The Quotient Rule

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WHAT'S COVERED

In this lesson, you will find derivatives of functions that are quotients of other functions (for example, $f(x)=\frac{2x}{x^2+1}$). Specifically, this lesson will cover:

1. The Quotient Rule
2. Combining Derivative Rules

1. The Quotient Rule

Let's say that $Q(x)=\frac{f(x)}{g(x)}$. The goal is to find $Q'(x)$.

$Q(x)=\frac{f(x)}{g(x)}$	Start with the original function.
$Q(x)\cdot g(x)=f(x)$	Multiply both sides by $g(x)$.
$Q'(x)\cdot g(x)+Q(x)\cdot g'(x)=f'(x)$	Take the derivative. Apply the product rule on the left-hand side.
$Q'(x)\cdot g(x)=f'(x)-Q(x)\cdot g'(x)$	Subtract $Q(x)\cdot g'(x)$ from both sides.
$Q'(x)=\frac{f'(x)-Q(x)\cdot g'(x)}{g(x)}$	Divide both sides by $g(x)$ to solve for $Q'(x)$.
$Q'(x)=\frac{f'(x)-\left[\frac{f(x)}{g(x)}\right]\cdot g'(x)}{g(x)}$	Replace $Q(x)$ with $\frac{f(x)}{g(x)}$ to get an expression that involves only $f(x)$ and $g(x)$.
$Q'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$	Multiply by $\frac{g(x)}{g(x)}$ to simplify the complex fraction.

We now have an expression for $Q'(x)$, the derivative of $\frac{f(x)}{g(x)}$.

Note how the numerator is very similar to the product rule, but there is a subtraction between the terms instead of an addition sign.

Another way to remember this is to use the following saying:

$$D\left[\frac{high}{low}\right] = \frac{low\ dee\ high - high\ dee\ low}{low\ low}$$

In words, this is “low dee high minus high dee low over low low.” The word *dee* is used for the derivative.

FORMULA TO KNOW

Quotient Rule for Derivatives

Using "Prime" Notation: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Using "D" Notation: $D\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot D[f(x)] - f(x) \cdot D[g(x)]}{[g(x)]^2}$

"High and Low" Version: $D\left[\frac{high}{low}\right] = \frac{low\ dee\ high - high\ dee\ low}{low\ low}$

EXAMPLE Find the derivative of $f(x) = \frac{2x}{x^2+1}$.

Apply the quotient rule formula and simplify:

$f'(x) = \frac{(x^2+1) \cdot D[2x] - 2x \cdot D[x^2+1]}{(x^2+1)^2}$	Apply the quotient rule.
$f'(x) = \frac{(x^2+1)(2) - 2x \cdot 2x}{(x^2+1)^2}$	$D[2x] = 2$, $D[x^2+1] = 2x$
$f'(x) = \frac{2x^2+2-4x^2}{(x^2+1)^2}$	Simplify on both sides of subtraction.
$f'(x) = \frac{-2x^2+2}{(x^2+1)^2}$	Simplify the numerator.

Thus, $f'(x) = \frac{-2x^2+2}{(x^2+1)^2}$. This is the simplest form. If you factor the numerator completely, you would get $f'(x) = \frac{-2(x+1)(x-1)}{(x^2+1)^2}$ and notice that there is no common factor between the numerator and denominator.

Now, try one for yourself. Be sure to simplify your answer as much as possible!

TRY IT

Consider the function $f(x)=\frac{3x^2-1}{2x^2+3}$.

Find the derivative. +

$$f'(x)=\frac{22x}{(2x^2+3)^2}$$

EXAMPLE Find the derivative of the function $f(x)=\frac{1}{3x^2+4}$.

Apply the quotient rule formula and simplify:

$$f'(x)=\frac{(3x^2+4)\cdot D[1]-(1)\cdot D[3x^2+4]}{(3x^2+4)^2}$$ Apply the quotient rule.

$$f'(x)=\frac{(3x^2+4)(0)-1\cdot 6x}{(3x^2+4)^2}$$ $D[1]=0,\ D[3x^2+4]=6x$

$$f'(x)=\frac{0-6x}{(3x^2+4)^2}$$ Simplify on both sides of subtraction.

$$f'(x)=\frac{-6x}{(3x^2+4)^2}$$ Simplify the numerator.

Thus, $f'(x)=\frac{-6x}{(3x^2+4)^2}$.

Let's look at an example that involves trigonometric functions.

WATCH

Here is a video to help illustrate the quotient rule by finding the derivative of $f(x)=\frac{\sin x}{x+1}$.

Here is an example for you to try on your own. Remember to simplify where possible!

TRY IT

Consider the function $f(x)=\frac{1+\cos x}{1-\cos x}$.

Find the derivative. +

$$f'(x)=\frac{-2\sin x}{(1-\cos x)^2}$$

Sometimes writing a derivative in simplest form requires the use of trigonometric identities. Here is an example.

➭ EXAMPLE Find the derivative of the function $f(x)=\dfrac{\sin x}{1+\cos x}$.

Now, apply the quotient rule formula and simplify:

$f'(x)=\dfrac{(1+\cos x)\cdot D[\sin x]-\sin x\cdot D[1+\cos x]}{(1+\cos x)^2}$	Apply the quotient rule.
$f'(x)=\dfrac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}$	$D[\sin x]=\cos x$, $D[1+\cos x]=-\sin x$
$f'(x)=\dfrac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$	Distribute $\cos x(1+\cos x)$ and simplify $-\sin x(-\sin x)$.
$f'(x)=\dfrac{\cos x+1}{(1+\cos x)^2}$	Trigonometric Identity: $\sin^2 x+\cos^2 x=1$
$f'(x)=\dfrac{1}{1+\cos x}$	Remove the common factor of $1+\cos x$.

Thus, $f'(x)=\dfrac{1}{1+\cos x}$.

Now, let's look at an example where the quotient rule could be used, but isn't required.

➭ EXAMPLE Find the derivative of the function $f(x)=\dfrac{6x^2+10x+1}{3x^2}$.

This is clearly a quotient, so the quotient rule can be used to find the derivative:

$f'(x)=\dfrac{3x^2\cdot D[6x^2+10x+1]-(6x^2+10x+1)\cdot D[3x^2]}{(3x^2)^2}$	Apply the quotient rule.
$f'(x)=\dfrac{(3x^2)(12x+10)-(6x^2+10x+1)(6x)}{9x^4}$	$D[6x^2+10x+1]=12x+10$, $D[3x^2]=6x$
$f'(x)=\dfrac{36x^3+30x^2-36x^3-60x^2-6x}{9x^4}$	Distribute on both sides of the subtraction sign.
$f'(x)=\dfrac{-30x^2-6x}{9x^4}$	Simplify the numerator.
$f'(x)=\dfrac{-30x^2}{9x^4}-\dfrac{6x}{9x^4}$	Write as single fractions.
$f'(x)=\dfrac{-10}{3x^2}-\dfrac{2}{3x^3}$	Simplify each fraction.
$f'(x)=-\dfrac{10}{3x^2}-\dfrac{2}{3x^3}$	Write negative in front of the fraction.

Thus, $f'(x) = -\frac{10}{3x^2} - \frac{2}{3x^3}$.

However, notice that the original expression can be rewritten by performing the division:

$f(x) = \frac{6x^2+10x+1}{3x^2} = \frac{6x^2}{3x^2} + \frac{10x}{3x^2} + \frac{1}{3x^2}$	Split into individual fractions.
$f(x) = 2 + \frac{10}{3x} + \frac{1}{3x^2}$	Simplify each fraction.
$f(x) = 2 + \frac{10}{3}x^{-1} + \frac{1}{3}x^{-2}$	Write each term as a single exponent (to get ready for differentiation).

Now, $f(x)$ is a sum/difference of powers of *x*, which is much easier to differentiate:

$f'(x) = D[2] + D\left[\frac{10}{3}x^{-1}\right] + D\left[\frac{1}{3}x^{-2}\right]$	Apply the sum/difference rules.
$f'(x) = 0 + \frac{10}{3}(-1)x^{-2} + \frac{1}{3}(-2)x^{-3}$	Apply the power and constant multiple rules.
$f'(x) = -\frac{10}{3}x^{-2} - \frac{2}{3}x^{-3}$	Simplify.
$f'(x) = -\frac{10}{3x^2} - \frac{2}{3x^3}$	Write using positive exponents.

As you can see, the results are identical.

BIG IDEA

When you need to find the derivative of a quotient of two functions, check to see if the function can be manipulated/simplified first. You could avoid having to use the quotient rule in exchange for an easier rule.

2. Combining Derivative Rules

As we continue to learn more derivative rules, it is important to see how the rules work together in more complicated functions.

EXAMPLE A company has determined that the total cost of producing *x* items is modeled by the function $C(x) = 20x + \frac{50x}{x+1}$, where $C(x)$ is measured in dollars.

The rate at which the total cost changes is called the marginal cost, and is found by computing $C'(x)$. The production manager is interested in knowing the marginal cost at the point when 9 units are produced.

First, find the derivative. Since $C(x)$ is the sum of two functions, we know $C'(x)=D[20x]+D\left[\frac{50x}{x+1}\right]$. By the power rule, $D[20x]=20$.

Now, we use the quotient rule to find $D\left[\frac{50x}{x+1}\right]$:

$D\left[\frac{50x}{x+1}\right]=\frac{(x+1)\cdot D[50x]-50x\cdot D[x+1]}{(x+1)^2}$	Use the quotient rule.
$D\left[\frac{50x}{x+1}\right]=\frac{(x+1)(50)-50x\cdot(1)}{(x+1)^2}$	$D[50x]=50,\ D[x+1]=1$
$D\left[\frac{50x}{x+1}\right]=\frac{50}{(x+1)^2}$	$50(x+1)-50x=50x+50-50x=50$

Thus, $C'(x)=D[20x]+D\left[\frac{50x}{x+1}\right]=20+\frac{50}{(x+1)^2}$.

Now, we seek the marginal cost at 9 units, which is $C'(9)=20+\frac{50}{(9+1)^2}=20.5$, which means the cost is rising at a rate of $20.50 per additional unit when 9 units are produced.

SUMMARY

In this lesson, you explored finding the derivative using **the quotient rule**, although it's important to check if the function can be manipulated or simplified first, as you may be able to use an easier rule.

At this point, you are able to find derivatives involving, sums, differences, products, and quotients. As you learn more derivative rules, you are able to take derivatives of more functions, including those that **combine these derivative rules**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

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