What Is L'Hopital's Rule?

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WHAT'S COVERED

In this lesson, you will evaluate limits in special forms by using L'Hopital's rule. Specifically, this lesson will cover:

1. A Brief Review of Frequently Used Limits

In this challenge, we will be seeing limits of exponential and logarithmic functions pretty often, so here is a chart that describes the behavior of some functions. To visualize these limits, use technology to graph the function.

Behavior of Common Functions	
$\lim\limits_{x \to \infty} e^x = \infty$	$\lim\limits_{x \to -\infty} e^x = 0$
$\lim\limits_{x \to \infty} e^{-x} = 0$	$\lim\limits_{x \to -\infty} e^{-x} = \infty$
$\lim\limits_{x \to \infty} \ln x = \infty$	$\lim\limits_{x \to 0^+} \ln x = -\infty$

2. Limits of the Form 0/0

Earlier in the course, we encountered limits such as $\lim\limits_{x \to 2} \frac{x-2}{x^2-4}$ and $\lim\limits_{x \to 3} \frac{5x-15}{7x-21}$.

Note that in each limit, the numerator and denominator are both zero when direct substitution is used. We say that these limits are in the form "$\frac{0}{0}$". This is an example of an **indeterminate form**, since its value is not known until further analysis is done. There is no way to determine its value just by being "$\frac{0}{0}$".

Case in point, if we evaluate these limits, we have:

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$$

$$\lim_{x \to 3} \frac{5x-15}{7x-21} = \lim_{x \to 3} \frac{5(x-3)}{7(x-3)} = \lim_{x \to 3} \frac{5}{7} = \frac{5}{7}$$

These limits both had the form $\frac{0}{0}$, but ended up having different values. This is why $\frac{0}{0}$ is called an indeterminate form.

While these limits were able to be manipulated easily using algebra, how would we go about evaluating $\lim_{x \to 0} \frac{e^{2x}-1}{x}$? This has the indeterminate form $\frac{0}{0}$, but there is no algebraic way to manipulate this expression. For limits like this, we have L'Hopital's rule.

BIG IDEA

Suppose $f(x)$ and $g(x)$ are differentiable on an open interval which contains $x=a$ and $g'(x) \neq 0$ except possibly at $x=a$. If $f(x)$ and $g(x)$ both approach 0 as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that the limit on the right exists.

Note: "a" can be replaced with either "$-\infty$" or "∞" to allow for limits as $x \to \pm\infty$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0} \frac{e^{2x}-1}{x}$

First check the requirements.

1. The numerator and denominator both approach 0 as $x \to 0$.
2. The numerator and denominator are both differentiable on any interval containing $x=0$.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to 0} \frac{e^{2x}-1}{x}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0} \frac{2e^{2x}}{1}$	$D[e^{2x}-1] = 2e^{2x}$ and $D[x]=1$
$= \lim_{x \to 0} \frac{2e^{2x}}{1} = \frac{2e^0}{1} = 2$	Use direct substitution.

We can conclude that $\lim_{x \to 0} \frac{e^{2x}-1}{x} = 2$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8}$

First, check the requirements.

1. The numerator and denominator both get closer to 0 as $x \to 2$.
2. The numerator and denominator are both differentiable on any interval containing $x = 2$.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 2} \frac{5x^4}{3x^2}$	$D[x^5 - 32] = 5x^4$ and $D[x^3 - 8] = 3x^2$
$= \lim_{x \to 2} \frac{5x^4}{3x^2} = \frac{5(2)^4}{3(2)^2} = \frac{20}{3}$	Use direct substitution.

Thus, $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8} = \frac{20}{3}$.

TRY IT

Consider the following limit: $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

Evaluate the limit. +

$\frac{3}{7}$

L'Hopital's rule can be applied more than once as long as the new limit is also in the form $\frac{0}{0}$.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^2}$

First, check the requirements. Let $f(x) = \cos(2x) - 1 + 2x^2$ and $g(x) = x^2$.

- $f(x)$ and $g(x)$ both get closer to 0 as $x \to 0$.
- $f'(x) = -2\sin(2x) + 4x$ and $g'(x) = 2x$
- $f(x)$ and $g(x)$ are both differentiable on any interval containing $x = 0$.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^2}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0} \frac{-2\sin(2x) + 4x}{2x}$	$f'(x) = -2\sin(2x) + 4x$ and $g'(x) = 2x$
$= \lim_{x \to 0} \frac{-4\cos(2x) + 4}{2}$	The numerator and denominator both approach 0 as $x \to 0$, and the numerator and denominator are both differentiable. Therefore, L'Hopital's rule can be applied again: $D[-2\sin(2x) + 4x] = -4\cos(2x) + 4$ and $D[2x] = 2$
$= \lim_{x \to 0} \frac{-4\cos(2x) + 4}{2} = \frac{-4\cos(0) + 4}{2} = 0$	Direct substitution gives 0.

Conclusion: After applying L'Hopital's rule twice, $\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^2} = 0$.

TERM TO KNOW

Indeterminate Form

A form of a limit, such as "$\frac{0}{0}$", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.

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3. Limits of the Form ∞/∞

Consider the limit $\lim_{x \to \infty} \frac{2x + 1}{3x + 4}$, whose value is $\frac{2}{3}$, and $\lim_{x \to \infty} \frac{2x + 1}{x^2 + 4}$, whose value is 0.

In each limit, the numerator and denominator go to ∞ as $x \to \infty$. This means that both limits are in the form $\frac{\infty}{\infty}$, which is another indeterminate form (since it leads to different values).

BIG IDEA

$\frac{\infty}{\infty}$ is another indeterminate form.

As it turns out, L'Hopital's rule can also be applied to this indeterminate form in a similar way:

BIG IDEA

Suppose $f(x)$ and $g(x)$ are differentiable on an open interval which contains $x = a$ and $g'(x) \neq 0$ except possibly at $x = a$. If $f(x)$ and $g(x)$ both tend toward ∞ or $-\infty$ as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that the limit on the right exists.

Note: "a" can be replaced with either "$-\infty$" or "∞" to allow for limits as $x \to \pm\infty$.

➦ EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} \frac{2x+1}{x^2+4}$

Note that both the numerator and denominator are tending toward ∞ as $x \to \infty$. And they are both differentiable.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to \infty} \frac{2x+1}{x^2+4}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to \infty} \frac{2}{2x}$	$D[2x+1]=2,\ D\left[x^2+4\right]=2x$
$= \lim_{x \to \infty} \frac{1}{x}=0$	Simplify the expression; then, the limit is 0 since $\lim_{x \to \infty} \frac{c}{x^n}=0$.

Thus, $\lim_{x \to \infty} \frac{2x+1}{x^2+4}=0$.

Let's look at an example that involves exponential functions.

➦ EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} \frac{e^x+x^2}{x^3+8x}$

Note that both the numerator and denominator are tending toward ∞ as $x \to \infty$. And they are both differentiable.

This means L'Hopital's rule can be used to evaluate the limit.

$\lim_{x \to \infty} \frac{e^x+x^2}{x^3+8x}$	Start with the limit that needs to be evaluated.
$= \lim_{x \to \infty} \frac{e^x+2x}{3x^2+8}$	$D\left[e^x+x^2\right]=e^x+2x,\ D\left[x^3+8x\right]=3x^2+8$
$= \lim_{x \to \infty} \frac{e^x+2}{6x}$	Numerator and denominator both tend toward ∞ as $x \to \infty$, and both are differentiable. Apply L'Hopital's rule again. $D\left[e^x+2x\right]=e^x+2,\ D\left[3x^2+8\right]=6x$
$= \lim_{x \to \infty} \frac{e^x}{6}$	Numerator and denominator both tend toward ∞ as $x \to \infty$, and both are differentiable.

Apply L'Hopital's rule again.
$D[e^x+2]=e^x$, $D[6x]=6$

$= \lim_{x\to\infty} \frac{e^x}{6} = \infty$

L'Hopital's rule no longer applies since the denominator is constant while the numerator tends toward ∞.
Since the numerator increases without bound, dividing by 6 doesn't affect this, and the limit is ∞.

Thus, $\lim_{x\to\infty} \frac{e^x+x^2}{x^3+8x} = \infty$.

TRY IT

Consider the following limit: $\lim_{x\to\infty} \frac{2x^3+8x+5}{5x^3+x}$

Evaluate the limit. +

$\frac{2}{5}$

WATCH

In this final example, we'll evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$.

SUMMARY

In this lesson, you began with **a brief review of frequently used limits**. You also learned that if $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ (except possibly when $x=a$), L'Hopital's rule is a very convenient way to evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$, that have the indeterminate **form 0/0** or the **form** ∞/∞.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Indeterminate Form

A form of a limit, such as "$\frac{0}{0}$", that doesn't always yield the same value. Further analysis is needed to determine its value, if it exists.