# Lecture 1

Probability Space $\{\Omega, \mathcal{A}, P\}$

1) $\Omega$ = set of outcomes (of the experiment)

2) $\mathcal{A}$ is a $\sigma$-algebra of events

a die – several dice

algebra of events

$\{1; 2; 3; 4; 5; 6\}$

complementary event

$\overline{X} = \Omega \setminus X$

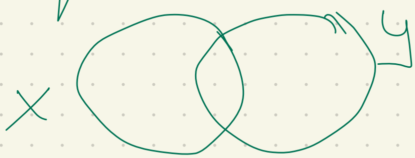
a) $\varnothing \in \mathcal{A}$

b) if $X \in \mathcal{A} \Rightarrow \overline{X} \in \mathcal{A}$ $\Big\} \Rightarrow \Omega \in \mathcal{A}$

c) $X \in \mathcal{A}, Y \in \mathcal{A} \Rightarrow \underbrace{X+Y}_{} \in \mathcal{A}$

a sum of events: at least one of events $X, Y$ happens

$X + Y = X \cup Y$




$XY = X \cap Y$ → a product of events: both events $X, Y$ happen

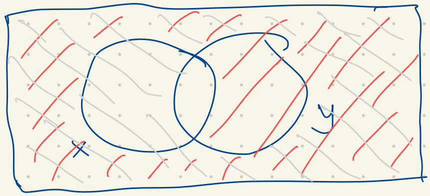
$\Rightarrow$ if $X \in \mathcal{A}, Y \in \mathcal{A} \Rightarrow XY \in \mathcal{A}$

$XY \in \mathcal{A} \Leftrightarrow \overline{XY} \in \mathcal{A} \Leftrightarrow \overline{X} + \overline{Y} \in \mathcal{A}$

$X \in \mathcal{A}, Y \in \mathcal{A} \Leftrightarrow \overline{X} \in \mathcal{A}, \overline{Y} \in \mathcal{A}$

$$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$$

$$\overline{X \cap Y} = \overline{X} \cup \overline{Y}$$




c') $X_1, X_2, \ldots, X_n, \ldots \in \mathcal{A} \Rightarrow$

$$\sum_{i=1}^{\infty} X_i \in \mathcal{A}$$

a) & b) & c') $\Longrightarrow$ $\sigma$-algebra of events

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$\Omega = \{1; 2; 3; 4; 5; 6\}$

$\{1; 2\}$ $\{1; 2; 5; 6\}$

$\emptyset$ $\{3; 4; 5; 6\}$ $\{3; 4\}$

$\{1; 2; 3; 4; 5; 6\}$ $\{1; 2; 3; 4\}$

$\{5; 6\}$

3) Probability is a function on set of events.

a) $\forall X \in \mathcal{A}$ : $P(X) \geq 0$

b) $P(\Omega) = 1$

c) $X$ and $Y$ are disjoint events (i.e. $XY = \emptyset$)

$\Longrightarrow P(X+Y) = P(X) + P(Y)$ additivity

c') $X_1, X_2, \ldots, X_n, \ldots$ are disjoints events

$\Longrightarrow P\left(\sum_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} P(X_i)$ $\sigma$-additivity

# Classical model

$\Omega = \{a_1, a_2, \dots, a_n\} \rightarrow$ n outcomes in total

$P(a_1) = P(a_2) = \dots = P(a_n) = \frac{1}{n}$

A is an event

$$P(A) = \frac{\text{number of outcomes in } A}{n}$$

$\cancel{1}, \cancel{2}, \cancel{3}, 4, \dots, 100$

50 numbers are taken at random

P( the smallest number is 4) = ?

$\binom{100}{50}$ is a total number of outcomes

$\binom{96}{49}$ favorable outcomes

$$P = \binom{96}{49} \Big/ \binom{100}{50}$$

10-digit numbers formed with digits 4, 5, 6, 7, 8 that follow in non-decreasing order (not all the digits have to be present)

4444778888

P(all 5 digits are "used" in this number)

$$* \;\; * \mid * \mid * \;\; * \;\; * \;\; * \mid * \;\; * \;\; * \mid$$

$$4 \;\; 4 \quad 5 \quad 6 \;\; 6 \;\; 6 \;\; 6 \quad 7 \;\; 7 \;\; 7$$

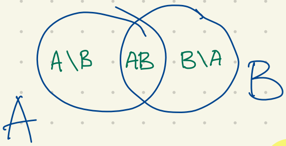
10 stars, 4 bars

$\binom{14}{4}$ is a total number of outcomes

$\binom{9}{4}$ is a number of favorable outcomes

$$P = \binom{9}{4} \Big/ \binom{14}{4}$$

$$P(A+B) = P(A) + P(B) \text{ if } A \text{ and } B \text{ are disjoint} \quad AB = \varnothing$$

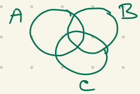



$$A+B = (A \setminus B) + (AB) + (B \setminus A)$$

$$P(A+B) = P(A \setminus B) + P(AB) + P(B \setminus A) =$$

$$= \underbrace{P(A \setminus B) + P(AB)} + \underbrace{P(B \setminus A) + P(AB)} - P(AB) =$$

$$= P(A) + P(B) - P(AB)$$




$AC+BC$

$$P((A+B)+C) = P(A+B) + P(C) - P((A+B)C) =$$

$$= P(A) + P(B) - P(AB) + P(C) - P(AC+BC) =$$

$$= P(A) + P(B) - P(AB) + P(C) - P(AC) - P(BC) +$$

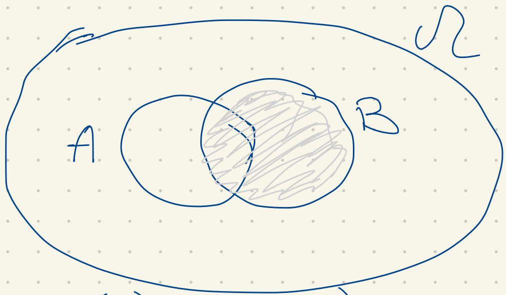
$$+ \underbrace{P(AC \cdot BC)}_{P(ABC)}$$

exclusion - inclusion formula

$$(A_1 + A_2 + \ldots + A_n)B =$$

$$= A_1 B + A_2 B + \ldots + A_n B$$

# Conditional probability




$P(B) \neq 0$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$P(AB) = P(B)\cdot P(A|B) = P(A)\cdot P(B|A)$$

$P(A) \neq 0$

$$P(ABC) = P(AB)\cdot P(C|AB) = P(A)\cdot P(B|A)\cdot P(C|AB)$$

$$P(ABCD) = P(A)\cdot P(B|A)\cdot P(C|AB)\cdot P(D|ABC)$$

A = getting an even number

B = getting "2"

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B)}{1/2} = \frac{1/6}{1/2} = \frac{1}{3}$$

MATHEMATICS $\quad 11$

$1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2$

$$\frac{\cancel{8}\ 1}{11\cdot 10\cdot 9\cdot \cancel{8}\cdot 7}$$

M A T H S

$P(X_1) \quad P(X_2|X_1) \quad P(X_3|X_1X_2) \quad P(X_4|X_1X_2X_3) \quad P(X_5|X_1X_2X_3X_4)$

$$\frac{2}{11}\cdot\frac{2}{10}\cdot\frac{2}{9}\cdot\frac{1}{8}\cdot\frac{1}{7}$$

$X_1,\ X_2,\ \ldots,\ X_5$

$X_i = \{$ the letter on position $i$ is the necessary one $\}$

$P(X_1 X_2 X_3 X_4 X_5)$

independent events

$$P(AB) = P(A) \cdot P(B)$$

$$\frac{P(AB)}{P(B)} = P(A)$$

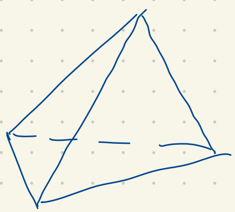
$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$A, B, C$

pairwise independence

$$\left.\begin{array}{l} \left\{\begin{array}{l} P(AB) = P(A)P(B) \\ P(AC) = P(A)P(C) \\ P(BC) = P(B)P(C) \end{array}\right. \\ P(ABC) = P(A)P(B)P(C) \end{array}\right\} \text{mutual independence}$$



green

blue

red

red blue green

$$P(A|BC) = P(A)$$

$$P(AB|C) = P(AB)$$

$$\frac{P(ABC)}{P(C)} = P(AB)$$

$G = \{$a face rolled contains green$\}$

$R = \{$a face rolled contains red$\}$

$B = \{$ ——— " ——— blue$\}$

$$P(B) = P(R) = P(G) = 1/2$$

$$P(BG) = P(BR) = P(GR) = 1/4$$

$$P(BGR) = 1/4$$

$$\underbrace{X_1, X_2, X_3, \ldots, X_n}_{\text{events}}$$ are (mutually) independent

$$P(X_{i_1} X_{i_2} \ldots X_{i_k} \mid X_{j_1} X_{j_2} \ldots X_{j_l}) = P(X_{i_1} X_{i_2} \ldots X_{i_k})$$

$$\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, 3, \ldots, n\}$$
$$\Rightarrow P(X_{i_1} X_{i_2} \ldots X_{i_k}) = P(X_{i_1}) \cdot P(X_{i_2}) \ldots P(X_{i_k})$$

def.

20 people; at least two of them have birthday on the same day

$365^{20}$ outcomes

$$1 - \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot \ldots \cdot 346}{365^{20}}$$