The General Power Rule for Functions

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WHAT'S COVERED

In this lesson, you will expand upon your derivative knowledge even further by examining powers of functions whose derivatives we know. For example, $f(x)=(3x+1)^5$ and $y=\sin^4 x$. This idea will also help in finding the derivatives of some other commonly used functions. Specifically, this lesson will cover:

1. Derivatives of Functions of the Form $y=[f(x)]^n$

Derivatives of powers of a function have several uses, as we will see once we get to applications of derivatives. To establish a pattern for this type of derivative, we'll consider the functions $y=f^2$, $y=f^3$, and $y=f^4$, where f is being used to represent some function $f(x)$.

First, consider the function $y=f^2=f\cdot f$.

By the product rule, we have:

$$\begin{aligned} y'=D[f^2]=D[f]\cdot f+f\cdot D[f] \\ =f'\cdot f+f\cdot f' \\ =2f\cdot f' \end{aligned}$$

Now consider the function $y=f^3=f^2\cdot f$.

By the product rule again, we have:

$y'=D[f^3]=D[f^2]\cdot f+f^2\cdot D[f]$	Apply the product rule.
$=(2f\cdot f')\cdot f+f^2\cdot f'$	Replace $D[f^2]$ with $=2f\cdot f'$.

$$= 2f^2 \cdot f' + f^2 \cdot f' \quad \text{Combine } f \cdot f = f^2.$$

$$= 3f^2 \cdot f' \quad \text{Combine like terms.}$$

Next, consider $y = f^4 = f^3 \cdot f$.

$$D[f^4] = D[f^3] \cdot f + f^3 \cdot D[f] \quad \text{Apply the product rule.}$$

$$= (3f^2 \cdot f') \cdot f + f^3 \cdot f' \quad \text{Replace } D[f^3] \text{ with } = 3f^2 \cdot f'.$$

$$= 3f^3 \cdot f' + f^3 \cdot f' \quad \text{Combine } f^2 \cdot f = f^3.$$

$$= 4f^3 \cdot f' \quad \text{Combine like terms.}$$

By looking at this pattern, it seems as though the derivative of f^n is $n \cdot f^{n-1}$ (looks like the power rule), but then also multiplied by f'.

FORMULA TO KNOW

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D[[f(x)]^n] = n \cdot [f(x)]^{n-1} \cdot f'(x)$.

EXAMPLE Earlier, we found the derivative of $f(x) = \cos^2 x$ by using the product rule. Let's use the power rule and compare.

First, note that this can be written as $f(x) = (\cos x)^2$.

By the power rule, we have the following:

$$f'(x) = 2(\cos x) \cdot D[\cos x] \quad \text{Apply the power rule.}$$

$$= 2(\cos x)(-\sin x) \quad D[\cos x] = -\sin x$$

$$= -2\sin x \cos x \quad \text{Combine and eliminate parentheses.}$$

This matches the answer obtained in challenge 3.2.4.

EXAMPLE Find the derivative of the function $f(x) = (5x+1)^{10}$.

By the power rule, we have the following:

$$f'(x) = 10(5x+1)^9 \cdot D[5x+1] \quad \text{Apply the power rule.}$$

$$= 10(5x+1)^9(5) \quad D[5x+1] = 5$$

$= 50(5x+1)^9$ Combine $10 \cdot 5$.

A common mistake to make here is to multiply $50(5x+1)$ to get $250x + 50$, and subsequently $(250x+50)^9$. This is not correct since the $(5x+1)$ is raised to the 9th power and the 50 is not; therefore, they cannot be combined this way. The final answer is $f'(x) = 50(5x+1)^9$.

TRY IT

Consider the function $y = (x^2 - 9x + 20)^4$.

Find the derivative. +

$$\frac{dy}{dx} = 4(2x-9)(x^2-9x+20)^3$$

Remember the other expressions that can be written as powers of *x*.

EXAMPLE Find the derivative of the function $f(x) = \sqrt{3x^2+8}$.

Remember that $\sqrt{u} = u^{1/2}$. Then the power rule can be used.

$f(x) = \sqrt{3x^2+8} = (3x^2+8)^{1/2}$	Rewrite the radical using a power.
$f'(x) = \frac{1}{2}(3x^2+8)^{-1/2} \cdot D[3x^2+8]$	Use the power rule for derivatives.
$f'(x) = \frac{1}{2}(3x^2+8)^{-1/2} \cdot 6x$	$D[3x^2+8] = 6x$
$f'(x) = 3x(3x^2+8)^{-1/2}$	$\frac{1}{2} \cdot 6x = 3x$
$f'(x) = \frac{3x}{(3x^2+8)^{1/2}}$	Rewrite with nonnegative exponents.

Thus, $f'(x) = \frac{3x}{(3x^2+8)^{1/2}}$, which could also be written $f'(x) = \frac{3x}{\sqrt{3x^2+8}}$ if radical notation is desired.

EXAMPLE Find the derivative of the function $f(x) = \frac{1}{(5x+\cos x)^3}$.

$f(x) = \frac{1}{(5x+\cos x)^3} = (5x+\cos x)^{-3}$	Rewrite so that the power rule can be used.
$f'(x) = -3(5x+\cos x)^{-4} \cdot D[5x+\cos x]$	Apply the power rule.

$f'(x) = -3(5x+\cos x)^{-4} \cdot (5-\sin x)$ — $D[5x+\cos x] = 5+(-\sin x) = 5-\sin x$

$f'(x) = -3(5-\sin x)(5x+\cos x)^{-4}$ — Rearrange the factors.

$f'(x) = \dfrac{-3(5-\sin x)}{(5x+\cos x)^4}$ — Rewrite with nonnegative exponents.

Thus, $f'(x) = \dfrac{-3(5-\sin x)}{(5x+\cos x)^4}$.

TRY IT

Consider the function $g(x) = \sqrt[3]{6x^4+5}$.

Find the derivative. +

$$g'(x) = \frac{8x^3}{(6x^4+5)^{2/3}}$$

EXAMPLE The distance (measured in feet) from a moving camera to an object positioned at the point (1, 4) is given by the function $f(t) = \sqrt{2t^2-2t+1}$, where t is measured in seconds. At what rate is the distance changing after 3 seconds?

Mathematically speaking, we want to compute $f'(3)$.

To find the derivative, we first need to rewrite $f(t)$:

$f(t) = \sqrt{2t^2-2t+1} = (2t^2-2t+1)^{1/2}$ — Write the radical as $\frac{1}{2}$ power.

$f'(t) = \frac{1}{2}(2t^2-2t+1)^{-1/2} \cdot D[2t^2-2t+1]$ — Apply the power rule.

$f'(t) = \frac{1}{2}(2t^2-2t+1)^{-1/2} \cdot (4t-2)$ — $D[2t^2-2t+1] = 4t-2$

$f'(t) = \frac{1}{2}(4t-2) \cdot (2t^2-2t+1)^{-1/2}$ — Rearrange the terms.

$f'(t) = (2t-1) \cdot (2t^2-2t+1)^{-1/2}$ — Distribute $\frac{1}{2}(4t-2) = 2t-1$

$f'(t) = \dfrac{2t-1}{(2t^2-2t+1)^{1/2}}$ — Rewrite with nonnegative exponents.

Now, we desire the rate of change when $t=3$, so we substitute 3.

$$f'(3)=\frac{2(3)-1}{(2(3)^2-2(3)+1)^{1/2}}=\frac{5}{(13)^{1/2}}\approx 1.39 \text{ feet per second}$$

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2. Combining Derivative Rules

Now that we are building up our derivative rules, we can find derivatives of more complex functions.

EXAMPLE Find the derivative of the function $f(x)=4x\sqrt{2x+1}$.

At this point, we are conditioned to write radicals as fractional powers (to use the power rule).

$f(x)=4x\sqrt{2x+1}=4x(2x+1)^{1/2}$	Rewrite the square root as $\frac{1}{2}$ power.
$f'(x)=D[4x]\cdot(2x+1)^{1/2}+4x\cdot D\left[(2x+1)^{1/2}\right]$	Apply the product rule.
$f'(x)=4\cdot(2x+1)^{1/2}+4x\cdot\frac{1}{2}(2x+1)^{-1/2}(2)$	$D[4x]=4$, $D\left[(2x+1)^{1/2}\right]=\frac{1}{2}(2x+1)^{-1/2}(2)$
$f'(x)=4(2x+1)^{1/2}+4x(2x+1)^{-1/2}$	$\frac{1}{2}\cdot 2=1$; remove excess symbols.
$f'(x)=4(2x+1)^{1/2}+\frac{4x}{(2x+1)^{1/2}}$	Rewrite with positive exponents.

At this point, $f'(x)$ is reasonably simplified. Thus, $f'(x)=4(2x+1)^{1/2}+\frac{4x}{(2x+1)^{1/2}}$.

It is possible to go further by forming a common denominator and combining the fractions. Let's see how this plays out:

$f'(x)=\frac{4(2x+1)^{1/2}}{1}\cdot\frac{(2x+1)^{1/2}}{(2x+1)^{1/2}}+\frac{4x}{(2x+1)^{1/2}}$	The common denominator is $(2x+1)^{1/2}$. Write $4(2x+1)^{1/2}$ over 1 so it "looks" like a fraction.
$f'(x)=\frac{4(2x+1)}{(2x+1)^{1/2}}+\frac{4x}{(2x+1)^{1/2}}$	Perform multiplication. $(2x+1)^{1/2}\cdot(2x+1)^{1/2}=(2x+1)^1=2x+1$
$f'(x)=\frac{8x+4}{(2x+1)^{1/2}}+\frac{4x}{(2x+1)^{1/2}}$	Distribute $4(2x+1)=8x+4$.
$f'(x)=\frac{12x+4}{(2x+1)^{1/2}}$	Combine the numerators.

As you can see, the expression simplified nicely to one single fraction. That said, writing

$f'(x)=4(2x+1)^{1/2}+\dfrac{4x}{(2x+1)^{1/2}}$ is equally acceptable.

WATCH

Sometimes factoring is very useful in obtaining a nicer form of the derivative. In the following video, we'll take the derivative of $f(x)=(4x-1)^3(2x+5)^4$ and write it in factored form.

TRY IT

Consider the function $f(x)=(x+1)^4(2x+1)^3$.

Find the derivative and write your final answer in factored form. +

$f'(x)=2(x+1)^3(2x+1)^2(7x+5)$

SUMMARY

In this lesson, you learned how to apply the general power rule for **derivatives of functions, such as the form $y=[f(x)]^n$**. As you develop your repertoire of derivative formulas, you are able to **combine derivative rules** to find derivatives of more complex functions, such as the ones explored in this unit.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

General Power Rule for Derivatives of Functions

If $f(x)$ is some function, then $D\left[[f(x)]^n\right]=n\cdot[f(x)]^{n-1}\cdot f'(x)$.