September 24, 2020

$\eta \backslash \xi$	-2	1	0	
-1	$0{,}1$	$0{,}4$	$0{,}05$	$0{,}55$
2	$0{,}1$	$0{,}15$	$0{,}2$	$0{,}45$

$$E\xi = \sum_{x_k} x_k P(\xi = x_k)$$

$$E(\xi|\eta)$$

$$E(\xi|\eta=-1) = \sum_{x_k} x_k P(\xi = x_k|\eta=-1) =$$

$$= \sum_{x_k} \frac{x_k P(\xi = x_k, \eta=-1)}{P(\eta=-1)} = \frac{1}{P(\eta=-1)} \cdot \sum_{x_k} x_k P(\xi = x_k, \eta=-1)$$

$$= \frac{1}{0{,}55} \cdot (-2 \cdot 0{,}1 + 1 \cdot 0{,}4 + 0 \cdot 0{,}05) = \frac{0{,}2}{0{,}55} = \frac{4}{11}$$

$$E(\xi|\eta=2) = \frac{1}{P(\eta=2)} \cdot \sum_{x_k} x_k P(\xi = x_k, \eta=2) =$$

$$= \frac{1}{0{,}45} \cdot (-2 \cdot 0{,}1 + 1 \cdot 0{,}15 + 0 \cdot 0{,}2) = \frac{-0{,}05}{0{,}45} = -\frac{1}{9}$$

$$E(\xi|\eta) \sim \begin{pmatrix} 4/11 & -1/9 \\ 0{,}55 & 0{,}45 \end{pmatrix}$$

$$E(E(\xi|\eta)) = \frac{4}{11} \cdot 0{,}55 + \left(-\frac{1}{9}\right) \cdot 0{,}45 =$$

$$= 4 \cdot 0{,}05 - 0{,}05 = 0{,}15$$

$$E\xi = 0{,}15$$

$\eta \backslash \xi$	-2	1	0
-1	$0,1$	$0,4$	$0,05$
2	$0,1$	$0,15$	$0,2$
	$0,2$	$0,55$	$0,25$

$$E(\eta|\xi)$$

$$E(\eta|\xi=-2)=\frac{1}{P(\xi=-2)}\cdot\sum_{y_k}P(\eta=y_k,\xi=-2)=$$

$$=\frac{1}{0,2}\cdot(-1\cdot 0,1+2\cdot 0,1)=\frac{1}{2}$$

$$E(\eta|\xi=1)=\frac{1}{P(\xi=1)}\cdot\sum_{y_k}P(\eta=y_k,\xi=1)=$$

$$=\frac{1}{0,55}\cdot(-1\cdot 0,4+2\cdot 0,15)=-\frac{10}{55}=-\frac{2}{11}$$

$$E(\eta|\xi=0)=\frac{1}{P(\xi=0)}\sum_{y_k}P(\eta=y_k,\xi=0)=$$

$$=\frac{1}{0,25}\cdot(-1\cdot 0,05+2\cdot 0,2)=\frac{35}{25}=\frac{7}{5}$$

$$E(\eta|\xi)\sim\begin{pmatrix}1/2 & -2/11 & 7/5\\ 0,2 & 0,55 & 0,25\end{pmatrix}$$

$$E(E(\eta|\xi))=0,1-0,1+0,35=0,35$$

$$E(\xi \mid \eta = y_k) = \underbrace{\frac{1}{P(\eta = y_k)} \cdot \sum_{x_j} x_j P(\xi = x_j, \eta = y_k)}_{P(\eta = y_k)}$$

$$E(\xi \mid \eta) \sim \begin{pmatrix} \ldots & \frac{1}{P(\eta = y_k)} \cdot \sum_{x_j} x_j P(\xi = x_j, \eta = y_k) & \ldots \\ \ldots & P(\eta = y_k) & \ldots \end{pmatrix}$$

$$E(E(\xi \mid \eta)) = \sum_{y_k} \cancel{P(\eta = y_k)} \cdot \cancel{\frac{1}{P(\eta = y_k)}} \sum_{x_j} x_j P(\xi = x_j, \eta = y_k)$$

$$= \sum_{x_j} \sum_{y_k} x_j P(\xi = x_j, \eta = y_k) = \sum_{x_j} x_j \cdot P(\xi = x_j) =$$

$$= E\xi$$

ξ = the number of sixes
η = the number of fives (out of n rolls)

$\rho_{\xi,\eta}$

$p=\frac{1}{6}$, $q=\frac{5}{6}$, n trials

$$E\xi = np = n/6, \quad Var\,\xi = npq = \frac{5n}{36}$$

$n/6 = E\eta$, $\quad \frac{5n}{36} = Var\,\eta$

$$E(\xi\eta|\xi) = \frac{n}{5}\xi - \frac{1}{5}\xi^2$$

$$E(\xi\eta|\xi=k) = k\,E(\eta|\xi=k) = k\cdot\frac{n-k}{5}$$

$$E(\xi\eta|\xi) = \xi\,E(\eta|\xi) = \xi\cdot\frac{n-\xi}{5}$$

$$E(\xi\eta) = E(E(\xi\eta|\xi)) = E\left(\frac{n}{5}\xi - \frac{1}{5}\xi^2\right) =$$
$$= \frac{n}{5}E\xi - \frac{1}{5}E\xi^2 = \frac{n}{5}E\xi - \frac{1}{5}\left(Var\,\xi + (E\xi)^2\right) =$$
$$= \frac{n}{5}\cdot\frac{n}{6} - \frac{1}{5}\left(\frac{5n}{36} + \frac{n^2}{36}\right) = \frac{n^2}{30} - \frac{n}{36} - \frac{n^2}{180} = \frac{n^2-n}{36}$$

$$Cov(\xi,\eta) = E(\xi\eta) - E\xi\cdot E\eta = \frac{n^2-n}{36} - \frac{n}{6}\cdot\frac{n}{6} =$$
$$= -\frac{n}{36}$$

$$\rho_{\xi,\eta} = \frac{Cov(\xi,\eta)}{\sqrt{Var\,\xi\cdot Var\,\eta}} = \frac{-n/36}{5n/36} = -\frac{1}{5}$$

$E(\xi\eta) \neq \xi \cdot E\eta$

$E(f(\xi)\eta \mid \xi) = f(\xi) \cdot E(\eta \mid \xi)$

S is a sum of all numbers rolled $\quad ES = m$

ξ is the result of the first roll

$E(S \mid \xi)$

$E(S \mid \xi = 1) = 1 + ES = 1 + m \qquad 1/6$

$E(S \mid \xi = 2) = 2 + m \qquad 1/6$

$E(S \mid \xi = 3) = 3 + m \qquad 1/6$

$E(S \mid \xi = 4) = 4 + m \qquad 1/6$

$E(S \mid \xi = 5) = 5 + m \qquad 1/6$

$E(S \mid \xi = 6) = 6 \qquad 1/6$

$$E(S \mid \xi) \sim \begin{pmatrix} 1+m & 2+m & 3+m & 4+m & 5+m & 6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

$$E(E(S \mid \xi)) = \frac{1}{6}(1+m) + \frac{1}{6}(2+m) + \frac{1}{6}(3+m) + \frac{1}{6}(4+m) + \frac{1}{6}(5+m) + 1$$

$\parallel$

$ES = m$

$$y_1 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$$

$m = ES = 21$

N = number of rolls $\Rightarrow N \sim Geom\left(\frac{1}{6}\right)$

$EN = \frac{1}{1/6} = 6$

$E(y_1 + \dots + y_N) = E(N-1)Ey_1 + 6 = 5 \cdot 3 + 6 = 21$

$y_1, y_2, \ldots, y_n, \ldots$ – independent, identically distributed

N – a r.v. with positive integer values, N is independent from $y_1, \ldots, y_n, \ldots$

$$S = \sum_{i=1}^{N} y_i$$

Ey_i, $Var\, y_i$, EN, $Var\, N$ exist

$$E(S|N) = \sum_{i=1}^{N} Ey_i = N \cdot Ey_1$$

$$ES = E(E(S|N)) = E(N\, Ey_1) = Ey_1 \cdot EN$$

$$E(S^2|N) = E\left(\sum_{i=1}^{N} y_i\right)^2 = E\left(\sum_{i=1}^{N} y_i^2 + \sum_{\substack{i \neq j \\ i,j \in [1;N]}} y_i y_j\right) =$$

$$= N\, Ey_1^2 + (N^2 - N)\, E(y_1 y_2) =$$

$$= N\left(Var\, y_1 + (Ey_1)^2\right) + (N^2 - N) \cdot (Ey_1)^2 = N\, Var\, y_1 +$$

$$+ N^2 \cdot (Ey_1)^2$$

$$ES^2 = E(E(S^2|N)) = Var\, y_1 \cdot EN + EN^2 \cdot (Ey_1)^2 =$$

$$= Var\, y_1 \cdot EN + (Var\, N + (EN)^2) \cdot (Ey_1)^2$$

$$Var\, S = ES^2 - (ES)^2 = Var\, y_1 \cdot EN + Var\, N \cdot (Ey_1)^2$$

$$y_i \equiv const = y$$

$$S = Ny \qquad Var\, S = y^2\, Var\, N$$

ξ is a random variable with non-negative integer values.

$$E\xi = \sum_{k=1}^{\infty} P(\xi \geq k) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P(\xi = n) =$$

$$m P(\xi = m)$$

$$\sum_{n=1}^{\infty} P(\xi = n), \sum_{n=2}^{\infty} P(\xi = n), \ldots,$$

$$\sum_{n=m}^{\infty} P(\xi = n)$$

$$= \sum_{m=1}^{\infty} m P(\xi = m) = E\xi$$

$$P(\xi = 1) + P(\xi = 2) + P(\xi = 3) + \ldots$$

$$P(\xi = 2) + P(\xi = 3) + \ldots$$

$$P(\xi = 3) + \ldots$$

$p = 0,96$ 25 bullets

$$E\xi = \sum_{k=1}^{\infty} P(\xi \geq k) = \sum_{k=0}^{24} 0,96^k = \frac{1-0,96^{25}}{1-0,96}$$

$$P(\xi \geq 1) = 1$$

$$P(\xi \geq 2) = 0,96$$

$$P(\xi \geq 3) = 0,96^2$$

$$P(\xi \geq 4) = 0,96^3$$

$$P(\xi \geq 25) = 0,96^{24}$$

$$P(\xi \geq 26) = 0$$

$$\sum_{k=1}^{\infty} kP(\xi = k)$$

$\varepsilon > 0$

$$E|\xi| \geq E(|\xi| \cdot I_{|\xi| \geq \varepsilon}) \geq$$

$$\geq E(\varepsilon \cdot I_{|\xi| \geq \varepsilon}) = \varepsilon E I_{|\xi| \geq \varepsilon} = \varepsilon P(|\xi| \geq \varepsilon)$$

$$\boxed{P(|\xi| \geq \varepsilon) \leq \frac{E|\xi|}{\varepsilon}}, \quad (\varepsilon > 0)$$ *Markov's inequality*

$\xi \to \xi^t$, $\varepsilon \to \varepsilon^t$

$(t > 0)$

$$\boxed{P(|\xi| \geq \varepsilon) \leq \frac{E|\xi|^t}{\varepsilon^t}}$$

$\xi \longrightarrow (\xi - E\xi)^2$ $\quad \varepsilon \longrightarrow \varepsilon^2$

$$P((\xi - E\xi)^2 \geq \varepsilon^2) \leq \frac{E(\xi - E\xi)^2}{\varepsilon^2}$$

$$\boxed{P(|\xi - E\xi| \geq \varepsilon) \leq \frac{Var\,\xi}{\varepsilon^2}}$$ *Chebyshev's inequality*

$$P(|\xi - E\xi| < 3\sqrt{Var\,\xi}) = 1 - P(|\xi - E\xi| \geq 3\sqrt{Var\,\xi})$$

$$\geq 1 - \frac{Var\,\xi}{(3\sqrt{Var\,\xi})^2} = \frac{8}{9}$$

$Var\,\xi = \sigma^2$

$$P(E\xi - 3\sigma < \xi < E\xi + 3\sigma) \geq \frac{8}{9}$$

If $\xi \sim N(\mu; \sigma^2)$ then $P \approx 0,9972$

$\xi \sim Exp(\lambda)$ $\quad P \approx 0,982$

$$\overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k \qquad X_1, X_2, \ldots, X_n \ \underline{i.i.d.}$$

$$EX_i = \mu, \quad Var\, X_i = \sigma^2$$

$$S^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X})^2 =$$

$$y_k = X_k - \mu$$

$$Var\, y_k = Var\, X_k = \sigma^2$$

$$E y_k = EX_k - \mu = 0$$

$$\overline{y} = \frac{1}{n}\sum_{k=1}^{n}(X_k - \mu) = \frac{1}{n}\left(\sum_{k=1}^{n} X_k - n\mu\right) = \overline{X} - \mu$$

$$= \frac{1}{n}\sum_{k=1}^{n}(y_k - \overline{y})^2 =$$

$$= \frac{1}{n}\sum_{k=1}^{n} y_k^2 - \frac{2}{n}\sum_{k=1}^{n} y_k \overline{y} + \frac{1}{n}\sum_{k=1}^{n}\overline{y}^2 =$$

$$= \frac{1}{n}\sum_{k=1}^{n} y_k^2 - \frac{2\overline{y}}{n}\cdot n\overline{y} + \frac{1}{n}\cdot n\overline{y}^2 = \frac{1}{n}\sum_{k=1}^{n} y_k^2 - \overline{y}^2$$

$$ES^2 = \frac{1}{n}\sum_{k=1}^{n} E y_k^2 - E\overline{y}^2 =$$

$$E(y_i y_j) = Ey_i\, Ey_j = 0$$

$$E y_k^2 = Var\, y_k + (Ey_k)^2 = \sigma^2$$

$$E\overline{y}^2 = \frac{1}{n^2}E\left(\sum_{k=1}^{n} y_k\right)^2 = \frac{1}{n^2}E\left(\sum_{k=1}^{n} y_k^2 + \sum_{\substack{i,j\in[1,n]\\ i\neq j}} y_i y_j\right)$$

$$= \frac{1}{n^2}(n\sigma^2 + 0) = \frac{\sigma^2}{n}$$

$$= \frac{1}{n}\cdot n\sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$