The Inverse Trigonometric Functions

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WHAT'S COVERED

In this lesson, you will learn about the inverse trigonometric functions and how they are evaluated.

1. The Inverse Trigonometric Functions

Recall the six basic trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, and $\cot x$.

For each of them, the input is some angle and the output is a real number.

The **inverse trigonometric functions** do just the reverse. The input is the real number, while the output is the angle that produces the ratio.

For example, we define the inverse sine function as $y = \sin^{-1} x$, which means $x = \sin y$. Looking at the equation $x = \sin y$, it's clear that x must be between -1 and 1 (inclusive) since the sine function only returns ratios between -1 and 1.

The six inverse trigonometric functions, with their domains and ranges, are summarized in the table below.

Function	Domain	Range
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	All real numbers	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = \sec^{-1}x$	$(-\infty, -1]\cup[1, \infty)$	$\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$
$y = \csc^{-1}x$	$(-\infty, -1]\cup[1, \infty)$	$\left[-\frac{\pi}{2}, 0\right)\cup\left(0, \frac{\pi}{2}\right]$
$y = \cot^{-1}x$	All real numbers	$(0, \pi)$

Note: when you use your calculator to evaluate an inverse trigonometric function, it will return the correct value.

HINT

The inverse trigonometric functions often go by other names. For example, $\sin^{-1}x$ can also be written as $\arcsin x$. This is sometimes more convenient since the “-1” in $\sin^{-1}x$ is often mistaken for an exponent of -1. Naturally, the other trigonometric functions follow suit. For example, $\tan^{-1}x$ is also known as $\arctan x$, etc.

TERM TO KNOW

Inverse Trigonometric Functions

A function that receives a real number as its input and returns an angle as its output.

2. Evaluating the Inverse Sine, Cosine, and Tangent Functions for Known Ratios

Recall that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Then, we can say $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ or $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

When evaluating inverse trigonometric functions, we need to keep the range in mind.

EXAMPLE $\sin^{-1}(1) = \frac{\pi}{2}$ since $\frac{\pi}{2}$ is inside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\left(\frac{\pi}{2}\right) = 1$.

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ since $\frac{2\pi}{3}$ is inside the interval $[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is inside the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\frac{\pi}{3} = \sqrt{3}$.

3. Evaluating the Inverse Cosecant, Secant, and Cotangent Functions for Known Ratios

Most calculators do not have dedicated buttons for $\csc^{-1}x$, $\sec^{-1}x$, or $\cot^{-1}x$; as you might suspect, these are related to their corresponding reciprocal functions.

For example, let's say we wish to find $\csc^{-1}(2)$.

This means that we want to find y so that $\csc y = 2$.

Since $\csc y$ and $\sin y$ are reciprocals, this is equivalent to writing $\sin y = \frac{1}{2}$, which means $y = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Through all this, something to notice is that $\csc^{-1}(2) = \sin^{-1}\left(\frac{1}{2}\right)$. This leads to some important identities.

FORMULA TO KNOW

Evaluating Inverse Cosecant

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Secant

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Cotangent

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

EXAMPLE Find $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$.

$$\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Note: $\frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{2\cdot 3} = \frac{\sqrt{3}}{2}$

EXAMPLE Find $\cot^{-1}(-1)$.

$$\cot^{-1}(-1) = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

EXAMPLE Find $\csc^{-1}\left(\frac{1}{2}\right)$.

$\csc^{-1}\left(\frac{1}{2}\right) = \sin^{-1}(2)$

Since $\sin^{-1}(2)$ is undefined, $\csc^{-1}\left(\frac{1}{2}\right)$ is undefined as well.

TRY IT

Consider the following inverse trigonometric function:

Inverse Trigonometric Function	Exact Value
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	?
$\sec^{-1}(\sqrt{2})$	?
$\tan^{-1}(-\sqrt{3})$	?

Find the exact value of each inverse trigonometric function. +

Inverse Trigonometric Function	Exact Value
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	$\frac{\pi}{4}$
$\sec^{-1}(\sqrt{2})$	$\frac{\pi}{4}$
$\tan^{-1}(-\sqrt{3})$	$-\frac{\pi}{3}$

SUMMARY

In this lesson, you learned that **the inverse trigonometric functions** provide a way to express the angle as a function of the trigonometric ratio. You also learned how to **evaluate the inverse trigonometric functions for known ratios**, noting that while not all of these inverse functions are available on most calculators, there are identities that can be used to relate to other more common functions.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 7 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Inverse Trigonometric Functions

A function that receives a real number as its input and returns an angle as its output.

FORMULAS TO KNOW

Evaluating Inverse Cosecant

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Cotangent

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Secant

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$