October 29, 2020

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \cdot I(x \geq 0)$$

$\alpha > 0, \ \lambda > 0$

$Gam(\alpha, \lambda)$

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \, dt$$

$$\int_0^{+\infty} x^{\alpha-1} e^{-\lambda x} dx = \qquad \lambda x = t \Rightarrow \lambda dx = dt$$

$$= \int_0^{+\infty} \frac{t^{\alpha-1}}{\lambda^{\alpha-1}} e^{-t} \frac{dt}{\lambda} = \frac{1}{\lambda^\alpha} \Gamma(\alpha)$$

$$EX = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^\alpha}{\lambda^\alpha} e^{-t} \frac{dt}{\lambda} =$$

$$= \frac{1}{\lambda \Gamma(\alpha)} \int_0^{+\infty} t^\alpha e^{-t} dt = \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} = \frac{\alpha \Gamma(\alpha)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda}$$

$$EX^2 = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^{\alpha+1}}{\lambda^{\alpha+1}} e^{-t} \frac{dt}{\lambda} =$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha) \cdot \lambda^2} = \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda^2} = \frac{(\alpha+1)\alpha}{\lambda^2}$$

$$Var\, X = \frac{\alpha^2 + \alpha}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \frac{\alpha}{\lambda^2}.$$

$$\varphi_\xi(t) = \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} e^{itx}\,dx =$$

$$-(it-\lambda)x = y \qquad (\lambda - it)\,dx = dy$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \frac{y^{\alpha-1}}{(\lambda-it)^{\alpha-1}} e^{-y} \frac{dy}{\lambda - it} = \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-it)^\alpha} \cdot$$

$$\cdot \int_0^{+\infty} y^{\alpha-1} e^{-y}\,dy = \frac{\lambda^\alpha}{(\lambda-it)^\alpha}$$

$\xi_1 \sim Gam(\alpha_1, \lambda)$, $\xi_2 \sim Gam(\alpha_2, \lambda)$
are independent

$\xi_1 + \xi_2 \sim ?$

$$\varphi_{\xi_1+\xi_2}(t) = \varphi_{\xi_1}(t) \cdot \varphi_{\xi_2}(t) = \frac{\lambda^{\alpha_1}}{(\lambda-it)^{\alpha_1}} \cdot \frac{\lambda^{\alpha_2}}{(\lambda-it)^{\alpha_2}} =$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{(\lambda-it)^{\alpha_1+\alpha_2}} \sim Gam(\alpha_1+\alpha_2, \lambda)$$

χ^2_n χ^2 distribution with n degrees of freedom

$X_1, X_2, \dots, X_n$ independent $N(0;1)$

$$\chi^2_n = X_1^2 + X_2^2 + \dots + X_n^2.$$

$$\chi^2_1 = X_1^2 \qquad X_1 \sim N(0;1)$$

$$F_{\chi^2_1}(t) = P(X_1^2 < t) = I(t>0)\cdot P(-\sqrt{t} < X_1 < \sqrt{t})$$

$$= I(t>0)\cdot 2P(0 < X_1 < \sqrt{t}) = 2I(t>0)\Phi_0(\sqrt{t})$$

$$f_{\chi^2_1}(t) = \cancel{2}I(t>0)\cdot f_{X_1}(\sqrt{t})\cdot \frac{1}{\cancel{2}\sqrt{t}} = I(t>0)\cdot$$

$$\cdot \frac{1}{\sqrt{2\pi}} e^{-t/2}\cdot \frac{1}{\sqrt{t}} \sim Gam\left(\tfrac{1}{2};\tfrac{1}{2}\right)$$

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\cdot I(x>0)$$

$$\chi^2_1 \sim Gam\left(\tfrac{1}{2};\tfrac{1}{2}\right)$$

$$\chi^2_n \sim Gam\left(\tfrac{n}{2};\tfrac{1}{2}\right)$$

$$f_{\chi^2_n}(x) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-x/2} I(x>0)$$

$$X \sim \chi^2_n \Rightarrow EX = \frac{n/2}{1/2} = n$$

$$Var X = \frac{n/2}{(1/2)^2} = 2n$$

Student's distribution (t-distribution)

$X_1, X_2, \dots, X_n, X_{n+1}$ are independent $N(0;1)$ r.v.

$$t_n = \frac{X_{n+1}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}}$$ Student's distribution with n degrees of freedom

$$t_n = \frac{X}{\sqrt{Y/n}} \qquad \left.\begin{array}{l} X \sim N(0;1) \\ Y \sim \chi_n^2 \end{array}\right\} \text{independent}$$

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{2^{-n/2}}{\Gamma(n/2)} y^{n/2-1} e^{-y/2} I(y>0)$$

$$t = \frac{x}{\sqrt{y/n}}, \quad u = y \Rightarrow y = u,\ x = t\sqrt{\frac{u}{n}}$$

$$\frac{\partial(x,y)}{\partial(t,u)} = \begin{vmatrix} \sqrt{\frac{u}{n}} & 0 \\ \frac{t}{2\sqrt{un}} & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

$$f_{T,U}(t,u) = f_{X,Y}(x(t,u), y(t,u)) \cdot \left|\frac{\partial(x,y)}{\partial(t,u)}\right| =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2 u}{2n}} \frac{2^{-n/2}}{\Gamma(n/2)} u^{n/2-1} e^{-u/2} I(u>0) \cdot \sqrt{\frac{u}{n}}$$

$$f_T(t) = \int_{-\infty}^{+\infty} f_{T,U}(t,u)\,du = \frac{2^{-n/2}}{\sqrt{2\pi n}\,\Gamma(n/2)} \int_0^{+\infty} u^{n/2-1/2} \cdot$$

$$\cdot\, e^{-\frac{t^2}{2n}u - \frac{u}{2}} du = \frac{2^{-n/2}}{\sqrt{2\pi n}\,\Gamma(n/2)} \cdot$$

$$\left(\frac{t^2}{2n}+\frac{1}{2}\right)u = v \Rightarrow du = \frac{dv}{\frac{t^2}{2n}+\frac{1}{2}}$$

$$= \frac{2^{-n/2}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\sqrt{2\pi n}\,\Gamma(n/2)\left(\frac{t^2}{2n}+\frac{1}{2}\right)^{n/2+1/2}} \xrightarrow{n\to+\infty} \underbrace{\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}$$

$$\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{t^2}{n}+1\right)^{n/2+1/2}}$$

$$\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}} = e^{\frac{n+1}{2}\ln\left(1+\frac{t^2}{n}\right)} =$$

$$= e^{\frac{n+1}{2}\cdot\frac{t^2}{n}+o(\ldots)}$$

$$f_T(t) = \int_{-\infty}^{+\infty} f_{t,u}(t,u)\,du = \frac{2^{-n/2}}{\sqrt{2\pi n}\,\Gamma(n/2)}\int_0^{+\infty} u^{n/2-1/2}\cdot$$

$$\cdot e^{-\frac{t^2}{2n}u-\frac{u}{2}}\,du = \frac{2^{-n/2}}{\sqrt{2\pi n}\,\Gamma(n/2)}\cdot\int_0^{+\infty}\frac{v^{n/2-1/2}}{\left(\frac{t^2}{2n}+\frac{1}{2}\right)^{n/2+1/2}}\,e^{-v}\,dv$$

$\vec{X}(X_1, X_2, X_3, \ldots, X_n)$ — i.i.d. r.v.

a simple sample

$\vec{x}(x_1, x_2, \ldots, x_n)$ — a realisation of a sample

$Bin(100, p)$

$f(\vec{X})$ is called statistics

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

θ is an unknown parameter.

$f(\vec{X})$ is an estimator of parameter θ.

$E f(\vec{X}) = \theta$ unbiasedness

$f(\vec{x})$ is an unbiased estimator of θ.

bias of the estimator is $E f(\vec{X}) - \theta$.

$X_1, X_2, \ldots, X_n \sim Poisson(\theta)$

$\overline{X} = \frac{1}{n} \sum_{j=1}^{n} X_j$ is an estimator for θ.

$$E\overline{X} = \frac{1}{n} \sum_{j=1}^{n} EX_j = \frac{1}{n} \cdot n\theta = \theta$$

$EX_1 = \theta$

$Var\, X_1 = \theta$

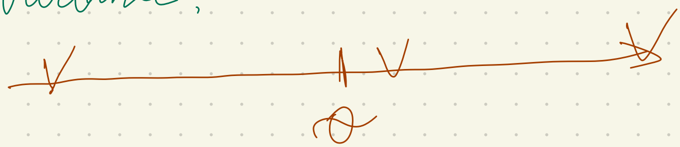
$$Var\, \overline{X} = \frac{1}{n^2} Var \sum_{j=1}^{n} X_j = \frac{1}{n^2} \cdot n\theta = \frac{\theta}{n}$$

Mean square error of the estimator

$$E\left(f(\vec{X}) - \theta\right)^2 = E\left(\left(f(\vec{X}) - Ef(\vec{X})\right) + \left(Ef(\vec{X}) - \theta\right)\right)^2$$

$$= E\left(f(\vec{X}) - Ef(\vec{X})\right)^2 + 2E\left(\left(f(\vec{X}) - Ef(\vec{X})\right)\left(Ef(\vec{X}) - \theta\right)\right) + E\left(Ef(\vec{X}) - \theta\right)^2 = Var\, f(\vec{X}) + bias^2 f(\vec{X})$$

$\Rightarrow$ for unbiased estimators mean square error is equal to their variance.



Consistent

$$f(\vec{X}) \xrightarrow{P} \theta, \quad n \to \infty$$

$$\forall \varepsilon > 0 \qquad P\left(|f(\vec{X}) - \theta| > \varepsilon\right) \xrightarrow{n \to \infty} 0$$

$$P\left(|f(\vec{X}) - \theta| > \varepsilon\right) < \frac{Var\, f(\vec{X})}{\varepsilon^2}$$

for unbiased estimators

If estimator $f(\vec{X})$ is unbiased and $Var\, f(\vec{X}) \to 0$, $n \to \infty$ then this estimator is consistent.