October 19, 2020

(1) $\begin{pmatrix}\xi\\ \eta\\ \varsigma\end{pmatrix} \sim N(\vec{\mu}, K)$ $\quad \vec{\mu} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$, $\quad 1 \to E\xi,\ 0 \to E\eta,\ -2 \to E\varsigma$

$\xi \sim N(1;2)$ $\quad K = \begin{pmatrix} 2 & 3 & -1\\ 3 & 6 & -1\\ -1 & -1 & 1\end{pmatrix}$

$\eta \sim N(0;6)$, $\varsigma \sim N(-2;1)$

$2 \to \mathrm{cov}(\xi,\xi)$, $\quad 3 \to \mathrm{cov}(\xi,\eta)$, $\quad 1 \to \mathrm{cov}(\varsigma,\varsigma) = \mathrm{Var}\,\varsigma$

(a) $f_{\xi,\eta,\varsigma}(x,y,z) =$

$$\det K = \begin{vmatrix} 1 & 2 & -1\\ 2 & 5 & -1\\ 0 & 0 & 1\end{vmatrix} = \begin{vmatrix} 1 & 2\\ 2 & 5\end{vmatrix} = 1$$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0\\ 3 & 6 & -1 & 0 & 1 & 0\\ -1 & -1 & 1 & 0 & 0 & 1\end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 1\\ 2 & 5 & 0 & 0 & 1 & 1\\ -1 & -1 & 1 & 0 & 0 & 1\end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & -2 & 1 & -1\\ 0 & 1 & 1 & 1 & 0 & 2\end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 3\\ 0 & 1 & 0 & -2 & 1 & -1\\ 0 & 0 & 1 & 3 & -1 & 3\end{array}\right)$$

$$(x-1,\ y,\ z+2)\begin{pmatrix}5&-2&3\\-2&1&-1\\3&-1&3\end{pmatrix}\begin{pmatrix}x-1\\y\\z+2\end{pmatrix}=$$

$$=(5x-2y+3z+1,\ -2x+y-z,\ 3x-y+3z+3)\cdot$$

$$\cdot\begin{pmatrix}x-1\\y\\z+2\end{pmatrix}=5x^2-2xy+3xz+x-5x+\cancel{2y}-\cancel{3z}-1-$$

$$-2xy+y^2-yz+3xz-yz+3z^2+\cancel{3z}+6x-\cancel{2y}+$$

$$+6z+6=5x^2+y^2+3z^2-4xy+6xz-2yz+2x+$$

$$+6z+5$$

$$f_{\xi,\eta,\varsigma}(x,y,z)=\frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}\left(5x^2+y^2+3z^2-4xy+\right.\right.$$

$$\left.\left.+6xz-2yz+2x+6z+5\right)\right)$$

$$\begin{pmatrix}\xi\\\varsigma\end{pmatrix}\sim N\left(\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}2&-1\\-1&1\end{pmatrix}\right)$$

$f_\xi(x)=$

$f_\eta(y)=$

$f_\varsigma(z)=$

$$f_{\xi,\varsigma}(x,z)=\int_{-\infty}^{+\infty}f_{\xi,\eta,\varsigma}(x,y,z)\,dy$$

$$(x-1,\ y,\ z+2)\begin{pmatrix}5&-2&3\\-2&1&-1\\3&-1&3\end{pmatrix}\begin{pmatrix}x-1\\y\\z+2\end{pmatrix}=$$

$$=(5x-2y+3z+1,\ -2x+y-z,\ 3x-y+3z+3)\cdot$$

$$\cdot\begin{pmatrix}x-1\\y\\z+2\end{pmatrix}=5x^2-2xy+3xz+x-5x+\cancel{2y}-\cancel{3z}-1-$$

$$-2xy+y^2-yz+3xz-yz+3z^2+\cancel{3z}+6x-\cancel{2y}+$$

$$+6z+6=5x^2+y^2+3z^2-4xy+6xz-2yz+2x+$$

$$+6z+5$$

$$f_{\xi,\eta,\varsigma}(x,y,z)=\frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{1}{2}\left(5x^2+y^2+3z^2-4xy+\right.\right.$$

$$\left.\left.+6xz-2yz+2x+6z+5\right)\right)$$

$$y^2-2y(2x+z)+(2x+z)^2-(2x+z)^2=$$

$$=(y-2x-z)^2-(2x+z)^2$$

$$\frac{1}{2\pi}\exp\left(-\frac{1}{2}\left(5x^2+3z^2+6xz+2x+6z+5-\right.\right.$$

$$\left.\left.-(2x+z)^2\right)\right)\cdot\underbrace{\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-2x-z)^2}dy}_{1}$$

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy=1 \qquad \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-a)^2}{2}}dy=1$$

$$P(2\xi - 3\eta - \varsigma < 9)$$

$$\iiint\limits_{2x-3y-z<9} f_{\xi,\eta,\varsigma}(x,y,z)\,dx\,dy\,dz$$

$$E(2\xi - 3\eta - \varsigma) = 2E\xi - 3E\eta - E\varsigma = 2 - 0 + 2 = 4$$

$$Var(2\xi - 3\eta - \varsigma) = Cov(2\xi - 3\eta - \varsigma, 2\xi - 3\eta - \varsigma) =$$

$$= 4Cov(\xi,\xi) - 12Cov(\xi,\eta) - 4Cov(\xi,\varsigma) +$$

$$+ 6Cov(\eta,\varsigma) + 9Var\eta + Var\varsigma = 4\cdot 2 - 12\cdot 3 - 4\cdot(-1) +$$

$$+ 6\cdot(-1) + 9\cdot 6 + 1 = 25$$

$$2\xi - 3\eta - \varsigma \sim N(4; 25)$$

$$\int_{-\infty}^{9} \frac{1}{5\sqrt{2\pi}} e^{-\frac{(x-4)^2}{50}}\,dx = \left/\!\!\left/ \frac{x-4}{5} = t \right/\!\!\right/ =$$

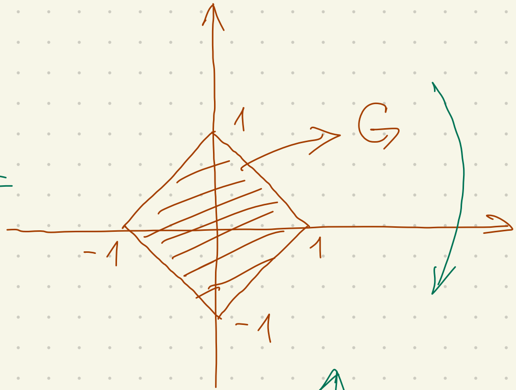
$$= \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = \Phi(1) = \frac{1}{2} + \Phi_0(1).$$

$$(2\ -3\ -1)\cdot K \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} =$$

8(a) ξ, η are independent $N(0;1)$

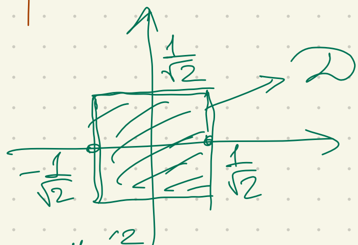
$$1 \le |x|+|y|$$

$$P_1 = \iint_G \frac{1}{2\pi} e^{-\frac{x^2}{2}-\frac{y^2}{2}}\,dx\,dy =$$



$$x = \frac{x'+y'}{\sqrt{2}}$$

$$y = \frac{-x'+y'}{\sqrt{2}}$$



$$= \iint_D \frac{1}{2\pi} e^{\frac{-x'^2-y'^2}{2}}\,dx'\,dy' =$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}}\,dx' \cdot \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y'^2}{2}}\,dy' =$$

$$= \left(2\Phi_0\left(\frac{1}{\sqrt{2}}\right)\right)^2$$

(c) $$\iint_{\sqrt{x^2+y^2}\le 2} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \cancel{\int_0^{2\pi} d\varphi} \int_0^2 \cancel{\frac{1}{2\pi}} e^{-\frac{r^2}{2}}\, r\,dr =$$

$$x = r\cos\varphi,\ y = r\sin\varphi$$

$$= -e^{-\frac{r^2}{2}}\Big|_0^2 = 1 - e^{-2}$$

(10) $\xi, \eta \sim N(0;1)$, $\rho_{\xi,\eta} = \rho$

$E(\xi^3\eta^3)$

1) Prove that ξ and $\eta - \rho\xi$ are independent.

$$Cov(\xi, \eta - \rho\xi) = E(\xi(\eta - \rho\xi)) - E\xi \cdot E(\eta - \rho\xi) =$$

$$= E(\xi\eta - \rho\xi^2) - 0 = E(\xi\eta) - \rho E\xi^2 = 0$$

$$E(\xi\eta) = E(\xi\eta) - E\xi \cdot E\eta = Cov(\xi, \eta) =$$

$$= \frac{Cov(\xi, \eta)}{\sqrt{Var\xi \cdot Var\eta}} = \rho$$

$$E\xi^2 = Var\xi + (E\xi)^2 = 1 + 0^2 = 1$$

$$E(\xi^3\eta^3) = E\left(\xi^3\left((\eta - \rho\xi) + \rho\xi\right)^3\right) =$$

$$= E(\xi^3 \cdot (\eta - \rho\xi)^3) + 3\rho E(\xi^4(\eta - \rho\xi)^2) +$$

$$+ 3\rho^2 E(\xi^5(\eta - \rho\xi)) + \rho^3 E\xi^6 =$$

$$= \cancel{E\xi^3 \cdot E(\eta - \rho\xi)^3}^{\,0} + 3\rho E\xi^4 \cdot E(\eta - \rho\xi)^2 +$$

$$+ \cancel{3\rho^2 E\xi^5 \cdot E(\eta - \rho\xi)}^{\,0} + \rho^3 E\xi^6 = \ldots$$

$$E\xi^{2n+1} = 0 \qquad n \in \mathbb{N}$$

$$\int_{-\infty}^{+\infty} x^{2n+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0$$

$$E\xi^{2n} = \int_{-\infty}^{+\infty} x^{2n} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^{2n-1} d e^{-x^2/2} =$$

$$x e^{-x^2/2} dx = -d e^{-x^2/2}$$

$$= -\frac{1}{\sqrt{2\pi}} x^{2n-1} e^{-x^2/2} \Big|_{-\infty}^{+\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} (2n-1) x^{2n-2} dx =$$

$$= (2n-1) \cdot E\xi^{2n-2}$$

$$E\xi^{2n} = (2n-1)!!$$

$$E\xi^2 = 1 \Rightarrow E\xi^4 = 3 \cdot E\xi^2 = 3,$$

$$E\xi^6 = 5 E\xi^4 = 15$$

$$E(\eta - \rho\xi)^2 = E\eta^2 + \rho^2 E\xi^2 - 2\rho E(\xi\eta) =$$

$$= 1 + \rho^2 - 2\rho^2 = 1 - \rho^2$$