October 15, 2020. Lecture 8

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(\sqrt{2\pi})^n \cdot \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right)$$

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ - & - & - & - \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}$$

$$\sigma_{ij} = \sigma_{ji} \quad \forall i, j$$

Σ is a positive definite matrix

$\forall \vec{x} \neq \vec{0}$: $\vec{x}^T \cdot \Sigma \vec{x} > 0$

an orthogonal matrix B: $B^{-1} \Sigma B = D =$

$$= \begin{pmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$$

$$\cancel{\det} B^{-1} \cdot \det \Sigma \cdot \cancel{\det} B = \prod_{i=1}^n \lambda_i^2$$

$$D^{-1} = \begin{pmatrix} \lambda_1^{-2} & & & 0 \\ & \lambda_2^{-2} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{-2} \end{pmatrix}$$

$$(B^{-1} \Sigma B)^{-1} = D^{-1}$$

$$B^{-1} \Sigma^{-1} B = D^{-1}$$

$$\Sigma^{-1} = B D^{-1} B^{-1}$$

$$\Sigma B = B D$$

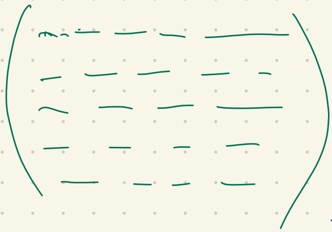
$$\Sigma = B D B^{-1} = B D B^T$$

$$\vec{y} = B^T\vec{x} \iff \vec{x} = (B^T)^{-1}\vec{y} = B\vec{y}$$

$B =$ 

$B^{-1} = B^T$

$\underbrace{B^TB = I}$ $\quad \det{}^2 B = 1$

$B^T =$ 

$$f_{\vec{y}}(\vec{y}) = f_{\vec{x}}(\vec{x}(\vec{y}))\cdot\left|\overbrace{\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)}}^{\det B = \pm 1}\right| =$$

$$= f_{\vec{x}}(B\vec{y}) = \frac{1}{(\sqrt{2\pi})^n\sqrt{\det\Sigma}}\cdot\exp\left(-\frac{1}{2}(B\vec{y}-\vec{\mu})^T\Sigma^{-1}\cdot(B\vec{y}-\vec{\mu})\right) =$$

$$D^{-1} = \begin{pmatrix} \lambda_1^{-2} & & & 0 \\ & \lambda_2^{-2} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{-2} \end{pmatrix}$$

$$(B(\vec{y}-B^{-1}\vec{\mu}))^T\Sigma^{-1}B(\vec{y}-B^{-1}\vec{\mu}) =$$

$$= (\vec{y}-B^T\vec{\mu})^T\underbrace{B^T\Sigma^{-1}B}(\vec{y}-B^T\vec{\mu}) =$$

$$= (\vec{y}-B^T\vec{\mu})^T D^{-1}(\vec{y}-B^T\vec{\mu}) = \sum_{k=1}^{n}\left((\vec{y}-B^T\vec{\mu})_k\right)^2\lambda_k^{-2}$$

$$= \frac{1}{(\sqrt{2\pi})^n\lambda_1\lambda_2\cdots\lambda_n}\cdot\exp\left(-\frac{1}{2}\sum_{k=1}^{n}\lambda_k^{-2}\left(y_k-(B^T\vec{\mu})_k\right)^2\right)$$

$$= \prod_{k=1}^{n}\frac{1}{\sqrt{2\pi}\lambda_k}\exp\left(-\frac{1}{2\lambda_k^2}\left(y_k-(B^T\vec{\mu})_k\right)^2\right)$$

$y_k \sim N\left((B^T\vec{\mu})_k ; \lambda_k^2\right)$ and are independent.

$$EX_j = \int\limits_{\mathbb{R}^n} \frac{x_j}{(2\pi)^{n/2}\sqrt{\det\Sigma}} \cdot \exp\left(-\frac{1}{2}(\vec{x}^T - \vec{\mu}^T)\Sigma^{-1}(\vec{x}-\vec{\mu})\right)d\vec{x}$$

$$\vec{x} = B\vec{y} \qquad d\vec{x} = \left|\frac{\partial(\vec{x})}{\partial(\vec{y})}\right| \cdot d\vec{y} = d\vec{y}$$

$$= \int\limits_{\mathbb{R}^n} \underbrace{(B\vec{y})_j}_{} \cdot \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\lambda_k} \cdot \exp\left(-\frac{1}{2\lambda_k^2}(y_k - (B^T\vec{\mu})_k)^2\right) d\vec{y} =$$

$$\longrightarrow (b_{j1}\ b_{j2} \dots b_{jn}) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} b_{ji} y_i$$

$$= \sum_{i=1}^{n} b_{ji} \underbrace{\int\limits_{\mathbb{R}^n} y_i \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\lambda_k} \cdot \exp\left(-\frac{1}{2\lambda_k^2}(y_k - (B^T\vec{\mu})_k)^2\right) d\vec{y}}_{} =$$

$$\prod_{k=1, k\neq i}^{n} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\lambda_k} \exp\left(-\frac{1}{2\lambda_k^2}(y_k - (B^T\vec{\mu})_k)^2\right) dy_k \cdot$$

$$\cdot \int_{-\infty}^{+\infty} \frac{y_i}{\sqrt{2\pi}\lambda_i} \exp\left(-\frac{1}{2\lambda_i^2}(y_i - (B^T\vec{\mu})_i)^2\right) dy_i =$$

$$= 1^{n-1} \cdot E y_i = (B^T\vec{\mu})_i$$

$$\begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} b_{ji} (B^T\vec{\mu})_i = \sum_{i=1}^{n} b_{ji} \cdot \sum_{\ell=1}^{n} b_{\ell i}\mu_\ell =$$

$$= \sum_{\ell=1}^{n} \mu_\ell \underbrace{\sum_{i=1}^{n} b_{\ell i} b_{ji}}_{\delta_{\ell j} = \begin{cases} 1, & \ell = j \\ 0, & \ell \neq j \end{cases}} = \sum_{\ell=1}^{n} \mu_\ell \delta_{\ell j} = \mu_j$$

$$Cov(X_i, X_j) = E\left((X_i - EX_i)(X_j - EX_j)\right) =$$

$$= \int\limits_{\mathbb{R}^n} \frac{(x_i - \mu_i)(x_j - \mu_j)}{(\sqrt{2\pi})^n \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}(\vec{x}^T - \vec{\mu}^T)\Sigma^{-1}(\vec{x} - \vec{\mu})\right) d\vec{x}$$

$$\underline{\vec{x} = B\vec{y} \Rightarrow d\vec{x} = d\vec{y}}$$

$$= \int\limits_{\mathbb{R}^n} \frac{((B\vec{y})_i - \mu_i)((B\vec{y})_j - \mu_j)}{(\sqrt{2\pi})^n \lambda_1 \lambda_2 \ldots \lambda_n} \prod_{k=1}^{n} \exp\left(-\frac{1}{2\lambda_k^2}(y_k - (B^T\vec{\mu})_k)^2\right) d\vec{y}$$

$$(B\vec{y} - \vec{\mu})_i (B\vec{y} - \vec{\mu})_j = \left(B(\vec{y} - B^{-1}\vec{\mu})\right)_i \left(B(\vec{y} - B^{-1}\vec{\mu})\right)_j =$$

$$= \sum_{k=1}^{n} b_{ik}(y_k - (B^{-1}\vec{\mu})_k) \cdot \sum_{\ell=1}^{n} b_{j\ell}(y_\ell - (B^{-1}\vec{\mu})_\ell) =$$

$$= \sum_{k,\ell \in [1,n]} b_{ik} b_{j\ell} (y_k - (B^{-1}\vec{\mu})_k)(y_\ell - (B^{-1}\vec{\mu})_\ell)$$

$$= \sum_{k,\ell \in [1,n]} b_{ik} b_{j\ell} \int\limits_{\mathbb{R}^n} (y_k - (B^{-1}\vec{\mu})_k)(y_\ell - (B^{-1}\vec{\mu})_\ell) \cdot \prod_{m=1}^{n} \frac{1}{\sqrt{2\pi}\lambda_m} \cdot$$

$$\cdot \exp\left(-\frac{1}{2\lambda_m^2}(y_m - (B^T\vec{\mu})_m)^2\right) d\vec{y} =$$

$$= \sum_{k,\ell \in [1,n]} b_{ik} b_{j\ell} \iint\limits_{\mathbb{R}^2} (y_k - (B^{-1}\vec{\mu})_k)(y_\ell - (B^{-1}\vec{\mu})_\ell) \cdot \frac{1}{2\pi \lambda_k \lambda_\ell} \cdot$$

$$\cdot \exp\left(-\frac{1}{2\lambda_k^2}(y_k - (B^T\vec{\mu})_k)^2 - \frac{1}{2\lambda_\ell^2}(y_\ell - (B^T\vec{\mu})_\ell)^2\right) dy_k dy_\ell$$

$$= \sum_{k,\ell \in [1,n]} b_{ik} b_{j\ell} \, Cov(Y_k, Y_\ell) = \sum_{k=1}^{n} b_{ik} b_{jk} \, Cov(Y_k, Y_k)$$

$$= \sum_{k=1}^{n} b_{ik} b_{jk} \lambda_k^2 = (BDB^T)_{ij} = \Sigma_{ij} = \sigma_{ij}$$

$$(BDB^T)_{ij} =$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ - & - & - & - \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$$

$$BD = \begin{pmatrix} b_{11}\lambda_1^2 & b_{12}\lambda_2^2 & \cdots & b_{1n}\lambda_n^2 \\ b_{21}\lambda_1^2 & b_{22}\lambda_2^2 & \cdots & b_{2n}\lambda_n^2 \\ - & - & - & - \\ b_{n1}\lambda_1^2 & b_{n2}\lambda_2^2 & \cdots & b_{nn}\lambda_n^2 \end{pmatrix}$$

$$\begin{pmatrix} b_{i1}\lambda_1^2 & b_{i2}\lambda_2^2 & \cdots & b_{in}\lambda_n^2 \end{pmatrix} \begin{pmatrix} b_{j1} \\ b_{j2} \\ \vdots \\ b_{jn} \end{pmatrix}$$

$X_i \sim N(-2; 4)$, $i = 1, 2, \dots, 100$; $\longrightarrow$ $EX_i^2 = (EX_i)^2 + Var\, X_i = 4+4$

X_i are independent.

$$Y_1 = \sum_{i=1}^{40} X_i, \quad Y_2 = \sum_{i=21}^{100} X_i$$

Probability density of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$$EY_1 = \sum_{i=1}^{40} EX_i = -80; \quad Var\, Y_1 = 160$$

$$EY_2 = -160; \quad Var\, Y_2 = 320$$

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - EY_1 \cdot EY_2 = 12880 - 80 \cdot 160 = 80$$

$$E\left(\sum_{i=1}^{40} \sum_{j=21}^{100} X_i X_j\right) = 3180\, E(X_i X_j) + 20\, EX_j^2 =$$

$$= 3180 \cdot (-2)^2 + 20 \cdot 8 =$$

3200

$$= 12880$$

$$\vec{\mu} = \begin{pmatrix} -80 \\ -160 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 160 & 80 \\ 80 & 320 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 160 & 80 & 1 & 0 \\ 80 & 320 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 4 & 0 & 1/80 \\ 2 & 1 & 1/80 & 0 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 4 & 0 & 1/80 \\ 0 & -7 & 1/80 & -2/80 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 4 & 0 & 7/560 \\ 0 & 1 & -1/560 & 2/560 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 4/560 & -1/560 \\ 0 & 1 & -1/560 & 2/560 \end{array}\right) \qquad \Sigma^{-1} = \frac{1}{560}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\sqrt{\det \Sigma} = \sqrt{160 \cdot 320 - 80^2} = 80\sqrt{2 \cdot 4 - 1} = 80\sqrt{7}$$

$$f_{(y_1, y_2)}(y_1, y_2) = \frac{1}{2\pi\sqrt{\det \Sigma}} \cdot$$

$$\cdot \exp\left(-\frac{1}{2}\begin{pmatrix} y_1+80 & y_2+160 \end{pmatrix} \Sigma^{-1} \begin{pmatrix} y_1+80 \\ y_2+160 \end{pmatrix}\right) =$$

$$\frac{1}{560}\begin{pmatrix} y_1+80 & y_2+160 \end{pmatrix} \cdot \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} y_1+80 \\ y_2+160 \end{pmatrix} =$$

$$= \frac{1}{560}\begin{pmatrix} 4y_1 - y_2 + 160 & -y_1 + 2y_2 + 240 \end{pmatrix} \begin{pmatrix} y_1+80 \\ y_2+160 \end{pmatrix} =$$

$$= \frac{1}{560}\Big(\underbrace{(4y_1 - y_2 + 160)(y_1+80) + (-y_1+2y_2+240)(y_2+160)}\Big)$$

$$= \frac{1}{160\pi\sqrt{7}} \cdot \exp\left(-\frac{1}{1120}(\ldots\ldots)\right)$$

Characteristic functions

$$\varphi_X(t) = E(e^{itX}) \xrightarrow{\text{cont.}} \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx$$

$t \in \mathbb{R}$

$\downarrow$ discr.

$$\sum_k e^{itx_k} P(X = x_k)$$

$$\int_{-\infty}^{+\infty} |e^{itx} f_X(x)|\,dx = \int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$|e^{itx}| = 1 \qquad \left|\int_a^b h(x)\,dx\right| \le \int_a^b |h(x)|\,dx$$

1) $|\varphi_X(t)| \le 1$

2) $\varphi_X(0) = 1$

3) there is one-to-one correspondence between characteristic functions and random variables.

4) X_1 and X_2 are independent then

$$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t) \cdot \varphi_{X_2}(t)$$

$$\varphi_{X_1+X_2}(t) = Ee^{i(X_1+X_2)t} = E(e^{iX_1t} \cdot e^{iX_2t}) =$$

$$= Ee^{iX_1t} \cdot Ee^{iX_2t}$$

5) $Y = aX + b \qquad (a = \text{const},\ b = \text{const})$

$$\boxed{\varphi_Y(t)} = Ee^{iYt} = Ee^{i(aX+b)t} = E(e^{iaXt} \cdot e^{ibt}) =$$

$$= e^{ibt} \cdot E(e^{i(at)X}) = \boxed{e^{ibt}\varphi_X(at)}$$

$$X \sim N(0;1)$$

$$\varphi_X(t) = \int_{-\infty}^{+\infty} e^{itx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \Rightarrow |\ldots| = \frac{|x|}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\varphi_X'(t) = \int_{-\infty}^{+\infty} ix\, e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

$$\left(itx - \frac{x^2}{2}\right)'_x = it - x$$

$$ix = \alpha(it - x) + \beta \Rightarrow \alpha = -i,\ \beta = -t$$

$$ix = -i(it - x) - t$$

$$= -i \int_{-\infty}^{+\infty} \frac{(it - x)}{\sqrt{2\pi}} e^{itx - \frac{x^2}{2}} dx - t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{itx - \frac{x^2}{2}} dx =$$

$$= -\frac{i e^{itx - \frac{x^2}{2}}}{\sqrt{2\pi}} \Bigg|_{x=-\infty}^{+\infty} - t\varphi_X(t) = -t\varphi_X(t)$$

$$\frac{d\varphi}{dt} = -t\varphi, \quad \frac{d\varphi}{\varphi} = -t\,dt, \quad \ln|\varphi| = -\frac{t^2}{2} + \tilde{C}$$

$$\underbrace{\varphi = c e^{-t^2/2}}$$

$$\varphi(0) = 1 \Rightarrow c = 1$$

$$\varphi_X(t) = e^{-t^2/2}$$

$$Y \sim N(\mu; \sigma^2) \Longrightarrow Y = \mu + \sigma X, \quad X \sim N(0;1)$$

$$\varphi_Y(t) = e^{i\mu t} \cdot \varphi_X(\sigma t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$$

$Y_1, Y_2, \dots, Y_n$ are independent,

$Y_j \sim N(\mu_j, \sigma_j^2)$

$\sum_{j=1}^{n} Y_j \sim ?$

$$\varphi_{\sum_{j=1}^{n} Y_j}(t) = \prod_{j=1}^{n} \varphi_{Y_j}(t) = \prod_{j=1}^{n} e^{i\mu_j t - \frac{\sigma_j^2 t^2}{2}} =$$

$$= e^{it\sum_{j=1}^{n}\mu_j - \frac{t^2}{2}\sum_{j=1}^{n}\sigma_j^2} \sim N\left(\sum_{j=1}^{n}\mu_j, \sum_{j=1}^{n}\sigma_j^2\right)$$

1) $X \sim \begin{pmatrix} -3 & 0 & 2 & 4 \\ 1/9 & 2/9 & 1/9 & 5/9 \end{pmatrix}$

$$\varphi_X(t) = \frac{1}{9}e^{-3it} + \frac{2}{9} + \frac{1}{9}e^{2it} + \frac{5}{9}e^{4it}$$

2) $\varphi_Y(t) = \cos t = \dfrac{e^{it} + e^{-it}}{2}$ $\qquad \dfrac{1}{1-q} = 1 + q + q^2 + \dots$ $\quad |q| < 1$

$Y \sim \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix}$

3) $$\varphi_Z(t) = \frac{1}{2 - e^{it}} = \frac{1}{2\left(1 - \frac{e^{it}}{2}\right)} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^{it}}{2}\right)^k =$$

$$= \sum_{k=0}^{\infty} \frac{e^{itk}}{2^{k+1}}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & \dots & k & \dots \\ 1/2 & 1/4 & 1/8 & 1/16 & \dots & 1/2^{k+1} & \dots \end{pmatrix}$$