Triple pendulum

Lagrangian (exact calculation)

Kinetic energy

In[1]:=
```
Clear["Global`*"]
```

In[2]:=
$$T1 = 3\,m\,l^2\,\dot{\phi}_1{}^2;$$
$$T2 = m\,l^2\left(\dot{\phi}_1{}^2 + \dot{\phi}_2{}^2 + 2\,\dot{\phi}_1\,\dot{\phi}_2\,\mathrm{Cos}[\phi_1 - \phi_2]\right);$$
$$T3 = \frac{1}{2}\,m\,l^2\left(\dot{\phi}_1{}^2 + \dot{\phi}_2{}^2 + \dot{\phi}_3{}^2 + 2\,\dot{\phi}_1\,\dot{\phi}_2\,\mathrm{Cos}[\phi_1 - \phi_2] + 2\,\dot{\phi}_1\,\dot{\phi}_3\,\mathrm{Cos}[\phi_1 - \phi_3] + 2\,\dot{\phi}_2\,\dot{\phi}_3\,\mathrm{Cos}[\phi_2 - \phi_3]\right);$$
$$T = T1 + T2 + T3;$$
$$T\;//\;\mathrm{Simplify}$$

Out[6]=
$$\frac{1}{2}\,l^2\,m\left(9\,\dot{\phi}_1^2 + 3\,\dot{\phi}_2^2 + 2\,\mathrm{Cos}[\phi_2 - \phi_3]\,\dot{\phi}_2\,\dot{\phi}_3 + \dot{\phi}_3^2 + 2\,\dot{\phi}_1\left(3\,\mathrm{Cos}[\phi_1 - \phi_2]\,\dot{\phi}_2 + \mathrm{Cos}[\phi_1 - \phi_3]\,\dot{\phi}_3\right)\right)$$

Potential Energy

In[7]:=
$$U1 = -6\,m\,g\,l\,\mathrm{Cos}[\phi_1];$$
$$U2 = -2\,m\,g\,l\,(\mathrm{Cos}[\phi_1] + \mathrm{Cos}[\phi_2]);$$
$$U3 = -m\,g\,l\,(\mathrm{Cos}[\phi_1] + \mathrm{Cos}[\phi_2] + \mathrm{Cos}[\phi_3]);$$
$$U = U1 + U2 + U3;$$
$$U\;//\;\mathrm{Simplify}$$

Out[11]=
$$-g\,l\,m\,(9\,\mathrm{Cos}[\phi_1] + 3\,\mathrm{Cos}[\phi_2] + \mathrm{Cos}[\phi_3])$$

Lagrangian

In[12]:=
$$L = \mathrm{Simplify}[T] - \mathrm{Simplify}[U]$$

Out[12]=
$$g\,l\,m\,(9\,\mathrm{Cos}[\phi_1] + 3\,\mathrm{Cos}[\phi_2] + \mathrm{Cos}[\phi_3]) + \frac{1}{2}\,l^2\,m\left(9\,\dot{\phi}_1^2 + 3\,\dot{\phi}_2^2 + 2\,\mathrm{Cos}[\phi_2 - \phi_3]\,\dot{\phi}_2\,\dot{\phi}_3 + \dot{\phi}_3^2 + 2\,\dot{\phi}_1\left(3\,\mathrm{Cos}[\phi_1 - \phi_2]\,\dot{\phi}_2 + \mathrm{Cos}[\phi_1 - \phi_3]\,\dot{\phi}_3\right)\right)$$

Small angle approximation

Kinetic energy

In[13]:=
$$T1Approx = 3\,m\,l^2\,\dot{\phi}_1{}^2;$$

In[14]:=
$$T2Approx = m\,l^2\left(\dot{\phi}_1{}^2 + \dot{\phi}_2{}^2 + 2\,\dot{\phi}_1\,\dot{\phi}_2\right);$$

In[15]:=
```
T3Approx = 1/2 m l^2 (φ̇1^2 + φ̇2^2 + φ̇3^2 + 2 φ̇1 φ̇2 + 2 φ̇1 φ̇3 + 2 φ̇2 φ̇3);
```

$$\texttt{T3Approx} = \frac{1}{2} m\, l^2 \left(\dot\phi_1^2 + \dot\phi_2^2 + \dot\phi_3^2 + 2\dot\phi_1\dot\phi_2 + 2\dot\phi_1\dot\phi_3 + 2\dot\phi_2\dot\phi_3\right);$$

In[16]:=
```
TApprox = T1Approx + T2Approx + T3Approx;
TApprox // Simplify
```

Out[17]=

$$\frac{1}{2} l^2 m \left(9\dot\phi_1^2 + 3\dot\phi_2^2 + 2\dot\phi_2\dot\phi_3 + \dot\phi_3^2 + 2\dot\phi_1\left(3\dot\phi_2 + \dot\phi_3\right)\right)$$

Potential energy

In[18]:=

$$\texttt{U1Approx} = 3\, m\, g\, l\, \phi_1^2;$$

In[19]:=

$$\texttt{U2Approx} = m\, g\, l\left(\phi_1^2 + \phi_2^2\right);$$

In[20]:=

$$\texttt{U3Approx} = \frac{1}{2} m\, g\, l\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right);$$

In[21]:=
```
UApprox = U1Approx + U2Approx + U3Approx;
UApprox // Simplify
```

Out[22]=

$$\frac{1}{2} g\, l\, m\left(9\phi_1^2 + 3\phi_2^2 + \phi_3^2\right)$$

“Mass” and “spring-constant” matrices

In[23]:=

$$\texttt{M} = m\, l^2 \{\{9, 3, 1\}, \{3, 3, 1\}, \{1, 1, 1\}\};$$

In[24]:=
```
M // MatrixForm
```

Out[24]//MatrixForm=

$$\begin{pmatrix} 9 l^2 m & 3 l^2 m & l^2 m \\ 3 l^2 m & 3 l^2 m & l^2 m \\ l^2 m & l^2 m & l^2 m \end{pmatrix}$$

In[25]:=
```
K = m g l {{9, 0, 0}, {0, 3, 0}, {0, 0, 1}};
K // MatrixForm
```

Out[26]//MatrixForm=

$$\begin{pmatrix} 9 g l m & 0 & 0 \\ 0 & 3 g l m & 0 \\ 0 & 0 & g l m \end{pmatrix}$$

Characteristic equation

In[27]:=
```
d = Det[(K - Ω M)]
```

Out[27]=

$$27 g^3 l^3 m^3 - 81 g^2 l^4 m^3 \Omega + 60 g l^5 m^3 \Omega^2 - 12 l^6 m^3 \Omega^3$$

In[28]:=
```
s = Solve[d == 0, Ω]
```

Out[28]=

$$\left\{\left\{\Omega \to \frac{g}{2 l}\right\}, \left\{\Omega \to \frac{3 g}{2 l}\right\}, \left\{\Omega \to \frac{3 g}{l}\right\}\right\}$$

Equations of motion

In[29]:=
```
NullSpace[(K - Ω M) /. s[[1]]]
```
Out[29]=

$$\left\{\left\{\frac{1}{3}, \frac{2}{3}, 1\right\}\right\}$$

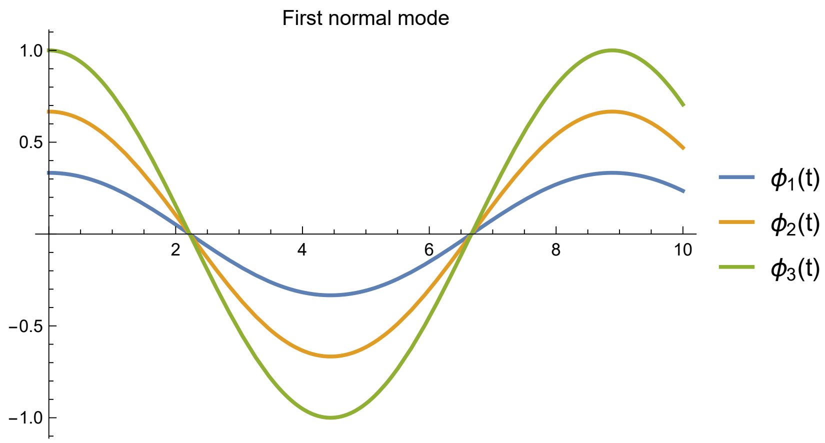
In[30]:=
```
NullSpace[(K - Ω M) /. s[[2]]]
```
Out[30]=

$$\left\{\left\{-\frac{1}{3}, 0, 1\right\}\right\}$$

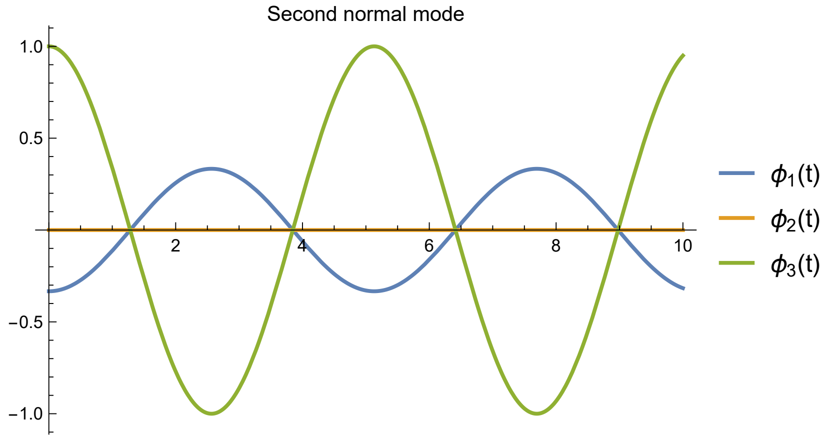
In[31]:=
```
NullSpace[(K - Ω M) /. s[[3]]]
```
Out[31]=

$$\left\{\left\{\frac{1}{3}, -1, 1\right\}\right\}$$

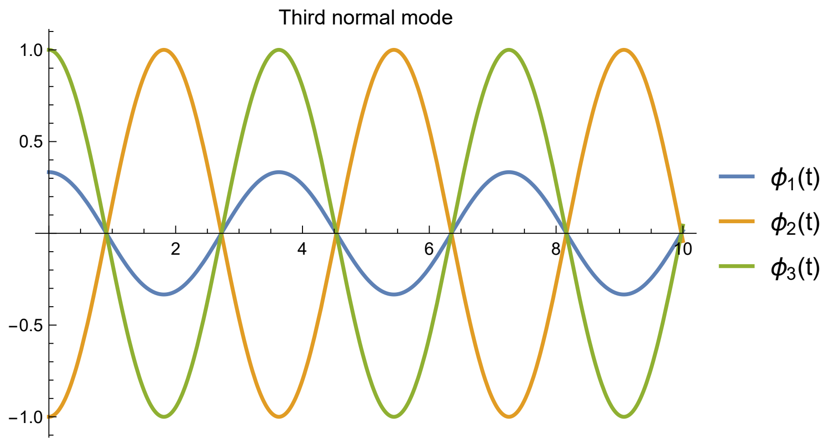
In[32]:=
```
a1 = NullSpace[(K - Ω M) /. s[[1]]][[1]];
a2 = NullSpace[(K - Ω M) /. s[[2]]][[1]];
a3 = NullSpace[(K - Ω M) /. s[[3]]][[1]];
```
In[35]:=
```
ω1 = Sqrt[Ω] /. s[[1]];
ω2 = Sqrt[Ω] /. s[[2]];
ω3 = Sqrt[Ω] /. s[[3]];
```
In[38]:=
```
ϕ1[t_, A_, δ_] = A a1 Cos[ω1 t - δ];
```
In[39]:=
```
legendToPlot = {"ϕ1(t)", "ϕ2(t)", "ϕ3(t)"};
tMax = 10;
```
In[41]:=
```
Plot[Evaluate@(ϕ1[t, A, δ] /. {g → 1, l → 1, A → 1, δ → 0}),
 {t, 0, tMax}, PlotLegends → legendToPlot, PlotLabel → "First normal mode"]
```
Out[41]=



In[42]:=
```
ϕ2[t_, A_, δ_] = A a2 Cos[ω2 t - δ];
Plot[Evaluate@(ϕ2[t, A, δ] /. {g → 1, l → 1, A → 1, δ → 0}),
 {t, 0, tMax}, PlotLegends → legendToPlot, PlotLabel → "Second normal mode"]
```
Out[43]=



In[44]:=
```
ϕ3[t_, A_, δ_] = A a3 Cos[ω3 t - δ];
Plot[Evaluate@(ϕ3[t, A, δ] /. {g → 1, l → 1, A → 1, δ → 0}),
 {t, 0, tMax}, PlotLegends → legendToPlot, PlotLabel → "Third normal mode"]
```
Out[45]=



Lagrange’s equations

In[46]:=
```
Lf[t_] =
  L /. {ϕ̇1 → ϕ1'[t], ϕ̇2 → ϕ2'[t], ϕ̇3 → ϕ3'[t], ϕ1 → ϕ1[t], ϕ2 → ϕ2[t], ϕ3 → ϕ3[t]};
```

In[47]:=
```
lhs1 = D[D[Lf[t], ϕ1'[t]], t];
lhs1 // Simplify
```
Out[48]=

$$l^2 m \left(-3 \,\mathrm{Sin}[\phi_1[t] - \phi_2[t]] \,(\phi_1'[t] - \phi_2'[t])\, \phi_2'[t] - \mathrm{Sin}[\phi_1[t] - \phi_3[t]] \,(\phi_1'[t] - \phi_3'[t])\, \phi_3'[t] + 9 \phi_1''[t] + 3 \,\mathrm{Cos}[\phi_1[t] - \phi_2[t]]\, \phi_2''[t] + \mathrm{Cos}[\phi_1[t] - \phi_3[t]]\, \phi_3''[t]\right)$$

In[49]:=
```
rhs1 = D[Lf[t], ϕ1[t]];
```

In[50]:=
```
lhs2 = D[D[Lf[t], ϕ2'[t]], t];
lhs2 // Simplify
```

Out[51]=

$$\frac{1}{2} l^2 m \left(-6 \operatorname{Sin}[\phi_1[t]-\phi_2[t]]\, \phi_1'[t]\, (\phi_1'[t]-\phi_2'[t]) - 2 \operatorname{Sin}[\phi_2[t]-\phi_3[t]]\, (\phi_2'[t]-\phi_3'[t])\, \phi_3'[t] + 6 \operatorname{Cos}[\phi_1[t]-\phi_2[t]]\, \phi_1''[t] + 6\, \phi_2''[t] + 2 \operatorname{Cos}[\phi_2[t]-\phi_3[t]]\, \phi_3''[t]\right)$$

In[52]:=
```
rhs2 = D[Lf[t], ϕ2[t]]
```

Out[52]=

$$-3\, g\, l\, m \operatorname{Sin}[\phi_2[t]] + \frac{1}{2} l^2 m \left(6 \operatorname{Sin}[\phi_1[t]-\phi_2[t]]\, \phi_1'[t]\, \phi_2'[t] - 2 \operatorname{Sin}[\phi_2[t]-\phi_3[t]]\, \phi_2'[t]\, \phi_3'[t]\right)$$

In[53]:=
```
lhs3 = D[D[Lf[t], ϕ3'[t]], t];
lhs3 // Simplify
```

Out[54]=

$$l^2 m \left(-\operatorname{Sin}[\phi_1[t]-\phi_3[t]]\, \phi_1'[t]\, (\phi_1'[t]-\phi_3'[t]) - \operatorname{Sin}[\phi_2[t]-\phi_3[t]]\, \phi_2'[t]\, (\phi_2'[t]-\phi_3'[t]) + \operatorname{Cos}[\phi_1[t]-\phi_3[t]]\, \phi_1''[t] + \operatorname{Cos}[\phi_2[t]-\phi_3[t]]\, \phi_2''[t] + \phi_3''[t]\right)$$

In[55]:=
```
rhs3 = D[Lf[t], ϕ3[t]]
```

Out[55]=

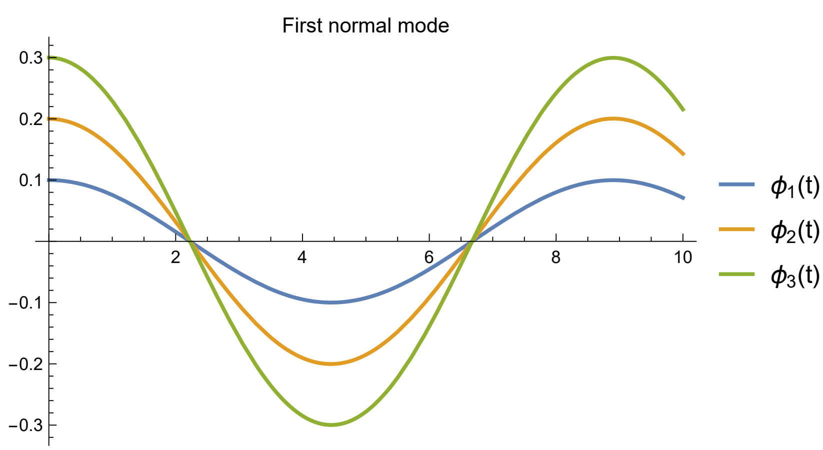
$$-g\, l\, m \operatorname{Sin}[\phi_3[t]] + \frac{1}{2} l^2 m \left(2 \operatorname{Sin}[\phi_1[t]-\phi_3[t]]\, \phi_1'[t]\, \phi_3'[t] + 2 \operatorname{Sin}[\phi_2[t]-\phi_3[t]]\, \phi_2'[t]\, \phi_3'[t]\right)$$

In[56]:=
```
subs = { l → 1, g → 1, m → 1};
eq1 = (lhs1 == rhs1) /. subs ;
eq2 = (lhs2 == rhs2) /. subs ;
eq3 = (lhs3 == rhs3) /. subs ;
tMax = 10;
```

In[61]:=
```
sl1 = NDSolve[{eq1, eq2, eq3, ϕ1[0] == 0.1, ϕ2[0] == 0.2,
    ϕ3[0] == 0.3, ϕ1'[0] == ϕ2'[0] == ϕ3'[0] == 0}, {ϕ1, ϕ2, ϕ3}, {t, 0, tMax}];
```

In[62]:=
```
Plot[Evaluate[{ϕ1[t], ϕ2[t], ϕ3[t]} /. sl1], {t, 0, tMax},
 PlotLegends → legendToPlot, PlotLabel → "First normal mode"]
```

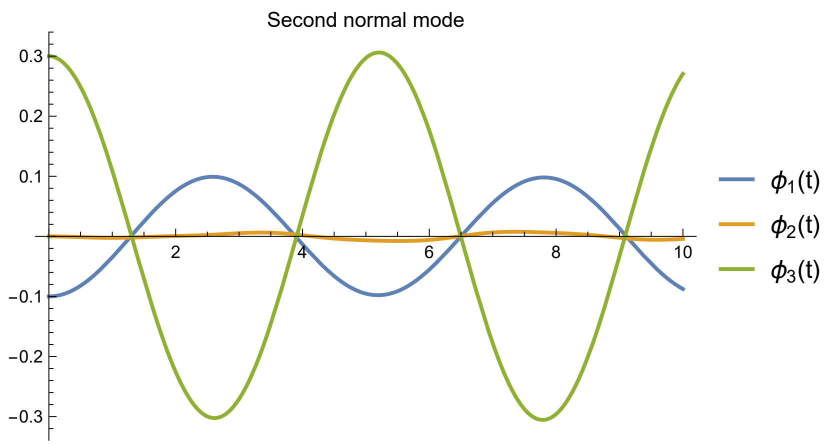
Out[62]=



In[63]:=
```
sl2 = NDSolve[{eq1, eq2, eq3, ϕ₁[0] == -0.1, ϕ₂[0] == 0,
    ϕ₃[0] == 0.3, ϕ₁'[0] == ϕ₂'[0] == ϕ₃'[0] == 0}, {ϕ₁, ϕ₂, ϕ₃}, {t, 0, tMax}];
```

In[64]:=
```
Plot[Evaluate[{ϕ₁[t], ϕ₂[t], ϕ₃[t]} /. sl2], {t, 0, tMax},
  PlotLegends → legendToPlot, PlotLabel → "Second normal mode"]
```

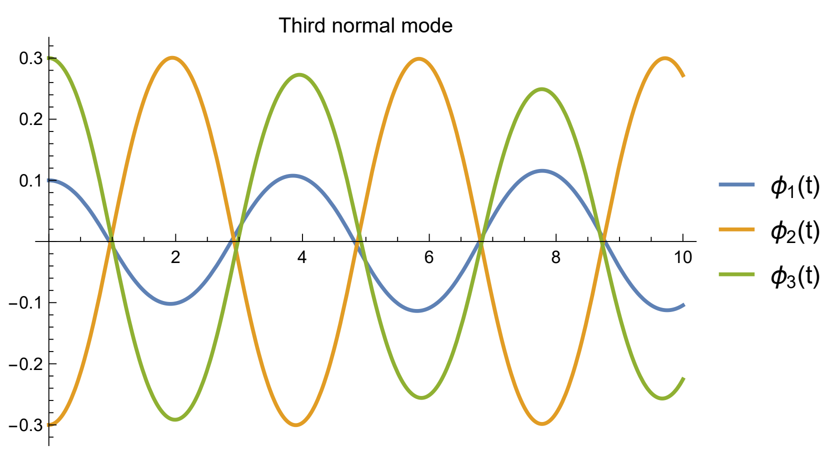
Out[64]=



In[65]:=
```
sl3 = NDSolve[{eq1, eq2, eq3, ϕ₁[0] == 0.1, ϕ₂[0] == -0.3,
    ϕ₃[0] == 0.3, ϕ₁'[0] == ϕ₂'[0] == ϕ₃'[0] == 0}, {ϕ₁, ϕ₂, ϕ₃}, {t, 0, tMax}];
```

In[66]:=
```
Plot[Evaluate[{ϕ₁[t], ϕ₂[t], ϕ₃[t]} /. sl3], {t, 0, tMax},
  PlotLegends → legendToPlot, PlotLabel → "Third normal mode"]
```

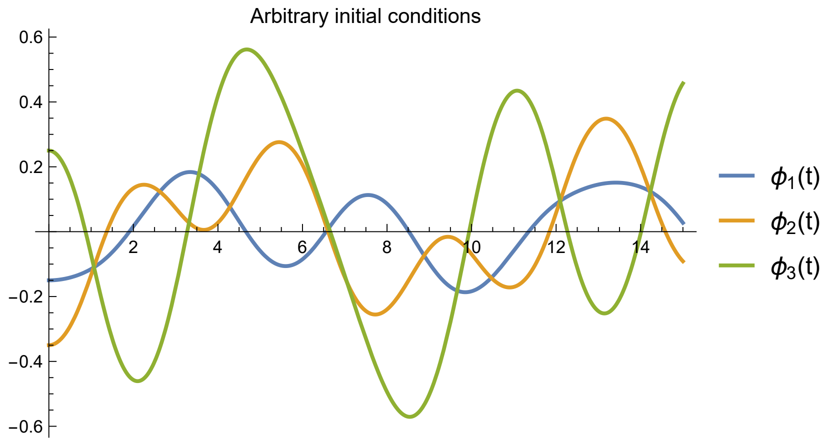
Out[66]=



In[67]:=
```
tMaxAnim = 15;
sl = NDSolve[{eq1, eq2, eq3, ϕ₁[0] == -0.15, ϕ₂[0] == -0.35, ϕ₃[0] == 0.25,
    ϕ₁'[0] == ϕ₂'[0] == ϕ₃'[0] == 0}, {ϕ₁, ϕ₂, ϕ₃}, {t, 0, tMaxAnim}];
```

In[69]:=
```
Plot[Evaluate[{ϕ1[t], ϕ2[t], ϕ3[t]} /. sl], {t, 0, tMaxAnim},
  PlotLegends → legendToPlot, PlotLabel → "Arbitrary initial conditions"]
```

Out[69]=

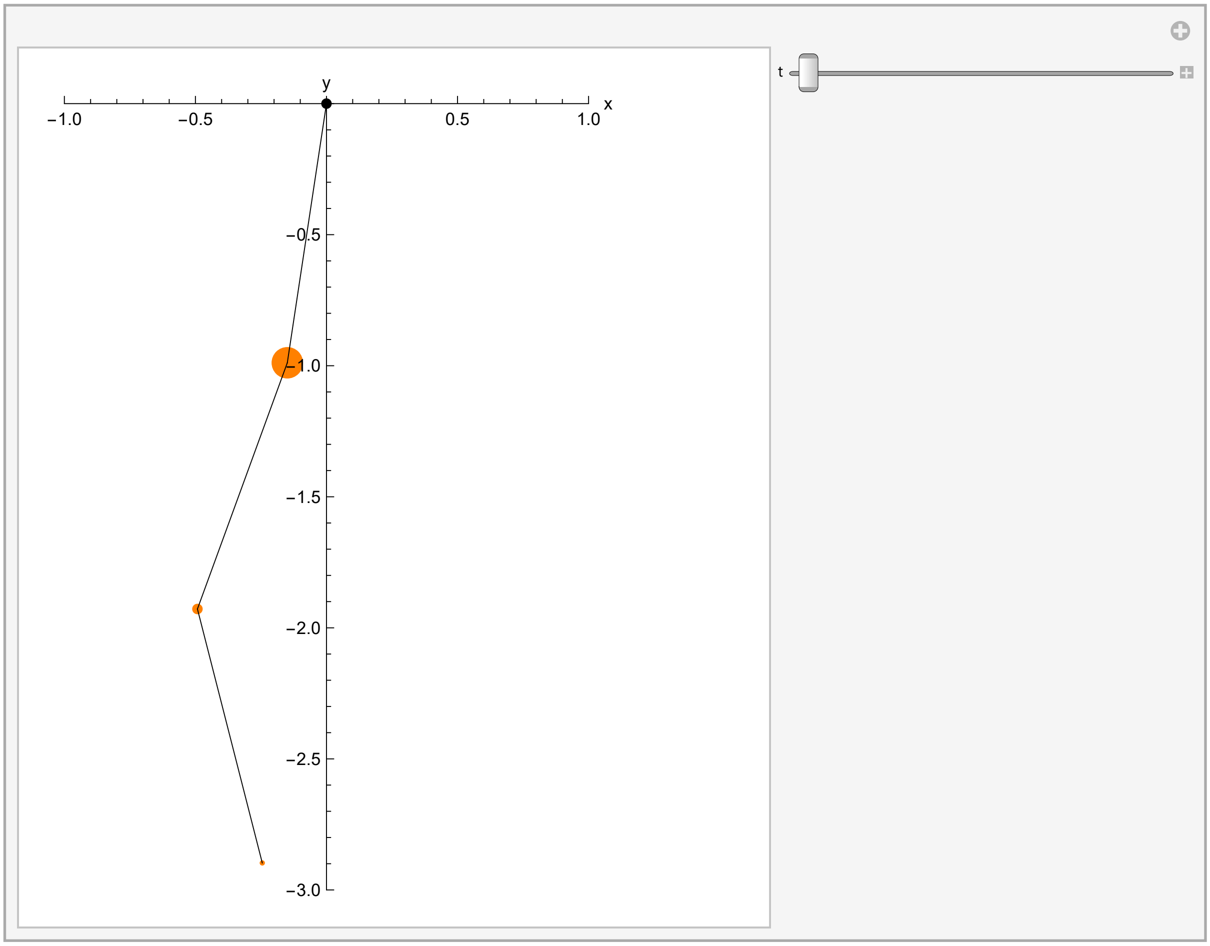


Animation

In[70]:=
```
x1[ϕ1_, ϕ2_, ϕ3_] = Sin[ϕ1];
y1 [ϕ1_, ϕ2_, ϕ3_] = - Cos[ϕ1];
x2[ϕ1_, ϕ2_, ϕ3_] = Sin[ϕ1] + Sin[ϕ2];
y2 [ϕ1_, ϕ2_, ϕ3_] = - Cos[ϕ1] - Cos[ϕ2];
x3[ϕ1_, ϕ2_, ϕ3_] = Sin[ϕ1 ] + Sin[ϕ2] + Sin[ϕ3];
y3 [ϕ1_, ϕ2_, ϕ3_] = - Cos[ϕ1] - Cos[ϕ2] - Cos[ϕ3];
```

In[75]:=
```
Manipulate[
 Module[
  {phiValues, x1Val, y1Val, x2Val, y2Val, x3Val, y3Val},
  (*Precompute values to avoid repeated calculations*)
  phiValues = {Subscript[ϕ, 1][t], Subscript[ϕ, 2][t], Subscript[ϕ, 3][t]} /. sl;
  x1Val = x1 @@ phiValues[[1]];
  y1Val = y1 @@ phiValues[[1]];
  x2Val = x2 @@ phiValues[[1]];
  y2Val = y2 @@ phiValues[[1]];
  x3Val = x3 @@ phiValues[[1]];
  y3Val = y3 @@ phiValues[[1]];
  Show[
   Graphics[{
     Orange,
     Disk[{x3Val, y3Val}, 0.01],
     Disk[{x2Val, y2Val}, 0.02],
     Disk[{x1Val, y1Val}, 0.06],
     Black,
     Line[{{x3Val, y3Val}, {x2Val, y2Val}}],
     Line[{{x2Val, y2Val}, {x1Val, y1Val}}],
     Line[{{x1Val, y1Val}, {0, 0}}],
     Disk[{0, 0}, 0.02]
    }],
   AxesLabel → {"x", "y"},
   Axes → True,
   PlotRange → {{-1, 1}, {0, -3}}
  ]
 ],
 {t, 0, tMaxAnim}
]
```

Out[75]=



In[76]:=
```
frames = Table[Module[
    {phiValues, x1Val, y1Val, x2Val, y2Val, x3Val, y3Val},
    (*Precompute values to avoid repeated calculations*)
    phiValues = {Subscript[ϕ, 1][t], Subscript[ϕ, 2][t], Subscript[ϕ, 3][t]} /. sl;
    x1Val = x1 @@ phiValues[[1]];
    y1Val = y1 @@ phiValues[[1]];
    x2Val = x2 @@ phiValues[[1]];
    y2Val = y2 @@ phiValues[[1]];
    x3Val = x3 @@ phiValues[[1]];
    y3Val = y3 @@ phiValues[[1]];
    Show[
     Graphics[{
       Orange,
       Disk[{x3Val, y3Val}, 0.01],
       Disk[{x2Val, y2Val}, 0.02],
       Disk[{x1Val, y1Val}, 0.06],
       Black,
       Line[{{x3Val, y3Val}, {x2Val, y2Val}}],
       Line[{{x2Val, y2Val}, {x1Val, y1Val}}],
       Line[{{x1Val, y1Val}, {0, 0}}],
       Disk[{0, 0}, 0.02]
      }],
     AxesLabel → {"x", "y"},
     Axes → True,
     PlotRange → {{-1, 1}, {0, -3}}
    ]
   ],
   {t, 0, tMaxAnim, 0.035} (*Adjust step size to control frame rate*)];

Export["animation.mp4", frames, "FrameRate" → 30,
 ImageResolution → 1000, Antialiasing → True]
```

Out[77]=

```
animation.mp4
```