<Z<1) + P(1<Z<2) = 1 \Rightarrow [a;b] \subseteq [0;2]$

$\begin{matrix} P_Z(a) = P_Z(0) \\ P_Z(b) = P_Z(2) \end{matrix}$, since $\begin{matrix} a \ge 0 \\ b \le 2 \end{matrix}$

$P(0<Z<1) = P_Z(1) - P_Z(0) = P_Z(1) - P_Z(a) = \frac{1-a}{b-a} - \frac{a-a}{b-a} = \frac{2}{3}$

$P(1<Z<2) = P_Z(2) - P_Z(1) = P_Z(b) - P_Z(1) = \frac{b-a}{b-a} - \frac{1-a}{b-a} = \frac{1}{3}$

Then we have the following conditions for a, b: $(a \le b)$

$\begin{cases} a = 3-2b \\ a \ge 0 \Rightarrow b \le \frac{3}{2} \\ b \le 2 \end{cases} \qquad a \le b \Leftrightarrow 3-2b \le b \Leftrightarrow b \ge 1$

$(EZ)(b) = \frac{b+a}{2} = \frac{3-b}{2}, \quad \min\limits_{[1;\frac{3}{2}]} (EZ)(b) = (EZ)(\frac{3}{2}) = \frac{3}{4}$, since $(EZ)(b) \searrow$ on $\mathbb{R}$

$(Var\,Z)(b) = \frac{(b-a)^2}{12} = \frac{3(b-1)^2}{4}$, a parabola with branches up

$\max\limits_{[1;\frac{3}{2}]} (Var\,Z)(b) = \max\{(Var\,Z)(1), (Var\,Z)(\frac{3}{2})\} = \max\{0, \frac{3}{16}\} = \frac{3}{16}$ for $b = \frac{3}{2}$

N11. $X \sim u[-a;a]$

$F_X(x) = \begin{cases} 0, & x \le -a \\ \frac{x+a}{2a}, & x \in (-a;a] \\ 1, & x > a \end{cases} \qquad F_{|X|}(x) = \begin{cases} 0, & x \le 0 \\ 2\frac{x}{2a}, & x \in (0;a] \\ 1, & x > a \end{cases}$ equal to the CDF of a variable distributed $u[0;a]$

N12. $\eta \sim u[a;b], \quad \xi = \frac{\eta - E\eta}{\sqrt{Var\,\eta}}$

$E\eta = \frac{b+a}{2}, \quad \sqrt{Var\,\eta} = \frac{b-a}{2\sqrt{3}}, \quad f_\xi\left(\frac{x-E\eta}{\sqrt{Var\,\eta}}\right) = f_\eta(x)$

$$f_\eta(x)=\begin{cases}\frac{1}{b-a}, & x\in[a,b]\\ 0, & \text{otherwise}\end{cases}$$

For $f_\xi(x)$, the range of non-zero values is scaled by a factor of $\frac{1}{\sqrt{Var\,\eta}}=\frac{2\sqrt{3}}{b-a}$

The length of that range is $(b-a)\cdot\frac{2\sqrt{3}}{b-a}=2\sqrt{3}$.

Following the scaling, the value of $f_\xi(x)$ over the non-zero range was also scaled.

Non-zero range boundaries for f_η are a, b, then for f_ξ the boundaries are c, d:

$$c=\frac{a-E\eta}{\sqrt{Var\,\eta}}=-\sqrt{3}, \qquad d=\frac{b-E\eta}{\sqrt{Var\,\eta}}=\sqrt{3}$$

Therefore, $\xi\sim U[-\sqrt{3};\sqrt{3}]$

<u>N13.</u> $\xi\sim U[-1;5]$, $E((\xi-1)(3-\xi))-?$

$E\xi=\frac{5+(-1)}{2}=2$, $Var\,\xi=\frac{(5+1)^2}{12}=3$

$E((\xi-1)(3-\xi))=E(-\xi^2+4\xi-3)=-E\xi^2+4E\xi-3=-Var\,\xi-(E\xi)^2+4E\xi-3=-2$

<u>N14.</u> $\theta\sim Exp(\lambda)$

$$f_\theta(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & x\le 0\end{cases} \Rightarrow F_\theta(x)=\int_{-\infty}^{x}f_\theta(x)dx=\begin{cases}\int_0^x\lambda e^{-\lambda t}dt, & x>0\\ 0, & x\le 0\end{cases}$$

$$F_\theta(x)=\begin{cases}1-e^{-\lambda x}, & x>0\\ 0, & x\le 0\end{cases}$$

$P(\theta\in(0;1))=F_\theta(1)-F_\theta(0)=1-e^{-\lambda}$

$P(\theta\in(1;2))=F_\theta(2)-F_\theta(1)=e^{-\lambda}-e^{-2\lambda}=e^{-\lambda}(1-e^{-\lambda})$

$P(\theta\in(k;k+1))=F_\theta(k+1)-F_\theta(k)=e^{-k\lambda}-e^{-(k+1)\lambda}=e^{-k\lambda}(1-e^{-\lambda})$

$\{P(\theta\in(k;k+1))\}_{k=0}^{\infty}$ is a geometric sequence with a ratio $e^{-\lambda}$

<u>N15.</u> $Z\sim Exp(\lambda)$

Using the calculations from N14, $F_Z(x)=\begin{cases}1-e^{-\lambda x}, & x>0\\ 0, & x\le 0\end{cases}$

$P(2<Z<3)=F_Z(3)-F_Z(2)=e^{-2\lambda}(1-e^{-\lambda})$

$P(2<Z<3)=\frac{4}{27}\Leftrightarrow\frac{e^{\lambda}-1}{e^{3\lambda}}=\frac{4}{27}\Leftrightarrow 4e^{3\lambda}-27e^{\lambda}+27=0 \quad (1)$

Let's substitute $t=e^{\lambda}$

$(1)\Leftrightarrow 4t^3-27t+27=0$, $t=-3$ is a root of the equation

$(t-3)(4t^2+12t+9)=0\Leftrightarrow 4(t-3)(t+\frac{3}{2})^2=0$

$\left[\begin{array}{l}e^{\lambda}=\frac{3}{2} \Rightarrow \lambda=\ln\frac{3}{2}\\ e^{\lambda}=-3 \quad \text{impossible}\end{array}\right.$ Let $Z\sim Exp(\ln\frac{3}{2})\Rightarrow EZ=\frac{1}{\ln\frac{3}{2}}$

N16. Let $\xi \sim Exp(\lambda)$, $E\xi = \sqrt{Var\,\xi} = \frac{1}{\lambda}$

$P(|\xi - \frac{1}{\lambda}| < \frac{3}{\lambda}) = P(-\frac{3}{\lambda} < \xi - \frac{1}{\lambda} < \frac{3}{\lambda}) = P(-\frac{2}{\lambda} < \xi < \frac{4}{\lambda})$

$$f_\xi(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad F_\xi(x) = \int_{-\infty}^{x} f_\xi(t)dt = \begin{cases} 0, & x \leq 0 \\ \int_0^x \lambda e^{-\lambda t}dt, & x > 0 \end{cases} = \begin{cases} 0, & x \leq 0 \\ 1-e^{-\lambda x}, & x > 0 \end{cases}$$

$P(-\frac{2}{\lambda} < \xi < \frac{4}{\lambda}) = F_\xi(\frac{4}{\lambda}) - F_\xi(-\frac{2}{\lambda}) = 1 - e^{-4}$

N17. $\xi \sim Exp(\lambda)$, $\eta = e^{-\xi}$.

$$f_\xi(x) = f_\eta(e^{-x}) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Let $\eta = y(\xi)$, then $y(x) = e^{-x}$, $y^{-1}(y) = x(y) = -\ln y$

$$P(a \leq \xi < b) = P(y(b) \leq y(\xi) < y(a)) = \int_a^b f_\xi(x)dx = \int_{y(a)}^{y(b)} f_\xi(x(y))\frac{dx}{dy}dy = \int_{y(b)}^{y(a)} -f_\xi(x(y))\frac{dx}{dy}dy$$

Therefore, $f_\eta(y) = -f_\xi(x(y)) \cdot \frac{dx}{dy} = \begin{cases} \frac{1}{y} \cdot \lambda e^{-\lambda \cdot (-\ln y)}, & -\ln y \geq 0 \\ 0, & -\ln y < 0 \end{cases}$

Since $\eta = e^{-\xi}$ and $\xi \in \mathbb{R}$, $\eta \in (0; +\infty)$

$$f_\eta(y) = \begin{cases} \frac{1}{y}\lambda \cdot y^\lambda, & 0 < y \leq 1 \\ 0, & y > 1 \end{cases} \Rightarrow E\eta = \int_0^{+\infty} y f_\eta(y)dy = \int_0^1 \lambda y^\lambda dy = \frac{\lambda}{\lambda+1}$$

$$E\eta^2 = \int_0^{+\infty} y^2 f_\eta(y)\,dy = \int_0^1 \lambda y^{\lambda+1}dy = \frac{\lambda}{\lambda+2}$$

$$Var\,\eta = E\eta^2 - (E\eta)^2 = \frac{\lambda}{(\lambda+2)(\lambda+1)^2}$$

N18. $\xi \sim Exp(\lambda)$

$$P(\xi > \tau) = \int_\tau^{+\infty} f_\xi(x)dx = \int_\tau^{+\infty} \lambda e^{-\lambda x} = -e^{-\lambda x}\Big|_\tau^{+\infty} = e^{-\tau\lambda}$$

Since τ and t are positive numbers, $\{\xi > t+\tau\} \subseteq \{\xi > t\}$, therefore,

$\{\xi > t+\tau\} \cap \{\xi > t\} = \{\xi > t+\tau\}$

$$P(\xi > t+\tau \mid \xi > t) = \frac{P(\xi > t+\tau)}{P(\xi > t)} = \frac{e^{-(\tau+t)\lambda}}{e^{-t\lambda}} = e^{-\tau\lambda} \quad \blacksquare$$

N19. $\xi \sim N(\mu; \sigma^2)$, $E\xi = \mu = 1$, $Var\ \xi = \sigma^2 = 4 \Rightarrow \sigma = 2$, since $\sigma > 0$

Let $\eta \sim N(0; 1)$, then $\xi = 2\eta + 1$

Let $\Phi_0(x) = P(-\infty < \eta < x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$

$P(a \leq \xi < b) = P(a \leq 2\eta + 1 < b) = P(\frac{a-1}{2} \leq \eta < \frac{b-1}{2}) = \Phi_0(\frac{b-1}{2}) - \Phi_0(\frac{a-1}{2})$

a) $P(-3 < \xi < 1) = \Phi_0(0) - \Phi_0(-2) = 0{,}5 - 0{,}0228 = 0{,}4772$, according to the standard normal table

b) $P(\xi < -2) = \Phi_0(-\frac{3}{2}) = 0{,}0668$

c) $P(\xi > 3) = 1 - P(\xi \leq 3) = 1 - \Phi_0(1) = 1 - 0{,}8413 = 0{,}1587$

N20. Let $\xi \sim N(-1; 1)$, $\eta \sim N(0; 1)$, then $\xi = \eta - 1$

$P(x < \xi < y) = P(x < \eta - 1 < y) = P(x+1 < \eta < y+1) = \Phi_0(y+1) - \Phi_0(x+1)$

a) $P(x < \xi < 1) = \Phi_0(2) - \Phi_0(x+1) = 0{,}8 \Rightarrow \Phi_0(x+1) = 0{,}9772 - 0{,}8 = 0{,}1772 \Rightarrow x+1 \approx -0{,}93$, $x \approx -1{,}93$

b) $P(0 < \xi < x) = \Phi_0(x+1) - \Phi_0(1) = 0{,}8 \Rightarrow \Phi_0(x+1) = 0{,}8 + 0{,}8413 = 1{,}6413 > 1 \Rightarrow x$ d.n.e.

c) $P(-1-x < \xi < -1+x) = \Phi_0(x) - \Phi_0(-x) = 0{,}8$

$\Phi_0(0) - \Phi_0(-x) = \Phi_0(x) - \Phi_0(0)$, since the PDF of η is symmetric about $x = 0$

$\begin{cases} \Phi_0(x) + \Phi_0(-x) = 1 \\ \Phi_0(x) - \Phi_0(-x) = 0{,}8 \end{cases} \Rightarrow \begin{matrix} \Phi_0(x) = 0{,}9 \\ \Phi_0(-x) = 0{,}1 \end{matrix} \Rightarrow x \approx 1{,}28$

N21. $\xi \sim N(0; \sigma^2)$, the PDF of ξ is symmetric w.r.t. 0, thus, $\int_{-a}^{-b} f_\xi(x)dx = \int_{b}^{a} f_\xi(x)dx$

$f_\xi(t) \searrow$ on $(0; +\infty)$

$P(\xi \in (0; 4)) = \int_0^4 f_\xi(x)dx = \int_0^3 f_\xi(x)dx + \int_3^4 f_\xi(x)dx$

$P(\xi \in (-1; 3)) = \int_{-1}^{3} f_\xi(x)dx = \int_{-1}^{0} f_\xi(x)dx + \int_0^3 f_\xi(x)dx = \int_0^3 f_\xi(x)dx + \int_0^1 f_\xi(x)dx$

$P(\xi \in (0; 4)) < P(\xi \in (-1; 3)) \Leftrightarrow \int_3^4 f_\xi(x)dx < \int_0^1 f_\xi(x)dx \Leftrightarrow \lim_{dx \to 0} \sum_{i=1}^{\frac{1}{dx}} dx \cdot f_\xi(3 + i \cdot dx) < \lim_{dx \to 0} \sum_{i=1}^{\frac{1}{dx}} dx \cdot f_\xi(i \cdot dx)$

Since the amount of terms in both sums is equal and $f_\xi(3 + i \cdot dx) < f_\xi(i \cdot dx)$ $\forall i, dx$, the above inequality holds.

$P(\xi \in (-1; 3)) = \int_{-1}^{3} f_\xi(x)dx = \int_{-1}^{2,5} f_\xi(x)dx + \int_{2,5}^{3} f_\xi(x)dx$

$P(\xi \in (-1{,}5; 2{,}5)) = \int_{-1,5}^{2,5} f_\xi(x)dx = \int_{-1,5}^{-1} f_\xi(x)dx + \int_{-1}^{2,5} f_\xi(x)dx = \int_{-1}^{2,5} f_\xi(x)dx + \int_{1}^{1,5} f_\xi(x)dx$

$P(\xi \in (-1; 3)) < P(\xi \in (-1{,}5; 2{,}5)) \Leftrightarrow \int_{2,5}^{3} f_\xi(x)dx < \int_{1}^{1,5} f_\xi(x)dx$, which is similarly proven true.

In the same way one can obtain $P(\xi \in (-1{,}5; 2{,}5)) < P(\xi \in (-2; 2))$.

Thus, $P(\xi \in (0; 4)) < P(\xi \in (-1; 3)) < P(\xi \in (-1{,}5; 2{,}5)) < P(\xi \in (-2; 2))$.

N22. $\xi \sim N(\mu; \sigma^2)$, $E\xi = \mu \Rightarrow \xi - E\xi = \xi - \mu$, $\xi - \mu \sim N(0; \sigma^2)$

$P(|\xi - \mu| < 1) = 0{,}3 \Leftrightarrow P(-1 < \xi - \mu < 1) = 0{,}3 \Leftrightarrow 2P(0 < \xi - \mu < 1) = 0{,}3$, since $\xi - \mu$ has a symmetric PDF w.r.t. 0

Let $\eta \sim N(0; 1)$, then $\xi - \mu = \sigma\eta$, $P(0 < \xi - \mu < x) = P(0 < \eta < \frac{x}{\sigma})$

$P(0 < \xi - \mu < 1) = 0{,}15 = P(0 < \eta < \frac{1}{\sigma}) \Rightarrow \Phi_0(\frac{1}{\sigma}) - \Phi_0(0) = 0{,}15 \Rightarrow \Phi_0(\frac{1}{\sigma}) = 0{,}65 \Rightarrow \frac{1}{\sigma} = 0{,}385$

$P(|\xi - \mu| < 2) = P(-2 < \xi - \mu < 2) = 2P(0 < \xi - \mu < 2) = 2P(0 < \eta < \frac{2}{\sigma}) = 2(\Phi_0(2 \cdot 0{,}385) - \Phi_0(0)) = 2 \cdot 0{,}2794 = 0{,}5588$

N23. $\xi \sim N(\mu;\sigma^2)$, $E\xi = \mu = 1$, $Var\,\xi = \sigma^2 = 5 \Rightarrow \sigma = \sqrt{5}$ since $\sigma > 0$.
The PDF of ξ is symmetric w.r.t. 1 and $\searrow$ on $(1;+\infty)$.
Therefore $\forall \varepsilon, a, b: \varepsilon > 0,\ 1 < a < b \Rightarrow \int_a^{a+\varepsilon} f_\xi(x)dx > \int_b^{b+\varepsilon} f_\xi(x)dx$ (refer to N21 for proof)
From this follows the fact that $P(\xi \in (1-\varepsilon;1+\varepsilon)) \geqslant P(\xi\in(a-\varepsilon;a+\varepsilon))\quad \forall a, \varepsilon > 0$
Let $\eta \sim N(0;1)$, then $\xi = \sqrt{5}\eta + 1$
$P(\xi \in (1-\varepsilon;1+\varepsilon)) = 2P(\xi\in(1;1+\varepsilon)) = 2P(\eta \in (0;\frac{\varepsilon}{\sqrt{5}})) = 0{,}95 = 2(\Phi_0(\frac{\varepsilon}{\sqrt{5}}) - \Phi_0(0))$
$\Phi_0(\frac{\varepsilon}{\sqrt{5}}) - 0{,}5 = \frac{0{,}95}{2} \Rightarrow \Phi_0(\frac{\varepsilon}{\sqrt{5}}) = 0{,}975 \Rightarrow \frac{\varepsilon}{\sqrt{5}} = 1{,}96$, according to the standard normal table
$\varepsilon = \sqrt{5}\cdot 1{,}96 = 4{,}38$
Thus, the shortest interval $(a;b)$ such that $P(\xi\in(a;b)) = 0{,}95$ is $(-3{,}38;5{,}38)$

N24. $\xi \sim N(\mu;\sigma^2)$, $P(1<\xi<7) = P(7<\xi<13) = 0{,}18$
The PDF for ξ is symmetric w.r.t. μ, thus $\int_{\mu-\varepsilon}^{\mu} f_\xi(x)dx = \int_{\mu}^{\mu+\varepsilon} f_\xi(x)dx \quad \forall \varepsilon > 0$
$$\left.\begin{aligned} P(1<\xi<7) &= \int_{7-6}^{7} f_\xi(x)dx \\ P(7<\xi<13) &= \int_{7}^{7+6} f_\xi(x)dx \end{aligned}\right\} \Rightarrow \mu = 7, \text{ since no other spot of } f_\xi \text{ exhibits symmetry.}$$
Let $\eta \sim N(0;1)$, then $\xi = \sigma\eta + 7$
$P(7<\xi<13) = P(0<\eta<\frac{6}{\sigma}) = \Phi_0(\frac{6}{\sigma}) - \Phi_0(0) = 0{,}18$
$\Phi_0(\frac{6}{\sigma}) = 0{,}18 + 0{,}5 \Rightarrow \frac{6}{\sigma} = 0{,}47 \Rightarrow \sigma = \frac{600}{47}$
$E\xi = \mu = 7$, $Var\,\xi = \sigma^2 = (\frac{600}{47})^2 = 163$

N25. $\eta \sim N(1;\sigma^2)$, $\sigma > 0$
Let $\xi \sim N(0;1)$, then $\eta = \sigma\xi + 1$
$P(2<\eta<4) = P(\frac{1}{\sigma} < \xi < \frac{3}{\sigma}) = \Phi_0(\frac{3}{\sigma}) - \Phi_0(\frac{1}{\sigma})$, where $\Phi_0(x) = \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}dt$
$h(\sigma) = \Phi_0(\frac{3}{\sigma}) - \Phi_0(\frac{1}{\sigma})$, $h'(\sigma) = \Phi'(\frac{3}{\sigma})\cdot(-\frac{3}{\sigma^2}) - \Phi'(\frac{1}{\sigma})\cdot(-\frac{1}{\sigma^2}) = (-\frac{1}{\sigma^2})\frac{1}{\sqrt{2\pi}}\cdot(3e^{-\frac{9}{2\sigma^2}} - e^{-\frac{1}{2\sigma^2}})$
$h'(\sigma) = 0 \Leftrightarrow 3e^{-\frac{9}{2\sigma^2}} = e^{-\frac{1}{2\sigma^2}} \Leftrightarrow e^{-\frac{4}{\sigma^2}} = \frac{1}{3} \Leftrightarrow \sigma = \frac{2}{\sqrt{\ln 3}}$
$h''(\sigma) = \frac{2}{\sigma^3}\cdot\frac{1}{\sqrt{2\pi}}((3-\frac{27}{2\sigma^2})e^{-\frac{9}{2\sigma^2}} - (1-\frac{1}{2\sigma^2})e^{-\frac{1}{2\sigma^2}})$
$h''(\frac{2}{\sqrt{\ln 3}}) = \frac{(\sqrt{\ln 3})^3}{4}\cdot\frac{1}{\sqrt{2\pi}}\cdot((3-\frac{27\ln 3}{8})e^{-\frac{9\ln 3}{8}} - (1-\frac{\ln 3}{8})e^{-\frac{\ln 3}{8}})$
$h''(\frac{2}{\sqrt{\ln 3}}) > 0 \Leftrightarrow (3-\frac{27\ln 3}{8})3^{-\frac{9}{8}} > (1-\frac{\ln 3}{8})3^{-\frac{1}{8}} \Leftrightarrow 3^{-\frac{1}{8}} - \frac{9\ln 3\cdot 3^{-\frac{1}{8}}}{8} > 3^{-\frac{1}{8}} - \frac{\ln 3\cdot 3^{-\frac{1}{8}}}{8} \Leftrightarrow 9 < 1$
$h''(\frac{2}{\sqrt{\ln 3}}) < 0 \Rightarrow \frac{2}{\sqrt{\ln 3}}$ is a local maximum for $h(\sigma) \Rightarrow \max\limits_{\sigma>0} P(2<\eta<4)$ is for $\sigma = \frac{2}{\sqrt{\ln 3}}$

N26. $\zeta \sim N(\mu;\sigma^2)$, $E\zeta = \mu = -2$, $Var\,\zeta = \sigma^2 = 9$
$E((3-\zeta)(\zeta+5)) = E(-\zeta^2 - 2\zeta + 15) = -E\zeta^2 - 2E\zeta + 15 = (-Var\,\zeta - (E\zeta)^2)\cdot 2E\zeta + 15 = 6$

N27. $\xi \sim N(\mu;\sigma^2)$, $a \neq 0$, $b \in \mathbb{R}$. $\eta = a\xi + b$

I. $a > 0$

$$P(\eta < x) = P\left(\xi < \frac{x-b}{a}\right)$$

$$P\left(\xi < \frac{x-b}{a}\right) = \int_{-\infty}^{\frac{x-b}{a}} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \overset{y = \frac{t-\mu}{\sigma}}{=} \int_{-\infty}^{\frac{x-b-a\mu}{a\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Let $\theta \sim N(a\mu + b; a^2\sigma^2)$, $P(\theta < x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{(t-a\mu-b)^2}{2a^2\sigma^2}} dt = \int_{0}^{\frac{x-b-a\mu}{a\sigma}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} dy$

Since there's a 1:1 correspondence with CDFs and random variables, $\theta = \eta \Rightarrow \eta \sim N(a\mu+b; a^2\sigma^2)$

II. $a < 0$

$$P(\eta < x) = P\left(\xi > \frac{x-b}{a}\right)$$

$P\left(\xi > \frac{x-b}{a}\right) = \int_{\frac{x-b}{a}}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \int_{\frac{x-b-a\mu}{a\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = P\left(\varphi > \frac{x-b-a\mu}{a\sigma}\right)$, where $\varphi \sim N(0;1)$

The PDF of φ is symmetric w.r.t. $0 \Rightarrow P(\varphi > x) = P(\varphi < -x)$

$$P(\eta < x) = P\left(\varphi < \frac{x-b-a\mu}{-a\sigma}\right)$$

Then for $\theta \sim N(a\mu + b; a^2\sigma^2)$ we have the same CDF $\Rightarrow \eta = \theta \Rightarrow \eta \sim N(a\mu + b; a^2\sigma^2)$

N28. Let $\xi \sim N(0;\sigma^2)$. $E|\xi|$ -?, $Var|\xi|$ -?

$$f_\xi(t) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{t^2}{2\sigma^2}}, \quad f_{|\xi|}(t) = \begin{cases} 0, & t < 0 \\ f_\xi(t) + f_\xi(-t), & \text{otherwise} \end{cases}$$

$$E|\xi| = \int_{-\infty}^{+\infty} t \cdot f_{|\xi|}(t)\, dt = \int_0^{+\infty} \frac{2t}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = -\frac{2\sigma}{\sqrt{2\pi}} \int_0^{+\infty} -\frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} dt = -\frac{2\sigma}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2\sigma^2}} \Big|_0^{+\infty} = \frac{\sqrt{2}\cdot\sigma}{\sqrt{\pi}}$$

$$E(|\xi|^2) = E(\xi^2) = \sigma^2 \Rightarrow Var|\xi| = \sigma^2\left(1 - \frac{2}{\pi}\right)$$

N29. Let $\eta = \sin\xi$

a) $\xi \sim U[-\frac{\pi}{2}; \frac{\pi}{2}]$

$$F_\xi(x) = \begin{cases} 0, & x < -\frac{\pi}{2} \\ \frac{x + \frac{\pi}{2}}{\pi}, & x \in (-\frac{\pi}{2}; \frac{\pi}{2}) \\ 1, & x > \frac{\pi}{2} \end{cases}$$

$\xi \in [-\frac{\pi}{2}; \frac{\pi}{2}] \Rightarrow \eta = \sin\xi \in [-1; 1]$

intersect

$F_\eta(x) = 0 \quad \forall x \leq -1$, $\quad F_\eta(x) = 1 \quad \forall x \geq 1$, since $\eta \in [-1; 1]$

for $x \in (-1; 1)$ $\quad F_\eta(x) = P(\eta < x) = P(\sin\xi < x) = P(\xi \in \bigcup_{k\in\mathbb{Z}} [\pi - \arcsin x + 2\pi k; \arcsin x + 2\pi + 2\pi k]) = P(\xi < \arcsin x) = F_\xi(\arcsin x)$

Thus $F_\eta(x) = \begin{cases} 0, & x \leq -1 \\ \frac{\arcsin x + \frac{\pi}{2}}{\pi}, & x \in (-1; 1) \\ 1, & x \geq 1 \end{cases} \Rightarrow f_\eta(x) = F'_\eta(x) = \begin{cases} 0, & x \notin (-1; 1) \\ \frac{1}{\pi\sqrt{1-x^2}}, & x \in (-1; 1) \end{cases}$

b) $\xi \sim U[0; \pi]$

$$F_\xi(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\pi}, & x \in (0; \pi) \\ 1, & x > \pi \end{cases}$$

$\xi \in [0; \pi] \Rightarrow \eta = \sin\xi \in [0; 1]$

$F_\eta(x) = 0 \quad \forall x \le 0, \qquad F_\eta(x) = 1 \quad \forall x \geq 1, \qquad$ since $\quad \eta \in [0;\ 1]$

for $x \in (0;1) \quad F_\eta(x) = P(\eta < x) = P(\sin\xi < x) = P(\xi \in \bigcup_{k \in \mathbb{Z}} [\pi - \arcsin x + 2\pi k;\ \arcsin x + 2\pi + 2\pi k]) =$

$= P(\xi \in [0;\ \arcsin x] \cup [\pi - \arcsin x;\ \pi])$, $\quad$ since $\xi \in [0;\ \pi]$

$= F_\xi(\arcsin x) + F_\xi(\pi) - F_\xi(\pi - \arcsin x)$

Thus, $F_\eta(x) = \begin{cases} 0, & x \le 0 \\ \frac{\arcsin x}{\pi} + 1 - \frac{\pi - \arcsin x}{\pi} = \frac{2\arcsin x}{\pi}, & x \in (0;1) \\ 1, & x \geq 1 \end{cases} \Rightarrow f_\eta(x) = F'_\eta(x) = \begin{cases} 0, & x \notin (0;1) \\ \frac{2}{\pi\sqrt{1-x^2}}, & x \in (0;1) \end{cases}$

<u>N30.</u> ξ is a random variable such that $f_\xi(x) = \frac{1}{\pi(1+x^2)}$, $\max\limits_{\lambda > 0} P(\xi \in (\lambda;\ 2\lambda))$ - ?

$$h(\lambda) = \int_\lambda^{2\lambda} \frac{dx}{\pi(1+x^2)} = \frac{1}{\pi} \operatorname{arctg} x \Big|_\lambda^{2\lambda} = \frac{1}{\pi}(\operatorname{arctg} 2\lambda - \operatorname{arctg} \lambda)$$

$h'(\lambda) = \frac{1}{\pi}\left(\frac{2}{1+4\lambda^2} - \frac{1}{1+\lambda^2}\right)$, $\qquad h'(\lambda) = 0 \Leftrightarrow 2 + 2\lambda^2 = 1 + 4\lambda^2 \Rightarrow \lambda = \frac{1}{\sqrt{2}}$, since $\lambda > 0$

$h''(\lambda) = \frac{1}{\pi}\left(-\frac{16\lambda}{(1+4\lambda^2)^2} + \frac{2\lambda}{(1+\lambda^2)^2}\right)$, $\quad h''(\frac{1}{\sqrt{2}}) = \frac{1}{\pi}\left(-\frac{16}{9\sqrt{2}} + \frac{8}{9\sqrt{2}}\right) < 0 \Rightarrow \lambda = \frac{1}{\sqrt{2}}$ is a local max for $h(\lambda)$

<u>N31.</u> ξ is a random variable such that $f_\xi(x) = \frac{1}{\pi(1+x^2)}$, $\quad \eta = \frac{1}{\xi}$

$$F_\xi(x) = \int_{-\infty}^x f_\xi(x)dx = \frac{1}{\pi}\int_{-\infty}^x \frac{dx}{1+x^2} = \frac{1}{\pi}\operatorname{arctg} x \Big|_{-\infty}^x = \frac{\operatorname{arctg} x}{\pi} + \frac{1}{2}$$

$$F_\eta(x) = P(\eta < x) = P(\tfrac{1}{\xi} < x) = P\left(\frac{1-\xi x}{\xi} < 0\right) = P(\xi(1-\xi x) < 0)$$

$$F_\eta(x) = P(\xi(1-\xi x) < 0) = \begin{cases} P(\xi < 0) + P(\xi > \frac{1}{x}), & x > 0 \\ P(\xi < 0), & x = 0 \\ P(\xi < 0) - P(\xi < \frac{1}{x}), & x < 0 \end{cases} = \begin{cases} \frac{\pi - \operatorname{arctg}\frac{1}{x}}{\pi}, & x > 0 \\ \frac{1}{2}, & x = 0 \\ -\frac{\operatorname{arctg}\frac{1}{x}}{\pi}, & x < 0 \end{cases} = \frac{\operatorname{arctg} x}{\pi} + \frac{1}{2}$$

Since a CDF uniquely identifies a random variable, we have a Cauchy distribution for η