Intermediate Value Theorem

by Sophia

WHAT'S COVERED

In this lesson, you will analyze functions using the intermediate value theorem. Specifically, this lesson will cover:

1. The Intermediate Value Theorem

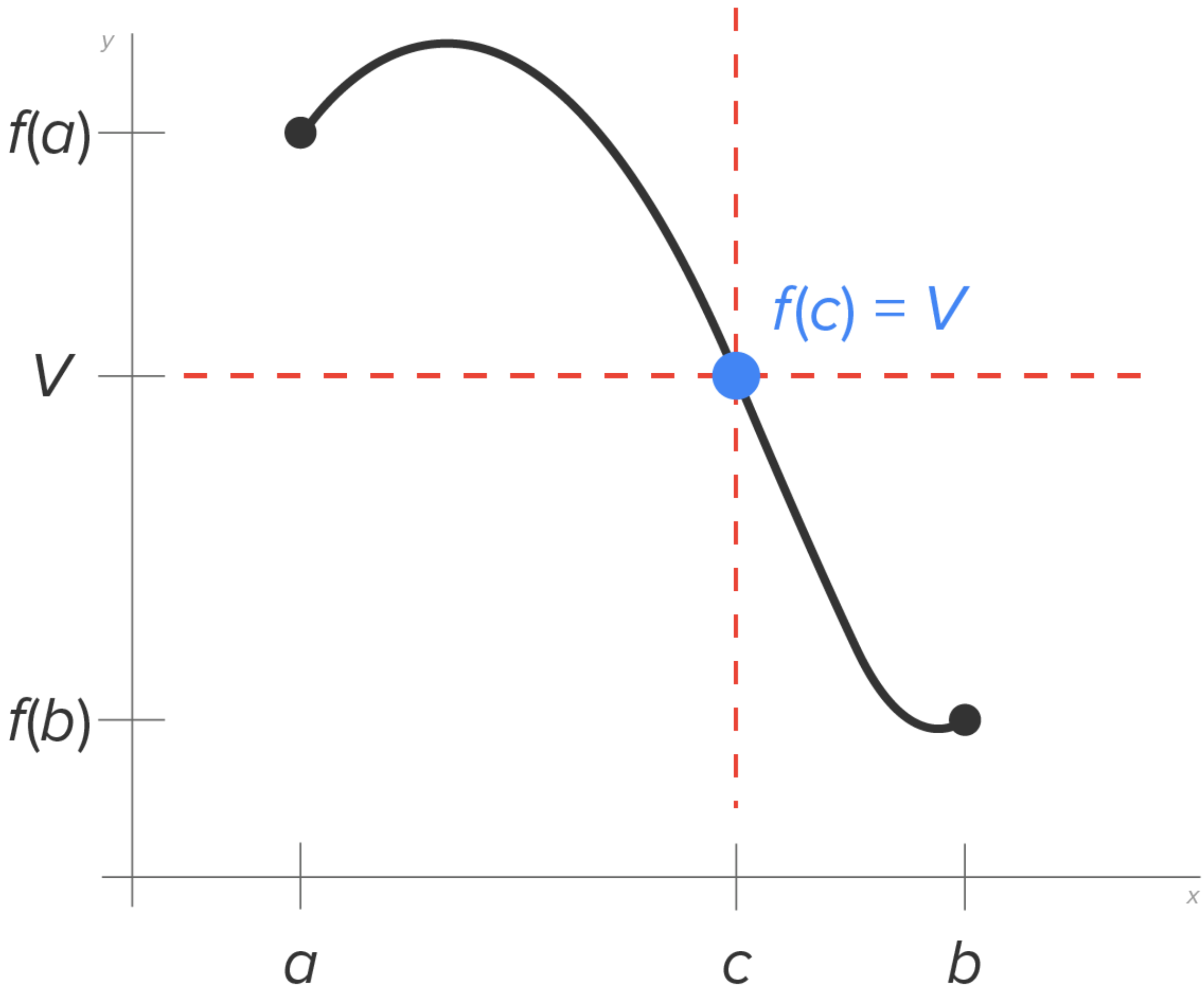
Suppose at 7 AM, you walk outside and it is 40°F. Then, at 11 AM, the temperature is 60°F. We know at some point between 7 AM and 11 AM, the temperature had to be 50°F. Why?

This is because temperature doesn't "jump" from one level to the next, meaning that the temperature is a continuous function of time.

Another way to visualize this:

1. Graph the points (7, 40) and (11, 60).
2. Connect the points with any continuous curve. Be creative.
3. Does your curve have a point where $y = 50$ between $x = 7$ and $x = 11$? The answer should be yes. Otherwise, your graph is not continuous.

This idea is generalized by the intermediate value theorem.
For the **intermediate value theorem (IVT)**, suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let V be a value between $f(a)$ and $f(b)$. Then, there is at least one value of c between a and b such that $f(c) = V$.



EXAMPLE Consider the continuous function $f(x) = x^2 + 1$ on the closed interval $[1, 4]$. Note that $f(1) = 1^2 + 1 = 2$ and $f(4) = 4^2 + 1 = 17$.

Choose a value between 2 and 17, say, the value 8. By the IVT, this means that there is at least one value of *c* between 1 and 4 such that $f(c) = 8$. Let's find this value.

Since we want $f(c) = 8$, this means $c^2 + 1 = 8$, which means $c^2 = 7$, or $c = \pm\sqrt{7}$. Since $\sqrt{7}$ is between 1 and 4, this illustrates the existence of the value of *c* in the theorem.

Note that $-\sqrt{7}$ is not in the interval $[1, 4]$, so this value is not considered.

WATCH

An example of the IVT for the function $f(x) = x^2 - 7x$ on $[-3, 1]$ is presented in this video.

TERM TO KNOW

Intermediate Value Theorem (IVT)

Suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let V be a value between $f(a)$ and $f(b)$. Then, there is at least one value of c between a and b such that $f(c) = V$.

2. Real-World Applications

Here is an example of a real-world application in which the IVT can be useful.

EXAMPLE Suppose a design requires a spherical shape with volume 200 in^3, but the radius of the sphere is to be between 3 and 4 inches. Is it possible to meet these requirements?

First, identify the function, which is the volume of a sphere: $V(r) = \frac{4}{3}\pi r^3$. This problem translates to: Is $V(r) = 200$ for some value in the interval $[3, 4]$?

Since this is a polynomial function, we know $V(r)$ is continuous. Now, evaluate $V(r)$ at the endpoints:

- $V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \approx 113.1 \text{ in}^3$
- $V(4) = \frac{4}{3}\pi(4)^3 = \frac{256}{3}\pi \approx 268.1 \text{ in}^3$

By the IVT, there is a value of r between 3 and 4 inches that produces a volume of 200 in^3.

One particularly useful application of the IVT is locating x-intercepts. Here is the important point:

BIG IDEA

If $f(a)$ and $f(b)$ have different signs (one is positive and one is negative), then there is a value of c in the interval (a, b) such that $f(c) = 0$.

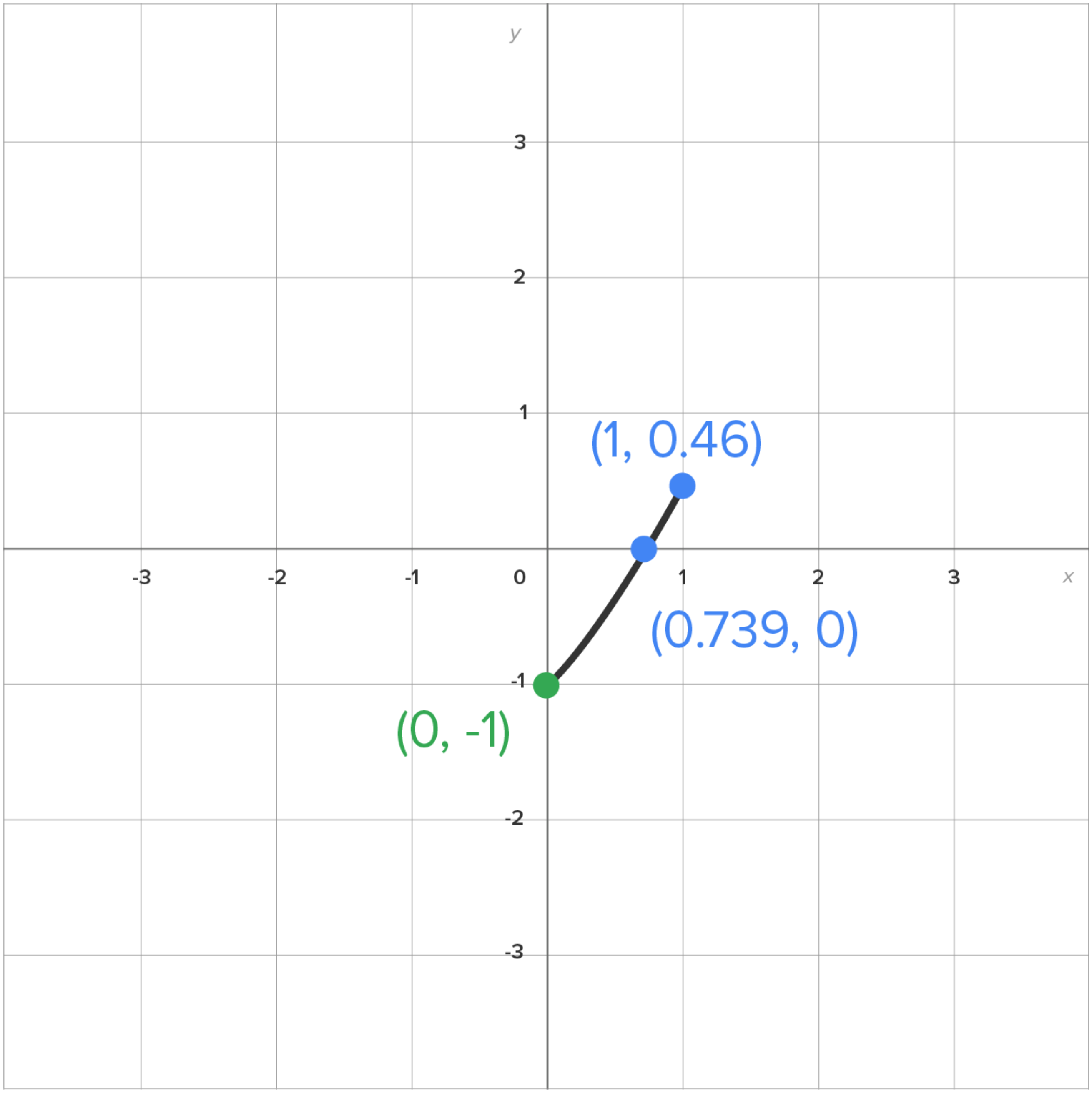
EXAMPLE Let $f(x) = x - \cos x$. Show that there is an x-intercept on the interval $[0, 1]$.

First, note that $f(x)$ is continuous. Next, evaluate the function at the endpoints:

- $f(0) = 0 - \cos 0 = -1$
- $f(1) = 1 - \cos 1 \approx 0.46$

Since $f(0)$ and $f(1)$ have opposite signs, it follows from the IVT that there is a value of x in the interval $[0, 1]$ such that $f(x) = 0$.

Here is a graph to help illustrate. As you can see, the x-intercept occurs when $x \approx 0.739$, which is inside the interval $[0, 1]$.



TRY IT

Let $f(x) = x - 5\sqrt{x}$.

Use the IVT to determine if there is a guaranteed value of *x* for which *f*(*x*) = 20 on the interval [36, 100].

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Since $f(x)$ is continuous on $[36, 100]$ with $f(36) = 6$ and $f(100) = 50$, there must be a value of *x* for which $f(x) = 20$ on the interval $[36, 100]$.

TRY IT

Let $f(x) = x - e^{-2x}$.

Use the IVT to determine if this function is guaranteed an x-intercept on the closed interval [0, 2]. +

Since $f(x)$ is continuous on $[0, 2]$ with $f(0) = -1$ and $f(2) \approx 1.98$, there must be a value of x for which $f(x) = 0$ on the interval $[0, 2]$.

SUMMARY

In this lesson, you learned about **the intermediate value theorem** (IVT), which is very useful in determining if an input is guaranteed in an interval (a,b) for which the output is V when you have a continuous function on a closed interval $[a,b]$. Specifically, the IVT states that if you have a continuous function on a closed interval $[a,b]$, and if V is between $f(a)$ and $f(b)$, you are guaranteed at least one input, c, in the interval $[a,b]$ for which $f(c) = V$.

You also learned about several useful **real-world applications** of the IVT, such as determining if x-intercepts exist on a closed interval. It is important to remember that if $f(a)$ and $f(b)$ have different signs (one is positive and one is negative), then there is a value of c in the interval (a, b) such that $f(c) = 0$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Intermediate Value Theorem (IVT)

Suppose $f(x)$ is a continuous function on the closed interval $[a, b]$. Let V be a value between $f(a)$ and $f(b)$. Then, there is at least one value of c between a and b such that $f(c) = V$.