November 12, 2020

$$y = \frac{1}{200} \sum_{k=1}^{200} (X_k - \overline{X})^2$$

$$EX_k = \lambda \Rightarrow \quad \xi_k = X_k - \lambda, \quad E\xi_k = 0$$

$$\overline{X} = \frac{1}{n} \sum_{k=1}^{n} X_k = \frac{1}{n} \sum_{k=1}^{n} (\xi_k + \lambda) = \overline{\xi} + \lambda$$

$$\frac{1}{n} \sum_{k=1}^{n} (\xi_k - \overline{\xi})^2 = \frac{1}{n} \sum_{k=1}^{n} \left(\xi_k^2 - 2\overline{\xi}\,\xi_k + \overline{\xi}^2\right) =$$

$$= \frac{1}{n} \sum_{k=1}^{n} \xi_k^2 - \frac{2\overline{\xi}}{n} \underbrace{\sum_{k=1}^{n} \xi_k}_{n\overline{\xi}} + \overline{\xi}^2 = \frac{1}{n} \sum_{k=1}^{n} \xi_k^2 - \overline{\xi}^2$$

$$E\ldots = E\xi_1^2 - E\left(\overline{\xi}^2\right) = Var\,\xi_1 + \cancel{(E\xi_1)^2}^{\,0} - \frac{1}{n} Var\,\xi_1 =$$

$$E\left(\frac{1}{n^2}\left(\sum_{k=1}^{n} \xi_k\right)^2\right) = \frac{1}{n^2}\left(\sum_{k=1}^{n} E\xi_k^2 + \sum_{\substack{1 \le j, k \le n \\ j \ne k}} E(\xi_j \xi_k)\right)$$

$$= \frac{1}{n} Var\,\xi_1$$

$$= \frac{n-1}{n} Var\,\xi_1 = \frac{n-1}{n} Var\,X_1$$

$$bias = \left(\frac{n-1}{n} Var\,X_1 - Var\,X_1\right) = -\frac{1}{n} Var\,X_1$$

$$y = \sum_{k=1}^{1054} \xi_k^2 \qquad \xi_k \sim N(0;9)$$

$$\left(\frac{\xi_k}{3}\right) \sim N(0;1)$$

$$\frac{y}{9} = \sum_{k=1}^{1054} \left(\frac{\xi_k}{3}\right)^2 \sim \chi^2_{1054}$$

$$Var\left(\frac{y}{9}\right) = 2 \cdot 1054$$

$$\frac{1}{81} Var\, y$$

$X_1, X_2, \dots, X_n$ — a simple sample

$T(X_1, X_2, \dots, X_n)$ — a sufficient statistics for parameter θ if

$\forall D \subset \mathbb{R}^n \to P(\vec{X} \in D \mid T(\vec{X}))$ does not depend on θ.

$X_1, X_2, \dots, X_n$ – a simple sample out of Poisson distribution with parameter θ.
Prove that $\overline{X}$ is sufficient statistics for θ.

$$P(\vec{X}=\vec{x} \mid \overline{X}=t) = \frac{P(\vec{X}=\vec{x}, \overline{X}=t)}{P(\overline{X}=t)} =$$

$$\begin{cases} X_1 = x_1 \\ X_2 = x_2 \\ \dots \\ X_n = x_n \\ \dfrac{X_1+\dots+X_n}{n} = t \end{cases} \Rightarrow x_1 + x_2 + \dots + x_n = nt$$

$\xi \sim \text{Poisson}(\lambda)$
$$P(\xi = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$=$ 1) $x_1 + x_2 + \dots + x_n \neq nt \Rightarrow 0$

2) $x_1 + x_2 + \dots + x_n = nt \Rightarrow \dfrac{P(\vec{X}=\vec{x})}{P(\overline{X}=t)} =$

$$= \frac{\prod_{j=1}^{n} P(X_j = x_j)}{P(\underbrace{X_1+\dots+X_n}_{\sim \text{Poisson}(n\theta)} = nt)} = \frac{\prod_{j=1}^{n} \cancel{e^{-\theta}} \cdot \frac{\theta^{x_j}}{x_j!}}{\cancel{e^{-n\theta}} \cdot \frac{(n\theta)^{nt}}{(nt)!}} =$$

$$= \frac{\cancel{\theta^{x_1+x_2+\dots+x_n}} (nt)!}{\prod_{j=1}^{n} x_j! \cdot n^{nt} \cdot \cancel{\theta^{nt}}}$$

$$P(\vec{X}=\vec{x} \mid \overline{X}=t) = \frac{(nt)!}{n^{nt} \prod_{j=1}^{n} x_j!} \cdot I\left(\sum_{j=1}^{n} x_j = nt\right)$$

Factorisation Criterion

$T(\vec{X})$ is sufficient for parameter $\theta \iff$

$$f_{\vec{X}}(\vec{x},\theta) = g(T(\vec{x}),\theta)\cdot h(\vec{x})$$

$\hookrightarrow$ probability density for continuous distributions / probability mass function for discrete distributions

$$\iff P(\vec{X}=\vec{x} \mid T(\vec{x})=t) \overset{\circledcirc}{=} \frac{P(\vec{X}=\vec{x}, T(\vec{x})=t)}{P(T(\vec{x})=t)} =$$

$$= \begin{cases} T(\vec{x})\neq t \Rightarrow 0 \\ T(\vec{x})=t \Rightarrow \dfrac{P(\vec{X}=\vec{x})}{P(T(\vec{x})=t)} = \end{cases}$$

$$= \frac{g(T(\vec{x}),\theta)\cdot h(\vec{x})}{\sum\limits_{\vec{x}:T(\vec{x})=t} P(\vec{X}=\vec{x})} = \frac{g(T(\vec{x}),\theta)\, h(\vec{x})}{\sum\limits_{\vec{x}:T(\vec{x})=t} g(T(\vec{x}),\theta) h(\vec{x})} =$$

$$= \frac{g(T(\vec{x}),\theta)\, h(\vec{x})}{g(t,\theta)\sum\limits_{\vec{x}:T(\vec{x})=t} h(\vec{x})} = \frac{h(\vec{x})}{\sum\limits_{\vec{x}:T(\vec{x})=t} h(\vec{x})}$$

$$= \frac{h(\vec{x})}{\sum\limits_{\vec{x}:T(\vec{x})=t} h(\vec{x})}\cdot I(T(\vec{x})=t)$$

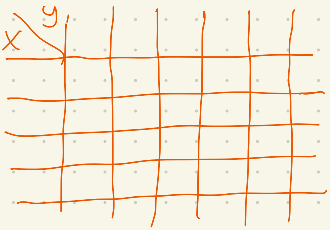
$\Rightarrow P(\vec{X}=\vec{x} \mid T=t)$ does not depend on θ

$$f_{\vec{X}}(\vec{x}) = P(\vec{X}=\vec{x}) = \underbrace{P(\vec{X}=\vec{x} \mid T=t)}_{\text{does not depend on } \theta}\cdot P(T=t)$$

$X_1, X_2, \dots, X_n \sim \text{Poisson}(\theta)$, independent

$$\Rightarrow P(\vec{X} = \vec{x}) = \prod_{j=1}^{n} P(X_j = x_j) = \prod_{j=1}^{n} e^{-\theta}\cdot\frac{\theta^{x_j}}{x_j!} =$$

$$= e^{-n\theta}\cdot\theta^{\sum_{j=1}^{n} x_j}\cdot\frac{1}{\prod_{j=1}^{n} x_j!} = \underbrace{e^{-n\theta}\cdot\theta^{n\bar{x}}}_{g(\bar{x},\theta)}\cdot\underbrace{\frac{1}{\prod_{j=1}^{n} x_j!}}_{h(\vec{x})}$$



$$E(Y|X) = \sum_j y_j P(Y=y_j|X)$$

Law of total variance

$$\operatorname{Var} Y = EY^2 - (EY)^2$$

$$\operatorname{Var}(Y|X) = E(Y^2|X) - (E(Y|X))^2$$

$$E(\operatorname{Var}(Y|X)) = E(E(Y^2|X)) - E\left((E(Y|X))^2\right) =$$

$$= EY^2 - E\left((E(Y|X))^2\right) = \underbrace{EY^2 - (EY)^2} +$$

$$+ (EY)^2 - E\left((E(Y|X))^2\right) = \operatorname{Var} Y + \left(E(E(Y|X))\right)^2 -$$

$$- E\left((\underbrace{E(Y|X)}_{\xi})^2\right) =$$

$$E(Y|X) = \xi$$

$$(E\xi)^2 - E\xi^2 = -\operatorname{Var}\xi$$

$$= \operatorname{Var} Y - \operatorname{Var}(E(Y|X))$$

$$\operatorname{Var} Y = E(\operatorname{Var}(Y|X)) + \operatorname{Var}(E(Y|X))$$

$$E(Y|X)$$

Rao–Blackwell theorem

T is sufficient statistics for θ

θ^* is an unbiased estimator of θ

$\hat{\theta}^* = E(\theta^* \mid T)$. Then

$E\hat{\theta}^* = E\theta^*$, i.e. $\hat{\theta}^*$ is also unbiased,

$Var\,\hat{\theta}^* \leq Var\,\theta^*$, and the equality is reached

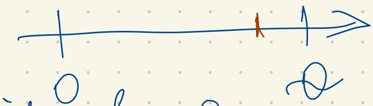
if and only if θ^* is a function of T.

$$Var\,\theta^* = E(Var(\theta^* \mid T)) + Var(E(\theta^* \mid T)) =$$

$$= \underbrace{E(Var(\theta^* \mid T))}_{\geq 0} + Var\,\hat{\theta}^*$$

$\theta^*(T)$

$$X_1, X_2, \dots, X_n \sim U[0;\theta].$$



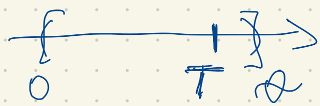
$$2\overline{X} = \theta^*$$

$$T(\vec{X}) = \max_{1\le j\le n} X_j$$ – sufficient statistics for θ

$$f_{\vec X}(\vec x) = \prod_{j=1}^{n} f_{X_j}(x_j) = \prod_{j=1}^{n} \frac{1}{\theta}\cdot I(0\le x_j\le\theta) =$$

$$= \frac{1}{\theta^n}\cdot I(0\le x_1\le\theta, \dots, 0\le x_n\le\theta) =$$

$$= \underbrace{\frac{1}{\theta^n}\cdot I\left(\max_{1\le j\le n} x_j \le\theta\right)}_{g(T(\vec x),\theta)} \cdot \underbrace{1}_{h(\vec x)}$$



$$\hat\theta^* = E(\theta^* | T) = E(2\overline X \mid \max X_j = T) =$$

$$= \frac{2}{n} E(X_1+\dots+X_n \mid \max X_j = T) =$$

$$= \frac{2}{n}\left((n-1)\cdot\frac{T}{2} + T\right) = \frac{n+1}{n} T$$