Composition of Functions

by Sophia

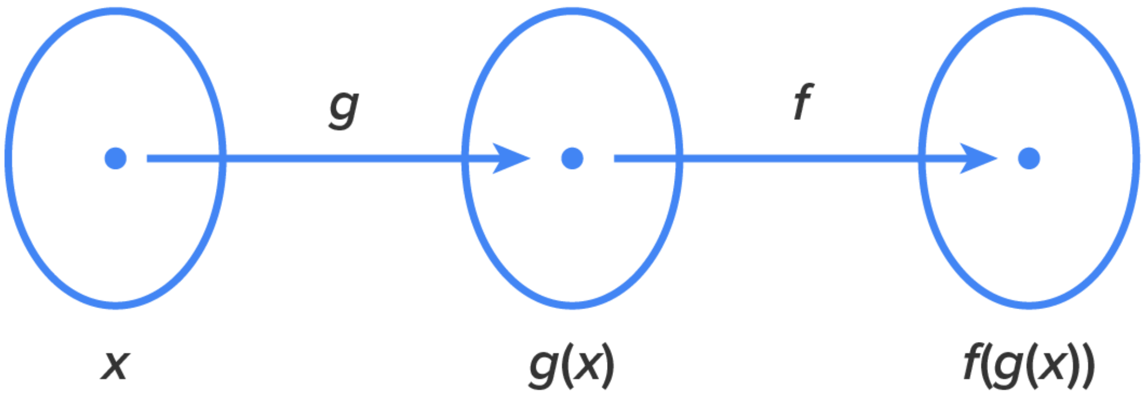
WHAT'S COVERED

In this lesson, you will find compositions of functions. Specifically, this lesson will cover:

1. The Definition of a Composition of Functions

Sometimes it is useful to use one function to get a result, then use that result in another function. This idea is called **composition of functions**.

Here is a picture to show how this works:



The original input is x, which is then substituted into $g(x)$. Then, $g(x)$ is input into function f, giving the result $f(g(x))$.

The notation used to represent a composition of functions is $(f \circ g)(x)$, which means $f(g(x))$. The expression $(f \circ g)$ means "f composed with g."

Using this notation, f is considered the outer function and g is the inner function. Notice that g is used first, then f is applied to the result. Therefore, the outer function is what is applied last. We will see how this works more closely in the next section when we evaluate compositions of functions.

TERM TO KNOW

Composition of Functions
Written $(f \circ g)(x)$, it is a function that is obtained by substituting one function into another function.

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2. Computing Compositions of Functions

2a. Computing Compositions for Specific Values

EXAMPLE Let $f(x) = 2x + 3$ and $g(x) = x^2 + 1$. Find and simplify $(f \circ g)(2)$.

$(f \circ g)(2) = f(g(2))$	Rewrite using the definition of composition.
$= f(5)$	Since $g(2)$ is the innermost expression, find that first: $g(2) = 2^2 + 1 = 5$
$= 2(5) + 3 = 13$	Evaluate $f(5)$.

TRY IT

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 1$.

Find and simplify $(g \circ f)(4)$. +

$(g \circ f)(4) = g(f(4))$	Rewrite using the definition of composition.
$= g(11)$	Since $f(4)$ is the innermost expression, find that first: $f(4) = 2(4) + 3 = 11$
$= 11^2 + 1 = 121 + 1 = 122$	Evaluate $g(11)$.

It is also possible to substitute a function into itself. If you keep the definition in mind, this follows the same format.

EXAMPLE Let $f(x) = 2x + 3$. Find and simplify $(f \circ f)(3)$.

$(f \circ f)(3) = f(f(3))$	Rewrite using the definition of composition.
$= f(9)$	Since $f(3)$ is the innermost expression, find that first: $f(3) = 2(3) + 3 = 9$
$= 2(9) + 3 = 21$	Evaluate $f(9)$.

2b. Computing the Expression for a Composition of Functions

The process for finding an expression for a composition is very similar to what we just did in the previous section, but this time there is no value to substitute first.

EXAMPLE Let $f(x) = 2x + 3$ and $g(x) = x^2 + 1$. Find and simplify $(f \circ g)(x)$.

$(f \circ g)(x) = f(g(x))$	Rewrite using the definition of composition.
$= f(x^2 + 1)$	Substitute $g(x) = x^2 + 1$.
$= 2(x^2 + 1) + 3$	Evaluate the function and simplify.
$= 2x^2 + 2 + 3$	
$= 2x^2 + 5$	

TRY IT

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 1$.

Find and simplify $(g \circ f)(x)$. +

$(g \circ f)(x) = g(f(x))$	Rewrite using the definition of composition.
$= g(2x + 3)$	Substitute $f(x) = 2x + 3$.

$= (2x+3)^2+1$ Evaluate the function and simplify.

$= (2x+3)(2x+3)+1$

$= 4x^2+12x+9+1$

$= 4x^2+12x+10$

HINT

Notice that $(f \circ g)(x)$ and $(g \circ f)(x)$ are not equal. In general, we can assume that $(f \circ g)(x) \neq (g \circ f)(x)$.

TRY IT

Consider the same function as above: $f(x) = 2x+3$.

Find and simplify $(f \circ f)(x)$.

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$(f \circ f)(x) = f(f(x))$ Rewrite using the definition of composition.

$= f(2x+3)$ Substitute $f(x) = 2x+3$.

$= 2(2x+3)+3$ Evaluate the function and simplify.

$= 4x+6+3$

$= 4x+9$

There are situations in which a composition can't be simplified.

EXAMPLE If $f(x) = \sqrt{x}$ and $g(x) = 4x+7$, find an expression for $(f \circ g)(x)$.

$(f \circ g)(x) = f(g(x))$ Rewrite using the definition of composition.

$= f(4x+7)$ Substitute $g(x) = 4x+7$.

$= \sqrt{4x+7}$ Evaluate the function.

There is no algebraic way to simplify $\sqrt{4x+7}$, so this is the final answer.

EXAMPLE If $f(x) = x^3$ and $g(x) = 2x-1$, find an expression for $(f \circ g)(x)$.

$(f \circ g)(x) = f(g(x))$ Rewrite using the definition of composition.

$= f(2x-1)$ Substitute $g(x) = 2x-1$.

$= (2x-1)^3$ Evaluate the function.

At this point, we could use multiplication to rewrite this expression, but this would be very time-consuming.

It is actually more useful to leave the answer as $(2x-1)^3$.

TRY IT

Suppose $f(x)=\sqrt[3]{x}$ and $g(x)=x^2+x+5$.

Find an expression for $(f\circ g)(x)$. +

$(f\circ g)(x)=f(g(x))$	Rewrite using the definition of composition.
$=f(x^2+x+5)$	Substitute $g(x)=x^2+x+5$.
$=\sqrt[3]{x^2+x+5}$	Evaluate the function.

3. Decomposing Composite Functions

Given a composition of functions, it is important to be able to identify the inner and outer functions.

For example, each of these functions are compositions of other functions:

$$h(x)=(3x-1)^2 \qquad j(x)=\sqrt{5x+6} \qquad m(x)=\frac{3}{(x+1)^2}$$

To decompose composite functions, identify the "inner" function first, then the "outer" function is apparent.

EXAMPLE Consider the expression $(3x+8)^2$, which is the result of a composition of functions. How can we find functions $f(x)$ and $g(x)$ so that $f(g(x))=f(3x+8)=(3x+8)^2$?

To answer this question, start with the expression inside the grouping symbols. Since $g(x)$ is the inside function, let $g(x)=3x+8$. Then, we have $f(g(x))=f(3x+8)=(3x+8)^2$.

Now, replace $3x+8$ with a symbol, say "?". We can write $f(?)=(?)^2$. As you can see, this tells us that $f(x)=x^2$.

Conclusion: Given $f(g(x))=(3x+8)^2$, $f(x)=x^2$ and $g(x)=3x+8$.

TRY IT

Suppose $f(g(x))=\sqrt[3]{x^2+4}$.

Find functions $f(x)$ and $g(x)$ to make the above composition of functions true. +

The inside function is $g(x)=x^2+4$ and the outside function is $f(x)=\sqrt[3]{x}$.

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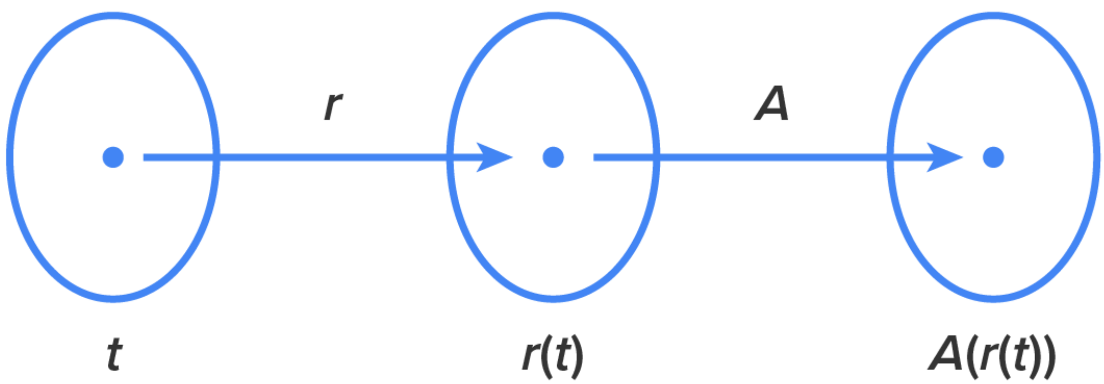
4. Applications of Compositions

When a stone is dropped into a lake, a circular ripple forms and continues to get larger until it dissipates. After t seconds, the radius (in inches) of the ripple is $r(t)=4t$. Recall also that the area of a circle with radius r is $A(r)=\pi r^2$.

EXAMPLE Suppose we want to find a function for the area inside the ripple, but as a function of time, t. Here is how the functions work together:



Therefore, the composition $(A\circ r)(t)$ will give the area enclosed by the ripple after t seconds.

$(A\circ r)(t)=A(r(t))$	Use the definition of composition.
$=A(4t)$	Replace $r(t)$ with $4t$.
$=\pi(4t)^2$	Evaluate the function.
$=16\pi t^2$	Simplify.

Notice that A is the outer function, which means that the result is an area. Notice also that using this function allows us to bypass knowing the radius in order to get the area.

EXAMPLE The radius of a circle is given by the function $r(C) = \frac{C}{2\pi}$, where C is the circumference of the circle. Recall also that the area of a circle is $A(r) = \pi r^2$. Using this information, we can find $(A \circ r)(C)$.

$A(r(C))$	Rewrite using the definition.
$= A\left(\frac{C}{2\pi}\right)$	Replace $r(C)$ with $\frac{C}{2\pi}$.
$= \pi\left(\frac{C}{2\pi}\right)^2$	Evaluate the function.
$= \pi\left(\frac{C^2}{4\pi^2}\right)$	Apply the exponent.
$= \frac{C^2}{4\pi}$	Remove the common factor of π.

This function gives the area of a circle when its circumference is known. This could be very useful since it is easier to measure the circumference of a circle than it is its radius.

SUMMARY

In this lesson, you learned **the definition of a composition of functions**, which is a function that is obtained by substituting one function into another function. Written $(f \circ g)(x)$, or $f(g(x))$, it means to substitute $g(x)$ into $f(x)$. You learned how to **compute compositions of functions**, including both **computing compositions for specific values** and **computing the expression for a composition of functions**. You learned how to **decompose composite functions** by identifying the "inner" function first, after which the "outer" function is apparent. Finally, you explored **applications of compositions** in real-world situations.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Composition of Functions
Written $(f \circ g)(x)$, it is a function that is obtained by substituting one function into another function.