Probability & Statistics. Assignment 3

1. Let us consider a circle of radius R centered at O. Point M is chosen at random inside this circle. Random variable ξ is equal to the length of OM. Find the expected value and variance of ξ.

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   Answer: $E\xi = \frac{2R}{3}$, $\operatorname{Var}\xi = \frac{R^2}{18}$.

2. Let us consider a sphere of radius R centered at O. Point M is chosen at random inside this circle. Random variable ξ is equal to the length of OM. Find the cumulative distribution function, probability density, expected value and variance of ξ.

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   Answer: $F_\xi(x) = \begin{cases} 0, & x \leqslant 0, \\ 1 - \frac{(R-x)^3}{R^3}, & 0 < x \leqslant R, \\ 1, & x > R; \end{cases}$ $f_\xi(x) = \begin{cases} \frac{3(R-x)^2}{R^3}, & 0 < x \leqslant R, \\ 0, & \text{otherwise}; \end{cases}$ $E\xi = \frac{R}{4}$, $\operatorname{Var}\xi = \frac{3R^2}{80}$.

3. Find all values of C such that function $F(x) = \begin{cases} 0, & x \leqslant 1, \\ 1 - \frac{C}{x}, & x > 1 \end{cases}$ can be a cumulative distribution function for some random variable ζ. Find the probability density and expected value of ζ.

   Answer: $C = 1$, $f(x) = \begin{cases} \frac{1}{x^2}, & x > 1, \\ 0, & x \leqslant 1; \end{cases}$ the expected value does not exist.

4. Is it possible that for some value of C the functions below are probability density functions of random variables? If it is so, indicate the values of C and find expected values and variances for these random variables.

   (a) $f(x) = \begin{cases} Ce^{-2x}, & x > 0, \\ 0, & x < 0; \end{cases}$; (b) $f(x) = Ce^{-|x|}$, $x \in \mathbb{R}$.

   Answer: (a) $C = 2$, $E\xi = 0.5$, $\operatorname{Var}\xi = 0.25$; (b) $C = 0.5$, $E\xi = 0$, $\operatorname{Var}\xi = 2$.

5. (a) Find such value of C that function $f(x) = \frac{C}{1+x^2}$ is probability density of a random variable.

   (b) Give an example of a random variable whose probability density function is even, but that does not meet the condition $E\xi = 0$.

   Answer: (a) $C = \pi$; (b) random variable from (a) satisfies this task.

6. Let ξ be uniformly distributed on $[0; 4]$. Calculate: (a) $P\{\xi < E\xi\}$; (b) $P\left\{\xi > \sqrt{D\xi}\right\}$; (c) $P\{-5 \leqslant \xi \leqslant 5\}$.

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   Answer: (a) 0.5; (b) $1 - \frac{1}{2\sqrt{3}}$; (c) 1.

7. Random variable Y has a uniform distribution on interval $(a; b)$, and $EY = \operatorname{Var}Y = 3$. Find a and b.

   Answer: $a = 0$, $b = 6$.

8. Random variable ξ is uniformly distributed on an interval, and $P\{\xi < 1\} = \frac{1}{2}$, $P\{\xi < 2\} = \frac{2}{3}$. Find the cumulative distribution function, probability density function, expected value and variance of ξ.

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   Answer: $F_\xi(x) = \begin{cases} 0, & x \leqslant -2, \\ \frac{x+2}{6}, & -2 < x \leqslant 4, \\ 1, & x > 4; \end{cases}$ $f_\xi(x) = \begin{cases} \frac{1}{6}, & -2 < x < 4, \\ 0, & \text{otherwise}; \end{cases}$ $E\xi = 1$; $\operatorname{Var}\xi = 3$.

9. What is the maximum value of variance of random variable Z uniformly distributed on an interval given that $F_Z(1) = \frac{1}{3}$, $F_Z(4) = 1$?

   Answer: $\frac{27}{16}$.

10. Random variable Z is uniformly distributed on an interval, and $P\{0 < Z < 1\} = \frac{2}{3}$, $P\{1 < Z < 2\} = \frac{1}{3}$. (a) What is the smallest possible value of EZ? (b) What is the largest possible value of $\operatorname{Var} Z$?

    Answer: (a) $\frac{3}{4}$, (b) $\frac{3}{16}$.

11. It is known that $X \sim U(-a; a)$. What is the distribution of $|X|$?

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    Answer: $U(0; a)$.

12. Random variable η is uniformly distributed on $[a; b]$. Find the distribution of $\xi = \frac{\eta - E\eta}{\sqrt{\operatorname{Var} \eta}}$.

    Answer: $U\left(-\sqrt{3}; \sqrt{3}\right)$.

13. Random variable ξ is uniformly distributed on $[-1; 5]$. Find $E\big((\xi - 1)(3 - \xi)\big)$.

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    Answer: -2.

14. Random variable θ is exponentially distributed with parameter λ. Calculate the probabilities that θ belongs to intervals $(0; 1), (1; 2), \ldots, (n-1; n), \ldots$ and show that these probabilities form a geometric sequence. What is the common ratio of this sequence?

    Answer: .

15. Is it possible that $P\{2 < Z < 3\} = \frac{4}{27}$ for an exponentially distributed random variable Z? If it is possible, specify the value of $E\xi$.

    Answer: $\frac{1}{\ln 1.5}$.

16. Calculate the probability $P\left\{|\xi - E\xi| < 3\sqrt{\operatorname{Var} \xi}\right\}$ for an exponentially distributed random variable ξ.

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    Answer: $1 - e^{-4} \approx 0.982$.

17. Let $\xi \sim \operatorname{Exp}(\lambda)$ and $\eta = e^{-\xi}$. Find $E\eta$ and $\operatorname{Var} \eta$.

    Answer: $E\eta = \frac{\lambda}{\lambda+1}$, $\operatorname{Var} \eta = \frac{\lambda}{(\lambda+1)^2(\lambda+2)}$.

18. Let ξ be exponentially distributed random variable and t and τ be positive numbers. Prove that $P\{\xi > t + \tau | \xi > t\} = P\{\xi > \tau\}$.

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19. Let ξ be a normally distributed random variable, and $E\xi = 1$, $\operatorname{Var} \xi = 4$. Find the probabilities that ξ belongs to the following intervals: (a) $(-3; 1)$; (b) $(-\infty; -2)$; (c) $3; +\infty)$.

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    Answer: (a) $\Phi_0(0) + \Phi_0(2) \approx 0.477$; (b) $0.5 - \Phi_0(1.5) \approx 0.067$; (c) $0.5 - \Phi_0(1) \approx 0.136$.

20. Let $\xi \sim N(-1; 1)$. Find the approximate values of x that satisfy the following equalities: (a) $P\{x < \xi < 1\} = 0.8$; (b) $P\{0 < \xi < x\} = 0.8$; (c) $P\{-1 - x < \xi < -1 + x\} = 0.8$.

    Answer: (a) $x \approx -1.92$; (b) the value of x does not exist; (c) $x \approx 1.28$.

21. Let $\xi \sim N(0; \sigma^2)$. Arrange the following probabilities in ascending order: $P\{-2 < \xi < 2\}$, $P\{-1 < \xi < 3\}$, $P\{0 < \xi < 4\}$, $P\{-1.5 < \xi < 2.5\}$.

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    Answer: $P\{0 < \xi < 4\} < P\{-1 < \xi < 3\} < P\{-1.5 < \xi < 2.5\} < P\{-2 < \xi < 2\}$.

22. It is known that ξ is normally distributed random variable, and $P\{|\xi - E\xi| < 1\} = 0.3$. Find the probability that $|\xi - E\xi| < 2$.

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    Answer: ≈ 0.56.

23. Random variable ξ is normally distributed, and $E\xi = 1$, $\operatorname{Var}\xi = 5$. Find the shortest interval $(a; b)$ such that $P\{a < \xi < b\} = 0.95$.

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    Answer: approximately $(-3.38; 5.38)$.

24. Find the expected value and variance of a normally distributed random variable ξ given that $P\{1 < \xi < 7\} = P\{7 < \xi < 13\} = 0.18$.

    Answer: $E\xi = 7$, $\operatorname{Var}\xi \approx 163$.

25. Let $\eta \sim N(1; \sigma^2)$ where $\sigma > 0$. Which value of σ yields maximum to $P\{2 < \xi < 4\}$.

    Answer: $\sigma = \frac{2}{\sqrt{\ln 3}}$.

26. Random variable ζ is normally distributed, and $E\zeta = -2$, $\operatorname{Var}\zeta = 9$. Find $E\big((3 - \zeta)(\zeta + 5)\big)$.

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    Answer: 6.

27. Let $\xi \sim N(\mu; \sigma^2)$, and $a \neq 0$ an arbitrary number. Find the distribution of $\eta = a\xi + b$.

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    Answer: $N(a\mu + b, a^2\sigma^2)$.

28. Let $\xi \sim N(0; \sigma^2)$. Find $E|\xi|$ and $\operatorname{Var}|\xi|$.

    Answer: $\sqrt{\frac{2}{\pi}}\sigma$, $\operatorname{Var}\left(1 - \frac{2}{\pi}\right)\sigma^2$.

29. Find the probability density function of random variable η if $\eta = \sin\xi$ and (a) $\xi \sim U\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$; (b) $\xi \sim U(0; \pi)$.

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    Answer: (a) $f(x) = \begin{cases} \frac{1}{\pi}, & |x| < \frac{\pi}{2}, \\ 0, & |x| > \frac{\pi}{2}; \end{cases}$; (b) $\begin{cases} \frac{1}{\pi\sqrt{1-x^2}}, & |x| < 1, \\ 0, & |x| \geqslant 1. \end{cases}$.

30. Find the positive value of λ that yields maximum to probability $P\{\lambda < \xi < 2\lambda\}$ given that ξ is a random variable with Cauchy distribution.

    Answer: $\lambda = \frac{1}{\sqrt{2}}$.

31. Prove that if random variable ξ has Cauchy distribution, then random variable $\eta = \frac{1}{\xi}$ also has Cauchy distribution.