Putting It All Together: Sketching a Graph

by Sophia

WHAT'S COVERED

In this lesson, you will use properties of $f(x)$, $f'(x)$, and $f''(x)$, and limits to sketch the graph of a function. Specifically, this lesson will cover:

1. Graphing Functions: A General Strategy

The following information is useful when graphing a function $y=f(x)$:

STEP BY STEP

1. Obtain the following information from $f(x)$:
   a. Find the domain of the function.
   b. Determine if there are any vertical, horizontal, slant, or nonlinear asymptotes by using limits.
   c. Find all x-intercepts (when convenient) and y-intercepts. Note that x-intercepts are not always easy to find without technology.
2. Obtain the following information from $f'(x)$:
   a. Find all critical numbers.
   b. Determine all intervals where $f(x)$ is increasing and decreasing.
   c. Use the first derivative test to locate all local maximum and minimum points.
3. Obtain the following information from $f''(x)$:
   a. Find all values where $f''(x)=0$ or is undefined.
   b. Determine all open intervals over which $f(x)$ is concave up or concave down.
   c. Determine any inflection points of $f(x)$.

2. Graphing Functions: Examples

EXAMPLE Use the techniques from this unit to sketch the graph of $f(x) = x^4 - 18x^2 + 32$.

Since both the first and second derivatives will be used, we'll find those first:

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36$

1. Information from $f(x)$:
    a. The domain of the function is all real numbers.
    b. Since $f(x)$ is a polynomial, there are no asymptotes.
    c. The y-intercept is (0, 32). To find the x-intercepts, set $f(x) = 0$ and solve.

        $x^4 - 18x^2 + 32 = 0$

        $(x^2 - 16)(x^2 - 2) = 0$

        $x^2 - 16 = 0$ or $x^2 - 2 = 0$

        $x^2 = 16$ or $x^2 = 2$

        $x = \pm 4$, $x = \pm\sqrt{2}$

        Thus, the graph of $f(x)$ has 4 x-intercepts: $(\pm 4, 0)$, $(\pm\sqrt{2}, 0)$
2. Information from $f'(x)$:
    a. Critical numbers:

        $4x^3 - 36x = 0$

        $4x(x^2 - 9) = 0$

        $4x(x + 3)(x - 3) = 0$

        $4x = 0$ or $x + 3 = 0$ or $x - 3 = 0$

        $x = 0, -3, 3$

For parts b and c, use a sign graph for $f'(x)$:

Interval	$(-\infty, -3)$	(-3, 0)	(0, 3)	$(3, \infty)$
Test Number	-4	-1	1	4
Value of $f'(x)$	-112	32	-32	112
Behavior	Decreasing	Increasing	Decreasing	Increasing
Visual	↘	↗	↘	↗

b. $f(x)$ is increasing on $(-3, 0) \cup (3, \infty)$.

$f(x)$ is decreasing on $(-\infty, -3) \cup (0, 3)$.

c. $f(x)$ has local minimums at $(-3, f(-3))$ and $(3, f(3))$. These points are (-3, -49) and (3, 49).

$f(x)$ has a local maximum at $(0, f(0))$. This point is (0, 32).

3. Information from $f''(x)$:

a. Set $f''(x) = 0$ and solve:

$$12x^2 - 36 = 0$$

$$12x^2 = 36$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

For parts b and c, use a sign graph for $f''(x)$:

(Note: $\sqrt{3} \approx 1.73$.)

Interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, \sqrt{3})$	$(\sqrt{3}, \infty)$
Test Number	-2	0	2
Value of $f''(x)$	12	-36	12
Behavior	Concave up	Concave down	Concave up

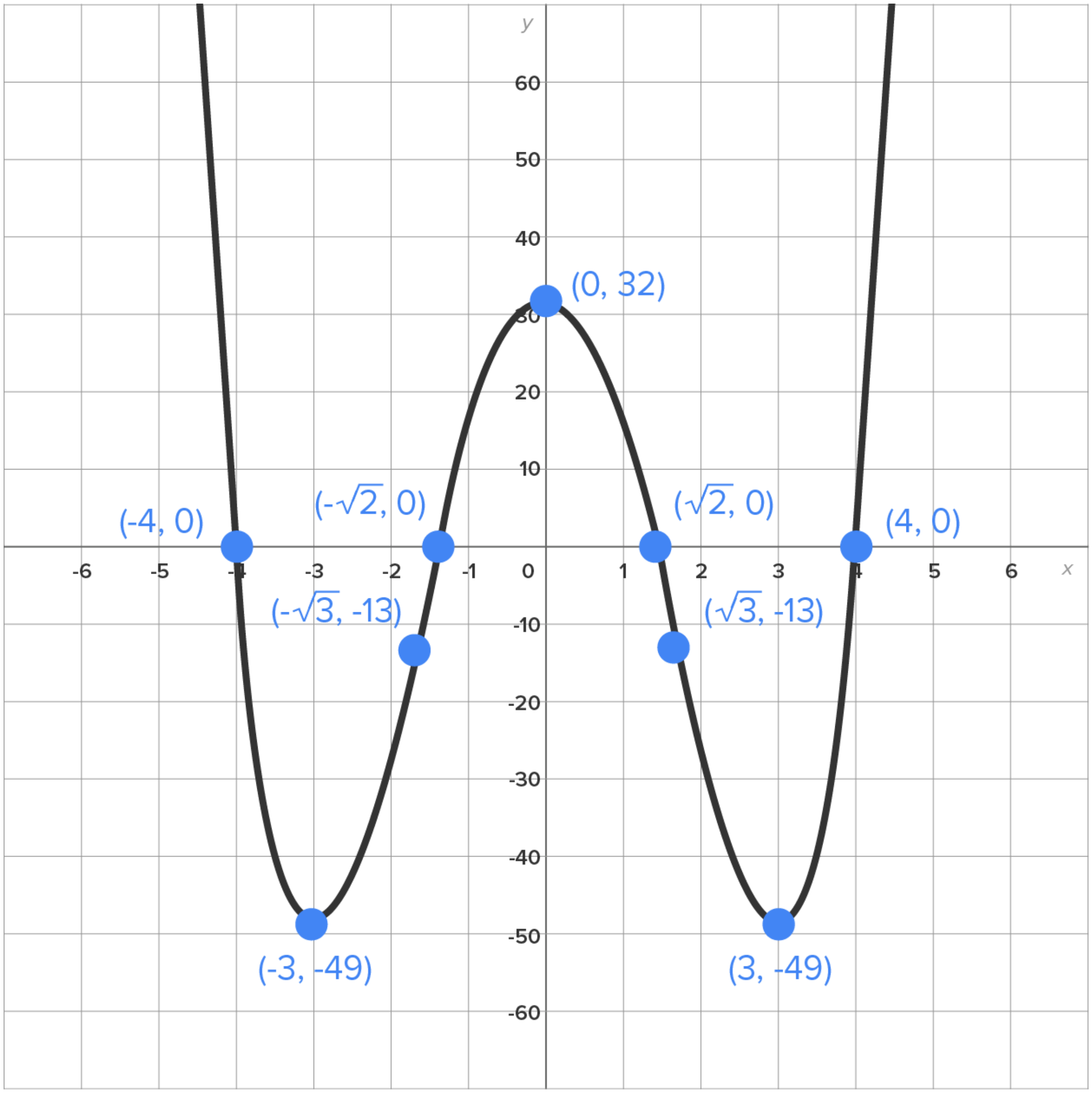
b. $f(x)$ is concave up on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.

$f(x)$ is concave down on the interval $(-\sqrt{3}, \sqrt{3})$.

c. There are two inflection points: $(-\sqrt{3}, f(-\sqrt{3}))$ and $(\sqrt{3}, f(\sqrt{3}))$

These are the points $(-\sqrt{3}, -13)$ and $(\sqrt{3}, -13)$.

Pulling all this information together, the graph of the function with all important points labeled is shown in the figure.



WATCH

In this video, we will use the techniques from this unit to sketch the graph of $f(x)=x-6\sqrt{x-1}$. (Note: This video is over 10 minutes long.)

WATCH

In this video, we will use the techniques from this unit to sketch the graph of $f(x)=10x^3-3x^5$.

Let's review one last example, involving a function that has asymptotes.

EXAMPLE Use the techniques from this unit to sketch the graph of $f(x)=x^2+\frac{8}{x}$.

First, let’s find all derivatives, rewriting $f(x)$ first:

$f(x) = x^2 + \frac{8}{x} = x^2 + 8x^{-1}$

$f'(x) = 2x - 8x^{-2}$

$f''(x) = 2 + 16x^{-3}$

1. Information from $f(x)$:
   a. Domain: $(-\infty, 0) \cup (0, \infty)$
   b. Asymptotes:
      Vertical asymptote: $x = 0$
      Nonlinear Asymptote: $y = x^2$ (Since $\frac{8}{x} \to 0$ as $x \to \infty$)
   c. Intercepts:
      There is no y-intercept since $x = 0$ is not in the domain of $f(x)$.
      To find x-intercepts, set $x^2 + \frac{8}{x} = 0$ and solve:

$x^3 + 8 = 0$	Multiply both sides by *x*.
$x^3 = -8$	Isolate x^3 to one side.
$x = -2$	Take the cube root of both sides.

Thus, there is an x-intercept at (-2, 0).

2. Information from $f'(x)$:
   a. Earlier, we calculated $f'(x) = 2x - 8x^{-2} = 2x - \frac{8}{x^2}$. $f'(x)$ is undefined when $x = 0$, which is not in the domain of *f*. Therefore, 0 is not a critical number.
   To find other critical numbers, solve $2x - \frac{8}{x^2} = 0$.

$2x^3 - 8 = 0$	Multiply both sides by x^2.
$2x^3 = 8$	Add 8 to both sides.
$x^3 = 4$	Divide both sides by 2.
$x = \sqrt[3]{4} \approx 1.59$	Take the cube root of both sides.

For parts b and c, use a sign graph for $f'(x)$. We have to consider possible changes in direction at $x = 0$ and $x = \sqrt[3]{4}$.

Interval	$(-\infty, 0)$	$(0, \sqrt[3]{4})$	$(\sqrt[3]{4}, \infty)$
Test Number	-1	1	2
Value of $f'(x) = 2x - \frac{8}{x^2}$	-10	-6	2
Behavior	Decreasing	Decreasing	Increasing

Visual	↘	↘	↗

b. Therefore, $f(x)$ is decreasing on $(-\infty, 0) \cup (0, \sqrt[3]{4})$ and increasing on $(\sqrt[3]{4}, \infty)$.

c. Remember that $f(x)$ is undefined when $x = 0$. Since $f(x)$ is defined when $x = \sqrt[3]{4}$, there is a local minimum point at $(\sqrt[3]{4}, f(\sqrt[3]{4}))$. Substituting, the local minimum point is approximately (1.59, 7.56).

3. Information from $f''(x)$:

a. Earlier, we computed $f''(x) = 2 + 16x^{-3} = 2 + \frac{16}{x^3}$. $f''(x)$ is undefined when $x = 0$, but $f(x)$ could still change concavity there.

To find possible inflection points, set $2 + \frac{16}{x^3} = 0$ and solve:

$2x^3 + 16 = 0$	Multiply both sides by x^3.
$x^3 = -8$	Subtract 16 from both sides, then divide both sides by 2.
$x = -2$	Take the cube root of both sides.

Now we make a sign graph for $f''(x)$, considering the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
Test Number	-3	-1	1
Value of $f''(x) = 2 + \frac{16}{x^3}$	$\frac{38}{27} \approx 1.41$	-14	18
Behavior	Concave up	Concave down	Concave up

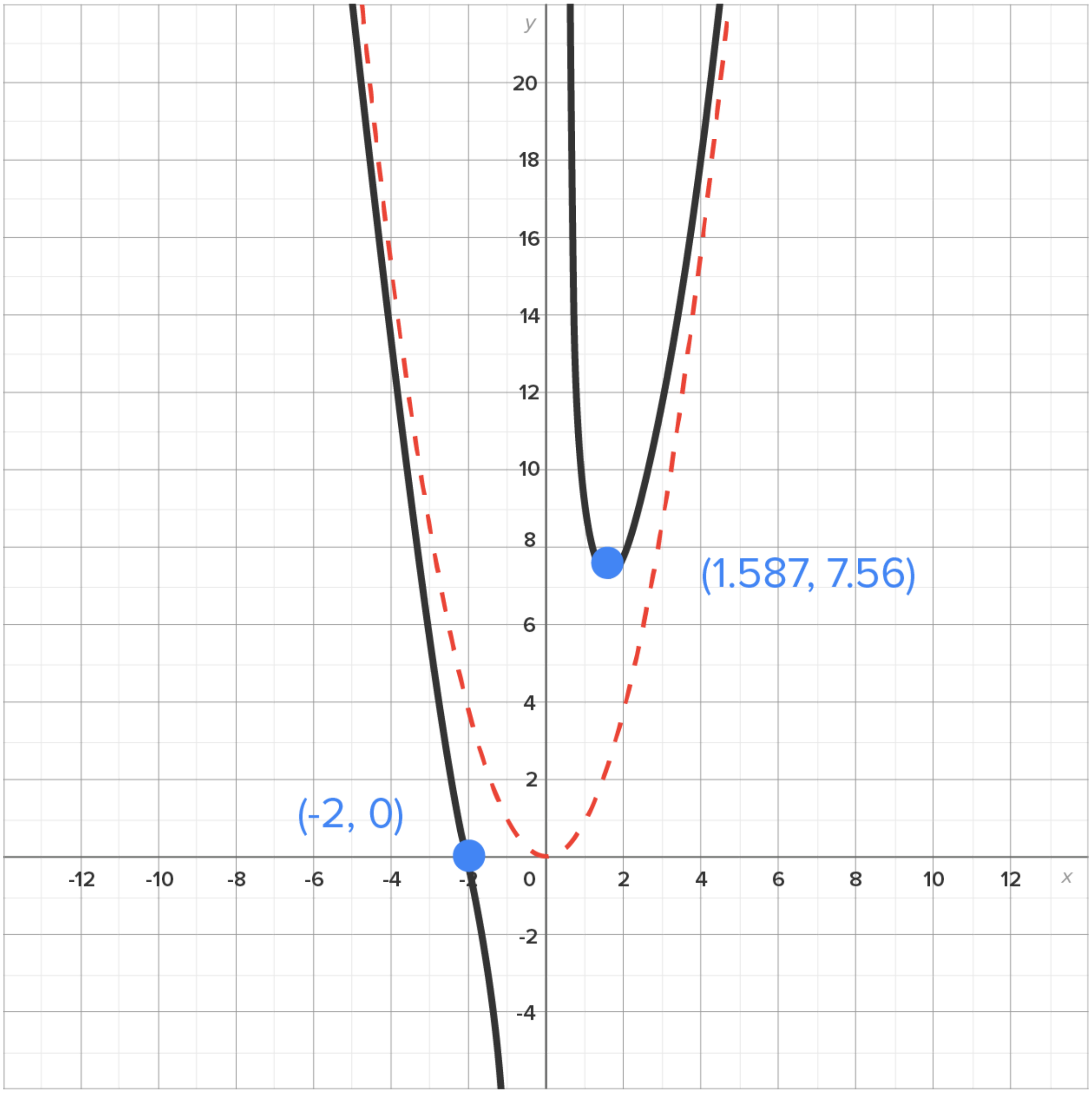
b. Thus, $f(x)$ is concave up on $(-\infty, -2) \cup (0, \infty)$ and concave down on the interval (-2, 0).

c. Since $f(-2)$ is defined, there is an inflection point at $(-2, f(-2))$, or (-2, 0), which we already found to be our x-intercept.

So, we know the following about the graph of $f(x)$:

- Vertical asymptote $x = 0$ and a nonlinear asymptote $y = x^2$
- x-intercept and inflection point at (-2, 0)
- Local minimum at (1.59, 7.56) (approximate coordinates)
- Decreasing on $(-\infty, 0) \cup (0, \sqrt[3]{4})$ and increasing on $(\sqrt[3]{4}, \infty)$
- Concave up on $(-\infty, -2) \cup (0, \infty)$ and concave down on (-2, 0)

Putting all the pieces together, here is the graph of $f(x)$, with a dashed curve to show the nonlinear asymptote.



SUMMARY

In this lesson, you learned **a general strategy** of using limits and properties of $f(x)$, $f'(x)$, and $f''(x)$ together to **graph a function** $y = f(x)$, followed by several **examples** of applying these techniques to **graph functions**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.