Notes to accompany 'The Theorem of Pythagoras and properties of $\sqrt{2}$'

Important Ideas and Useful Facts:

(i) **Square roots**: If $\alpha \in \mathbb{R}$ then $\sqrt{\alpha} = \beta$ if and only if $\beta \geq 0$ and $\beta^2 = \alpha$. In particular, $\sqrt{0} = 0$. If $\alpha > 0$ then we call $\sqrt{\alpha}$ the *positive square root* and $-\sqrt{\alpha}$ the *negative square root* of α. For example, 2 is the positive square root of 4, and -2 is the negative square root of 4.

We also write $\alpha^{\frac{1}{2}}$ for $\sqrt{\alpha}$, and note the following laws hold, where $\alpha, \beta \geq 0$ and $\gamma > 0$:

$$\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}} = \sqrt{\alpha}\sqrt{\beta} = \sqrt{\alpha\beta} = (\alpha\beta)^{\frac{1}{2}}$$

and

$$\frac{\alpha^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} = \frac{\sqrt{\alpha}}{\sqrt{\gamma}} = \sqrt{\frac{\alpha}{\gamma}} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}} .$$

Negative real numbers do not have square roots within the real number system $\mathbb{R}$ (though they do have square roots in the larger *complex number* system $\mathbb{C}$).

(ii) **Surds**: Any expression involving a square root symbol $\sqrt{}$ is called a *surd expression*. For example,

$$\sqrt{2}, \quad 2\sqrt{3}, \quad \sqrt{2} + \sqrt{3}, \quad 4 - \sqrt{2}, \quad \frac{1}{2 + \sqrt{5}}, \quad \sqrt{a^2 + b^2}, \quad \frac{\sqrt{a + \sqrt{b + 2}}}{2\sqrt{c}}$$

are all surd expressions. (More generally, surd expressions can involve other types of roots, such as cube roots, or tenth roots, but simple square roots are the most common.)

(iii) **The Distributive Law (expanding brackets)**: If $a, b, c \in \mathbb{R}$ then

$$(a + b)c = ac + bc \qquad \text{and} \qquad a(b + c) = ab + ac .$$

It follows that, if $a, b, c, d \in \mathbb{R}$, then

$$(a + b)(c + d) = ac + ad + bc + bd .$$

(iv) **Squares and differences of squares**: As a consequence of distributivity, we have the following identities, for all $a, b, \in \mathbb{R}$:

$$(a + b)^2 = a^2 + 2ab + b^2 , \quad (a - b)^2 = a^2 - 2ab + b^2 , \quad a^2 - b^2 = (a + b)(a - b) .$$

The last identity is called the *difference of two squares formula*, and it features in many algebraic tricks or manipulations. For example,

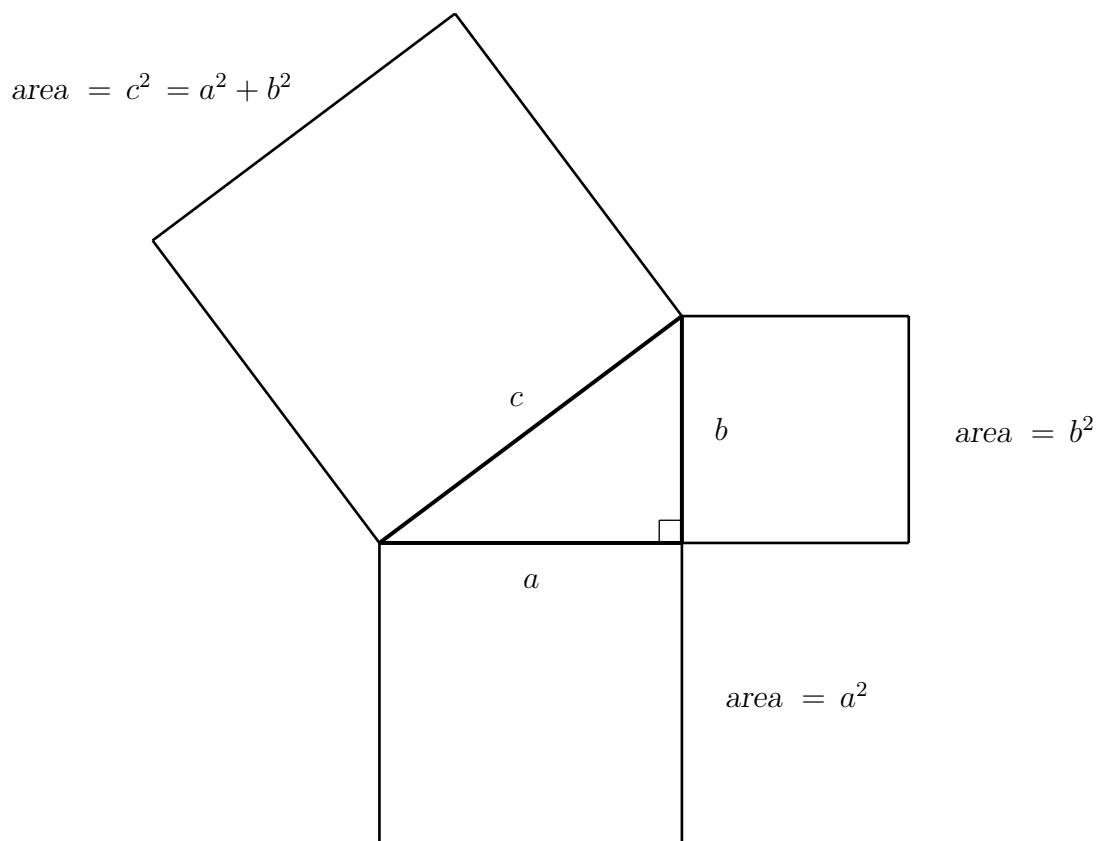$$77 \times 83 = (80 - 3)(80 + 3) = 80^2 - 3^2 = 6400 - 9 = 6391 ,$$

and, for any positive reals a and b,

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b .$$

(v) **Theorem of Pythagoras**: If a and b are the lengths of the shorter sides of a right angled triangle, and c is the length of the hypotenuse (the longest side), then

$$a^2 + b^2 = c^2 ,$$

so that $c = \sqrt{a^2 + b^2}$, $a = \sqrt{c^2 - b^2}$ and $b = \sqrt{c^2 - a^2}$.

area $= c^2 = a^2 + b^2$

c

b

area $= b^2$

a

area $= a^2$

(vi) **Irrational surds**: If $n \in \mathbb{N}$ then $\sqrt{n}$ is rational if and only if n is a perfect square, that is $n = m^2$ for some $m \in \mathbb{N}$. Thus, for example, $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and $\sqrt{6}$ are irrational.

Examples and proofs:

1. The decimal expansion of the square root of two begins as follows (if you type it into your calculator):

$$\sqrt{2} = 1.414213562\ldots$$

In fact, there is no point at which the infinite decimal expansion becomes recurring, because $\sqrt{2}$ turns out to be irrational, that is, cannot be expressed as a fraction or ratio of two integers. Here is a proof of that fact, which can easily be adapted to prove that the square root of any positive integer that is not a perfect square is also irrational (such as $\sqrt{3}, \sqrt{5}$ and $\sqrt{6}$).

Proof that the square root of two is irrational:

We argue by contradiction, supposing to the contrary that $\sqrt{2}$ is a fraction, and looking towards manufacturing impossible nonsense. (This is analogous to methods used by a detective in trying to eliminate suspects in a criminal investigation.)

We suppose that
$$\sqrt{2} = \frac{a}{b},$$
where a and b are positive integers. We may suppose further that all common factors in the numerator and denominator have been cancelled out. Hence a and b have no common divisors. (This is a crucial property of the fraction that we need later to obtain a contradiction.)

Squaring both sides gives
$$2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2},$$
so that, multiplying through by b^2, we get
$$2b^2 = a^2.$$

Hence we conclude that a^2 is an even integer (twice an integer).

If a turns out to be an odd integer, say one more than $2k$, for some integer k, then
$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$
which is also odd, contradicting that a^2 is even. (No integer can be both even and odd.)

Hence a must also be an even integer, say $a = 2c$, for some integer c. But then
$$2b^2 = a^2 = (2c)^2 = 4c^2,$$
so that, dividing through by 2,
$$b^2 = 2c^2.$$

Hence b^2 is even, so, by the same argument as before, b must also be even. But then both a and b have 2 as a common factor. This contradicts that a and b have no common divisors, achieving nonsense, and completing the proof that $\sqrt{2}$ is irrational.

2. We give a proof of the Theorem of Pythagoras, by adapting the argument implied by the diagram on the Yale Tablet (which essentially proves the Theorem of Pythagoras in the case where the right-angled triangle is isosceles, see Beery and Swetz (2012)).

Suppose we have a right-angled triangle with shorter side lengths a and b and hypotenuse c. We may suppose that $a \geq b$, though in the diagram below $a > b$.
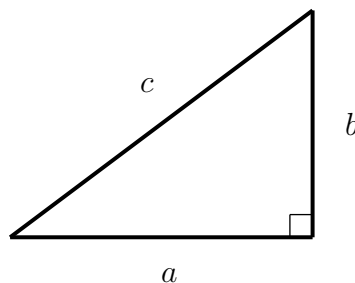
(If you follow through the following argument when $a = b$ then you get the argument essentially presented in the video when $a = b = 1$.)

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Note that the area of this triangle is half of the area of a rectangle of side lengths a and b, that is, $\frac{ab}{2}$.

We can make copies of the triangle after rotating it three times, ninety degrees about the centre of a middle square of side-length $a - b$, as in the following diagram. (In the isosceles case this middle square beomes just a single point.)
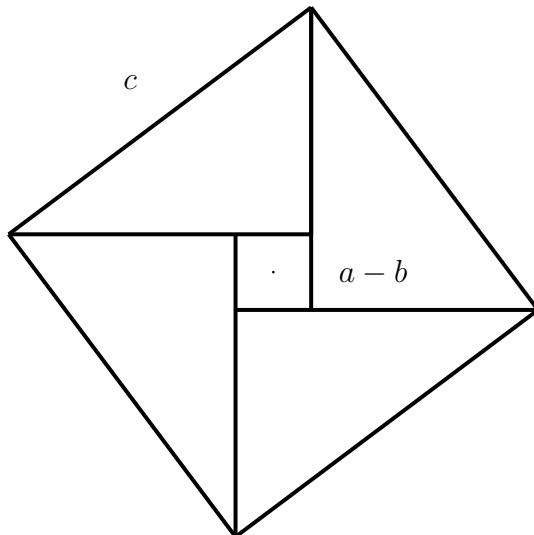
The larger square has area c^2. Recall that the original triangle has area $\frac{ab}{2}$. Also, the middle square has area $(a - b)^2$. Adding everything up gives

$$c^2 = 4\left(\frac{ab}{2}\right) + (a - b)^2 = 2ab + a^2 - 2ab + b^2 = a^2 + b^2 \, ,$$

completing the proof of the Theorem of Pythagoras.

Reference: Beery, Janet L. and Swetz, Frank J., "The best known old Babylonian tablet?", *Convergence*, Mathematical Association of America, July 2012:

https://www.maa.org/press/periodicals/convergence/the-best-known-old-babylonian-tablet