# Instantaneous Rate of Change

*by Sophia*

WHAT'S COVERED

In this lesson, you will compute and visualize the instantaneous rate of change of a function. Specifically, this lesson will cover:



# 1. Instantaneous Rate of Change

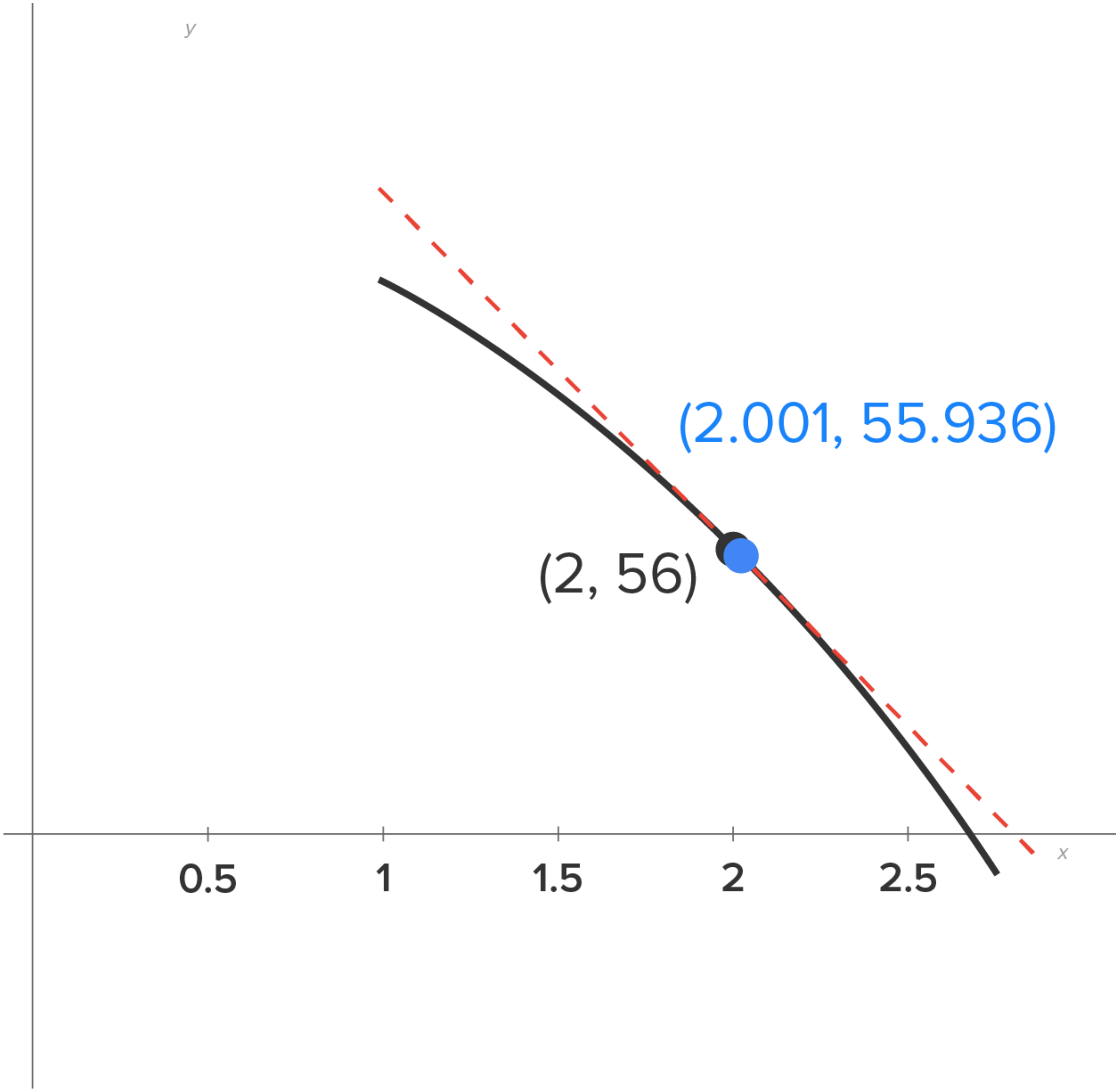
In the last section, we computed average rates of change, which require an interval of time. What if we want to compute the **instantaneous rate of change**, which is the rate of change at one specific point?

Let's say a tennis ball is dropped off the top of a building. Its height after $t$ seconds is described by the function $y(t) = -16t^2 + 120$. Suppose we want to find the instantaneous rate of change at the instant that 2 seconds have passed.

To see if we can find a pattern, let's find average rates of change on small intervals of time starting with $t = 2$.

| Interval | Length of Interval | Average Rate of Change |
|---|---|---|
| $[2, 2.1]$ | 0.1 seconds | $\frac{y(2.1)-y(2)}{2.1-2} = \frac{49.44-56}{0.1} = -65.6$ ft/sec |
| $[2, 2.01]$ | 0.01 seconds | $\frac{y(2.01)-y(2)}{2.01-2} = \frac{55.3584-56}{0.01} = -64.16$ ft/sec |
| $[2, 2.001]$ | 0.001 seconds | $\frac{y(2.001)-y(2)}{2.001-2} = \frac{55.935984-56}{0.001} = -64.016$ ft/sec |

Notice that as the length of the interval decreases, the average rate of change seems to be approaching a value. It is safe to estimate the instantaneous rate of change as -64 ft per second (we will see later that this is the actual answer).




Let's examine the curve $y(t) = -16t^2 + 120$ and the secant line for the interval $[2, 2.001]$. (The curve is solid; the secant line is dashed).

Notice that the two points are nearly indistinguishable.

Furthermore, the secant line appears to only pass through one point instead of two (remember, because they are so close together), so the secant line actually looks like a tangent line!

BIG IDEA

The instantaneous rate of change of a function is represented graphically by the slope of the tangent line.

TERM TO KNOW

**Instantaneous Rate of Change**

The rate of change of a function at a specific point.

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# 2. Computing Instantaneous Rate of Change

The problem we just completed in the previous section can be generalized by finding the average rate of change in $y(t) = -16t^2 + 120$ over the interval $[2, 2+h]$ for small values of *h*.

Since the instantaneous rate of change is the result of letting *h* get smaller and smaller, here is our plan to find the instantaneous rate of change for $y(t)$:

STEP BY STEP

1. Calculate the average rate of change on the interval $[2, 2+h]$.

   The expression for the average rate of change is $\dfrac{y(2+h)-y(2)}{(2+h)-2} = \dfrac{y(2+h)-y(2)}{h}$.

   First, find $y(2+h)$ and $y(2)$.

   $$\begin{aligned} y(2+h) &= -16(2+h)^2 + 120 \\ &= -16(4+4h+h^2) + 120 \\ &= -64 - 64h - 16h^2 + 120 \\ &= 56 - 64h - 16h^2 \end{aligned}$$

   $$\begin{aligned} y(2) &= -16(2)^2 + 120 \\ &= -16(4) + 120 \\ &= -64 + 120 \\ &= 56 \end{aligned}$$

   Then, the average rate of change is $\dfrac{(56-64h-16h^2)-56}{h} = \dfrac{-64h-16h^2}{h} = -64 - 16h$.

2. The instantaneous rate of change is the average rate of change as the value of *h* gets smaller and smaller. In the simplified expression, substitute $h = 0$. This gives $-64 - 16(0) = -64$ ft/sec.

   Conclusion: the instantaneous rate of change is -64 ft/sec.

BIG IDEA

To find the instantaneous rate of change of $f(x)$ at $x = a$, follow these steps.

1. Find and simplify $\dfrac{f(a+h)-f(a)}{h}$.

   Note, this should look familiar...it looks like a difference quotient!

2. Once simplified, substitute $h = 0$ to get the instantaneous rate of change.

EXAMPLE Compute the instantaneous rate of change of $f(x)=2x^2-3x+10$ when $x=1$. We know the average rate of change is $\frac{f(1+h)-f(1)}{h}$.

$$\begin{aligned} f(1+h) &= 2(1+h)^2-3(1+h)+10 \\ &= 2(1+2h+h^2)-3-3h+10 \\ &= 2+4h+2h^2-3-3h+10 \\ &= 2h^2+h+9 \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^2-3(1)+10 \\ &= 2-3+10 \\ &= 9 \end{aligned}$$

Then, $\frac{f(1+h)-f(1)}{h}=\frac{(2h^2+h+9)-9}{h}=\frac{2h^2+h}{h}=2h+1$. The instantaneous rate of change is $2(0)+1=1$.

Reminder: This is also the slope of the tangent line when $x=1$.

TRY IT

Consider the function $f(x)=x^3$.

**Find the instantaneous rate of change of the function when $x=4$.** +

First, find $f(4+h)$ and $f(4)$.

$$\begin{aligned} f(4+h) &= (4+h)^3 \\ &= (4+h)(4+h)(4+h) \\ &= (16+8h+h^2)(4+h) \\ &= h^3+12h^2+48h+64 \end{aligned}$$

$$\begin{aligned} f(4) &= 4^3 \\ &= 64 \end{aligned}$$

Then, $\frac{f(4+h)-f(4)}{h}=\frac{(h^3+12h^2+48h+64)-64}{h}=\frac{h^3+12h^2+48h}{h}=h^2+12h+48$.

The instantaneous rate of change is $0^2+12(0)+48=48$.

WATCH

The following video walks you through the process of calculating the instantaneous rate of change of $f(x)=\frac{1}{x+3}$ when $x=2$.

## SUMMARY

In this lesson, you learned that the **instantaneous rate of change** of a function gives the rate of change of the function at a single point (as opposed to average rate of change, which requires two points). The geometric interpretation of instantaneous rate of change is that it is the slope of the line tangent to $y = f(x)$ at that specific point. You also learned how to **compute the instantaneous rate of change**, which enables you to calculate instantaneous velocity at a specific point in time.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Instantaneous Rate of Change**

The rate of change of a function at a specific point.