October 22, 2020

$X_1, X_2, \dots, X_n, \dots$ independent identically distributed r.v. (i.i.d. r.v.)

$EX_j = \mu$

$Var\, X_j = \sigma^2$

$$\forall \varepsilon > 0 \quad P\left(\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \mu\right| > \varepsilon\right) \xrightarrow{n\to\infty} 0$$

law of large numbers

$\lim\limits_{n\to\infty} Y_n = Z$ in probability, if

$$\forall \varepsilon > 0 \quad P(|Y_n - Z| > \varepsilon) \xrightarrow{n\to\infty} 0$$

$$Y_n \xrightarrow{P} Z, \quad n\to\infty$$

$\lim\limits_{n\to\infty} Y_n = Z$ in distribution if

$$F_{Y_n}(x) \longrightarrow F_Z(x), \quad \underset{n\to\infty}{x \in \mathbb{R}}$$

$$P(Y_n < x) \longrightarrow P(Z < x), \quad n\to\infty$$

(weak convergence)

$$P\left(\lim_{n\to\infty} Y_n = Z\right) = 1 \qquad Y_n \xrightarrow[n\to\infty]{a.s.} Z$$

almost sure convergence

$$\lim_{n\to\infty} Y_n(\omega) = Z(\omega)$$

Convergence in mean

$$Y_n \xrightarrow[\text{square}]{\text{in mean}} Z$$

$$\lim_{n\to\infty} E(Y_n - Z)^2 = 0$$

$$\begin{tikzcd}
\text{convergence in mean square} \arrow[dr] & & \text{almost sure convergence} \arrow[dl] \\
& \text{convergence in probability} \arrow[d] & \\
& \text{convergence in distribution} &
\end{tikzcd}$$

$$Y_n \to Z$$

X = a number rolled on a die

$$EX = \frac{1+2+3+4+5+6}{6} = 3,5$$

$$\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{P} 3,5, \; n \to \infty$$

1	2	3	4	5	6
30%	10%	10%	10%	10%	30%

$$\% = \frac{1}{100}$$

$$E\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) = \mu$$

$$Var\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n^2} Var\left(\sum_{j=1}^{n} X_j\right) =$$

$$= \frac{1}{n^2}\sum_{j=1}^{n} Var X_j = \frac{\sigma^2}{n}.$$

$$P(|Y - EY| \geq \varepsilon) \leq \frac{Var Y}{\varepsilon^2}$$

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \mu\right| \geq \varepsilon\right) \leq \frac{Var\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)}{\varepsilon^2} =$$

$$= \frac{\sigma^2}{n\varepsilon^2}$$

Central limit theorem

$X_1, X_2, \dots, X_n, \dots$ — i.i.d r.v.

$E X_j = \mu$, $Var X_j = \sigma^2$

$$S_n = \sum_{j=1}^{n} X_j$$

$$\frac{S_n - n\mu}{\sqrt{Var S_n}} \xrightarrow[n \to \infty]{\mathcal{D}} \xi \sim N(0;1)$$

convergence in distribution

$$\sqrt{Var S_n} = \sigma \sqrt{n}$$

$$\frac{S_n - n\mu}{\sqrt{Var S_n}} = \frac{\sum_{j=1}^{n}(X_j - \mu)}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{X_j - \mu}{\sigma}$$

$$\varphi_{\frac{S_n - n\mu}{\sqrt{Var S_n}}}(t) = \prod_{j=1}^{n} \varphi_{\frac{1}{\sqrt{n}}\left(\frac{X_j-\mu}{\sigma}\right)}(t) =$$

$$E e^{it \cdot \frac{1}{\sqrt{n}}\left(\frac{X_j - \mu}{\sigma}\right)} = \varphi_{\frac{X_j - \mu}{\sigma}}\left(\frac{t}{\sqrt{n}}\right)$$

$$= \left(\varphi_{\frac{X_1 - \mu}{\sigma}}\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$\varphi(\tau) = \varphi(0) + \varphi'(0)\tau + \frac{\varphi''(0)}{2}\tau^2 + o(\tau^2)$$

$$\varphi_{\frac{X_1-\mu}{\sigma}}(0) = 1; \quad \varphi'_{\frac{X_1-\mu}{\sigma}}(0) = iE\frac{X_1-\mu}{\sigma} = 0$$

$$\varphi''_{\frac{X_1-\mu}{\sigma}}(0) = -E\left(\frac{X_1-\mu}{\sigma}\right)^2 = -Var\left(\frac{X_1-\mu}{\sigma}\right) = -1$$

$$\varphi_X(t) = E e^{itX}$$

$$\varphi_X'(t) = E\left(iX e^{itX}\right) \Rightarrow \varphi_X'(0) = iEX$$

$$\varphi_X''(t) = E\left(-X^2 e^{itX}\right) \Rightarrow \varphi_X''(0) = -EX^2$$

$$E\xi^2 = Var\,\xi + (E\xi)^2$$

$$\varphi_{\frac{X_1-\mu}{\sigma}}\left(\frac{t}{\sqrt{n}}\right) = 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)$$

$$\left(\varphi_{\frac{X_1-\mu}{\sigma}}\left(\frac{t}{\sqrt{n}}\right)\right)^n = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n =$$

$$= e^{n \ln\left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)} = e^{n\left(-\frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)} =$$

$$= e^{-t^2/2 + o(1)} \longrightarrow e^{-t^2/2}, \quad n \to \infty.$$

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \approx \xi \qquad \xi \sim N(0;1)$$

$$S_n \approx \underbrace{\xi \cdot \sqrt{n}\,\sigma + n\mu}_{\sim N(n\mu;\, n\sigma^2)}$$

$P(\text{boy}) = 0{,}52$

$P(\text{girl}) = 0{,}48$

1000 babies

ξ is a number of boys out of 1000 babies

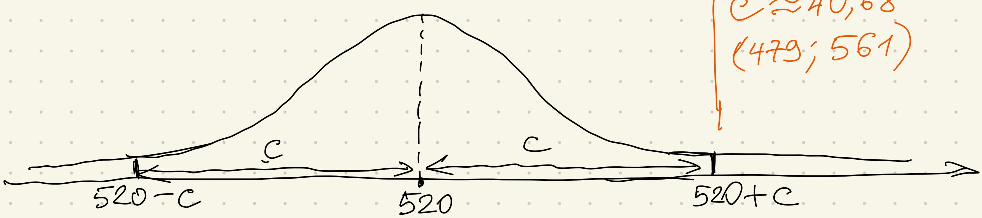
95%

(confidence interval)

$P(a < \xi < b) = 0{,}95$ $\underline{\underline{0{,}99}}$

$c \approx 40{,}68$

$(479; 561)$



$p = 0{,}52,\ q = 0{,}48 \Rightarrow E\xi = np = 520$

$n = 1000 \qquad \mathrm{Var}\,\xi = npq = 249{,}6$

$\xi \sim N(520;\ 249{,}6)$

0,99

$$\int_{520}^{520+c} \frac{1}{\sqrt{2\pi}\sqrt{249{,}6}} \cdot e^{-\frac{(x-520)^2}{2\cdot 249{,}6}}\,dx = \frac{0{,}95}{2} \qquad \frac{x-520}{\sqrt{249{,}6}} = t$$

$(489;\ 551)$

$$\int_0^{c/\sqrt{249{,}6}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = 0{,}475 \qquad \frac{c}{\sqrt{249{,}6}} \approx 1{,}96$$

0,495   2,575

$c \approx 30{,}97$

$$\frac{\xi - E\xi}{\sqrt{Var\,\xi}} \sim N(0;1)$$

$$520 - C < \xi < 520 + C$$

$$-C < \xi - 520 < C$$

$$P\left(-\frac{C}{\sqrt{249{,}6}} < \frac{\xi - 520}{\sqrt{249{,}6}} < \frac{C}{\sqrt{249{,}6}}\right) = 0{,}95$$

$$2\Phi_0\left(\frac{C}{\sqrt{249{,}6}}\right) = 0{,}95$$

$$C = \ldots$$

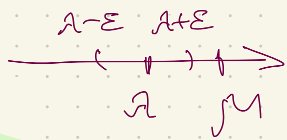
$X_1, X_2, \ldots, X_n, \ldots$ — i.i.d. r.v. $\sim \text{Poisson}(\lambda)$

$$\lim_{n\to\infty} \underbrace{P(S_n \le \mu n)}_{\to\, P\left(\frac{S_n}{n} \le \mu\right)} = ?$$

$$S_n = \sum_{k=1}^{n} X_k, \qquad \mu = \text{const} > 0.$$

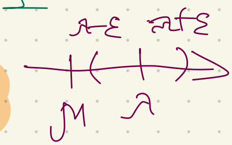
$$S_n \sim \text{Poisson}(n\lambda)$$

$$P\left(\left|\frac{S_n}{n} - \lambda\right| \le \varepsilon\right) \xrightarrow{n\to\infty} 1$$



$$P\left(\lambda - \varepsilon \le \frac{S_n}{n} \le \lambda + \varepsilon\right) \xrightarrow{n\to\infty} 1$$

$$P\left(n\lambda - \varepsilon n \le S_n \le n\lambda + \varepsilon n\right) \xrightarrow{n\to\infty} 1$$



$$\mu > \lambda \Rightarrow P\left(\frac{S_n}{n} \le \mu\right) \xrightarrow{n\to\infty} 1$$

$$\mu < \lambda \Rightarrow P\left(\frac{S_n}{n} \le \mu\right) \xrightarrow{n\to\infty} 0$$

$$\mu = \lambda \Rightarrow P(S_n \le \lambda n) = P(S_n - \lambda n \le 0) =$$

$$= P\left(\frac{S_n - \lambda n}{\sqrt{\operatorname{Var} S_n}} \le 0\right) \xrightarrow{n\to\infty} \Phi(0) = \frac{1}{2}$$

$X_{m,n}$ independent $\quad \underline{\lambda > 0}$

$$F_{X_{m,n}}(x) = \left(1 - e^{-\lambda n x}\right) I_{x \geq 0}$$

$$X_n = \sum_{m=1}^{n} X_{m,n} \qquad \lim_{n \to \infty} X_n = y$$

$$y \sim ?$$

$$f_{X_{m,n}}(x) = \lambda n e^{-\lambda n x} I_{x \geq 0} \sim Exp(\lambda n)$$

$$\varphi_{X_{m,n}}(x) = E e^{it X_{m,n}} = \int_{-\infty}^{+\infty} e^{itx} f_{X_{m,n}}(x) dx =$$

$$= \int_0^{+\infty} e^{itx} \lambda n e^{-\lambda n x} dx = \frac{\lambda n}{it - \lambda n} e^{(it - \lambda n)x} \Big|_{x=0}^{+\infty} =$$

$$= \frac{\lambda n}{\lambda n - it}$$

$$\varphi_{X_n}(t) = \left(\frac{\lambda n}{\lambda n - it}\right)^n = e^{n \ln \frac{\lambda n}{\lambda n - it}} =$$

$$= e^{n \ln \frac{1}{1 - it/\lambda n}} = e^{-n \ln\left(1 - \frac{it}{\lambda n}\right)} = e^{-n\left(-\frac{it}{\lambda n} + o\left(\frac{1}{n}\right)\right)} =$$

$$= e^{\frac{it}{\lambda} + o(1)} \xrightarrow{n \to \infty} e^{it/\lambda}$$

$$y \sim \begin{pmatrix} 1/\lambda \\ 1 \end{pmatrix} \qquad P\left(y = \frac{1}{\lambda}\right) = 1$$

$X_1, X_2, \dots, X_n \dots$ — i.i.d.r.v.

$P(X_j = 4^r) = 2^{-r}, \; r \in \mathbb{N}$

$$X_j \sim \begin{pmatrix} 4 & 4^2 & 4^3 & \dots \\ 1/2 & 1/2^2 & 1/2^3 & \dots \end{pmatrix}$$

$EX_j = \sum_{r=1}^{\infty} 2^r$ — divergent

→ does not exist

$$P\left(\frac{1}{2^n}\sum_{k=1}^{2^n} X_k \geq C\right) \not\longrightarrow 0, \; n \to \infty$$

$\alpha_n = P($at least one random variable of $X_1, X_2, \dots, X_{2^n}$ is equal to $4^n) = 1 - \prod_{k=1}^{2^n} P(X_k \neq 4^n) = 1 - (P(X_1 \neq 4^n))^{2^n} = 1 - (1 - P(X_1 = 4^n))^{2^n} = 1 - (1-2^{-n})^{2^n} \xrightarrow{n\to\infty} 1 - \frac{1}{e}$

$$\sum_{k=1}^{2^n} X_k \geq C \cdot 2^n$$

$\exists n : \quad 2^n \geq C \Longleftrightarrow 4^n \geq C \cdot 2^n$

$$P\left(\sum_{k=1}^{2^n} X_k \geq 4^n\right) \geq \alpha_n \longrightarrow 1 - \frac{1}{e}$$