Risk, rare events and extremes: a brief overview on Multivariate Extremes

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Structure of semester project

1. Learning main results about extreme theory

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   - Lectures of the course *Risk, rare events and extremes*
   - Slides
   - Exercises and solutions
   - Weekly meetings

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2. Project

Univariate extremes

Let $X_1, \dots, X_n$ be a sequence of *iid* random variables with common distribution function F. One way of characterising extremes is by considering maxima $M_n = \max(X_1, \dots, X_n)$.

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\begin{aligned}
P(M_n \leq x) &= P(X_1 \leq x, \dots, X_n \leq x) \\
&= P(X_1 \leq x) \cdot \dots \cdot P(X_1 \leq x) \\
&= F(x)^n.
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But, as $n \to \infty$,

$$
F(x)^n \longrightarrow \begin{cases} 0 & \textit{if } F(x) < 1 \\ 1 & \textit{if } F(x) = 1, \end{cases}
$$

so $M_n \xrightarrow{\text{D}} x^*$, where $x^* = \sup\{x \ : \ F(x) < 1\}$ is the upper support point of F.

Extremal Types Theorem

Theorem (Extremal Types Theorem)

Let $M_n = \max(X_1, \ldots, X_n)$ be the maximum of a random sample $X_1, \ldots, X_n$. If there exist sequences of real numbers $\{a_n\} > 0$ and $\{b_n\} > 0$ such that the centered and scaled sample maximum $(M_n - b_n)/a_n$, has a non degenerate limiting distribution G, then this must be the **generalized extreme-value distribution** *(GEV), i.e.*

$$G(x) = \begin{cases} \exp\left[-\{1+\xi(x-\eta)/\tau\}_+^{-1/\xi}\right] & \text{if } \xi \neq 0 \\ \exp\left[-\exp\{(x-\eta)/\tau\}\right] & \text{if } \xi = 0 \end{cases} \qquad \forall x \in \mathbb{R},$$

for any real a and with $\xi, \eta \in \mathbb{R}$ and $\tau > 0$. Equivalently, $(M_n - b_n)/a_n \xrightarrow{D} Z$ as $n \to \infty$, where Z has distribution function G.

Multivariate extremes

- Let $X_1, \dots, X_n$ be a sequence of *iid* r.v., with $X_i = (X_{i,1}, \dots, X_{i,D}) \in \mathbb{R}^D$. We define the vector of **componentwise maxima**

$$M_n = (M_{X_1,n}, \dots, M_{X_D,n}),$$

where

$$M_{X_j,n} = \max_{i=1,\dots,n} \{X_{i,j}\}.$$

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- The aim is to extend the univariate case, asking if there exist a non-degenerate limiting distributions describing the joint behaviour of maxima. This presupposes that limiting distributions exist also for the rescaled margins individually, otherwise any limiting joint distribution will be degenerate.

Copulas, margins and extremes

- As limiting distribution, we will use the **simple** joint distribution, i.e. with **unit Fréchet margins**.

Copulas, margins and extremes

- As limiting distribution, we will use the **simple** joint distribution, i.e. with **unit Fréchet margins**.
- Firstly, take $X = (X_1, \ldots, X_D)$, we transform the components to unit Fréchet variables via

$$Z_d = -\frac{1}{\log F_d(X_d)} \quad d = 1, \ldots, D, \quad X_d \sim F_d.$$

- If there are n independent replicates of X, the corresponding transformed marginal variables satisfy

$$\begin{aligned} P\left\{n^{-1} \max_{j=1,\ldots,n} Z_{j,d} \leq z_d\right\} &= P\left\{Z_{1,d} \leq nz_d, \ldots, Z_{n,d} \leq nz_d\right\} \\ &= P(Z_{1,d} \leq nz_d)^n \\ &= \exp(-1/z_d). \end{aligned}$$

Copulas, margins and extremes

- The class of limiting multivariate distributions does not depend on this marginal transformation, but it simply modifies the limiting variable by the transformation

$$Z_d \mapsto t_d(Z_d) = (1 + \xi_d Z_d)_+^{1/\xi_d} \sim \textit{unit Fréchet},$$

replacing the limiting joint extreme-value distribution $G(z_1, \ldots, z_D)$ by $C_F\{t_1(z_1), \ldots, t_D(z_D)\}$, where the copula C_F has unit Fréchet margins.

Limit distribution of componentwise maxima

Theorem

If $X_1, X_2, \ldots,$ are independent copies of a D-dimensional random variable whose componentwise maxima can be linearly renormalised to converge as $n \to \infty$ to a r.v. $Z = (Z_1, \ldots, Z_D)$ that has a non-degenerate distribution with unit Fréchet margins, then for any $z_1, \ldots, z_D > 0$

$$P(Z_1 \leq z_1, \ldots, Z_D \leq z_D) = \exp\left[-DE\left\{\max_{d=1,\ldots,D}(W_d/z_D)\right\}\right],$$

where $W = (W_1, \ldots, W_D)$ is called **angular variable** *with* **angular distribution** ν *on the $(D-1)$-dimensional simplex, i.e.*

$$W \in \mathcal{S}_{D-1} = \left\{(w_1, \ldots, w_D) \ : \ w_d \geq 0, \ \sum_d w_d = 1\right\}$$

and satisfies $E(W_d) = 1/D$ for any $d = 1, \ldots, D$.

Limit distribution of componentwise maxima

- We refer the following as the **exponent function**

$$V(z_1, \dots, z_D) = DE\left\{\max_{d=1,\dots,D}(W_d/z_D)\right\}.$$

- The function V is **homogeneous of order -1**, i.e.

$$V(tz_1, \dots, tz_D) = t^{-1}V(z_1, \dots, z_D), \quad z_1, \dots, z_D > 0,\ t > 0.$$

- The event $Z \leq z$ is equivalent to the set $A_z = \{x \in \mathbb{R}_+^D : x \not\leq z\}$ containing no points of Poisson process $\mathcal{P} = \{Q_j : j = 1, 2, \dots\}$ on $\mathcal{E} = [0, \infty)^D - \{0\}$ with mean measure

$$\mu(A_z) = V(z), \ \ z \in \mathcal{E},$$

where $Q_j = R_j W_j$ are called **extremal functions** and $R_j \in (0, \infty)$ have intensity D/r^2 independent of $W_j \overset{iid}{\sim} \nu$.

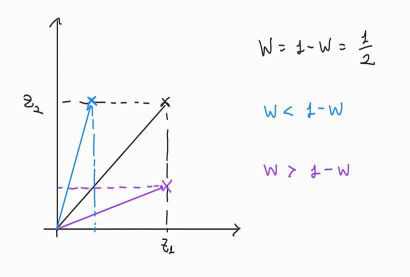
Bivariate maxima

Consider the case $D = 2$.

- We make a **pseudo-polar transformation** from (Z_1, Z_2) to a scalar variable $R > 0$ and the angular variable (W_1, W_2) with $W_1 = 1 - W_2 =: W$, since it lies on the $1-$dimensional simplex. I.e. consider the transformation $T : \mathcal{E} \longrightarrow (0, \infty) \times \mathcal{S}_2$ defined by

$$T(z_1, z_2) = (r, (w, 1 - w)), \quad r = z_1 + z_2, \quad w = z_1/r.$$

and with inverse $T^{-1}(r, (w, 1 - w)) = (rw, r(1 - w))$.



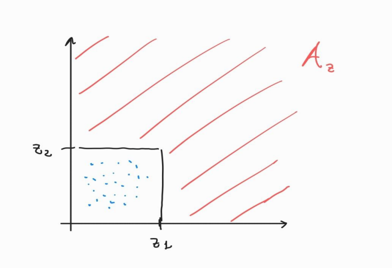
Bivariate maxima

- $$(rw, r(1-w)) \in A_z \;\Leftrightarrow\; \max\left(\frac{rw}{z_1}, \frac{r(1-w)}{z_2}\right) > 1$$

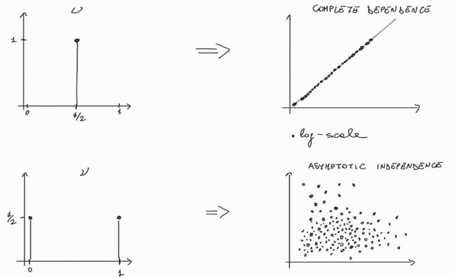
- By homogeneity

$$\mu(A_z) = V((z_1, z_2)) = V((rw, r(1-w)) = \frac{1}{r} V((w, 1-w)) = \frac{1}{r}\nu(w)$$

which means that the intensity is given by $\frac{1}{r^2} dr \times \nu(dw)$ and so R and W are independent.



Summary of dependence



- An important summary of strength of dependence is the **extremal coefficient**

$$\theta = V(1,1),$$

that can interpreted as the "number of independent maxima" contribuiting to Z and $\theta = 1$ for perfectly dependent data, $\theta = 2$ for independent data.

- θ can also be interpreted in terms of limiting conditional probabilities of rare events, since

$$2 - \theta = \lim_{z \to \infty} P(Z_1 > z \mid Z_2 > z) = \chi,$$

where χ is the **extremal correlation** and if $\chi > 0$ Z_1 and Z_2 are **asymptotically dependent**, if $\chi = 0$ they are **asymptotically independent**.

- References: Lecture notes of the course *"Risk, Rare events and extremes"*, prof. Anthony Davison.

THANK YOU FOR YOUR ATTENTION

Bivariate Hüsler–Reiss model

- The bivariate Hüsler–Reiss model depends on a scalar parmeter $\lambda > 0$ and exponent function

$$V(z_1, z_2) = \frac{1}{z_1}\Phi\left\{\frac{\lambda}{2} + \frac{1}{\lambda} log\left(\frac{z_2}{z_1}\right)\right\} + \frac{1}{z_2}\Phi\left\{\frac{\lambda}{2} + \frac{1}{\lambda} log\left(\frac{z_1}{z_2}\right)\right\}, \quad z_1, z_2 > 0,$$

  where Φ denotes the standard normal distribution function.

- If $\lambda \to \infty$, then

$$V(z_1, z_2) \to \frac{1}{z_1} + \frac{1}{z_2} \qquad (indipendence),$$

  While, if $\lambda \to 0$, then

$$V(z_1, z_2) \to \frac{1}{min(z_1, z_2)} \qquad (total\ dependence).$$

Bivariate Hüsler–Reiss model

- Using the following formula for angular density

$$\dot{\nu}(w) = -\frac{r^{D+1}}{D}\frac{\partial^D V(z)}{\partial z_1 \cdots \partial z_D}\bigg|_{z=rw},$$

it turns out that

$$\dot{\nu}(w) = \frac{e^{-\lambda^2/8}}{2\lambda\{w(1-w)\}^{3/2}}\phi\left\{\frac{1}{\lambda}log\left(\frac{w}{1-w}\right)\right\}, \quad 0 < w < 1.$$

- Taking $I = \pm 1$ with equal probabilities, and taking $\epsilon \sim \mathcal{N}(0,1)$ independent of I, we have that $W = \{1 + \exp(\lambda\epsilon + \lambda^2 I/2)\}^{-1}$ has density $\dot{\nu}$ and it is possible to use this result to simulate bivariate Hüsler–Reiss variables.