Derivatives of Trigonometric Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn the derivatives of the remaining four trigonometric functions, and then incorporate these rules to find derivatives of combinations of functions. Specifically, this lesson will cover:

1. The Derivatives of tan*x*, sec*x*, cot*x*, and csc*x*

Recall the following identities:

- $\tan x = \frac{\sin x}{\cos x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$
- $\csc x = \frac{1}{\sin x}$

Notice that all four functions are quotients related to sin*x* and/or cos*x*. Since the derivatives of sin*x* and cos*x* are known, establishing rules for the other four functions should be fairly straightforward. The derivatives of each of the remaining trigonometric functions follows.

FORMULA TO KNOW

Derivative of Tangent

$D[\tan x] = \sec^2 x$

$$D[\tan x] = D\left[\frac{\sin x}{\cos x}\right]$$ Use the identity for tan*x*.

$$= \frac{\cos x \cdot D[\sin x] - \sin x \cdot D[\cos x]}{(\cos x)^2}$$ Apply the quotient rule.

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2}$$ $D[\sin x] = \cos x$, $D[\cos x] = -\sin x$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$ Simplify the numerator and rewrite the denominator.

$$= \frac{1}{\cos^2 x}$$ Use another trigonometric identity: $\sin^2 x + \cos^2 x = 1$

$$= \sec^2 x$$ Since $\sec x = \frac{1}{\cos x}$, $\sec^2 x = \frac{1}{\cos^2 x}$.

FORMULA TO KNOW

Derivative of Cotangent

$D[\cot x] = -\csc^2 x$

$$D[\cot x] = D\left[\frac{\cos x}{\sin x}\right]$$ Use the identity for cot*x*.

$$= \frac{\sin x \cdot D[\cos x] - \cos x \cdot D[\sin x]}{(\sin x)^2}$$ Apply the quotient rule.

$$= \frac{(\sin x)(-\sin x) - \cos x \cdot (\cos x)}{(\sin x)^2}$$ $D[\sin x] = \cos x$, $D[\cos x] = -\sin x$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$ Simplify the numerator and rewrite the denominator.

$$= \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x}$$ Factor out -1 from the numerator.

$$= \frac{-1}{\sin^2 x}$$ Use another trigonometric identity: $\sin^2 x + \cos^2 x = 1$

$$= -\csc^2 x$$ Since $\csc x = \frac{1}{\sin x}$, $\csc^2 x = \frac{1}{\sin^2 x}$.

FORMULA TO KNOW

Derivative of Secant

$D[\sec x] = \sec x \tan x$

$$D[\sec x] = D\left[\frac{1}{\cos x}\right]$$ Use the identity for sec*x*.

$= \dfrac{\cos x \cdot D[1] - 1 \cdot D[\cos x]}{(\cos x)^2}$	Apply the quotient rule.
$= \dfrac{\cos x(0) - 1 \cdot (-\sin x)}{(\cos x)^2}$	$D[1] = 0, D[\cos x] = -\sin x$
$= \dfrac{\sin x}{\cos^2 x}$	Simplify the numerator and rewrite the denominator.

At this point, it might appear that we should stop. However, there is a slightly simpler way to express this by using some trigonometric identities:

$= \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\cos x}$	Split the fraction into two factors.
$= \tan x \cdot \sec x$	Use these trigonometric identities: $\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$
$= \sec x \tan x$	It's more common to place $\sec x$ first.

The final result of $\sec x \tan x$ is easier to remember than $\dfrac{\sin x}{\cos^2 x}$, even though they are equivalent.

FORMULA TO KNOW

Derivative of Cosecant

$D[\csc x] = -\csc x \cot x$

$D[\csc x] = D\left[\dfrac{1}{\sin x}\right]$	Use the identity for $\csc x$.
$= \dfrac{\sin x \cdot D[1] - 1 \cdot D[\sin x]}{(\sin x)^2}$	Apply the quotient rule.
$= \dfrac{\sin x(0) - 1 \cdot (\cos x)}{(\sin x)^2}$	$D[1] = 0, D[\sin x] = \cos x$
$= \dfrac{-\cos x}{\sin^2 x}$	Simplify the numerator and rewrite the denominator.

At this point, it might appear that we should stop. However, there is a slightly simpler way to express this by using some trigonometric identities:

$= -\dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\sin x}$	Split the fraction into two factors.
$= -\cot x \csc x$	Use these trigonometric identities: $\cot x = \dfrac{\cos x}{\sin x}$, $\csc x = \dfrac{1}{\sin x}$
$= -\csc x \cot x$	It's more common to place $\csc x$ first.

The final result of $-\csc x \cot x$ is easier to remember than $\dfrac{-\cos x}{\sin^2 x}$, even though they are equivalent.

2. Summary of the Derivatives of the Trigonometric Functions

Here are all the trigonometric functions and their derivatives:

- $D[\sin x] = \cos x$
- $D[\cos x] = -\sin x$
- $D[\tan x] = \sec^2 x$
- $D[\sec x] = \sec x \tan x$
- $D[\csc x] = -\csc x \cot x$
- $D[\cot x] = -\csc^2 x$

Notice these similarities:

- $\tan x$ and $\cot x$ have similar derivatives.
- $\sec x$ and $\csc x$ also have similar derivatives.
- If the function name begins with a “c”, then its derivative has a negative sign.

---

3. Derivatives of Combinations of Functions with $\tan x$, $\sec x$, $\cot x$, and $\csc x$

Now that we have more derivative rules, we can see how they are used when combined with other functions.

EXAMPLE Find the derivative: $y = 4x^2 - 2\csc x + \frac{6}{x^2}$

First, rewrite $\frac{6}{x^2}$ as $6x^{-2}$ so that the power rule can be used.

Then, your function is: $y = 4x^2 - 2\csc x + 6x^{-2}$

$y' = D[4x^2] - D[2\csc x] + D[6x^{-2}]$	Use the sum/difference rules.
$y' = 4D[x^2] - 2D[\csc x] + 6D[x^{-2}]$	Apply the constant multiple rules.
$y' = 4(2x) - 2(-\csc x \cot x) + 6(-2)x^{-3}$	Apply the power rule and $D[\csc x] = -\csc x \cot x$.
$y' = 8x + 2\csc x \cot x - 12x^{-3}$	Simplify.
$y' = 8x + 2\csc x \cot x - \frac{12}{x^3}$	Write the last term with a positive exponent.

Thus, $y' = 8x + 2\csc x \cot x - \frac{12}{x^3}$.

EXAMPLE Find the derivative of the function $f(x) = 3x\tan x$.

This is the product of two functions, so the product rule will be used:

$f'(x) = D[3x] \cdot \tan x + 3x \cdot D[\tan x]$	Apply the product rule.
$f'(x) = 3 \cdot \tan x + 3x \cdot \sec^2 x$	$D[3x] = 3$, $D[\tan x] = \sec^2 x$
$f'(x) = 3\tan x + 3x\sec^2 x$	Eliminate unnecessary symbols.

Thus, $f'(x) = 3\tan x + 3x\sec^2 x$.

EXAMPLE Find the derivative of the function $f(x) = 4\sec^2 x$.

Since $4\sec^2 x = 4(\sec x)^2$, we can use the general power rule.

$f'(x) = 4 \cdot D\left[(\sec x)^2\right]$	Use the constant multiple rule.
$f'(x) = 4 \cdot 2(\sec x)^1 \cdot \sec x\tan x$	Apply the general power rule: $D\left[u^2\right] = 2u \cdot u'$ $D[\sec x] = \sec x\tan x$
$f'(x) = 8\sec x \cdot \sec x\tan x$ $f'(x) = 8\sec^2 x\tan x$	Simplify.

Thus, $f'(x) = 8\sec^2 x\tan x$.

TRY IT

Consider the function $f(x) = 4x + x\cot x$.

Find the derivative. +

$f'(x) = 4 + \cot x - x\csc^2 x$

SUMMARY

In this lesson, you learned **the derivatives of tan*x*, sec*x*, cot*x*, and csc*x***, the four remaining trigonometric functions, followed by a **summary of all six derivatives of the trigonometric functions**. By adding these to the mix of functions you can take derivatives of using basic rules instead the limit definition, you explored how they can be used to find the **derivatives of combinations of functions involving tan*x*, sec*x*, cot*x*, and csc*x*.**

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of Cosecant

$D[\csc x] = -\csc x \cot x$

Derivative of Cotangent

$D[\cot x] = -\csc^2 x$

Derivative of Secant

$D[\sec x] = \sec x \tan x$

Derivative of Tangent

$D[\tan x] = \sec^2 x$