Limits As x Becomes Arbitrarily Large ("Approaches Infinity")

by Sophia

WHAT'S COVERED

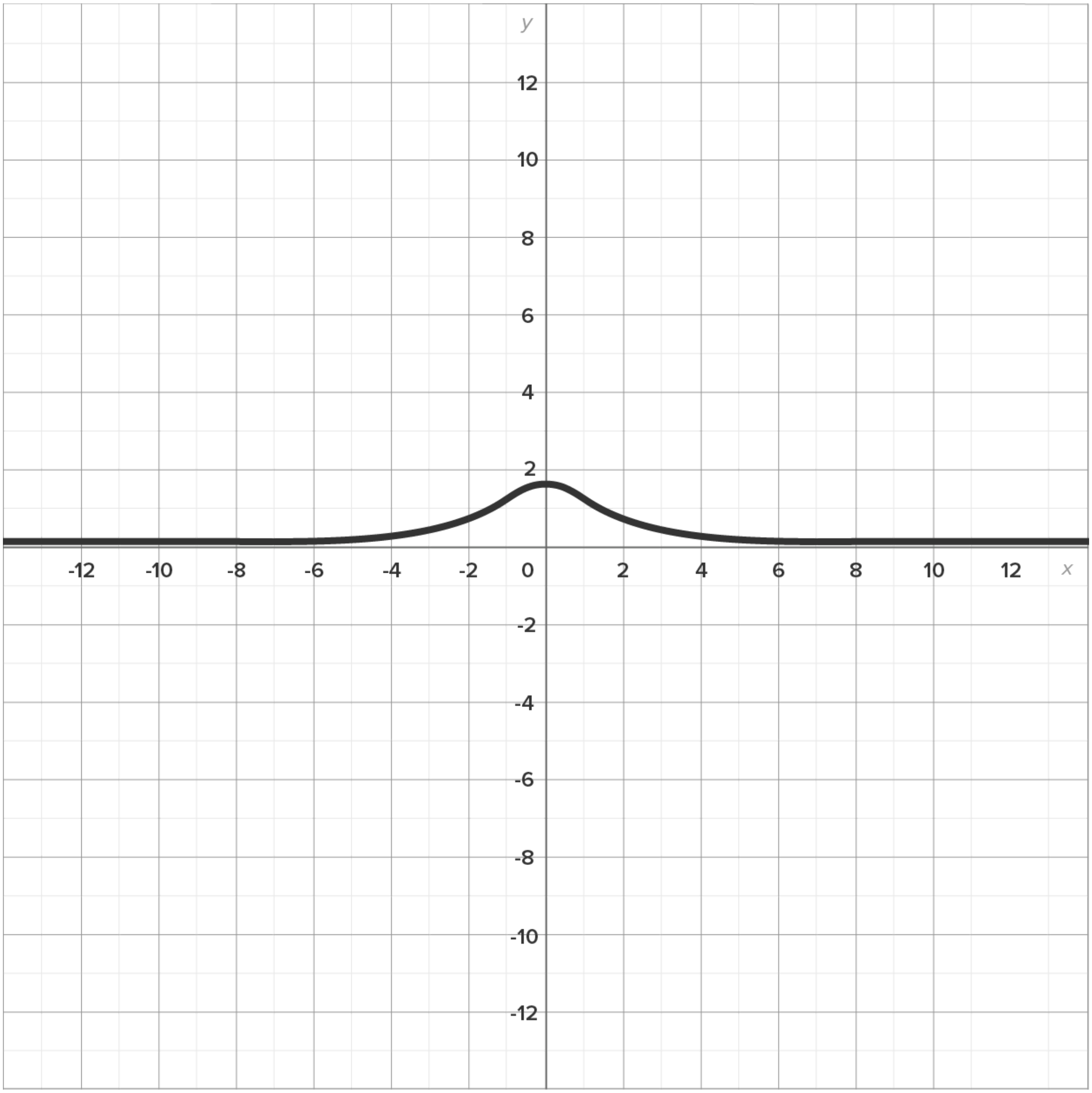
In this lesson, you will investigate the behavior of a function as *x* gets arbitrarily large. Specifically, this lesson will cover:

1. Graphically Finding Limits As $x \to \infty$ and As $x \to -\infty$

The notation $\lim_{x \to \infty} f(x)$ is used when we want to know what the value of $f(x)$ is approaching as *x* continues to increase, which we sometimes say increases without bound. The notation $\lim_{x \to -\infty} f(x)$ is used when we want to know what the value of $f(x)$ is approaching as *x* continues to decrease, which we sometimes say decreases without bound. In this tutorial, we will see how to find limits of the form $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ graphically, numerically, and algebraically.

Let's look at finding this graphically first.

EXAMPLE Consider the graph of $f(x) = \frac{4}{x^2+3}$ shown below:



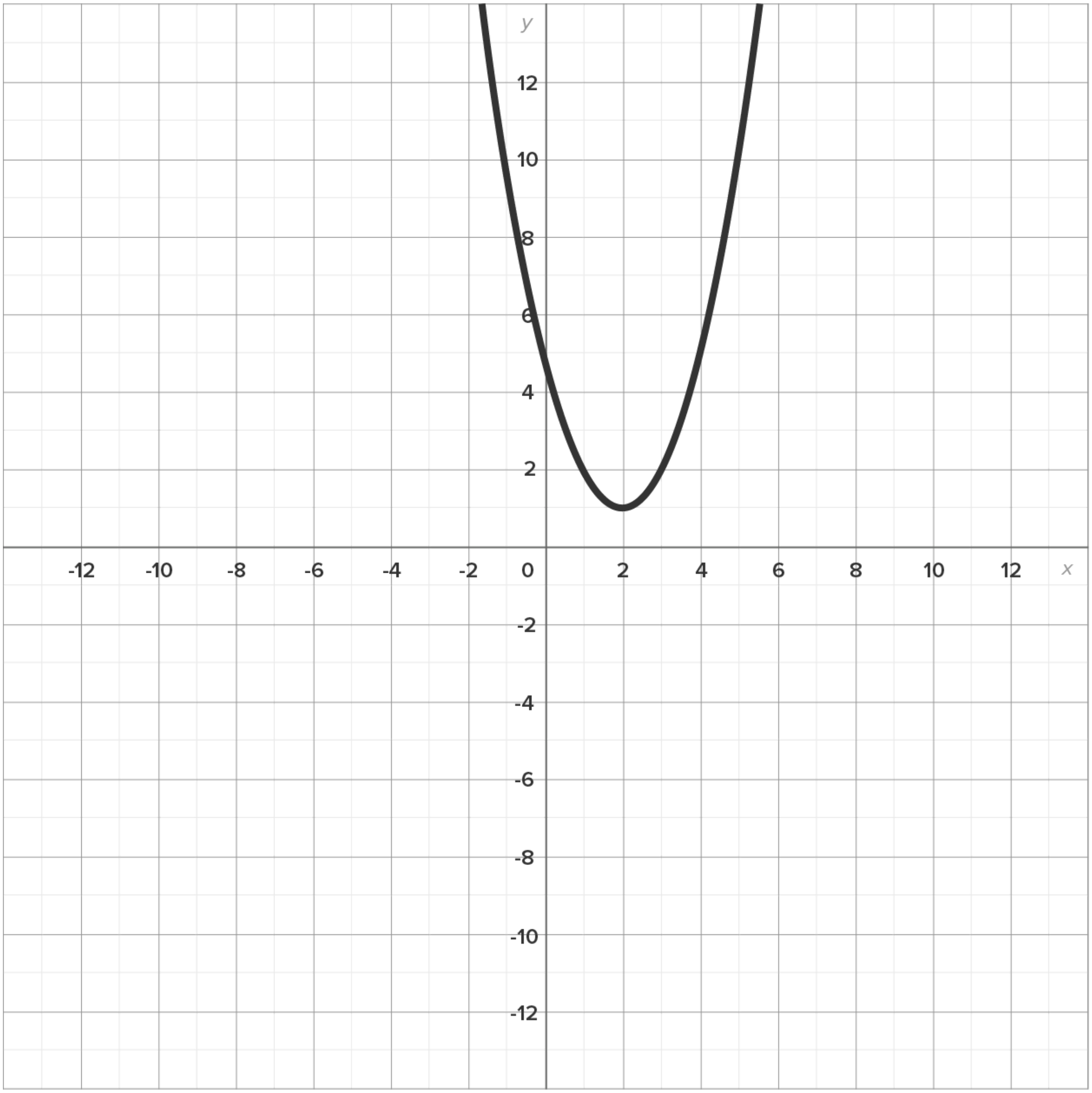
Notice that as x gets larger (as you move to the extreme right on the graph), the value of $f(x)$ appears to get closer to 0.

Notice also that as x gets smaller (as you move to the extreme left on the graph), the value of $f(x)$ appears to get closer to 0 as well.

Thus, we write $\lim_{x \to \infty} \frac{4}{x^2+3} = 0$ and $\lim_{x \to -\infty} \frac{4}{x^2+3} = 0$.

Let's now look at the limit of a quadratic function.

EXAMPLE Consider the graph of $f(x) = x^2 - 4x + 5$.



Notice that as x gets larger (as you move to the extreme right on the graph), the value of $f(x)$ appears to increase indefinitely.

Notice also that as x gets smaller (as you move to the extreme left on the graph), the value of $f(x)$ appears to increase indefinitely.

Thus, we could say $\lim_{x \to \infty} (x^2 - 4x + 5) = \infty$ and $\lim_{x \to -\infty} (x^2 - 4x + 5) = \infty$.

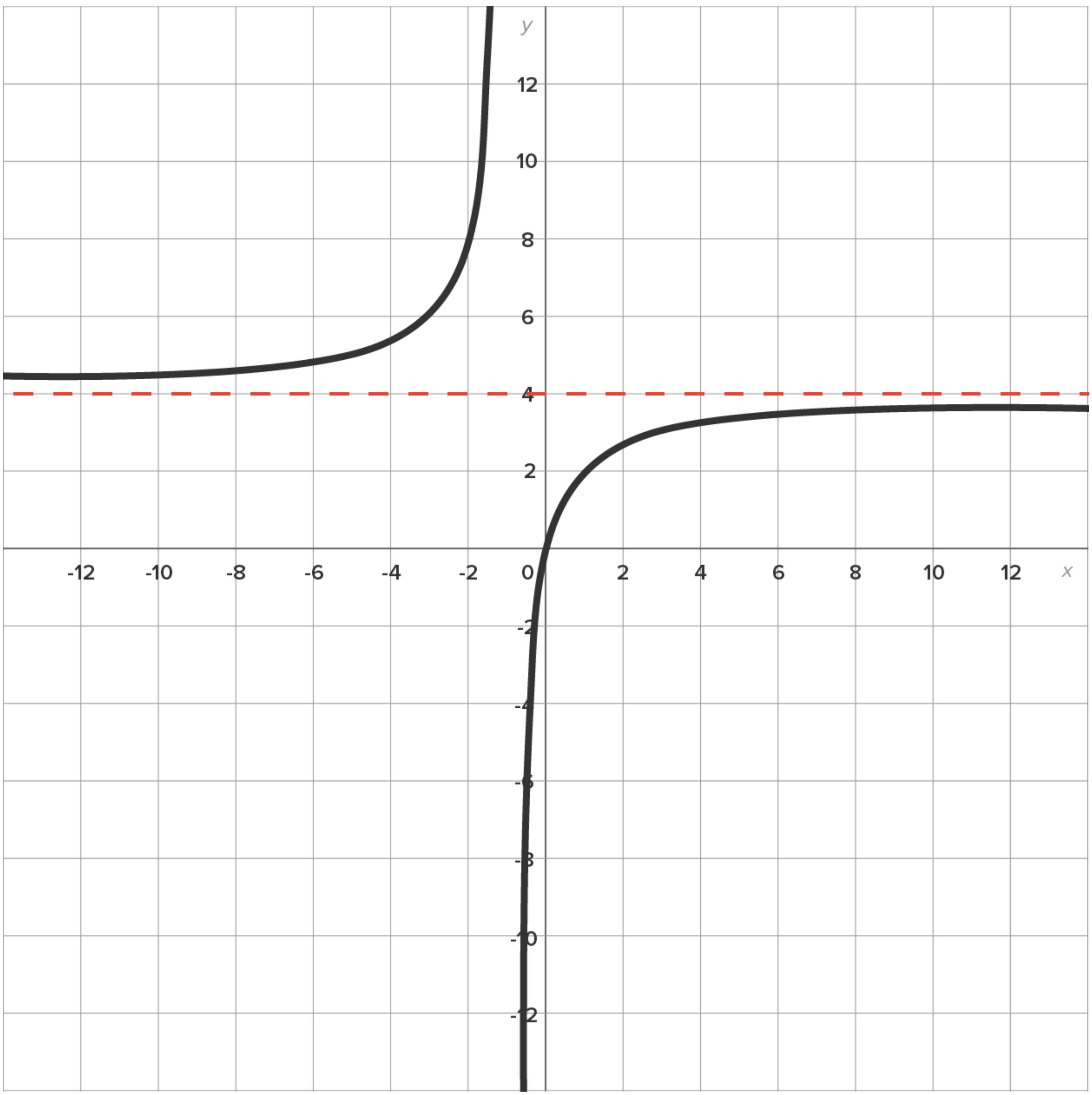
HINT

Stating that a limit is equal to infinity is contradictory since there actually is no limit.

There are cases where knowing the limit is "∞" or "$-\infty$" is useful, but when this happens, we can also say that the “limit does not exist” or “there is no limit.”

For the purposes of this tutorial and this course, we will say "∞" and "$-\infty$" are acceptable answers. Now let’s look at a graph in which the limit is not as apparent.

EXAMPLE Consider the graph of $f(x)=\frac{4x}{x+1}$ shown below. The graph of $y=4$ (dashed line) is also shown for reference:



Notice that as x gets larger (as you move to the extreme right on the graph), the value of $f(x)$ appears to get closer to 4.

Notice also that as x gets smaller (as you move to the extreme left on the graph), the value of $f(x)$ appears to get closer to 4 as well.

Thus, we write $\lim_{x \to \infty} \frac{4x}{x+1} = 4$ and $\lim_{x \to -\infty} \frac{4x}{x+1} = 4$.

You may note the use of the words “appears to get closer to” in the last two examples.

For instance, how do we really know that the limit in the last example is 4 and not 3.9 or something else “close to 4”?

When using a graph, we can only really estimate limits. This is why other techniques are used.

---

2. Numerically Finding Limits As $x \to \infty$ and As $x \to -\infty$

Recall that to find a limit numerically means to use a table of values to determine a pattern in the values of $f(x)$ in order to evaluate the limit.

EXAMPLE Consider the function $f(x) = \frac{4}{x^2+3}$.

When evaluating $\lim_{x \to \infty} \frac{4}{x^2+3}$, we want to choose x-values that get progressively larger. While there is no set rule for choosing x-values, consider using powers of 10:

x	10	100	1,000	10,000
$f(x) = \frac{4}{x^2+3}$	0.03883495	0.00039988	0.00000400	0.00000004

By looking at the table, we can conclude that $\lim_{x \to \infty} \frac{4}{x^2+3} = 0$.

Let’s look at what happens when the limit itself is infinite.

EXAMPLE Consider the function $f(x) = x^2 - 4x + 5$.

We’ll use the same tables to evaluate $\lim_{x \to \infty} (x^2 - 4x + 5)$.

x	10	100	1,000	10,000
$f(x) = x^2 - 4x + 5$	65	9,605	996,005	99,960,005

As we can see, the values of $f(x)$ are increasing rather quickly as x gets larger. We can conclude that $\lim_{x \to \infty} (x^2 - 4x + 5) = \infty$.

To find $\lim_{x \to -\infty} (x^2 - 4x + 5)$, we use a similar table:

x	-10	-100	-1,000	-10,000
$f(x) = x^2 - 4x + 5$	145	10,405	1,004,005	100,040,005

Once again, the values of $f(x)$ are convincingly increasing rather quickly as x gets smaller (more negative). We can conclude that $\lim_{x \to -\infty} (x^2 - 4x + 5) = \infty$.

TRY IT

Earlier, we found $\lim_{x \to \infty} \frac{4}{x^2 + 3}$ and concluded that the limit is equal to 0. Complete the following table and evaluate $\lim_{x \to -\infty} \frac{4}{x^2 + 3}$. (Notice that this is for negative infinity.)

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4}{x^2 + 3}$				

Round each value to 8 decimal places. +

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4}{x^2 + 3}$	0.03883495	0.00039988	0.00000400	0.00000004

What can we conclude about the limit? +

We can conclude that $\lim_{x \to -\infty} \frac{4}{x^2 + 3} = 0$.

Here is another for you to try.

TRY IT

Consider the function $f(x) = \frac{4x}{x+1}$.

Complete each table in order to evaluate $\lim_{x \to \infty} \frac{4x}{x+1}$ and $\lim_{x \to -\infty} \frac{4x}{x+1}$.

x	10	100	1,000	10,000
$f(x) = \frac{4x}{x+1}$				

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4x}{x+1}$				

Round each value to 6 decimal places. +

x	10	100	1,000	10,000
$f(x) = \frac{4x}{x+1}$	3.636364	3.960396	3.996004	3.999600

x	-10	-100	-1,000	-10,000
$f(x) = \frac{4x}{x+1}$	4.444444	4.040404	4.004004	4.000400

What can we conclude about the limit? +

We can conclude that $\lim_{x \to \infty} \frac{4x}{x+1} = 4$ and $\lim_{x \to -\infty} \frac{4x}{x+1} = 4$.

3. Analytically Finding Limits As $x \to \infty$ and As $x \to -\infty$

Evaluating limits numerically may be more convincing than using a graph, but it is still not as precise as using algebraic facts.

To get us started, here are a few properties of limits, if c is a constant and n is positive:

Limit as $x \to \infty$	Limit as $x \to -\infty$	Explanation
$\lim_{x \to \infty} \frac{c}{x^n} = 0$	$\lim_{x \to -\infty} \frac{c}{x^n} = 0$	As $x \to \pm\infty$, the denominator grows, making the overall value smaller and closer to 0.
$\lim_{x \to \infty} c = c$	$\lim_{x \to -\infty} c = c$	The limit of a constant is a constant.
$\lim_{x \to \infty} x = \infty$	$\lim_{x \to -\infty} x = -\infty$	As x itself increases or decreases without bound, the limit of x also does.
$\lim_{x \to \infty} x^n = \infty$	$\lim_{x \to -\infty} x^n = \infty$ if n is even $\lim_{x \to -\infty} x^n = -\infty$ if n is odd	A negative number raised to an even power is positive; a negative number raised to an odd power is negative.
If $\lim_{x \to \infty} f(x) = \pm\infty$, then $\lim_{x \to \infty} \frac{c}{f(x)} = 0$	If $\lim_{x \to -\infty} f(x) = \pm\infty$, then $\lim_{x \to -\infty} \frac{c}{f(x)} = 0$	If $f(x)$ grows larger and larger as $x \to \pm\infty$, then $\frac{c}{f(x)}$ gets smaller and tends toward 0. (Dividing by a larger number gives a smaller result.)

EXAMPLE Evaluate the following limit: $\lim_{x \to \infty} e^{-3x}$

Since $e^{-3x} = \frac{1}{e^{3x}}$ and $e^{3x} \to \infty$ as $x \to \infty$, it follows that $\lim_{x \to \infty} e^{-3x} = \lim_{x \to \infty} \frac{1}{e^{3x}} = 0$.

STEP BY STEP

If $f(x)$ is a rational function, we can use the following technique to evaluate the limit:

1. Divide the numerator and denominator by x^n, where n is the highest power of x in the denominator.
2. Use properties of limits (including those shown above) to evaluate the limit.

EXAMPLE Evaluate $\lim_{x \to \infty} \frac{4}{x^2+3}$ analytically.

$\lim_{x \to \infty} \frac{\left(\frac{4}{x^2}\right)}{\frac{(x^2+3)}{x^2}}$	Divide the numerator and denominator by the highest power of x in the denominator, which is x^2.
$\lim_{x \to \infty} \frac{\left(\frac{4}{x^2}\right)}{\left(\frac{x^2}{x^2}+\frac{3}{x^2}\right)} = \lim_{x \to \infty} \frac{\left(\frac{4}{x^2}\right)}{\left(1+\frac{3}{x^2}\right)}$	Expand the denominator, then simplify.

$$\frac{\lim_{x\to\infty}\left(\frac{4}{x^2}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{3}{x^2}\right)} \quad \lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\lim_{x\to\infty}f(x)}{\lim_{x\to\infty}g(x)}$$

$$\frac{0}{1+0}=0 \quad \lim_{x\to\infty}\left(\frac{4}{x^2}\right)=0,\ \lim_{x\to\infty}\left(\frac{3}{x^2}\right)=0,\ \lim_{x\to\infty}1=1$$

Thus, analytically, $\lim\limits_{x\to\infty}\frac{4}{x^2+3}=0$.

TRY IT

Consider the following limit: $\lim\limits_{x\to\infty}\frac{3x+2}{7x+1}$.

Evaluate the limit analytically.

+

$\lim\limits_{x\to\infty}\frac{3x+2}{7x+1}=\frac{3}{7}$ (Divide numerator and denominator by *x*.)

THINK ABOUT IT

In the problem you just tried, how would you have arrived at this answer either graphically or numerically without using an approximation?

Here is a limit in which the squeeze theorem is used.

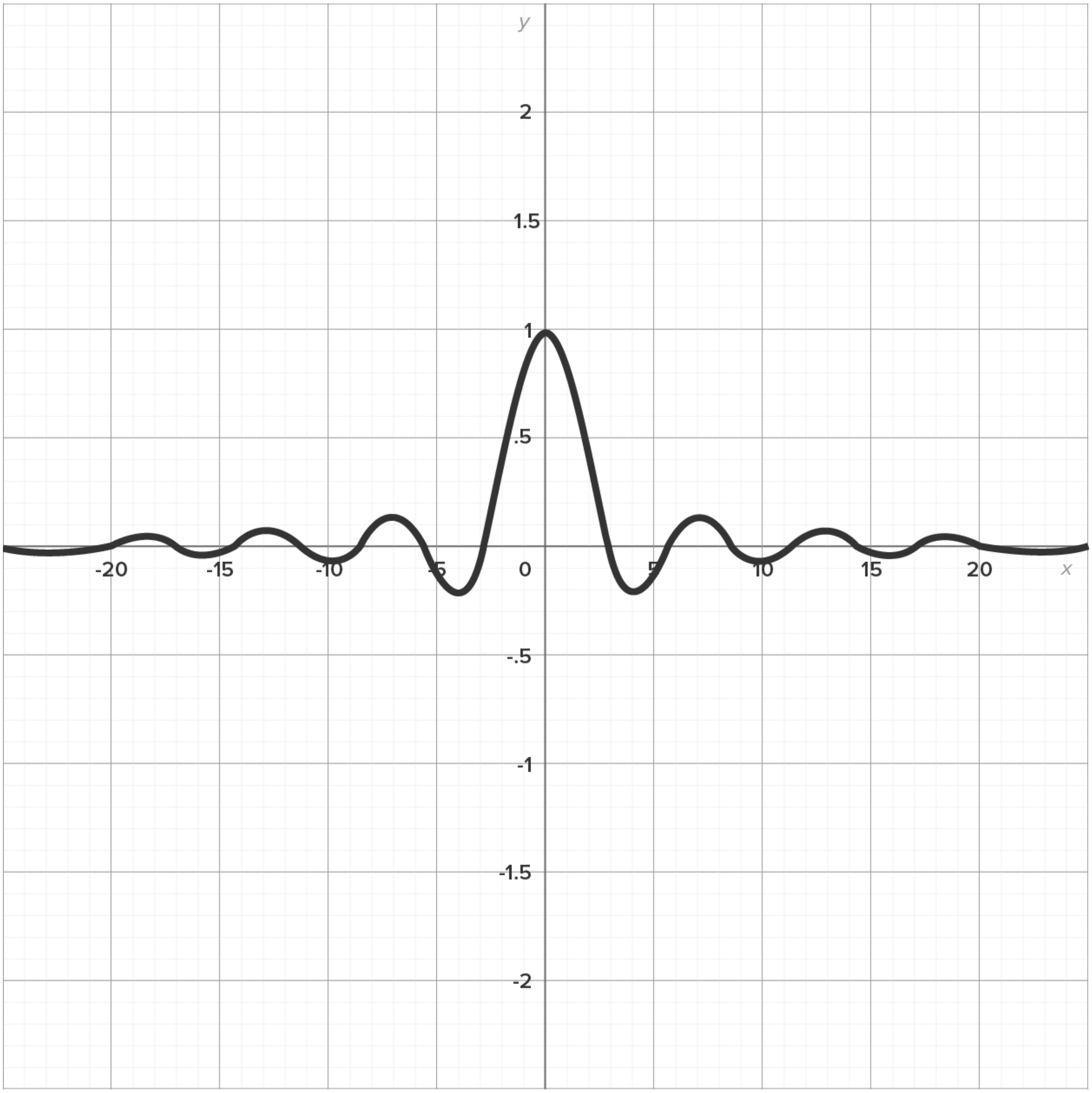
EXAMPLE Evaluate $\lim\limits_{x\to\infty}\frac{\sin x}{x}$. The method outlined earlier will not work since $\sin x$ is not a polynomial.

Recall that $-1\le\sin x\le 1$ for all real numbers *x*. As $x\to\infty$, *x* is a positive number, so it is possible to divide all parts of the inequality by *x*: $\frac{-1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$

Using properties of limits, we know $\lim\limits_{x\to\infty}\frac{-1}{x}=0$ and $\lim\limits_{x\to\infty}\frac{1}{x}=0$.

Since $\frac{-1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$, it follows that $\lim\limits_{x\to\infty}\frac{\sin x}{x}=0$.

Here is the graph of $f(x)=\frac{\sin x}{x}$, which helps to confirm this result:



WATCH

In this video, we'll evaluate $\lim_{x \to \infty} \frac{2x+1}{\sqrt{9x^2+1}}$.

SUMMARY

In this lesson, you learned that **evaluating limits as $x \to \infty$ and as $x \to -\infty$ can be done graphically, numerically, and analytically**. To evaluate graphically is probably the easiest, but it is sometimes difficult to get a precise answer. Using a numerical approach helps to see the patterns in how the values of $f(x)$ change, but again, it is sometimes difficult to get a precise answer. The analytical approach, while more lengthy, will produce a precise answer.

The graphical approach is best to use when the limit is either ∞ or $-\infty$ since this is fairly simple to spot from a graph. As long as the expression can be manipulated algebraically, the analytical approach will give a more precise and convincing answer.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.