# November 19, 2020, Lab

$$\tilde{\beta} = \sum_{i=1}^{n} b_i y_i \qquad E\tilde{\beta} = \beta$$

mean square error $(\tilde{\beta}) \longrightarrow \min$

$$y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i$$

$$E\tilde{\beta} = \sum_{i=1}^{n} b_i E y_i = \sum_{i=1}^{n} b_i(\alpha + \beta(x_i - \bar{x})) =$$

$$= \alpha \sum_{i=1}^{n} b_i + \beta \sum_{i=1}^{n} (x_i - \bar{x}) b_i = \beta$$

$$\sum_{i=1}^{n} b_i = 0, \quad \sum_{i=1}^{n} (x_i - \bar{x}) b_i = 1$$

$$\mathrm{Var}\,\tilde{\beta} = \sum_{i=1}^{n} b_i^2 \,\mathrm{Var}\, y_i = \sigma^2 \sum_{i=1}^{n} b_i^2 \longrightarrow \min$$

$$\sum_{i=1}^{n} b_i^2 \to \min; \quad \sum_{i=1}^{n} b_i = 0, \quad \sum_{i=1}^{n} b_i (x_i - \bar{x}) = 1$$

$$L = \sum_{i=1}^{n} b_i^2 + \lambda \sum_{i=1}^{n} b_i + \mu\left(\sum_{i=1}^{n} b_i (x_i - \bar{x}) - 1\right)$$

$$\frac{\partial L}{\partial b_i} = 2b_i + \lambda + \mu(x_i - \bar{x}) = 0 \qquad | \cdot (x_i - \bar{x})$$

$$\boxed{\mu = -\frac{2}{S_{xx}} \quad \lambda = 0}$$

$$2\sum_{i=1}^{n} b_i + n\lambda + \sum_{i=1}^{n} \mu(x_i - \bar{x}) = 0 \Rightarrow \lambda = 0$$

$$2\sum_{i=1}^{n} b_i(x_i - \bar{x}) + \lambda \cancel{\sum_{i=1}^{n}(x_i - \bar{x})}^{0} + \mu \sum_{i=1}^{n} (x_i - \bar{x})^2 = 0$$

$$2 + \mu S_{xx} = 0$$

$$b_i = -\frac{\mu}{2}(x_i - \bar{x}) = \frac{1}{S_{xx}}(x_i - \bar{x})$$

$$\tilde{\beta} = \sum_{i=1}^{n} b_i y_i = \sum_{i=1}^{n} \frac{(x_i - \bar{x})y_i}{S_{xx}}$$

$$\left.\begin{array}{l} \dfrac{\partial^2 L}{\partial b_i \partial b_j} = 0, \quad i \neq j \\[2ex] \dfrac{\partial^2 L}{\partial b_i^2} = 2 \end{array}\right\} d^2L = 2(db_1^2 + db_2^2 + \ldots + db_n^2)$$

$\underbrace{X_1, X_2, \ldots, X_n}_{\text{independent}} \sim N(\mu; \sigma^2)$ /both parameters are unknown/

$$\frac{1}{n-1}\underbrace{\sum_{i=1}^{n}(X_i-\bar{X})^2}_{S_{xx}} = S^2$$ (an unbiased estimator for variance)

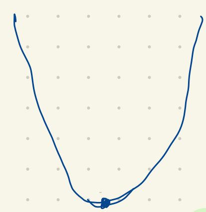
$\lambda S_{xx}$ is some estimator for variance
What is the value of $\lambda$ that the estimator has the least square error possible?

$$\frac{S_{xx}}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow E\left(\frac{S_{xx}}{\sigma^2}\right) = n-1$$

$$Var\left(\frac{S_{xx}}{\sigma^2}\right) = 2n-2$$

$$Var\, S_{xx} = \sigma^4(2n-2)$$

$$E(\lambda S_{xx} - \sigma^2)^2 = E(\lambda^2 S_{xx}^2) - 2\lambda\sigma^2 E S_{xx} +$$
$$+\sigma^4 = \lambda^2\left((ES_{xx})^2 + Var\, S_{xx}\right) - 2\lambda\sigma^2(n-1)\sigma^2 +$$
$$+\sigma^4 = \lambda^2\left((n-1)^2\sigma^4 + \sigma^4(2n-2)\right) - 2\lambda(n-1)\sigma^4 + \sigma^4 =$$
$$= \sigma^4\left(\lambda^2(n^2-1) - 2\lambda(n-1) + 1\right)$$



$$\lambda^* = \frac{(n-1)}{n^2-1} = \frac{1}{n+1}$$

$$\sigma^4\left(\frac{n^2-1}{(n+1)^2} - \frac{2(n-1)}{(n+1)} + 1\right) = \sigma^4\left(1 - \frac{n-1}{n+1}\right) = \boxed{\frac{2\sigma^4}{n+1}}$$

$$\mathrm{Var}\left(\frac{1}{n-1}S_{xx}\right)=\frac{1}{(n-1)^2}\mathrm{Var}\,S_{xx}=\frac{2\sigma^4}{n-1}$$

$$E\left(\frac{1}{n+1}\sum_{i=1}^{n}(X_i-\overline{X})^2\right)=\frac{n-1}{n+1}\sigma^2$$

$$\text{bias}=\frac{n-1}{n+1}\sigma^2-\sigma^2=\frac{-2\sigma^2}{n+1}\xrightarrow{n\to\infty}0$$

$X_1, X_2, \ldots, X_n$ — a simple sample out of Poisson $(\theta)$ $\theta > 0$

$$P(X=k) = e^{-\theta} \cdot \frac{\theta^k}{k!}$$

an unbiased estimator of $\frac{1}{\theta}$.

$n=1$ a sample consists of 1 r. v.

$T(X)$ is an estimator of $\frac{1}{\theta}$

$$E\,T(X) = \frac{1}{\theta}$$

$$\sum_{k=0}^{\infty} T(k) e^{-\theta} \cdot \frac{\theta^k}{k!} = \frac{1}{\theta} \quad \Big| \cdot \theta \cdot e^{\theta}$$

$$\sum_{k=0}^{\infty} T(k) \frac{\theta^{k+1}}{k!} = e^{\theta}$$

$$\sum_{k=0}^{\infty} T(k) \frac{\theta^{k+1}}{k!} = \sum_{p=0}^{\infty} \frac{\theta^p}{p!}$$

$k+1=p$

$$\sum_{p=1}^{\infty} T(p-1) \frac{\theta^p}{(p-1)!} = \sum_{p=0}^{\infty} \frac{\theta^p}{p!}$$

$X_1, X_2, \ldots, X_n$ — a simple sample

$H_0$ — hypothesis

$H_1$ — an alternative hypothesis

$$\mathbb{R}^n \supset (x_1, x_2, \ldots, x_n)$$

$$\mathbb{R}^n = C \cup (\mathbb{R}^n \setminus C)$$

if $\vec{x} \in \mathbb{R}^n \setminus C$ then $H_0$

critical domain

if $\vec{x} \in C$ then $H_1$

$$H_0 : X \sim N(0; 1)$$