Assignment 3, #19

$$\left.\begin{aligned} EX_1 &= 1 \\ EX_2 &= \frac{1}{5/6} = 1{,}2 \\ EX_3 &= \frac{1}{4/6} = 1{,}5 \\ &\ldots \\ EX_6 &= \frac{1}{1/6} = 6 \end{aligned}\right\} E(X_1+\ldots+X_6) = EX_1+EX_2+\ldots+EX_6 = 1+1{,}2+1{,}5+2+3+6 = 14{,}7$$

Assignment 4, #4(a)

$$f(x) = \begin{cases} ce^{-2x}, & x>0 \\ 0, & x<0 \end{cases} \qquad f(x) = ce^{-2x} I_{x>0}$$

$$\int_0^{+\infty} ce^{-2x}\,dx = 1 \qquad -\frac{c}{2}e^{-2x}\Big|_{x=0}^{+\infty} = \frac{c}{2} = 1, \quad c=2$$

$$f(x) = 2e^{-2x} I_{x>0}$$

$$E\xi = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^{+\infty} x\cdot 2e^{-2x}\,dx =$$

$$= \cancelto{0}{-xe^{-2x}\Big|_{x=0}^{+\infty}} + \int_0^{+\infty} e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_{x=0}^{+\infty} = \frac{1}{2}$$

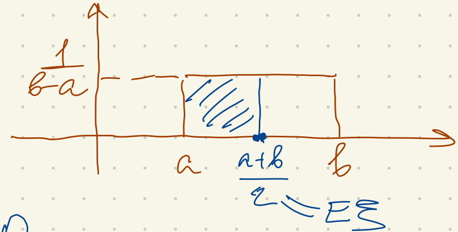
$$E\xi^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^{+\infty} x^2\, d(-e^{-2x}) = \cancelto{0}{-x^2e^{-2x}\Big|_{x=0}^{+\infty}} +$$

$$+ \int_0^{+\infty} e^{-2x}\, 2x\,dx = \frac{1}{2} \Rightarrow \mathrm{Var}\,\xi = E\xi^2 - (E\xi)^2 = \frac{1}{4}$$

⑥ $\xi \sim U[a;b]$

$$f_\xi(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & x \notin [a;b] \end{cases}$$

$$f_\xi(x) = \frac{1}{b-a} I_{a \le x \le b}$$



$\xi \sim U[0;10] \quad P(\xi = 5) = 0$

$$E\xi^2 = \int_a^b \frac{1}{b-a} \cdot x^2 dx = \frac{x^3}{3(b-a)} \Big|_{x=a}^{b} = \frac{b^3 - a^3}{3(b-a)} =$$

$$= \frac{a^2 + ab + b^2}{3}$$

$$Var\,\xi = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} = \frac{(a-b)^2}{12}$$

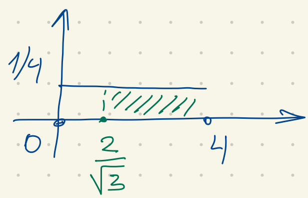
$\xi \sim U[0;4]$

$Var\,\xi = \frac{16}{12} = \frac{4}{3}$

(a) $P(\xi < E\xi) = \frac{1}{2}$

(b) $P(\xi > \sqrt{Var\,\xi}) = P\left(\xi > \frac{2}{\sqrt{3}}\right) =$

$$= \frac{4 - 2/\sqrt{3}}{4} = 1 - \frac{1}{2\sqrt{3}}$$



(12) $\eta \sim U[a;b]$ $\quad \xi = \dfrac{\eta - E\eta}{\sqrt{Var\,\eta}}$ $\quad f_\eta(t) = \dfrac{1}{b-a} I_{a \le t \le b}$

$F_\xi(t) = P(\xi < t) = P\left(\dfrac{\eta - E\eta}{\sqrt{Var\,\eta}} < t\right) =$

$= P(\eta < t\sqrt{Var\,\eta} + E\eta) = P\left(\eta < t \cdot \dfrac{b-a}{2\sqrt{3}} + \dfrac{a+b}{2}\right) =$

$= F_\eta\left(t \cdot \dfrac{b-a}{2\sqrt{3}} + \dfrac{a+b}{2}\right)$ $\quad F_\eta(x) = P(\eta < x)$

$f_\xi(t) = \dfrac{b-a}{2\sqrt{3}} f_\eta\left(t \cdot \dfrac{b-a}{2\sqrt{3}} + \dfrac{a+b}{2}\right) =$

$= \dfrac{b-a}{2\sqrt{3}} \cdot \dfrac{1}{b-a} \cdot I\left(a \le t \cdot \dfrac{b-a}{2\sqrt{3}} + \dfrac{a+b}{2} \le b\right) =$

$\dfrac{a-b}{2} \le t \cdot \dfrac{b-a}{2\sqrt{3}} \le \dfrac{b-a}{2}$

$-\sqrt{3} \le t \le \sqrt{3}$

$= \dfrac{1}{2\sqrt{3}} \cdot I(-\sqrt{3} \le t \le \sqrt{3})$ $\quad U[-\sqrt{3}; \sqrt{3}]$

#13 $\xi \sim U[-1; 5]$

$$\int_{\mathbb{R}} (x-1)(3-x) f_{\xi}(x)\,dx$$

$$E\left((\xi-1)(3-\xi)\right) =$$

$$= E\left(-\xi^2+4\xi-3\right) = -E\xi^2 + 4E\xi - 3 =$$

$$= -\left(Var\,\xi + (E\xi)^2\right) + 4E\xi - 3 =$$

$$E\xi = \frac{-1+5}{2} = 2, \quad Var\,\xi = \frac{(5-(-1))^2}{12} = 3$$

$$= -3-4+8-3 = -2$$

$\xi \sim Exp(\lambda) \qquad (\lambda > 0)$

$$f_\xi(x) = \lambda e^{-\lambda x} I_{x>0}$$

$$E\xi = \int_0^{+\infty} \lambda e^{-\lambda x} \cdot x\, dx = \cancelto{0}{-x e^{-\lambda x}\Big|_{x=0}^{+\infty}} + \int_0^{+\infty} e^{-\lambda x} dx =$$

$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_{x=0}^{+\infty} = \frac{1}{\lambda}$$

$$E\xi^2 = \int_0^{+\infty} \lambda e^{-\lambda x} \cdot x^2 dx = \cancelto{0}{-x^2 e^{-\lambda x}\Big|_{x=0}^{+\infty}} + \int_0^{+\infty} e^{-\lambda x} 2x\, dx =$$

$$= \frac{2}{\lambda^2} \qquad Var\,\xi = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

16 $P\left(|\xi - E\xi| < 3\sqrt{Var\,\xi}\right) = P\left(|\xi - \frac{1}{\lambda}| < \frac{3}{\lambda}\right) =$

$$= P\left(-\frac{3}{\lambda} < \xi - \frac{1}{\lambda} < \frac{3}{\lambda}\right) = P\left(-\frac{2}{\lambda} < \xi < \frac{4}{\lambda}\right) =$$

$$= \int_{-2/\lambda}^{4/\lambda} f_\xi(t)\, dt = \int_0^{4/\lambda} \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_{t=0}^{4/\lambda} = 1 - e^{-4}$$

#17 $\eta = e^{-\xi}$, $\xi \sim Exp(\lambda)$

$$E\eta = \int_{\mathbb{R}} e^{-x} \cdot f_{\xi}(x)dx = \int_0^{+\infty} e^{-x} \underbrace{\lambda e^{-\lambda x}}_{\lambda e^{-x(\lambda+1)}} dx =$$

$$= -\frac{\lambda}{\lambda+1} e^{-x(\lambda+1)} \Big|_{x=0}^{+\infty} = \frac{\lambda}{\lambda+1}$$

$\eta^2 = e^{-2\xi}$

$$E\eta^2 = \int_{\mathbb{R}} e^{-2x} f_{\xi}(x) dx = \int_0^{+\infty} \lambda e^{-(\lambda+2)x} dx =$$

$$= -\frac{\lambda}{\lambda+2} e^{-x(\lambda+2)} \Big|_{x=0}^{+\infty} = \frac{\lambda}{\lambda+2}$$

$$Var\,\eta = \frac{\lambda}{\lambda+2} - \frac{\lambda^2}{(\lambda^2+2\lambda+1)} = \frac{\lambda}{(\lambda+2)(\lambda+1)^2}$$

$$F_\eta(x) = P(\eta < x) = P(e^{-\xi} < x) =$$

$$P(e^{-\xi} < x) \cdot I_{x>0} = P(-\xi < \ln x) I_{x>0} =$$

$$= P(\xi > -\ln x) \cdot I_{x>0} = (1 - P(\xi \le -\ln x)) \cdot I_{x>0} =$$

$$= (1 - F_{\xi}(-\ln x)) \cdot I_{x>0}$$

$$f_\eta(x) = -f_{\xi}(-\ln x) \cdot \left(-\frac{1}{x}\right) \cdot I_{x>0} = \frac{1}{x} f_{\xi}(-\ln x) \cdot I_{x>0}$$

$$= \frac{1}{x} \cdot \lambda e^{-\lambda(-\ln x)} \cdot I_{-\ln x > 0} \, I_{x>0} =$$

$$= \frac{1}{x} \lambda x^{\lambda} I_{0<x<1} = \lambda x^{\lambda-1} I_{0<x<1}.$$