GENERAL TOPOLOGY
EXERCISES FOR SESSION 4 (WED 6.3)

Exercise 1. *Show that any continuous function $f : X \to Y$ is sequentially continuous.*

Exercise 2. *Suppose that $\mathcal{B}$ is a basis for the topology on Y and $f : X \to Y$ is such that $f^{-1}(B)$ is open for every $B \in Y$. Prove that f is continuous.*

Exercise 3. *Let $\chi_A : X \to \mathbb{R}$ denote the **characteristic function** of a set $A \subset X$, meaning that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Show that χ_A is discontinuous at a point x if and only if $x \in \partial A$.*

Exercise 4. *Let $f, g : X \to Y$ be continuous and Y be Hausdorff.*

- *Show that the set $\{x \in X : f(x) = y_0\}$ is closed for all $y_0 \in Y$.*
- *Show that the set $\{x \in X : f(x) = g(x)\}$ is closed.*

Exercise 5. *Show that $f : X \to Y$ is continuous if and only if for every $x \in X$ and $A \subset X$ such that $x \in \overline{A}$, we have that $f(x) \in \overline{f(A)}$.*

Exercise 6. *Come up with*

- *A continuous map which is not open.*
- *A continuous open map which is not closed, and vice versa.*

Exercise 7. *Find a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at exactly one point.*

Exercise 8. *Show that $\mathbb{R}$ is homeomorphic to its own subspace $(0, 1)$.*

Exercise 9. *Show that the pointwise limit of continuous functions does not need to be continuous (in contrast to the Uniform Limit Theorem).*

Exercise 10. *Let us say that $f : X \to \mathbb{R}$ is **lower semicontinuous** if the following holds: For every $a \in X$ and every $\epsilon > 0$ there exists a neighbourhood U of a such that $f(x) > f(a) - \epsilon$ for all $x \in U$. Find a topology $\mathcal{T}$ on $\mathbb{R}$ such that $f : X \to \mathbb{R}$ is continuous w.r.t. the topology $\mathcal{T}$ if and only if it is lower semicontinuous.*