GENERAL TOPOLOGY
EXERCISES FOR SESSION 1 (TUE 27.2)

Exercise 1. *Let* $X = \mathbb{R}$ *and let* $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. *Prove that* $\mathcal{T}$ *is a topology.*

Exercise 2. *Find a collection of open sets in* $\mathbb{R}^2$ *(in the standard metric topology) whose intersection is not open.*

Exercise 3. *Let* $\mathbb{R}$ *be equipped with the discrete topology. Find the boundary of the open interval* $(0, 1)$. *How does the answer change if we use the mini-topology* $\mathcal{T}_{mini} = \{\emptyset, \mathbb{R}\}$ *instead?*

Exercise 4. *Suppose that* $B \subset A \subset X$, *and* B *is an open set with* $\operatorname{int} A \subset B$. *Show that* $B = \operatorname{int} A$.

Exercise 5. *Find the interior and boundary of the following sets in the standard metric topology of* $\mathbb{R}^2$*:*

(1) $\{(x, y) : y = 0\}$.
(2) $\{(x, y) : x > 0 \text{ and } y \neq 0\}$.
(3) $\{(x, y) : x \in \mathbb{Q}\}$.
(4) $\{(x, y) : 0 < x^2 + y^2 \leq 1\}$.
(5) $\{(x, y) : x, y \in \mathbb{Q}, x > y\}$.

Exercise 6. *Does it always hold that* $\operatorname{int}(A \cup B) = \operatorname{int} A \cup \operatorname{int} B$*? If not, what about an inclusion* $\subset$ *in either direction?*

Exercise 7. *Let* $A \subset B \subset X$. *Suppose that* B *is closed and* A *is a closed set in* B *(with the relative topology). Show that* A *is closed in* X.

Exercise 8. *Show that the relative topology on* $\mathbb{Z}$ *as a subset of* $\mathbb{R}$ *is the discrete topology.*

Exercise 9. *Let* $A, B \subset X$ *be two subsets. What is the coarsest topology on* X *which contains both the sets* A *and* B*?*

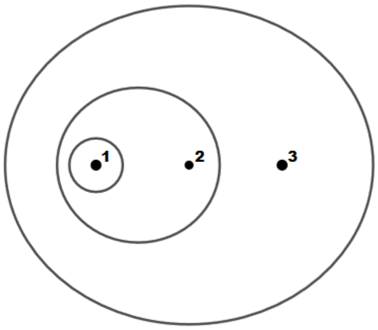
Exercise 10. *Let* $A \subset B \subset X$. *We denote by* $\operatorname{int} A$ *the interior of* A *as a subset of* X, *and by* $\operatorname{int}_B A$ *the interior of* A *in the relative topology on* B. *Show that* $\operatorname{int} A \subset \operatorname{int}_B A$.

Exercise 11. *Let* $X = \mathbb{R}$ *and* $\mathcal{T}_{co} = \{\mathbb{R}\} \cup \{A \subset \mathbb{R} : \mathbb{R} \setminus A \text{ is countable.}\}$. *Is* $\mathcal{T}_{co}$ *a topology on* $\mathbb{R}$*?*

Exercise 12. *Suppose that X is an infinite set with a topology in which every infinite set is open. Show that X has the discrete topology.*

Exercise 13. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Show that $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology. What about $\mathcal{T}_1 \cup \mathcal{T}_2$?*

Exercise 14. *Let $X = \{1, 2, 3\}$. We have drawn one topology $\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, X\}$ for X in the attached picture:*



How many topologies of X exist in total?