

Horizontal and Vertical Asymptotes

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WHAT'S COVERED

In this lesson, you will connect limits with horizontal and vertical asymptotes. Specifically, this lesson will cover:

1. Horizontal Asymptotes and Limits
2. Vertical Asymptotes and Limits

1. Horizontal Asymptotes and Limits

The graph of $f(x)$ has a **horizontal asymptote** $y = c$ if either $\lim_{x \rightarrow \infty} f(x) = c$ or $\lim_{x \rightarrow -\infty} f(x) = c$.

⇒ **EXAMPLE** Find all horizontal asymptotes of the graph of $f(x) = \frac{7x+1}{8x+3}$.

To find horizontal asymptotes, evaluate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

$$\begin{aligned}
 & \lim_{x \rightarrow \infty} \frac{7x+1}{8x+3} && \text{Start with the required limit to evaluate.} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{7x}{x} + \frac{1}{x}}{\frac{8x}{x} + \frac{3}{x}} && \text{Divide each term by the highest power of } x \text{ in the denominator, which is } x. \\
 &= \lim_{x \rightarrow \infty} \frac{7 + \frac{1}{x}}{8 + \frac{3}{x}} && \text{Simplify.} \\
 &= \frac{\lim_{x \rightarrow \infty} \left(7 + \frac{1}{x}\right)}{\lim_{x \rightarrow \infty} \left(8 + \frac{3}{x}\right)} && \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow \infty} f(x)}{\lim_{x \rightarrow \infty} g(x)}
 \end{aligned}$$

$$= \frac{7}{8} \quad \lim_{x \rightarrow \infty} \left(7 + \frac{1}{x} \right) = \lim_{x \rightarrow \infty} 7 + \lim_{x \rightarrow \infty} \frac{1}{x} = 7 + 0 = 7$$

$$\lim_{x \rightarrow \infty} \left(8 + \frac{3}{x} \right) = \lim_{x \rightarrow \infty} 8 + \lim_{x \rightarrow \infty} \frac{3}{x} = 8 + 0 = 8$$

Thus, the graph of $f(x)$ has a horizontal asymptote at $y = \frac{7}{8}$.

Now, check $\lim_{x \rightarrow -\infty} \frac{7x+1}{8x+3}$.

$$\lim_{x \rightarrow -\infty} \frac{7x+1}{8x+3} \quad \text{Start with the required limit to evaluate.}$$

$$= \lim_{x \rightarrow -\infty} \frac{\frac{7x}{x} + \frac{1}{x}}{\frac{8x}{x} + \frac{3}{x}} \quad \text{Divide each term by the highest power of } x \text{ in the denominator, which is } x.$$

$$= \lim_{x \rightarrow -\infty} \frac{7 + \frac{1}{x}}{8 + \frac{3}{x}} \quad \text{Simplify.}$$

$$= \frac{\lim_{x \rightarrow -\infty} \left(7 + \frac{1}{x} \right)}{\lim_{x \rightarrow -\infty} \left(8 + \frac{3}{x} \right)} \quad \lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow -\infty} f(x)}{\lim_{x \rightarrow -\infty} g(x)}$$

$$= \frac{7}{8} \quad \lim_{x \rightarrow -\infty} \left(7 + \frac{1}{x} \right) = \lim_{x \rightarrow -\infty} 7 + \lim_{x \rightarrow -\infty} \frac{1}{x} = 7 + 0 = 7$$

$$\lim_{x \rightarrow -\infty} \left(8 + \frac{3}{x} \right) = \lim_{x \rightarrow -\infty} 8 + \lim_{x \rightarrow -\infty} \frac{3}{x} = 8 + 0 = 8$$

This produces the same result as the other limit, so there is no additional horizontal asymptote.

We can conclude that the graph of $f(x)$ has one horizontal asymptote at $y = \frac{7}{8}$. If you graph this function, you would see that the graph approaches the horizontal line $y = \frac{7}{8}$ as $x \rightarrow \pm \infty$.



Consider the function $f(x) = \frac{3x}{x^2+3}$.

Find all horizontal asymptotes of the graph of the function.

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$f(x)$ has one horizontal asymptote, and its equation is $y = 0$.

If $f(x) = \frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are both polynomials, the following are results of limits:

- If $N(x)$ and $D(x)$ have the same degree, then the horizontal asymptote is $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If the degree of $N(x)$ is less than the degree of $D(x)$, then the horizontal asymptote is $y = 0$.
- If the degree of $N(x)$ is more than the degree of $D(x)$, then there is no horizontal asymptote, and this case is discussed in the next tutorial.

⇒ EXAMPLE The function $f(x) = \frac{2x^2 + 4x + 1}{3x^2 + 5x}$ has the horizontal asymptote $y = \frac{2}{3}$ since the degrees are the same.

⇒ EXAMPLE The function $f(x) = \frac{3x}{x^2 + 3}$ has the horizontal asymptote $y = 0$, as you saw in the Try It above, since the degree of the numerator is less than the degree of the denominator.



In this video, we'll use limits to find the equations of all horizontal asymptotes of the function $f(x) = \frac{|x|}{2x + 3}$.



Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = c$ or $\lim_{x \rightarrow -\infty} f(x) = c$.

2. Vertical Asymptotes and Limits

The graph of $f(x)$ has a **vertical asymptote** $x = a$ if either $\lim_{x \rightarrow a^-} f(x) = \pm \infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm \infty$.

The " $\pm \infty$ " in the above definition means that the limit could be either $-\infty$ or ∞ for there to be a vertical asymptote when $x = a$.



If $f(x)$ is a rational function, the only values of x where a vertical asymptote could occur are those values where the denominator is equal to 0.

⇒ EXAMPLE Determine the vertical asymptotes of $f(x) = \frac{x^2 - 2x}{x^2 - 4x}$.

First, find all values of x for which the denominator is 0:

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0, x = 4$$

Thus, the possible vertical asymptotes are $x = 0$ and $x = 4$. To determine which are vertical asymptotes, we need to evaluate a one-sided limit for each x -value. For this example, we'll choose right-sided limits.

Is $x = 0$ a vertical asymptote?

$$\lim_{x \rightarrow 0^+} \frac{x^2 - 2x}{x^2 - 4x}$$
 Start with the required limit to evaluate.

$$\lim_{x \rightarrow 0^+} \frac{x(x-2)}{x(x-4)} = \lim_{x \rightarrow 0^+} \frac{x-2}{x-4}$$
 Factor, then remove the common factor.

$$= \frac{0-2}{0-4} = \frac{1}{2}$$
 Direct substitution works!

Since the limit is not $\pm \infty$, there is no vertical asymptote at $x = 0$. (Note: The left-sided limit would produce the same result.)

Is $x = 4$ a vertical asymptote?

$$\lim_{x \rightarrow 4^+} \frac{x^2 - 2x}{x^2 - 4x}$$
 Start with the required limit to evaluate.

$$\lim_{x \rightarrow 4^+} \frac{x(x-2)}{x(x-4)} = \lim_{x \rightarrow 4^+} \frac{x-2}{x-4}$$
 Factor, then remove the common factor.

$$= \infty$$

As x approaches 4 from the right, $x - 2$ is around 2, and $x - 4$ is a small positive number.

$\frac{\text{around } 2}{\text{small positive number}} =$ a large number, so the limit is ∞ .

We can conclude that there is a vertical asymptote at $x = 4$, but not at $x = 0$. If you were to graph the function, this would confirm this result.



Consider the function $f(x) = \frac{2x+1}{x^2-6x+5}$.

Find the equations of all vertical asymptotes of the function.

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The vertical asymptotes are $x = 1$ and $x = 5$.



TERM TO KNOW

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \rightarrow a^-} f(x) = \pm \infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm \infty$.



SUMMARY

In this lesson, you learned that **the horizontal and vertical asymptotes of a function are related to limits** of a function where infinity is involved. Specifically, a function $f(x)$ has a horizontal asymptote at $y = c$ if

$\lim_{x \rightarrow \infty} f(x) = c$ or $\lim_{x \rightarrow -\infty} f(x) = c$, and a function $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x \rightarrow a^-} f(x) = \pm \infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm \infty$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.



TERMS TO KNOW

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = c$ or $\lim_{x \rightarrow -\infty} f(x) = c$.

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \rightarrow a^-} f(x) = \pm \infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm \infty$.