

# The Chain Rule

by Sophia



## WHAT'S COVERED

In this lesson, you will learn how to find derivatives of general composite functions by using the chain rule. In a previous tutorial, you learned how to find derivatives of  $y = (f(x))^n$ , which is a composite function, but this doesn't cover all composite functions. For example, we are still unable to find the derivative of functions such as  $f(x) = \sin(3x)$  and  $y = \tan(x^2)$ . In this tutorial, we'll learn the techniques necessary to find derivatives of said functions. Specifically, this lesson will cover:

### 1. Motivation for the Chain Rule

#### 1a. Composite Functions

#### 1b. Examining Rates of Change

### 2. Applying the Chain Rule

#### 2a. Basic Functions

#### 2b. Applying the Chain Rule Twice

#### 2c. Combining the Chain Rule With Other Rules

## 1. Motivation for the Chain Rule

### 1a. Composite Functions

Recall that a composite function has the form  $y = f(g(x))$ . We call  $g(x)$  the "inner function" since it is plugged into the function  $f$ .

To find derivatives of composite functions, it will be helpful to first identify the inner function. Given  $y = f(g(x))$ , let  $u = g(x)$ . Then  $y = f(u)$ , a less complicated function. Here are a few examples:

Function	Inner Function	Written in Terms of $u$
$y = \sqrt{2x^2 + 5x}$	$u = 2x^2 + 5x$	$y = \sqrt{u}$
$y = \sin(3x)$	$u = 3x$	$y = \sin u$

$$y = \frac{3}{(2 + \sin x)^4}$$

$$u = 2 + \sin x$$

$$y = \frac{3}{u^4}$$

## 1b. Examining Rates of Change

The chain rule is a derivative technique that uses several rates of change in one problem. Here is a real-life consideration:

A factory can produce 30 units of a certain item per hour at a cost of \$20 per item. What is the cost per hour of producing the items?

If you think the answer is \$600, great! That is correct. But let's look at this more closely so that we can understand the rates of change.

Let  $x$  = the number of hours,  $u$  = the number of units produced, and  $C$ , the cost.

We can translate the given information into rates of change:

- The factory can produce 30 units per hour:  $\text{Slope} = \frac{\Delta u}{\Delta x} = 30$
- The units cost \$20 each to produce:  $\text{Slope} = \frac{\Delta C}{\Delta u} = 20$

Then, the cost per hour is  $\frac{\Delta C}{\Delta x}$ , which can be written as  $\frac{\Delta C}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = (20)(30) = \$600$  per hour.

Generally speaking, let's say that  $C$  is a function of  $u$ , and  $u$  is a function of  $x$ . Then,  $C$  is also a function of  $x$ , and  $\frac{\Delta C}{\Delta x} = \frac{\Delta C}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$ .

While not a proof, this idea can be extended to derivatives (instantaneous rates of change). That is, given  $C$  is a function of  $u$  and  $u$  is a function of  $x$ ,  $\frac{dC}{dx} = \frac{dC}{du} \cdot \frac{du}{dx}$ . This derivative rule applies to composite functions and is called the chain rule.

## 2. Applying the Chain Rule

### 2a. Basic Functions

The chain rule can be expressed with the following formula:



#### FORMULA TO KNOW

##### Chain Rule

Suppose  $y = f(u)$ , a composite function, where  $u$  is a function of  $x$ .

$$\text{Then, } \frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.$$

Using “prime” notation, we can write  $y' = f'(u) \cdot u'$ .

Using “D” notation, we can write  $D[y] = f'(u) \cdot D[u]$ .



#### BIG IDEA

With the chain rule in mind, we can write the derivative of each basic function.

- $D[u^n] = n \cdot u^{n-1} \cdot D[u]$
- $D[\sin u] = \cos u \cdot D[u]$
- $D[\cos u] = -\sin u \cdot D[u]$
- $D[\tan u] = \sec^2 u \cdot D[u]$
- $D[\cot u] = -\csc^2 u \cdot D[u]$
- $D[\sec u] = \sec u \tan u \cdot D[u]$
- $D[\csc u] = -\csc u \cot u \cdot D[u]$



#### HINT

$D[u^n] = n \cdot u^{n-1} \cdot D[u]$  is the power rule from [The General Power Rule for Functions](#).

⇒ **EXAMPLE** Consider the function  $y = \sin(3x)$ . Find its derivative.

$$y = \sin u \quad \text{Let } u = 3x, \text{ the inner function.}$$

$$\frac{dy}{dx} = \cos u \cdot D[u] \quad \text{Apply the chain rule: } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \cdot 3 \quad D[u] = D[3x] = 3$$

$$\frac{dy}{dx} = 3\cos 3x \quad \text{Rewrite “3” in front and replace } u \text{ with } 3x.$$

$$\text{Thus, } \frac{dy}{dx} = 3\cos 3x.$$

In challenge 3.2, you took derivatives of functions of the form  $y = [f(x)]^n$ , which is a specific form of the chain rule ( $y = u^n$  where  $u = f(x)$ ). Here is a reminder of a power function that requires the chain rule.

⇒ **EXAMPLE** Consider the function  $y = \sqrt[3]{4x^2 + \sin x}$ . Find its derivative.

$$y = \sqrt[3]{u} = u^{1/3} \quad \text{Let } u = 4x^2 + \sin x, \text{ the inner function.}$$

$$\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \cdot D[u] \quad \text{Apply the chain rule: } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \cdot (8x + \cos x) \quad D[u] = D[4x^2 + \sin x] = 8x + \cos x$$

$$\frac{dy}{dx} = \frac{1}{3} (4x^2 + \sin x)^{-2/3} (8x + \cos x) \quad \text{Replace } u \text{ with } 4x^2 + \sin x.$$

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{(4x^2 + \sin x)^{2/3}} (8x + \cos x) \quad \text{Rewrite with positive exponents.}$$

$$\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}} \quad \text{Combine fractions.}$$

$$\text{Thus, } \frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}.$$

⇒ **EXAMPLE** Consider the function  $y = \tan(x^2 + 1)$ . Find its derivative.

$$y = \tan u \quad \text{Let } u = x^2 + 1, \text{ the inner function.}$$

$$\frac{dy}{dx} = \sec^2 u \cdot D[u] \quad \text{Apply the chain rule: } \frac{dy}{dx} = f'(u) \cdot D[u]$$

$$\frac{dy}{dx} = (\sec^2 u) \cdot 2x \quad D[u] = D[x^2 + 1] = 2x \text{ (parentheses added to show separation)}$$

$$\frac{dy}{dx} = 2x \cdot \sec^2(x^2 + 1) \quad \text{Rewrite "2x" in front and replace } u \text{ with } x^2 + 1.$$

$$\text{Thus, } \frac{dy}{dx} = 2x \cdot \sec^2(x^2 + 1).$$



TRY IT

Consider the function  $y = \cos(2x^3)$ .

Find its derivative.

+

$$\frac{dy}{dx} = -6x^2 \sin(2x^3)$$



WATCH

Here is a video in which we find the derivative of  $y = \sec(3x + 1)$ .

## 2b. Applying the Chain Rule Twice

When the inner function is itself a composite function, the chain rule is applied more than once. Rather than assign a letter to each inside function, we'll present another way to organize this.

⇒ **EXAMPLE** Consider the function  $y = \sin^2 3x$ . Find its derivative.

$$y = (\sin(3x))^2 \quad \text{Rewrite as a quantity squared.}$$

$$\frac{dy}{dx} = 2(\sin(3x)) \cdot D[\sin(3x)] \quad \text{Apply the chain rule: } D[u^2] = 2u \cdot u'$$

$$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot D[3x] \quad \text{Apply the chain rule again: } D[\sin u] = \cos u \cdot u'$$

$$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot 3 \quad D[3x] = 3$$

$$\frac{dy}{dx} = 6\sin 3x \cos 3x \quad 2 \cdot 3 = 6; \text{ omit unnecessary parentheses.}$$

Thus,  $\frac{dy}{dx} = 6\sin 3x \cos 3x$ .



TRY IT

Consider the function  $y = \tan^4 2x$ .

Find its derivative.



$$\frac{dy}{dx} = 8\tan^3 2x \sec^2 2x$$

## 2c. Combining the Chain Rule With Other Rules

Sometimes sums, differences, products, and quotients contain composite functions. The key is to approach the derivative carefully.

⇒ **EXAMPLE** Consider the function  $y = x \sin 4x$ . Find its derivative.

$$y = x \sin 4x \quad \text{Start with the given function.}$$

$$\frac{dy}{dx} = D[x] \cdot \sin 4x + x \cdot D[\sin 4x] \quad \text{Apply the product rule.}$$

$$\frac{dy}{dx} = 1 \cdot \sin 4x + x \cdot \cos 4x \cdot D[4x] \quad D[x] = 1, D[\sin u] = \cos u \cdot D[u]$$

$$\frac{dy}{dx} = 1 \cdot \sin 4x + x \cdot \cos 4x \cdot (4) \quad D[4x] = 4$$

$$\frac{dy}{dx} = \sin 4x + 4x \cos 4x \quad \text{Omit unnecessary grouping symbols.}$$

Thus,  $\frac{dy}{dx} = \sin 4x + 4x \cos 4x$ .



## SUMMARY

In this lesson, you began by understanding the **motivation for the chain rule**, a derivative technique used to compute derivatives of **composite functions**, namely that it expands your ability to find **rates of change**. You learned how to **apply the chain rule**, exploring the formula and the derivative of each **basic function**. You also learned that when the inner function is itself a composite function, you need to **apply the chain rule twice**. Lastly, you examined an example of finding a derivative by **combining the chain rule with other rules**, such as the product rule.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.



## FORMULAS TO KNOW

### Chain Rule

Suppose  $y = f(u)$ , a composite function, where  $u$  is a function of  $x$ .

Then,  $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ .

Using “prime” notation, we can write  $\frac{dy}{dx} = f'(u) \cdot u'$ .

Using “D” notation, we can write  $\frac{dy}{dx} = f'(u) \cdot D[u]$ .