GENERAL TOPOLOGY
EXERCISES FOR SESSION 11 (TUE 1.4)

X is a topological vector space in all of the exercises.

Exercise 1. *Let $x_0 \in X$. Show that the single translation map $T_{x_0} : X \to X$ given by $T_{x_0}(x) = x + x_0$ is continuous (and also a homeomorphism).*

Exercise 2. *Prove that if $U \subset X$ is open, then $U + A$ is open for any $A \subset X$.*

Exercise 3. *Prove that if $A, B \subset X$, then $\overline{A} + \overline{B} \subset \overline{A + B}$.*

Exercise 4. *Give an example of closed sets $A, B \subset \mathbb{R}^2$ such that $A+B$ is not closed.*

Exercise 5. *A subset $A \subset X$ is called **balanced** if $\lambda A \subset A$ whenever $|\lambda| \leq 1$. Prove that for every neighbourhood U of 0, there is a balanced neighbourhood V of 0 such that $V \subset U$.*

Hint: Use the continuity of scalar multiplication $\mathcal{M} : \mathbb{R} \times X \to X$.

Exercise 6. *Prove that $\overline{\{0\}}$ is a linear subspace of X.*

Exercise 7. *We say that a subset $A \subset X$ is **bounded** (in the topological vector space sense) if, for every neighbourhood V of 0, there is $\lambda > 0$ such that $A \subset \lambda V$.*

Prove that every compact subset of X is bounded.

Exercise 8. *Let $F : X \to \mathbb{R}$ be a continuous linear functional. Show that if $A \subset X$ is bounded (in the topological vector space sense), then $F(A)$ is bounded in $\mathbb{R}$ (in the usual sense).*

Exercise 9. *Let $T : X \to Y$ be a linear function between two topological vector spaces. Show that T is continuous if and only if it is continuous at 0.*

Exercise 10. *Let X be a normed space, in which case X^* is also a topological vector space with the topology given by the operator norm $||F|| := \sup_{||x|| \leq 1} |F(x)|$. Which topology is finer on X^*, the weak*-topology or the weak topology (of X^*, not X)?*