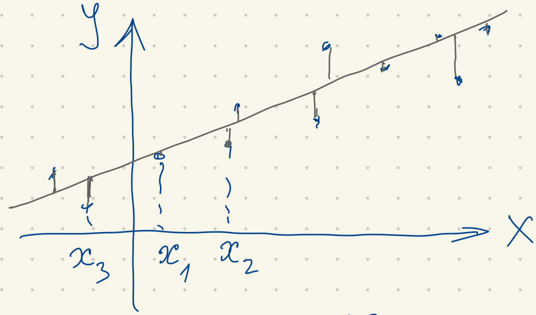
# November 19, 2020

## Linear regression




$(X, Y)$ $\quad Y = aX + b$

$y_i = \beta x_i + \gamma + \varepsilon_i$ — a random variable (values are unknown)

$x_i$ — numbers (known)

$\beta, \gamma$ — unknown constants

$E\varepsilon_i = 0$

$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are identically distributed & independent

$\operatorname{Var} y_i = \operatorname{Var} \varepsilon_i = \sigma^2$

$\alpha - \beta \bar{x} = \gamma$

$$y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i$$

$$S = \sum_{i=1}^{n} \left(y_i - \alpha - \beta(x_i - \bar{x})\right)^2 \longrightarrow \min_{\alpha, \beta}$$

$$\begin{cases} \dfrac{\partial S}{\partial \alpha} = -2 \sum_{i=1}^{n} \left(y_i - \alpha - \beta(x_i - \bar{x})\right) = 0 \\ \dfrac{\partial S}{\partial \beta} = -2 \sum_{i=1}^{n} (x_i - \bar{x})\left(y_i - \alpha - \beta(x_i - \bar{x})\right) = 0 \end{cases}$$

$$\sum_{i=1}^{n} y_i - n\alpha - \beta \cancel{\sum_{i=1}^{n} (x_i - \bar{x})}^{\,0} = 0 \qquad \underbrace{\sum x_i}_{} - n\bar{x}$$

$$\alpha = \frac{1}{n} \sum_{i=1}^{n} y_i = \bar{y}$$

$$\sum_{i=1}^{n} (x_i - \bar{x}) y_i - \alpha \cancel{\sum_{i=1}^{n} (x_i - \bar{x})}^{\,0} - \beta \sum_{i=1}^{n} (x_i - \bar{x})^2 = 0$$

$$\beta = \sum_{i=1}^{n} (x_i - \bar{x}) y_i \Big/ \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{S_{xy}}{S_{xx}}$$

$$S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} (x_i - \bar{x}) y_i - \cancelto{0}{\sum_{i=1}^{n} (x_i - \bar{x})}\, \bar{y}$$

$$\hat{\alpha} = \bar{y}, \quad \hat{\beta} = \frac{S_{xy}}{S_{xx}}$$

$$E\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} E y_i = \frac{1}{n} \sum_{i=1}^{n} \left(\alpha + \beta(x_i - \bar{x})\right) =$$

$$= \frac{1}{n} \cdot n\alpha + \frac{\beta}{n} \sum_{i=1}^{n} (x_i - \bar{x}) = \alpha$$

$$Var\,\hat{\alpha} = \frac{1}{n^2} \sum_{i=1}^{n} Var\, y_i = \frac{\sigma^2}{n}$$

$$E\hat{\beta} = \frac{1}{S_{xx}} E S_{xy} = \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i - \bar{x}) E y_i =$$

$$= \frac{1}{S_{xx}} \sum_{i=1}^{n} (x_i - \bar{x})\left(\alpha + \beta(x_i - \bar{x})\right) =$$

$$= \frac{1}{S_{xx}} \left( \cancelto{0}{\sum_{i=1}^{n} \alpha (x_i - \bar{x})} + \beta \sum_{i=1}^{n} (x_i - \bar{x})^2 \right) = \beta$$

$$Var\,\hat{\beta} = \frac{1}{S_{xx}^2} Var\, S_{xy} = \frac{1}{S_{xx}^2} \sum_{i=1}^{n} Var\left((x_i - \bar{x}) y_i\right) =$$

$$= \frac{1}{S_{xx}^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \sigma^2 = \frac{\sigma^2}{S_{xx}}$$

$$Cov(\hat{\alpha}, \hat{\beta}) = Cov\left(\bar{y}, \frac{S_{xy}}{S_{xx}}\right) =$$

$$= \frac{1}{n S_{xx}} Cov\left(\sum_{i=1}^{n} y_i , \sum_{j=1}^{n} (x_j - \bar{x}) y_j\right) =$$

$$= \frac{1}{n S_{xx}} \sum_{i,j=1}^{n} Cov\left(y_i, (x_j - \bar{x}) y_j\right) =$$

$$= \frac{1}{n S_{xx}} \sum_{i=1}^{n} (x_i - \bar{x}) \underbrace{Cov(y_i, y_i)}_{Var\, y_i = \sigma^2} = 0$$

$$\tilde{\alpha} = \sum_{i=1}^{n} a_i y_i \quad \text{is unbiased}$$

$$E\tilde{\alpha} = \alpha$$

$$E\tilde{\alpha} = \sum_{i=1}^{n} a_i E y_i = \sum_{i=1}^{n} a_i \left(\alpha + \beta(x_i - \overline{x})\right) =$$

$$= \alpha \sum_{i=1}^{n} a_i + \beta \sum_{i=1}^{n} a_i (x_i - \overline{x}) = \alpha$$

$$\sum_{i=1}^{n} a_i = 1, \quad \sum_{i=1}^{n} a_i (x_i - \overline{x}) = 0$$

$$\operatorname{Var} \tilde{\alpha} = \sum_{i=1}^{n} a_i^2 \operatorname{Var} y_i = \sigma^2 \sum_{i=1}^{n} a_i^2 \longrightarrow \min$$

$$\sum_{i=1}^{n} a_i^2 \longrightarrow \min \quad \sum_{i=1}^{n} a_i = 1, \quad \sum_{i=1}^{n} a_i (x_i - \overline{x}) = 0$$

$$L = \sum_{i=1}^{n} a_i^2 + \lambda \left(\sum_{i=1}^{n} a_i - 1\right) + \mu \sum_{i=1}^{n} a_i (x_i - \overline{x})$$

$$\frac{\partial L}{\partial a_i} = 2a_i + \lambda + \mu (x_i - \overline{x}) = 0, \quad i \in [1, n] \quad 1 \cdot (x_i - \overline{x})$$

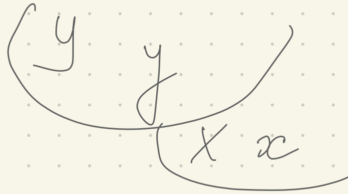
$$2\sum_{i=1}^{n} a_i + n\lambda = 0 \qquad \lambda = -\frac{2}{n} \sum_{i=1}^{n} a_i = -\frac{2}{n}$$

$$2\sum_{i=1}^{n} a_i (x_i - \overline{x}) + \mu \sum_{i=1}^{n} (x_i - \overline{x})^2 = 0 \Rightarrow \mu = 0$$

$$2a_i - \frac{2}{n} = 0 \Rightarrow a_i = \frac{1}{n}$$

$$\tilde{\alpha} = \sum_{i=1}^{n} \frac{1}{n} y_i = \overline{y}$$

$$\varepsilon_i \sim N(0,\sigma^2) \qquad y=\hat{\alpha}+\hat{\beta}(x-\bar{x})$$




$$R=\sum_{i=1}^{n}\left(y_i-\hat{\alpha}-\hat{\beta}(x_i-\bar{x})\right)^2$$

1) $\hat{\alpha}\sim N(\alpha;\ \sigma^2/n)$; $\hat{\beta}\sim N(\beta;\ \sigma^2/S_{xx})$,

2) $\dfrac{R}{\sigma^2}\sim\chi^2_{n-2}$; 3) $R,\hat{\alpha},\hat{\beta}$ are independent

$$A=\begin{pmatrix} 1/\sqrt{n} & (x_1-\bar{x})/\sqrt{S_{xx}} & \vdots & \vdots & a_{1j} & \vdots \\ 1/\sqrt{n} & (x_2-\bar{x})/\sqrt{S_{xx}} & \vdots & \vdots & a_{2j} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/\sqrt{n} & (x_n-\bar{x})/\sqrt{S_{xx}} & \vdots & \vdots & a_{nj} & \vdots \end{pmatrix}$$
— an orthogonal matrix

$$\sum_{i=1}^{n}\frac{x_i-\bar{x}}{\sqrt{n}\sqrt{S_{xx}}}=0$$

$$\vec{Z}=A^T\vec{y}$$

$$Z_1=\sum_{i=1}^{n}\frac{1}{\sqrt{n}}y_i=\sqrt{n}\sum_{i=1}^{n}\frac{y_i}{n}=\sqrt{n}\,\bar{y}=\sqrt{n}\,\hat{\alpha}$$

$$Z_2=\sum_{i=1}^{n}\frac{(x_i-\bar{x})y_i}{\sqrt{S_{xx}}}=\sqrt{S_{xx}}\sum_{i=1}^{n}\frac{(x_i-\bar{x})y_i}{S_{xx}}=\sqrt{S_{xx}}\,\hat{\beta}$$

$$\sum_{i=1}^{n}Z_i^2=\sum_{i=1}^{n}y_i^2 \quad \text{(since } A \text{ is orthogonal)}$$

$$R=\sum_{i=1}^{n}\left(y_i^2+\hat{\alpha}^2+\hat{\beta}^2(x_i-\bar{x})^2-2\hat{\alpha}y_i-2\hat{\beta}(x_i-\bar{x})y_i+\right.$$
$$\left.+2\hat{\alpha}\hat{\beta}(x_i-\bar{x})\right)=\sum_{i=1}^{n}y_i^2\underline{+n\hat{\alpha}^2}+\hat{\beta}^2S_{xx}\underbrace{-2n\hat{\alpha}\bar{y}}_{\hat{\alpha}}-$$
$$-2\hat{\beta}\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})y_i}=\sum_{i=1}^{n}y_i^2-n\hat{\alpha}^2-\hat{\beta}^2S_{xx}$$

$$\frac{S_{xy}}{S_{xx}}\cdot S_{xx}=\hat{\beta}S_{xx}$$

$$\sum_{i=3}^{n} z_i^2 = \sum_{i=1}^{n} z_i^2 - z_1^2 - z_2^2 = \sum_{i=1}^{n} y_i^2 - z_1^2 - z_2^2 =$$

$$= \sum_{i=1}^{n} y_i^2 - n\hat{\alpha}^2 - S_{xx}\hat{\beta}^2 = R$$

$$j \geq 3 \Rightarrow Ez_j = E\left(\sum_{i=1}^{n} a_{ij} y_i\right) =$$

$$= \sum_{i=1}^{n} a_{ij} Ey_i = \sum_{i=1}^{n} a_{ij}(\alpha + \beta(x_i - \bar{x})) =$$

$$= \alpha \sum_{i=1}^{n} a_{ij} + \beta \sum_{i=1}^{n} a_{ij}(x_i - \bar{x}) = 0 + 0 = 0$$

$$\operatorname{Var} z_j = \sum_{i=1}^{n} a_{ij}^2 \operatorname{Var} y_i = \sigma^2 \sum_{i=1}^{n} a_{ij}^2 = \sigma^2$$

For $j \geq 3 \Rightarrow z_j \sim N(0; \sigma^2)$

$$z_j/\sigma \sim N(0; 1)$$

$$\underbrace{\sum_{j=3}^{n} \left(\frac{z_j}{\sigma}\right)^2}_{= \frac{R}{\sigma^2}} \sim \chi^2_{n-2}$$

$$E\left(\frac{R}{\sigma^2}\right) = n-2$$

$$ER = \sigma^2(n-2)$$

$X_1, X_2, \dots, X_n \sim N(\mu; \sigma^2)$, independent

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\frac{S_{xx}}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

<u>Fisher's lemma:</u> $\overline{X}$ and $\frac{S_{xx}}{\sigma^2}$ are independent, $\overline{X} \sim N(\mu; \sigma^2/n)$

$$\frac{S_{xx}}{\sigma^2} \sim \chi^2_{n-1}$$

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}\left((X_i-\overline{X})+(\overline{X}-\mu)\right)^2 = \sum_{i=1}^{n}(X_i-\overline{X})^2 +$$

$$+\sum_{i=1}^{n}(\overline{X}-\mu)^2 + 2\sum_{i=1}^{n}(X_i-\overline{X})(\overline{X}-\mu) = \sum_{i=1}^{n}(X_i-\overline{X})^2 +$$

$$+n(\overline{X}-\mu)^2$$

$$A = \begin{pmatrix} 1/\sqrt{n} & \vdots & a_{1j} & \vdots \\ 1/\sqrt{n} & \vdots & a_{2j} & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1/\sqrt{n} & \vdots & a_{nj} & \vdots \end{pmatrix}$$ — an orthogonal matrix

$$\vec{\mu} \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}$$

$$y_1 = \sum_{i=1}^{n}\frac{1}{\sqrt{n}}(X_i-\mu) = \sqrt{n}\,\overline{X} - \sqrt{n}\,\mu$$

$$\vec{y} = A^T(\vec{X} - \vec{\mu})$$

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\overline{X})^2 + \overbrace{n(\overline{X}-\mu)^2}^{y_1^2} =$$

$$= S_{xx} + y_1^2 \Longrightarrow \sum_{i=2}^{n} y_i^2 = S_{xx}$$

$$j \geq 2 \Rightarrow \quad y_j = \sum_{i=1}^{n} a_{ij} (x_i - \mu)$$

$$E y_j = \sum_{i=1}^{n} a_{ij} (E x_i - \mu) = 0$$

$$Var\, y_j = \sum_{i=1}^{n} a_{ij}^2 \, Var(x_i - \mu) = \sigma^2 \sum_{i=1}^{n} a_{ij}^2 = \sigma^2$$

$$\frac{S_{xx}}{\sigma^2} = \sum_{i=2}^{n} \left(\frac{y_i}{\sigma}\right)^2 \sim \chi^2_{n-1}$$