One-Sided Limits

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WHAT'S COVERED

In this lesson, we will explore limits by examining the left and right sides separately. Specifically, this lesson will cover:

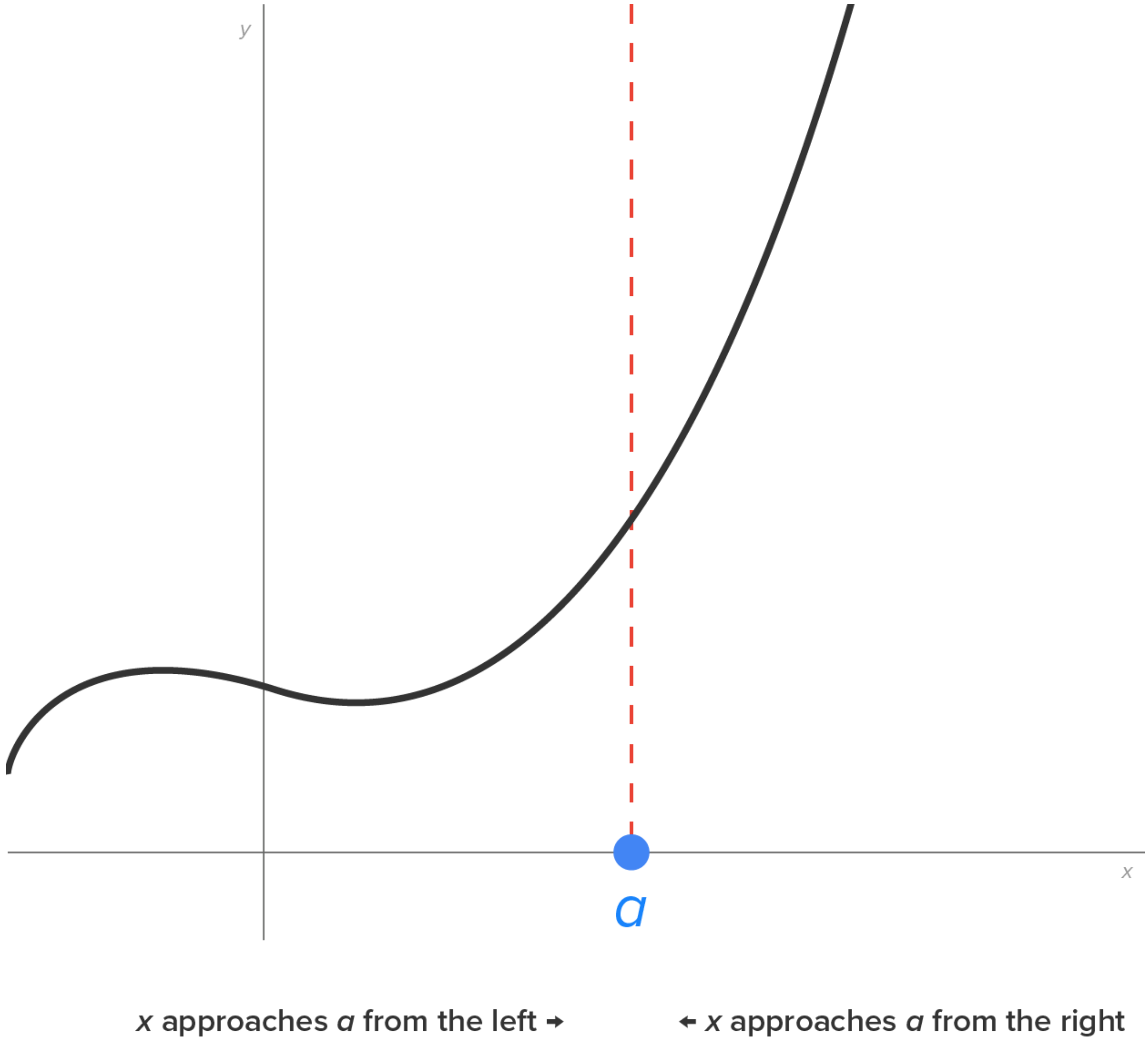
1. Notation Used for One-Sided Limits and Evaluating Them Graphically
2. Evaluating One-Sided Limits

1. Notation Used for One-Sided Limits and Evaluating Them Graphically

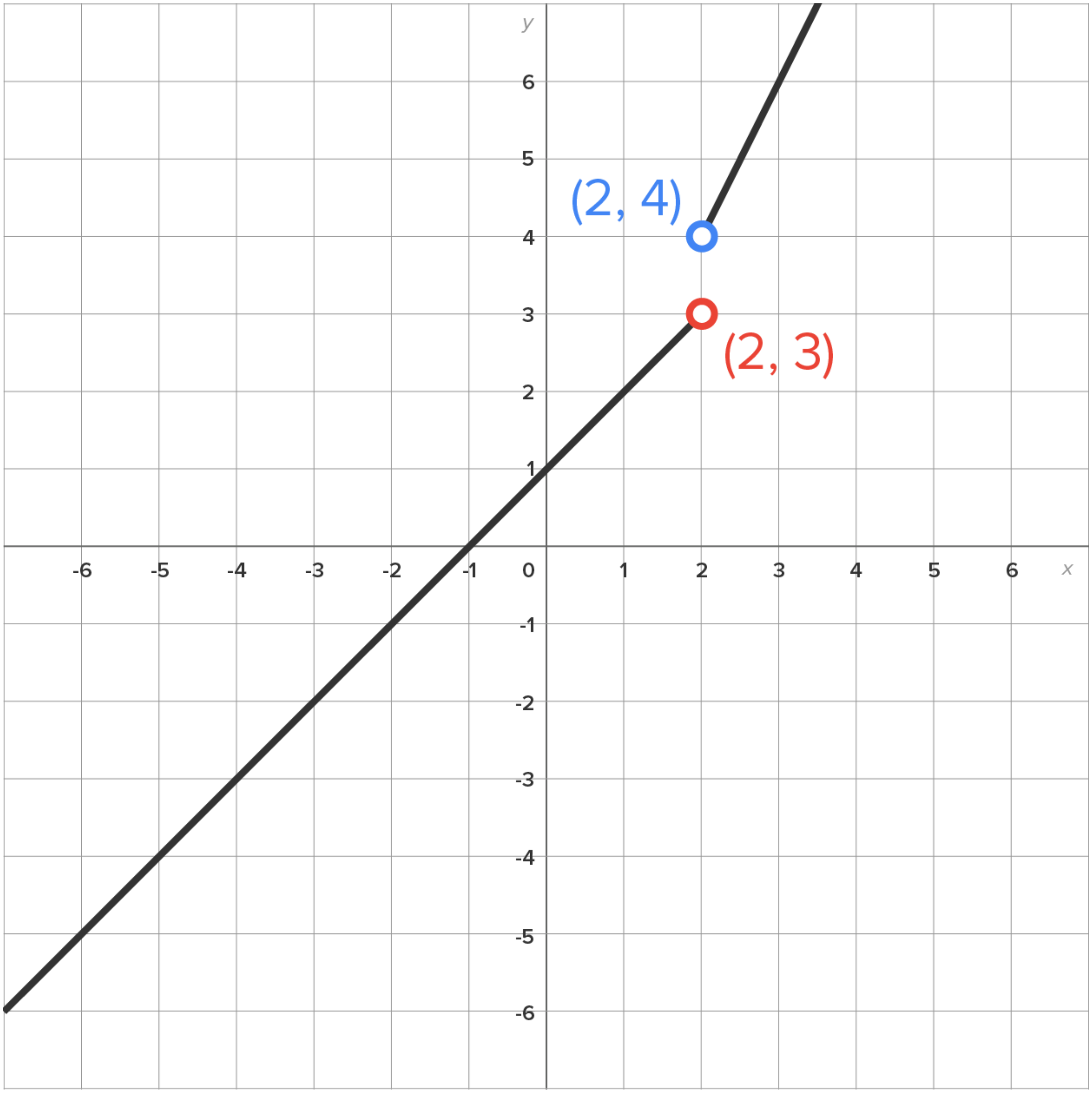
Recall that the notation $\lim_{x \to a} f(x)$ means to evaluate the limit of some function $f(x)$ as x gets closer to a from both sides. If the values from both sides don't match, $\lim_{x \to a} f(x)$ does not exist.

To that end, we define one-sided limits, meaning limits that focus on one side of $x = a$.

- Limit from the left: $\lim_{x \to a^-} f(x)$
- Limit from the right: $\lim_{x \to a^+} f(x)$



EXAMPLE Consider the graph below, which shows some function $f(x)$.



We can say the following:

Statement	Description
$\lim_{x \to 2^-} f(x) = 3$	As x approaches 2 from the left, $f(x)$ gets closer to 3.
$\lim_{x \to 2^+} f(x) = 4$	As x approaches 2 from the right, $f(x)$ gets closer to 4.
$\lim_{x \to 2} f(x)$ does not exist	Since the left-hand and right-hand limits are not equal, $\lim_{x \to 2} f(x)$ does not exist.

BIG IDEA

If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L$, then $\lim\limits_{x \to a} f(x) = L$. This means that if the left-hand and right-hand limits are both equal to the same value (L), the limit of the function is also equal to L as $x \to a$.

If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $\lim\limits_{x \to a} f(x)$ does not exist.

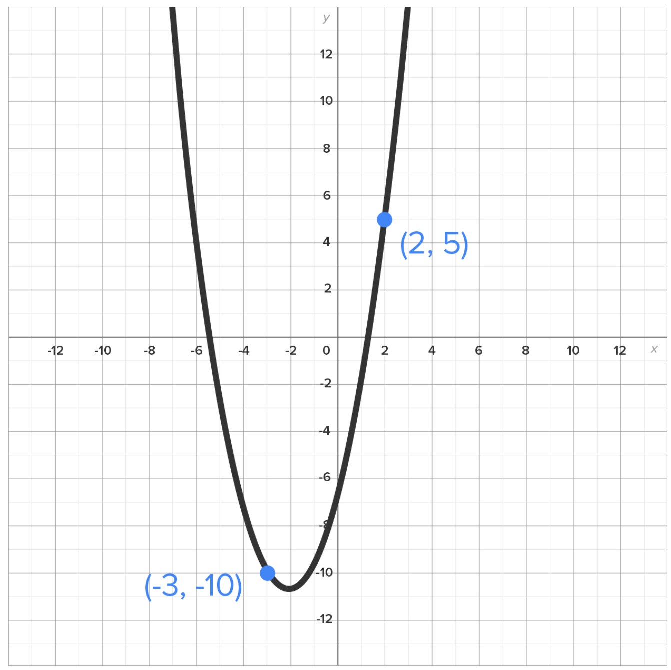
2. Evaluating One-Sided Limits

It turns out that one-sided limits can be evaluated using direct substitution.

Consider the graph of $f(x) = x^2 + 4x - 7$ shown in the graph to the right.

Notice the following:

- $\lim\limits_{x \to -3^-} (x^2 + 4x - 7) = -10$
- $\lim\limits_{x \to -3^+} (x^2 + 4x - 7) = -10$
- $\lim\limits_{x \to 2^-} (x^2 + 4x - 7) = 5$
- $\lim\limits_{x \to 2^+} (x^2 + 4x - 7) = 5$



Notice that in each case, the limits could have also been evaluated by direct substitution. Refer back to the "Direct Substitution" section earlier in this challenge to see which limits can be evaluated using this method.

EXAMPLE Evaluate the one-sided limits below by using the direct substitution method.

Limit	Solution
$\lim\limits_{x \to 4^+} (x^2 + 4x + 5)$	$\lim\limits_{x \to 4^+} (x^2 + 4x + 5) = 4^2 + 4(4) + 5 = 37$
$\lim\limits_{x \to 1^-} \dfrac{x+2}{x^2+3}$	$\lim\limits_{x \to 1^-} \dfrac{x+2}{x^2+3} = \dfrac{1+2}{1^2+3} = \dfrac{3}{4}$

TRY IT

Consider the function $\sqrt{2x^2 + 5x + 7}$.

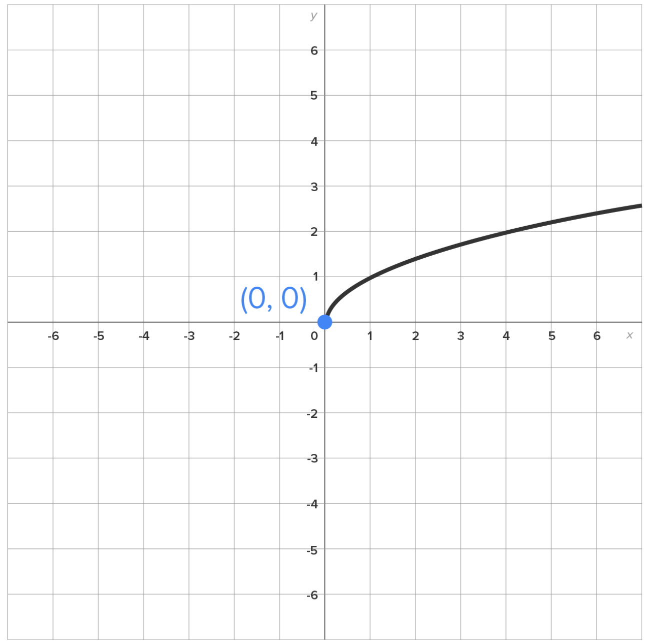
Evaluate the limit as x approaches 2 from the left. +

$$\lim_{x \to 2^-} \sqrt{2x^2+5x+7} = 5$$

Sometimes it is tempting to use direct substitution, even though it is technically not applicable.

EXAMPLE

Consider $f(x)=\sqrt{x}$. Let's find the one-sided limits as $x \to 0$. To help visualize this, the graph is pictured to the right.



Left-Sided Limit: $\lim_{x \to 0^-} \sqrt{x}$ does not exist since $\sqrt{x}$ is undefined to the left of $x=0$.

Right-Sided Limit: $\lim_{x \to 0^+} \sqrt{x} = 0$ since as x gets closer to 0 from the right, the value of $\sqrt{x}$ gets closer to 0.

Since the one-sided limits are not equal, this also means that $\lim_{x \to 0} \sqrt{x}$ does not exist.

It might have been tempting to use direct substitution since $\sqrt{0}=0$. However, since $x=0$ is also the endpoint of the domain of $f(x)=\sqrt{x}$, more care has to be taken when analyzing this function near $x=0$.

BIG IDEA

When evaluating one-sided limits, be sure that $f(x)$ is defined on that side before using direct substitution.

- To evaluate $\lim_{x \to a^-} f(x)$, make sure that $f(x)$ is defined for $x < a$.
- To evaluate $\lim_{x \to a^+} f(x)$, make sure that $f(x)$ is defined for $x > a$.

When $f(x)$ is a piecewise function, one-sided limits are very useful in examining $f(x)$ at the x-values where the function changes definition.

EXAMPLE Consider the function $f(x)=\begin{cases} 2x+1 & \text{if } x<3 \\ 10-x & \text{if } x \geq 3 \end{cases}$. Evaluate $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$.

Since $f(x)$ changes definitions when at $x=3$, evaluating $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$ takes some extra thought.

When evaluating $\lim_{x \to 3^-} f(x)$, we can replace $f(x)$ with $2x+1$ and evaluate the limit: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x+1) = 7$

When evaluating $\lim_{x \to 3^+} f(x)$, we can replace $f(x)$ with 10 - x and evaluate the limit: $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (10-x) = 7$

WATCH

The following video walks you through the process of evaluating $\lim_{x \to 5^-} f(x)$ and $\lim_{x \to 5^+} f(x)$ for

$f(x) = \begin{cases} 4-x^2 & \text{if } x<5 \\ x^2+x-1 & \text{if } x>5 \end{cases}$.

TRY IT

Consider the function $f(x) = \begin{cases} 5x-x^2 & \text{if } x \le 1 \\ 2x+3 & \text{if } x>1 \end{cases}$.

Evaluate the limit as x approaches 1 from the left. +

$\lim_{x \to 1^-} f(x) = 4$

Evaluate the limit as x approaches 1 from the right. +

$\lim_{x \to 1^+} f(x) = 5$

SUMMARY

In this lesson, you recalled that the notation $\lim_{x \to a} f(x)$ means to evaluate the limit of some function $f(x)$ as x gets closer to a from both sides. You also learned about the **notation used for one-sided limits**, specifically the notation $\lim_{x \to a^-} f(x)$ means to find the value $f(x)$ is approaching as x gets closer to a from values smaller than a, and $\lim_{x \to a^+} f(x)$ means to find the value of $f(x)$ is approaching as x gets closer to a from values larger than a. In general, **evaluating one-sided limits** is very similar to evaluating limits; one-sided limits can be **evaluated by graphing**, by tables, or by direct substitution, but be sure that $f(x)$ is defined on that side of x if you are using direct substitution.

When evaluating an overall limit, you learned that if $\lim_{x \to a^-} f(x)$ is not the same value as $\lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist. When dealing with piecewise functions and other functions that have restricted domains (for example, $f(x)=\sqrt{x}$ around $x=0$), more care needs to be taken when evaluating both one-sided and overall limits.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

