f" and Extreme Values of f

by Sophia

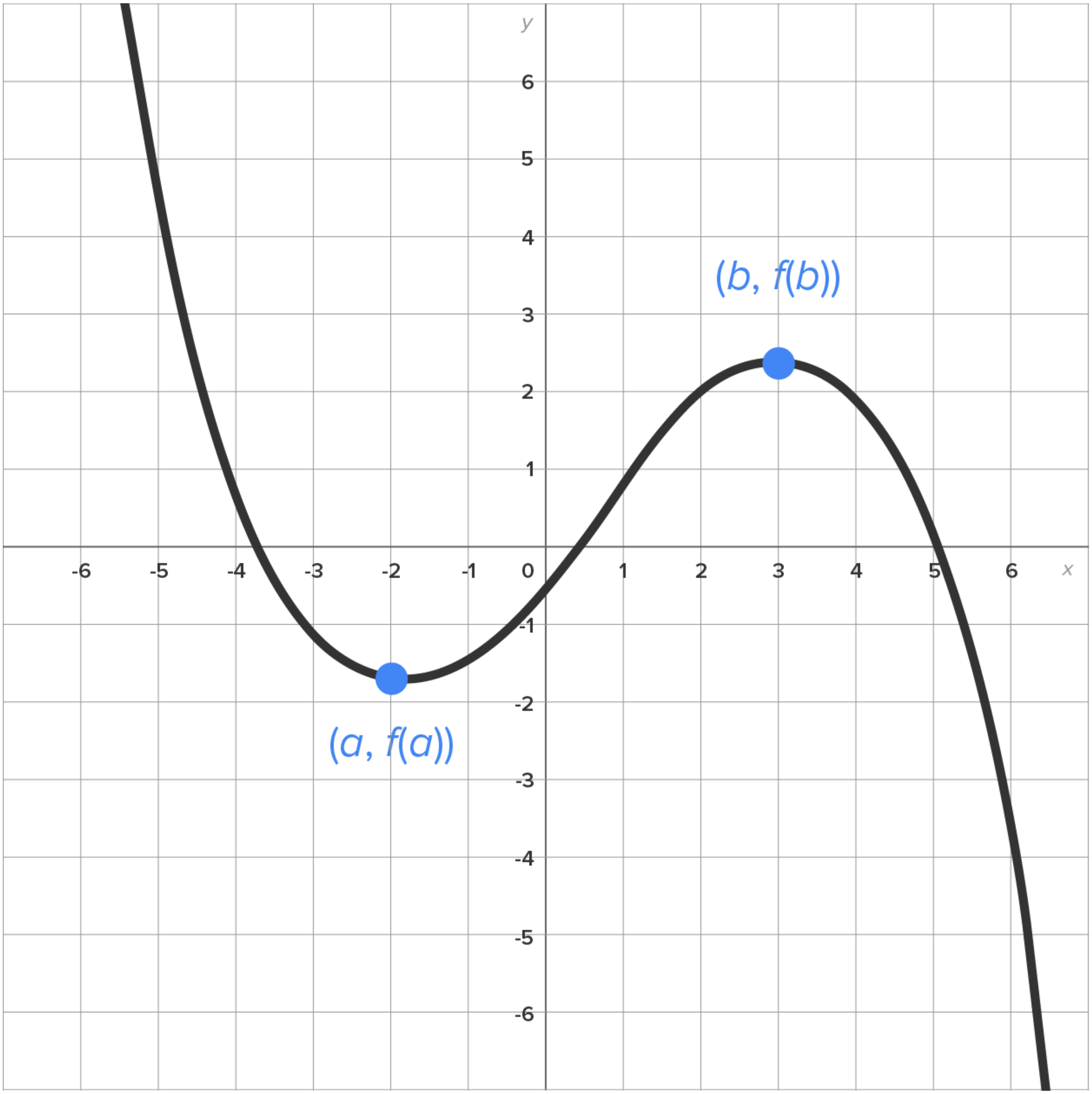
WHAT'S COVERED

In this lesson, you will use the second derivative to determine if a maximum or minimum occurs at a critical number. Specifically, this lesson will cover:

1. The Second Derivative Test

When a function has several critical numbers, and when the second derivative is relatively easy to get, the second derivative test is much more efficient to use than the first derivative test when locating maximum and minimum values. However, there are conditions under which the second derivative doesn't give enough information, and we need to use information from the first derivative to determine maximum and minimum points of a function.

Consider the graph shown below:



Observe the following at the local extreme points:

The point $(a, f(a))$ is a local minimum.

- Since there is a horizontal tangent at $x = a$, we know $f'(a) = 0$.
- Since the graph is concave up at $x = a$, we know that $f''(a) > 0$.

The point $(b, f(b))$ is a local maximum.

- Since there is a horizontal tangent at $x = b$, we know $f'(b) = 0$.
- Since the graph is concave down at $x = b$, we know that $f''(b) < 0$.

Based on this graph, there is a connection between the concavity of $f(x)$ when there is a horizontal tangent line and the type of local extrema at that point. Formally stated, this is called the **second derivative test**.

HINT

Only critical numbers where $f'(c)=0$ are considered for the second derivative test. If $f'(c)$ is undefined, then $f''(c)$ is also undefined, which means we cannot determine if $f''(c)$ is positive, negative, or zero.

That said, if $f(x)$ has critical numbers where $f'(c)$ is undefined, then the first derivative test will need to be used to determine if any local extrema occur at $x=c$.

TERM TO KNOW

Second Derivative Test

Suppose $f'(c)=0$, which means $f(x)$ has a horizontal tangent at $x=c$.

- If $f''(c)<0$, this means $f(x)$ is concave down around *c*, which means there is a local maximum at *c*.
- If $f''(c)>0$, this means $f(x)$ is concave up around *c*, which means there is a local minimum at *c*.
- If $f''(c)=0$, the test is inconclusive, and the first derivative test needs to be used to determine the behavior at *c*.

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2. Locating Maximum/Minimum With the Second Derivative Test

Let's look at a few examples of how the second derivative test is implemented.

EXAMPLE Determine the local maximum and minimum values of $f(x)=-2x^3+6x^2+15$.

First, find the values of *c* for which $f'(c)=0$.

$f(x)=-2x^3+6x^2+15$	Start with the original function; the domain is all real numbers.
$f'(x)=-6x^2+12x$	Take the derivative.
$-6x^2+12x=0$ $-6x(x-2)=0$ $-6x=0,\ x-2=0$ $x=0,\ x=2$	Set $f'(x)=0$ and solve.

The critical numbers are $x = 0$ and $x = 2$.

Now, take the second derivative using $f'(x) = -6x^2 + 12x$ and substitute $x = 0$ and $x = 2$.

$f'(x) = -6x^2 + 12x$	Take the first derivative.
$f''(x) = -12x + 12$	Take the second derivative.
$f''(0) = -12(0) + 12 = 12$	Substitute the critical number, $x = 0$. Since $f''(0)$ is positive, $f(0)$ is a local minimum.
$f''(2) = -12(2) + 12 = -12$	Substitute the critical number, $x = 2$. Since $f''(2)$ is negative, $f(2)$ is a local maximum.

Therefore, the local minimum value is $f(0) = 15$ and the local maximum value is $f(2) = 23$. On the graph, the local minimum is located at (0, 15) and the local maximum is located at (2, 23).

Let's look at another example.

➦ EXAMPLE Determine the local maximum and minimum values of $f(x) = 5x + \frac{20}{x}$.

First, find the values of *c* for which $f'(c) = 0$.

$f(x) = 5x + \frac{20}{x} = 5x + 20x^{-1}$	Start with the original function; rewrite to use the power rule. The domain is $(-\infty, 0) \cup (0, \infty)$.
$f'(x) = 5 - 20x^{-2}$	Take the derivative.
$5 - 20x^{-2} = 0$	Set $f'(x) = 0$ and solve.
$5 - \frac{20}{x^2} = 0$	
$5x^2 - 20 = 0$	
$5(x^2 - 4) = 0$	
$5(x+2)(x-2) = 0$	
$x = 2, x = -2$	

The critical numbers are $x = -2$ and $x = 2$.

Now, take the second derivative using $f'(x) = 5 - 20x^{-2}$, then substitute $x = -2$ and $x = 2$.

$f'(x) = 5 - 20x^{-2}$	Take the first derivative.

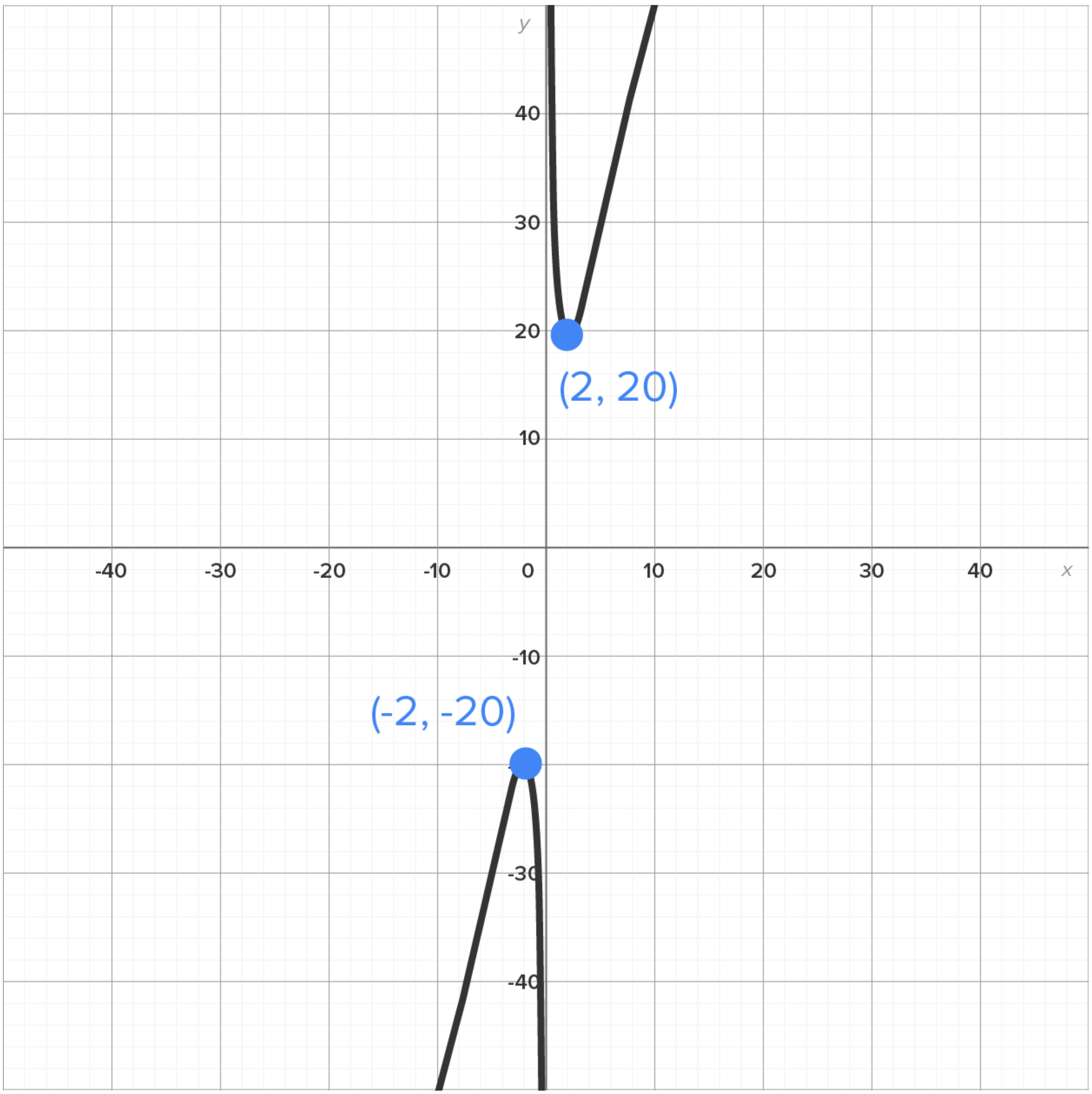
$f''(x) = 40x^{-3} = \dfrac{40}{x^3}$ Take the second derivative.

$f''(-2) = \dfrac{40}{(-2)^3} = -5$ Substitute the critical number, $x = -2$. Since $f''(-2)$ is negative, $f(-2)$ is a local maximum.

$f''(2) = \dfrac{40}{2^3} = 5$ Substitute the critical number, $x = 2$. Since $f''(2)$ is positive, $f(2)$ is a local minimum.

Therefore, the local maximum value is $f(-2) = -20$ and the local minimum value is $f(2) = 20$. On the graph, the local maximum is located at (-2, -20) and the local minimum is located at (2, 20).

Here is a graph of the function:



Let's now look at an example where the second derivative test cannot be used.

EXAMPLE Determine the local maximum and minimum values of $f(x)=x^3-6x^2+12x+10$.

First, find the values of *c* for which $f'(c)=0$.

$f(x)=x^3-6x^2+12x+10$	Start with the original function; the domain is all real numbers.
$f'(x)=3x^2-12x+12$	Take the derivative.
$3x^2-12x+12=0$	Set $f'(x)=0$ and solve.
$3(x^2-4x+4)=0$	
$3(x-2)(x-2)=0$	
$x=2$	

There is only one critical number, $x=2$.

Now, take the second derivative using $f'(x)=3x^2-12x+12$, then substitute $x=2$.

$f'(x)=3x^2-12x+12$	Take the first derivative.
$f''(x)=6x-12$	Take the second derivative.
$f''(2)=6(2)-12=0$	Substitute the critical number, $x=2$.

This means that the second derivative cannot be used to determine if $f(x)$ attains a local maximum or minimum at $x=2$.

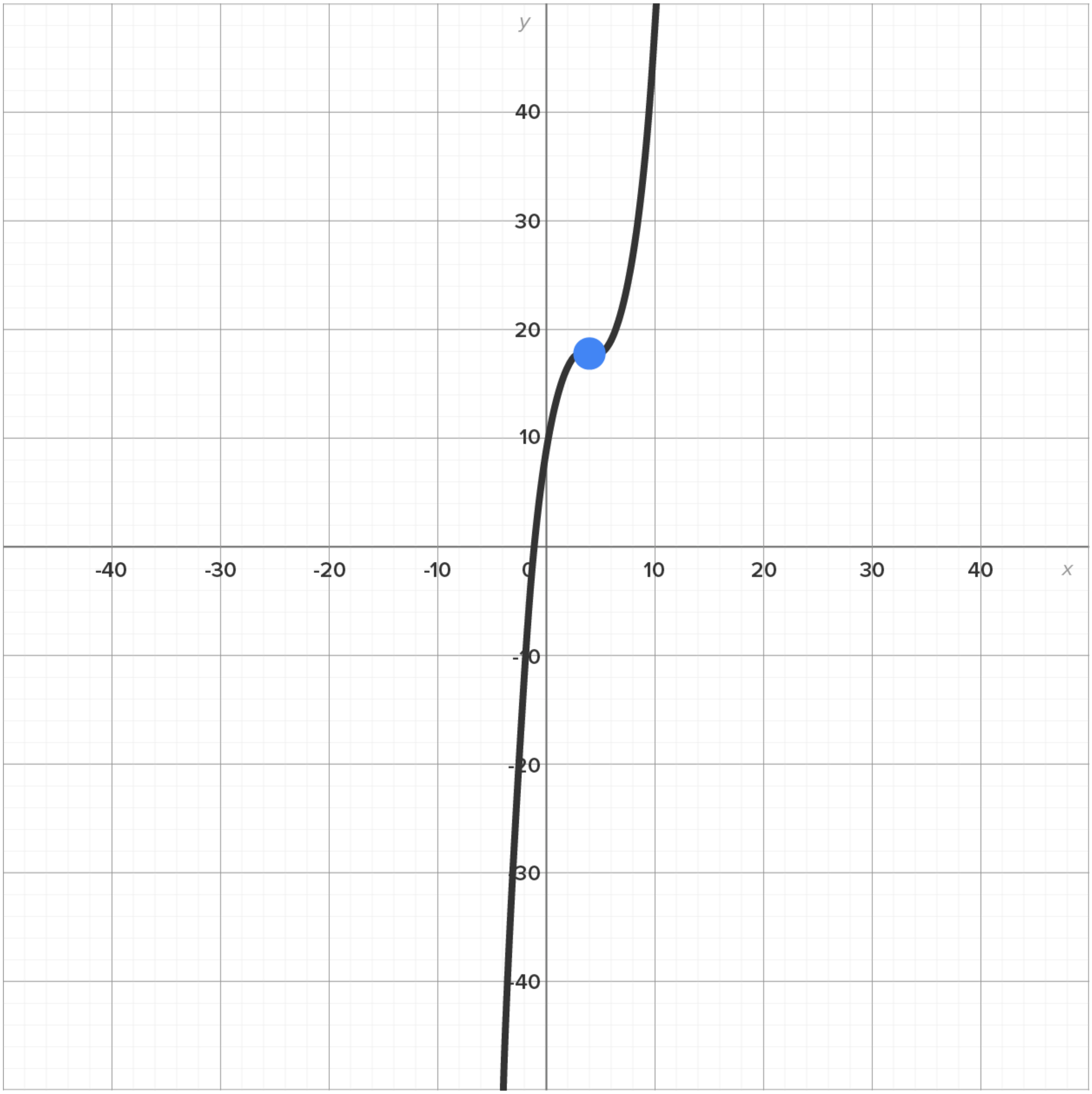
To determine the behavior of $f(x)$ at $x=2$, we now turn back to the first derivative, $f'(x)=3x^2-12x+12$.

Evaluate on either side by finding $f'(1)$ and $f'(3)$:

- $f'(1)=3(1)^2-12(1)+12=3$ (increasing)
- $f'(3)=3(3)^2-12(3)+12=3$ (increasing)

Since $f(x)$ is increasing on both sides of $x=2$, this means that there is no minimum or maximum when $x=2$.

The graph of $f(x)$ is shown in the figure:



WATCH

In this video, we'll use the second derivative test to locate local maximum and minimum values of the function $f(x) = xe^{-0.5x}$.

SUMMARY

In this lesson, you learned that under certain conditions, the second derivative can be used to determine if $f(x)$ has a local maximum or minimum at a critical value c for which $f'(c) = 0$, known as **the second derivative test**. Next, you explored several examples **locating the maximum and minimum**

values with the second derivative test; however, when those conditions aren't met, the first derivative test is used to determine the locations of local maximum and minimum values.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Second Derivative Test

Suppose $f'(c) = 0$, which means $f(x)$ has a horizontal tangent at $x = c$.

• If $f''(c) < 0$, this means $f(x)$ is concave down around *c*, which means there is a local maximum at *c*.

• If $f''(c) > 0$, this means $f(x)$ is concave up around *c*, which means there is a local minimum at *c*.

• If $f''(c) = 0$, the test is inconclusive, and the first derivative test needs to be used to determine the behavior at *c*.