Probability & Statistics. Assignment 1

1. A cube with all of its faces painted is sawn into 343 smaller cubes of equal sizes. One of the small cubes is chosen at random. Find the probability that this cube:

   (a) has exactly one painted face;

   (b) has exactly two painted faces;

   (c) has at least two painted faces.

   Answer:

2. m white balls and n black balls are placed at random in a row. Find the probability that k^{th} place is occupied with a black ball. 1.10 from 1

   Answer: $\frac{n}{m+n}$.

3. Six volumes are placed on a bookshelf in a random order. Find the probability that at least one of the volumes is not on its correct position.

   Answer: $\frac{719}{720}$.

4. There were 26 white balls and 14 black balls in the urn. One of the balls has disappeared. After that one ball is taken at random from the urn. Determine the probability that the ball is black. 1.20 from 1

   Answer: 0.35.

5. There are ten different pairs of shoes in a closet. Four shoes are chosen at random. Find the probability that at least one pair can be formed out of them.

   Answer: $\frac{99}{323}$.

6. Four people enter the lift at the ground floor of a nine-storey building[1]. Find the probability that at least two of them get off at the same floor.

   Answer: $\frac{151}{256}$.

7. Each game of a match between two equal players can end with a victory of one of them with probability 0.5 independently of the other games. Each victory yields one point, and the match is played until one of the players scores 6 points. Due to technical reasons the match was interrupted when the score was $5:3$ in favour of the first player. What do you think is a fair way to distribute the prize between the players?

   Answer: $7:1$.

8. Seventy numbers are chosen at random from integers $1, 2, 3, \cdot, 100$. What is the probability that the largest number chosen is 98? 1.43 from 1

   Answer: $\frac{29}{462}$.

9. m white balls, n black balls and k yellow balls are placed at random in a row. Determine the probability that if looking at the balls from left to right, we will see a white ball before we see a black one? 1.46 from 1

   Answer: $\frac{m}{m+n}$.

[1] We use the following conventions for problems with lifts if it is not stated otherwise. 1) The floors are counted this way: ground floor, first floor, second floor, etc. Thus, the upper floor of a nine-storey building is the eighth floor. 2) We assume that all people who have entered the lift at the ground floor can go out of it on each floor starting with the first one with equal probabilities and independently from the others. 3) No other people enter the lift as it goes up.

10. Six letters are chosen at random from the ones of the word PROPORTION. Determine the probabilities that it is possible to form the following words from these letters:

    (a) OPTION;

    (b) PORT;

    (c) RIOT.

    Answer:

11. Twenty participants of a chess tournament are casting lots in order to get into one of two groups (each of the groups has 10 players in it). Find the probability that two of the strongest players are in the same group. 1.58 from 1

    Answer: $\frac{9}{19}$.

12. Two different numbers are chosen at random from the set $\{1, 2, \ldots, 100\}$. What is the probability that the first one is greater than the second? 1.65 from 1

    Answer: 0.5.

13. What is the probability to get two "threes" when rolling two dice given that the sum of the two numbers is a multiple of 3?

    Answer: $\frac{1}{12}$.

14. There are three children in the family. What is the probability that all of them are boys if it is known that at least two of them are boys?

    Answer: 0.5.

15. An urn contains 6 white balls, 4 black balls and 2 orange balls. Three balls are taken at random out of the urn. What is the probability that the balls are of the same colour? 3.1 from 1

    Answer: $\frac{6}{55}$.

16. One card is drawn at random from a deck of 36 cards. Let $A = \{\text{the card is a jack}\}$ and $B = \{\text{the card has a black suit}\}$. Find the probability of $A + B$. 3.5 from 1

    Answer: $\frac{5}{9}$.

17. Six letters are chosen at random from the ones of the word MASTERPIECE. What is the probability that

    (a) at least one of the words REST, STRIP, PEST can be formed out of the letters chosen;

    (b) all three words REST, STRIP, PEST can be formed out of the letters chosen?

    Answer:

18. It is known that some events A and B satisfy the equality $P(A) = P(B) = 0.5$. Find the difference $P\{AB\} - P\left\{\overline{A}\,\overline{B}\right\}$.

    Answer: 0.

19. Prove that

    (a) $P\{AB\} \geqslant P\{A\}P\{B\} - 1$;

    (b) $P\{A_1 A_2 \ldots A_n\} \geqslant P\{A_1\} + P\{A_2\} + \ldots + P\{A_n\} - (n-1)$.

20. Find the largest and the smallest values of $P\{ABC\}$ if $P\{A\} = 0.7$, $P\{B\} = 0.8$, $P\{C\} = 0.9$.

21. Prove that if $P\{A\} = 0$ then $P\{A \triangle B\} = P\{B\}$. Is the opposite statement also true?

22. It is known that $P\{A\} = \frac{2}{3}$, $P\{B\} = \frac{3}{4}$. Prove that $P\{A|B\} \geqslant 59$. 3.30 from 1

23. A child is playing with the letters of word "SUPERPOSITION". He takes five letters and places them in an arbitrary order. What is the probability that he gets word "NOISE"?

    Answer:

24. Is it possible for disjoint events to be independent?

25. Let A and B be independent events. Prove that (a) A and $\overline{B}$ are independent; (b) $\overline{A}$ and $\overline{B}$ are independent.

26. Three snipers are shooting at a target. The probabilities that they hit the target are equal to 0.7, 0.6 and 0.5 respectively. For the target to be destroyed it is sufficient to hit it once. Find the probability that the target is destroyed given that the snipers shoot independently of each other.

27. Give an example of such three events that are pairwise independent but not mutually independent.

28. Three marksmen are shooting at a target, their probabilities of hitting the target are equal to 0.5, 0.4 and 0.3 respectively (the results are independent). Each of them makes exactly one shot. What is the probability that (a) the target has not been hit; (b) the target has been hit exactly two times; (c) the target has been hit at least once? 3.65 from 1

29. Two players are making turns in flipping a (symmetric) coin. The first of them to get tails wins the game. Find the probability for the first player to lose. 3.71 from 1

    Answer: $\frac{1}{3}$.

30. Two players are taking turns in drawing cards from a deck of 52 cards. The first player starts with drawing one random card. If it is an ace, he wins the game. Otherwise he returns the card back to the deck, shuffles it properly, and then the second player makes his turn. The game continues until the winner is determined (the first one to draw an ace). Is this a fair game (i.e. the probability for each of the players to win are equal)? Does the game become more fair if the winner is the first to draw the ace of clubs?

    Answer: No. Yes.

31. A coin is flipped until one does not get a sequence of "tails-tails-tails". What is the probability that the coin has been flipped (a) exactly 6 times; (b) exactly seven times?

    Answer: (a) $\frac{1}{16}$; (b) $\frac{7}{128}$.

32. A coin is flipped until one does not get a sequence of "heads-tails-heads". What is the probability that the coin has been flipped (a) exactly 6 times; (b) exactly seven times?

    Answer: (a) $\frac{5}{64}$; (b) $\frac{9}{128}$.

33. Ten people enter the lift at the ground floor of an eight-storey building. Find the probability that

    (a) the lift stops neither on the fourth nor on the fifth floor;

    (b) the lift stops on the fourth floor and does not stop on the fifth;

    (c) the lift stops both on the fourth and on the fifth floors;

    (d) the lift stops at least twice, including the stop on the fourth floor;

    (e) the lift stops at least twice.

    Answer: (a) $\left(\frac{3}{4}\right)^{10}$; (b) $\left(\frac{7}{8}\right)^{10} - \left(\frac{6}{8}\right)^{10}$; (c) $1 + \left(\frac{6}{8}\right)^{10} - 2\left(\frac{7}{8}\right)^{10}$; (d) $1 - \left(\frac{1}{8}\right)^{10} - \left(\frac{7}{8}\right)^{10}$; (e) $1 - \left(\frac{1}{8}\right)^{9}$.