MAT 230 EXAM TWO

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

This question has 2 parts.

Part 1: Suppose that F and X are events from a common sample space with $P(F) \neq 0$ and $P(X) \neq 0$.

(a) Prove that $P(X) = P(X|F)P(F) + P(X|\bar{F})P(\bar{F})$. Hint: Explain why $P(X|F)P(F) = P(X \cap F)$ is another way of writing the definition of conditional probability, and then use that with the logic from the proof of Theorem 4.1.1.

My answer: A proof that, given F and X are events in the sample space of ω, and that their respective probabilities are non-zero, the theorem:

$$P(X) = P(X|F)P(F) + P(X|\neg F)P(\neg F)$$

is true:[1]

(a) Assume that $P(X) + P(\neg X) = 1$
(b) Assume that F and X are events from a common sample space ω— or that

$$(F, X) \in \omega$$

(c) Assume that $P(F) \neq 0 \wedge P(X) \neq 0$
(d) Because F and X are non-zero events (line c) from a common sample space (line b) the definition of conditional probability can be used to make sense its properties,

$$P(X|F) = \frac{P(X \cap F)}{P(F)}$$

(e) With algebra, it can be worked out from the definition shown in line (d) that,

$$P(X \cap F) = P(X|F)P(F)$$

(f) Theorem 4.1.1 posits that

$$P(X|F) + P(X|\neg F) = 1$$

(g) From theorem 4.1.1 expressed in line (f), I can write the discovered relation expressed in line (e) as

$$P(X) = P(X \cap F) + P(\neg(X \cap F))$$

or, when substituting $P(X \cap E)$ with its equivalence of $P(X|F)P(F)$,

$$P(X) = P(X|F)P(F) + P(X|\neg F)P(\neg F)$$

(h) Thus, I have proven that when the assumptions posited in lines (a, b, c) are made, and when theorem 4.1.1 is accepted, that

$$P(X) = P(X|F)P(F) + P(X|\neg F)P(\neg F)$$

[1] For negation, I replaced the bar with a "$\neg$" symbol.

(b) Explain why $P(F|X) = P(X|F)P(F)/P(X)$ is another way of stating Theorem 4.2.1 Bayes Theorem.

My answer: A generalised version of Bayes' theorem states that:

$$P(F|X) = \frac{P(X|F)P(F)}{P(X|F)P(F) + P(X|\neg F)P(\neg F)}$$

The following is a proof that

$$P(F|X) = \frac{P(X|F)P(F)}{P(X)} = \frac{P(X|F)P(F)}{P(X|F)P(F) + P(X|\neg F)P(\neg F)}$$

is correct:

(a) Assume that $(F, X) \in \omega$

(b) Assume that $P(F) \neq 0 \wedge P(X) \neq 0$

(c) Now, starting with the definition of a generalised version of Bayes theorem,

$$P(F|X) = \frac{P(X|F)P(F)}{P(X|F)P(F) + P(X|\neg F)P(\neg F)}$$

(d) Knowing, from a previous proof, that $P(X \cap F) = P(X|F)P(F)$, I will rewrite Bayes' theorem in the form of the left-hand of the relation:

$$P(F|X) = \frac{P(X \cap F)}{P(X \cap F) + P(X|\neg F)P(\neg F)}$$

(e) Likewise, $P(X \cap \neg F) = P(X|\neg F)P(\neg F)$ — and the theorem works out to:

$$P(F|X) = \frac{P(X \cap F)}{P(X \cap F) + P(X \cap \neg F)}$$

(f) As discussed in the previous proof, $P(X \cap F) + P(X \cap \neg F) = P(X)$ — so, the theorem can be rewritten as:

$$P(F|X) = \frac{P(X \cap F)}{P(X)}$$

(g) Therefore, with basic algebraic manipulation, I have proven that:

$$P(F|X) = \frac{P(X \cap F)}{P(X)} = \frac{P(X|F)P(F)}{P(X|F)P(F) + P(X|\neg F)P(\neg F)}$$

Part 2: A website reports that 70% of its users are from outside a certain country. Out of their users from outside the country, 60% of them log on every day. Out of their users from inside the country, 80% of them log on every day.

(a) What percent of all users log on every day? Hint: Use the equation from Part 1 (a).

My answer: First, I need to identify the components of the problem that should be formalised into Bayes' theorem:

- The conditional of logging in given being from an outside country is: $P(L|N) = .6$
- The conditional of **not logging in** given being from an outside country is: $P(L|\neg N) = 1 - P(L|N) = .4$
- The probability of being from an outside country is $P(N) = .7$
- The probability of being from **not an outside country** is $P(\neg N) = 1 - P(N) = .3$

With the variables at hand, I just need to plug them into the derived theorem from 1a, and chug out a result:

$$P(L) = P(L|N)P(L) + P(L|\neg N)P(\neg N) = .6 \cdot .7 + .4 \cdot .3 = .54$$

Or, 54 per cent.

(b) Using Bayes Theorem, out of users who log on every day, what is the probability that they are from inside the country?

My answer: I will start by identifying the key components that makes up this application of Bayes' theorem. That is, the probability of logging onto the website from an inside country is $P(L|\neg N) = .4$ — the other components of this application of Bayes' theorem have been identified in the previous problem.

To work out the $P(\neg N|L)$, an application of Bayes' theorem is needed:

$$P(\neg N|L) = \frac{P(L|\neg N)P(\neg N)}{P(L|\neg N)P(\neg N) + P(L|\neg\neg N)P(\neg\neg N)}$$
$$= \frac{P(L|\neg N)P(\neg N)}{P(L|\neg N)P(L) + P(L|N)P(N)} = \frac{.4 \cdot .3}{.4 \cdot 3 + .6 \cdot .7} \approx 0.22\bar{2}$$

Or, about 22.2 per cent.

Problem 2

This question has 2 parts.

Part 1: The drawing below shows a Hasse diagram for a partial order on the set:
$\{A, B, C, D, E, F, G, H, I, J\}$

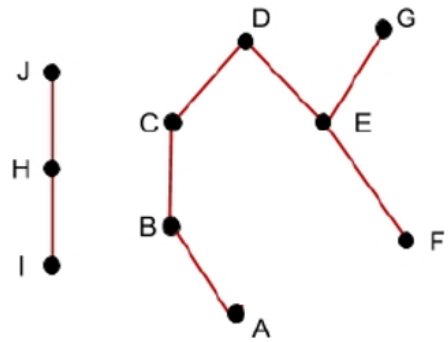


Figure 1: *A Hasse diagram shows 10 vertices and 8 edges. The vertices, represented by dots, are as follows: vertex J is upward of vertex H; vertex H is upward of vertex I; vertex B is inclined upward to the left of vertex A; vertex C is upward of vertex B; vertex D is inclined upward to the right of vertex C; vertex E is inclined upward to the left of vertex F; vertex G is inclined upward to the right of vertex E. The edges, represented by line segments between the vertices are as follows: 3 vertical edges connect the following vertices: B and C, H and I, and H and J; 5 inclined edges connect the following vertices: A and B, C and D, D and E, E and F, and E and G.*

Determine the properties of the Hasse diagram based on the following questions:

(a) What are the minimal elements of the partial order?

My answer: From my reading of the *Hasse diagram* depicted by figure 1, it appears that the nodes I, A, and F are its minimal elements. My justification for this conclusion is that I, A, and F are all $\preceq$ other nodes which are of a higher value as depicted by the Hasse diagram.

(b) What are the maximal elements of the partial order?

My answer: From my reading of the *Hasse diagram* depicted by figure 1, it appears that the nodes J, D and G are of its maximal elements. My justification for this conclusion is that all other nodes in the Hasse diagram are $\preceq$ the J, D and G nodes.

(c) Which of the following pairs are comparable?

$$(A,\ D),\ (J,\ F),\ (B,\ E),\ (G,\ F),\ (D,\ B),\ (C,\ F),\ (H,\ I), (C,\ E)$$

My answer: From my reading of the *Hasse diagram* depicted by figure 1, it appears that the following pairs are comparable:

- (A, D) because I can traverse its graph vertically and *not horizontally*.
- (G, F) for the same reasons as (A, D).
- (D, B) for the same reasons as (A, D).
- (H, I) for the same reasons as (A, D).

The pairs (J, F), (B, E), (C, F) and (C, E) did not meet the criteria for pairs that can are comparable.

Part 2: Consider the partial order with domain {3, 5, 6, 7, 10, 14, 20, 30, 60, 70} and with $x \leq y$ if x evenly divides y. Select the correct Hasse diagram for the partial order.

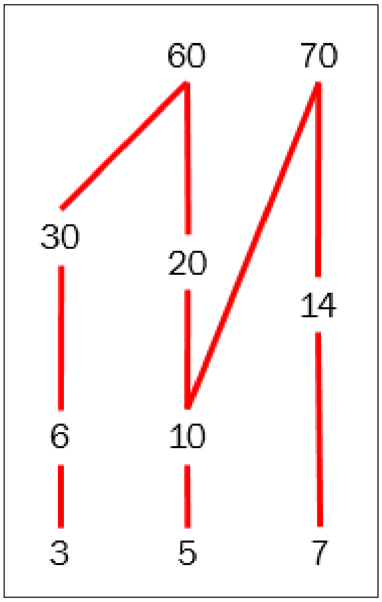
(a) 

Figure 2: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 20, 20 and 60, 10 and 70, 7 and 14, 14 and 70.*

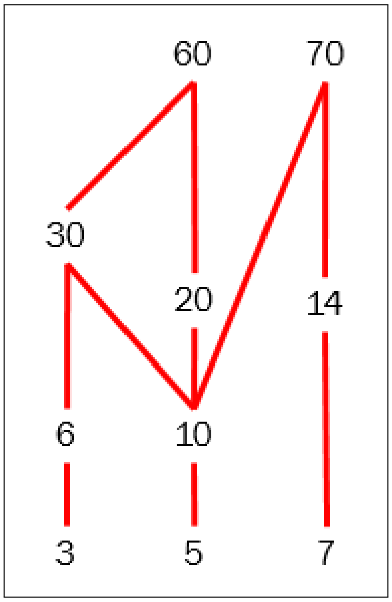
(b) 

Figure 3: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 30, 10 and 20, 20 and 60, 10 and 70, 7 and 14, 14 and 70.*

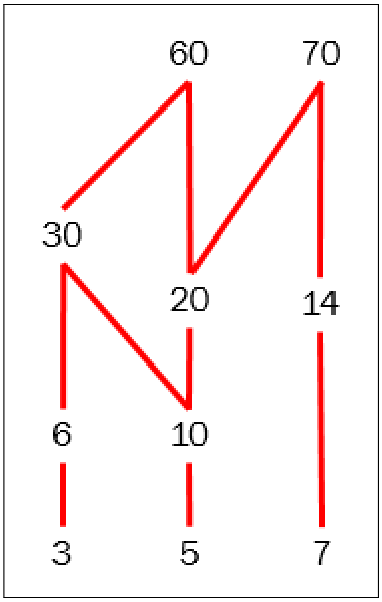
(c) 

Figure 4: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 30, 10 and 20, 20 and 60, 20 and 70, 7 and 14, 14 and 70.*

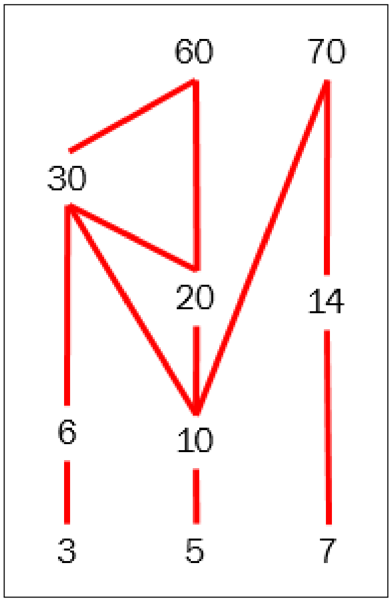
(d) 

Figure 5: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 30, 10 and 20, 20 and 30, 20 and 60, 10 and 70, 7 and 14, 14 and 70.*

My answer: Four different *Hasse diagrams* that claim to accurately display the partial order $3, 5, 6, 7, 10, 14, 20, 30, 60, 70$ with the rules that $x < y$ and $x|y$ are presented. The given problem is to select the correct one out of all the alternatives.

In this case, I will select figure 3 as the *Hasse diagram* that best represents the relation of $x \leq y$ and $x|y$. To explain why, I will use the process of elimination:

- I eliminated figure 5 because it incorrectly links 30 to 20, and 30 does not evenly divide 20.
- I eliminated figure 2 because it did not link all of the integers such that x is divisible by y— e.x. it did not link the pair $(30, 10)$
- I eliminated figure 4 because it incorrectly links 70 to 20, and 70 does not evenly divide 20.

Therefore, the "most right," and most likely correct Hasse diagram, is depicted by figure 3.

PROBLEM 3

A car dealership sells cars that were made in 2015 through 2020. Let the cars for sale be the domain of a relation R where two cars are related if they were made in the same year.

(a) Prove that this relation is an equivalence relation.

My answer: I will proceed by defining the relation described in the opening of the problem:

$$x_1 R x_2 \text{ if } M(x_1) = M(x_2)$$

where $M(x)$ is a relational operator that returns the given year for car x. An *equivalence* relation is a kind of relation that exhibits the properties of *being reflexive*, *symmetry* and *transitive*:

- A relation R is reflexive iff
$$\text{Reflexive}(R) \equiv (\forall x_1 \in A : x_1 R x_1)$$
- A relation R is symmetric iff
$$\text{Symmetric}(R) \equiv (\forall x_1, x_2 : x_1 R x_2 \to x_2 R x_1)$$
- A relation R is symmetric iff
$$\text{Transitive}(R) \equiv (\forall x_1, x_2, x_3 : x_1 R x_2 \wedge x_2 R x_3 \to x_1 R x_3)$$

$M(x)$ takes a car as an input, and outputs the year that it was manufactured. The domain $\mathbb{D}$ is:

$$\mathbb{D} = \text{ any kind of car manufactured from 2015 - 2020}$$

and its range $\mathbb{R}$ is:

$$\mathbb{R} = \{2015, 2016, 2017, 2018, 2019, 2020\}$$

Starting with the property of reflexiveness, the xRx is valid if $M(x_1) = M(x_2)$. It is tautological to proclaim that a car x that was manufactured in the year y— s.t.[2]

$$y \in \{2015, 2016, 2017, 2018, 2019, 2020\}$$

will result in the same y if the same x is supplied to the $M(x)$ function.

Moving on to the property of *symmetry*, given two cars x_1 and x_2, if, when subjected to the $M(x)$ function map to a year that satisfies the expression $M(x_1) = M(x_2)$, then the property of symmetry is met. It is an accepted fact that given two integers x and y,

$$(x = y) \equiv (y = x)$$

The resulting years $M(x_1)$ and $M(x_2)$ are integers, so the notion that

$$M(x_1) = M(x_2) \to M(x_2) = M(x_1)$$

is justified if one accepts the aforementioned "math fact."

[2]s.t. = such that

Finally, the notion of transitivity posits that given three cars x_1, x_2, and x_3, given that one car x_1 has a manufacture year $M(x_1)$ that is equal to the other car's x_2 manufacture year $M(x_2)$, and that car x_2 has a manufacture year $M(x_2)$ that is equal to the third car's x_3 manufacture year $M(x_3)$, then it can be said that:

$$M(x_1) = M(x_3)$$

.

Given the *idempotent* nature of the binary relation, that is, if $M(x_1) = M(x_2) = c$ and $M(x_2) = M(x_3) = c$, then I can confidently say that $M(x_1) = M(x_3) = c$. It is an accepted fact that

$$\forall x, y, z \in \mathbb{Z} : (x = y) \wedge (y = z) \rightarrow x = z$$

The $M(x)$ function maps a car onto an integer. If we accept the stated math fact, then we can conclude that:

$$\forall x_1, x_2, x_3 : [M(x_1) = M(x_2)] \wedge [M(x_2) = M(x_3)] \rightarrow [M(x_1) = M(x_3)]$$

Given that it has been proven that all three sufficient conditions of an equivalence relation is met, in turn it has been proven that the relation

$$x_1 R x_2 \text{ if } M(x_1) = M(x_2)$$

is an equivalence relation.

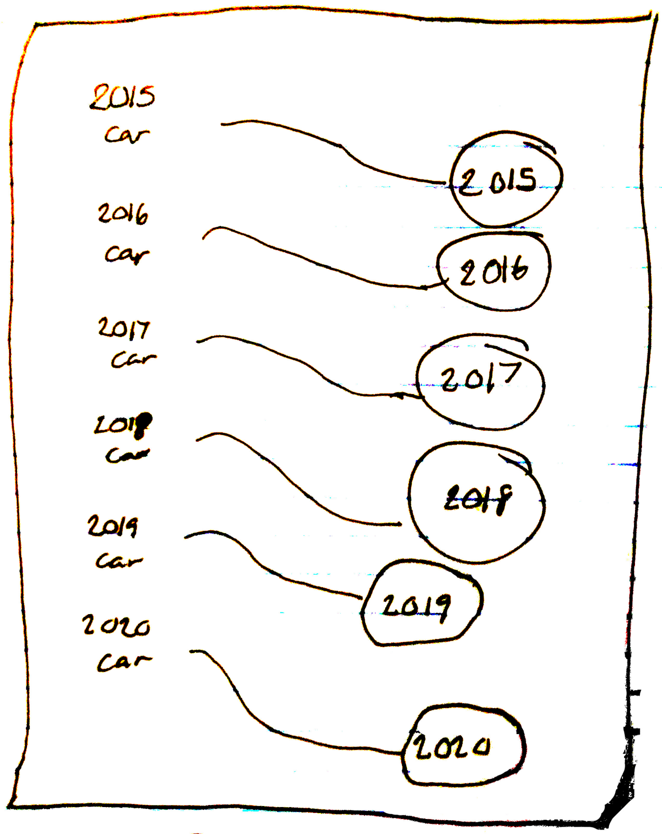
(b) Describe the partition defined by the equivalence classes.

My answer: Before attacking this question, I must first define what a "partition" is and describe some of its properties:

- A *partition* of some set A is a set of non-empty subsets of A s.t. they are pairwise disjoint and whose union is A.[3]
- Theorem 5.14.2[4] states that given a relation of set A, the set of distinct equivalence classes that defines a partition of A.

[3]Quoted almost verbatim from zyBooks §5.14.
[4]Quoted almost verbatim from Ibid.

So, for the relation $x_1 R x_2$, the partition of the sets are:



Cars sold from
2015 - 2020

The figure of the partition for the $x_1 R x_2$ relation shows how a car made in some year, which is written as "[year] car," and is then mapped out to the integer of the year. The figure depicts each input for cars that maps to an integer representing the year that car was made. The image depicted above shows disjoint sets that, when subjected to the operation of a set union, forms the entire "universe" of cars that this relation concerns itself with.

PROBLEM 4

Analyze each graph below to determine whether it has an Euler circuit and/or an Euler trail.

- If it has an Euler circuit, specify the nodes for one.
- If it does not have an Euler circuit, justify why it does not.
- If it has an Euler trail, specify the nodes for one.
- If it does not have an Euler trail, justify why it does not.

(a)

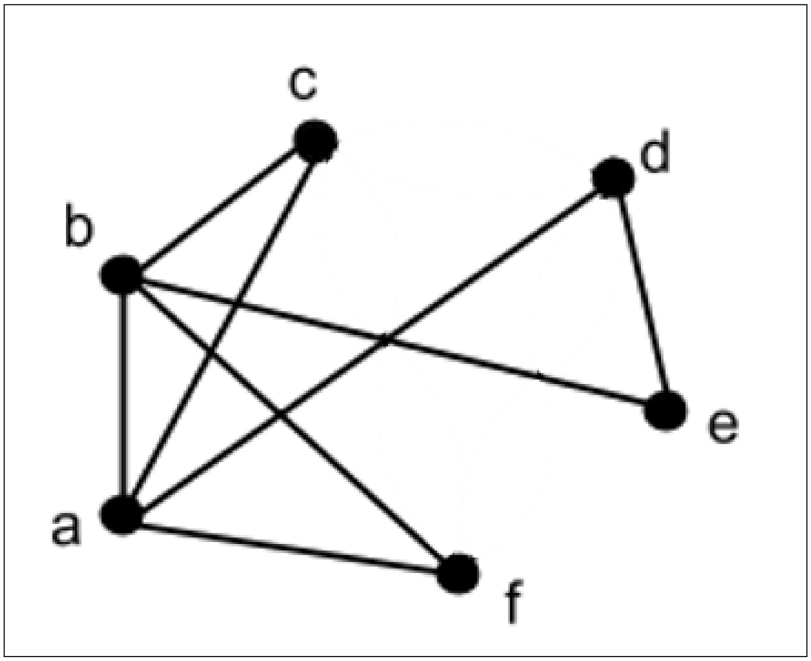


Figure 6: *An undirected graph has 6 vertices, a through f. There are 8-line segments that are between the following vertices: a and b, a and c, a and d, a and f, b and c, b and e, b and f, d and e.*

My answer: Before I can attack the problem, I need to list out the criteria[5] for what makes for a *Euler circuit* and a *Euler trail*:

An Euler circuit ...

- in the context of an **undirected graph**
- is a **circuit** that
- **connects** every **vertex** and every **edge**
- without traversing an edge more than once.

Likewise, an Euler trail ...

- in the context of an **undirected graph**
- is an **open trail**
- that **includes each edge**
- **exactly once**.

[5] After zyBooks §6.6

Some potentially useful theorems for Euler circuits and Euler trails are:

- Theorem 6.6.1:[6] An undirected graph G having an Euler circuit implies that G is connected and all vertices in G have an even degree.
- Theorem 6.6.2:[7] An undirected graph G being connected and each vertex in G having an even degree implies that an undirected graph G has an Euler circuit.
- Theorem 6.6.3:[8] An undirected graph G has an Euler circuit $\equiv$ G is connected and exactly two vertices in G has an odd number of connections.

Finally, the zyBooks (fig. 6.6.2 and fig. 6.6.3) describe a method to discover Euler circuits.[9]

Regarding figure 6, I can say that it has an *Euler circuit*. Each vertex is connected to the graph by some edge, and each vertex has an even number of connections:

- Vertex a has 4 connections.
- Vertex b has 4 connections.
- Vertex c has 2 connections.
- Vertex d has 2 connections.
- Vertex e has 2 connections.
- Vertex f has 2 connections.

The *Euler circuit* that I have discovered is:

$$\begin{aligned} W = <&(f,(f,a)),(a,(a,b)),(b,(b,c)),\\ &(c,(c,a)),(a,(a,d)),(d,(d,e))> \end{aligned}$$

Or, a more intuitive picture of the evolution of the walk is shown as:

$$f \to a \to b \to c \to a \to d \to e$$

However, the graph depicted in figure 6 *does not have an Euler trail* because there does not exist exactly two vertices that has an odd number of connections.

[6]After zyBooks §6.6
[7]After Ibid.
[8]After Ibid.
[9]Which I will not write down in this paper as to keep things succinct.

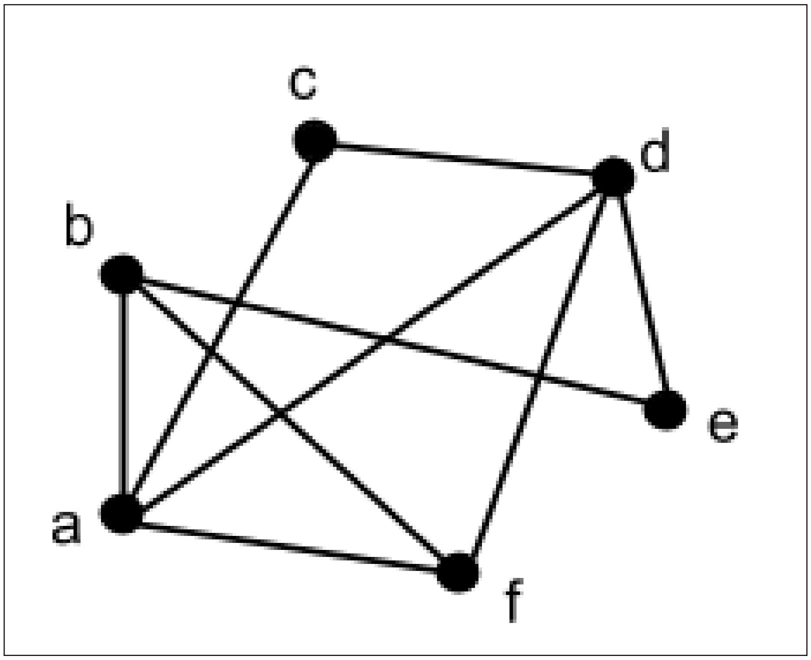
(b) 

Figure 7: *An undirected graph has 6 vertices, a through f. There are 9-line segments that are between the following vertices: a and b, a and c, a and d, a and f, b and e, b and f, c and d, d and e, d and f.*

My answer: The criteria that *Euler circuits* and *Euler trails* must meet have been discussed in a previous variant of this question. In the case for figure 7, it *does have an Euler trail* because there are two vertices[10] with an odd number of connections.[11]

Unfortunately, I was not able to identify either the *Euler circuit* or *Euler trail* for figure 7. I know that there exists an Euler tail because the criteria for an Euler trail are met, but I must assume *prima facie* that there exists no Euler circuit unless one can be shown — something that I was unable to do.

[10]Specifically, vertex b and vertex f.
[11]Both b and f have a degree of the odd numbered 3.

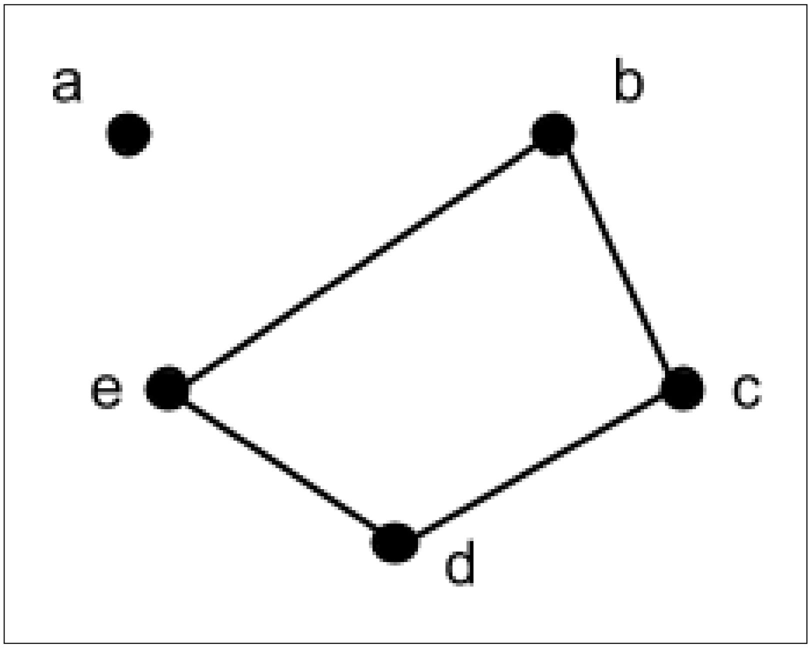
(c) 

Figure 8: *An undirected graph has 5 vertices, a through e. There are 4-line segments that are between the following vertices: b and c, b and e, c and d, d and e.*

My answer: There can be no Euler circuit nor an Euler trail for figure 8, as the vertex a is disconnected from the graph. Therefore, no Euler circuit, nor Euler trail, can be made that connects all vertices.

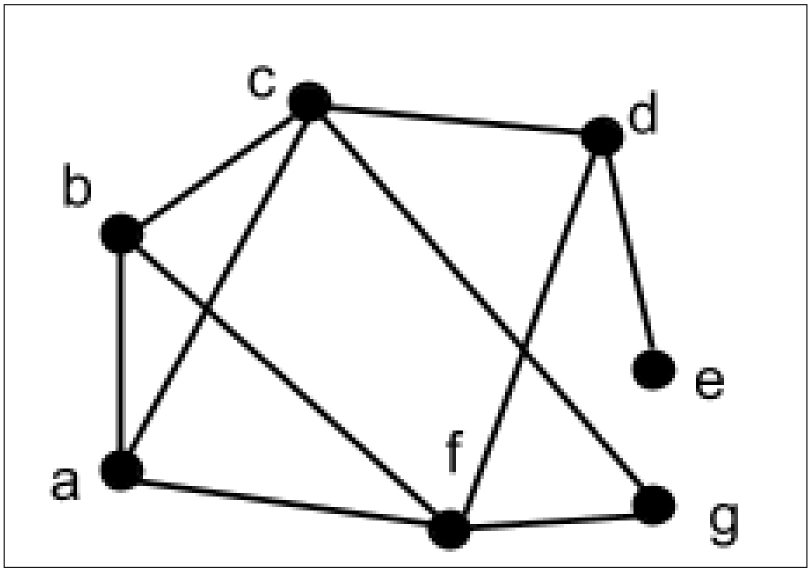
(d) 

Figure 9: *An undirected graph has 7 vertices, a through g. There are 10-line segments that are between the following vertices: a and b, a and c, a and f, b and c, b and f, c and d, c and g, d and e, d and f, f and g.*

My answer: The criteria for an *Euler circuit* and a *Euler trail* have previously been discussed in a previous variant of this question, so I will not repeat myself here. Unlike the previous variant of this problem, *there does exist an Euler trail because there are exactly two vertices with an odd number of connections*, but I was not able to identify it.

Nonetheless, I was able to identify an *Euler circuit.* It has the walk of:

$$W = <(e,(e,d)),(d,(d,f)),(f,(f,g)),(g,(g,c)),(c,(c,b)),(b,(b,a))>$$

Or, to put the evolution of the Euler circuit in a more intuitive manner:

$$e \to d \to f \to g \to c \to b \to a$$

Problem 5

Use Prim's algorithm to compute the minimum spanning tree for the weighted graph. Start the algorithm at vertex A. Explain and justify each step as you add an edge to the tree.

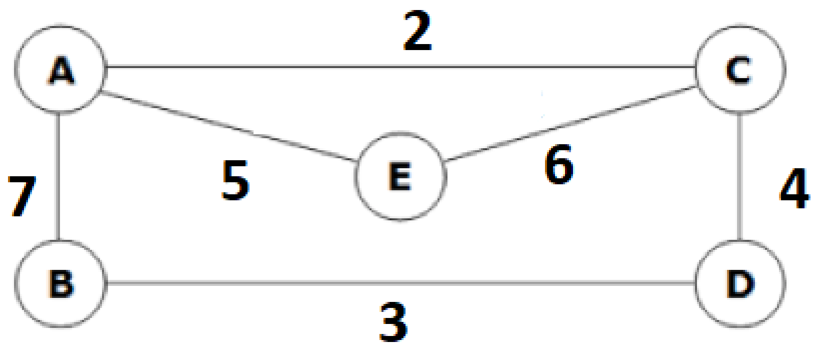


Figure 10: *A weighted graph shows 5 vertices, represented by circles, and 6 edges, represented by line segments. Vertices A, B, C, and D are placed at the corners of a rectangle, whereas vertex E is at the center of the rectangle. The edges, A B, B D, A C, C D, A E, and E C, have the weights, 7, 3, 2, 4, 5, and 6, respectively.*

My answer: *Prim's Algorithm* is a procedure for working out a "minimum spanning tree," or a kind of tree in which all a connection to all of the vertices are made with edges of the smallest possible weights. A formal description[12] of Prim's Algorithm is as follows:

```
1. Function Prim(Graph G, Node v1)
2.  Let T = an empty set
3.  Add v1 to T

4.  While G has not been fully traversed
5.    Let C = a list of edges that connect V to its respective neighbours
6.    Let min(C) = the edge with the smallest weight
7.    Add min(C) to T
8.    Traverse all nodes and repeat the process
        until all nodes have been added to T
9.  End While

10. Return T
11. End Function
```

[12] After zyBooks §6.12, fig. 6.12.13

I will set the initial $v1$ to node A. The following are the results of a manual execution of Prim's algorithm to the graph depicted in figure 10:

- Declare new Graph $T = \emptyset$
- Starting with node A, I am given the edges of (A, C), (A, B), and (A, E), with respective weights of 2, 7, and 5. From this, I will append edge (A, C) to the tree because it has the smallest weight. Now, $T = \{(A, C)\}$
- Going to node C, I have identified edges (C, D) and (C, E) in the graph, with respective weights of 4 and 6.[13] The (C, D) edge is of minimum weight, ergo should be appended to graph T. Now, $T = \{(A, C), (C, D)\}$
- Going to node D, the only edge is (D, B) with a weight of 3. Ergo, the only edge to add to graph T is (D, B). Now, $T = \{(A, C), (C, D), (D, B)\}$
- Finally, I need to find an edge that allows vertex E to be included in the final result of graph T. Faced with the alternatives edges of (A, E) of weight 5 and (E, C) of weight 6, I will append the edge (A, E) for its smaller weight. The resulting graph T is now:

$$T = \{(A, C), (C, D), (D, B), (A, E)\}$$

Working out the weight of this graph is done through the following formula:

$$\begin{aligned}\sum_{T_e \in \text{ T weights}} &= w((A, C)) + w((C, D)) + w((D, B)) + w((A, E)) \\ &= 2 + 4 + 3 + 7 + 5 = 21\end{aligned}$$

where $w(T_e)$ is a function that maps T_e to a weight, and T_e is a specific edge in T.

So, I can confidently say that the weight of the minimum spanning tree derived from the graph depicted in figure 10 is "21 units."

[13] I did not include (A, C) since that edge is already in T

Problem 6

A lake initially contains 1000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 10% each month. However, factoring in all causes, 80 fish are lost each month.

Give a recurrence relation for the population of fish after n months. How many fish are there after 5 months? If your fish model predicts a non-integer number of fish, round down to the next lower integer.

My answer: Let n_0 be the initial number of fish in the lake, r be the rate at which the population of fish increases, c be a constant by which the fish population decreases. A general recurrence relation model of this situation would be:

$$n_0 = \text{ initial population}$$
$$n_t = n_{t-1} + r \cdot n_{t-1} - c, \ \forall t \geq 1$$

Specifically, with an initial fish population of $n_0 = 1000$, growth rate of $r = 10\% = .1$ and a loss constant of $c = 80$, the recursion relation then becomes:

$$n_0 = 1000$$
$$n_t = n_{t-1} + .1n_{t-1} - 80, \ \forall t \geq 1$$

A "simulation" with this model gives the following estimate of fish up to $n_{t=5}$ generations:

(1) $n_0 = 1000$
(2) $n_1 = 1000 + (.1 \times 1000) - 80 = 1020$
(3) $n_2 = 1020 + (.1 \times 1020) - 80 = 1042$
(4) $n_3 = 1042 + (.1 \times 1042) - 80 = 1066.2 \approx 1066$
(5) $n_4 = 1066 + (.1 \times 1066) - 80 = 1090.2 \approx 1090$
(6) $n_5 = 1090 + (.1 \times 1090) - 80 = 1119$

From my recursive relational model, I predict that the $n_{t=5}$ population will be a total of 1119 fish. Of course, it is worth noting *that this is just a mathematical model* which makes simplified assumptions about population dynamics, and that Nature will likely produce a result *that is different to my model's predictions*.