

GENERAL TOPOLOGY

EXERCISES FOR SESSION 2 (WED 28.2)

Exercise 1. Prove that $\bar{A} = \bigcap \{F : A \subset F \text{ and } F \text{ is closed}\}$.

Exercise 2. Determine the closure of the set $\{0\}$ in the topology of \mathbb{R} given by $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$.

Exercise 3. Prove Lemma 1.3 in the lecture notes.

Exercise 4. Provide an example where $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Exercise 5. Prove that $\bar{A} = A \cup \text{acc } A$.

Exercise 6. Prove Theorem 1.4 in the lecture notes.

Exercise 7. Let \mathcal{B} and \mathcal{B}' be bases for two topologies \mathcal{T} and \mathcal{T}' on X . Prove that \mathcal{T}' is finer than \mathcal{T} if and only if the following holds:

For each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Exercise 8. Show that $\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{Q}\}$ provides a basis for the standard topology on \mathbb{R} .

Exercise 9. Given a collection \mathcal{S} of subsets of X . Let

$$\mathcal{T} = \bigcap \{\mathcal{T}' : \mathcal{S} \subset \mathcal{T}' \text{ and } \mathcal{T}' \text{ is a topology of } X\}.$$

Check that this defines a topology \mathcal{T} on X .

Exercise 10. Let $A \subset X$. Prove that

- $\partial\partial A \subset \partial A$.
- $\partial\partial A = \partial A$ if A is closed.
- $\partial\partial\partial A = \partial\partial A$.

Exercise 11. Let us consider the following collection of subsets of the real line, $\mathcal{B} = \{\mathbb{R}\} \cup \{\mathbb{Q} \cap (a, b) : a, b \in \mathbb{R}\}$. Show that \mathcal{B} is a basis for a topology on \mathbb{R} . What is the interior and boundary of $(0, 1)$ in this topology?

Exercise 12. Let us look at the subsets of \mathbb{R}^2 which are open half-spaces. The precise definition of an open half-space is a set of the form

$$A = \{(x, y) \in \mathbb{R}^2 : ax + by > c\} \quad \text{with} \quad a, b, c \in \mathbb{R}, a \neq 0 \text{ or } b \neq 0.$$

What topology is defined by taking the open half-spaces as a subbasis on \mathbb{R}^2 ? Geometric reasoning is fine here. What about if we use closed half-spaces instead?