Higher-Order Derivatives

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WHAT'S COVERED

In this lesson, you will take all the rules you have learned about derivatives and apply them to higher-order derivatives. There are situations in which the change in the rate of change is important and useful. Specifically, this lesson will cover:

1. Higher-Order Derivatives: Definitions and Notation

When you take the derivative of a function, you actually are taking the first derivative of a function. That said, the second derivative of a function is the derivative of the first derivative. Then, the third derivative is the derivative of the second derivative, and so on.

Here is a table that shows past notations used with first derivatives, as well as the corresponding notation used for higher-order derivatives.

Function Representation	y-notation		$f(x)$-notation	
1st Derivative	y'	$\frac{dy}{dx}$	$f'(x)$	$D[f(x)]$
2nd Derivative	y''	$\frac{d^2y}{dx^2}$	$f''(x)$	$D^2[f(x)]$
3rd Derivative	y'''	$\frac{d^3y}{dx^3}$	$f'''(x)$	$D^3[f(x)]$
4th Derivative	$y^{(4)}$	$\frac{d^4y}{dx^4}$	$f^{(4)}(x)$	$D^4[f(x)]$

n^{th} Derivative	$y^{(n)}$	$\frac{d^n y}{dx^n}$	$f^{(n)}(x)$	$D^n[f(x)]$

HINT

Note that in the “prime” notations, the order of the derivative is written as a number enclosed in parentheses. Anything more than three prime symbols would be difficult to read.

2. Finding Higher-Order Derivatives

To find a second derivative, we need to find the first derivative, then differentiate a second time.

EXAMPLE Find the second derivative of the function $f(x)=3x^4-9x^3+3x^2+8x-12$.

First derivative:

$f'(x)=3(4x^3)-9(3x^2)+3(2x)+8(1)-0$	Apply the sum/difference and constant multiple rules.
$f'(x)=12x^3-27x^2+6x+8$	Simplify.

Second derivative:

$f''(x)=D[f'(x)]=D\left[12x^3-27x^2+6x+8\right]$	Substitute the first derivative.
$f''(x)=12(3x^2)-27(2x)+6+0$	Apply the sum/difference and constant multiple rules.
$f''(x)=36x^2-54x+6$	Simplify.

Thus, $f''(x)=36x^2-54x+6$.

EXAMPLE Find the 5th derivative of the function $y=4\cos x$.

Note that in each step, the constant multiple rule is used.

$y=4\cos x$	Start with the original function, $f(x)$.
$y'=4(-\sin x)=-4\sin x$	Find the first derivative.
$y''=-4(\cos x)=-4\cos x$	Find the second derivative.
$y'''=-4(-\sin x)=4\sin x$	Find the third derivative.
$y^{(4)}=4(\cos x)=4\cos x$	Find the fourth derivative (notice this is the same as $f(x)$).

$y^{(5)} = 4(-\sin x) = -4\sin x$ Lastly, find the fifth derivative (notice this is the same as $f'(x)$).

Thus, $y^{(5)} = 4(-\sin x) = -4\sin x$.

TRY IT

Consider the function $y = 10x^2 + 2\sin x$.

Find the 6th derivative of the above formula. +

$y^{(6)} = -2\sin x$

EXAMPLE Find the third derivative of $f(x) = 4x^2 - \frac{4}{x^3}$.

First, rewrite using negative exponents to make use of the power rule: $f(x) = 4x^2 - \frac{4}{x^3} = 4x^2 - 4x^{-3}$

Now, take the appropriate derivatives:

$f'(x) = 4(2x) - 4\left(-3x^{-4}\right)$	Apply the sum/difference and power rules.
$f'(x) = 8x + 12x^{-4}$	Simplify.

Since we are finding more derivatives, there is no need to rewrite with positive exponents just yet. We will save this for when all derivatives are taken.

$f''(x) = 8 + 12\left(-4x^{-5}\right)$	Apply the sum/difference and power rules.
$f''(x) = 8 - 48x^{-5}$	Simplify.

And now the third derivative:

$f'''(x) = 0 - 48\left(-5x^{-6}\right)$	Apply the sum/difference and power rules.
$f'''(x) = 240x^{-6}$	Simplify.
$f'''(x) = \frac{240}{x^6}$	Rewrite with positive exponents.

Thus, $f'''(x) = \frac{240}{x^6}$.

Sometimes different rules are needed to find higher-order derivatives.

EXAMPLE Find the second derivative of the function $y = 5\tan x$.

First derivative: $\frac{dy}{dx} = D[5\tan x] = 5\sec^2 x$

Second derivative:

$\frac{d^2y}{dx^2} = D[5\sec^2 x] = 5D[\sec^2 x]$	Apply the constant multiple rule.
$\frac{d^2y}{dx^2} = 5(2\sec x \cdot D[\sec x])$	Apply the power rule.
$\frac{d^2y}{dx^2} = 5(2\sec x \cdot \sec x \tan x)$	$D[\sec x] = \sec x \tan x$
$\frac{d^2y}{dx^2} = 10\sec^2 x \tan x$	Perform multiplications.

Thus, $\frac{d^2y}{dx^2} = 10\sec^2 x \tan x$.

TRY IT

Consider the function $f(x) = 2\sec x$.

Find the second derivative. +

$f''(x) = 2\sec x \tan^2 x + 2\sec^3 x$

3. Interpreting Higher-Order Derivatives

Recall that $f'(x)$ is the (instantaneous) rate of change of $f(x)$. Then:

- $f''(x)$ is the rate of change of $f'(x)$.
- $f'''(x)$ is the rate of change of $f''(x)$.
- And so on...

The interpretation of higher-order derivatives depends on the meaning of the original function. For example, if $h(t)$ measures the distance that an object has traveled t seconds after being set into motion, then $h'(t)$ is the change in distance with respect to time. This is called the **velocity** of the object.

Furthermore, we are also interested in how the velocity changes, which is $h''(t)$. This is called the **acceleration** of the object. The rate of change in acceleration, called **jerk**, is $h'''(t)$.

EXAMPLE A tennis ball is launched off the top of a building. Its height (in feet) after t seconds is modeled by the function $h(t) = -16t^2 + 20t + 40$. Let's find the height, velocity, and acceleration at the precise moment the object was in the air for 2 seconds.

Remember units:
- $h(t)$ = height after t seconds
- $h'(t)$ = velocity = change in height per second (feet/sec)
- $h''(t)$ = acceleration = change in velocity per second (ft/sec)/sec

Height: $h(2) = -16(2)^2 + 20(2) + 40 = 16$ feet

Velocity: $h'(t) = -32t + 20$
Then, $h'(2) = -32(2) + 20 = -44$ feet per second

Acceleration: $h''(t) = -32$
This means that $h''(t) = -32$ regardless of t. Therefore, $h''(2) = -32$ feet per second per second, meaning that the velocity is changing by -32 feet per second each second.

WATCH

In this video, we will examine the rate of change in the derivative (slope) of $f(x) = \sin^2 x$ at $x = 0$.

TERMS TO KNOW

Velocity
An object's change in distance with respect to time.

Acceleration
An object's change in velocity with respect to time.

Jerk
An object's change in acceleration with respect to time.

SUMMARY

In this lesson, you learned the **definitions and notation of higher-order derivatives**, understanding that **finding higher-order derivatives** involves using rules of derivatives repeatedly. You learned that the **interpretation of higher-order derivatives** depends on the meaning of the original function. For example, if $h(t)$ measures the distance that an object has traveled t seconds after being set into motion, then the first derivative, $h'(t)$, is the velocity of the object; the second derivative, $h''(t)$, is the acceleration of the object; and the third derivative, $h'''(t)$, is the rate of change in acceleration, or jerk. We will explore the geometrical meanings of the first and second derivatives in unit 4.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Acceleration
An object's change in velocity with respect to time.

Jerk
An object's change in acceleration with respect to time.

Velocity
An object's change in distance with respect to time.