GENERAL TOPOLOGY
EXERCISES FOR SESSION 10 (WED 26.3)

This exercise session has some rehearsal questions. Practice your abilities in any of the previous topics of the course by picking the appropriate exercises:

Exercise 1. (Set theory) *Let $f : X \to Y$ and $A, B \subset Y$. Prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.*

Exercise 2. (Topology basics) *Prove that a set $A \subset X$ is closed if and only if $\partial A \subset A$.*

Exercise 3. (Relative topology) *Let $A \subset B \subset X$. We consider B as a topological space with the relative topology. Prove that the relative topology of A as a subspace of B is the same as the relative topology of A as a subspace of X.*

Exercise 4. (Bases) *Show that the collection of sets $\{[a, b) : a, b \in \mathbb{Q}\}$ is a basis for some topology on $\mathbb{R}$. Is this topology the same as the topology of $\mathbb{R}_\ell$?*

Exercise 5. (Continuous functions) *Let $f, g : X \to \mathbb{R}$ be continuous. Prove that the function $S(x) = f(x) + g(x)$ is continuous.*

Hint: *For example, first show that if $f(x)+g(x) > a$, then also $f(y)+g(y) > a$ for y in some neighbourhood of x.*

Exercise 6. (Homeomorphisms) *Fill in the details of this result used in the proof of Urysohn's metrization theorem: If $F : X \to Y$ is a homeomorphism and Y is metrizable, then X is metrizable.*

Exercise 7. (Product topology) *Show that the projection map $\mathrm{proj}_X : X \times Y \to X$ is an open map.*

Exercise 8. (Product topology) *Let $A_i \subset X_i$ be nonempty subsets for all $i \in I$. Prove that $\prod_{i \in I} A_i$ is closed in the product space $\prod_{i \in I} X_i$ if and only if A_i is closed in X_i for every $i \in I$.*

Exercise 9. (Connectedness) *Let S be a connected and dense subset in X, and let $S \subset A \subset X$. Show that A is connected.*

Exercise 10. (Connectedness) *Check whether the following subsets of $\mathbb{R}^2$ are connected:*

- $\mathbb{Q}^2$
- $\mathbb{R}^2 \setminus \mathbb{Q}^2$
- $\mathbb{Q}^2 \cup (\mathbb{R} \setminus \mathbb{Q})^2$ *(a bit more difficult)*

Exercise 11. (Countability axioms) *Prove that a separable metric space is second-countable.*