A modulus pattern formula for working out the derivative, $f^{(w)}$, for sine and cosine functions given an arbitrarily large w

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Abstract

An elementary calculus problem asks the following: given a function $f(x) = k \cdot f_{\text{trig}}(ax + b)$, work out its w^{th}-order derivative:

$$\frac{d^w}{dx^w}\left[k \cdot f_{\text{trig}}(ax + b)\right]$$

For arbitrarily large w-values, I propose the *modulus pattern formula* as a solution: [TODO]

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1 Introduction

In *differential calculus*, one is sometimes faced with having to work out solutions to derivatives of a higher-order. Such a derivative, with an order w, may be arbitrarily large. In particular, I am interested in working out derivatives for trigonometric functions in the general form of:

$$\begin{aligned} f_1 &= k \cdot \sin(ax+b) \\ &\xrightarrow{\text{Additive Inverse}} \quad f_{-1} = -k \cdot \sin(ax+b) \\ f_2 &= k \cdot \cos(ax+b) \\ &\xrightarrow{\text{Additive Inverse}} \quad f_{-2} = -k \cdot \cos(ax+b) \end{aligned} \tag{1}$$

The kind of problem that is the subject of this paper is expressed in equation 2:

$$f^{(w)} = \frac{d^w}{dx^w}\Big[k \cdot f_{\text{trig}}(ax+b)\Big] \tag{2}$$

where a, b, and k are constants, f_{trig} can $=\sin(x)$, $\cos(x)$, $-\sin(x)$ or $-\cos(x)$, and w is an arbitrarily large integer ≥ 0.

The w could be of a magnitude in the thousands,[1] tens of thousands,[2] and of higher magnitudes where is upper limit is infinite. The basic method to work out derivatives of higher orders is through iteration: the $\dfrac{d^w f}{dx^w}$ is solved by applying the rules of limits and derivatives to the given $f(x)$, and then taking its solution and working out its derivative. This process is repeated until the w^{th} is worked out, and then the complete expression is finally worked.

However, when w becomes large enough, this kind basic method becomes economically infeasible, and new strategies, or set of "mathematical tricks,"

[1]Formally put, $w \sim a \cdot 10^4$ where $(\forall a \in \mathbb{Z}) : 1 \leq a < 10$.

[2]Formally put, $w \sim a \cdot 10^5$ where a is in the same bounds as the previous example.

that are less computationally expensive is called for. My proposed solution is to devise a "pattern formula" to reduce the w down to a smaller number that will give an equivalent, which can be easily solved with the method of iteration.

2 Finding a Pattern Formula

2.1 First and second derivatives of $k \cdot f_{\textbf{tan}}(ax+b)$

I will start by working out derivatives to the general form of trigonometric functions shown in equations 1 and 2. For the first and second derivatives, equation 3 gives solutions for the sine function, and equation 4 gives solutions for the cosine function.[3]

$$
\begin{aligned}
&\frac{d}{dx}\Big[k \cdot \sin(ax+b)\Big] \\
&= k \cdot \frac{d}{dx}\big[\sin(ax+b)\big] \because \frac{d}{dx}\big[c \cdot f(x)\big] = c \cdot \frac{d}{dx}\big[f(x)\big] \\
&= k \cdot \Bigg[\cos(ax+b) \cdot \frac{d}{dx}\big[ax+b\big]\Bigg] \because \frac{d}{dx}\big[\sin(f_{\text{in}})\big] = \cos(f_{\text{in}}) \cdot f'_{\text{in}}(x) \\
&= k \cdot \big[\cos(ax+b) \cdot a\big] \because \frac{d}{dx}\big[ax+b\big] = a \\
&= ak \cdot \cos(ax+b) \because \text{ Simplifications}
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&\frac{d^2}{dx^2}\Big[k \cdot \sin(ax+b)\Big] = \frac{d}{dx}\big[ak \cdot \cos(ax+b)\big] \\
&= -ka^2 \cdot \sin(x)
\end{aligned}
$$

[3]The reader may consult appendix A for (some of) the rules of differential calculus applied in equations 3 and 4 to work out their general solution.

$$
\begin{aligned}
&\frac{d}{dx}\Big[k \cdot \cos(ax+b)\Big] \\
&= k \cdot \frac{d}{dx}\big[\cos(ax+b)\big] \because \frac{d}{dx}\big[c \cdot f(x)\big] = c \cdot \frac{d}{dx}\big[f(x)\big] \\
&= k \cdot \Bigg[-\sin(ax+b) \cdot \frac{d}{dx}\big[ax+b\big]\Bigg] \\
&\quad \because \frac{d}{dx}\big[\cos(f_{\text{in}})\big] = -\sin(f_{\text{in}}) \cdot f'_{\text{in}}(x) \\
&= k\big[-\sin(ax+b) \cdot a\big] \because \frac{d}{dx}\big[ax+b\big] = a \\
&= -ka \cdot \sin(ax+b) \because \text{ Simplifications}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&\frac{d^2}{dx^2}\Big[k \cdot \cos(ax+b)\Big] = \frac{d}{dx}\big[-ka \cdot \sin(ax+b)\big] \\
&= -ka^2 \cdot \cos(ax+b)
\end{aligned}
$$

From this, the general procedure to work out the derivative of a function in the form $f(x) = k \cdot f_{\text{trig}}(ax+b)$ is as follows:

1. The following problem setup is given: $\frac{d}{dx}\big[k \cdot f_{\text{trig}}(ax+b)\big]$
2. Begin by extracting out the k constant from the derivative: $\frac{d}{dx}\big[k \cdot f_{\text{trig}}(ax+b)\big] \Rightarrow k \cdot \frac{d}{dx}\big[f_{\text{trig}}(ax+b)\big]$
3. Apply the chain rule to the $f_{\text{trig}}(ax+b)$ function: $k \cdot \frac{d}{dx}\big[f_{\text{trig}}(ax+b)\big] = k \cdot \left(\frac{df_{\text{trig}}}{d[ax+b]} \cdot \frac{d[ax+b]}{dx}\right)$
4. Solve the derivatives for f_{trig} and $ax+b$:

$$
k \cdot a \cdot \begin{cases} \cos(x) & \text{if } f_{\text{trig}} = \sin(x) \\ -\sin(x) & \text{if } f_{\text{trig}} = \cos(x) \\ -\cos(x) & \text{if } f_{\text{trig}} = -\sin(x) \\ \sin(x) & \text{if } f_{\text{trig}} = -\cos(x) \end{cases}
$$

Equations 3 and 4 demonstrates the solution to higher derivatives by iteration— which can serve as a basis for working out higher-order derivatives

$\mathbf{w}$	$\frac{d^w}{dx^w}\Big[k \cdot \sin(ax+b)\Big]$	$\frac{d^w}{dx^w}\Big[k \cdot \cos(ax+b)\Big]$
$0 \longrightarrow f(x)$	$= k \cdot \sin(ax+b)$	$= k \cdot \cos(ax+b)$
$1 \longrightarrow \frac{df}{dx}$	$= ka \cdot \cos(ax+b)$	$= -ka \cdot \sin(ax+b)$
$2 \longrightarrow \frac{d^2 f}{dx^2}$	$= -ka^2 \cdot \sin(ax+b)$	$= -ka^2 \cdot \cos(ax+b)$
$3 \longrightarrow \frac{d^3 f}{dx^3}$	$= -ka^3 \cdot \cos(ax+b)$	$= ka^3 \cdot \sin(ax+b)$
$4 \longrightarrow \frac{d^4 f}{dx^4}$	$= ka^4 \cdot \sin(ax+b)$	$= ka^4 \cdot \cos(ax+b)$
$5 \longrightarrow \frac{d^5 f}{dx^5}$	$= ka^5 \cdot \cos(ax+b)$	$= -ka^5 \cdot \sin(ax+b)$
$6 \longrightarrow \frac{d^6 f}{dx^6}$	$= -ka^6 \cdot \sin(ax+b)$	$= -ka^6 \cdot \cos(ax+b)$
$7 \longrightarrow \frac{d^7 f}{dx^7}$	$= -ka^7 \cdot \cos(ax+b)$	$= ka^7 \cdot \sin(ax+b)$
$8 \longrightarrow \frac{d^8 f}{dx^8}$	$= ka^8 \cdot \sin(ax+b)$	$= ka^8 \cdot \cos(ax+b)$
$9 \longrightarrow \frac{d^9 f}{dx^9}$	$= ka^9 \cdot \cos(ax+b)$	$= -ka^9 \cdot \sin(ax+b)$
$10 \longrightarrow \frac{d^{10} f}{dx^{10}}$	$= -ka^{10} \cdot \sin(ax+b)$	$= -ka^{10} \cdot \cos(ax+b)$
$11 \longrightarrow \frac{d^{11} f}{dx^{11}}$	$= -ka^{11} \cdot \cos(ax+b)$	$= ka^{11} \cdot \sin(ax+b)$
$12 \longrightarrow \frac{d^{12} f}{dx^{12}}$	$= ka^{12} \cdot \sin(ax+b)$	$= ka^{12} \cdot \cos(ax+b)$

Table 1: Higher derivatives w from orders $0 \leq w \leq 12$.

($\forall w > 2$) by applying the general procedure described above to solve for derivatives, taking the resulting expression, and repeating the process w times with the resulting expression as an input to the derivative operator.

Another thing worth noting is that doing higher derivatives by hand, given an arbitrary large w, will allow one to behold the immense amount of tedious effort in "clerical mathematics" that is expended to work out derivatives of a higher order, and perhaps desire a technique by which to escape the unnecessary mental labour.

2.2 Patterns in higher derivatives of the given expression

Equations 3 and 4 demonstrate examples of calculating derivatives at a higher order for $w = \{1, 2\}$, with the method's details laid out with something of a "pseudo-proof table." Mathematics is about discovering patterns, and the objective of this paper is to exploit patterns in the solutions of w^{th}-order higher derivatives to work out a "pattern formula" by which to reduce the iterations needed to work out derivatives of a large w-order.

To find a pattern, I have calculated higher derivatives of the given $f_1(x)$ and $f_2(x)$ formulæ as shown in equation 1 for w-orders from 0 to 12, and stored their results in table 2.1— and I have observed the following in all of the results as the w-order increases:

- k is always multiplied by f_{trig}.
- a in the slope expression inside the f_{trig} is multiplied by f_{trig}, and is equal to: a^w.
- $f_{\text{trig}}(x)$ is always parameterised by the expression: $ax + b$.
- Starting from $\sin(x)$, the pattern is as follows:

$$f_{\text{trig}} = \sin(x) \xrightarrow{\frac{d}{dx}} \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x) \xrightarrow{\frac{d}{dx}} -\cos(x) \xrightarrow{\frac{d}{dx}} \sin(x)$$

- Starting from $\cos(x)$, the pattern is as follows:

$$f_{\text{trig}} = \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x) \xrightarrow{\frac{d}{dx}} -\cos(x) \xrightarrow{\frac{d}{dx}} \sin(x) \xrightarrow{\frac{d}{dx}} \cos(x)$$

Regarding the last two observations, I can posit the following piecewise function that takes the derivative of f_{trig} based on how what trigonometric function that it is set to:

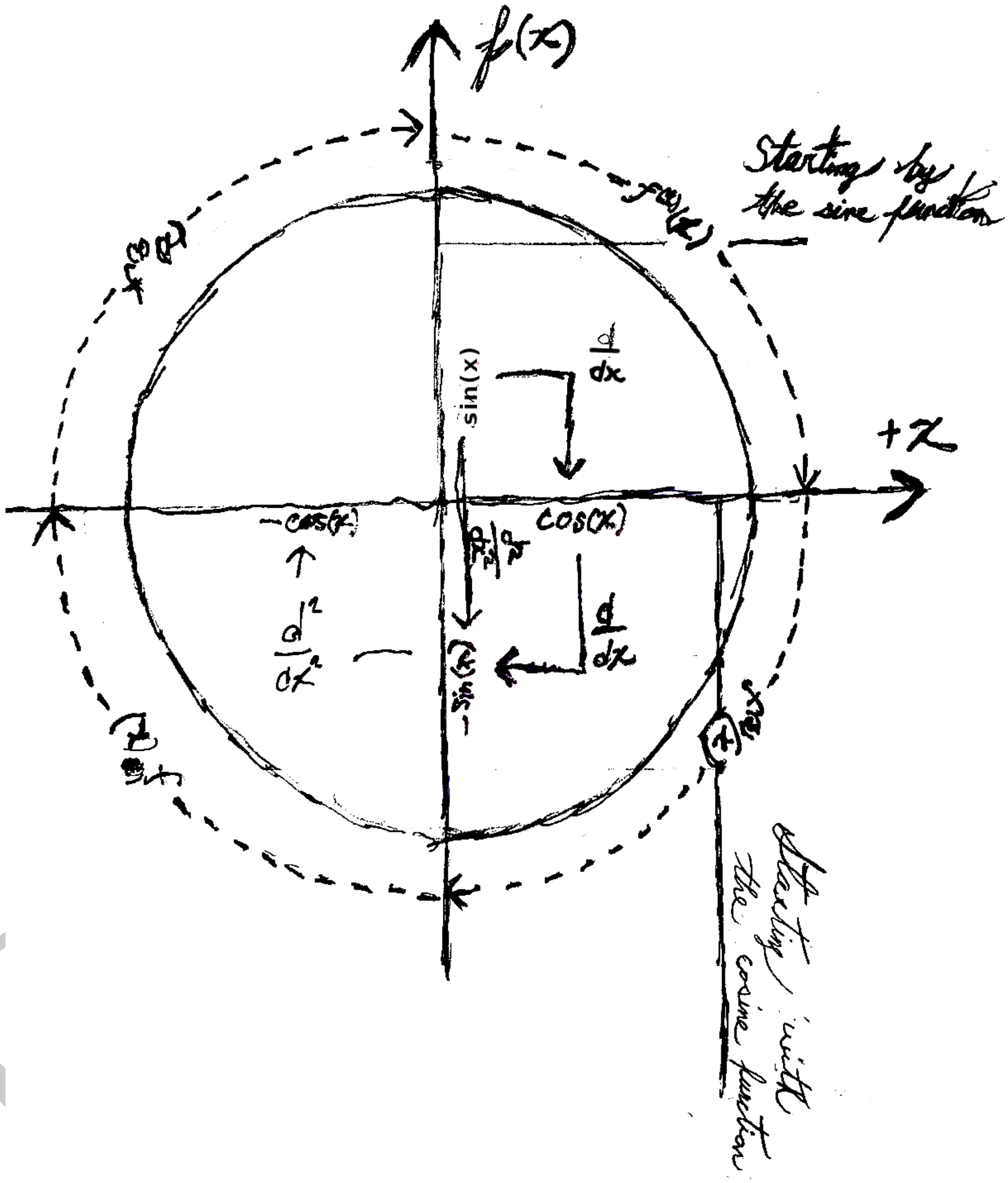

Figure 1: The “clock” to visualise for higher derivatives of $\sin(x)$ and $\cos(x)$.

$$f'_{\text{trig}} = \begin{cases} \cos(x) & \text{if } f_{\text{trig}} = \sin(x) \\ -\sin(x) & \text{if } f_{\text{trig}} = \cos(x) \\ -\cos(x) & \text{if } f_{\text{trig}} = -\sin(x) \\ \sin(x) & \text{if } f_{\text{trig}} = -\cos(x) \end{cases} \tag{5}$$

2.3 Intuition and a "clockwork analogy"

3 The Proposed Modulus Pattern Formula

The strategy is to reduce the arbitrarily large w down to a lesser value so that it can be evaluated. The pattern formula, which is depicted in table 2, depicts a possible solution.

Table 2: Proposed $f^{(w)}$ Pattern Formula

Given the general form: $f(x) = k \cdot f_{\text{trig}}(ax + b)$ and the given problem setup of: $f^{(w)} = \frac{d^w f}{dx^w}$, the proposed pattern formula is given as:

$$\frac{d^w f}{dx^w} = \begin{cases} ka^w \cdot f_{\text{trig}}(ax + b) & \text{if } m = 0 \\ ka^{w-m(w)} \cdot \frac{d^{m(w)} \ [f_{\text{trig}}(ax+b)]}{dx^{m(w)}} & \text{if } m \neq 0 \end{cases}$$

where a, b and k are constants, $f_{\text{trig}} = \sin(x)$ or $= \cos(x)$, w is an integer ≥ 0, and $m(w) = w \bmod 3$.

The procedure to apply the pattern formula to a derivative problem of the form $\frac{d^w f}{dx^w} = k \cdot f_{\text{trig}}(ax + b)$ is as follows:

1. Start by identifying $f_{\text{trig}}(x)$, which will be either the sine variant: $f(x) = k \cdot \sin(ax + b)$, or the cosine variant: $f(x) = k \cdot \cos(ax + b)$.

2. Proceed by identifying the a, b, and k constants, and the w-order, or the amount of iterations to apply the derivative to the given expression.

3. Compute the numeric solution to the $m(w)$ formula:

$$m(w) = w \text{ mod } 3$$

If $m(w) = 0$, then substitute the values for a, b, k, and w into the following expression:

$$ka^w \cdot f_{\text{trig}}(ax + b)$$

where $f_{\text{trig}} = \sin(x)$ or $f_{\text{trig}} = \cos(x)$.

4. If, it is instead the case that $m(w) \neq 0$, then substitute the values for a, b, k, and $m(w)$ into the following expression:

$$ka^w \cdot \frac{d^{[w \text{ mod } 3]} \; f_{\text{trig}}}{dx^{[w \text{ mod } 3]}}$$

where $f_{\text{trig}} = \sin(x)$ or $f_{\text{trig}} = \cos(x)$. Perform the iteration of the derivative operation on the $f_{\text{trig}}(ax + b)$ expression $m(w)$ times.

5. For the case of $m(w) = 0$, the final worked derivative is:

$$\frac{d^w f}{dx^w} = ka^w \cdot \begin{cases} \sin(ax + b) & \text{if } \; f_{\text{trig}} = \sin(x) \\ \cos(ax + b) & \text{if } \; f_{\text{trig}} = \cos(x) \end{cases} \tag{6}$$

and for the case that `w mod 3` $\neq$ `0`, the final worked derivative is:

$$\frac{d^w f}{dx^w} = ka^w \cdot f_{\text{trig}}^{(w \text{ mod } 3)} \tag{7}$$

where $f_{\text{trig}}^{w-m} = \dfrac{d^m \; f_{\text{trig}}}{dx^m}$

3.1 Example 1: $\dfrac{d^w}{dx^w}\big[k \cdot \sin(ax+42)\big]$

3.2 Example 2: $\dfrac{d^w}{dx^w}\big[k \cdot \cos(ax+b)\big]$

4 Reliability of the Modulus Pattern Formula

5 Discussion

5.1 On Conjectures and Theorems

A Some rules of differential calculus

The following formulæ are (some of) the solutions to the derivatives of common algebraic expressions:

- The "constant rule," $\dfrac{d}{dx}\big[c \cdot f(x)\big] = c \cdot \dfrac{d}{dx}\big[f(x)\big]$
- The derivative of $\sin(x)$, $\dfrac{d}{dx}\big[\sin(x)\big] = \cos(x)$
- The derivative of $\cos(x)$, $\dfrac{d}{dx}\big[\cos(x)\big] = -\sin(x)$
- The derivative of $-\sin(x)$, $\dfrac{d}{dx}\big[-\sin(x)\big] = -\cos(x)$
- The derivative of $-\cos(x)$, $\dfrac{d}{dx}\big[-\cos(x)\big] = \sin(x)$
- The "chain rule," $\dfrac{d}{dx}\big[f_{\text{outer}}(x) \circ f_{\text{inner}}(x)\big] = \dfrac{df_{\text{outer}}}{df_{\text{inner}}} \cdot \dfrac{df_{\text{inner}}}{dx}$
- The derivative of a linear function, $\dfrac{d}{dx}\big[ax+b\big] = a$

When solving problems involving the derivative, the mathematics student learns these given rules, and to apply them to a given expression or formula. Some "creative thinking" is needed to know which rules are applied to transform the expression, and in what order to apply them.