November 12, 2020. Lab

$X_1, X_2, \ldots, X_n$ – a simple sample, $Bin(k, p)$, k is known. Then $\overline{X}$ is sufficient for p.

$P(\vec{X} = \vec{x} \mid \overline{X} = t)$ – does not depend on p.

$$= \frac{P(\vec{X} = \vec{x}, \overline{X} = t)}{P(\overline{X} = t)} \quad \xrightarrow{\bar{x} \neq t} 0$$

$$\xrightarrow{\bar{x} = t} \frac{P(\vec{X} = \vec{x})}{P(\overline{X} = t)} =$$

$$\begin{cases} X_1 = x_1 \\ X_2 = x_2 \\ \ldots \\ X_n = x_n \\ X_1 + \ldots + X_n = nt \end{cases}$$

$$= \frac{P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)}{P(X_1 + \ldots + X_n = nt)} =$$

$$= \frac{\prod_{j=1}^{n} \frac{k!}{x_j!(k-x_j)!} \cdot \cancel{p^{x_j}(1-p)^{k-x_j}}}{\frac{(nk)!}{(nt)!(nk-nt)!} \cdot \cancel{p^{nt}(1-p)^{nk-nt}}}$$

$X_1 + \ldots + X_n \sim$

$\sim Bin(nk, p)$

$$p^{\sum_{j=1}^{n} x_j - nt} (1-p)^{nk - \sum_{j=1}^{n} x_j - nk + nt}$$

$\underbrace{X_1, X_2, \dots, X_n}_{\rightarrow \text{independent}} \sim N(\mu; \sigma^2)$

1) σ^2 is known $\Rightarrow \overline{X}$ is sufficient for μ

2) μ is known $\Rightarrow \sum_{i=1}^{n} (X_i - \mu)^2$ is sufficient for σ^2

3) $\left(\overline{X}, \sum_{i=1}^{n} X_i^2\right)$ is sufficient for $(\mu; \sigma^2)$.

$$f_{\vec{X}}(\vec{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{n/2}\sigma^n} \cdot$$

$$\cdot \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i^2 - 2x_i\mu + \mu^2)\right) =$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} \cdot \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right) \cdot \exp\left(\frac{n\overline{x}\mu}{\sigma^2}\right) \cdot \exp\left(\frac{-n\mu^2}{2\sigma^2}\right)$$

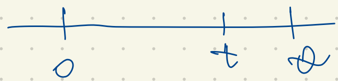
$$f_{\vec{X}}(\vec{x}) = g(T(\vec{x}), \theta) \cdot h(\vec{x})$$

$$X_1, X_2, \dots, X_n \sim U[0;\theta]$$

$$\theta^* = 2\overline{X}, \quad \hat{\theta}^* = \frac{n+1}{n} X_{max}$$

$$\mathrm{Var}\,\theta^* = \mathrm{Var}(2\overline{X}) = \frac{4}{n^2}\mathrm{Var}\left(\sum_{j=1}^{n} X_j\right) =$$

$$= \frac{4}{n^2}\sum_{j=1}^{n}\mathrm{Var}\,X_j = \frac{4}{n^2}\cdot n\cdot\frac{\theta^2}{12} = \frac{\theta^2}{3n}$$



$$\mathrm{Var}\,\hat{\theta}^* = \frac{(n+1)^2}{n^2}\mathrm{Var}(X_{max})$$

$$F_{X_{max}}(t) = P(X_{max} < t) = P(X_1 < t, X_2 < t, \dots, X_n < t)$$

$$= \frac{t^n}{\theta^n}, \ 0 \le t \le \theta; \quad f_{X_{max}}(t) = \frac{n t^{n-1}}{\theta^n} I_{0 \le t \le \theta}$$

$$E X_{max} = \int_0^{\theta} t \cdot \frac{n t^{n-1}}{\theta^n} dt = \frac{n}{\theta^n (n+1)} t^{n+1}\Big|_{t=0}^{\theta} = \frac{n}{n+1}\theta$$

$$E X_{max}^2 = \int_0^{\theta} t^2 \cdot \frac{n t^{n-1}}{\theta^n} dt = \frac{n}{\theta^n (n+2)} t^{n+2}\Big|_{t=0}^{\theta} = \frac{n}{n+2}\theta^2$$

$$\mathrm{Var}\,X_{max} = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \theta^2 \cdot \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2} =$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$\mathrm{Var}\,\hat{\theta}^* = \frac{\theta^2}{n(n+2)}$$

$X_1, X_2, \dots, X_n \sim \mathcal{U}[\theta, \theta+1]$ — a simple sample

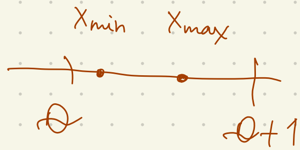
$$E\overline{X} = \theta + \frac{1}{2} \Rightarrow \theta^* = \overline{X} - \frac{1}{2}$$

$$f_{\vec{X}}(\vec{x}) = \prod_{j=1}^{n} I(\theta \le x_j \le \theta + 1) =$$

$$= I(\theta \le x_{min},\ x_{max} \le \theta + 1) =$$

$$= I(x_{max} - 1 \le \theta \le x_{min})$$

(X_{min}, X_{max}) is sufficient for θ.



$$\hat{\theta}^* = E\left(\overline{X} - \frac{1}{2} \,\middle|\, X_{min}, X_{max}\right) =$$

$$= \frac{1}{n} E(X_1 + X_2 + \dots + X_n \mid X_{min}, X_{max}) - \frac{1}{2} =$$

$$= \frac{1}{n}\left(X_{max} + X_{min} + (n-2)\cdot\frac{X_{min} + X_{max}}{2}\right) - \frac{1}{2} =$$

$$= \frac{1}{2}(X_{max} + X_{min} - 1)$$