The University of Sydney

MOOC Introduction to Calculus

Notes for 'Algebraic expressions, surds and approximations'

Important Ideas and Useful Facts:

(i) Differences of two squares formula: In any well-behaved arithmetic we have the identity

$$a^2 - b^2 \; = \; (a+b)(a-b) \, .$$

known as the *difference of two squares formula*. It is exploited, for example, in the next important idea (rationalising the denominator).

(ii) Rationalising the denominator: If $a \in \mathbb{Z}$, $b \in \mathbb{N}$ and $a \pm \sqrt{b} \neq 0$ then

$$\frac{1}{a \pm \sqrt{b}} \; = \; \left(\frac{1}{a \pm \sqrt{b}}\right)\left(\frac{a \mp \sqrt{b}}{a \mp \sqrt{b}}\right) \; = \; \frac{a \mp \sqrt{b}}{a^2 - b} \, .$$

For example,

$$\frac{1}{1+\sqrt{5}} \; = \; \left(\frac{1}{1+\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right) \; = \; \frac{1-\sqrt{5}}{1-5} \; = \; \frac{\sqrt{5}-1}{4} \, ,$$

and

$$\frac{1}{3-\sqrt{7}} \; = \; \left(\frac{1}{3-\sqrt{7}}\right)\left(\frac{3+\sqrt{7}}{3+\sqrt{7}}\right) \; = \; \frac{3+\sqrt{7}}{9-7} \; = \; \frac{3+\sqrt{7}}{2} \, .$$

It is a fact, discussed below, that if $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, such that b is not a perfect square, then $a \pm \sqrt{b}$ is irrational. The terminology *rationalising the denominator* comes about, as in the examples above, where the original fraction has an irrational number in the denominator. The fraction is then transformed by this technique, so that the denominator becomes *rational* (an integer, in fact, in each of these examples).

(iii) Continued fractions: An expression obtained by taking a number and adding a reciprocal, where the denominator itself is another expression involving a number added to a reciprocal, and allowing this process to repeat, is called a *continued fraction*.

If the entire expression is just a reciprocal, but the denominator can be expressed symbolically in terms of the entire expression, then the expression can be fed into itself indefinitely, to obtain an *infinite continued fraction*. For example, the expression

$$X \; = \; \frac{1}{1+X}$$

leads to the infinite continued fraction

$$X \; = \; \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

By truncating such an expression at some place, one obtains a *finite continued fraction*, which then evaluates to an ordinary fraction.

Examples and proofs:

1. Consider the infinite continued fraction mentioned above:

$$X = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}$$

If we replace the three dots in the above expression by zero, we get the following finite continued fraction Y, which is a rough approximation to X:

$$X \approx Y = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+0}}}}$$

We can evaluate Y by evaluating the expression "inside-out" as follows:

$$Y = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+0}}}} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}$$

$$= \cfrac{1}{1+\cfrac{1}{3/2}} = \cfrac{1}{1+\cfrac{2}{3}} = \frac{1}{5/3} = \frac{3}{5}\,.$$

It is indeed a fact, by no means obvious, that X represents a real number (the reciprocal of the *golden ratio*, explored in a future video). To see that this expression can be evaluated in a sensible way involves the *theory of limits*, introduced in **Module 3**. The proof that there is convergence to the reciprocal of the golden ratio, or indeed any real number at all, is an advanced argument involving the *theory of convergent sequences*, which is beyond the scope of this course.

2. We explain why $\alpha = a + \sqrt{b}$ is irrational if a is any integer and b is a natural number that is not a perfect square. We mentioned in earlier notes that $\sqrt{b}$ is irrational.

Suppose, by way of contradiction, that α is rational, say $\alpha = \frac{m}{n}$ where m and n are integers. Then

$$a + \sqrt{b} = \frac{m}{n}\,,$$

so that

$$\sqrt{b} = \frac{m}{n} - a = \frac{m - an}{n}\,.$$

But this expression on the right is a ratio of two integers, $m - an$ and n, expressing $\sqrt{b}$ as a fraction, which is impossible. Hence α is irrational.

An almost identical argument shows that $a - \sqrt{b}$ is also irrational.