October 1, 2020

$$F_{\vec{X}}(\vec{x}) = \qquad \vec{X}(X_1, X_2, \dots, X_n)$$
$$\vec{x}(x_1, x_2, \dots, x_n)$$

$$= P(X_1 < x_1, X_2 < x_2, \dots, X_n < x_n) \qquad X_i < x_i$$

$$\lim_{x_i \to +\infty} F_{\vec{X}}(\vec{x}) = P(X_1 < x_1, X_2 < x_2, \dots X_{i-1} < x_{i-1}, X_{i+1} < x_{i+1}, \dots, X_n < x_n)$$

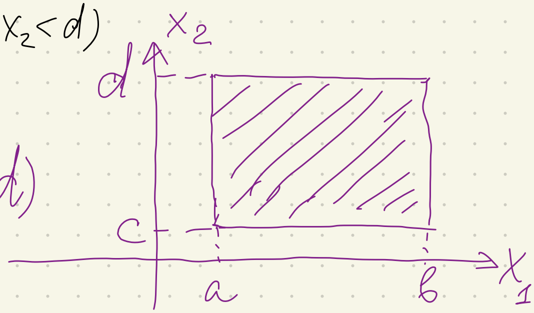
$$\lim_{\substack{x_1 \to +\infty \\ x_2 \to +\infty \\ \dots \\ x_{n-1} \to +\infty}} F_{\vec{X}}(\vec{x}) = P(X_n < x_n) = F_{X_n}(x_n)$$

$$P(X_1 < x_1, \dots, X_n < x_n) \leq P(X_i < x_i)$$

$$\lim_{x_i \to -\infty} F_{\vec{X}}(\vec{x}) = 0$$

$$P\left(\begin{array}{c}\text{inside the}\\ \text{rectangle}\end{array}\right) = F(b, d) + F(a, c) - F(b, c) - F(a, d)$$

$$F(b,d) \to P(X_1 < b, X_2 < d)$$



$$f_{\vec{X}}(\vec{x}) = \frac{\partial^n F_{\vec{X}}(\vec{x})}{\partial x_1 \partial x_2 \dots \partial x_n}$$

$$f_{X_1, X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1, X_2}(x_1, x_2)}{\partial x_1 \partial x_2}$$

$$P(\text{inside the rectangle}) = \iint\limits_{\substack{a \leq x_1 \leq b \\ c \leq x_2 \leq d}} f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$IQ = -80\ln(1-x) = X \qquad x \sim U(0;1)$$

$$IQ = -70\ln\left(1-\frac{4}{3}y\right) = Y \qquad y \sim U\left(0;\frac{3}{4}\right)$$

$$f_x(t_1) = 1\cdot I(0<t_1<1)$$

$$f_y(t_2) = \frac{4}{3}\cdot I\left(0<t_2<\frac{3}{4}\right)$$

x, y are independent

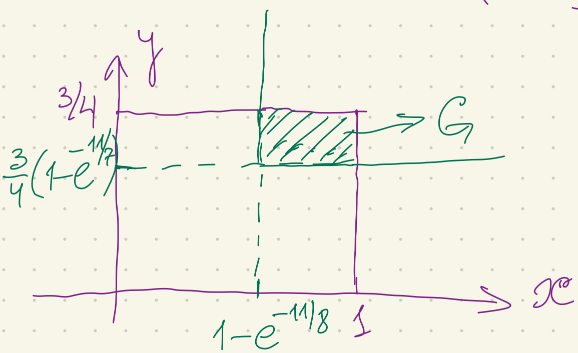
$$P(X \geq 110 \ \& \ Y \geq 110) = ?$$

$$X \geq 110 \qquad \ln(1-x) \leq -\frac{11}{8}, \quad 1-x \leq e^{-11/8}$$

$$x \geq 1-e^{-11/8}$$

$$Y \geq 110 \qquad \ln\left(1-\frac{4}{3}y\right) \leq -\frac{11}{7}, \quad 1-\frac{4}{3}y \leq e^{-11/7}$$

$$y \geq \frac{3}{4}\left(1-e^{-11/7}\right)$$



$f_{x,y}(t_1,t_2) = \frac{4}{3}$, (t_1,t_2) is inside the rectangle

$f_{x,y}(t_1,t_2) = 0$, otherwise

$$f_{x,y}(t_1,t_2) = \frac{4}{3} I\left(0 \leq t_1 \leq 1,\ 0 \leq t_2 \leq \frac{3}{4}\right)$$

For independent random variables

$$f_{x,y}(t_1,t_2) = f_x(t_1)\, f_y(t_2)$$

$$P = \iint_G f_{x,y}(t_1,t_2)\,dt_1\,dt_2 = \frac{4}{3}\iint_G dt_1\,dt_2 =$$

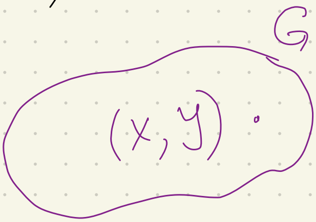
$$= \frac{4}{3}\cdot e^{-11/8}\cdot \frac{3}{4}e^{-11/7}$$

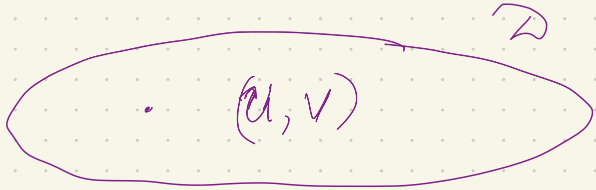
$$U = U(X, Y) \qquad u = u(x, y)$$

$$V = V(X, Y) \qquad v = v(x, y)$$

$$f_X(x) = \lambda e^{-\lambda x} I_{x>0}$$

$$f_{U,V}(u, v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$





$$\underline{(X, Y) \in G} \iff \underline{(U, V) \in D}$$

$$\iint\limits_D \underbrace{f_{U,V}(u, v)} \, du\, dv = \iint\limits_G f_{X,Y}(x, y)\, dx\, dy$$

$$x = x(u, v)$$

$$y = y(u, v)$$

$$\iint\limits_D \underline{f_{X,Y}(x(u,v), y(u,v))} \cdot \underline{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|} \cdot du\, dv$$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$f_{U,V}(u,v) \qquad f_{U+V}(t) = ?$$

$$\begin{cases} X = U + V \\ Y = V \end{cases} \qquad \begin{cases} U = X - Y \\ V = Y \end{cases} \qquad \begin{matrix} u = x - y \\ v = y \end{matrix}$$

$$\underline{f_{X,Y}(x,y)} = f_{U,V}(u(x,y), v(x,y)) \cdot \left| \frac{\partial(u,v)}{\partial(x,y)} \right| =$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right| = 1$$

$$= \underline{f_{U,V}(x-y, y)}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{+\infty} f_{U,V}(x-y, y)\,dy$$

$$\boxed{f_{U+V}(x) = \int_{-\infty}^{+\infty} f_{U,V}(x-y, y)\,dy} \text{ convolution formula}$$

$$\begin{cases} X = U + V \\ Y = U - V \end{cases} \qquad \begin{cases} U = \frac{X+Y}{2} \\ V = \frac{X-Y}{2} \end{cases} \qquad \frac{\partial(u,v)}{\partial(x,y)} = \left| \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \right| = \frac{1}{2}$$

$$f_{X,Y}(x,y) = f_{U,V}\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \cdot \frac{1}{2}$$

$$f_X(x) = \frac{1}{2} \int_{-\infty}^{+\infty} f_{U,V}\left(\frac{x+y}{2}, \frac{x-y}{2}\right) dy$$

$X_1, X_2, \dots, X_n \sim Exp(\lambda)$, independent

$X_1 + X_2 \sim ?$

$$f_{X_1+X_2}(t) = \int_{-\infty}^{+\infty} f_{X_1,X_2}(t-y, y)\,dy =$$

$$f_{X_1}(x_1) = \lambda e^{-\lambda x_1} I_{x_1>0}$$

$$f_{X_2}(x_2) = \lambda e^{-\lambda x_2} I_{x_2>0}$$

$$\Rightarrow f_{X_1,X_2}(x_1,x_2) = \lambda^2 e^{-\lambda(x_1+x_2)} I_{x_1>0} I_{x_2>0}$$

$$= \int_{-\infty}^{+\infty} \lambda^2 e^{-\lambda t} I_{t-y>0} I_{y>0}\,dy = \int_0^{+\infty} \lambda^2 e^{-\lambda t} I_{y<t}\,dy =$$

$$= I_{t>0} \cdot \int_0^t \lambda^2 e^{-\lambda t}\,dy = \lambda^2 t e^{-\lambda t} I_{t>0}$$

$T = X_1 + X_2$ $\quad f_T(t) = \lambda^2 t e^{-\lambda t} I_{t>0}$

X_3 $\quad f_{X_3}(x_3) = \lambda e^{-\lambda x_3} I_{x_3>0}$

$$f_{T+X_3}(y) = \int_{-\infty}^{+\infty} f_{T,X_3}(y-u, u)\,du = \int_0^{+\infty} I_{u<y}\,du$$

$$= \int_{-\infty}^{+\infty} \lambda^2 (y-u) e^{-\lambda(y-u)} I_{y-u>0} \cdot \lambda e^{-\lambda u} I_{u>0}\,du =$$

$$= I_{y>0} \cdot \int_0^y \lambda^3 (y-u) e^{-\lambda y}\,du = I_{y>0} \lambda^3 e^{-\lambda y} \cdot$$

$$\cdot \left(yu - \frac{u^2}{2}\right)\Big|_{u=0}^{y} = I_{y>0} \lambda^3 e^{-\lambda y} \cdot \frac{y^2}{2}$$

$$f_{X_1+X_2+\dots+X_n}(y) = \lambda^n e^{-\lambda y} \cdot \frac{y^{n-1}}{(n-1)!} I_{y>0}$$

$f_{X,Y}(x,y)$ $\qquad f_{X-Y}(t) = ?$

$(u = y - x,\ v = y \quad x = v - u)$

$u = x - y, \quad v = y$

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| =$$

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1 \qquad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$= f_{X,Y}(u+v, v) \Rightarrow f_{X-Y}(u) = \int_{-\infty}^{+\infty} \underbrace{f_{X,Y}(u+v,v)}_{f_X(u+v)\cdot f_Y(v)}\, dv$$

$X \sim Exp(\lambda), \quad Y \sim Exp(\mu)$, independent

$$f_{X-Y}(u) = \int_{-\infty}^{+\infty} \lambda e^{-\lambda(u+v)} I_{u+v>0} \cdot \mu e^{-\mu v} I_{v>0}\, dv =$$

$$= \lambda\mu \int_0^{+\infty} e^{-\lambda u - (\lambda+\mu)v} I_{v>-u}\, dv$$

$$1)\ u > 0 \Rightarrow \lambda\mu e^{-\lambda u} \int_0^{+\infty} e^{-(\lambda+\mu)v}\, dv = \lambda\mu e^{-\lambda u} \cdot \frac{e^{-(\lambda+\mu)v}}{\lambda+\mu}\Bigg|_{+\infty}^{0} =$$

$$= \lambda\mu e^{-\lambda u} \cdot \frac{1}{\lambda+\mu}$$

$$2)\ u \leq 0 \Rightarrow \lambda\mu e^{-\lambda u} \int_{-u}^{+\infty} e^{-(\lambda+\mu)v}\, dv = \lambda\mu e^{-\lambda u} \cdot \frac{e^{-(\lambda+\mu)v}}{\lambda+\mu}\Bigg|_{+\infty}^{-u} =$$

$$= \lambda\mu e^{-\lambda u} \cdot \frac{e^{(\lambda+\mu)u}}{\lambda+\mu} = \frac{\lambda\mu}{\lambda+\mu} e^{\mu u}$$

$$f_{X-Y}(u) = \begin{cases} \dfrac{\lambda\mu}{\lambda+\mu} e^{-\lambda u}, & u > 0 \\[2ex] \dfrac{\lambda\mu}{\lambda+\mu} e^{\mu u}, & u \leq 0 \end{cases}$$

$X_1, X_2 \sim Exp(\lambda)$ and independent

$y_1 = X_1 + X_2, \quad y_2 = \frac{X_1}{X_2}$ $\quad f_{X_1,X_2}(x_1,x_2) = \lambda^2 e^{-\lambda(x_1+x_2)} \cdot I(x_1>0, x_2>0)$

$$f_{y_1,y_2}(y_1,y_2) - ? \quad f_{X_1,X_2}\big(x_1(y_1,y_2),\ x_2(y_1,y_2)\big) \cdot \left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right| =$$

$$\begin{cases} x_1 = x_2 y_2 \\ y_1 = x_2 y_2 + x_2 \end{cases} \qquad x_2 = \frac{y_1}{y_2+1}, \quad x_1 = \frac{y_1 y_2}{y_2+1}$$

$$\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{vmatrix} \frac{y_2}{y_2+1} & \frac{1}{y_2+1} \\ \frac{y_1}{(y_2+1)^2} & -\frac{y_1}{(y_2+1)^2} \end{vmatrix} = -\frac{y_1 y_2}{(y_2+1)^3} - \frac{y_1}{(y_2+1)^3} = -\frac{y_1}{(y_2+1)^2}$$

$$\frac{\partial x_1}{\partial y_2} = \frac{y_1(y_2+1) - y_1 y_2}{(y_2+1)^2} = \frac{y_1}{(y_2+1)^2}$$

$$= \lambda^2 \cdot e^{-\lambda\left(\frac{y_1 y_2}{y_2+1} + \frac{y_1}{y_2+1}\right)} \cdot I_{\frac{y_1}{y_2+1}>0} \, I_{\frac{y_1 y_2}{y_2+1}>0} \cdot \frac{|y_1|}{(y_2+1)^2} =$$

$$= \lambda^2 \cdot e^{-\lambda y_1} I_{y_2>0} \cdot I_{y_1>0} \cdot \frac{y_1}{(y_2+1)^2} =$$

$$= \underbrace{\lambda \cdot e^{-\lambda y_1} \cdot y_1 \cdot I_{y_1>0}} \cdot \underbrace{\frac{\lambda}{(y_2+1)^2} \cdot I_{y_2>0}}$$

$$f_{y_1}(y_1) = C \cdot y_1 e^{-\lambda y_1} I_{y_1>0}$$

$$f_{y_2}(y_2) = \widetilde{C} \cdot I_{y_2>0} \frac{1}{(y_2+1)^2}$$

$$A = \left(Cov(X_i, X_j)\right) \qquad Cov(X_i, X_i) = Var\, X_i$$

$$\begin{pmatrix} Cov(X_1, X_1) & Cov(X_1, X_2), & \dots, & Cov(X_1, X_n) \\ - & - & - & - \\ Cov(X_n, X_1) & Cov(X_n, X_2), & \dots, & Cov(X_n, X_n) \end{pmatrix}$$

$$Cov(a_1X_1 + a_2X_2 + \dots + a_nX_n,\ b_1X_1 + b_2X_2 + \dots + b_nX_n) =$$

$$= \sum_{i=1}^{n} a_i\, Cov(X_i,\ b_1X_1 + b_2X_2 + \dots + b_nX_n) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j\, Cov(X_i, X_j) =$$

$$= (a_1\ a_2 \dots a_n) \begin{pmatrix} Cov(X_1, X_1) & \dots & Cov(X_1, X_n) \\ - & - & - \\ Cov(X_n, X_1) & \dots & Cov(X_n, X_n) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$Var(a_1X_1 + \dots + a_nX_n) =$$

$$= (a_1\ a_2 \dots a_n) \cdot \begin{pmatrix} - & - & - \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \geqslant 0$$

$$\forall (a_1, a_2, \dots, a_n)$$