BS19-F20-ProbStat Homework 4

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October 6, 2020

Question 1.

Given a circle of radius R and center O with random point M inside. Let $\xi = OM$.

$$F_\xi(x) = P(\xi < x) = \begin{cases} 0, & x \le 0 \\ \frac{x^2}{R^2}, & x \in (0; R) \\ 1, & x \ge R \end{cases} \Rightarrow f_\xi(x) = F'_\xi(x) = \begin{cases} 0, & x \le 0 \\ \frac{2x}{R^2}, & x \in (0; R) \\ 0, & x \ge R \end{cases}$$

$$E\xi = \int_{-\infty}^{+\infty} x f_\xi(x) dx = \int_{-\infty}^{0} x \cdot 0 \cdot dx + \int_0^R x \cdot \frac{2x}{R^2} \cdot dx + \int_R^{+\infty} x \cdot 0 \cdot dx = \frac{2}{R^2} \int_0^R x^2 dx = \frac{2}{R^2} \cdot \frac{x^3}{3} \Big|_0^R = \mathbf{\frac{2}{3}R}$$

$$E\xi^2 = \int_{-\infty}^{+\infty} x^2 f_\xi(x) dx = \int_0^R \frac{2x^3}{R^2} dx = \frac{1}{2}R^2$$

$$Var\xi = E\xi^2 - (E\xi)^2 = \frac{1}{2}R^2 - \frac{4}{9}R^2 = \mathbf{\frac{1}{18}R^2}$$

Question 2.

Given a sphere of radius R and center O with a random point M inside. Let $\xi = R - OM$.

$$F_\xi(x) = P(\xi < x) = P(OM > R - x) = \begin{cases} 0, & x < 0 \\ \frac{R^3 - (R-x)^3}{R^3}, & x \in (0; R) \\ 1, & x > R \end{cases} \Rightarrow f_\xi(x) = F'_\xi(x) = \begin{cases} 0, & x < 0 \\ \frac{3(R-x)^2}{R^3}, & x \in (0; R) \\ 0, & x > 0 \end{cases}$$

$$E\xi = \int_0^R x \cdot \frac{3(R-x)^2}{R^3} dx = \int_0^R \frac{3x}{R} dx - \int_0^R \frac{6x^2}{R^2} dx + \int_0^R \frac{3x^3}{R^3} dx = \frac{3}{2}R - 2R + \frac{3}{4}R = \mathbf{\frac{R}{4}}$$

$$E\xi^2 = \int_0^R x^2 \cdot \frac{3(R-x)^2}{R^3} dx = \int_0^R \frac{3x^2}{R} dx - \int_0^R \frac{6x^3}{R^2} dx + \int_0^R \frac{3x^4}{R^3} dx = R^2 - \frac{6}{4}R^2 + \frac{3}{5}R^2 = \frac{R^2}{10}$$

$$Var\xi = E\xi^2 - (E\xi)^2 = \mathbf{\frac{3}{80}R^2}$$

Question 3.

$$F_\zeta(x) = \begin{cases} 0, & x \le 1 \\ 1 - \frac{C}{x}, & x > 1 \end{cases} \Rightarrow f_\zeta(x) = F'{}_\zeta(x) = \begin{cases} 0, & x \le 1 \\ \frac{C}{x^2}, & x > 1 \end{cases}$$

For $F_\zeta(x)$ to be a valid cumulative distribution function(**CDF**), $f_\zeta(x)$ needs to be a valid probability density function(**PDF**), so:

$$\begin{cases} f_\zeta(x) \ge 0,\ \forall x, & \text{true for } C \ge 0 \\ \int_{-\infty}^{\infty} f_\zeta(x)dx = 1, & \int_1^{+\infty} \frac{C}{x^2}dx = -\frac{C}{x}\Big|_1^{+\infty} = \lim_{x\to+\infty}(-\frac{C}{x}) + C = C, \text{ equal to 1 for } C = 1 \end{cases}$$

$$E\zeta = \int_1^{+\infty} x \cdot \frac{C}{x^2}dx = C \cdot \ln x|_1^{+\infty}, \text{the integral diverges } \Rightarrow \mathbf{E\zeta \text{ d.n.e.}}$$

Question 4.

a)

$$f(x) = \begin{cases} Ce^{-2x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

For $f(x)$ to be a **PDF**, two conditions must be met:

$$\begin{cases} f(x) \ge 0, \forall x \in R, & \text{true for } C \ge 0 \\ \int_{-\infty}^{\infty} f(x)dx = 1, & \int_0^{+\infty} Ce^{-2x}dx = -\frac{C}{2}\int_0^{+\infty} e^{-2x}d(-2x) = -\frac{C}{2}e^{-2x}\Big|_0^{+\infty} = \frac{C}{2}, \text{ equal to 1 for } C = 2 \end{cases}$$

$f(x)$ can be a **PDF** for $C = 2$, let ξ be a random variable which has it as a **PDF**.

$$E\xi = \int_{-\infty}^{+\infty} xf(x)dx = \int_0^{+\infty} x \cdot 2e^{-2x} = -(xe^{-2x}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-2x}dx) = -\frac{1}{2}e^{-2x}\Big|_0^{+\infty} - xe^{-2x}\Big|_0^{+\infty}$$
$$= \frac{1}{2} - \lim_{x\to+\infty} xe^{-2x} = \frac{1}{2}$$

$$\lim_{x\to+\infty} xe^{-2x} = \lim_{x\to+\infty}\frac{x}{e^{2x}} = \lim_{x\to+\infty}\frac{(x)'}{(e^{2x})'} = \lim_{x\to+\infty}\frac{1}{2e^{2x}} = 0$$

$$E\xi^2 = \int_0^{+\infty} x^2 2e^{-2x}dx = -(x^2e^{-2x}\Big|_0^{+\infty} - \underbrace{\int_0^{+\infty} 2x \cdot e^{-2x}dx}_{E\xi=\frac{1}{2}}) = \frac{1}{2} - \lim_{x\to\infty} x^2e^{-2x} = \frac{1}{2}$$

$$\lim_{x\to+\infty} x^2e^{-2x} = \lim_{x\to+\infty}\frac{x^2}{e^{2x}} = \lim_{x\to+\infty}\frac{(x^2)'}{(e^{2x})'} = \lim_{x\to+\infty}\frac{2x}{2e^{2x}} = 0, \text{ as found previously.}$$

$$Var\xi = E\xi^2 - (E\xi)^2 = \mathbf{\frac{1}{4}}$$

b)

$$f(x) = Ce^{-|x|}$$

For $f(x)$ to be a **PDF**, two conditions must be met:

$$\begin{cases} f(x) \geq 0, \forall x \in R, & \text{true for } C \geq 0 \\ \displaystyle\int_{-\infty}^{\infty} f(x)dx = 1, & \displaystyle\int_{-\infty}^{0} Ce^{x}dx + \int_{0}^{+\infty} Ce^{-x}dx = C \cdot (e^{x} \Big|_{-\infty}^{0} - e^{-x} \Big|_{0}^{+\infty}) = 2C; \text{ equal to 1 for } C = \frac{1}{2} \end{cases}$$

$f(x)$ can be a **PDF** for $C = \frac{1}{2}$, let ζ be a random variable which has it as a **PDF**.

$$E\zeta = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{0} \frac{1}{2} \cdot x \cdot e^{x}dx + \int_{0}^{+\infty} \frac{1}{2} \cdot x \cdot e^{-x}dx$$

$$\int_{-\infty}^{0} \frac{1}{2}x \cdot e^{x}dx = \frac{1}{2}(xe^{x} \Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}dx) = -\frac{1}{2}\int_{-\infty}^{0} e^{x}dx = -\frac{1}{2}e^{x} \Big|_{-\infty}^{0} = -\frac{1}{2}$$

$$\lim_{x\to-\infty} xe^{x} = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{(x)'}{(e^{-x})'} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$$

$$\int_{0}^{+\infty} \frac{1}{2}x \cdot e^{-x}dx = -\frac{1}{2}(xe^{-x} \Big|_{0}^{+\infty} - \int_{0}^{+\infty} e^{-x}dx) = \frac{1}{2}\int_{0}^{+\infty} e^{-x}dx = -\frac{1}{2}e^{-x} \Big|_{0}^{+\infty} = \frac{1}{2}$$

$$\lim_{x\to+\infty} xe^{-x} = \lim_{x\to+\infty} \frac{x}{e^{-x}} = \lim_{x\to+\infty} \frac{(x)'}{(e^{x})'} = \lim_{x\to+\infty} \frac{1}{e^{x}} = 0$$

$$E\zeta = -\frac{1}{2} + \frac{1}{2} = 0$$

$$E\zeta^2 = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{-\infty}^{0} \frac{1}{2}x^2e^{x}dx + \int_{0}^{+\infty} \frac{1}{2}x^2e^{-x}dx$$

$$\int_{-\infty}^{0} \frac{1}{2}x^2e^{x}dx = \frac{1}{2}(x^2e^{x} \Big|_{-\infty}^{0} - \int_{-\infty}^{0} 2xe^{x}dx) = -\int_{-\infty}^{0} xe^{x}dx = 1, \text{as computed previously}$$

$$\lim_{x\to-\infty} x^2e^{x} = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \lim_{x\to-\infty} \frac{(x^2)'}{(e^{-x})'} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{(2x)'}{-(e^{-x})'} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$$

$$\int_{0}^{+\infty} \frac{1}{2}x^2e^{-x}dx = -\frac{1}{2}(x^2e^{-x} \Big|_{0}^{+\infty} - \int_{0}^{+\infty} 2xe^{-x}dx) = \int_{0}^{+\infty} xe^{-x}dx = 1$$

$$\lim_{x\to+\infty} x^2e^{-x} = \lim_{x\to+\infty} \frac{x^2}{e^{x}} = \lim_{x\to+\infty} \frac{2x}{e^{x}} = \lim_{x\to+\infty} \frac{2}{e^{x}} = 0$$

$$E\zeta^2 = 1 + 1 = 2$$

$$Var\zeta = E\zeta - (E\zeta)^2 = 2 - 0 = \mathbf{2}$$

Question 5.

a)

$$f(x) = \frac{C}{1+x^2}$$

For $f(x)$ to be a probability density function, two conditions must be met:

$$\begin{cases} f(x) \geq 0 & ,\forall x \in \mathbb{R} \ true \ for \ C \geq 0 \\ \int_{-\infty}^{\infty} f(x)dx = C\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = C \cdot arctg(x) \Big|_{-\infty}^{\infty} = C\pi = 1 & ,true \ for \ C = \frac{1}{\pi} \end{cases}$$

b)

$$f(x) = \begin{cases} 0 & ,|x| \leq 1 \\ \frac{1}{2x^2} & ,|x| > 1 \end{cases}$$

It is valid PDF, since its satisfies the conditions below:

$$\begin{cases} f(x) \geq 0 & ,\forall x \in \mathbb{R} \\ \int_{-\infty}^{\infty} f(x)dx = 1 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} \frac{dx}{2x^2} + \int_{1}^{\infty} \frac{dx}{2x^2} = \int_{1}^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{1}^{\infty} = 1$$

However, $E\zeta$ d.n.e., where ζ is a random variable with $f(x)$ as PDF.

$$E\zeta = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} \frac{1}{x}dx + \int_{1}^{\infty} \frac{1}{x}dx = lnx\Big|_{-\infty}^{-1} + lnx\Big|_{1}^{\infty}$$

d.n.e.

Question 6.

$\xi \sim u[0;4], \ f_\xi(x) = \begin{cases} \frac{1}{4} & ,x \in [0;4] \\ 0 & ,x \notin [0;4] \end{cases}$

$$E\xi = \frac{0+4}{2} = 2, \ Var\xi = \frac{4^2}{12} = \frac{4}{3}$$

$$F_\xi(x) = \int_{-\infty}^{x} f_\xi(t)dt = \begin{cases} 0, & x \leq 0 \\ \frac{1}{4}x, & x \in (0;4] \\ 1, & x > 4 \end{cases}$$

$$P(\xi < E\xi) = F_\xi(E\xi) = \frac{1}{2}$$

$$P(\xi > \sqrt{Var\xi}) = 1 - P(\xi \leq \sqrt{Var\xi}) = 1 - F_\xi(\sqrt{Var\xi}) = 1 - \frac{1}{2\sqrt{3}}$$

$$P(-5 \leq \xi \leq 5) = F_\xi(5) - F_\xi(-5) = 1$$

Question 7.

$$Y \sim u[a;b],\ EY = 3,\ VarY = 3$$

$$\left\{\begin{array}{ll} \frac{a+b}{2} = 3 & \Rightarrow a = 6 - b \\ \frac{(b-a)^2}{12} = 3 & \Rightarrow b - a = \pm 6 \\ a \leq b & \end{array}\right\} \Rightarrow a = 0,\ b = 6$$

Question 8.

$\zeta \sim u[a;b]$

$P(\zeta \in [1;2)) = P(\zeta < 2) - P(\zeta < 1) = F_\zeta(2) - F_\zeta(1) = \frac{1}{6}$, where $F_{\zeta_{CDF}}(x) = \begin{cases} 0, x \leq a \\ \frac{x-a}{b-a}, x \in (a;b] \\ 1, x > b \end{cases}$

$F_\zeta(2) - F_(1) = \frac{1}{b-a} = \frac{1}{6} \Rightarrow b - a = 6,$

$F_\zeta(1) = \frac{1-a}{b-a} = \frac{1}{2} \Rightarrow a = -2, b = 4 \Rightarrow F_\zeta(x) = \begin{cases} 0, x \leq -2 \\ \frac{x+2}{6}, x \in (-2;4] \\ 1, x > 4 \end{cases}$

$f_\zeta(X) = F'_\zeta(x) = \begin{cases} 0, x \leq -2 \\ \frac{1}{6}, x \in (-2;4] \\ 0, x > 4 \end{cases}$

$E_\zeta = \frac{4-2}{2} = 1,\ Var\ \zeta = \frac{6^2}{12} = 3$

Question 9.

$Z \sim u[a;b], F_Z(1) = \frac{1}{3}, F_Z(4) = 1$
$P(1 \leq Z < 4) = F_Z(4) - F_Z(1) = \frac{2}{3}$
Since $P(Z < 4) = F_Z(4) = 1, b \leq 4$, since $P(Z < 1) = F_Z(1) > 0, a < 1$
$F_Z(4) = F_Z(b)$, because $b \leq 4$.

Then we have the following conditions for a, b:
$\begin{cases} F_Z(b) - F_Z(1) = \frac{2}{3} \Leftrightarrow \frac{b-a}{b-a} - \frac{1-a}{b-a} = \frac{2}{3} \Leftrightarrow \frac{b-1}{b-a} = \frac{2}{3} \Leftrightarrow b = 3 - 2a, \\ a < 1 \\ b \leq 4 \Rightarrow 3 - 2a \leq 4 \Rightarrow a \geq -\frac{1}{2} \end{cases}$
$Var\ Z = \frac{(b-a)^2}{12} = \frac{(3-3a)^2}{12}, (Var\ Z)(a) = \frac{3(1-a)^2}{4}$, a parabola with branches up
$max(Var\ Z)(a) = (Var\ Z)(-\frac{1}{2}) = \frac{27}{16}$, since $a = 1$ is the apex of the parabola $[-\frac{1}{2}; 1)$

Question 10.

$Z \sim u[a;b], P(0 < Z < 1) = \frac{2}{3}, P(1 < Z < 2) = \frac{1}{3}$
$P(0 < Z < 2) = P(0 < Z < 1) + P(1 < Z < 2) = 1 \Rightarrow [a;b] \subseteq [0;2]$

$$\begin{matrix} P_Z(a) = P_Z(0) \\ P_Z(b) = P_Z(2) \end{matrix}, \; since \; \begin{matrix} a \geq 0 \\ b \leq 2 \end{matrix}$$
$P(0 < Z < 1) = P_Z(1) - P_Z(0) = P_Z(1) - P_Z(a) = \frac{1-a}{b-a} - \frac{a-a}{b-a} = \frac{2}{3}$
$P(1 < Z < 2) = P_Z(2) - P_Z(1) = P_Z(b) - P_Z(1) = \frac{b-a}{b-a} - \frac{1-a}{b-a} = \frac{1}{3}$

Then we have the following conditions for a,b: $(a \leq b)$
$$\begin{cases} a = 3 - 2b \\ a \geq - \Rightarrow b \leq \frac{3}{2} \\ b \leq 2 \end{cases} \qquad a \leq b \Leftrightarrow 3 - 2b \leq b \Leftrightarrow b \geq 1$$

$(E\ Z)(b) = \frac{b+a}{2} = \frac{3-b}{2}, min_{[1;\frac{3}{2}]}(E\ Z)(b) = (E\ Z)(\frac{3}{2}) = \frac{3}{4}$, since $(E\ Z)(B) \searrow on\ R$

$(Var\ Z)(b) = \frac{(b-a)^2}{12} = \frac{3(b-1)^2}{4}$, a parabola with branches up

$max_{[1;\frac{3}{2}]}(Var\ Z)(b) = max\{(Var\ Z)(1), (Var\ Z)(\frac{3}{2})\} = max\{0, \frac{3}{16}\} = \frac{3}{16}$, for $b = \frac{3}{2}$

Question 11.

$$X \sim u[-a;a]; \quad F_x(X) = \begin{cases} 0, & x \leq -a \\ \frac{x+a}{2a}, & x \in (-a;a] \\ 1, & x > a \end{cases}; \quad F_{|x|}(X) = \begin{cases} 0, & x \leq 0 \\ 2\frac{x}{2a}, & x \in (0;a] \\ 1, & x > a \end{cases}$$

Equal to the CDF of a variable distributed $u[0,a]$.

Question 12.

$\eta \sim u[a;b]$, $\xi = \frac{\eta - E\eta}{\sqrt{Var\ \eta}}$
$E\eta = \frac{b+a}{2}$, $\sqrt{Var\ \eta} = \frac{b-a}{2\sqrt{3}}$, $f_\xi(\frac{x-E\ \eta}{\sqrt{Var\ \eta}}) = f_\eta(x)$
$f_\eta(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & otherwise \end{cases}$
For $f_\xi(x)$, the range of non-zero values is scaled by a factor of $\frac{1}{\sqrt{Var\ \eta}} = \frac{2\sqrt{3}}{b-a}$.
The length of that range is $(b-a) \cdot \frac{2\sqrt{3}}{b-a} = 2\sqrt{3}$.
Following the scaling, the value of $f_\xi(x)$ over the non-zero range was also scaled.
Non-zero range boundaries for f_η are a, b, then for f_ξ the boundaries are c, d:
$c = \frac{a-E\eta}{\sqrt{Var\ \eta}} = -\sqrt{3}$, $d = \frac{b-E\eta}{\sqrt{Var\ \eta}} = \sqrt{3}$.
Therefore, $\xi \sim u[-\sqrt{3};\sqrt{3}]$.

Question 13.

$\xi \sim u[-1;5]$, $E((\xi-1)(3-\xi)) =?$
$E\xi = \frac{5+(-1)}{2} = 2$, $Var\ \xi = \frac{(5+1)^2}{12} = 3$
$E((\xi-1)(3-\xi)) = E(-\xi^2+4\xi-3) = -E\xi^2+4E\xi-3 = -Var\ \xi - (E\xi)^2 + 4E\xi - 3 = -2$

Question 14.

$\theta \sim Exp(\lambda)$

$$f_\theta(X) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \implies F_\theta(x) = \int_{-\infty}^{x} f_\theta(x)dx = \begin{cases} \int_0^x \lambda e^{-\lambda t}dt, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$F_\theta(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$P(\theta \in (0;1)) = F_\theta(1) - F_\theta(0) = 1 - e^{-\lambda}$
$P(\theta \in (1;2)) = F_\theta(2) - F_\theta(1) = e^{-\lambda} - e^{-2\lambda} = e^{-\lambda}(1-e^{-\lambda})$
$P(\theta \in (k;k+1)) = F_\theta(k+1) - F_\theta(k) = e^{-k\lambda} - e^{-(k+1)\lambda} = e^{-k\lambda}(1-e^{-\lambda})$
$\{P(\theta \in (k;k+1))\}_{k=0}^{\infty}$ is a geometric sequence with a ratio $e^{-\lambda}$.

Question 15.

$Z \sim Exp(\lambda)$, $P\{2 < Z < 3\} = \frac{4}{27}$
Since $P\{2 < Z < 3\} = F_Z(3) - F_Z(2)$, we can write the following:

$$\int_2^3 \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_2^3 = e^{-2\lambda} - e^{-3\lambda} = \frac{4}{27}$$

Let's try a substitution $t = e^{\lambda}, t > 0$:

$$\frac{t-1}{t^3} = \frac{4}{27} \Rightarrow -4t^3 + 27t - 27 = 0 \Leftrightarrow -(t+3)(2t-3)^2 = 0 \Rightarrow t = 1.5 \Leftrightarrow \lambda = \ln 1.5$$

$$\text{Therefore, } EZ = \int_0^{+\infty} \ln 1.5 x e^{-\ln 1.5 x} dx = -xe^{-\ln 1.5 x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\ln 1.5 x} dx =$$

$$= -\frac{e^{-\ln 1.5 x}}{\ln 1.5} \Big|_0^{+\infty} = \frac{1}{\ln 1.5}$$

Answer: $\frac{1}{\ln 1.5}$

Question 16.

$\xi \sim Exp(\lambda)$, $P\{|\xi - E\xi| < 3\sqrt{Var\xi}\} = ?$

$$E\xi = \int_0^{+\infty} \lambda x e^{-\lambda x} dx = -xe^{-\lambda x} \Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x} dx = -\frac{e^{-\lambda x}}{\lambda} \Big|_0^{+\infty} = \frac{1}{\lambda}$$

$$E\xi^2 = \int_0^{+\infty} \lambda x^2 e^{-\lambda x} dx = -x^2 e^{-\lambda x} \Big|_0^{+\infty} + 2\int_0^{+\infty} x e^{-\lambda x} dx = -\frac{2e^{-\lambda x}}{\lambda^2} \Big|_0^{+\infty} = \frac{2}{\lambda^2} \Rightarrow$$

$$Var\xi = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \Rightarrow$$

$$P\{|\xi - E\xi| < 3\sqrt{Var\xi}\} = P\{|\xi - \frac{1}{\lambda}| < \frac{3}{\lambda}\} = P\{-\frac{2}{\lambda} < \xi < \frac{4}{\lambda}\} = F_\xi(\frac{4}{\lambda}) - F_\xi(0) \text{ since } -\frac{2}{\lambda} < 0$$

$$F_\xi(\frac{4}{\lambda}) - F_\xi(0) = \int_0^4 \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_0^4 = 1 - e^{-4}$$

Answer: $1 - e^{-4}$

Question 17.

$\xi \sim Exp(\lambda)$, $\eta = e^{-\xi}$, $E\eta = ?$, $Var\eta = ?$

$$E\eta = \int_0^{+\infty} e^{-x} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda + 1}$$

$$E\eta^2 = \int_0^{+\infty} e^{-2x} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda + 2} \Rightarrow$$

$$Var\eta = \frac{\lambda}{\lambda + 2} - \frac{\lambda^2}{(\lambda+1)^2} = \frac{\lambda}{(\lambda+1)^2(\lambda+2)}$$

Answer: $E\eta = \frac{\lambda}{\lambda+1}$, $Var\eta = \frac{\lambda}{(\lambda+1)^2(\lambda+2)}$

Question 18.

$\xi \sim Exp(\lambda)$, $t, \tau > 0$. Prove that $P\{\xi > t+\tau \,\Big|\, \xi > t\} = P\{\xi > \tau\}$

$$P\{\xi > t+\tau \,\Big|\, \xi > t\} = \frac{P\{\xi > t+\tau\}}{P\{\xi > t\}} \text{ since t } + \tau > t$$

$$\frac{P\{\xi > t+\tau\}}{P\{\xi > t\}} = \frac{\int_{t+\tau}^{+\infty} \lambda e^{-\lambda x} dx}{\int_{t}^{+\infty} \lambda e^{-\lambda x} dx} = \frac{-e^{-\lambda x}\Big|_{t+\tau}^{+\infty}}{-e^{-\lambda x}\Big|_{t}^{+\infty}} = \frac{e^{-\lambda(t+\tau)}}{e^{-\lambda t}} = e^{-\lambda\tau}$$

$$P\{\xi > \tau\} = \int_{\tau}^{+\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_{\tau}^{+\infty} = e^{-\lambda\tau}$$

Hence, $P\{\xi > t+\tau \,\Big|\, \xi > t\} = P\{\xi > \tau\}$

Question 19.

$\xi \sim N(\mu; \sigma^2), E\xi = \mu = 1, Var\xi = \sigma^2 = 4 \quad \Rightarrow \sigma = 2, \quad$ since $\sigma > 0$

Let $\eta \sim N(0;1)$, then $\xi = 2\eta + 1$

Let $\Phi_0(x) = P(-\infty < \eta < x) = \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$

$P(a \leq \xi < B) = P(a \leq 2\eta + 1 < b) = P\left(\frac{a-1}{2} \leqslant \eta < \frac{b-1}{2}\right) = \Phi_0\left(\frac{b-1}{2}\right) - \Phi_0\left(\frac{a-1}{2}\right).$

a) $P(-3 < \xi < 1) = \Phi_0(0) - \Phi_0(-2) = 0,5 - 0,0228 = 0,4772,$ according to the standard normal table

b) $P(\xi < -2) = \Phi_0\left(-\frac{3}{2}\right) = 0,0668$

c) $P(\xi > 3) = 1 - P(\xi \leq 3) = 1 - \Phi_0(1) = 1 - 0,8413 = 0,1587$

Question 20.

Let $\xi \sim N(-1;1), \eta \sim N(0;1)$, then $\xi = \eta - 1$

$P(x < \xi < y) = P(x < \eta - 1 < y) = P(x+1 < \eta < y+1) = \Phi_0(y+1) - \Phi_0(x+1)$

a) $P(x < \xi < 1) = \Phi_0(2) - \Phi_0(x+1) = 0,8 \Rightarrow \Phi_0(x+1) = 0,9772 - 0,8 = 0,1772 \Rightarrow x+1 \approx -0,93 \Rightarrow x \approx -1,93$

b) $P(0 < \xi < x) = \Phi_0(x+1) - \Phi_0(1) = 0,8 \Rightarrow \Phi_0(x+1) = 0,8 + 0,8413 = 1,6413 > 1 \Rightarrow x$ **d.n.e**

c) $P(-1-x < \xi < -1+x) = \Phi_0(x) - \Phi_0(-x) = 0,8$

$\Phi_0(0) - \Phi_0(-x) = \Phi_0(x) - \Phi_0(0),$ since the PDF of η is symmetric w.r.t. $x = 0$

$$\begin{cases} \Phi_0(x) + \Phi_0(-x) = 1 \\ \Phi_0(x) - \Phi_0(-x) = 0,8 \end{cases} \Rightarrow \begin{matrix} \Phi_0(x) = 0,9 \\ \Phi_0(-x) = 0,1 \end{matrix} \Rightarrow x \approx 1,28$$

$\theta_0(x) - \Phi_0(-x) = 0,8$

Question 21.

$\xi \sim N(0;\sigma^2)$, the PDF of ξ is symmetric w.r.t. 0, thus, $\int_{-a}^{-b} f_\xi(x)dx = \int_b^a f_\xi(x)dx$
$f_\xi(t) \searrow$ on $(0;+\infty)$

$$P(\xi \in (0;4)) = \int_0^4 f_\xi(x)dx = \int_0^3 f_\xi(x)dx + \int_3^4 f_\xi(x)dx$$
$$P(\xi \in (-1;3)) = \int_{-1}^3 f_\xi(x)dx = \int_{-1}^0 f_\xi(x)dx + \int_0^3 f_\xi(x)dx = \int_0^3 f_\xi(x)dx + \int_0^1 f_\xi(x)dx$$
$$P(\xi \in (0;4)) < P(\xi \in (-1;3)) \Leftrightarrow \int_3^4 f_\xi(x)dx < \int_0^1 f_\xi(x)dx \Leftrightarrow \lim_{dx\to 0}\sum_{i=1}^{\frac{1}{dx}} dx f_\xi(3+i*dx) < \lim_{dx\to 0}\sum_{i=1}^{\frac{1}{dx}} dx f_\xi(i*dx)$$

Since the amount of terms in both sums is equal and $f_\xi(3+i*dx) < f_\xi(i*dx)\ \forall i,\ dx$, the above inequality holds.

$$P(\xi \in (-1;3)) = \int_{-1}^3 f_\xi(x)dx = \int_{-1}^{2.5} f_\xi(x)dx + \int_{2.5}^3 f_\xi(x)dx$$
$$P(\xi \in (-1.5;2.5)) = \int_{-1.5}^{2.5} f_\xi(x)dx = \int_{-1.5}^{-1} f_\xi(x)dx + \int_{-1}^{2.5} f_\xi(x)dx = \int_{-1}^{2.5} f_\xi(x)dx + \int_1^{1.5} f_\xi(x)dx$$
$$P(\xi \in (-1;3)) < P(\xi \in (-1.5;2.5)) \Leftrightarrow \int_{2.5}^3 f_\xi(x)dx < \int_1^{1.5} f_\xi(x)dx, \text{which is similarly proven true.}$$

In the same way one can obtain $P(\{\xi \in (-1,5;2,5)) < P(\xi \in (-2;2))$
Thus, $P(\xi \in (0;4)) < P(\{\in (-1;3)) < P(\xi \in (-1,5;2,5)) < P(\xi \in (-2;2))$

Question 22.

$\xi \sim N(\mu;\sigma^2), E\xi = \mu \Rightarrow \xi - E\xi = \xi - \mu, \xi - \mu \sim N(0;\sigma^2)$.
$P(|\xi-\mu| < 1) = 0,3 \Leftrightarrow P(-1 < \xi - \mu < 1) = 0,3 \Leftrightarrow 2P(0 < \xi-\mu < 1) = 0,3$ since $\xi - \mu$ has a symmetric PDF w.r.t 0
Let $\eta \sim M(0;1)$, then $\xi - \mu = \sigma\eta$, $P(0 < \xi - \mu < x) = P(0 < \eta < \frac{x}{\sigma})$

$$P(0 < \xi-\mu < 1) = 0.15 = P(0 < \eta < \frac{x}{\sigma}) \Rightarrow \Phi_0(\frac{1}{\sigma}) - \Phi_0(0) = 0.15 \Rightarrow \Phi_0(\frac{1}{\sigma}) = 0.15 \Rightarrow \frac{1}{\sigma} \approx 0.385$$
$$P(|\xi-\mu| < 2) = P(-2 < \xi-\mu < 2) = 2P(0 < \xi-\mu < 2) = 2P(0 < \eta < \frac{2}{\sigma}) =$$
$$= 2*(\Phi_0(2*0.385) - \Phi_0(0)) = 2*0.2794 = 0.5588$$

Question 23.

$\xi \sim N\left(\mu;\sigma^2\right), \quad E\xi = \mu = 1, \quad \text{Var}\ \xi = \sigma^2 = 5 \Rightarrow \sigma = \sqrt{5}$ since $\sigma > 0$
The PDF of ξ is symmetric w.r.t to 1 and $\searrow$ on $(1;+\inf)$
Therefore, $\forall a,b : \epsilon > 0, 1 < a < b \Rightarrow \int_a^{a+\epsilon} f_\xi(x)dx > \int_b^{b+\epsilon} f_\xi(x)dx$ (refer to №21 for proof)
From this follows the fact that $P(\xi \in (1-\varepsilon;1+\varepsilon)) \geqslant P(\xi \in (a-\varepsilon;a+\varepsilon)) \quad \forall a,\varepsilon > 0$
Let $\eta \sim N(0;1)$, then $\xi = \sqrt{5}\eta + 1$

$$P(\xi \in (1-\varepsilon; 1+\varepsilon)) = 2P(\xi \in (1; 1+\varepsilon)) = 2P(\eta \in (0; \frac{\varepsilon}{\sqrt{5}}) = 0,95 = 2(\Phi_0(\frac{\varepsilon}{\sqrt{5}}) - \Phi_0(0))$$

$$\Phi_0(\frac{\varepsilon}{\sqrt{5}}) = \frac{0,95}{2} \Rightarrow \Phi_0\left(\frac{\varepsilon}{\sqrt{5}}\right) = 0,475 \Rightarrow \frac{\varepsilon}{\sqrt{5}} = 1,96, \text{ according to the standard normal table}$$

$$\varepsilon = \sqrt{5} \cdot 1,96 = 4,38$$

Thus, the shortest interval $(a;b)$ such that $P(\xi \in (a;b)) = 0,95$ is $(-3,38; 5,38)$

Question 24.

$\xi \sim N\left(\mu; \sigma^2\right), \quad P(1 < \xi < 7) = P(7 < \xi < 13) = 0,18$

The PDF for ξ is symmetric w.r.t. μ, thus $\int_{\mu-\varepsilon}^{\mu} f_\xi(x)dx = \int_{\mu}^{\mu+\varepsilon} f_\xi(x)dx \ \forall \varepsilon > 0$

$$\left.\begin{array}{l} P(1 < \xi < 7) = \int_{7-6}^{7} f_\xi(x)dx \\ P\left(7 < \xi < 13\right) = \int_{7}^{7+6} f_\xi(x)dx \end{array}\right\} \Rightarrow \mu = 7, \text{ since no other spot of } f_\xi \text{ exhibits symmetry.}$$

Let $\eta \sim N(0;1)$, then $\xi = \sigma\eta + 7$

$P\left(7 < \xi < 13\right) = P\left(0 < \eta < \frac{6}{\sigma}\right) = \Phi_0\left(\frac{6}{\sigma}\right) - \Phi_0(0) = 0,18$

$\Phi_0\left(\frac{6}{\sigma}\right) = 0,18 + 0,5 \Rightarrow \frac{6}{\sigma} = 0,47 \Rightarrow \sigma = \frac{600}{47}$

$E\xi = \mu = 7, \quad \operatorname{Var}\xi = \sigma^2 = \left(\frac{600}{47}\right)^2 = 163$

Question 25.

$\eta \sim N\left(1; \sigma^2\right), \sigma > 0$

Let $\xi \sim N(0;1)$, then $\eta = \sigma\xi + 1$

$$P(2 < \eta < 4) = P\left(\frac{1}{\sigma} < \xi < \frac{3}{\sigma}\right) = \Phi_0\left(\frac{3}{\sigma}\right) - \Phi_0\left(\frac{1}{\sigma}\right), \text{ where } \Phi_0(x) = \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}dt$$

$$h(\sigma) = \Phi_0\left(\frac{3}{\sigma}\right) - \Phi_0\left(\frac{1}{\sigma}\right), \quad h'(\sigma) = \Phi'\left(\frac{3}{\sigma}\right)\cdot\left(-\frac{3}{\sigma^2}\right) - \Phi'\left(\frac{1}{\sigma}\right)\cdot\left(-\frac{1}{\sigma^2}\right) = \left(-\frac{1}{\sigma^2}\right)\frac{1}{\sqrt{2\pi}}\cdot\left(3e^{-\frac{9}{2\sigma^2}} - e^{-\frac{1}{2\sigma^2}}\right)$$

$$h'(\sigma) = 0 \Leftrightarrow 3e^{-\frac{9}{2\sigma^2}} = e^{-\frac{1}{2\sigma^2}} \Leftrightarrow e^{-\frac{4}{\sigma^2}} = \frac{1}{3} \Leftrightarrow \sigma = \frac{2}{\sqrt{\ln 3}}$$

$$h''(\sigma) = \frac{2}{\sigma^3}\cdot\frac{1}{\sqrt{2\pi}}\left(\left(3 - \frac{27}{2\sigma^2}\right)e^{-\frac{9}{2\sigma^2}} - \left(1 - \frac{1}{2\sigma^2}\right)e^{-\frac{1}{2\sigma^2}}\right)$$

$$h''\left(\frac{2}{\sqrt{\ln 3}}\right) = \frac{(\sqrt{\ln 3})^3}{4}\cdot\frac{1}{\sqrt{2\pi}}\cdot\left(\left(3 - \frac{27\ln 3}{8}\right)e^{-\frac{9\ln 3}{8}} - \left(1 - \frac{\ln 3}{8}\right)e^{-\frac{\ln 3}{2}}\right)$$

$$h''\left(\frac{2}{\sqrt{\ln 3}}\right) > 0 \Leftrightarrow \left(3 - \frac{27\ln 3}{8}\right)3^{-\frac{9}{8}} > \left(1 - \frac{\ln 3}{8}\right)3^{-\frac{1}{8}} \Leftrightarrow 3^{-\frac{1}{8}} - \frac{9\ln 3\cdot 3^{-\frac{1}{8}}}{8} > 3^{-\frac{1}{8}} - \frac{\ln 3\cdot 3^{-\frac{1}{8}}}{8} \Leftrightarrow 9 < 1$$

$$h''\left(\frac{2}{\sqrt{\ln 3}}\right) < 0 \Rightarrow \frac{2}{\sqrt{\ln 3}} \text{ is a local maximum for } h(\sigma) \Rightarrow \max_{\sigma>0} P(2 < \eta < 4) \text{ is for } \sigma = \frac{2}{\sqrt{\ln 3}}$$

Question 26.

$\zeta \sim N\left(\mu; \sigma^2\right), \quad E\zeta = \mu = -2, \quad \operatorname{Var}\zeta = \sigma^2 = 9$

$E\left((3-\zeta)(\zeta+5)\right) = E\left(-\zeta^2 - 2\zeta + 15\right) = -E\zeta^2 - 2E\zeta + 15 = \left(-\operatorname{Var}\zeta - (E\zeta)^2\right)\cdot 2E\zeta + 15 = 6$

Question 27.

Let $\xi \sim N(\mu;\sigma^2)$, and $a \neq 0$ is an arbitrary number. Find the distribution of $\eta = a\xi + b$.

Solution.

1. Let $a > 0$. So,

$$P(\eta < x) = P(\xi < \frac{x-b}{a}) = \int_{-\infty}^{\frac{x-b}{a}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = //y = \frac{t-\mu}{\sigma}// = \int_{-\infty}^{\frac{x-b-a\mu}{a\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Let $\theta \sim N(a\mu + b; a^2\sigma^2)$ then

$$P(\theta < x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-a\mu-b)^2}{2a^2\sigma^2}} dt = \int_{0}^{\frac{x-b-a\mu}{a\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Since there is a one to one correspondence with CDFs and random variables, $\theta = \eta \Rightarrow \eta \sim N(a\mu + b; a^2\sigma^2)$

2. Let now $a < 0$. Then,

$$P(\eta < x) = P(\xi > \frac{x-b}{a}) = \int_{\frac{x-b}{a}}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = //y = \frac{t-\mu}{\sigma}// = \int_{\frac{x-b-a\mu}{a\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = P(\varphi > \frac{x-b-a\mu}{a\sigma})$$

Where $\varphi \sim N(0;1)$.

Let $\theta \sim N(a\mu + b; a^2\sigma^2)$ then the PDF of φ is symmetric w.r.t. $0 \Rightarrow P(\varphi > x) = P(\varphi < -x)$, and $P(n < x) = P\left(\varphi < \frac{x-b-a\mu}{-a\sigma}\right)$

Then for $\theta \sim N\left(a\mu + B; a^2\sigma^2\right)$ we have the same $CDF \Rightarrow \eta = \theta \Rightarrow \eta \sim N\left(a\mu + b; a^2\sigma^2\right)$

Question 28.

Let $\xi \sim N\left(0;\sigma^2\right)$. Find $E|\xi|$ and $\operatorname{Var}|\xi|$.

Solution.

$$f_\xi(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}, \quad f_{|\xi|}(t) = \begin{cases} 0, & t < 0 \\ f_{|\xi|}(t) + f_{|\xi|}(-t), & \text{otherwise} \end{cases}$$

$$E|\xi| = \int_{-\infty}^{\infty} t \cdot f_{|\xi|}(t) dt = \int_{0}^{+\infty} \frac{2t}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = -\frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{+\infty} -\frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} dt = -\frac{2\sigma}{\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \Big|_0^{+\infty} = \frac{\sqrt{2}\sigma}{\sqrt{\pi}}$$

And

$$E\left(|\xi|^2\right) = E\left(\xi^2\right) = \sigma^2 \Rightarrow \operatorname{Var}|\xi| = \sigma^2\left(1 - \frac{2}{\pi}\right)$$

Question 29.

Let $\eta = \sin\xi$

a) $\xi \sim u[-\frac{\pi}{2};\frac{\pi}{2}]$.

So, $\xi \in [-\frac{\pi}{2};\frac{\pi}{2}] \implies \eta \in [-1;1]$.

$$F_\xi(x) = \begin{cases} 0, & x \le -\frac{\pi}{2} \\ \frac{x+\frac{\pi}{2}}{\pi}, & x \in (-\frac{\pi}{2};\frac{\pi}{2}) \\ 1, & x \ge \frac{\pi}{2} \end{cases} \implies F_\eta(x) = \begin{cases} 0, & x \le -1 \\ \frac{\arcsin x+\frac{\pi}{2}}{\pi}, & x \in (-1;1) \\ 1, & x \ge 1 \end{cases}$$

Finally,

$$f_\eta = F'_\eta(x) = \begin{cases} 0, & x \notin (-1;1) \\ \frac{1}{\pi\sqrt{1-x^2}}, & \text{otherwise} \end{cases}$$

b) $\xi \sim u[0;\pi]$.

Then, $\xi \in [0;\pi] \Rightarrow \eta = \sin\xi \in [0;1]$.

$$F_\xi(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\pi}, & x \in (0;\pi) \\ 1, & x > \pi \end{cases}$$

We have $F_\eta(x) = 0, \forall x \le 0$ and $F_\eta(x) = 1, \forall x \ge 1$.

For $x \in (-1;1)$,

$$F_\eta(x) = P(\eta < x) = P(\sin\xi < x) = P(\xi \in \bigcup_{k\in\mathbb{Z}}[\pi - \arcsin x + 2\pi k; 2\pi + \arcsin x + 2\pi k]) =$$

$$= //\xi \in [-\frac{\pi}{2};\frac{\pi}{2}]// = P(\xi \in [0;\arcsin x] \cup [\pi - \arcsin x;\pi]) = F_\xi(\arcsin x) + F_\xi(\pi) + F_\xi(\pi - \arcsin x)$$

Thus,

$$F_\eta(x) = \begin{cases} 0, & x < 0 \\ \frac{\arcsin x}{\pi} + 1 - \frac{\pi - \arcsin x}{\pi} = \frac{2\arcsin x}{\pi}, & x \in (0;\pi) \\ 1, & x > \pi \end{cases} \implies f_\eta = F'_\eta(x) = \begin{cases} 0, & x \notin (-1;1) \\ \frac{2}{\pi\sqrt{1-x^2}}, & \text{otherwise} \end{cases}$$

Question 30.

Nao. $\{$ is a random variable such that $f_f(x) = \dfrac{1}{\pi\left(1+x^2\right)}$, $\quad \max\limits_{\lambda>0} P(\{\in(\lambda;2\lambda))-?$

$$h(\lambda)=\int_{\lambda}^{2\lambda}\frac{dx}{\pi\left(1+x^2\right)}=\frac{1}{\pi}\operatorname{arctg}x\Big|_{\lambda}^{2\lambda}=\frac{1}{\pi}(\operatorname{arctg}2\lambda-\operatorname{arctg}\lambda)$$

$$h'(\lambda)=\frac{1}{\pi}\left(\frac{2}{1+4\lambda^2}-\frac{1}{1+\lambda^2}\right),\quad h'(\lambda)=0\Leftrightarrow 2+2\lambda^2=1+4\lambda^2\Rightarrow\lambda=\frac{1}{\sqrt{2}},\text{since}\,\lambda>0$$

$$h''(\lambda)=\frac{1}{\pi}\left(-\frac{16\lambda}{\left(1+4\lambda^2\right)^2}+\frac{2\lambda}{\left(1+\lambda^2\right)^2}\right),\quad h''\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{\pi}\left(-\frac{16}{3\sqrt{2}}+\frac{8}{9\sqrt{2}}\right)<0\Rightarrow\lambda=\frac{1}{\sqrt{2}}\text{ is }a\log(\max,\text{for}\,h(\lambda)$$

Question 31.

N_{31} is a random variable such that $f_g(x)=\frac{1}{\pi(1+x^2)}$, $\quad \eta=\frac{1}{3}$
$F_\xi(x)=\int f_f(x)dx=\frac{1}{\pi}\int\frac{\alpha x}{1+x^2}=\frac{1}{\pi}\operatorname{arctg}x\big|_{-\infty}^{x}=\frac{\operatorname{arctg}x}{\pi}+\frac{1}{2}$
$F_\eta(x)=P(\eta<x)=P\left(\frac{1}{\xi}<x\right)=P\left(\frac{1-\xi x}{3}<0\right)=P(\xi(1-\xi x)<0)$

$$F_\eta(x)=P(\xi(1-\xi x)<0)=\begin{cases}P(\xi<0)+P\left(\xi>\frac{1}{x}\right), & x>0\\ P(\xi<0) & x=0\\ P(\xi<0)-P\left(\xi<\frac{1}{x}\right), & x<0\end{cases}$$

$$=\begin{cases}\frac{\pi-\operatorname{arctg}\frac{1}{x}}{\pi}, & x>0\\ \frac{1}{2} & x=0=\frac{\operatorname{arctg}x+1}{\pi}+\frac{1}{2}\\ \frac{-\operatorname{arctg}\frac{1}{x}}{\pi} & x<0\end{cases}$$

since a CDF uniquely identifies a vandoin variable, we luve a Corucly distributicn for η