**Probability & Statistics. Assignment 8**

1. A book of 600 hundred pages has on average one typo per page. Evaluate the following probabilities using Poisson approximation for binomial distribution:

   (a) page 13 does not have a single typo;

   (b) there are exactly two typos on page 13;

   (c) there are no more than three typos on page 13.

   **Answer**: $\frac{1}{e}$; $\frac{1}{2e}$; $\frac{8}{3e}$.

2. When baking raisin scones, it sometimes (with probability $0.003$) happens that a scone does not contain any raisins at all. Estimate the probability that in a batch of $1\,000$ scones

   (a) there are no scones without raisins;

   (b) there are exactly three scones without raisins;

   (c) there are at least three scones without raisins.

   **Answer:** (a) $\frac{1}{e^3}$; (b) $\frac{9}{2e^3}$; (c) $1 - \frac{17}{2e^3}$.

3. How many raisins do the scones described in the previous problem have to contain on average so that the probability that a scone does not have a single raisin does not exceed $0.01$?

   **Answer:** $\lambda \geq ln(100)$.

4. Each of the electors votes for candidate $A$ with probability $0.7$, and for candidate $B$ with probability $0.3$, and they vote independently of each other. There are $5\,000$ electors in town $N$. Estimate the probability that candidate $A$ (a) gets exactly $1\,900$ votes more than $B$; (b) gets at least $1\,900$ votes more than $B$?

   **Answer:** $\frac{1}{\sqrt{2\pi}\sqrt{1050}} e^{-\frac{25}{21}}$; $\Phi\left(\frac{10}{\sqrt{42}}\right)$.

5. There are $1\,000$ places in a theatre. The theatre has two entrances, each of them having its own cloakroom. How many places should each cloakroom have in order that each of the visitors can give his or her coat to the cloakroom near the entrance where they came in. Please use 99% confidence interval. Consider the following situations:

   (a) the visitors come in pairs, and each of the pairs chooses one of the entrances at random;

   (b) the visitors come alone, and each of the visitors chooses a random entrance.

   **Answer:** (a) $\approx 557.46$; (b) $\approx 540.63$.

6. Find the minimum quantity of times one has to flip a fair coin in order that the percentage of tails differs from the probability of getting tails no more than by $0.01$ with $95\%$ probability.

   **Answer:** $N \geq 9604$.

   **Hint:** use one of these facts:

   - $P(|\mu_n - np| < \lambda\sqrt{npq}) \approx 2\Phi_0(\lambda)$.
   - $P(|\frac{\mu}{n} - p| \leq \Delta) \approx 2\Phi_0\left(\Delta\sqrt{\frac{n}{pq}}\right)$.

7. Find the minimum number of times one has to roll a fair die in order that the percentage of fours differs from the probability of getting a four no more than by $0.01$ with probability (a) $0.95$; (b) $0.99 >$

   **Answer:** $N \approx 5336$; $N > 9173$.

8. On a dry road I cycle at 20 mph; when the road is wet at 10 mph. The distance from home to the lecture building is three miles, and the 9:00 am course consists of 24 lectures. The probability that on a given morning the road is dry is $0.5$, but there is no reason to believe that dry and wet mornings follow independently. Find the expected time to cycle to a single lecture and the expected time for the whole course.

   **Answer:**$13.5$ minutes; $5$ hours $24$ minutes.

9. Let $X_1, \ldots X_n$ be an **IID**[1] sample from $N\left(\mu_1; \sigma_1^2\right)$, and $Y_1, \ldots Y_m$ be an IID sample from $N\left(\mu_2; \sigma_2^2\right)$ and assume that the samples are independent of each other. What is the joint distribution of $U = \overline{X} - \overline{Y}$[2] and $V = \frac{1}{\sigma_1^2} \sum\limits_{i=1}^{n} X_i + \frac{1}{\sigma_2^2} \sum\limits_{y=1}^{m} Y_j$?

   **Answer:**

   $f_U(t) = (\frac{mn}{2\pi(m\sigma_1^2 + n\sigma_2^2)})^{\frac{1}{2}} exp(-\frac{1}{2} \frac{mn}{m\sigma_1^2 + n\sigma_2^2} (t - (\mu_1 - \mu_2)^2));$

   $f_V(t) = \left(\frac{\sigma_1^2 \sigma_2^2}{2\pi(m\sigma_1^2 + n\sigma_2^2)}\right)^{\frac{1}{2}} exp\left(-\frac{1}{2} \frac{\sigma_1^2 \sigma_2^2}{m\sigma_1^2 + n\sigma_2^2} (t - (\frac{n}{\sigma_1^2}\mu_1 + \frac{m}{\sigma_2^2}\mu_2)^2)\right).$

10. Let $X$ and $Y$ be **IID** random variables with a standard normal distributions. Find the joint probability density of $U = X + Y$ and $V = X - Y$. Show that $U$ and $V$ are independent and find marginal distributions for $U$ and $V$. Let $Z = |Y|(-1)^{I_{X<0}}$. Show that $Z$ has a standard normal distribution. Is joint distribution of $X$ and $Z$ a bivariate normal distribution?

    **Answer:** $f_{UV}(u, v) = \frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}} \cdot \frac{1}{2\sqrt{\pi}} e^{-\frac{v^2}{4}}.$

11. Let $X$ and $Y$ be independent random variables with a standard normal distribution. Find probability densities of $X + Y$, $X^2$, $X^2 + Y^2$.

    **Answer:**

    $f_{X+Y}(t) = \frac{1}{\sqrt{4\pi}} e^{\frac{t^2}{4}} \sim N(0, 2);$

    $f_{X^2}(t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{t}{2}};$

    $f_{X^2+Y^2}(t) = \frac{1}{2} e^{-\frac{t}{2}} \sim Exp\left(\frac{1}{2}\right).$

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1. independent & identically distributed ↩

2. $\overline{X}$ is called *sample mean* and $\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$. ↩