Probability & Statistics. Assignment 9

1. It is given that $(\xi, \eta, \zeta)^T \sim N(\mu, K)$ where $\mu = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 6 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

   (a) Find probability density $f(x, y, z)$ of the given random vector as well as the marginal probability densities $f_1(x), f_2(y), f_3(z), f_{13}(x, z)$.

   (b) Calculate probabilities $P(2\xi - 3\eta - \zeta < 9)$ and $P(|2\eta - 5\zeta| < 16)$.

2. Let us consider a random vector from the previous task. Find probability densities and characteristic functions of the following:

   (a) $u = 2\xi - 3\eta - \zeta$;

   (b) $v = 2\eta - 5\zeta - 7$.

   Hint: the characteristic function of a normally distributed random variable is $\varphi(t) = \exp(\mu i t - \frac{1}{2}\sigma^2 t^2)$.

3. Probability density $f(x_1, x_2, x_3)$ of random vector $\xi = (\xi_1, \xi_2, \xi_3)^T$ is
   $f(x_1, x_2, x_3) = a \exp\{-\frac{1}{2}(x_1 + 3)^2 - (x_2 - 2)^2 - 3(x_3 + 1)^2 + (x_1 + 3)(x_2 - 2) + 2(x_1 + 3)(x_3 + 1) - 3(x_2 - 2)(x_3 + 1)\}$

   (a) Find constant a, mathematical expectation μ, covariance matrix K of random vector ξ;

   (b) Calculate probabilities $P(0 < \xi_1 + 2\xi_2 - 3\xi_3 < 8)$ and $P(-10 < 3\xi_1 + 2\xi_2 - \xi_3 < 2)$.

4. Probability density $f(x_1, x_2, x_3)$ of random vector $\xi = (\xi_1, \xi_2, \xi_3)^T$ is
   $f(x_1, x_2, x_3) = a \exp\{-2(x_1 + 1)^2 - 3(x_2 - 1)^2 - 9x_3^2 + 4(x_1 + 1)(x_2 - 1) + 2(x_2 - 1)x_3\}$.

   (a) Find constant a, mathematical expectation μ and covariance matrix K of random vector ξ;

   (b) Find the value of λ which satisfies the relation $P(2\xi_1 - \xi_2 - 3\xi_3 < \lambda) = \frac{1}{4}$.

5. Random vector $(\xi, \eta)^T \sim N(\mu, K)$ where $\mu = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $K = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$ is given.

   Calculate conditional probabilities: (a) $P(\eta < 3 | \xi = 0)$; (b) $P(\eta < 0 | \xi = 1)$; (c) $P(|\eta + 4| < 2 | \xi = 2)$.

   Hint: conditional pdf induces a random variable L from a normal distribution
   $L(\eta|\xi = a) \equiv N\left(\nu + \rho\frac{\sigma_\eta}{\sigma_\xi}(a - \mu), \sigma_\eta^2\left(1 - \rho^2\right)\right)$, where a is a fixed value.

6. Let us consider random vector $(\xi, \eta, \zeta)^T \sim N(\mu, K)$ where $\mu = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 6 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

   Calculate conditional probabilities: (a) $P(|\zeta| > 1 | \xi = 1)$; (b) $P(|\zeta + 3| < 1 | \eta = 6)$.

7. It is known that $\xi_1, \xi_2, \ldots \xi_{100}$ are *i.i.d.* random variables with $N(0, 1)$ distribution; $\eta_1 = \xi_1 + \xi_2 + \ldots \xi_{40}$, $\eta_2 = \xi_1 + \xi_2 + \ldots \xi_{100}$. Find the probability density function of vector $\eta = (\eta_1, \eta_2)^T$.

8. Let ξ and η be independent random variables with $N(0; 1)$ distribution. Find the probability that a point with coordinates $(\xi; \eta)$ is situated within figure:

   (a) $1 \le |x| + |y|$;

   (b) $1 \le |x| + |y| \le 2$;

   (c) $\sqrt{x^2 + y^2} \le 2$.

   Hint: tasks 10 and 11 from Assignment 7 could be helpful.

9. Let ξ_1 and ξ_2 be independent random variables with $N(0, 1)$ distribution. Find probabilities

   (a) $P(\xi_1 < 3\xi_2)$;

   (b) $P(|\xi_1| \le \sqrt{3}|\xi_2|)$.

10. Let us consider two random variables ξ and η with $N(0, 1)$ distribution. And it is known that their correlation coefficient is equal to ρ.

    Calculate the expected value of $\xi^3\eta^3$.

    Hint: random variables ξ and $\eta - \rho\xi$ are independent.

11. Let us consider a sequence of independent random variables $\xi_1, \xi_2, \ldots$ such that $\frac{Var\ \xi_n}{n} \to 0$, when $n \to \infty$. Prove that law of large numbers (LLN) holds.

    Hint: use Chebyshev's inequality.

12. We are given a sequence of independent random variables $\xi_1, \xi_2, \ldots$ where
    $\xi_n \sim \begin{pmatrix} -\sqrt{n} & 0 & \sqrt{n} \\ \frac{1}{2n} & 1 - \frac{1}{n} & \frac{1}{2n} \end{pmatrix}$ for any $n \in \mathbb{N}$.
    Determine if LLN holds.

13. Let us consider a sequence of independent random variables $\xi_1, \xi_2, \ldots$ where
    $\xi_n \sim \begin{pmatrix} -n & 0 & n \\ 2^{-n} & 1 - 2^{-n+1} & 2^{-n} \end{pmatrix}$ for any $n \in \mathbb{N}$.
    Determine if LLN holds.