Lab 2

27 августа 2020 г. 12:44

Formulae Solutions

1) $A_1, A_2, \dots, A_n$ is exhaustive $\Leftrightarrow A_1 + A_2 + \dots + A_n = \Omega$

2) $A_1, A_2, \dots, A_n$ are mutually exclusive $\Leftrightarrow i \neq j \Rightarrow A_i A_j = \emptyset$

$(A_1 \cup A_2 \cup \dots A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \dots \cup (A_n \cap B)$

B is some event

$P(B) = P(B\Omega) = P(B(A_1 + A_2 + \dots + A_n)) =$

$= P(BA_1 + BA_2 + \dots + BA_n) = P(BA_1) + \dots + P(BA_n)$

$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_n)P(A_n)$

Law of total probability — for unconditional probability

$$P(A_i|B) = \frac{P(A_i B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \dots + P(B|A_n)P(A_n)}$$

Bayes rule — for conditional probability; $(A_i|B)$ – how good evidence B is for premise A_i

Bernoulli's trials

p – a probability of success

$q = 1 - p$ – a probability of failure

n trials

M_n = number of successes in n trials

$$P(M_n = k) = \binom{n}{k} p^k q^{n-k}$$

$\underbrace{11\dots1}_{k}\underbrace{00\dots0}_{n-k}$ $\quad p \cdot \dots \cdot p \cdot \underbrace{q \cdot \dots \cdot q}_{n-k} = p^k \cdot q^{n-k}$

Bernoulli: shift (scheme)

each trial can have several outcomes

k diff. outcomes; their probabilities are $p_1, p_2, \dots, p_k$; $\sum_{i=1}^{k} p_i = 1$

n experiments

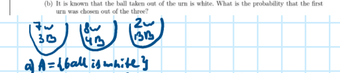
$P(n_1, n_2, n_3, \dots, n_k)$ = where $n_1 + n_2 + \dots + n_k = n$

$$= p_1^{n_1} p_2^{n_2} \dots p_k^{n_k} \frac{n!}{n_1! n_2! \dots n_k!}$$

$\underbrace{11\dots1}_{n_1}\underbrace{22\dots2}_{n_2}\dots\underbrace{kk\dots k}_{n_k}$ (n)

Probability & Statistics. Assignment 2

1. There are 7 white balls and 3 black balls in the first urn, 8 white balls and 4 black balls in the second urn, 2 white balls and 13 black balls in the third urn. One of the urns is chosen at random, and a random ball is taken out of it.
   (a) Determine the probability that the ball is white.
   (b) It is known that the ball taken out of the urn is white. What is the probability that the first urn was chosen out of the three?



a) $A = \{$ball is white$\}$

$P(A) = \frac{1}{3}\left(\frac{7}{10} + \frac{8}{12} + \frac{2}{15}\right) = \frac{42+40+8}{180} = \frac{1}{2}$

b) $A_1 = \{$ball is taken from urn 1$\}$

$B = \{$ball is white$\}$

$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B)} = \frac{\frac{1}{3} \cdot \frac{7}{10}}{\frac{1}{2}} = \frac{7}{15}$

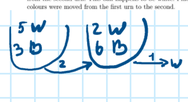
2. A hospital specializes in curing three types of diseases: A, B and C. On average, there are 50% of patients who suffer from disease A, 30% of patients with disease B, and 20% of patients with disease C (each of the patients has exactly one of these diseases). The probabilities to fully recover from the diseases are equal to 0.95, 0.9 and 0.85 respectively. A patient who came to the hospital recovered completely. What is the probability that he had disease B?

sick with	A	B	C
	0.5	0.3	0.2
recover from	0.95	0.9	0.85

$D = \{$recovered$\}$

$P(B|D) = \frac{P(D|B) \cdot P(B)}{P(D)} = \frac{0.3 \cdot 0.9}{0.5 \cdot 0.95 + 0.3 \cdot 0.9 + 0.2 \cdot 0.85} = \frac{0.27}{0.915} = \frac{270}{915} = \frac{18}{61}$

3. There are 5 white and 3 black balls in the first urn, 2 white and 6 black balls in the second urn. Two random balls are transferred from the first urn to the second, and after that some ball is taken from the second urn. This ball happens to be white. Find the probability that two balls of different colours were moved from the first urn to the second.



$B_2 = \{$ball from urn 2 is white$\}$

$A_1 = \{$2 black balls taken from urn 1$\}$: urn 2: 2 W, 8 B

$A_2 = \{$2 white ——— 11 ——— $\}$: 4 W, 6 B

$A_3 = \{$1 white & 1 black — « ——— $\}$: 3 W, 7 B

$$P(A_3|B_2) = \frac{P(B_2|A_3) \cdot P(A_3)}{P(B_2|A_1) \cdot P(A_1) + P(B_2|A_2) \cdot P(A_2) + P(B_2|A_3) \cdot P(A_3)} =$$

$$= \frac{\frac{3}{10} \cdot \left(\frac{5}{8} \cdot \frac{3}{7} \cdot 2\right)}{\frac{2}{10} \cdot \left(\frac{3}{8} \cdot \frac{2}{7}\right) + \frac{4}{10} \left(\frac{5}{8} \cdot \frac{4}{7}\right) + \frac{3}{10}\left(\frac{5}{8} \cdot \frac{3}{7} \cdot 2\right)} = \frac{90}{12 + 80 + 60} = \frac{90}{152} = \frac{45}{84}$$

(symmetry: take 1 black, take white ⇔ vice-versa)

4. The quantity of passenger cars that pass along the gas station is on average four times greater than that of trucks. The probability that a car refuels is equal to 0.05 for trucks and 0.15 for passenger cars. A car has just left the gas station. What is the probability that it is a truck?

type	car	truck
refuels	0,8	0,2
	0,15	0,05

$A = \{$car is a truck$\}$

$B = \{$car refuelled$\}$

$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{0.05 \cdot 0.2}{0.8 \cdot 0.15 + 0.2 \cdot 0.05} = \frac{0.01}{0.12 + 0.01} = \frac{1}{13}$

5. Three marksmen are shooting at a target. The probabilities for them to hit the target are equal to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. Each of the marksmen fires one shot, and the target is hit exactly two times. Find the probability that the third marksman has hit the target.

$A_1 = \{$III $A_i + \bar{A}_i\}$

$A_2 = \overline{A_1}$

$B = \{$target is hit 2 times$\}$

$$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2)} \quad (*)$$

$P(A_1) = \frac{1}{2}$ $\quad P(B|A_1) = \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{4}{24}$

$P(B|A_2) = \frac{1}{3} \cdot \frac{3}{4} = \frac{3}{12}$

$(*) \quad \frac{\frac{4}{24} \cdot \frac{1}{2}}{\frac{4}{24} \cdot \frac{1}{2} + \frac{3}{12} \cdot \frac{1}{2}} = \frac{14}{14+12} = \frac{14}{26}$

6. Three marksmen are shooting at a target. The probabilities for them to hit the target are equal to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. Each of the marksmen fires one shot each, and the target is destroyed. What is the probability that the third marksman has hit the target?

$A_1 = \{$III $A_i + \bar{A}_i\}$

$A_2 = \overline{A_1}$

$B = \{$target is destroyed$\}$

$$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} \quad (*)$$

$P(A_1) = \frac{1}{2}$ $\quad P(B|A_1) = 1 - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{19}{20}$

$P(A_2) = \frac{1}{2}$ $\quad P(B|A_2) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{12}{20}$

$(*) \quad \frac{\frac{19}{20} \cdot \frac{3}{4}}{\frac{19}{20} \cdot \frac{3}{4} + \frac{12}{20} \cdot \frac{1}{2}} = \frac{38}{38+12} = \frac{38}{50} = \frac{19}{25}$

7. A white ball is added into an urn that initially contained n balls. It is known that the probabilities of having $0, 1, 2, \dots, n$ white balls (at the start) in the urn are equal to each other. (a) One ball is taken at random from the urn. What is the probability that the ball is white? (b) The ball taken from the urn has turned out to be white. Find the most probable number of white balls that were in the urn from the start.

(a) $A = \{$the ball is white$\}$

$$\frac{1}{n+1}\left(\frac{1}{n+1} + \frac{2}{n+1} + \dots + \frac{n+1}{n+1}\right) = \frac{1}{n+1} \cdot \frac{(1+n)}{2} \cdot \frac{(n+1)}{(n+1)} = \frac{n+2}{2n+2}$$

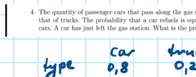
(b) A_i – urn contains exactly i white balls

$B = \{$white ball is taken$\}$

$$\frac{P(A_i|B)}{P(A_{i+1}|B)} = \frac{\frac{P(B|A_i)P(A_i)}{P(B)}}{\frac{P(B|A_{i+1})P(A_{i+1})}{P(B)}} = \frac{P(B|A_i)}{P(B|A_{i+1})} = \frac{\frac{i}{n+1}}{\frac{i+1}{n+1}} = \frac{i}{i+1}$$

$\Rightarrow$ n

8. There are 7 white balls and 3 black balls in the first urn; 8 white balls and 4 black balls in the second; 2 white and 13 black balls in the third. One of the urns is chosen at random and a ball is taken out of this urn. The ball has happened to be white. It is returned into the urn, and another ball is randomly chosen from the urn. What is the probability that this ball is also white?



$A_i = \{$ball taken from urn $i\}$

$B = \{$white ball taken from this urn$\}$

$C = \{$another ball from the same urn is white$\}$

$P(B) = P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + P(B|A_3) \cdot P(A_3)$

$= \frac{1}{3}\left(\frac{7}{10} + \frac{8}{12} + \frac{2}{15}\right) = \frac{1}{3} \cdot \frac{42+40+8}{60} = \frac{90}{180} = \frac{1}{2}$

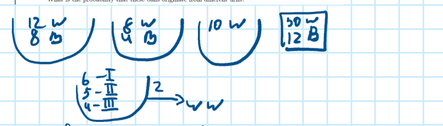
$P(A_1|B) = \frac{P(B|A_1) \cdot P(A_1)}{P(B)} = \frac{\frac{7}{10} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{14}{30} = \frac{21}{45}$

$P(A_2|B) = \frac{\frac{8}{12} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{4}{36} = \frac{4}{8} = \frac{20}{45}$

$P(A_3|B) = \frac{\frac{2}{15} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{4}{45}$

$= \frac{21}{45} \cdot \frac{2}{10} + \frac{20}{45} \cdot \frac{2}{3} + \frac{4}{45} \cdot \frac{2}{15} = \frac{42+120+16}{45 \cdot 30} = \frac{88}{1350}$

9. There were 12 white and 8 black balls in the first urn, 8 white and 4 black balls in the second urn, 10 white balls in the third urn. The fourth urn, that had been empty, was filled with the balls: 6 balls from the first urn, 3 balls from the second, and 4 balls from the third (all the balls were chosen at random). After that two balls were taken out of the fourth urn, and they happened to be white. What is the probability that these balls originate from different urns?



$A_{ij} = \{$one ball from urn i, another from urn $j\}$

$B = \{$2 balls are white$\}$

$P(A_{11}) = \frac{6}{15} \cdot \frac{5}{14} = \frac{1}{7}$ $\quad P(A_{12}) = \frac{6}{15} \cdot \frac{5}{14} \cdot 2 = \frac{2}{7}$

$P(A_{13}) = \frac{6}{15} \cdot \frac{4}{14} \cdot 2 = \frac{8}{35}$ $\quad P(A_{22}) = \frac{5}{15} \cdot \frac{4}{14} = \frac{2}{21}$

$P(A_{23}) = \frac{5}{15} \cdot \frac{4}{14} \cdot 2 = \frac{4}{21}$ $\quad P(A_{33}) = \frac{4}{15} \cdot \frac{3}{14} = \frac{2}{35}$

$P(B) = P(B|A_{11}) \cdot P(A_{11}) + \dots + P(B|A_{33})$

$= \frac{1}{7} \cdot \frac{12}{20} \cdot \frac{11}{19} + \frac{2}{7} \cdot \frac{12}{20} \cdot \frac{8}{12} + \frac{8}{35} \cdot \frac{12}{20} \cdot 1 + \frac{2}{21} \cdot \frac{8}{12} \cdot \frac{7}{11} +$

$+ \frac{4}{21} \cdot \frac{8}{12} \cdot 1 + \frac{2}{35} \cdot 1 = \frac{33}{665} + \frac{4}{35} + \frac{24}{175} + \frac{4}{99} + \frac{8}{63} + \frac{2}{35} =$

$= \frac{18009}{32917.5} \approx 0.5256$

$P(A_{13}|B) = \frac{P(B|A_{13})P(A_{13})}{P(B)} = \frac{\frac{8}{35} \cdot \frac{12}{20} \cdot 1}{P(B)} = \frac{\frac{24}{175}}{P(B)}$

$P(A_{23}|B) = \frac{\frac{4}{21} \cdot \frac{8}{12} \cdot 1}{P(B)} = \frac{\frac{8}{63}}{P(B)}$

$P(A_{12}|B) = \frac{\frac{2}{7} \cdot \frac{12}{20} \cdot \frac{8}{12}}{P(B)} = \frac{\frac{4}{35}}{P(B)}$

$P(A_{13} + A_{12} + A_{23}) = \left(\frac{24}{175} + \frac{8}{63} + \frac{4}{35}\right) : \frac{32917.5}{173009} \approx 0.72$

10. Five people entered the lift at the ground floor of a nine-storey building. Find the probability that
    (a) none of them went out on the fifth floor;
    (b) (exactly) one of them went out at the fifth floor;
    (c) (exactly) three of them went out at the fifth floor.

(a) $P_1 = \frac{7}{8} = P_2 = P_3 = P_4$

$P(M_5 = 5) = \binom{5}{5} P^5 (1-P)^{8-5} = 1 \cdot \left(\frac{7}{8}\right)^5 \left(\frac{1}{8}\right)^0 = 0,513$

(b) $P(M_5 = 1) = \binom{5}{1}\left(\frac{1}{8}\right)^1 \left(\frac{7}{8}\right)^4 = \frac{5 \cdot 7^4}{8^5} \approx 0,366$

(c) $P(M_5 = 3) = \binom{5}{3}\left(\frac{1}{8}\right)^3 \left(\frac{7}{8}\right)^2 = \frac{10 \cdot 49}{8^5} \approx 0,015$

11. What is more probable in case of two equally strong players: to win at least three games out of four or to win at least six games out of eight (No draws are possible)?

$P(M_4 = 3) + P(M_4 = 4) = \binom{4}{3}\left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^4 = \frac{1}{2}$

$P(M_8 = 6) + P(M_8 = 7) + P(M_8 = 8) = 3 \cdot \frac{1}{2^8}$

to win at least three games out of four

12. There are ten devices installed on a workbench. During the trials each of them may break down with probability 0.15 independently from other devices. Find the probability that (exactly) three devices have broke down given that not all the devices went through the trials successfully.

$A = \{$neither device broke$\}$

$$P(M_{10} = 3 | \bar{A}) = \frac{P(\bar{A}|M_{10} = 3) \cdot P(M_{10} = 3)}{P(\bar{A})}$$

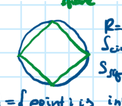
$$= \frac{1 \cdot \binom{10}{3}(0.15)^3 \cdot (0.85)^7}{1 - (0.85)^{10}} \approx 0,162$$

13. On average 25% students subscribe to the newsletter. Determine the most probable number of subscribers out of (a) 100 students; (b) 103 students.

$0,25 \cdot 100 = 25$

$0,25 \cdot 103 = 25$ or 26

14. A square is inscribed into a circle. Four points are chosen randomly inside the circle. What is the probability that the chosen only one point inside the square? What is the most probable number of points inside the first square



R = radius of circle

$S_{circle} = \pi R^2$

$S_{square} = 2R^2$

a) $A_i = \{$point i is inside the square$\}$

$P(A_i) = \frac{S_{square}}{S_{circle}} = \frac{2R^2}{\pi R^2} = \frac{2}{\pi}$

$P(M_4 = 1) = \binom{4}{1}\left(\frac{2}{\pi}\right)\left(1 - \frac{2}{\pi}\right)^3 = \frac{8}{\pi}\left(1 - \frac{2}{\pi}\right)^3$

b) $A_i = \{i$ points are inside the circle$\}$

$$\frac{P(A_i)}{P(A_{i+1})} = \frac{\binom{4}{i}\left(\frac{2}{\pi}\right)^i \left(1 - \frac{2}{\pi}\right)^{4-i}}{\binom{4}{i+1}\left(\frac{2}{\pi}\right)^{i+1}\left(1 - \frac{2}{\pi}\right)^{3-i}} = \frac{\frac{4!}{i!(4-i)!}\left(1 - \frac{2}{\pi}\right)}{\frac{4!}{(i+1)!(3-i)!} \cdot \frac{2}{\pi}} =$$

$$= \frac{(i+1)(\pi - 2)}{(4-i) \cdot 2}, \quad i \in \{0, 1, 2, 3\}$$

i	0	1	2	3
ratio	$\frac{\pi-2}{8}$	$\frac{\pi-2}{4}$	$\frac{3(\pi-2)}{8}$	$\frac{2(\pi-2)}{4}$
	<1	<1	<1	>1

3 is the most probable number of points inside the square

15. A marksman is shooting at a target until he misses three times. The probability to miss is equal to 0.2 for each of the shots he makes. What is the probability that (a) the marksman uses exactly seven bullets; (b) five bullets are sufficient for him?

(a) $\binom{6}{2}(0.8)^4 (0.2)^2 \cdot (0.2) = 0,049152$

(b) $0,2^3 + \binom{4}{1} 0,8 \cdot 0,2^3 + \binom{4}{2} 0,8^2 \cdot 0,2^3 =$

$= 0,2^3 (1 + 4 + 3,84) = 0,05792$

16. Two players are playing a match (that consists of several games), each of the games can finish in favour of the younger player with probability 0.6 and in favour of the older player with probability 0.4 The younger player has won exactly five games in the first eight games. What is the probability that he started the match with a debut?

$A = \{$ young lost the first game$\}$

$$P(A|M_8 = 5) = \frac{P(M_8 = 5|A)P(A)}{P(M_8 = 5)} = \frac{P(M_7 = 5) \cdot P(A)}{P(M_8 = 5)} =$$

$$= \frac{\binom{7}{5}(0.6)^5 (0.4)^2 (0.4)}{\binom{8}{5}(0.6)^5 (0.4)^3} = \frac{\frac{7!}{5!2!}}{\frac{8!}{5!3!}} = \frac{3}{8}$$

17. Three dice are rolled. What is the probability that the product of all numbers rolled is 24?

18. Nine people enter the lift at the ground floor of a nine-storey building. What is the probability that the lift stops at each floor as it goes up?

17) $24 = 2 \cdot 2 \cdot 6 = 2 \cdot 3 \cdot 4 = 1 \cdot 4 \cdot 6$

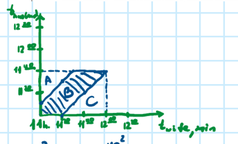
$P = \frac{15}{6^3} = \frac{5}{72}$

18) $P(2, 1, \dots, 1) = \underbrace{\left(\frac{1}{8}\right)^2}_{8} \cdot \underbrace{\left(\frac{1}{8}\right)^{\cdot 7}}_{} \cdot \underbrace{\frac{9!}{2! \cdot 1! \cdot \dots \cdot 1!}}_{9} = \frac{9!}{2^{28}}$

$P(2, 1, \dots, 1) = P(1, 2, 1, \dots, 1) = \dots = P(1, 1, \dots, 1, 2)$

$P(\text{diff stops at each floor}) = \frac{9!}{2^{28}} \cdot 8 = \frac{9!}{2^{25}}$

19. A woman arrives at the airport between 11 and 12 o'clock, waits for her husband for ten minutes, and if he does not arrive, she hails a taxi and leaves the airport alone (and the husband has a lot of trouble!) Her husband shows up at the airport at some time between 11 and 11 : 40. He waits for twenty minutes, and if he does not meet his wife, he goes home (and he is in a lot of trouble!) What is the probability that the husband ends up in having lots of trouble?



$A = \frac{20^2}{2} = 200$, $C = \frac{40^2}{2} = 800$, $A + B + C = 40 \cdot 60 = 2400$

$P = \frac{A + C}{A + B + C} = \frac{1000}{2400} = \frac{25}{48}$