GENERAL TOPOLOGY
EXERCISES FOR SESSION 1 (TUE 27.2)

Exercise 1. *Let $X = \mathbb{R}$ and let $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. Prove that $\mathcal{T}$ is a topology.*

Exercise 2. *Find a collection of open sets in $\mathbb{R}^2$ (in the standard metric topology) whose intersection is not open.*

Exercise 3. *Let $\mathbb{R}$ be equipped with the discrete topology. Find the boundary of the open interval $(0, 1)$. How does the answer change if we use the mini-topology $\mathcal{T}_{mini} = \{\emptyset, \mathbb{R}\}$ instead?*

Exercise 4. *Suppose that $B \subset A \subset X$, and B is an open set with $\operatorname{int} A \subset B$. Show that $B = \operatorname{int} A$.*

Exercise 5. *Find the interior and boundary of the following sets in the standard metric topology of $\mathbb{R}^2$:*

(1) $\{(x, y) : y = 0\}$.
(2) $\{(x, y) : x > 0$ *and* $y \neq 0\}$.
(3) $\{(x, y) : x \in \mathbb{Q}\}$.
(4) $\{(x, y) : 0 < x^2 + y^2 \leq 1\}$.
(5) $\{(x, y) : x, y \in \mathbb{Q}, x > y\}$.

Exercise 6. *Does it always hold that $\operatorname{int}(A \cup B) = \operatorname{int} A \cup \operatorname{int} B$? If not, what about an inclusion $\subset$ in either direction?*

Exercise 7. *Let $A \subset B \subset X$. Suppose that B is closed and A is a closed set in B (with the relative topology). Show that A is closed in X.*

Exercise 8. *Show that the relative topology on $\mathbb{Z}$ as a subset of $\mathbb{R}$ is the discrete topology.*

Exercise 9. *Let $A, B \subset X$ be two subsets. What is the coarsest topology on X which contains both the sets A and B?*

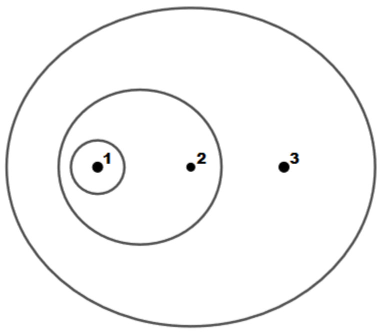
Exercise 10. *Let $A \subset B \subset X$. We denote by $\operatorname{int} A$ the interior of A as a subset of X, and by $\operatorname{int}_B A$ the interior of A in the relative topology on B. Show that $\operatorname{int} A \subset \operatorname{int}_B A$.*

Exercise 11. *Let $X = \mathbb{R}$ and $\mathcal{T}_{co} = \{\emptyset\} \cup \{A \subset \mathbb{R} : \mathbb{R} \setminus A$ is countable.$\}$. Is $\mathcal{T}_{co}$ a topology on $\mathbb{R}$?*

Exercise 12. *Suppose that X is an infinite set with a topology in which every infinite set is open. Show that X has the discrete topology.*

Exercise 13. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology. What about $\mathcal{T}_1 \cup \mathcal{T}_2$?*

Exercise 14. *Let $X = \{1, 2, 3\}$. We have drawn one topology $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, X\}$ for X in the attached picture:*



How many topologies of X exist in total?