BS19-F20-ProbStat Homework 6

Brave people (@)

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Question 1.

Question 2.

Question 3.

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a) $\xi \sim N(\mu; \sigma^2)$
$P(|\xi - E\xi| \leq 3\sqrt{Var\xi})$ - ?

$$P(|\xi - E\xi| \leq 3\sqrt{Var\xi}) = P(|\xi - \mu| \leq 3\sigma) = P(\mu - 3\sigma \leq \xi \leq \mu + 3\sigma) =$$
$$= \int_{\mu-3\sigma}^{\mu+3\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 2\int_{0}^{3} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 2\Phi_0(3) \approx 0.9972$$

Answer: $2\Phi_0(3) \approx 0.9972$

b) $\xi \sim Exp(\lambda)$
$P(|\xi - E\xi| \leq 3\sqrt{Var\xi})$ - ?

$$P(|\xi - E\xi| \leq 3\sqrt{Var\xi}) = P(|\xi - \frac{1}{\lambda}| \leq \frac{3}{\lambda}) = P(-\frac{2}{\lambda} \leq \xi \leq \frac{4}{\lambda}) =$$
$$= \int_{0}^{\frac{4}{\lambda}} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_{0}^{\frac{4}{\lambda}} = 1 - e^{-4}$$

Answer: $1 - e^{-4}$

c) $\xi \sim U(a; b)$
$P(|\xi - E\xi| \leq 3\sqrt{Var\xi})$ - ?
$P(|\xi - E\xi| \leq 3\sqrt{Var\xi}) = P(|\xi - \frac{a+b}{2}| \leq \frac{(b-a)\sqrt{3}}{2}) = P(\frac{a(\sqrt{3}+1)+b(1-\sqrt{3})}{2} \leq \xi \leq \frac{b(\sqrt{3}+1)+a(1-\sqrt{3})}{2})$
Since $\frac{a(\sqrt{3}+1)+b(1-\sqrt{3})}{2} \leq \frac{a(\sqrt{3}+1)+a(1-\sqrt{3})}{2} = a$ and $\frac{b(\sqrt{3}+1)+a(1-\sqrt{3})}{2} \geq \frac{b(\sqrt{3}+1)+b(1-\sqrt{3})}{2} = b$,
$P(\frac{a(\sqrt{3}+1)+b(1-\sqrt{3})}{2} \leq \xi \leq \frac{b(\sqrt{3}+1)+a(1-\sqrt{3})}{2}) = 1$
Answer: 1

Question 4.

ξ - water consumption per day
$E\xi = 50000$
$P(\xi \leq 3E\xi)$ - ?
Since $\xi \geq 0$, $|\xi| = \xi$
Let's use Markov's inequality:
$P(\xi \geq 3E\xi) \leq \frac{E\xi}{3E\xi} = \frac{1}{3} \Leftrightarrow P(\xi > 3E\xi) \leq \frac{1}{3} \Leftrightarrow P(\xi \leq 3E\xi) = 1 - P(\xi > 3E\xi) \geq \frac{2}{3}$
Answer: $P(\xi \leq 3E\xi) \geq \frac{2}{3}$

Question 5.

$\xi > 0$, $0 < p < \frac{1}{2}, \xi \sim \begin{pmatrix} -\epsilon & 0 & \epsilon \\ p & 1-2p & p \end{pmatrix}$
$E\xi = 0$
$E\xi^2 = 2\epsilon^2 p \Rightarrow Var\xi = 2\epsilon^2 p$ $P(|\xi - E\xi| \geq \epsilon) = P(|\xi| \geq \epsilon) = P(|\xi| = \epsilon) = 2p = \frac{Var\xi}{\epsilon^2} = \frac{2\epsilon^2 p}{\epsilon^2} = 2p$

Question 6.

Question 7.

Question 8.

Question 9.

Solution.

First, let us assume that the very first floor is a ground one, and the floors in the building are enumerated as $0, 1, 2, \ldots, 8$. So after entering a lift, a person may arrive to the floors $1-8$ which are actually $2-9$. This is done for the simplicity.

Let us evaluate

$$\mathrm{P}(\xi \geq 1) = 1,\ \mathrm{P}(\xi \geq 2) = (\frac{7}{8})^{10},\ \mathrm{P}(\xi \geq 3) = (\frac{6}{8})^{10},\ \ldots,\ \mathrm{P}(\xi \geq 8) = (\frac{1}{8})^{10}$$

and

$$\mathrm{P}(\eta \geq 8) = 1 - \mathrm{P}(\eta < 8) = 1 - (\frac{7}{8})^{10},\ \mathrm{P}(\eta \geq 7) = 1 - \mathrm{P}(\eta < 7) = 1 - (\frac{6}{8})^{10},$$

$$\mathrm{P}(\eta \geq 6) = 1 - \mathrm{P}(\eta < 6) = 1 - (\frac{5}{8})^{10},\ \ldots,\ \mathrm{P}(\eta \geq 1) = 1$$

Then

$$\mathrm{E}(\xi) = \sum_{k=1}^{8} \mathrm{P}(\xi \geq k) \approx 1.3295,\ \mathrm{E}(\eta) = \sum_{k=1}^{8} \mathrm{P}(\eta \geq k) \approx 7.6705$$

Question 10.

Solution.

a). Consider $\mathrm{Var}(\eta|\xi = 1)$ first

$$\mathrm{E}(\eta|\xi = 1) = \frac{1}{\frac{1}{3} + \frac{1}{6} + 0}(-1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} + 1 \cdot 0) = -\frac{2}{3}$$

$$\mathrm{E}(\eta^2|\xi = 1) = \frac{1}{\frac{1}{3} + \frac{1}{6} + 0}((-1)^2 \cdot \frac{1}{3} + 0^2 \cdot \frac{1}{6} + 1^2 \cdot 0) = \frac{2}{3}$$

So, $\mathrm{Var}(\eta|\xi = 1) = \mathrm{E}(\eta^2|\xi = 1) - \mathrm{E}(\eta|\xi = 1)^2 = \frac{2}{3} - (-\frac{2}{3})^2 = \frac{2}{9}$.

In the same manner we can calculate $\mathrm{Var}(\eta|\xi = 1) = \frac{8}{9}$.

b). Here we are asked to find just the distribution of $\xi + \eta$

$$\xi + \eta \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{8} & \frac{1}{12} & \frac{1}{3} + \frac{7}{24} & \frac{1}{6} & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{3}{24} & \frac{2}{24} & \frac{15}{24} & \frac{4}{24} & 0 \end{pmatrix}$$

c). Here we are asked to find just the distribution of $\xi \cdot \eta$

$$\xi \cdot \eta \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{3} + \frac{7}{24} & \frac{1}{12} + \frac{1}{6} & \frac{1}{8} + 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ \frac{15}{24} & \frac{6}{24} & \frac{3}{24} \end{pmatrix}$$

d). `TODO`

Question 11.

Solution.

Let $p = 0,8$ be the probability of the marksman hitting the target.

Let δ be the amount of shots until the marksman hits and misses at least once.

Let φ be the indicator random variable that shows it the first shot hit the target.

Let $H \sim \text{Geo}(p)$, $M \sim \text{Geo}(1-p)$ be the amount of misses and hits before a hit and a miss respectively

$$\begin{cases} E(\delta \mid \varphi = 1) = 1 + EM \\ E(\delta \mid \varphi = 0) = 1 + EH \end{cases} \implies E(\delta \mid \varphi) \sim \begin{pmatrix} 1+EM & 1+EH \\ P & 1-p \end{pmatrix}$$

where $EM = \frac{1}{1-p} = 5$, and $EH = \frac{1}{p} = \frac{5}{4}$.

$E\delta = E(E(\delta|\varphi)) = p(1+5) + (1-p)(1+\frac{5}{4}) = 5.25$

Question 12.

Question 13.

Question 14.

Question 15.

$$\mu = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix},\ K = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 35 \end{bmatrix},\ \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$
$$\mu = \begin{bmatrix} E\xi_1 \\ E\xi_2 \\ E\xi_3 \end{bmatrix},\ K = \begin{bmatrix} Cov(\xi_1,\xi_1) & Cov(\xi_1,\xi_2) & Cov(\xi_1,\xi_3) \\ Cov(\xi_2,\xi_1) & Cov(\xi_2,\xi_2) & Cov(\xi_2,\xi_3) \\ Cov(\xi_3,\xi_1) & Cov(\xi_3,\xi_2) & Cov(\xi_3,\xi_3) \end{bmatrix}$$

a)

$$\begin{aligned} \eta_a &= \xi_1 - \xi_3 \\ E\eta_a &= E(\xi_1 - \xi_3) = E\xi_1 - E\xi_3 = 0 - 1 = 1 \\ Var\eta_a &= Var(\xi_1 - \xi_3) = Var\xi_1 - 2Cov(\xi_1,\xi_3) + Var\xi_3 = 5 + 2 + 35 = 42 \end{aligned}$$

b)

$$\begin{aligned} \eta_b &= 2\xi_1 - \xi_2 + 3\xi_3 \\ E\eta_b &= E(2\xi_1 - \xi_2 + 3\xi_3) = 2E\xi_1 - E\xi_2 + 3E\xi_3 = 0 + 3 + 3 = 6 \\ Cov((A \pm B), C) &= E((A \pm B)C) - E(A \pm B) \cdot EC = E(AC) \pm E(BC) - EA \cdot EC \mp EB \cdot EC = \\ &= Cov(A, C) \pm Cov(B, C) \\ Var(A \pm B \pm C) &= Var(A \pm B) \pm 2Cov((A \pm B), C) + VarC = \\ &= VarA + VarB + VarC \pm 2Cov(A, B) \pm 2Cov(A, C) \pm (\pm 1) \cdot 2Cov(B, C) \\ Var\eta_b &= Var(2\xi_1) + Var\xi_2 + Var(3\xi_3) - 2Cov(2\xi_1, \xi_2) + 2Cov(2\xi_1, 3\xi_3) - 2Cov(\xi_2, 3\xi_3) = \\ &= 2^2 \cdot 5 + 1 + 3^2 \cdot 35 - 2 \cdot 2 \cdot (-2) + 2 \cdot 2 \cdot 3 \cdot (-1) - 2 \cdot 3 \cdot 3 = 314 \end{aligned}$$

c)

$$\begin{aligned} \eta_c &= -2\xi_1 + 3\xi_2 - \xi_3 \\ E\eta_c &= E(-2\xi_1 + 3\xi_2 - \xi_3) = -2E\xi_1 + 3E\xi_2 - E\xi_3 = -2 \cdot 0 - 3 \cdot 3 - 1 = -10 \\ Var\eta_c &= Var(3\xi_2 - 2\xi_1 - \xi_3) = Var(3\xi_2) + Var(2\xi_1) - Var(\xi_3) - \\ &- 2Cov(3\xi_2, 2\xi_1) - 2Cov(3\xi_2, \xi_3) + 2Cov(2\xi_1, \xi_3) = \\ &= 3^2 \cdot 1 + 2^2 \cdot 5 + 35 - 3 \cdot 2 \cdot 2 \cdot (-2) - 3 \cdot 2 \cdot 3 + 2 \cdot 2 \cdot (-1) = 66 \end{aligned}$$

Question 16.

$$K = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & \lambda \\ 1 & \lambda & 2 \end{bmatrix},\ \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix},\ l_1 = \lambda\xi_1 + 2\xi_2 - \xi_3,\ argmin_\lambda Varl_1 - ?$$

$$Varl_1 = Var(\lambda\xi_1) + Var(2\xi_2) + Var(\xi_3) + 2Cov(\lambda\xi_1, 2\xi_2) - 2Cov(\lambda\xi_1, \xi_3) - 2Cov(2\xi_2, \xi_3) =$$
$$= \lambda^2 \cdot 1 + 2^2 \cdot 3 + 2 + 2 \cdot 2 \cdot \lambda \cdot (-1) - 2 \cdot \lambda \cdot 1 - 2 \cdot 2 \cdot \lambda = \lambda^2 - 10\lambda + 14$$

$Since\ Varl_1\ must\ be\ non-negative,\ we\ only\ consider\ \lambda : \lambda^2 - 10\lambda + 14 \geq 0 \Rightarrow argmin_\lambda Varl_1 = 5 \pm \sqrt{11}$

Question 17.

$\xi_1, \xi_2, \ldots, \xi_{100}$ are independent and identically distributed, $\xi_i \sim Exp(\lambda)$

$$\text{Let } \eta_1 = \sum_{k=1}^{100} \xi_k, \quad \eta_2 = \sum_{k=1}^{30} \xi_k$$

$$E\eta_1 = E\left(\sum_{k=1}^{100} \xi_k\right) = 100E\xi_1 = \frac{100}{\lambda}, \quad E\eta_2 = \frac{30}{\lambda} \Rightarrow \mu = \begin{bmatrix} \frac{100}{\lambda} \\ \frac{30}{\lambda} \end{bmatrix}$$

$$\operatorname{Var}\eta_1 = \operatorname{Var}\left(\sum_{k=1}^{100} \xi_k\right) = \operatorname{Var}\xi_1 \cdot \underbrace{E(100)}_{100} + E(\xi_1)^2 \cdot \underbrace{\operatorname{Var}(100)}_{0} = 100 \operatorname{Var}\xi_1 = \frac{100}{\lambda^2}$$

$$\operatorname{Var}\eta_2 = \frac{30}{\lambda^2}$$

$$\operatorname{Cov}(\eta_1, \eta_2) = E(\eta_1\eta_2) - E\eta_1 \cdot E\eta_2 = E\left(\sum_{k=1}^{100}\sum_{l=1}^{30} \xi_k\xi_l\right) - \frac{3000}{\lambda^2} = \sum_{k=1}^{100}\sum_{l=1}^{30} E(\xi_k\xi_l) - \frac{3000}{\lambda^2}$$
$$= \sum_{k=1}^{100}\sum_{l=1}^{30} E_{\xi_k}E_{\xi_l} - \frac{3000}{\lambda^2} = \frac{3000}{\lambda^2} - \frac{3000}{\lambda^2} = 0 = \operatorname{Cov}(\eta_2, \eta_1)$$

$$k = \begin{bmatrix} \frac{100}{\lambda^2} & 0 \\ 0 & \frac{30}{\lambda^2} \end{bmatrix}$$

Question 18.

Let ξ be the amount of letters (out of N) that have reached their destination.

Let ξ_i be the indicator that the i-th letter has reached its destination.

$$\xi = \sum_{i=1}^{N} \xi_i \Rightarrow \operatorname{Var}\xi = \operatorname{Var}\left(\sum_{i=1}^{N} \xi_i\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} \operatorname{Cov}(\xi_i, \xi_j)$$

$$\operatorname{Cov}(\xi_i, \xi_j) = E(\xi_i\xi_j) - E\xi_i E\xi_j = P(\xi_i = 1, \xi_j = 1) - P(\xi_i = 1)P(\xi_j = 1)$$

$$\left.\begin{array}{r} P(\xi_i = 1) = \frac{(N-1)!}{N!} = \frac{1}{N}, \qquad \forall i \in [1; N] \\ P(\xi_i = 1, \xi_j = 1) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)}, \forall i, j \in [1; N], i \neq j \end{array}\right\} \Rightarrow \operatorname{Cov}(\xi_i, \xi_j) = \begin{cases} \frac{N-1}{N^2}, & i = j \\ \frac{1}{N^2(N-1)}, & i \neq j \end{cases}$$

$$\operatorname{Var}\xi = \sum_{i=1}^{N}\left(\frac{N-1}{N^2} + \sum_{\substack{j=1 \\ i\neq j}}^{N} \frac{1}{N^2(N-1)}\right) = \sum_{i=1}^{N}\left(\frac{N-1}{N^2} + \frac{1}{N^2}\right) = \frac{N}{N} = 1$$

Question 19.

Let ξ be the sum of the ten random numbers from the set $\{1, 2, \dots, 100\}$

Let ξ_i i be the indicator that the number i was taken.

$$E\xi_i = P(\xi_i = 1) = \frac{10 \cdot A_{99}^{9}}{A_{100}^{10}} = \frac{10 \cdot 99! \cdot 90!}{100! \cdot 90!} = \frac{1}{10}, \text{ where } A_n^k \text{ is an arrangement of } k \text{ from } n.$$

$$E(\xi_i\xi_j) = P(\xi_i = 1, \xi_j = 1) = \frac{A_{10}^{2} \cdot A_{98}^{8}}{A_{100}^{10}} = \frac{10 \cdot 9 \cdot 98! \cdot 90!}{100! \cdot 90!} = \frac{90}{9900} = \frac{1}{110}, \quad i \neq j$$

$$\xi = \sum_{i=1}^{100} i \cdot \xi_i \Rightarrow E\xi = E\left(\sum_{i=1}^{100} i \cdot \xi_i\right) = \sum_{i=1}^{100} i \cdot E\xi_i = \frac{1}{10}\sum_{i=1}^{100} i = \frac{100 \cdot 101}{2 \cdot 10} = 505$$

$$\Rightarrow \operatorname{Var}\xi = \operatorname{Var}\left(\sum_{i=1}^{100} i \cdot \xi_i\right) = \sum_{i=1}^{100}\sum_{j=1}^{100} \operatorname{Cov}(i \cdot \xi_i, j \cdot \xi_j) = \sum_{i=1}^{100}\sum_{j=1}^{100} i \cdot j \cdot \operatorname{Cov}(\xi_i, \xi_j)$$

$$= \sum_{i=1}^{100} i \cdot \sum_{j=1}^{100} j \cdot (E(\xi_i\xi_j) - E\xi_i \cdot E\xi_j) = \sum_{i=1}^{100} i \cdot \left(i \cdot \left(E\xi_i - E\xi_i^2\right) + \sum_{j=1}^{100} j \cdot (E(\xi_i\xi_j) - E\xi_i \cdot E\xi_j)\right)$$

$$= \sum_{i=1}^{100} i \cdot \left(i \cdot \left(\frac{1}{10} - \frac{1}{100}\right) + \sum_{j=1}^{100} j \cdot \left(\frac{1}{110} - \frac{1}{100}\right)\right) = \sum_{i=1}^{100} \frac{9}{100} i^2 + \sum_{i=1}^{100} i \cdot \left(-\frac{1}{1100}\right) \cdot \left(\sum_{j=1}^{100}(j) - i\right)$$

$$= \frac{9}{100} \cdot \sum_{i=1} i^2 + \left(-\frac{1}{1100}\right)\sum_{i=1}^{100} i \cdot (50 \cdot 101 - i) = \frac{100}{1100}\sum_{i=1}^{100} i^2 - \frac{5050}{1100} \cdot \sum_{i=1}^{100} i =$$

$$= \frac{100 \cdot 101 \cdot 201}{6 \cdot 11} - \frac{50^2 \cdot 101^2}{1100} = \frac{101}{11}(3350 - 2525) = 7575$$