Limits with Variable Bases and Exponents

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WHAT'S COVERED

In this lesson, you will learn strategies to evaluate indeterminate forms that have both variable bases and exponents. Specifically, this lesson will cover:

1. The Strategy for Evaluating Limits With Variable Bases and Exponents

Consider a function that has the form $y = f(x)^{g(x)}$. Of all the possible behaviors of $f(x)$ and $g(x)$ that could occur in a limit, there are three situations that lead to indeterminate forms.

Form	Explanation
0^0	The base and exponent both approach 0.
∞^0	The base grows without bound and at the same time, the exponent approaches 0.
1^∞	When the base approaches 1 and at the same time, the exponent increases without bound.

Since L'Hopital's rule can only be applied to limits with indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, limits with the indeterminate forms 0^0, ∞^0, or 1^∞ will need to be manipulated in order to use L'Hopital's rule.

To see how to start, consider the identity $a = e^{\ln a}$, which is valid as long as $a > 0$.

Replacing a with $f(x)^{g(x)}$, we can write $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}}$.

By the property of logarithms, we know that $\ln(f(x)^{g(x)}) = g(x) \cdot \ln f(x)$, which allows us to write $f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)}$.

This also means that $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)}$.

The limit on the right-hand side suggests that we can focus on the exponent $g(x) \cdot \ln f(x)$, which is a product, something that we have already handled using L'Hopital's rule.

BIG IDEA

If $\lim_{x \to a} g(x) \cdot \ln f(x) = L$, then the limit we seek is $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)} = e^L$.

To summarize, these steps will help to evaluate limits with indeterminate forms 0^0, ∞^0, or 1^∞.

STEP BY STEP

To evaluate a limit with an indeterminate form 0^0, 1^∞, or ∞^0:

1. Let $y = f(x)^{g(x)}$. Then, $\ln y = g(x) \cdot \ln f(x)$.
2. Find $\lim_{x \to a} \ln y$.
3. Assuming that $\lim_{x \to a} \ln y = L$, we know $\lim_{x \to a} y = e^L$, where $y = f(x)^{g(x)}$.

Let's see how this methodology is applied to specific examples.

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2. Evaluating Limits With Variable Bases and Exponents

Now that we have a strategy, let's evaluate a few limits that have one of these indeterminate forms.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0^+} x^x$

Note that this is a limit of the form 0^0, which will use our new strategy:

1. Take the natural logarithm of x^x: $\ln x^x = x \ln x$
2. Now find the limit:

$\lim_{x \to 0^+} x^x$	Start with the limit that needs to be evaluated.
$\lim_{x \to 0^+} x \ln x$	Evaluate the limit of the natural logarithm of the function. This has the form $0 \cdot (-\infty)$, which is another indeterminate form.
$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}$	The strategy here is to rewrite as either $\frac{x}{\left(\frac{1}{\ln x}\right)}$ or $\frac{\ln x}{\left(\frac{1}{x}\right)}$. The latter is preferable.

$$= \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

The limit has the form $\frac{\infty}{\infty}$ and both numerator and denominator are differentiable, so L'Hopital's rule can be used.

$D\left[\frac{1}{x}\right] = D[x^{-1}] = -x^{-2} = \frac{-1}{x^2}$, $D[\ln x] = \frac{1}{x}$

$$= \lim_{x \to 0^+} (-x)$$

Simplify $\frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \frac{1}{x} \cdot \frac{x^2}{-1} = -x$.

$$= 0$$

Use direct substitution.

3. Then, the limit of the original function is $e^0 = 1$.

Thus, $\lim_{x \to 0^+} x^x = 1$.

WATCH

In this video, we will evaluate the limit $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$.

TRY IT

Consider the following limit: $\lim_{x \to \infty} x^{3/x}$

Evaluate the limit. +

$$\lim_{x \to \infty} x^{3/x} = 1$$

SUMMARY

In this lesson, you learned **the strategy for evaluating limits with variable bases and exponents**. For instance, when evaluating $\lim_{x \to a} f(x)^{g(x)}$ and the limit results in one of the indeterminate forms (0^0, 1^∞, and ∞^0), the limit will need to be manipulated using logarithms in order to use L'Hopital's rule.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.