$\xi \in \{-1; 1\}$, $\eta \in \{-1; 0; 1\}$ $\Rightarrow$ $\xi+\eta \in \{-2; -1; 0; 1; 2\}$, $\xi\eta \in \{-1; 0; 1\}$

$\xi+\eta \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{8} & \frac{1}{12} & \frac{7}{24}+\frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix}$, $\quad \xi\eta \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{7}{24}+\frac{1}{3} & \frac{1}{12}+\frac{1}{6} & \frac{1}{8}+0 \end{pmatrix}$

$\xi\eta$ \ $\xi+\eta$	-2	-1	0	1	2
-1	0	0	$\frac{1}{3}+\frac{7}{24}$	0	0
0	0	$\frac{1}{12}$	0	$\frac{1}{6}$	0
1	$\frac{1}{8}$	0	0	0	0

, where each cell is $\sum\limits_{k,l} P(\xi=k, \eta=l)$ and kl, $k+l$ define the cell

The joint distribution of $\xi\eta$, $\xi+\eta$ is the distribution of $\begin{bmatrix}\xi\eta \\ \xi+\eta\end{bmatrix}$.

<u>N11.</u> Let $p=0{,}8$ be the probability of the marksman hitting the target.
Let δ be the amount of shots until the marksman hits and misses at least once.
Let φ be the indicator random variable that shows if the first shot hit the target.
Let $H \sim Geo(p)$, $M \sim Geo(1-p)$ be the amount of misses and hits before a hit and a miss respectively ($Geo(p)$ is a geometrical distribution)

$E(\delta \mid \varphi=1) = 1 + EM$
$E(\delta \mid \varphi=0) = 1 + EH$
$\Rightarrow E(\delta \mid \varphi) \sim \begin{pmatrix} 1+EM & 1+EH \\ p & 1-p \end{pmatrix}$

$EM = \frac{1}{1-p} = 5$
$EH = \frac{1}{p} = \frac{5}{4}$

$E\delta = E(E(\delta \mid \varphi)) = p(1+5) + (1-p)(1+\frac{5}{4}) = 5{,}25$

<u>N12.</u> $\zeta \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac13 & \frac13 & \frac13 \end{pmatrix}$, $\quad \xi = 1-\zeta^{1000}$, $\quad \eta = 1-\zeta^{1001}$

$\xi \sim \begin{pmatrix} 0 & 1 \\ \frac13+\frac13 & \frac13 \end{pmatrix}$, $\quad \eta \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac13 & \frac13 & \frac13 \end{pmatrix}$

ξ \ η	0	1	2
0	$\frac13$	0	$\frac13$
1	0	$\frac13$	0

$P(\xi=0, \eta=0) = \frac13$ (from the table, $\zeta=1$)
$P(\xi=0)\cdot P(\eta=0) = \frac23 \cdot \frac13 = \frac29 \neq \frac13$ (from the marginal distributions)
Therefore, ξ and η are not independent.

$\xi\eta \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac23(\frac13+\frac13+\frac13)+\frac13\cdot\frac13 & \frac13\cdot\frac13 & \frac13\cdot\frac13 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ \frac79 & \frac19 & \frac19 \end{pmatrix}$

$Cov(\xi,\eta) = E(\xi\eta) - E\xi\cdot E\eta = 1\cdot\frac19 + 2\cdot\frac19 - (\frac13\cdot 1)(\frac13\cdot 1 + \frac13\cdot 2) = 0$

$Var\,\xi = E\xi^2 - (E\xi)^2 = \frac29$, $\quad Var\,\eta = E\eta^2 - (E\eta)^2 = \frac23$, $\quad \rho_{\xi,\eta} = \frac{Cov(\xi,\eta)}{\sqrt{Var\xi\cdot Var\eta}} = 0$

<u>N13.</u> Let $\xi \sim Geo(p)$, $\eta \sim Geo(p)$, where $Geo(p)$ is a geometric distribution.

$P(\xi=k \mid \xi+\eta=n) = \frac{P(\xi=k, \xi+\eta=n)}{P(\xi+\eta=n)} = \frac{P(\xi=k, \eta=n-k)}{P(\xi+\eta=n)}$, since $AB = A(AB)$ for events A, B

$P(\xi=k, \eta=n-k) \overset{\text{independent}}{=} P(\xi=k)\cdot P(\eta=n-k) = p\cdot(1-p)^{k-1}\cdot p\cdot(1-p)^{n-k-1} = p^2(1-p)^{n-2}$ for $k \in [1; n-1]$, since $\xi, \eta > 0$

$P(\xi+\eta=n) = \sum\limits_{i=1}^{n-1} P(\xi=i, \eta=n-i) = \sum\limits_{i=1}^{n-1} p^2(1-p)^{n-2} = (n-1)p^2(1-p)^{n-2}$

Thus, $P(\xi=k \mid \xi+\eta=n) = \frac{p^2(1-p)^{n-2}}{(n-1)p^2(1-p)^{n-2}} = \frac{1}{n-1}$ ∎

N14. Let $\xi_1 \sim Geo(p_1)$, $\xi_2 \sim Geo(p_2)$, where $Geo(p)$ is a geometric distribution.

Let $\zeta = \min\{\xi_1, \xi_2\}$

$$F_\zeta(x) = P(\zeta < x) = 1 - P(\zeta \geq x) = 1 - P(\xi_1 \geq x, \xi_2 \geq x) \overset{\text{indep.}}{=} 1 - P(\xi_1 \geq x)\cdot P(\xi_2 \geq x) = 1 - ((1-p_1)(1-p_2))^{x-1}$$

$$= 1 - (1-(p_1+p_2-p_1p_2))^{x-1} \Rightarrow \zeta \sim Geo(p_1+p_2-p_1p_2)$$

N15.

$$\mu = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 35 \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$\mu = \begin{bmatrix} E\xi_1 \\ E\xi_2 \\ E\xi_3 \end{bmatrix}, \quad K = \begin{bmatrix} Cov(\xi_1,\xi_1) & Cov(\xi_1,\xi_2) & Cov(\xi_1,\xi_3) \\ Cov(\xi_2,\xi_1) & Cov(\xi_2,\xi_2) & Cov(\xi_2,\xi_3) \\ Cov(\xi_3,\xi_1) & Cov(\xi_3,\xi_2) & Cov(\xi_3,\xi_3) \end{bmatrix}$$

a) $\eta_a = \xi_1 - \xi_3$

$E\eta_a = E(\xi_1 - \xi_3) = E\xi_1 - E\xi_3 = 0 - 1 = -1$

$Var\,\eta_a = Var(\xi_1 - \xi_3) = Var\,\xi_1 - 2Cov(\xi_1, \xi_3) + Var\,\xi_3 = 5 + 2 + 35 = 42$

б) $\eta_б = 2\xi_1 - \xi_2 + 3\xi_3$

$E\eta_б = E(2\xi_1 - \xi_2 + 3\xi_3) = 2E\xi_1 - E\xi_2 + 3E\xi_3 = 0 + 3 + 3 = 6$

$Var\,\eta_б = Var(2\xi_1 - \xi_2 + 3\xi_3)$

$$Cov((A \pm B), C) = E((A\pm B)C) - E(A\pm B)\cdot EC = E(AC) \pm E(BC) - EA\cdot EC \mp EB\cdot EC =$$
$$= Cov(A,C) \pm Cov(B,C)$$

$$Var(A \overset{(1)}{\pm} B \overset{(2)}{\pm} C) = Var(A \overset{(1)}{\pm} B) \overset{(2)}{\pm} 2Cov((A \overset{(1)}{\pm} B), C) + Var\,C =$$
$$= Var\,A + Var\,B + Var\,C \overset{(1)}{\pm} 2Cov(A,B) \overset{(2)}{\pm} 2Cov(A,C) \overset{(2)}{\pm}(\overset{(1)}{\pm}1)\cdot 2Cov(B,C)$$

$$Var\,\eta_б = Var(2\xi_1) + Var\,\xi_2 + Var(3\xi_3) - 2Cov(2\xi_1, \xi_2) + 2Cov(2\xi_1, 3\xi_3) - 2Cov(\xi_2, 3\xi_3) =$$
$$= 2^2\cdot 5 + 1 + 3^2\cdot 35 - 2\cdot 2\cdot(-2) + 2\cdot 2\cdot 3\cdot(-1) - 2\cdot 3\cdot 3 = 314$$

c) $\eta_c = -2\xi_1 + 3\xi_2 - \xi_3$

$E\eta_c = E(-2\xi_1 + 3\xi_2 - \xi_3) = -2E\xi_1 + 3E\xi_2 - E\xi_3 = -2\cdot 0 - 3\cdot 3 - 1 = -10$

$$Var\,\eta_c = Var(3\xi_2 - 2\xi_1 - \xi_3) = Var(3\xi_2) + Var(2\xi_1) + Var(\xi_3) - 2Cov(3\xi_2, 2\xi_1) - 2Cov(3\xi_2, \xi_3) + 2Cov(2\xi_1, \xi_3)$$
$$= 3^2\cdot 1 + 2^2\cdot 5 + 35 - 3\cdot 2\cdot 2\cdot(-2) - 3\cdot 2\cdot 3 + 2\cdot 2\cdot(-1) = 66$$

N16.

$$K = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & \lambda \\ 1 & \lambda & 2 \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad \zeta = \lambda\xi_1 + 2\xi_2 - \xi_3 \quad \arg\min_\lambda Var\,\zeta - ?$$

$$Var\,\zeta = Var(\lambda\xi_1) + Var(2\xi_2) + Var(\xi_3) + 2Cov(\lambda\xi_1, 2\xi_2) - 2Cov(\lambda\xi_1, \xi_3) - 2Cov(2\xi_2, \xi_3) =$$
$$= \lambda^2\cdot 1 + 2^2\cdot 3 + 2 + 2\cdot 2\cdot\lambda\cdot(-1) - 2\cdot\lambda\cdot 1 - 2\cdot 2\cdot\lambda = \lambda^2 - 10\lambda + 14$$

Since $Var\,\zeta$ must be non-negative, we only consider $\lambda: \lambda^2 - 10\lambda + 14 \geq 0 \Rightarrow \arg\min_\lambda Var\,\zeta = 5 \pm \sqrt{11}$

N17. $\xi_1, \xi_2, \dots, \xi_{100}$ are independent and identically distributed, $\xi_i \sim Exp(\lambda)$

Let $\eta_1 = \sum_{k=1}^{100} \xi_k$, $\eta_2 = \sum_{k=1}^{30} \xi_k$

$$E\eta_1 = E\left(\sum_{k=1}^{100} \xi_k\right) = 100\,E\xi_1 = \frac{100}{\lambda}, \quad E\eta_2 = \frac{30}{\lambda} \Rightarrow \mu = \begin{bmatrix} \frac{100}{\lambda} \\ \frac{30}{\lambda} \end{bmatrix}$$

$$Var\,\eta_1 = Var\Big(\sum_{k=1}^{100}\xi_k\Big) = Var\,\xi_1\cdot\underbrace{E(100)}_{100} + E(\xi_1)^2\cdot\underbrace{Var(100)}_{0} = 100\,Var\,\xi_1 = \frac{100}{\lambda^2}$$

$$Var\,\eta_2 = \frac{30}{\lambda^2}$$

$$Cov(\eta_1,\eta_2) = E(\eta_1\eta_2) - E\eta_1\cdot E\eta_2 = E\Big(\sum_{k=1}^{100}\sum_{l=1}^{30}\xi_k\xi_l\Big) - \frac{3000}{\lambda^2} = \sum_{k=1}^{100}\sum_{l=1}^{30}E(\xi_k\xi_l) - \frac{3000}{\lambda^2} =$$

$$= \sum_{k=1}^{100}\sum_{l=1}^{30}E\xi_k E\xi_l - \frac{3000}{\lambda^2} = \frac{3000}{\lambda^2} - \frac{3000}{\lambda^2} = 0 = Cov(\eta_2,\eta_1)$$

$$K = \begin{bmatrix}\frac{100}{\lambda^2} & 0\\ 0 & \frac{30}{\lambda^2}\end{bmatrix}$$

N18. Let ξ be the amount of letters (out of N) that have reached their destination.
Let ξ_i be the indicator that the ith letter has reached its destination.

$$\xi = \sum_{i=1}^{N}\xi_i \Rightarrow Var\,\xi = Var\Big(\sum_{i=1}^{N}\xi_i\Big) = \sum_{i=1}^{N}\sum_{j=1}^{N}Cov(\xi_i,\xi_j)$$

$$Cov(\xi_i,\xi_j) = E(\xi_i\xi_j) - E\xi_i E\xi_j = P(\xi_i=1,\ \xi_j=1) - P(\xi_i=1)P(\xi_j=1)$$

$$\left.\begin{aligned}&P(\xi_i=1) = \frac{(N-1)!}{N!} = \frac{1}{N} \quad \forall\, i\in[1;N]\\ &P(\xi_i=1,\ \xi_j=1) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)} \quad \forall\, i,j\in[1;N],\ i\neq j\end{aligned}\right\}\Rightarrow Cov(\xi_i,\xi_j) = \begin{cases}\frac{N-1}{N^2}, & i=j\\ \frac{1}{N^2(N-1)}, & i\neq j\end{cases}$$

$$Var\,\xi = \sum_{i=1}^{N}\Big(\frac{N-1}{N^2} + \sum_{\substack{j=1\\ i\neq j}}^{N}\frac{1}{N^2(N-1)}\Big) = \sum_{i=1}^{N}\Big(\frac{N-1}{N^2} + \frac{1}{N^2}\Big) = \frac{N}{N} = 1$$

N19. Let ξ be the sum of the ten random numbers from the set $\{1, 2, \dots, 100\}$.
Let ξ_i be the indicator that the number i was taken.

$$E\xi_i = P(\xi_i=1) = \frac{10\cdot A_{99}^{9}}{A_{100}^{10}} = \frac{10\cdot 99!\cdot 90!}{100!\cdot 90!} = \frac{1}{10}$$, where A_n^k is an arrangement of k from n.

$$E(\xi_i\xi_j) = P(\xi_i=1,\ \xi_j=1) = \frac{A_{10}^{2}\cdot A_{98}^{8}}{A_{100}^{10}} = \frac{10\cdot 9\cdot 98!\cdot 90!}{100!\cdot 90!} = \frac{90}{9900} = \frac{1}{110},\quad i\neq j$$

$$\xi = \sum_{i=1}^{100} i\cdot\xi_i \Rightarrow E\xi = E\Big(\sum_{i=1}^{100} i\cdot\xi_i\Big) = \sum_{i=1}^{100} i\cdot E\xi_i = \frac{1}{10}\sum_{i=1}^{100} i = \frac{100\cdot 101}{2\cdot 10} = 505$$

$$\Rightarrow Var\,\xi = Var\Big(\sum_{i=1}^{100} i\cdot\xi_i\Big) = \sum_{i=1}^{100}\sum_{j=1}^{100}Cov(i\cdot\xi_i, j\cdot\xi_j) = \sum_{i=1}^{100}\sum_{j=1}^{100} i\cdot j\cdot Cov(\xi_i,\xi_j) =$$

$$= \sum_{i=1}^{100} i\cdot\sum_{j=1}^{100} j\cdot(E(\xi_i\xi_j) - E\xi_i\cdot E\xi_j) = \sum_{i=1}^{100} i\cdot\Big(i\cdot(E\xi_i - E\xi_i^2) + \sum_{\substack{j=1\\ j\neq i}}^{100} j\cdot(E(\xi_i\xi_j) - E\xi_i\cdot E\xi_j)\Big) =$$

$$= \sum_{i=1}^{100} i\cdot\Big(i\cdot\Big(\frac{1}{10} - \frac{1}{100}\Big) + \sum_{\substack{j=1\\ j\neq i}}^{100} j\cdot\Big(\frac{1}{110} - \frac{1}{100}\Big)\Big) = \sum_{i=1}^{100}\frac{9}{100}i^2 + \sum_{i=1}^{100} i\cdot\Big(-\frac{1}{1100}\Big)\cdot\Big(\sum_{j=1}^{100}(j) - i\Big) =$$

$$= \frac{9}{100}\cdot\sum_{i=1}^{100} i^2 + \Big(-\frac{1}{1100}\Big)\sum_{i=1}^{100} i\cdot(50\cdot 101 - i) = \frac{100}{1100}\sum_{i=1}^{100} i^2 - \frac{5050}{1100}\cdot\sum_{i=1}^{100} i =$$

$$= \frac{100\cdot 101\cdot 201}{6\cdot 11} - \frac{50^2\cdot 101^2}{1100} = \frac{101}{11}(3350 - 2525) = 7575$$