Probability & Statistics. Assignment 7

1. Random vector $(\xi;\eta)^T$ is uniformly distributed inside ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

   (a) Find marginal distributions of ξ and η;

   (b) Determine if ξ and η are independent;

   (c) Find covariation matrix of this random vector;

   (d) Find conditional expectations $E(\xi|\eta)$ and $E(\eta|\xi)$.

   Answer:

   (a) $f_\xi(x)=\frac{2}{\pi a}\sqrt{1-\frac{x^2}{a^2}}\cdot I_{x\in[-a,a]}$, $f_\eta(y)=\frac{2}{\pi b}\sqrt{1-\frac{y^2}{b^2}}\cdot I_{y\in[-b,b]}$;

   (b) dependent. *Hint*: $P(0<x<\epsilon, 0<y<\epsilon)$;

   (c) $K=\frac{1}{4}\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$;

   (d) $H(x)=E(\eta|\xi=x)=0$, $G(y)=E(\xi|\eta=y)=0$.

2. Random vector $(\xi;\eta)^T$ is uniformly distributed in a triangle with vertices $(-5;0)$, $(5;0)$ and $(0;4)$.

   (a) Find $E(\eta|\xi=2)$, $\mathrm{Var}(\eta|\xi=2)$;

   (b) Find $E(\xi|\eta=2)$, $\mathrm{Var}(\xi|\eta=2)$.

   Answer: (a) $\frac{6}{5}$, $\frac{12}{25}$; (b) 0, $\frac{25}{12}$.

3. Random vector $(\xi;\eta)^T$ is given by its cumulative distribution function:
   $F_{\xi,\eta}(x;y)=\left(1-e^{-\lambda x}-e^{-\mu y}+e^{-\lambda x-\mu y}\right)I_{x>0}I_{y>0}$.

   (a) Find marginal distributions of ξ and η;

   (b) Determine if ξ and η are independent;

   (c) Find covariation matrix of this random vector;

   (d) Find conditional expectations $E(\xi|\eta)$ and $E(\eta|\xi)$.

   Answer:

   (a) $f_\xi(x)=(1-e^{-\lambda x})\cdot I_{x>0}$, $f_\eta(y)=(1-e^{-\mu y})\cdot I_{y>0}$;

   (b) independent;

   (c) $K=\begin{pmatrix} \frac{1}{\lambda^2} & 0 \\ 0 & \frac{1}{\mu^2} \end{pmatrix}$;

   (d) $H(x)=E(\eta|\xi=x)=\frac{1}{\mu}$, $G(y)=E(\xi|\eta=y)=\frac{1}{\lambda}$.

4. Probability density of random vector $(\xi;\eta)^T$ is given by $f(x;y)=\frac{a}{1+x^2+y^2+x^2y^2}$. Find a and marginal densities of ξ and η. Determine whether ξ and η are independent and correlated.

   Answer: $a=\frac{1}{\pi^2}$, $f_\xi(x)=\frac{1}{\pi}\frac{1}{1+x^2}$, $f_\eta(y)=\frac{1}{\pi}\frac{1}{1+y^2}$, independent.

5. Let $\xi\sim N(0;1)$ and $\eta=\xi^2$. Determine if ξ and η are (a) independent, (b) correlated.

Answer: (a) dependent; (b) uncorrelated.

6. The covariance matrix of random vector $(\xi;\eta)^T$ is equal to $\begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}$. Find $\mathrm{Var}(-\xi-2\eta)$ and $\mathrm{Var}(3\xi-\eta+2)$.

   Answer: 14, 42.

7. Let a, b, c, d be constants, and $ac \neq 0$. Show that $\rho(a\xi+b, c\eta+d) = \pm\rho(\xi,\eta)$. How do we determine the sign at the right side of the equality?

   Answer: $\rho(a\xi+b, c\eta+d) = \begin{cases} +\rho(\xi,\eta), & ac > 0 \\ -\rho(\xi,\eta), & ac < 0 \end{cases}$.

8. Let $\zeta \sim \mathrm{Exp}(\lambda)$. Find the correlation coefficient between

   (a) $2\zeta+3$ and $3\zeta-1$;

   (b) ζ^2 and $\zeta^2-\zeta$.

   Answer: (a) 1; (b) $\frac{10-2\lambda}{\sqrt{5(\lambda^2-8\lambda+20)}}$.

9. Let $\zeta \sim N(0;1)$. Find the correlation coefficient between

   (a) 2ζ and ζ^3;

   (b) $3\zeta^2-2$ and $2\zeta^2+3$.

   Answer: (a) $\frac{3}{\sqrt{15}}$; (b) 1.

10. Let ξ and η be independent random variables with $N(0;1)$ distribution. Find the probability that a point with coordinates $(\xi;\eta)$ is situated within a rectangle centered at the origin whose sides are equal to $2a$ and $2b$. (Note that the sides of a rectangle do not have to be parallel to coordinate axes.)

    Answer: $(2\Phi(\frac{a}{2})-1)(2\Phi(\frac{b}{2})-1)$.

11. Let ξ and η be independent random variables with $N(0;1)$ distribution. Find the probability that a point with coordinates $(\xi;\eta)$

    (a) is situated within figure $\{|x| \leq 1, |y| \leq 1\}$;

    (b) is situated within figure $\{|x|+|y| \leqslant 1\}$.

    Answer: (a) $(2\Phi(1)-1)^2$; (b) $(2\Phi(\frac{\sqrt{2}}{2})-1)^2$.

12. Random variables ξ_1 and ξ_2 are independent and uniformly distributed on $[0;1]$. Find the distributions of $\eta = \xi_1\xi_2$ and $\zeta = \frac{\xi_2}{\xi_1}$.

    Answer: $f_{\xi_1\xi_2}(x) = -ln(x)$, $f_{\frac{\xi_2}{\xi_1}}(x) = \begin{cases} \frac{1}{2}, & x \in (0,1) \\ \frac{1}{2x^2}, & x \geq 1 \end{cases}$.

13. Random variables ξ_1 and ξ_2 are independent and exponentially distributed with parameter λ. Find the distributions of $\zeta = \frac{\xi_2}{\xi_1}$ and $\zeta = \frac{\xi_2}{\xi_1+\xi_2}$.

    Answer: $f_{\frac{\xi_2}{\xi_1}}(x) = \frac{1}{(1+y)^2}$; $f_{\frac{\xi_2}{\xi_1}}(x) = 1 \cdot I_{x\in(0,1)}$.

14. Random variables ξ_1 and ξ_2 are independent and have a standard normal distribution. Find the distributions of $\zeta = \frac{\xi_2}{\xi_1}$, $\eta = \frac{|\xi_2|}{\xi_1}$ and $\gamma = \frac{\xi_2}{|\xi_1|}$.

Answer: $f_\zeta(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $f_\eta(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $f_\gamma(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

15. Random vector $(\xi;\eta)^T$ is uniformly distributed within a circle given by $x^2 + y^2 \leqslant 25$. Find the distribution of $\zeta = \frac{\xi}{\eta}$.

    Answer: $f_\zeta(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

16. Probability density of random vector $(\xi;\eta)$ is given by $f_{\xi,\eta}(x;y) = \frac{C\cdot I_{x>0} I_{y>0}}{(1+x+y)^3}$. Find probability density of $\zeta = \xi + \eta$.

    Answer: $f_\zeta(x) = \frac{2x}{(1+x)^3} \cdot I_{x>0}$.

17. Find probability density of $\xi + \eta$ where ξ and η are independent random variables uniformly distributed on $(0;1)$.

    Answer: $f_{\xi+\eta}(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2-x, & 1 < x < 2 \\ 0, & otherwise \end{cases}$.

18. The altitude of a cylinder η and the radius of its base ξ are independent random variables whose distributions are $U[a;b]$ and $\mathrm{Exp}(\lambda)$ respectively. Find expectation and variance of a volume of such a cylinder.

    Answer: $\frac{\pi}{\lambda^2}(a+b)$; $\frac{\pi^2}{\lambda^4}(7a^2 + 6ab + 7b^2)$.

19. Prove that $\mathrm{Var}(\xi\eta) = \mathrm{Var}\,\xi \cdot \mathrm{Var}\,\eta + \mathrm{Var}\,\xi \cdot (E\eta)^2 + \mathrm{Var}\,\eta \cdot (E\xi)^2$ for arbitrary independent random variables ξ and η.