Lecture 4, September 10, 2020

geometric distribution $\quad 0 < p < 1$

$\xi \sim G(p) \quad q = 1 - p$

$$\xi \sim \begin{pmatrix} 1 & 2 & 3 & \ldots & k & \ldots \\ p & qp & q^2p & \ldots & q^{k-1}p & \ldots \end{pmatrix}$$

$$p + qp + q^2p + \ldots = \frac{p}{1-q} = 1$$

$$E\xi = \sum_{k=1}^{\infty} k q^{k-1} p = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$\sum_{k=1}^{\infty} q^k = \frac{q}{1-q} \qquad -1 < q < 1$$

$$\checkmark \quad \sum_{k=1}^{\infty} k q^{k-1} = \frac{1-q+q}{(1-q)^2} = \frac{1}{(1-q)^2}$$

$$Var\,\xi = E\xi^2 - (E\xi)^2 = \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{q}{p^2}$$

$$E\xi^2 = \sum_{k=1}^{\infty} k^2 q^{k-1} p = \frac{1+q}{(1-q)^2} = \frac{1+q}{p^2} \qquad \sum_{k=1}^{\infty} k^2 q^{k-1} = \frac{q+1}{(1-q)^3}$$

$$\sum_{k=1}^{\infty} k(k-1) q^{k-2} = \frac{2}{(1-q)^3}$$

$$\checkmark \quad \sum_{k=1}^{\infty} (k^2-k) q^{k-1} = \frac{2q}{(1-q)^3}$$

$\xi \sim G(p)$

$P(\xi > k+n \mid \xi > n) \ominus$ $\qquad k, n \in \mathbb{N}$

$P(\xi > n) = P(\xi = n+1) + P(\xi = n+2) + \dots =$

$= pq^n + pq^{n+1} + pq^{n+2} + \dots = \frac{pq^n}{1-q} = q^n$

$\ominus \frac{P(\xi > k+n \ \& \ \xi > n)}{P(\xi > n)} = \frac{P(\xi > k+n)}{P(\xi > n)} =$

$= \frac{q^{k+n}}{q^n} = q^k = P(\xi > k)$

Independent random variables

X, Y

$\forall A, B \longrightarrow \quad P(X\in A, Y\in B) = P(X\in A)\cdot P(Y\in B)$.

$$P(X=a, Y=b) = P(X=a)\cdot P(Y=b) \quad \forall a,b\in\mathbb{R}$$

$$A = \{a_1, a_2, \dots, a_n\}$$

$$P(X=a_1, Y=b) = P(X=a_1)\cdot P(Y=b)$$

$$+ \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---}$$

$$P(X=a_n, Y=b) = P(X=a_n)\cdot P(Y=b)$$

$$\overline{P(X\in A, Y=b) = P(X\in A, Y=b)}$$

$$P(X_1\in A_1, X_2\in A_2, \dots, X_n\in A_n) =$$

$$= P(X_1\in A_1)\cdot P(X_2\in A_2)\cdot \ldots \cdot P(X_n\in A_n).$$

$\forall A_1, A_2, \dots, A_n$

$E(XY) = EX \cdot EY$ if X and Y are independent.

$$\sum_{i,j} x_i y_j P(X = x_i, Y = y_j) \stackrel{\text{indep.}}{=}$$

$$= \sum_{i,j} x_i y_j P(X = x_i) P(Y = y_j) =$$

$$= \sum_i x_i P(X = x_i) \cdot \sum_j y_j P(Y = y_j) = EX \cdot EY$$

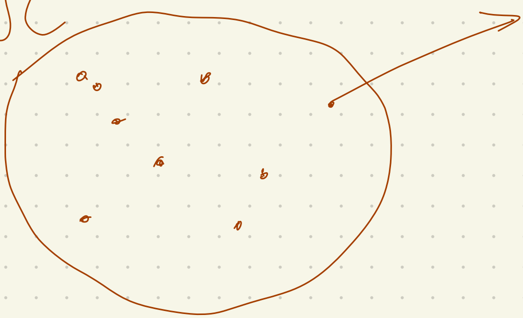
$X \backslash Y$	0	1
0	0,5	0,2
1	0,1	0,2

$P(X=0, Y=1) = 0,2$

$P(X=0) = 0,7$

$P(Y=1) = 0,4$

Ω



$X(\omega)$
$Y(\omega)$

$X(\omega) \cdot Y(\omega)$

$$\sum_{i=1}^{n} a_i \cdot \sum_{j=1}^{k} b_j = \sum_{\substack{i \in [1,n] \\ j \in [1,k]}} a_i b_j$$

$$(a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_k)$$

$$E(x^2)$$

$$Cov(X,Y) = E((X-EX)(Y-EY)) =$$ covariance

$$= E(XY - YEX - XEY + EXEY) =$$

$$= E(XY) - E(YEX) - E(XEY) + E(EXEY) =$$

$$= E(XY) - EXEY - \cancel{EXEY} + \cancel{EXEY}$$

If X and Y are independent then $Cov(X,Y)=0$.

$$Cov(X,X) = E((X-EX)^2) = Var\,X.$$

$$Var(X \pm Y) = E((X \pm Y)^2) - (E(X \pm Y))^2 =$$

$$= E(X^2 \pm 2XY + Y^2) - (EX \pm EY)^2 = EX^2 \pm$$

$$\pm 2E(XY) + EY^2 - (EX)^2 \mp 2EX\,EY - (EY)^2 =$$

$$= Var\,X + Var\,Y \pm 2\,Cov(X,Y)$$

If X, Y are independent then $Var(X \pm Y) =$

$$= Var\,X + Var\,Y$$

Probability generating functions

$g_X(t) = Et^X$ (probability generating function of random variable X)

X takes nonnegative integer values only

$$X \sim \begin{pmatrix} 0 & 1 & 3 & 6 \\ 1/7 & 1/7 & 2/7 & 3/7 \end{pmatrix}$$

$$t^X \sim \begin{pmatrix} 1 & t & t^3 & t^6 \\ 1/7 & 1/7 & 2/7 & 3/7 \end{pmatrix} \quad g_X(t) = \frac{1}{7} + \frac{1}{7}t + \frac{2}{7}t^3 + \frac{3}{7}t^6$$

polynomial / power series, the sum of all coefficients is equal to 1

$g_X'(t) = E(X t^{X-1})$ $\quad t=1 \Rightarrow g_X'(1) = EX$

$g_X''(t) = E(X(X-1)t^{X-2})$ $\quad t=1 \Rightarrow g_X''(1) = E(X^2 - X)$

X and Y are independent r.v.

$$g_{X+Y}(t) = Et^{X+Y} = E(t^X \cdot t^Y) = Et^X \cdot Et^Y = g_X(t) \cdot g_Y(t)$$

Poisson distribution

$X \sim Po(\lambda)$ if $X \in \mathbb{N} \cup \{0\}$, $\quad \lambda > 0$

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$$g_X(t) = Et^X = \sum_{k=0}^{\infty} t^k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} =$$

$$= e^{-\lambda} \cdot e^{t\lambda} = e^{\lambda(t-1)}$$

$$g_X'(t) = \lambda e^{\lambda(t-1)}, \qquad g_X''(t) = \lambda^2 e^{\lambda(t-1)}$$

$$g_X'(1) = \underbrace{\lambda = EX}, \qquad g_X''(1) = \underbrace{\lambda^2 = E(X^2 - X)}$$

$$\lambda^2 + \lambda = EX^2$$

$$\operatorname{Var} X = EX^2 - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$EX = \operatorname{Var} X = \lambda$$

$X \sim Po(\lambda)$, $Y \sim Po(\mu)$, X and Y are independent

$X + Y \sim ?$

$$g_{X+Y}(t) = g_X(t) \cdot g_Y(t) = e^{\lambda(t-1)} \cdot e^{\mu(t-1)} =$$

$$= e^{(\lambda+\mu)(t-1)} \Rightarrow X + Y \sim Po(\lambda + \mu)$$

Continuous distributions

$$\underbrace{F_X(x)}_{\parallel} = \int_{-\infty}^{x} \underbrace{f_X(t)}_{} dt, \quad x \in \mathbb{R}$$

$\rightarrow$ probability density

$P(X<x)$

$$F_X'(x) = f_X(x) \geq 0$$

$$P(a \leq X < b) = P(X<b) - P(X<a) = F_X(b) - F_X(a) =$$

$$= \int_{-\infty}^{b} f_X(t)\,dt - \int_{-\infty}^{a} f_X(t)\,dt = \int_{a}^{b} f_X(t)\,dt$$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

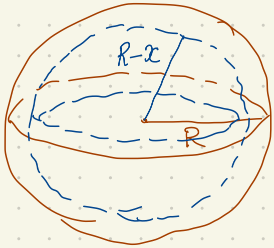
$$EX = \int_{-\infty}^{+\infty} x \underbrace{f_X(x)\,dx}_{}$$

$$E\,h(X) = \int_{-\infty}^{+\infty} h(x) f_X(x)\,dx$$

$$E(CX) = C\,EX$$

$$E(X \pm Y) = EX \pm EY$$

$$E(C) = C$$



a random point (M) is taken inside the ball. ξ is equal to the distance from M to the sphere.

$E\xi = ?$ $Var\,\xi = ?$

$$F_\xi(x) = P(\xi < x) = \begin{cases} 0, & x \le 0 \\ \frac{R^3 - (R-x)^3}{R^3} & 0 < x \le R \\ 1, & x > R \end{cases}$$

$$f_\xi(x) = \begin{cases} 0, & x \le 0 \\ \frac{3(R-x)^2}{R^3}, & 0 < x \le R \\ 0, & x > R \end{cases} \qquad f_\xi(x) = \frac{3(R-x)^2}{R^3} I_{0<x\le R}$$

$$E\xi = \int_{-\infty}^{+\infty} x\, f_\xi(x)dx = \int_{-\infty}^{+\infty} x \cdot \frac{3(R-x)^2}{R^3} I_{0<x\le R}\, dx =$$

$$= \int_0^R \frac{3}{R^3}(R^2x - 2Rx^2 + x^3)dx = \frac{3}{R^3}\left(\frac{R^2x^2}{2} - \frac{2Rx^3}{3} + \frac{x^4}{4}\right)\Big|_0^R =$$

$$= \frac{3}{R^3}\left(\frac{R^4}{2} - \frac{2R^4}{3} + \frac{R^4}{4}\right) = \frac{R}{4}$$

$$E\xi^2 = \int_{-\infty}^{+\infty} x^2 f_\xi(x)dx = \int_0^R \frac{3}{R^3}(R^2x^2 - 2Rx^3 + x^4)dx =$$

$$= \frac{3}{R^3}\left(\frac{R^2x^3}{3} - \frac{Rx^4}{2} + \frac{x^5}{5}\right)\Big|_0^R = \frac{3}{R^3}\left(\frac{R^5}{3} - \frac{R^5}{2} + \frac{R^5}{5}\right) =$$

$$= \frac{R^2}{10} \qquad Var\,\xi = \frac{R^2}{10} - \left(\frac{R}{4}\right)^2 = \frac{3R^2}{80}$$

$$I_A = \begin{cases} 1, & A \text{ has happened} \\ 0, & A \text{ has not happened} \end{cases}$$

$$\parallel$$

$$I(A)$$