The Algorithm for Newton's Method

by Sophia

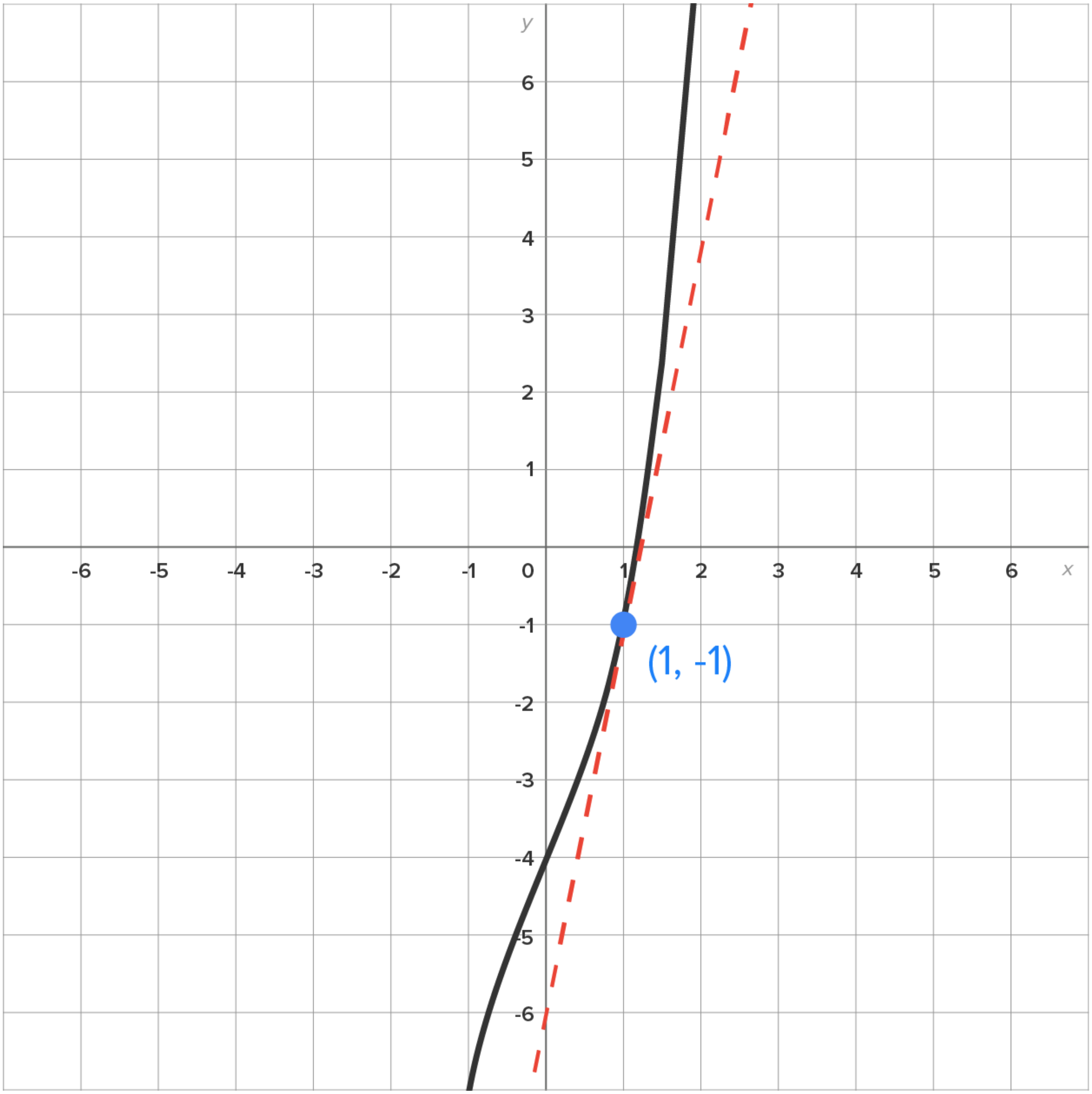
WHAT'S COVERED

As you have seen in this challenge, the tangent line to a function at $x = a$ provides a good estimate to $f(x)$ near $x = a$. Another way to use the tangent line is to find its x-intercept to approximate the x-intercept of $f(x)$. In this lesson, you will learn Newton's method, which uses successive tangent lines to approximate an x-intercept. Most graphing utilities use Newton's method to locate x-intercepts and points of intersections of graphs. Specifically, this lesson will cover:

1. The Idea Behind Newton's Method

The goal of Newton's method is to use tangent lines to approximate an x-intercept of the graph of $y = f(x)$. In other words, the goal is to solve the equation $f(x) = 0$.

Consider the function $f(x) = x^3 + 2x - 4$.



Now, consider the picture shown above, which has two graphs:

- The solid curve is the graph of $f(x)$.
- The dashed line is the tangent line at $x = 1$ (this corresponds to our “guess”).

To start the process for Newton’s method, we’re going to “guess” $x = 1$ as the x-intercept.
Notice that the x-intercept of $f(x)$ is very close to the x-intercept of the tangent line. The advantage of using the tangent line is that it is much easier to solve a linear equation than it is a cubic equation.

First step: Find the equation of the tangent line at $x = 1$.

Given $f(x) = x^3 + 2x - 4$, the derivative is $f'(x) = 3x^2 + 2$. Then, the slope of the tangent line is $f'(1) = 3(1)^2 + 2 = 5$.

Then, the equation of the tangent line is:

$$\begin{aligned} y &= f(1) + f'(1)(x-1) \\ y &= -1 + 5(x-1) \\ y &= -1 + 5x - 5 \\ y &= 5x - 6 \end{aligned}$$

Then, the x-intercept of the tangent line is found by letting $y = 0$ and solving for *x*:

$$\begin{aligned} 0 &= 5x - 6 \\ 6 &= 5x \\ \frac{6}{5} &= x \text{ (or 1.2 in decimal form)} \end{aligned}$$

Thus, our approximation for the x-intercept is (1.2, 0).
So, where would we go from here?

We now have a new “guess” for the x-intercept of the graph of $f(x)$. To continue with this process, find the equation of the tangent line to $f(x)$ at $x = 1.2$, then find its x-intercept. We’ll formalize this process and then complete this problem in the next part of this challenge.

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2. Applying Newton’s Method

Consider a function $y = f(x)$ and let x_0 be the first guess for its x-intercept.

Write the equation of the tangent line at $x = x_0$: $y = f(x_0) + f'(x_0)(x - x_0)$.

Find the x-intercept of the tangent line, which means $y = 0$:

$0 = f(x_0) + f'(x_0)(x - x_0)$	Replace *y* with 0.
$-f(x_0) = f'(x_0)(x - x_0)$	Subtract $f(x_0)$ from both sides.
$-\frac{f(x_0)}{f'(x_0)} = x - x_0$	Divide both sides by $f'(x_0)$.
$x_0 - \frac{f(x_0)}{f'(x_0)} = x$	Add x_0 to both sides.

Now, this x-intercept is the next guess for the intercept, which under normal conditions, is a closer estimate than x_0. Since this process will continue, let’s call the x-intercept of the tangent line x_1. Then, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Now, suppose we want to continue this process:

- Find the equation of the tangent line at $x = x_1$.
- Find the x-intercept of the tangent line and call it x_2. Then, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

If we continue this process, we get a sequence of estimates x_0, x_1, x_2, ... for the estimates of the x-intercept that get closer to some number (which would be the actual x-intercept). Performing these iterations is what is known as Newton's method.

2a. Newton's Method: The Algorithm

Suppose the goal is to find an approximation to an x-intercept of a function $y = f(x)$, which is equivalent to finding a solution to $f(x) = 0$. Starting with an initial guess at $x = x_0$, the sequence of guesses x_1, x_2, x_3, ... is generated by the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

The process stops when one of two things occurs:

- Two consecutive x-values are "close enough" together.
- The x-values are jumping around to the point where they aren't getting closer to a common number.

FORMULA TO KNOW

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

2b. Approximating x-Intercepts with Newton's Method

EXAMPLE Let's pick back up with the function $f(x) = x^3 + 2x - 4$. When we left off, we had $x_0 = 1$ and $x_1 = 1.2$. Let's perform two more iterations of Newton's Method to get a better approximation of the x-intercept. To use Newton's method, it is best to organize the information into a table:

Note: $f(x) = x^3 + 2x - 4$ and $f'(x) = 3x^2 + 2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	-1	5	1.2
1	1.2	0.128	6.32	1.179746835
2	1.179746835	0.001468379	6.175407787	1.179509057
3	1.179509057	0.0000002	6.173724847	1.179509025

The last two estimates are identical to 6 decimal places, so we conclude that the x-intercept to six decimal places of $f(x)$ is (1.179509, 0). This also means that the equation $x^3+2x-4=0$ has the solution $x \approx 1.179509$.

WATCH

Use Newton's method to find the approximate solution to $x-\cos x=0$.

SUMMARY

In this lesson, you learned **the idea behind Newton's method**, which is to use tangent lines to approximate an x-intercept of the graph of $y=f(x)$. Newton's method is a very straightforward approximation method designed to solve equations of the form $f(x)=0$ (equivalent to finding the x-intercepts of the graph of $y=f(x)$). You learned how to **apply Newton's method** using its **algorithm**, by starting with an initial guess at $x=x_0$, then generating a sequence of guesses x_1, x_2, x_3, ... to arrive at a close **approximation of the x-intercept**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$.