# Differentiability

*by Sophia*

WHAT'S COVERED

In this lesson, you will investigate the differentiability of a function by using analytical techniques, which include a determination of continuity. Specifically, this lesson will cover:



## 1. Defining Differentiability

Differentiability is an important concept in calculus since it pertains to the "smoothness" of a curve. A function $y = f(x)$ is said to be **differentiable** at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

TERM TO KNOW

**Differentiable**
A function $y = f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

## 2. Determining Differentiability at $x = a$ Analytically

The following statements are equivalent:

- If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.
- If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

How to interpret these statements:

- If $f(x)$ is not continuous at $x = a$, then it is never differentiable at $x = a$.
- If $f(x)$ is differentiable at $x = a$, then it is always continuous at $x = a$.

Note: This means that if $f(x)$ is continuous at $x = a$, $f(x)$ may or may not be differentiable at $x = a$.

HINT

Recall that the definition of continuity of $f(x)$ at $x = a$ is $\lim_{x \to a} f(x) = f(a)$.

## 2a. Continuous but Not Differentiable

Here is an example of a function that is continuous but not differentiable at a point.

EXAMPLE Determine if $f(x) = \sqrt[3]{x}$ is differentiable at $x = 0$. First, check for continuity at $x = 0$:

$f(0) = \sqrt[3]{0} = 0$

$\lim_{x \to 0} \sqrt[3]{x} = 0$

Since $\sqrt[3]{x}$ is defined for positive and negative real numbers, there is no need to use one-sided limits. Since the limit and $f(0)$ are equal, $f(x)$ is continuous at $x = 0$.

Now, let's check the derivative. Note that $f(x) = \sqrt[3]{x} = x^{1/3}$. Then, $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$.

Since $f'(0)$ is undefined (0 in the denominator), $f(x)$ is not differentiable at $x = 0$.

Note: In this case, this means that the slope of the tangent line is undefined, and that the tangent line is vertical.

## 2b. Differentiable for All Real Numbers

Here is an example of a function that is differentiable for all real numbers.

EXAMPLE Show that $f(x) = x^3$ is differentiable for all real numbers.

Check continuity: Since $f(x)$ is a polynomial function, it is continuous for all real numbers (this was established in Challenge 2.3).

Check the derivative: $f'(x) = 3x^2$, which is defined for all real numbers.

Thus, $f(x) = x^3$ is differentiable for all real numbers.

## 2c. Not Continuous

Here is an example of a function that is not continuous at a point, which means that it is also not differentiable at the point.

EXAMPLE Consider the function $f(x) = \frac{2}{x-1}$.

Since $f(x)$ is not continuous at $x = 1$, it is also not differentiable at $x = 1$.

TRY IT

Consider the following functions and x-values.

| Function | Given x-value | Differentiable (Yes or No)? |
|---|---|---|
| $f(x) = \cos x$ | $x = 0$ | ? |
| $g(x) = \sqrt{x}$ | $x = 4$ | ? |
| $h(x) = \frac{x}{2x-1}$ | $x = \frac{1}{2}$ | ? |

**Determine if each function is differentiable at the given x-value in the table.** +

| Function | Given x-value | Differentiable (Yes or No)? |
|---|---|---|
| $f(x) = \cos x$ | $x = 0$ | Yes |
| $g(x) = \sqrt{x}$ | $x = 4$ | Yes |
| $h(x) = \frac{x}{2x-1}$ | $x = \frac{1}{2}$ | No (not continuous, therefore not differentiable at $x = \frac{1}{2}$) |

SUMMARY

In this lesson, you explored the first of two ways to **define differentiability**, noting that if a function is to be differentiable at $x = a$, it must be continuous at $x = a$ and $f'(a)$ needs to be defined. You learned how to **determine differentiability at** $x = a$ **analytically**, exploring examples of functions that are **continuous but not differentiable**, **differentiable for all real numbers**, and also **not continuous** and therefore not differentiable at the points of discontinuity.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

## TERMS TO KNOW

**Differentiable**

A function $y = f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.