GENERAL TOPOLOGY
EXERCISES FOR SESSION 2 (WED 25.2)

Exercise 1. *Determine the closure of the set* $\{0\}$ *in the topology of* $\mathbb{R}$ *given by* $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$.

Exercise 2. *Provide an example where* $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Exercise 3. *Prove that* $\bar{A} = \bigcap \{F : A \subset F \text{ and } F \text{ is closed}\}$ *(Second part of Theorem 1.2).*

Exercise 4. *Let* $X = \{1, 2, 3\}$ *and* $S = \{\{1, 2\}, \{2, 3\}\}$. *Show that* S *is not a basis of any topology on* X. *Find the basis* $\mathcal{B}$ *generated by using* S *as a subbasis.*

Exercise 5. *Prove Lemma 1.3 in the lecture notes.*

Exercise 6. *Prove that* $\bar{A} = A \cup \operatorname{acc} A$.

Exercise 7. *Prove Theorem 1.4 in the lecture notes.*

Exercise 8. *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be bases for two topologies* $\mathcal{T}$ *and* $\mathcal{T}'$ *on* X. *Prove that* $\mathcal{T}'$ *is finer than* $\mathcal{T}$ *if and only if the following holds:*

For each $x \in X$ *and each* $B \in \mathcal{B}$ *such that* $x \in B$, *there is a* $B' \in \mathcal{B}'$ *such that* $x \in B' \subset B$.

Exercise 9. *Show that* $\mathcal{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{Q}\}$ *provides a basis for the standard topology on* $\mathbb{R}$.

Exercise 10. *Given a collection* $\mathcal{S}$ *of subsets of* X. *Let*

$$\mathcal{T} = \bigcap \{\mathcal{T}' : \mathcal{S} \subset \mathcal{T}' \text{ and } \mathcal{T}' \text{ is a topology of } X\}.$$

Check that this defines a topology $\mathcal{T}$ *on* X.

Exercise 11. *Let* $A \subset X$. *Prove that*

- $\partial\partial A \subset \partial A$.
- $\partial\partial A = \partial A$ *if* A *is closed.*
- $\partial\partial\partial A = \partial\partial A$.

Exercise 12. *Let us consider the following collection of subsets of the real line,* $\mathcal{B} = \{\mathbb{R}\} \cup \{\mathbb{Q} \cap (a, b) : a, b \in \mathbb{R}\}$. *Show that* $\mathcal{B}$ *is a basis for a topology on* $\mathbb{R}$. *What is the interior and boundary of* $(0, 1)$ *in this topology?*

Exercise 13. *Let us look at the subsets of* $\mathbb{R}^2$ *which are open half-spaces. The precise definition of an open half-space is a set of the form*

$$A = \{(x, y) \in \mathbb{R}^2 : ax + by > c\} \qquad \text{with} \qquad a, b, c \in \mathbb{R}, a \neq 0 \text{ or } b \neq 0.$$

What topology is defined by taking the open half-spaces as a subbasis on $\mathbb{R}^2$? *Geometric reasoning is fine here. What about if we use closed half-spaces instead?*