Apply L'Hopital's Rule to the Indeterminate Forms "∞ - ∞" and "∞ * 0"

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WHAT'S COVERED

In this lesson, you will learn strategies to use when evaluating limits that have other indeterminate forms. Specifically, this lesson will cover:

1. The Indeterminate Form ∞ – ∞

The form $\infty - \infty$ occurs when there is a difference between two expressions that are both tending toward ∞ as $x \to a$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

Since $\frac{1}{x} \to \infty$ and $\frac{1}{x^2} \to \infty$ as $x \to 0^+$, we have a limit of the form $\infty - \infty$. One strategy is to write it as a single fraction, since this is a more familiar scenario.

Since $\frac{1}{x} - \frac{1}{x^2} = \frac{x}{x^2} - \frac{1}{x^2} = \frac{x-1}{x^2}$, we have the following:

$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2}\right)$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0^+} \left(\frac{x-1}{x^2}\right)$	Replace the expression with a single fraction.
$= \frac{\text{close to } -1}{\text{small positive number}}$	As x approaches 0 from the right, $x - 1$ approaches -1 and x^2 is a small positive number.

$= -\infty$ A negative number divided by a small positive number is a large negative number.

Thus, $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2} \right) = -\infty$.

BIG IDEA

Many might think that a limit of the form $\infty - \infty$ should be 0 since you are "subtracting something from itself." As we can see, this is not the case. Once we see $\infty - \infty$ produce another value, we will see why it is an indeterminate form.

WATCH

In this video, we'll evaluate $\lim_{x \to \infty} \left(\sqrt{x^2 + 9x} - x \right)$.

Considering the results from these last two examples, it is clear now why $\infty - \infty$ is an indeterminate form. In one case, the result was $-\infty$, and in another case, the result was $\frac{9}{2}$.

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2. The Indeterminate Form ∞ · 0

The indeterminate form $\infty \cdot 0$ is handled in one of two ways.

Loosely speaking, we can say that a limit of the form $\frac{1}{\infty}$ will approach 0 and a limit of the form $\frac{1}{0}$ will approach $\pm\infty$.

That said, we can treat "0" and "∞" as reciprocals as far as limits are concerned.

This means that the indeterminate form $\infty \cdot 0$ could be rewritten as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, whichever is more convenient.

EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} x^2 e^{-2x}$

If we look at each factor separately, we see that $x^2 \to \infty$ and $e^{-2x} \to 0$ as $x \to \infty$. Thus, this limit has the form $\infty \cdot 0$.

To rewrite, consider the fact that $e^{-2x} = \frac{1}{e^{2x}}$, which means $\lim_{x \to \infty} x^2 e^{-2x} = \lim_{x \to \infty} x^2 \frac{1}{e^{2x}} = \lim_{x \to \infty} \frac{x^2}{e^{2x}}$, which now has the form $\frac{\infty}{\infty}$.

To evaluate, use L'Hopital's rule.

$\lim\limits_{x \to \infty} \frac{x^2}{e^{2x}}$	Start with the limit that needs to be evaluated.
$= \lim\limits_{x \to \infty} \frac{2x}{2e^{2x}}$	Since x^2 and e^{2x} are differentiable and the limit has the form $\frac{\infty}{\infty}$, L'Hopital's rule is used. $D[x^2] = 2x$, $D[e^{2x}] = 2e^{2x}$
$= \lim\limits_{x \to \infty} \frac{x}{e^{2x}}$	Remove the common factor of 2.
$= \lim\limits_{x \to \infty} \frac{1}{2e^{2x}}$	Since $\lim\limits_{x \to \infty} \frac{x}{e^{2x}}$ has the form $\frac{\infty}{\infty}$, continue to use L'Hopital's rule. $D[x] = 1$, $D[e^{2x}] = 2e^{2x}$
$= 0$	Since the denominator grows very large as $x \to \infty$, the limit is 0.

Thus, $\lim\limits_{x \to \infty} x^2 e^{-2x} = 0$.

BIG IDEA

If $\lim\limits_{x \to a} f(x) \cdot g(x)$ has the form $\infty \cdot 0$, write $\lim\limits_{x \to a} \frac{f(x)}{\left(\frac{1}{g(x)}\right)}$ or $\lim\limits_{x \to a} \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$, then use L'Hopital's rule.

WATCH

In this video, we'll evaluate $\lim\limits_{x \to 0^+} x^3 \cdot \ln x$.

WATCH

In this video, we'll evaluate $\lim\limits_{x \to \infty} x \cdot \sin\left(\frac{1}{x}\right)$.

SUMMARY

In this lesson, you learned that with the addition of new indeterminate forms, more strategies need to be used. Specifically, you learned that for **the indeterminate form** $\infty - \infty$, combining the fractions or rationalizing are the most common strategies; for **the indeterminate form** $\infty \cdot 0$, rewriting the expression using reciprocals then using L'Hopital's rule is the main strategy.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.