October 1, 2020

$f_\xi(x) = I_{0<x<1}$

$\xi \sim U(0;1)$; $\eta = -\ln\xi$ $\quad \eta \sim ?$

$$F_\eta(x) = P(\eta < x) = P(-\ln\xi < x) = P(\ln\xi > -x) = P(\xi > e^{-x}) = 1 - P(\xi < e^{-x}) = 1 - F_\xi(e^{-x})$$

$$f_\eta(x) = f_\xi(e^{-x}) \cdot e^{-x} = I_{0<e^{-x}<1} \cdot e^{-x} = e^{-x} \cdot I_{x>0}$$

$\eta \sim Exp(1)$

$X, Y, Z \sim U(0;1)$, independent

Prove that $(XY)^Z \sim U(0;1)$

$$-\ln(XY)^Z \sim Exp(1)$$

$$\underbrace{Z}_{\sim U(0;1)}\underbrace{(\underbrace{-\ln X}_{\substack{U \\ Exp(1)}} \underbrace{-\ln Y}_{\substack{V \\ Exp(1)}})}_{T} \sim Exp(1)$$

$$f_T(t) = \int_{-\infty}^{+\infty} f_{U,V}(t-u, u)\,du = \int_{-\infty}^{+\infty} e^{-(t-u)} I_{t-u>0} \cdot e^{-u} \cdot I_{u>0}\,du = I_{t>0} \int_{-\infty}^{+\infty} e^{-t}\,du = te^{-t} I_{t>0}.$$

$$f_{x,y}(x,y) = \qquad xy = u \quad y = v \qquad \begin{cases} x = u/v \\ y = v \end{cases}$$

$$f_{u,v}(u,v) = f_{x,y}\left(\frac{u}{v}, v\right) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{|v|} f_{x,y}\left(\frac{u}{v}, v\right)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/v & 0 \\ -u/v^2 & 1 \end{vmatrix} = \frac{1}{v}$$

z

z

$$f_{xy}(u) = \int_{-\infty}^{+\infty} \frac{1}{|v|} f_{x,y}\left(\frac{u}{v}, v\right) dv$$

$$f_{Z,T}(z,t) = I_{0<z<1} \cdot t e^{-t} I_{t>0}$$

$$f_{ZT}(u) = \int_{-\infty}^{+\infty} \frac{1}{|v|} \cdot I_{0<\frac{u}{v}<1} \cdot v e^{-v} I_{v>0}\, dv =$$

$$= \int_{0}^{+\infty} I_{0<\frac{u}{v}<1} \cdot e^{-v} dv = \int_{0}^{+\infty} I_{0<u} I_{u<v} e^{-v} dv =$$

$$= I_{u>0} \int_{u}^{+\infty} e^{-v} dv = I_{u>0} \cdot \left(-e^{-v}\right)\Big|_{v=u}^{+\infty} = I_{u>0} e^{-u}$$

$Exp(1)$

④ $f_{\xi,\eta}(x,y) = \dfrac{a}{1+x^2+y^2+x^2y^2}$, $(x,y) \in \mathbb{R}$

$$\iint\limits_{\mathbb{R}^2} \frac{a}{1+x^2+y^2+x^2y^2}\,dx\,dy = 1$$

$$a\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}\int_{-\infty}^{+\infty}\frac{dy}{1+y^2} = a\cdot\left(\arctan x\Big|_{x=-\infty}^{+\infty}\right)^2 = a\pi^2 = 1$$

$a = \dfrac{1}{\pi^2}$

$$\underbrace{\frac{1}{\pi(1+x^2)}}_{\xi}\cdot\underbrace{\frac{1}{\pi(1+y^2)}}_{\eta}$$

$E\xi = ?$

$f_{\xi,\eta}(x,y) = f_\xi(x)\cdot f_\eta(y) \Rightarrow$ independent.

$$\rho_{\xi,\eta} = \frac{E(\xi\eta) - E\xi\cdot E\eta}{\sqrt{Var\xi\cdot Var\eta}}$$

$$\int_{y=-\infty}^{+\infty} \underbrace{f_{X,Y}(x,y)}_{\downarrow}\,dy = f_X(x)$$

$\underbrace{g(x)\cdot h(y)}$

$\dfrac{g(x)}{c}\cdot h(y)c$

$$\int_{-\infty}^{+\infty} g(x)\,dx = c > 0$$

$$\int_{-\infty}^{+\infty} h(y)\,dy = \frac{1}{c}$$

$$f_{X,Y} = \frac{|x|}{1+x^4}\cdot\frac{1}{1+y^2}\cdot c$$

⑥ $\begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}$

$$Var(-\xi-2\eta) = \begin{pmatrix} -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -9 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} = 14$$

$$Var(3\xi-\eta+2) = Var(3\xi-\eta) = \dots$$

(3) $\xi \sim N(0;1)$ (a) $\rho_{2\xi,\,\xi^3}$

$$Cov(2\xi,\xi^3) = E(2\xi\cdot\xi^3) - E(2\xi)\cdot E(\xi^3) =$$
$$= 2E(\xi^4) = 2\cdot 3 = 6$$

$$Var(2\xi) = 4\,Var\,\xi = 4$$

$$Var\,\xi^3 = E(\xi^6) - (E\xi^3)^2 = 15$$

$$te^{-t^2/2}dt = $$
$$= -de^{-t^2/2}$$

$$\int\limits_{-\infty}^{+\infty} t^4 \cdot e^{-t^2/2}dt = \underbrace{\cancel{-t^3 \cdot e^{-t^2/2}\Big|_{t=-\infty}^{+\infty}}}_{0} + \int\limits_{-\infty}^{+\infty} e^{-t^2/2}\,3t^2dt =$$
$$= \underbrace{\cancel{-3te^{-t^2/2}\Big|_{-\infty}^{+\infty}}}_{0} + \int\limits_{-\infty}^{+\infty} e^{-t^2/2}\,3dt = 3\sqrt{2\pi}$$

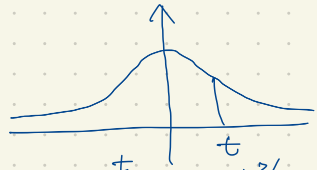
$$E\xi^4 = 3$$

$$E\xi^6 = \int\limits_{-\infty}^{+\infty} t^6 e^{-t^2/2}\cdot\frac{1}{\sqrt{2\pi}}dt = -\int\limits_{-\infty}^{+\infty} t^5\frac{1}{\sqrt{2\pi}}d\left(e^{-t^2/2}\right) =$$
$$= -\frac{t^5e^{-t^2/2}}{\sqrt{2\pi}}\Bigg|_{t=-\infty}^{+\infty} + \int\limits_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt^5 = 5E\xi^4 = 15$$

$$\rho_{2\xi,\,\xi^3} = \frac{6}{\sqrt{4\cdot 15}} = \sqrt{\frac{3}{5}}$$

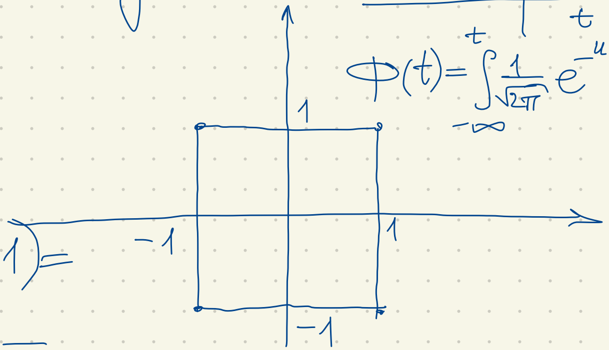
(11) $\xi \sim N(0;1)$, $\eta \sim N(0;1)$

$(\xi, \eta) \in$ (a) $\{|x+y| + |x-y| \le 2\}$



$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$\begin{cases} -1 \le x \le 1 \\ -1 \le y \le 1 \end{cases}$$

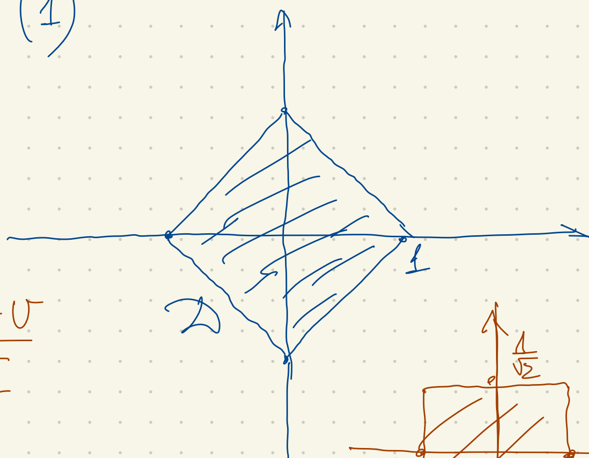


$$P(-1 \le \xi \le 1) \cdot P(-1 \le \eta \le 1) =$$

$$= \left(\Phi(1) - \Phi(-1)\right)^2 =$$

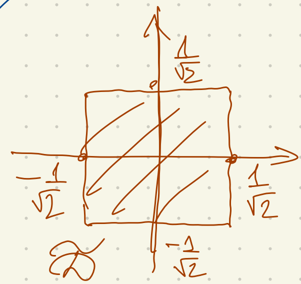
$$= \left(2\Phi_0(1)\right)^2 = 4\Phi_0^2(1)$$

(b) $|x| + |y| \le 1$



$$\iint_{D} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx dy =$$

$x = \frac{u-v}{\sqrt{2}}$, $y = \frac{u+v}{\sqrt{2}}$



$$= \frac{1}{2\pi} \iint_{\tilde{D}} e^{-\frac{u^2+v^2}{2}} du dv =$$

$$= \iint_{\tilde{D}} \left(\frac{1}{\sqrt{2\pi}} e^{-u^2/2}\right) \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-v^2/2}\right) du dv = 4\Phi_0^2\left(\frac{1}{\sqrt{2}}\right)$$

$\xi_1 \sim N(0;1)$, $\xi_2 \sim N(0;1)$ independent

$\varsigma = \xi_2/\xi_1$, $\eta = \xi_1$ $\quad z = x_2/x_1$, $y = x_1$

$$f_{\varsigma,\eta}(z,y) = f_{\xi_1,\xi_2}(x_1(z,y), x_2(z,y)) \cdot \left|\frac{\partial(x_1,x_2)}{\partial(z,y)}\right| =$$

$x_2 = yz$, $x_1 = y$

$$\frac{\partial(x_1,x_2)}{\partial(y,z)} = \begin{vmatrix} 1 & z \\ 0 & y \end{vmatrix} = y$$

$$f_{\xi_1,\xi_2}(x_1,x_2) = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}$$

$$= f_{\xi_1,\xi_2}(y, zy) \cdot |y|$$

$$f_\varsigma(z) = \int_{-\infty}^{+\infty} |y| f_{\xi_1,\xi_2}(y, zy)\, dy =$$

$$= \int_{-\infty}^{+\infty} |y| \cdot \frac{1}{2\pi} e^{-\frac{y^2+z^2y^2}{2}} dy = \frac{1}{2\pi} \cdot 2 \underbrace{\int_0^{+\infty} y e^{-\frac{y^2+z^2y^2}{2}} dy}_{} =$$

$$= \frac{1}{2\pi} \int_0^{+\infty} e^{-y^2 \cdot \frac{1+z^2}{2}} dy^2 = \frac{1}{2\pi} \cdot \exp\left(-y^2 \cdot \frac{1+z^2}{2}\right) \cdot \frac{2}{1+z^2}\Bigg|_{+\infty}^{0} =$$

$$= \frac{1}{\pi(1+z^2)}.$$