Assignment 8.

N1. An average of 1 typo per page in a 600-page book means there's 600 typos in the book.

Let T_{13} be the amount of typos on page 13, $T_{13} \sim Bin(600, \frac{1}{600})$

Then $\lambda = 600 \cdot \frac{1}{600} = 1 \Rightarrow P(T_{13} = k) \approx \frac{\lambda^k}{k!} e^{-\lambda} \Rightarrow P(T_{13} = 0) \approx \frac{1}{1} e^{-1} = \frac{1}{e} = 0{,}368$ (a)

(b)

$P(T_{13} = 2) = \frac{1^2}{2} e^{-1} = \frac{1}{2e} = 0{,}184$

$P(T_{13} \leq 3) = P(T_{13} = 0) + P(T_{13} = 1) + P(T_{13} = 2) + P(T_{13} = 3) = \frac{1}{e} + \frac{1}{e} + \frac{1}{2e} + \frac{1}{6e} = \frac{16}{6e} = 0{,}981$ (c)

N2. Let $p=0{,}003$ be the probability that a scone doesn't have any raisins.
Let S be the amount of scones without raisins within a batch of 1000 scones.

$S \sim Bin(1000,\ 0{,}003)$, $\lambda = 1000 \cdot 0{,}003 = 3$

Therefore, $P(S=k) \approx \frac{\lambda^k}{k!}e^{-\lambda} = \frac{3^k}{k!e^3}$

a) $P(S=0) \approx \frac{3^0}{0!e^3} = \frac{1}{e^3} = 0{,}05$

b) $P(S=3) \approx \frac{3^3}{3!e^3} = \frac{27}{6e^3} = 0{,}224$

c) $P(S \geqslant 3) = 1 - P(S<3) = 1-(P(S=0)+P(S=1)+P(S=2)) \approx 1-(0{,}05 + \frac{3}{e^3} + \frac{9}{2e^3}) = 0{,}576$

N3. Let R be the amount of raisins in a scone.
Assuming there is x raisins among 1000 scones, we have $R \sim Bin(x, \frac{1}{1000})$

$\lambda = x \cdot \frac{1}{1000} = ER$ - the average amount of raisins in a scone.

$P(R=0) \approx \frac{\lambda^0}{0!e^\lambda} = \frac{1}{e^\lambda}$, we want λ: $\frac{1}{e^\lambda} \leqslant 0{,}01 \Rightarrow \lambda \geqslant -\ln(0{,}01) = 4{,}6$

Therefore, we need 4,6 raisins per scone on average.

N4. Let $N=5000$ be the amount of electors, $p=0{,}7$, $q=0{,}3$
Let ξ be the amount of electors who voted for candidate A, then $N-\xi$ people voted for B.

$\xi \sim Bin(N,p)$, $\xi = \sum_{i=1}^{N} \xi_i$, where $\xi_i \sim Bin(1,p)$ - indicator that the i-th elector voted for A.

Since ξ_i are independent and identically distributed, one can apply the Central Limit Theorem:

$P(\frac{\xi - E\xi}{\sqrt{Var\,\xi}} < y) = P(\frac{\xi - Np}{\sqrt{Npq}} < y) \approx \Phi(y) \Rightarrow P(\xi < x) \approx \Phi(\frac{x-Np}{\sqrt{Npq}})$

a) $P(\xi-(N-\xi)=1900) = P(\xi = \frac{1900+N}{2}) = P(\xi = 3450) \approx \frac{P(\xi \leqslant 3450{,}5) - P(\xi \leqslant 3449{,}5)}{1} \approx$

$\approx \Phi(\frac{3450{,}5-Np}{\sqrt{Npq}}) - \Phi(\frac{3449{,}5-Np}{\sqrt{Npq}}) = \Phi(-1{,}527) - \Phi(-1{,}558) = 0{,}0037$

b) $P(\xi-(N-\xi) \geqslant 1900) \approx P(\mu \geqslant -1{,}54) = 1 - \Phi(-1{,}54) = 1-0{,}063 = 0{,}937$

N5. Let ξ be the amount of people that went in through the first entrance.

Case (a): $\xi_a \sim 2\,Bin(500, \frac{1}{2})$, $E\xi_a = 2 \cdot 500 \cdot \frac{1}{2} = 500$, $Var\,\xi_a = 2^2 \cdot 500 \cdot \frac{1}{2} \cdot \frac{1}{2} = 500$

Case (b): $\xi_b \sim Bin(1000, \frac{1}{2})$, $E\xi_b = 1000 \cdot \frac{1}{2} = 500$, $Var\,\xi_b = 1000 \cdot \frac{1}{2} \cdot \frac{1}{2} = 250$

$\xi_a = \sum_{i=1}^{500} \xi_{ai}$, where $\xi_{ai} \sim \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ - i.i.d.

$\xi_b = \sum_{i=1}^{1000} \xi_{bi}$, where $\xi_{bi} \sim Bin(1, \frac{1}{2})$ - i.i.d.

$\Rightarrow$ for ξ_a and ξ_b one can apply the Central Limit Theorem

$P(\frac{\xi_a - E\xi_a}{\sqrt{Var\,\xi_a}} < y) = P(\frac{\xi_a - 500}{\sqrt{500}} < y) \approx \Phi(y) \Rightarrow P(\xi_a < y) \approx \Phi(\frac{x-500}{\sqrt{500}})$, $P(\xi_b < x) \approx \Phi(\frac{x-500}{\sqrt{250}})$

Assuming that k_a and k_b are the amounts of cloakroom places in cases (a) and (b), one needs to find k_a, k_b:

$P(\xi_a \leqslant k_a) = 0{,}99$

$P(\xi_b \leqslant k_b) = 0{,}99$

Note: since $\frac{\xi_a}{2}$ and ξ_b are binomially distributed, and $p = \frac{1}{2} = 1-p$, $P(\xi_b \leqslant k_b) = 0{,}99$ implies $P(1000-\xi_b \leqslant k_b) = 0{,}99$, since $P(\xi_b \leqslant x) = P(\xi_b \geqslant 1000-x)$ $\forall x \in [0; 1000]$

$P(\xi_a \le k_a) \approx \Phi\left(\frac{k_a-500}{\sqrt{500}}\right) = 0,99 \qquad \frac{k_a-500}{\sqrt{500}} = 2,575 \Rightarrow k_a = 2,575\sqrt{500}+500 = 557,6$

$P(\xi_B \le k_B) \approx \Phi\left(\frac{k_B-500}{\sqrt{250}}\right) = 0,99 \Rightarrow \frac{k_B-500}{\sqrt{250}} = 2,575 \Rightarrow k_B = 2,575\sqrt{250}+500 = 540,7$

N6. Since a coin is fair, the probability of getting tails is $p=0,5$

Let ξ_k be the percentage of tails among k throws (expressed as a ratio)

Then $\xi_k \sim \frac{1}{k}Bin(k, \frac{1}{2})$, $E\xi_k = \frac{1}{k}\cdot k \cdot \frac{1}{2} = \frac{1}{2}$, $Var\,\xi_k = (\frac{1}{k})^2 \cdot k \cdot \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4k}$

One needs to find min k: $P(p-0,01 \le \xi_k \le p+0,01) = 0,95$

$P(\xi_k \in [0,49;\ 0,51]) \approx \Phi\left(\frac{0,51-\frac{1}{2}}{\frac{1}{\sqrt{4k}}}\right) - \Phi\left(\frac{0,49-\frac{1}{2}}{\frac{1}{\sqrt{4k}}}\right) = \Phi(0,02\sqrt{k}) - \Phi(-0,02\sqrt{k}) = 0,95$

$\Phi(x) - \Phi(-x) = (0,5+\Phi_0(x)) - (0,5+\Phi_0(-x)) = \Phi_0(x) - \Phi_0(-x) = 2\Phi_0(x)$, since Φ_0 is odd

Thus, $2\Phi_0(0,02\sqrt{k}) = 0,95 \Rightarrow \Phi(0,02\sqrt{k}) = 0,5 + \frac{0,95}{2} = 0,975$

According to the standard normal table,

$\Phi(0,02\sqrt{k}) = 0,975 \Leftrightarrow 0,02\sqrt{k} = 1,96 \Rightarrow k = 9604$

Therefore, the minimum amount of flips is 9604.

N7. Since a die is fair, the probability of getting a "4" is $\frac{1}{6}$.

Let ξ_k be the percentage of "4"s among k rolls (expressed as a ratio)

Then $\xi_k \sim \frac{1}{k}Bin(k, \frac{1}{6})$, $E\xi_k = \frac{1}{k}\cdot k\cdot\frac{1}{6} = \frac{1}{6}$, $Var\,\xi_k = (\frac{1}{k})^2\cdot k\cdot\frac{1}{6}\cdot\frac{5}{6} = \frac{5}{36k}$

One needs to find min k_a: $P(\frac{1}{6}-0,01 \le \xi_{k_a} \le \frac{1}{6}+0,01) = 0,95$

$P(\xi_{k_a} \in [\frac{1}{6}-0,01;\ \frac{1}{6}+0,01]) = \Phi\left(\frac{0,01}{\sqrt{\frac{5}{36k_a}}}\right) - \Phi\left(\frac{-0,01}{\sqrt{\frac{5}{36k_a}}}\right) = 0,95$

As shown in the previous problem, $P(\xi_{k_a} \in [\frac{1}{6}-0,01;\ \frac{1}{6}+0,01]) = 2\Phi_0(\frac{0,06}{\sqrt{5}}\sqrt{k_a})$

According to the standard normal table, $\Phi(\frac{0,06}{\sqrt{5}}\sqrt{k_a}) = 0,5 + \frac{0,95}{2}$ for $\frac{0,06}{\sqrt{5}}\sqrt{k_a} = 1,96$

Thus, $k_a = \left(\frac{1,96\sqrt{5}}{0,06}\right)^2 = 5335,5$ the minimum amount of rolls is $\lceil 5335,5 \rceil = 5336$

One also needs to find min k_B: $P(\frac{1}{6}-0,01 \le \xi_{k_B} \le \frac{1}{6}+0,01) > 0,99$

$2\Phi_0(\frac{0,06}{\sqrt{5}}\sqrt{k_B}) > 0,99$ for $\frac{0,06}{\sqrt{5}}\sqrt{k_B} > 2,575$ (Φ is $\nearrow$)

Thus, $k_B > \left(\frac{2,575\sqrt{5}}{0,06}\right)^2 = 9209,2$ so the minimum amount of rolls is $\lceil 9209,2 \rceil = 9210$

differs from the answer because 2,575 was not rounded to 2,57

Common note for N6 and N7: the Central Limit Theorem requires ξ_k to be a sum of i.i.d. random variables. To satisfy this condition, one could express ξ_k as a sum of ξ_{k_i} for i-th throw/roll, then the distribution of ξ_{k_i} will have 1 experiment instead of k.

N8. The time to cycle 3 miles on a wet day is $\frac{3}{10} = 0,3$ hours

— " — dry — " — $\frac{3}{20} = 0,15$ hours

Let T_i be the time to cycle on the i-th day of the course.

$T_i \sim \begin{pmatrix} 0,3 & 0,15 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \forall i \Rightarrow ET_i = \frac{0,45}{2} = 0,225$ hours

Let T be the total cycling time over the whole course.

$T = \sum_{i=1}^{24} T_i \Rightarrow ET = \sum_{i=1}^{24} ET_i = 24\cdot\frac{0,45}{2} = 12\cdot 0,45 = 5,4$ hours

N9. Let $X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}$ be independent multivariate normal

Let $\bar{\mu}_X = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \end{bmatrix}$ and $\Sigma_X = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_1^2 \end{bmatrix}$ be the mean vector and covariance matrix of $\bar{X}$,

$\bar{\mu}_Y = \begin{bmatrix} \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix}$ and $\Sigma_Y = \begin{bmatrix} \sigma_2^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_2^2 \end{bmatrix}$ be the mean vector and covariance matrix of $\bar{Y}$.

$\Sigma_X[i,j] = \Sigma_Y[i,j] = 0$ for $i \neq j$ since X_i and Y_i are all mutually independent.

$f_X(x_1, x_2, \dots, x_n) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det \Sigma_X}} e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{1}{\sigma_1^2}(x_i-\mu_1)^2}$, $f_Y(y_1, y_2, \dots, y_m) = \frac{1}{(\sqrt{2\pi})^m \sqrt{\det \Sigma_Y}} e^{-\frac{1}{2}\sum_{i=1}^{m} \frac{1}{\sigma_2^2}(y_i-\mu_2)^2}$

Since X and Y are independent, $f_{X,Y}(x_1, \dots, x_n, y_1, \dots, y_m) = f_X(x_1, \dots, x_n) \cdot f_Y(y_1, \dots, y_m)$

Then we can introduce $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$, $\bar{\mu}_Z = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix}$, $\Sigma_Z = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_1^2 & & \\ & & \sigma_2^2 & \\ 0 & & & \sigma_2^2 \end{bmatrix}$

Z is multivariate normal since $f_Z(\bar{z}) = f_{X,Y}(\bar{x}, \bar{y}) = \frac{1}{(\sqrt{2\pi})^{n+m} \sqrt{\det \Sigma_X \det \Sigma_Y}} e^{-\frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{\sigma_1^2}(x_i-\mu_1)^2 + \sum_{j=1}^{m} \frac{1}{\sigma_2^2}(y_j-\mu_2)^2\right)}$

According to the properties of a multivariate normal distribution, any linear combination of Z with coefficients $c_1, \dots, c_{m+n}$ is normally distributed like:

$[c_1 \cdots c_{m+n}] \times Z \sim N\left([c_1 \cdots c_{m+n}] \times \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \end{bmatrix}; [c_1 \cdots c_{m+n}] \times \Sigma_Z \times \begin{bmatrix} c_1 \\ \vdots \\ c_{m+n} \end{bmatrix}\right)$

Therefore, $\bar{X} - \bar{Y} = \sum_{i=1}^{n} \frac{1}{n} X_i - \sum_{j=1}^{m} \frac{1}{m} Y_j \sim N\left(n \cdot \frac{1}{n}\mu_1 - m \cdot \frac{1}{m}\mu_2; n\left(\frac{1}{n}\right)^2 \cdot \sigma_1^2 + m \cdot \left(-\frac{1}{m}\right)^2 \cdot \sigma_2^2\right)$

$\sim N\left(\mu_1 - \mu_2; \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)$

Let $U = \bar{X} - \bar{Y}$, then $f_U(t) = \frac{1}{\sqrt{2\pi\left(\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)}} e^{-\frac{1}{2} \cdot \frac{1}{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}(t - \mu_1 + \mu_2)^2}$

Let $V = \frac{1}{\sigma_1^2}\sum_{i=1}^{n} X_i + \frac{1}{\sigma_2^2}\sum_{j=1}^{m} Y_j \sim N\left(n \cdot \frac{\mu_1}{\sigma_1^2} + m \cdot \frac{\mu_2}{\sigma_2^2}; n \cdot \left(\frac{1}{\sigma_1^2}\right)^2 \cdot \sigma_1^2 + m \cdot \left(\frac{1}{\sigma_2^2}\right)^2 \cdot \sigma_2^2\right)$

$\sim N\left(\frac{n\mu_1}{\sigma_1^2} + \frac{m\mu_2}{\sigma_2^2}; \frac{n}{\sigma_1^2} + \frac{m}{\sigma_2^2}\right)$

Then $f_V(t) = \frac{1}{\sqrt{2\pi\left(\frac{n}{\sigma_1^2} + \frac{m}{\sigma_2^2}\right)}} e^{-\frac{1}{2} \cdot \frac{1}{\frac{n}{\sigma_1^2} + \frac{m}{\sigma_2^2}} \cdot \left(t - \frac{n\mu_1}{\sigma_1^2} - \frac{m\mu_2}{\sigma_2^2}\right)^2}$

N10. Let $X, Y \sim N(0; 1)$ be independent RVs

Treating them as univariate normal, we have that $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ is bivariate normal,

$Z \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ (similarly to N9)

According to the properties of a multivariate normal distribution, a vector composed of linear combinations of a multivariate normal vector is also multivariate normal:

$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \times Z \sim N\left(\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\right)$

$W = \begin{bmatrix}1&1\\1&-1\end{bmatrix} \times Z = \begin{bmatrix}X+Y\\X-Y\end{bmatrix}$, $W \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}; \begin{bmatrix}2&0\\0&2\end{bmatrix}\right)$

$f_W(x,y) = \frac{1}{2\pi \cdot 2} e^{-\frac{x^2+y^2}{4}}$

$X+Y = [1\ 1] \times Z$, $X-Y = [1\ -1] \times Z$, both are normally distributed:

$f_{X+Y}(u) = \frac{1}{\sqrt{4\pi}} e^{-\frac{u^2}{4}}$, $f_{X-Y}(v) = \frac{1}{\sqrt{4\pi}} e^{-\frac{v^2}{4}}$

$f_W(x,y) = f_{X+Y}(x) \cdot f_{X-Y}(y) \Rightarrow X+Y$ and $X-Y$ are independent

Let $Z' = |Y| \cdot (-1)^{I_{X<0}} = \begin{cases} |Y|, & X \geq 0 \\ -|Y|, & X<0 \end{cases}$

$P(Z'<t) = P(X \geq 0) \cdot P(|Y|<t) + P(X<0) \cdot P(|Y|>-t) = 0{,}5(P(|Y|<t) + P(|Y|>-t))$

$$P(Z'<t) = \begin{cases} 0{,}5(P(|Y|<t)+1), & t \geq 0 \quad (1) \\ 0{,}5(P(|Y|>-t)), & t<0 \quad (2) \end{cases}$$

$(1) = 0{,}5 + 0{,}5P(-t<Y<t) = 0{,}5 + 0{,}5\Phi(t) - 0{,}5\Phi(-t) = 0{,}5 + 0{,}5 \cdot (0{,}5 + \Phi_0(t)) - 0{,}5(0{,}5+\Phi_0(-t)) =$

$= 0{,}5 + 0{,}5\Phi_0(t) - 0{,}5\Phi_0(-t) = 0{,}5 + \Phi_0(t) = \Phi(t)$, since Φ_0 is odd

$(2) = 0{,}5(\Phi(t) + (1-\Phi(-t))) = 0{,}5 + 0{,}5\Phi(t) - 0{,}5\Phi(-t) = \ldots = \Phi(t)$ (similarly)

Thus, $P(Z'<t) = \Phi(t) \quad \forall t \in \mathbb{R} \Rightarrow Z' \sim N(0;1)$

To show that $\begin{bmatrix}X\\Z'\end{bmatrix}$ is not bivariate normal, lets consider the joint CDF:

$$P(X<u, Z'<v) = \begin{cases} P(X<u, -|Y|<v), & u<0 \\ P(0<X<u, |Y|<v) + P(X<0, -|Y|<v), & u \geq 0 \end{cases}$$

(X and Y are independent) $$= \begin{cases} P(X<u)P(|Y|>-v), & u<0,\ v \geq 0 \\ P(X<u)P(Y>-v) + P(X<u)P(Y<v), & u<0,\ v<0 \\ P(X<0)P(Y>-v) + P(X<0)P(Y<v), & u \geq 0,\ v<0 \\ P(X<0)P(|Y|>-v) + P(0<X<u)P(-v<Y<v), & u \geq 0,\ v \geq 0 \end{cases}$$

For $u<0$, $v \geq 0$ $P(X<u, Z'<v) = P(X<u)P(|Y|>-v) = P(X<u)$, not dependent on v.

Therefore, there is no $\bar{\mu}$ and Σ that would form a bivariate normal PDF $\Rightarrow$

$\Rightarrow X$ and Z' are not bivariate normal.

<u>N11.</u> $X, Y \sim N(0;1)$, independent $\Rightarrow Z = \begin{bmatrix}X\\Y\end{bmatrix}$ is bivariate normal

Therefore, $X+Y \sim N\left([1\ 1] \times \begin{bmatrix}0\\0\end{bmatrix}; [1\ 1]\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right) = N(0;2)$

$P(X^2<u) = P(-\sqrt{u}<X<\sqrt{u}) \cdot I_{u>0} = (2\Phi(\sqrt{u})-1) \cdot I_{u>0}$

$f_{X^2}(u) = (P(X^2<u))'_u = I_{u>0} \cdot 2\Phi(\sqrt{u}) \cdot \frac{1}{2\sqrt{u}} = I_{u>0} \cdot \frac{1}{\sqrt{2\pi u}} e^{-\frac{u}{2}}$

$$F_{X^2,Y^2}(u,v) = P(X^2<u,\ Y^2<v) = P(-\sqrt{u}<X<\sqrt{u},\ -\sqrt{v}<Y<\sqrt{v})\cdot I_{u>0}\cdot I_{v>0}$$

X and Y are independent $\longrightarrow$ $= \underbrace{P(-\sqrt{u}<X<\sqrt{u})\cdot I_{u>0}}_{F_{X^2}(u)} \cdot \underbrace{P(-\sqrt{v}<Y<\sqrt{v})\cdot I_{v>0}}_{F_{Y^2}(v)}$

Therefore, X^2 and Y^2 are also independent.

$$f_{X^2,Y^2}(u,v) = f_{X^2}(u)\cdot f_{Y^2}(v) = \frac{1}{2\pi\sqrt{uv}}e^{-\frac{u}{2}-\frac{v}{2}}\cdot I_{u>0}\cdot I_{v>0}$$

According to the convolution formula: $f_{X^2+Y^2}(w) = \int_{-\infty}^{+\infty} f_{X^2,Y^2}(w-v,\ v)\,dv$

$$f_{X^2+Y^2}(w) = \int_{-\infty}^{+\infty}\frac{1}{2\pi}\cdot\frac{1}{\sqrt{(w-v)v}}e^{-\frac{w}{2}}\cdot I_{w>v}\cdot I_{v>0}\,dv = \frac{1}{2\pi}e^{-\frac{w}{2}}\cdot\int_0^w\frac{dv}{\sqrt{(w-v)v}} \underset{t=v-\frac{w}{2}}{\overset{\uparrow}{=}} \frac{1}{2\pi}e^{-\frac{w}{2}}\cdot\int_{-\frac{w}{2}}^{\frac{w}{2}}\frac{d(t+\frac{w}{2})}{\sqrt{(\frac{w}{2}-t)(\frac{w}{2}+t)}} =$$

$$= \frac{1}{2\pi}e^{-\frac{w}{2}}\cdot\int_{-\frac{w}{2}}^{\frac{w}{2}}\frac{dt}{\sqrt{\frac{w^2}{4}-t^2}} = \frac{1}{2\pi}e^{-\frac{w}{2}}\cdot\arcsin\frac{2t}{w}\Big|_{t=-\frac{w}{2}}^{t=\frac{w}{2}} = \frac{1}{2\pi}e^{-\frac{w}{2}}\pi = \frac{1}{2}e^{-\frac{w}{2}}$$