MAT 230 EXAM ONE

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

(a) The domain for all variables in the expressions below is the set of real numbers. **Determine whether each statement is true or false.**

(i) $\forall\, x\; \exists\, y\; (x\, +\, y\; \geq\; 0)$

My answer: the above expression reads: "assuming that x and y are real numbers, it is the case that for all cases of x, there exists a y such that their sum is greater than or equal to 0.

In the sense of completeness, this statement is *true*. Given the case of:

- $x \geq 0$, the y that satisfies the expression $(x + y) \geq 0$ is in the bound $y \in [0, +\infty)$
- $x < 0$, the y that satisfies the expression $(x+y) \geq 0$ is in the bound $y \in [abs(x), +\infty)$

(ii) $\exists\, x\; \forall\, y\; (x\, \cdot\, y\; >\; 0)$

My answer: the above expression reads: "assuming that x and y are real numbers, it is the case that there exists an x such that for all possible y values, their product is greater than 0.

In the sense of completeness, this statement is *false*. Given the case of:

(i) $x < 0$, the product of xy is < 0 if $y > 0$
(ii) $x = 0$, the product of xy is $= 0$
(iii) $x > 0$, the product of xy is < 0 if $y < 0$

There is no possible setting for x where $x \cdot y > 0$

(b) **Translate each of the following English statements into logical expressions.**

(i) There are two numbers whose ratio is less than 1.

My answer:

$$\exists(x, y)\ s.t.\ x/y < 1$$

(ii) The reciprocal of every positive number is also positive.

My answer:

$$\forall x \in \mathbb{R}\ (x > 0) \rightarrow (x^{-1} > 0)$$

PROBLEM 2

Prove the following using the specified technique:

(a) Let x and y be two real numbers such that $x + y$ is rational. Prove by contrapositive that if x is irrational, then $x - y$ is irrational.

My answer: A proof that I have devised for the notion that

$$(x, y \in \mathbb{R}) \wedge (x + y \in \mathbb{Q}) \wedge (x \in \mathbb{R} \backslash \mathbb{Q}) \rightarrow (x - y \in \mathbb{R} \backslash \mathbb{Q})$$

is as follows:

(a) Suppose that $x, y \in \mathbb{R}$ and that $x + y \in \mathbb{Q}$. It can be demonstrated that if $x \in \mathbb{R}\backslash\mathbb{Q}$, then $(x - y) \in \mathbb{R}\backslash\mathbb{Q}$.

(b) By definition,

$$(x \in \mathbb{Q}) \equiv \left[x = \frac{a}{b}\right], \text{ where } a, b \in \mathbb{Z} \wedge b \neq 0$$

(c) By extension, if x cannot be expressed in terms of a/b, then $x \in \mathbb{R}\backslash\mathbb{Q}$.

(d) This proof invokes the method of a *proof-by-contradiction*, I will write this statement in its logical equivalent: if $(x - y) \in \mathbb{Q}$, then $x \in \mathbb{Q}$.

This is founded on the notion that $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$. In this case:

- $p \equiv (x \in \mathbb{R}\backslash\mathbb{Q})$
- $\neg p \equiv \neg(x \in \mathbb{R}\backslash\mathbb{Q}) \equiv (x \in \mathbb{Q})$
- $q \equiv (x - y \in \mathbb{R}\backslash\mathbb{Q})$
- $\neg q \equiv \neg(x - y \in \mathbb{R}\backslash\mathbb{Q}) \equiv (x - y \in \mathbb{Q})$

(e) Knowing that $(x - y) \in \mathbb{Q}$, x and y can be expressed as:

$$x = \frac{x_a}{x_b}, y = \frac{y_a}{y_b}, \text{ where } (x_a, x_b, y_a, y_b) \in \mathbb{Z} \wedge (y_a, y_b) \in \mathbb{Z}$$

(f) By definition, the subtraction of two ratios is:

$$\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1 \cdot b_2 - b_1 \cdot a_2}{b_1 \cdot b_2}$$

(g) Applying the definition of the subtraction of two ratios shown in line (f) to the expression of x and y as ratios shown in line (e), I have worked out that:

$$x - y = \frac{x_a}{y_a} - \frac{x_b}{y_b} = \frac{x_a \cdot y_b - y_a \cdot x_b}{y_a \cdot y_b}$$

(h) Knowing that $(y_a, y_b) \neq 0$, the products of the integers that make up the difference of the two ratios are in the set of rational numbers. Or,

$$(y_a y_b, x_a y_b, -y_a x_b) \in \mathbb{Z}$$

(i) Therefore,

$$((x - y) \in \mathbb{Q}) \rightarrow (x \in \mathbb{Q})$$

which is equivalent to the theorem statement that:

$$(x \in \mathbb{R}\backslash\mathbb{Q}) \rightarrow ((x - y) \in \mathbb{R}\backslash\mathbb{Q})$$

This argument proves the theorem statement.

(b) Prove by contradiction that for any positive two real numbers, x and y, if $x \cdot y \leq 50$, then either $x < 8$ or $y < 8$. .

My answer: My proof for the notion that

$$\{x, y \in \mathbb{R} \mid x > 0\} \wedge (xy \leq 50) \rightarrow (x < 8 \vee y < 8)$$

is as follows:

(a) Suppose that $(x, y) \in \mathbb{R}$, that $x > 0$ and that $y > 0$. It can be proven that if $x \cdot y \leq 50$ then $x < 8 \vee y < 8$

(b) I will start by assuming that $(x \cdot y \leq 50) \rightarrow (x \geq 8 \vee y \geq 8)$.
   - This is equivalent to writing $(x \cdot y \leq 50) \rightarrow \neg(x < 8 \vee y < 8)$

(c) Assuming the lower bound for both x and y, that is, $x = 8$ and $y = 8$, its product works out to $x \cdot y = 64$.

(d) This contradicts the antecedent of the theorem, which is $x \cdot y \leq 50$.

(e) Therefore, either x, y or both are < 8. This proves the theorem statement that $x \cdot y \leq 50$ implies that either x, y or both, must be, to some magnitude, < 8.

Please note that: this “magnitude” is not my concern. I just care that the expression

$$(x \cdot y \leq 50) \rightarrow (x < 8 \vee y < 8)$$

can be justified.

Problem 3

Let $n \geq 1$, x be a real number, and $x \geq -1$. **Prove the following statement using mathematical induction.**

$$(1 + x)^n \geq 1 + nx$$

My answer: I will prove the notion that

$$(1+x)^n \geq 1 + nx$$

like so:

(1) Suppose that $n \geq 1$, that $x \geq -1$, and that $x \in \mathbb{R}$. It can be proven that

$$(1+x)^n \geq 1 + nx$$

(2) The method by which to prove the completeness of the statement will be *proof-by-induction*. Two criteria must be shown to be the case:

- $P(1)$ is true.
- $\forall k \geq 1 : (P(k) \to P(k+1))$

(3) First, the base case must be demonstrated— this is done by "splitting up" the inequality into two functions: with $f_L(x)$ standing for the left-hand expression of the inequality, and $f_R(x)$ standing for the right-hand expression of the inequality:

- The base case: $f_L(x, n) = (1+x)^n$
- The inductive step: $f_R(x, n) = 1 + nx$

I will proceed by setting $x = 1$ and $n = 1$, which results in:

$$f_L(x = 1, n = 1) = (1+1)^1 = 2,$$
$$f_R(x = 1, n = 1) = 1 + 1 \times 1 = 2$$

It is the case that $f_L(x = 1, n = 1) \geq f_R(x = 1, n = 1)$, and therefore the base case has been demonstrated.

(4) Next, the inductive step must be proven:

(a) The theorem states that:

$$(1+x)^n \geq 1 + nx$$

(b) In accordance with the setup of the inductive step, the theorem is be expressed as:

$$[(1 + k_1)^{k_2}] \geq [1 + k_1 k_2]$$
$$\to [1 + (k_1 + 1)^{k_2+1}] \geq [1 + (k_1 + 1)(k_2 + 1)]$$

(c) On the consequent of the implication shown in line $(4b)$, I will use algebraic manipulations to rewrite the inequality into something that is clearly true:

- $1 + (k_1 + 1)^{k_2+1} \geq 1 + (k_1 + 1)(k_2 + 1)$
- $\Rightarrow 1 + (k_1 + 1)^{k_2+1} \geq k_1 k_2 + k_1 + k_2 + 2$

- $\Rightarrow 1 + \log_{k_2+1}(k_1 + 1) \leq \log_{k_2+1}(k_1k_2 + k_1 + k_2 + 2)$
- $\Rightarrow 1 \leq \log_{k_2+1}(k_1k_2 + k_1 + k_2 + 2) - \log_{k_2+1}(k_1 + 1)$
- $\Rightarrow 1 \leq \log_{k_2+1} \frac{k_1k_2+k_1+k_2+2}{k_1+1}$
- $\Rightarrow 1 \leq \log_{k_2+1}(k_1k_2 + k_2 + 1)$

Unfortunately, due to time constraints, I was not able to fully verify the correctness of this proof. But when set to their bounds, I have confirmed with experimental methods $x\ or\ k_1$ and the $n\ or\ k_2$ values always increase beyond 1 — thus offering some justification of the theorem.

Problem 4

Solve the following problems:

(a) How many ways can a store manager arrange a group of 1 team leader and 3 team workers from his 25 employees?

My answer: This appears to be a problem involving permutations — as order matters in this situation. I will use the following counting formula:

$$P(n, r) = \frac{n!}{(n-r)!}$$

The problem is asking that, given a set of $n = 25$ employees, how many ways can a subset of $r = 1$ group leader and $r = 3$ team workers be organised. This is asking "what the permutations of $P(n = 25, r = 1)$ groups of team leaders and the permutations of $P(n = 24, r = 3)$ team workers.[1] The following is the solution to this kind of problem:

$$P(n = 25, r = 1) \times P(n = 24, r = 3) =$$
$$\frac{25!}{(25-1)!} \times \frac{24!}{(24-3)!} = 25 \times 12,144 = 303,600$$

Assuming that this is a problem of the number of permutations, where the order of the employees matters, then the answer to this problem is 303,600 different possible ways to sort a (1) store manager and their respective group of three (3) employees.

(b) A states license plate has 7 characters. Each character can be a capital letter $(A - Z)$, or a non-zero digit $(1 - 9)$. How many license plates start with 3 capital letters and end with 4 digits with no letter or digit repeated?

My answer: Given an alphabet of twenty-six (26) capital letters, and nine (9) digits, the formula that I used to work out the number of possibilities for a string with three letters at the start (assuming no repeating letters), and four digits in the end (assuming no repeating digits) is:

$$\begin{aligned} C = [n_{\text{letters}} \times (n_{\text{letters}} - 1) \times (n_{\text{letters}} - 2)] \\ \times [n_{\text{digits}} \times (n_{\text{digits}} - 1) \times (n_{\text{digits}} - 2) \times (n_{\text{digits}} - 3)] \\ = [26 \times (26-1) \times (26-2)] \times [9 \times (9-1) \times (9-2) \times (9-3)] \\ = 47,174,400 \end{aligned}$$

Or, nearly 47.2 million possible combinations of three (3) letters in the start, and four (4) letters in the end of the string, with no repeating letters.

[1]For team workers, $n = 24$ because the question is sampling without replacement.

(c) How many binary strings of length 5 have at least 2 adjacent bits that are the same ("00" or "11") somewhere in the string?

My answer: The only way that I can think of to work out the solution to this problem is to enumerate all the possible strings for an alphabet of 0 and 1 characters, and then count up the strings that meet the criteria.

I used the following script to exhaust the possible ways to make a string:

```
alphabet = ["0", "1"]

for a in alphabet:
    for b in alphabet:
        for c in alphabet:
            for d in alphabet:
                for e in alphabet:
                    print("{0}{1}{2}{3}{4}{5}".format(a,b,c,d,e))
```

And was given the following as output:

0	0	0	0	0	✓
0	0	0	0	1	✓
0	0	0	1	0	✓
0	0	0	1	1	✓
0	0	1	0	0	✓
0	0	1	0	1	✓
0	0	1	1	0	✓
0	0	1	1	1	✓
0	1	0	0	0	✓
0	1	0	0	1	✓
0	1	0	1	0	
0	1	0	1	1	✓
0	1	1	0	0	✓
0	1	1	0	1	✓
0	1	1	1	0	✓
0	1	1	1	1	✓
1	0	0	0	0	✓
1	0	0	0	1	✓
1	0	0	1	0	✓
1	0	0	1	1	✓

1	0	1	0	0	✓
1	0	1	0	1	
1	0	1	1	0	✓
1	0	1	1	1	✓
1	1	0	0	0	✓
1	1	0	0	1	✓
1	1	0	1	0	✓
1	1	0	1	1	✓
1	1	1	0	0	✓
1	1	1	0	1	✓
1	1	1	1	0	✓
1	1	1	1	1	✓
					$\sum_{\text{Criteria Met}} = 30$ (out of 32)

From this "brute force" method, I have worked out that given a binary string of length five (5), the number of times where two adjacent characters equal one another is thirty (30) out of a possible thirty-two (32) combinations.

PROBLEM 5

A class with n kids lines up for recess. The order in which the kids line up is random with each ordering being equally likely. There are two kids in the class named Betty and Mary. The use of the word "*or*" in the description of the events, should be interpreted as the inclusive or. That is "A *or* B" means that A is true, B is true, or both A and B are true.

What is the probability that Betty is first in line or Mary is last in line as a function of n? Simplify your final expression as much as possible and include an explanation of how you calculated this probability.

My answer: The probability that Betty is first in line, or Mary is last in line, as a function p of n, are:

$$\begin{aligned} P(n) = P(\text{Betty will be first in line}) + P(\text{Mary will be last in line}) \\ -P(\text{Betty will be first in line, and Mary will be last in line}) \\ = \frac{1}{n} + \frac{1}{n} - \frac{1}{n^2} \end{aligned}$$

This kind of problem involves an injunction of events that are not mutually exclusive: so the addition rule for non mutually exclusive events is applied to the given problem.

PROBLEM 6

The general manager, marketing director, and 3 other employees of Company A are hosting a visit by the vice president and 2 other employees of Company B. The eight people line up in a random order to take a photo. Every way of lining up the people is equally likely.

(a) What is the probability that the general manager is next to the vice president?

My answer: Given that there are eight people in the photo shoot, there are 2×7 possible ways that general manager and vice president are next to each other, or 14 ways. The sample space of the lineup is $\omega = 8! = 40,320$, and the probability of the target event is $E/\omega = 14/40,320 = 3.47 \times 10^{-4}$

(b) What is the probability that the marketing director is in the leftmost position?

The probability of this event is

$$\frac{1}{8!} = 2.48 \times 10^{-5}$$

(c) Determine whether the two events are independent. Prove your answer by showing that one of the conditions for independence is either true or false.