Which Functions Are Continuous?

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WHAT'S COVERED

In this lesson, you will explore how continuity can be applied to combinations of functions. Specifically, this lesson will cover:

1. Functions That Are Continuous for All Real Numbers

To gain an understanding of continuity, let's recall some non-piecewise functions that contain no breaks or holes in their graphs (i.e., no points where the function is undefined). In other words, these functions are continuous for every real number.

- Polynomial functions
- Sine and cosine functions
- Absolute value functions

Moreover, given that $f(x)$ and $g(x)$ are continuous at $x = a$, it follows that these functions are also continuous for $x = a$.

Function	Operations
$f \pm g$	Sum/difference
$f \cdot g$	Product
$\frac{f}{g}$	Quotient, provided $g(a) \neq 0$

$[f(x)]^n$, $n = 0, 1, 2, \ldots$	Raise $f(x)$ to a whole number power
$f(g(x))$	Composition of f and g

It follows that any of these combinations of polynomial, sine, cosine, and absolute value functions are continuous for all real numbers (provided that we do not create a denominator that could be zero).

The following functions are continuous for all real numbers.

Functions	Combinations
$f(x) = -2x^3 + 12x^2 + 5x - 8$	Polynomial
$g(x) = \sin^2 x - 2\cos x$	Powers of sine and cosine
$h(x) = x^3 \sin 2x$	Product of polynomial and sine, with composition
$j(x) = \frac{2x}{x^2+4}$	Quotient of two functions, but no value of x will make the denominator equal to 0

2. Rational Functions

Let's explore the quotient of two functions a bit more. We have to be careful though, since a function is undefined when its denominator is equal to 0.

A **rational function** has the form $f(x) = \frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

EXAMPLE $f(x) = \frac{2x}{x-1}$ is continuous for every real number except where $x - 1 = 0$, which means $x = 1$. The intervals over which $f(x)$ is continuous are $(-\infty, 1) \cup (1, \infty)$.

EXAMPLE $g(x) = \frac{2x^2-3}{x^2-5x+6}$ is continuous for every real number except where $x^2 - 5x + 6 = 0$.

Writing in factored form, we have $(x-2)(x-3) = 0$, which has solutions $x = 2$ and $x = 3$. Thus, $g(x)$ is continuous for all real numbers except 2 and 3.

Using interval notation, $g(x)$ is continuous on the intervals $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

TERM TO KNOW

Rational Function

A function in the form $f(x) = \frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

3. Radical Functions

Recall that the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$ when *n* is even (even root), and $(-\infty, \infty)$ when *n* is odd (odd root).

It follows that $f(x) = \sqrt[n]{x}$ is continuous on its domain, which depends on the type of root (even or odd).

EXAMPLE $f(x) = \sqrt[3]{x-1}$ is continuous for all real numbers.

EXAMPLE $g(x) = \sqrt{x-4}$ is continuous when $x - 4 \geq 0$, which means the interval $[4, \infty)$.

4. Combinations of Functions

Consider the two functions $f(x)$ and $g(x)$. Given that we know where $f(x)$ and $g(x)$ are continuous, the following can be said for some combinations of $f(x)$ and $g(x)$:

- The functions $f(x) + g(x)$, $f(x) - g(x)$, and $f(x) \cdot g(x)$ are all continuous on the interval(s) over which $f(x)$ and $g(x)$ are both continuous.
- The function $\frac{f(x)}{g(x)}$ is continuous on the interval(s) over which $f(x)$ and $g(x)$ are both continuous, with the added condition that $g(x) \neq 0$.

EXAMPLE Consider the function $f(x) = x^3\sqrt{x-4}$, which is a product of $y = x^3$ and $y = \sqrt{x-4}$.

- $y = x^3$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- $y = \sqrt{x-4}$ is continuous where $x - 4 \geq 0$, which is the interval $[4, \infty)$.

Therefore, $f(x) = x^3\sqrt{x-4}$ is continuous on the interval $[4, \infty)$.

EXAMPLE Consider the function $f(x) = \frac{\cos x}{x-5}$, which is the quotient of $y = \cos x$ and $y = x - 5$.

- $y = \cos x$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- $y = x - 5$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- Since the denominator is $x - 5$, $f(x)$ is not continuous when $x - 5 = 0$, which means $x = 5$.

Therefore, $f(x)$ is continuous for all real numbers except $x = 5$, which means $(-\infty, 5) \cup (5, \infty)$ in interval notation.

EXAMPLE Consider the function $f(x) = \frac{2x}{\sqrt{x-3}}$, which is the quotient of $y = 2x$ and $y = \sqrt{x-3}$.

- $y = 2x$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
- $y = \sqrt{x-3}$ is continuous where $x - 3 \geq 0$, which is the interval $[3, \infty)$.
- Since the denominator is $\sqrt{x-3}$, $f(x)$ is also not continuous when $x - 3 = 0$, which means $x = 3$.

Therefore, $f(x)$ is continuous when $x > 3$, which means the interval $(3, \infty)$.

TRY IT

Consider the function $f(x) = 3x + \sqrt{2x-5}$.

Determine the interval(s) over which this function is continuous. +

$\left[\frac{5}{2}, \infty\right)$

WATCH

The following video goes through the examples determining the intervals of continuity for the functions $f(x) = 4x^3\sqrt{x+15}$, $f(x) = \frac{x^2-16}{x+4}$, and $f(x) = \frac{9\sin x}{\sqrt{x-8}}$.

TRY IT

Consider the following table:

Function	Continuous Interval
$f(x) = 3x\cos(x^2)$	?
$g(x) = \frac{x^3-8}{x-2}$	?
$h(x) = \sqrt[5]{x^3 - 3x^2 + 10x - 4}$	?

Determine the intervals over which each function is continuous. +

Function	Continuous Interval
$f(x) = 3x\cos(x^2)$	All real numbers or $(-\infty, \infty)$
$g(x) = \frac{x^3-8}{x-2}$	All real numbers except 2 or $(-\infty, 2) \cup (2, \infty)$

$h(x)=\sqrt[5]{x^3-3x^2+10x-4}$	All real numbers or $(-\infty, \infty)$

SUMMARY

In this lesson, you learned about some non-piecewise **functions that are continuous for all real numbers**, including polynomial functions, sine and cosine functions, and absolute value functions. You explored the quotient of two functions, noting that you have to be careful since a function is undefined when its denominator is equal to 0. You learned about **rational functions**, which have the form $f(x)=\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials; a rational function is continuous at all real numbers except for those where $D(x)=0$. You also learned about **radical functions**, understanding that $f(x)=\sqrt[n]{x}$ is continuous on its domain, which depends on the type of root (even or odd). Lastly, you learned that for **combinations of functions**, take special care with radical and rational functions. If $f(x)$ has no values where it is undefined, then $f(x)$ is continuous for all real numbers.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Rational Function

A function in the form $f(x)=\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x)=0$.