Lecture 3

Random variables

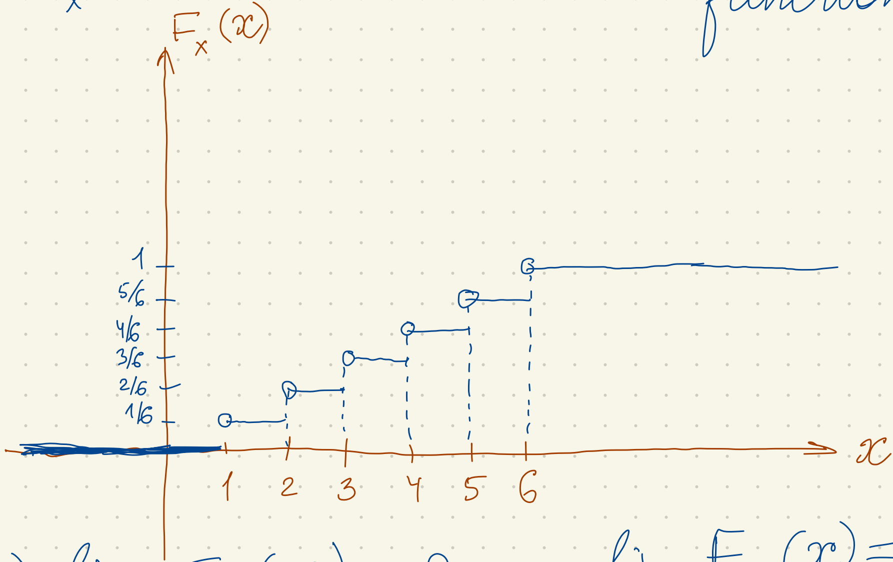
Ω

ω	1	2	3	4	5	6
X	1	2	3	4	5	6
y	-9	-6	-1	6	15	26

$F_X(x) = P(X < x)$ cumulative distribution function

$P(X<3) = \frac{2}{6}$

$P(X<2{,}5) = \frac{2}{6}$



1) $\lim\limits_{x \to -\infty} F_X(x) = 0$ $\quad \lim\limits_{x \to +\infty} F_X(x) = 1$

2) $F_X(x)$ is increasing

$x_1 < x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$

3) $F_X(x)$ is continuous on the left, $x \in \mathbb{R}$

Discrete random variables

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & x_3 & & & \end{array} \\ X \sim & \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix} \begin{matrix} x_i \\ p_i \end{matrix} \\ & \begin{array}{cccccc} p_1 & p_2 & p_3 & - & - & - \end{array} \end{array}$$

ω	1	2	3	4	5	6	7	8	9	10	11	12
X	1	1	2	2	3	3	4	4	5	5	6	6

$$EX = \sum_i x_i p_i$$

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3{,}5$$

expected value/expectation

$$EX = \sum_{\omega \in \Omega} X(\omega) p(\omega)$$

$$1 \cdot \frac{1}{12} + 1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + \ldots$$

$$1\left(\frac{1}{12} + \frac{1}{12}\right) + 2\left(\frac{1}{12} + \frac{1}{12}\right) + \ldots$$

EX exists if the series on the right converges absolutely.

$$Y = f(X)$$

e.g. $Y = X^2$

$$EY = \sum_{\omega \in \Omega} f(X(\omega))\,p(\omega) = \sum_{x_i} f(x_i)\,p(X = x_i)$$

x_i ← all possible values of X

$$= \sum_{y_i} y_i\, P(f(X) = y_i)$$

y_i → all possible values of Y

$\underline{Y = X^2}$

ω	1	2	3	4	5	6
$p(\omega)$	0,1	0,2	0,1	0,2	0,3	0,1
X	−2	−1	0	1	2	−1
Y	4	1	0	1	4	1

$$EY = 4\cdot 0{,}1 + \underline{1\cdot 0{,}2} + 0\cdot 0{,}1 + 1\cdot 0{,}2 + 4\cdot 0{,}3 + \underline{1\cdot 0{,}1}$$

$$X \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0{,}1 & 0{,}3 & 0{,}1 & 0{,}2 & 0{,}3 \end{pmatrix}$$

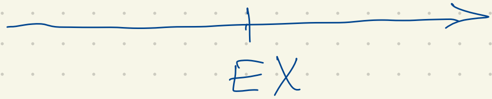
$$EY = (-2)^2\cdot 0{,}1 + \underline{(-1)^2\cdot 0{,}3} + 0^2\cdot 0{,}1 + 1^2\cdot 0{,}2 + 2^2\cdot 0{,}3$$

$$Y \sim \begin{pmatrix} 4 & 1 & 0 \\ 0{,}4 & 0{,}5 & 0{,}1 \end{pmatrix}$$

$$EY = 4\cdot 0{,}4 + 1\cdot 0{,}5 + 0\cdot 0{,}1$$

Variance

$$Var\, X = E(X-EX)^2$$



$$E(\alpha X) = \alpha EX, \quad \alpha = const$$

$$E(X \pm Y) = EX \pm EY$$

$$E(X+Y) = \sum_{\omega\in\Omega} \big(X(\omega)+Y(\omega)\big)p(\omega) = \sum_{\omega\in\Omega} X(\omega)p(\omega) + \sum_{\omega\in\Omega} Y(\omega)p(\omega) = EX + EY$$

$$EC = C, \quad C = const$$

$$\Big(\forall \omega\in\Omega : C(\omega) = C$$

$$EC = \sum_{\omega\in\Omega} C(\omega)p(\omega) = C\Big)$$

$$EX^{\alpha} = E(X^{\alpha})$$

$$(EX)^{\alpha}$$

$$Var\, X = EX^2 - (EX)^2$$

$$E(X-EX)^2 = E(X^2 - 2X\,EX + (EX)^2) = EX^2 - E(2X\,EX) + E((EX)^2) = EX^2 - 2EX\cdot EX + (EX)^2 = EX^2 - (EX)^2$$

$$Var(\alpha X) = E(\alpha X - E(\alpha X))^2 = \alpha^2\, Var\, X$$

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix} \qquad EX = \frac{7}{2}$$

$$X^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 & 36 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

$$EX^2 = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$$

$$\operatorname{Var} X = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{182-147}{12} = \frac{35}{12} \,//$$

$$(X-EX)^2 \sim \begin{pmatrix} 25/4 & 9/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\operatorname{Var} X = E(X-EX)^2 = \frac{25}{4}\cdot\frac{1}{3} + \frac{9}{4}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{35}{12} \,//$$

$$E(X-EX) = EX - E(EX) = EX - EX = 0$$

$$E|X-EX|$$

EX^n n-th moment of X

$E(X-EX)^n$ n-th central moment of X

$$EX^3 = \sum_{x_i} x_i^3 p_i$$ converges absolutely

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & k & \dots \\ c/1^3 & c/2^3 & c/3^3 & c/4^3 & \dots & c/k^3 & \dots \end{pmatrix}$$

$\frac{1}{c} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent $\qquad \sum_{k=1}^{\infty} \frac{c}{k^3} = 1$

$EX = \sum_{k=1}^{\infty} \frac{c}{k^2}$ converges

$EX^2 = \sum_{k=1}^{\infty} \frac{c}{k}$ diverges

Indicator random variable

an event

$$I_A = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} \qquad I_A \sim \begin{pmatrix} 1 & 0 \\ P(A) & 1-P(A) \end{pmatrix}$$

$$E I_A = 1 \cdot P(A) + 0 \cdot (1 - P(A)) = P(A)$$

$$E I_A^2 = E I_A = P(A)$$

$$Var\, I_A = P(A) - P(A)^2 = P(A)(1 - P(A))$$

Binomial distribution $\quad X \sim Bin(n, p)$ $\quad p+q=1$

$n \in \mathbb{N}, \quad 0 < p < 1$

$x \in \mathbb{Z}, \quad 0 \le x \le n$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

$$(p + (1-p))^n$$

$$X_i = \begin{cases} 1, & \text{there is a success in } i\text{-th trial} \\ 0, & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i$$

$$EX = \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} p = np$$

$$X(0101000100) = 3$$

If random variables $X_1, X_2, \dots, X_n$ are independent then $Var(X_1 + X_2 + \dots + X_n) = Var X_1 + Var X_2 + \dots + Var X_n$

$$Var X = \sum_{i=1}^{n} Var X_i = np(1-p) = npq$$

$$\begin{pmatrix} 0 & \dots & k & \dots & n \\ \dots & & \binom{n}{k} p^k q^{n-k} & \dots & \end{pmatrix}$$