Assignment 5.

N1.

$\eta \backslash \xi$	-1	0	1	
-2	$\frac{3}{17}$	$\frac{4}{17}$	$\frac{1}{17}$	$\frac{8}{17}$
2	$\frac{1}{17}$	$\frac{5}{17}$	$\frac{3}{17}$	$\frac{9}{17}$
	$\frac{4}{17}$	$\frac{9}{17}$	$\frac{4}{17}$	

Marginal distributions:

$$\eta \sim \begin{pmatrix} -2 & 2 \\ \frac{3}{17}+\frac{4}{17}+\frac{1}{17} & \frac{1}{17}+\frac{5}{17}+\frac{3}{17} \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ \frac{8}{17} & \frac{9}{17} \end{pmatrix}$$

$$\xi \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{3}{17}+\frac{1}{17} & \frac{4}{17}+\frac{5}{17} & \frac{1}{17}+\frac{3}{17} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ \frac{4}{17} & \frac{9}{17} & \frac{4}{17} \end{pmatrix}$$

$E\eta = (-2)\cdot\frac{8}{17} + 2\cdot\frac{9}{17} = \frac{2}{17}$, $\quad E\eta^2 = 4\cdot\frac{8}{17} + 4\cdot\frac{9}{17} = 4 \Rightarrow Var\,\eta = 4 - \frac{4}{17^2} = \frac{1152}{289}$

$E\xi = (-1)\cdot\frac{4}{17} + 1\cdot\frac{4}{17} = 0$, $\quad E\xi^2 = 1\cdot\frac{4}{17} + 1\cdot\frac{4}{17} = \frac{8}{17} \Rightarrow Var\,\xi = \frac{8}{17} - 0 = \frac{8}{17}$

$\xi\eta \sim \begin{pmatrix} -2 & 0 & 2 \\ \frac{1}{17}+\frac{1}{17} & \frac{4}{17}+\frac{5}{17} & \frac{3}{17}+\frac{3}{17} \end{pmatrix}$, $\quad E(\xi\eta) = (-2)\cdot\frac{2}{17} + 2\cdot\frac{6}{17} = \frac{8}{17} \neq E\xi\cdot E\eta = 0 \Rightarrow Cov(\xi,\eta) \neq 0$,

ξ and η are not independent

$$\rho_{\xi,\eta} = \frac{Cov(\xi,\eta)}{\sqrt{Var\,\xi \cdot Var\,\eta}} = \frac{E(\xi\eta) - E\xi\cdot E\eta}{\sqrt{Var\,\xi\cdot Var\,\eta}} = \frac{8}{17}\cdot\frac{289}{96\sqrt{17}} = \frac{\sqrt{17}}{12}$$

$E(E(\xi|\eta)) = E\xi = 0$, $\quad E(E(\eta|\xi)) = E\eta = \frac{2}{17}$

N2. A coin is flipped thrice, ξ is the amount of tails, η is the amount of changes in the sequence.

	---	+-- / --+	++- / -++	+++
$\eta \backslash \xi$	0	1	2	3
0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
1	0	$\frac{2}{8}$	$\frac{2}{8}$	0
2	0	$\frac{1}{8}$	$\frac{1}{8}$	0
		-+-	+-+	

Marginal distributions:

$$\xi \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{2}{8}+\frac{1}{8} & \frac{2}{8}+\frac{1}{8} & \frac{1}{8} \end{pmatrix}, \quad \eta \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{8}+\frac{1}{8} & \frac{2}{8}+\frac{2}{8} & \frac{1}{8}+\frac{1}{8} \end{pmatrix}$$

$$E\xi = 1\cdot\frac{3}{8}+2\cdot\frac{3}{8}+3\cdot\frac{1}{8} = \frac{3}{2}, \quad E\xi^2 = 1\cdot\frac{3}{8}+4\cdot\frac{3}{8}+9\cdot\frac{1}{8} = \frac{6}{2}, \quad Var\,\xi = \frac{6}{2}-\frac{9}{4} = \frac{3}{4}$$

$$E\eta = 1\cdot\frac{4}{8}+2\cdot\frac{2}{8} = 1, \quad E\eta^2 = 1\cdot\frac{4}{8}+4\cdot\frac{2}{8} = \frac{3}{2}, \quad Var\,\eta = \frac{3}{2}-1 = \frac{1}{2}$$

$$\xi\eta \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 6 \\ \frac{1}{8}+\frac{1}{8} & \frac{2}{8} & \frac{1}{8}+\frac{2}{8} & 0 & \frac{1}{8} & 0 \end{pmatrix}, \quad E(\xi\eta) = 1\cdot\frac{2}{8}+2\cdot\frac{3}{8}+4\cdot\frac{1}{8} = \frac{3}{2} = E\xi\cdot E\eta$$

$P(\eta=0, \xi=0) = \frac{1}{8} \neq P(\eta=0)\cdot P(\xi=0) = \frac{2}{64} \Rightarrow \xi, \eta$ are dependent

$$\rho_{\xi,\eta} = \frac{Cov(\xi,\eta)}{\sqrt{Var\,\xi\cdot Var\,\eta}} = 0 \quad \text{since } Cov(\xi,\eta) = E(\xi\eta) - E\xi\cdot E\eta = 0$$

$$E(E(\xi|\eta)) = E\xi = \frac{3}{2}, \qquad E(E(\eta|\xi)) = E\eta = 1$$

N3. Let ω be the amount of days to see both kinds of weather

$$\left.\begin{matrix} P(\omega \geq 1) = 1 \\ P(\omega \geq 2) = 1 \end{matrix}\right\} \text{it takes at least two days}$$

$P(\omega \geq 3) = P(\{$the weather on the first two days was the same$\}) = 0{,}4^2 + 0{,}6^2$

$P(\omega \geq 4) = P(\{$ —"— the first three days —"— $\}) = 0{,}4^3 + 0{,}6^3$

$\vdots$

$P(\omega \geq k+1) = P(\{$ —"— the first k days —"— $\}) = 0{,}4^k + 0{,}6^k$

$$E\omega = \sum_{k=2}^{\infty} k\cdot P(\omega=k) = \sum_{k=2}^{\infty} k\cdot(P(\omega<k+1) - P(\omega<k)) = \sum_{k=2}^{\infty} k\cdot((1-P(\omega\geq k+1)) - (1-P(\omega\geq k))) =$$

$$= \sum_{k=2}^{\infty} k\cdot(P(\omega\geq k) - P(\omega\geq k+1)) = \sum_{k=2}^{\infty} k\cdot(0{,}4^{k-1}+0{,}6^{k-1}-0{,}4^k-0{,}6^k) = \sum_{k=2}^{\infty} k\cdot(0{,}4^{k-1}\cdot 0{,}6 + 0{,}6^{k-1}\cdot 0{,}4) =$$

$$= 0{,}6\sum_{k=2}^{\infty} k\cdot 0{,}4^{k-1} + 0{,}4\sum_{k=2}^{\infty} k\cdot 0{,}6^{k-1}$$

Let $x_0 \in (0;1)$, $\frac{1}{1-x_0} = \sum_{k=0}^{\infty} x_0^k$

Since x_0 is inside the interval of convergence,

$$\left(\frac{1}{1-x_0}\right)' = \sum_{k=0}^{\infty}(x_0^k)' \Rightarrow \frac{1}{(1-x_0)^2} = \sum_{k=0}^{\infty} k x_0^{k-1}$$

$$\sum_{k=2}^{\infty} k\cdot 0{,}4^{k-1} = \frac{1}{(1-0{,}4)^2} - 1 = \frac{16}{9}, \qquad \sum_{k=2}^{\infty} k\cdot 0{,}6^{k-1} = \frac{1}{(1-0{,}6)^2} - 1 = \frac{21}{4}$$

$$E\omega = 0{,}6\cdot\frac{16}{9} + 0{,}4\cdot\frac{21}{4} = \frac{19}{6} = 3{,}167$$

N4. Identical to N3, but 0,6 and 0,4 are replaced for 0,5 and 0,5.

Let ω be the amount of flips to get heads and tails at least once.

$$E\omega = 0{,}5\cdot\sum_{k=2}^{\infty} k\cdot 0{,}5^{k-1} + 0{,}5\cdot\sum_{k=2}^{\infty} k\cdot 0{,}5^{k-1} = \frac{1}{(1-0{,}5)^2} - 1 = 3$$

N5. Let F_{HH} be the amount of flips of a coin until we get heads twice (including those two heads)

Let FF_T be an indicator that the first flip was tails,
FF_{HH} —‖— the first two flips were heads,
FF_{HT} —‖— the first flip was heads and the second was tails

$FF_T \cup FF_{HH} \cup FF_{HT} = 1$, $P(FF_T=1) = \frac{1}{2}$, $P(FF_{HH}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, $P(FF_{HT}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

Let's introduce a random variable FF such that

$$\begin{array}{lll} FF = 0 & \text{if} & FF_T = 1 \\ FF = 1 & \text{if} & FF_{HH} = 1 \\ FF = 2 & \text{if} & FF_{HT} = 1 \end{array} \Rightarrow FF \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$E(F_{HH} | FF = 0) = 1 + EF_{HH}$, since the first flip didn't get us closer to the goal

$E(F_{HH} | FF=1) = 2$, since the goal is reached

$E(F_{HH} | FF=2) = 2 + EF_{HH}$, since the second flip broke our streak

$$EF_{HH} = E(E(F_{HH}|FF)) = \frac{1}{2}(1+EF_{HH}) + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot(2+EF_{HH}) \Leftrightarrow \frac{1}{4}EF_{HH} = \frac{3}{2} \Rightarrow EF_{HH} = 6$$

Let F_{TH} be the amount of flips of a coin until we get "heads"-"tails" (inclusive)

Let's introduce a random variable FF' such that

$$\begin{array}{lll} FF' = 0 & \text{if} & FF_H = 1 \\ FF' = 1 & \text{if} & FF_{TT} = 1 \\ FF' = 2 & \text{if} & FF_{TH} = 1 \end{array} \Rightarrow FF \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$ where FF_H, FF_{TT}, FF_{TH} are similar to the previous point

$E(F_{TH} | FF'=0) = 1 + EF_{TH}$

$E(F_{TH} | FF'=2) = 2$

$E(F_{TH} | FF'=1) = 2 + EF_H$, where F_H is the amount of flips to get heads once, because as long as we keep getting tails, the streak is preserved

F_H has a geometric distribution with $p = \frac{1}{2}$, so $EF_H = \frac{1}{p} = 2$

$$EF_{TH} = E(E(F_{TH}|FF')) = \frac{1}{2}(1+EF_{TH}) + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot(2+2) \Leftrightarrow \frac{1}{2}EF_{TH} = 2 \Rightarrow EF_{TH} = 4$$

N6. Let R_{66} be the amount of rolls of a die to get "6" twice (inclusive)

Let FR_k be an indicator that the first roll yielded "k"
FR_{kL} —‖— the first roll yielded "k" and the second yielded "L"

Let's introduce a random variable FR such that:

$FR = k$ if $FR_k = 1$, $FR = \overline{kL}$ if $FR_{kL} = 1$

$$FR \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 61 & 62 & 63 & 64 & 65 & 66 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \end{pmatrix}$$

$$\left.\begin{array}{ll} E(R_{66} | FR = k) = 1 + ER_{66} & \text{for } k \in [1;5] \\ E(R_{66} | FR = L) = 2 + ER_{66} & \text{for } L \in [61;65] \\ E(R_{66} | FR = 66) = 2 & \end{array}\right\} \Rightarrow ER_{66} = E(E(R_{66}|FR)) = \frac{5}{6}(1+ER_{66}) + \frac{5}{36}(2+ER_{66}) + \frac{2}{36}$$

$$\frac{1}{36}ER_{66} = \frac{7}{6} \Rightarrow ER_{66} = 42$$

Let R_{666} be the amount of rolls of a die to get "6" thrice (inclusive)

Defining FR similarly, we have

$E(R_{666}|FR=k) = 1+ER_{666}$ for $k\in[1;5]$

$E(R_{666}|FR=l) = 2+ER_{666}$ for $k\in[61;65]$

$E(R_{666}|FR=m) = 3+ER_{666}$ for $k\in[661;665]$

$E(R_{666}|FR=666) = 3$

Thus, $ER_{666} = E(E(R_{666}|FR)) = \frac{5}{6}(1+ER_{666}) + \frac{5}{36}(2+ER_{666}) + \frac{5}{6\cdot 36}(3+ER_{666}) + \frac{3}{36\cdot 6}$

$$\frac{1}{6\cdot 36}ER_{666} = \frac{86}{2\cdot 36} \Rightarrow ER_{666} = 3\cdot 86 = 258$$

N7. Let S_4 be the sum of die rolls until we get a "4" (including 4)

Let FR be a random variable that is equal to the first roll

$$\left.\begin{aligned}
E(S_4|FR=1) &= ES_4+1\\
E(S_4|FR=2) &= ES_4+2\\
E(S_4|FR=3) &= ES_4+3\\
E(S_4|FR=4) &= 4\\
E(S_4|FR=5) &= ES_4+5\\
E(S_4|FR=6) &= ES_4+6
\end{aligned}\right\}
\begin{aligned}
&ES_4 = E(E(S_4|FR)) = \sum_{\substack{i=1\\ i\neq 4}}^{6}\frac{1}{6}\cdot(k+ES_4) + \frac{1}{6}\cdot 4\\
&\frac{1}{6}ES_4 = \frac{21}{6} \Rightarrow ES_4 = 21
\end{aligned}$$

N8. Let ξ_k be the quantity of sixes in K die rolls,

η_k —||— fives —||—

$\xi_k \sim Bin(K,\frac{1}{6})$, $\eta_k \sim Bin(k,\frac{1}{6}) \Rightarrow E\xi_k = E\eta_k = \frac{K}{6}$, $Var\,\xi_k = Var\,\eta_k = \frac{5K}{36}$

$E(\xi_k|\eta_k=n) = \frac{k-n}{5}$ (like $E\xi_{k-n}$, where $\xi_{k-n}\sim Bin(k-n,\frac{1}{5})$)

$E(\xi_k\eta_k|\eta_k=n) = E(n\xi_k|\eta_k=n) = nE(\xi_k|\eta_k=n) = n\frac{k-n}{5} \Rightarrow E(\xi_k\eta_k|\eta_k) = \eta_k\frac{k-\eta_k}{5}$

$E(\xi_k\eta_k) = E(E(\xi_k\eta_k|\eta_k)) = E(\frac{k}{5}\eta_k - \frac{1}{5}\eta_k^2) = \frac{k^2}{30} - \frac{1}{5}(Var\,\eta_k + (E\eta_k)^2) = \frac{K^2}{36} - \frac{K}{36}$

$Cov(\xi_k,\eta_k) = E(\xi_k\eta_k) - E\xi_k\cdot E\eta_k = \frac{K^2}{36} - \frac{K}{36} - \frac{K^2}{36} = -\frac{K}{36}$

$$\rho_{\xi_k,\eta_k} = \frac{Cov(\xi_k,\eta_k)}{\sqrt{Var\,\xi_k\cdot Var\,\eta_k}} = \frac{-\frac{K}{36}}{\frac{5K}{36}} = -\frac{1}{5}$$

N9. Let ς be the quantity of threes in K die rolls,

η —||— odd digits —||—

$\varsigma \sim Bin(K,\frac{1}{6}) \Rightarrow E\varsigma = \frac{k}{6}$, $Var\,\varsigma = \frac{5K}{36}$

$\eta \sim Bin(K,\frac{1}{2}) \Rightarrow E\eta = \frac{k}{2}$, $Var\,\eta = \frac{k}{4}$

$E(\varsigma|\eta=n) = \frac{n}{3}$ (like $E\xi$, where $\xi\sim Bin(n,\frac{1}{3})$)

$E(\varsigma\eta|\eta=n) = E(n\varsigma|\eta=n) = nE(\varsigma|\eta=n) = \frac{n^2}{3} \Rightarrow E(\varsigma\eta|\eta) = \frac{\eta^2}{3}$

$E(\varsigma\eta) = E(E(\varsigma\eta|\eta)) = E(\frac{\eta^2}{3}) = \frac{1}{3}(Var\,\eta + (E\eta)^2) = \frac{k^2+k}{12}$

$Cov(\varsigma,\eta) = E(\varsigma\eta) - E\varsigma\cdot E\eta = \frac{k^2+k}{12} - \frac{k^2}{12} = \frac{k}{12}$

$$\rho_{\varsigma,\eta} = \frac{Cov(\varsigma,\eta)}{\sqrt{Var\,\varsigma\cdot Var\,\eta}} = \frac{\frac{k}{12}}{\frac{k\sqrt{5}}{12}} = \frac{1}{\sqrt{5}}$$