Applied Maximum and Minimum Problems

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WHAT'S COVERED

In this lesson, you will apply your knowledge of derivatives to real-world maximization and minimization problems (which collectively are called *optimization problems*). Specifically, this lesson will cover:

1. Strategy for Solving Optimization Problems

An **optimization problem** is a problem in which the maximum or minimum value is sought, whichever is relevant.

STEP BY STEP

To solve an optimization problem:

1. Identify the function to be optimized. This is called the primary equation.
   a. If the goal is to maximize the area, the primary equation expresses the area as a function.
   b. If the goal is to minimize the amount of material used, then the primary equation gives the total amount of material used.
2. If your primary equation has more than one variable (for example: $A = xy$), you will need to form a secondary equation based on other information that is given in the problem.
3. If applicable, use the secondary equation in Step 2 to write the primary equation in Step 1 in terms of one independent variable. Also, state the domain of the function.
4. Find critical numbers.
5. Keeping the requirements in mind, use one of methods covered in this challenge to determine where the extreme points are located:
   a. Extreme Value Theorem
   b. First Derivative Test

c. Second Derivative Test

Now that we have a game plan, let's solve a few optimization problems.

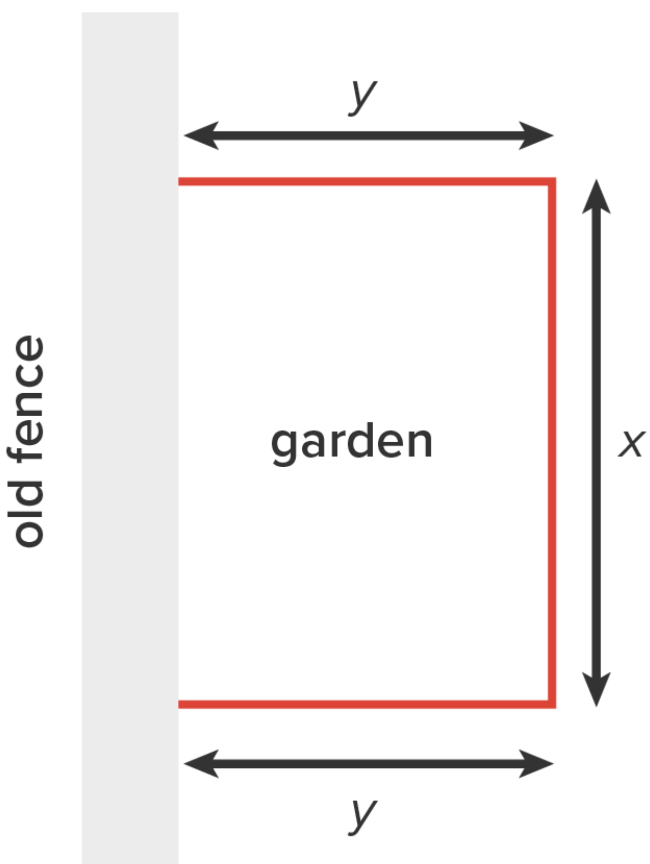
TERM TO KNOW

Optimization Problem

A problem in which the maximum or minimum value is sought, whichever is relevant.

2. Solving Applied Optimization Problems

EXAMPLE A garden is to be constructed against an old fence, as shown in the figure. Trim is to be placed around the garden on the three remaining sides but is not needed on the fence side. If 24 feet of trim is to be used, what is the largest area that can be enclosed?



As the figure suggests, let x = the side parallel to the fence and let y = the length of the other two sides.

We want to maximize the area of the garden, which means our primary equation is $A = xy$, but this equation has too many variables for us to use calculus just yet. Thus, there should be a secondary equation we can use from information in the problem.

We also know there is 24 feet of trim available, which means $y+y+x=24$, or $2y+x=24$. This is the secondary equation.

Since it is easier to solve for *x*, the equation can be written $x=24-2y$.

Now, the area equation can be written $A=xy=(24-2y)y=24y-2y^2$, which leads us to:

The function to optimize (maximize) is $A(y)=24y-2y^2$.

The next thing we should look at is the domain of the function. Since *y* is a side of the rectangle, it must be nonnegative and can be no more than 12 since the total amount of fencing is 24 feet, and there are two sides with length *y*.

Thus, the domain is $0 \le y \le 12$.

To determine the maximum value, we first take the derivative (with respect to *y*) and find critical points. Since $A(y)$ is continuous on the closed interval $[0, 12]$, we can apply the extreme value theorem, which means evaluating $A(y)$ at its endpoints and at any critical numbers.

First, find the derivative and critical numbers:

$A(y)=24y-2y^2$	Start with the original function.
$A'(y)=24-4y$	Take the derivative.
$24-4y=0$ $24=4y$ $6=y$	Set $A'(y)=0$, then solve.

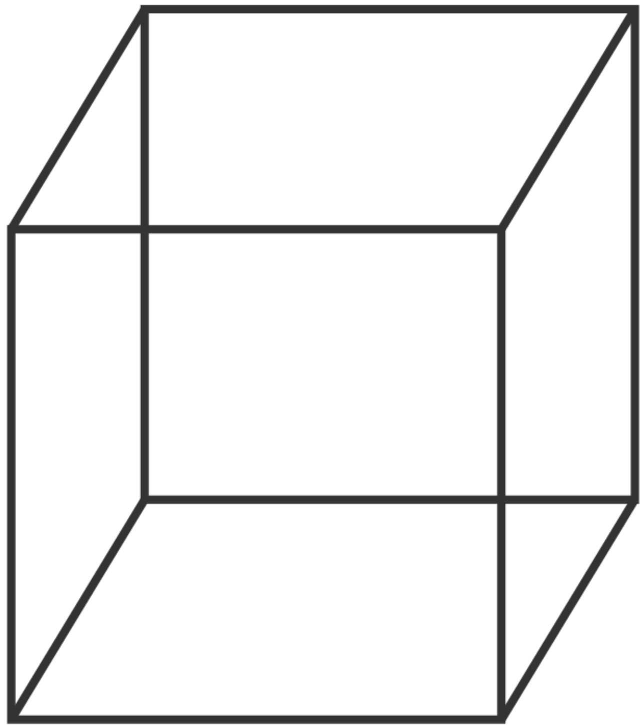
The critical number is $y=6$, which is inside the interval $[0, 12]$. Now, evaluate $A(y)$ at each endpoint and at $y=6$.

y	0	6	12
$A(y)=24y-2y^2$	0	72	0

Thus, the maximum area is 72 ft^2, which occurs when $y=6$.

Let's now look at an example where we minimize the surface area of a rectangular box with known volume.

➦ EXAMPLE A rectangular box with a square base and no lid has volume 500 in^3. What is the least amount of material that could be used to construct such a box? (In other words, what is the minimum surface area?)



- Let x = the length of the base.
- Let h = the height of the box.

Primary Equation:
The surface area is the sum of the areas of all sides. Since this box has no lid, we do not count the area of the top.

The base has area x^2 and each of the four sides have area *xh*. This means that the surface area is $S = x^2 + 4xh$. This is the primary (optimization) equation.

Secondary Equation:
Since volume is (length)(width)(height), this translates to volume $= x \cdot x \cdot h = x^2 h$.

We are given that the volume is 500 in^3, so this is written $x^2 h = 500$. This is the secondary equation that will be used to write the primary equation in terms of one variable.

To do so, it is easiest to replace *h* with an expression in terms of *x*. Using the volume (secondary) equation, rewrite $x^2h = 500$ as $h = \frac{500}{x^2}$.

Substituting into the surface area (primary) equation gives $S = x^2 + 4xh = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + \frac{2000}{x}$, which leads us to:

The function we want to optimize (minimize) is $S(x) = x^2 + \frac{2000}{x}$.

Now, let's discuss the domain of this function. Looking at the equation $h = \frac{500}{x^2}$, the value of *h* will be positive (and therefore valid) for any positive value of *x*. Therefore, the domain is $(0, \infty)$.

Since this is an open interval (no endpoints), we will only focus on finding the minimum value by investigating the function at the critical values in the interval, not the endpoints.

Now, we find the derivative and all critical numbers.

$S(x) = x^2 + \frac{2000}{x} = x^2 + 2000x^{-1}$	Start with the original function; rewrite so the power rule can be used.
$S'(x) = 2x - 2000x^{-2}$	Take the derivative.
$2x - 2000x^{-2} = 0$	Set $S'(x) = 0$, then solve.
$2x - \frac{2000}{x^2} = 0$	Rewrite with positive exponents.
$2x = \frac{2000}{x^2}$	Add $\frac{2000}{x^2}$ to both sides.
$2x^3 = 2000$	Multiply both sides by x^2.
$x^3 = 1000$	Divide both sides by 2.
$x = 10$	Take the cube root of both sides.

Thus, there is a critical number at $x = 10$. To determine if it is a minimum, use the second derivative test.

$S'(x) = 2x - 2000x^{-2}$	Take the first derivative.
$S''(x) = 2 + 4000x^{-3}$	Take the second derivative.
$S''(10) = 2 + 4000(10)^{-3} = 6$	Evaluate $S''(10)$.

Since $S''(10)$ is positive, there is a minimum when $x = 10$.

Thus, the minimum amount of material to build the box is $S(10) = 10^2 + \frac{2000}{10} = 300 \text{ in}^2$.

WATCH

In this video, we will determine the optimal route to lay cable across two terrains (underground and underwater, with water being more expensive).

EXAMPLE A 30-inch rod is to be cut into at most two pieces. One piece will be made into a square, and the other piece will be made into an equilateral triangle. How long should each piece be in order to maximize the combined area, and what is the maximum area?

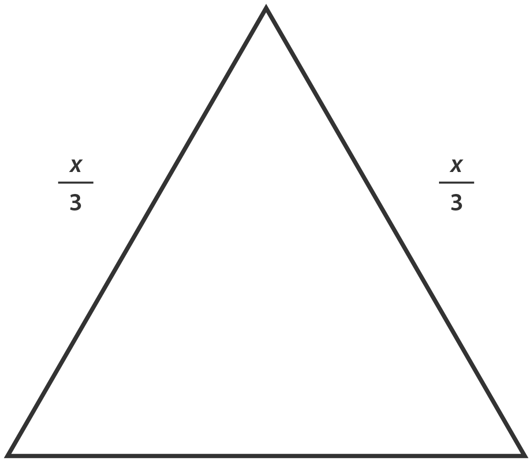
First, start by drawing a picture. Here is the rod being divided into two pieces:

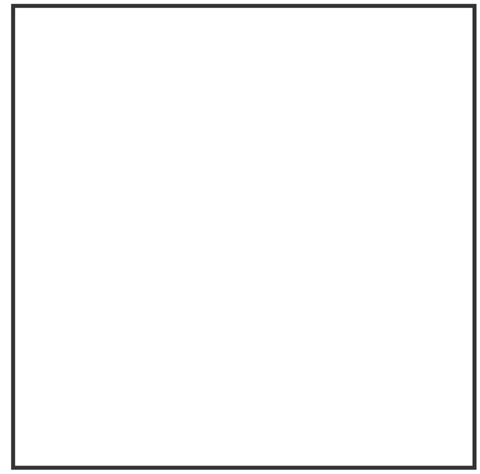
- Let $x =$ the length of the piece that will be used for the triangle.
- Then, $30 - x =$ the length of the piece that will be used for the square.

It is clear that x must be between 0 and 30 inches, therefore the domain is $[0, 30]$, a closed interval.

Now, let’s form the expressions for the lengths of the sides.

- Triangle: Since the piece of the rod has length x, each of its 3 sides has length $\frac{x}{3}$.
- Square: Since the piece of the rod has length $30 - x$, each of its 4 sides has length $\frac{30 - x}{4}$.





Forming the area function:

The area of an equilateral triangle with a side of length *s* is $A = \frac{\sqrt{3}}{4}s^2$. Then, the area of our equilateral triangle is $A_T = \frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2 = \frac{\sqrt{3}}{36}x^2$.

The area of a square with sides of length *s* is $A = s^2$. Then, the area of the square is $A_S = \left(\frac{30-x}{4}\right)^2 = \frac{900-60x+x^2}{16} = \frac{225}{4} - \frac{15}{4}x + \frac{1}{16}x^2$.

Since we want to find the maximum combined area, the optimization function is $A_T + A_S$, which leads us to: $A(x) = \frac{\sqrt{3}}{36}x^2 + \frac{225}{4} - \frac{15}{4}x + \frac{1}{16}x^2$ on the interval $[0, 30]$.

Now, take the derivative and find all critical numbers on the interval $[0, 30]$. Since $A(x)$ is continuous on the closed interval $[0, 30]$, the extreme value theorem can be used to determine the minimum and maximum values of $A(x)$.

$A(x) = \frac{\sqrt{3}}{36}x^2 + \frac{1}{16}x^2 + \frac{225}{4} - \frac{15}{4}x$	Start with the original function (place like terms next to each other).
$A'(x) = \frac{\sqrt{3}}{36}(2x) + \frac{1}{16}(2x) + 0 - \frac{15}{4}$	Take the derivative.
$A'(x) = \frac{\sqrt{3}}{18}x + \frac{1}{8}x - \frac{15}{4}$	Simplify.

$\frac{\sqrt{3}}{18}x+\frac{1}{8}x-\frac{15}{4}=0$	Set $A'(x)=0$. There is no possibility for $A'(x)$ to be undefined since it is a linear function.
$72\left(\frac{\sqrt{3}}{18}x+\frac{1}{8}x-\frac{15}{4}\right)=72(0)$	Multiply both sides by the LCD (72) to clear the fractions.
$4\sqrt{3}x+9x-270=0$	Simplify.
$4\sqrt{3}x+9x=270$ $(4\sqrt{3}+9)x=270$ $x=\frac{270}{4\sqrt{3}+9}\approx 16.95$	Solve for *x*. Since the exact value is complicated, use the approximate value.

Now, make a table to compare the values of $A(x)$ at the critical number as well as the endpoints.

x	0	30	16.95
$A(x)$	$\frac{225}{4}=56.25$	$25\sqrt{3}\approx 43.30$	24.47 (approx.)

The maximum area occurs when $x=0$ (which means the entire 30 inches will be used to make the square and none of it will be used to make the triangle).

Thus, the maximum possible area is 56.25 in^2.

SUMMARY

In this lesson, you learned about the **strategy for solving optimization problems**, which are problems in which the maximum or minimum value is sought (whichever is relevant). As you learned by examining several real-world examples, **solving applied optimization problems** can be particularly challenging since you have to come up with the function on your own. This takes practice, and drawing pictures or making tables is often helpful.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Optimization Problem

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