Lab. Week 6

K rolls

5-9 $\xi = \{\text{quantity of "3"}\}$

$\eta = \{\text{quantity of "1", "3", "5"}\}$

$\xi \sim Bin\left(K, \frac{1}{6}\right)$; $\eta \sim Bin\left(K; \frac{1}{2}\right)$

$E\xi = \frac{k}{6}$; $Var\,\xi = k \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5k}{36}$

$E\eta = \frac{k}{2}$; $Var\,\eta = k \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{k}{4}$

ξ "3"

$k-\xi$ others

$1\,2\,\cancel{3}\,4\,5\,6$

$\underbrace{\frac{2}{5}(k-\xi)+\xi}$

$E(\eta\xi|\xi) = \xi \cdot E(\eta|\xi) =$

$= \xi \cdot \left(\frac{2}{5}(k-\xi)+\xi\right) = \frac{2k}{5}\xi + \frac{3}{5}\xi^2$

$E(\eta\xi) = E\left(E(\eta\xi|\xi)\right) = \frac{2k}{5}E\xi + \frac{3}{5}E\xi^2 = \frac{2k}{5} \cdot \frac{k}{6} +$

$+ \frac{3}{5}\left(Var\,\xi + (E\xi)^2\right) = \frac{k^2}{15} + \frac{3}{5} \cdot \left(\frac{5k}{36} + \frac{k^2}{36}\right) = \frac{k^2}{15} + \frac{k}{12} +$

$+ \frac{k^2}{60} = \frac{k^2+k}{12}$

$Cov(\eta, \xi) = \frac{k^2+k}{12} - \frac{k}{6} \cdot \frac{k}{2} = \frac{k}{12}$

$\rho_{\eta,\xi} = \frac{k/12}{\sqrt{5k/36 \cdot k/4}} = \frac{1}{\sqrt{5}}$

5-5 (a) HH (b) TH

N is a number of flips required

$$\xi \sim \begin{pmatrix} T & HT & HH \\ 1/2 & 1/4 & 1/4 \end{pmatrix} \qquad EN = m$$

$$E(N|\xi) \qquad E(N|\xi) \sim \begin{pmatrix} 1+m & 2+m & 2 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$$

$$E(N|\xi = T) = 1 + m$$

$$E(N|\xi = HT) = 2 + m$$

$$E(N|\xi = HH) = 2$$

$$m = EN = E(E(N|\xi)) =$$

$$= (1+m)\frac{1}{2} + (2+m)\frac{1}{4} + 2\cdot\frac{1}{4}$$

$$m = 6$$

$$\xi \sim \begin{pmatrix} H & TH & TTH & TTTH, & \dots \\ 1/2 & 1/4 & 1/8 & 1/16, & \dots \end{pmatrix} \qquad EM = m$$

$$E(M|\xi = H) = 1 + m$$

$$E(M|\xi = TH) = 2$$

$$E(M|\xi = TTH) = 3$$

$$E(M|\xi = TTTH) = 4$$

$$E(M|\xi) \sim \begin{pmatrix} 1+m & 2 & 3 & 4 & \dots \\ 1/2 & 1/4 & 1/8 & 1/16 & \dots \end{pmatrix}$$

$$m = EM = E(E(M|\xi)) =$$

$$= \frac{1+m}{2} + \sum_{k=2}^{\infty} \frac{k}{2^k}$$

$$m = 1 + \sum_{k=2}^{\infty} \frac{k}{2^{k-1}} = 4$$

$|q|<1$ $$\sum_{k=2}^{\infty} q^k = q^2 \cdot \frac{1}{1-q} = \frac{q^2}{1-q}$$

$$\sum_{k=2}^{\infty} k q^{k-1} = \frac{2q(1-q)+q^2}{(1-q)^2} = \frac{2q-q^2}{(1-q)^2}$$

$$\sum_{k=2}^{\infty} \frac{k}{2^{k-1}} = \frac{2 \cdot 1/2 - 1/4}{(1-1/2)^2} = \frac{3/4}{1/4} = 3$$

3) $P(\text{sunny}) = 0,4$

$P(\text{cloudy}) = 0,6$

N = number of days

ξ = the weather on the 1st day

$\xi \sim \begin{pmatrix} S & C \\ 0,4 & 0,6 \end{pmatrix}$

$P(\text{sunny on the 1st day}) = 0,4$

$$1 + \frac{1}{0,6} = 1 + \frac{5}{3} = \frac{8}{3}$$

$P(\text{cloudy on the 1st day}) = 0,6$

$$1 + \frac{1}{0,4} = 3,5$$

$$E(N|\xi) \sim \begin{pmatrix} 8/3 & 3,5 \\ 0,4 & 0,6 \end{pmatrix}$$

$$EN = E(E(N|\xi)) = 0,4 \cdot \frac{8}{3} + 0,6 \cdot 3,5 = \frac{16}{15} + \frac{21}{10} =$$

$$= \frac{32+63}{30} = \frac{19}{6}.$$

6–10

$\xi \backslash \eta$	-1	0	1
-1	$3/24$	$2/24$	$7/24$
1	$8/24$	$4/24$	0

$Var(\eta \mid \xi = 1)$

$Var(\xi \mid \eta = 0)$

$\xi + \eta \sim ?$

$\xi\eta \sim ?$

$\begin{pmatrix} \xi\eta \\ \xi+\eta \end{pmatrix} \sim ?$

$$E(\eta \mid \xi = 1) = \sum_k y_k P(\eta = y_k \mid \xi = 1) =$$

$$= \frac{1}{P(\xi=1)} \sum_k y_k P(\eta = y_k, \xi = 1) =$$

$$= 2 \cdot \left(-1 \cdot \frac{8}{24} + 0 \cdot \frac{4}{24}\right) = -\frac{2}{3}$$

$$E(\eta^2 \mid \xi = 1) = \sum_k y_k^2 P(\eta = y_k \mid \xi = 1) =$$

$$= \frac{1}{P(\xi=1)} \cdot \sum_k y_k^2 P(\eta = y_k, \xi = 1) =$$

$$= 2 \cdot \left((-1)^2 \cdot \frac{8}{24} + 0^2 \cdot \frac{4}{24}\right) = \frac{2}{3}$$

$$Var(\eta \mid \xi = 1) = E(\eta^2 \mid \xi = 1) - \left(E(\eta \mid \xi = 1)\right)^2 =$$

$$= \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$$

(6–10)

$\xi \backslash \eta$	-1	0	1
-1	$3/24$	$2/24$	$7/24$
1	$8/24$	$4/24$	0

$Var(\eta \mid \xi = 1)$

$Var(\xi \mid \eta = 0)$

$\xi + \eta \sim ?$

$\xi\eta \sim ?$

$\begin{pmatrix} \xi\eta \\ \xi+\eta \end{pmatrix} \sim ?$

$$\xi+\eta \sim \begin{pmatrix} -2 & -1 & 0 & 1 \\ 3/24 & 2/24 & 15/24 & 4/24 \end{pmatrix}$$

$$\xi\eta \sim \begin{pmatrix} -1 & 0 & 1 \\ 15/24 & 6/24 & 3/24 \end{pmatrix}$$

$\xi\eta \backslash \xi+\eta$	-2	-1	0	1
-1	0	0	$15/24$	0
0	0	$2/24$	0	$4/24$
1	$3/24$	0	0	0

6-5 $\quad \varepsilon > 0, \quad 0 < p < \frac{1}{2}$

$$\xi \sim \begin{pmatrix} -\varepsilon & 0 & \varepsilon \\ p & 1-2p & p \end{pmatrix}$$

$$\underbrace{P(|\xi - E\xi| \geq \varepsilon) = \frac{Var\,\xi}{\varepsilon^2}}_{2p = \frac{2\varepsilon^2 p}{\varepsilon^2}}$$

$$E\xi = -\varepsilon p + \varepsilon p = 0$$

$$E\xi^2 = \varepsilon^2 p + \varepsilon^2 p = 2\varepsilon^2 p$$

$$Var\,\xi = 2\varepsilon^2 p$$

$$P(|\xi - E\xi| \geq \varepsilon) = P(|\xi| \geq \varepsilon) = p + p = 2p$$

12) $\varsigma \sim \begin{pmatrix} -1 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

$\xi = 1 - \varsigma^{1000}$, $\eta = 1 - \varsigma^{1001}$

$\xi \sim \begin{pmatrix} 0 & 1 \\ 2/3 & 1/3 \end{pmatrix}$ $\qquad \eta \sim \begin{pmatrix} 0 & 1 & 2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

$\xi \backslash \eta$	0	1	2	
0	1/3	0	1/3	2/3
1	0	1/3	0	1/3
	1/3	1/3	1/3	

$P(\xi = 1, \eta = 2) \neq$
$\neq P(\xi = 1)\,P(\eta = 2)$
$0 \neq \frac{1}{3} \cdot \frac{1}{3}$
$\Rightarrow$ dependent

$E\xi = \frac{1}{3}$, $E\eta = 1$

$E(\xi\eta) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}$

$Cov(\xi, \eta) = E(\xi\eta) - E\xi \cdot E\eta = 0$